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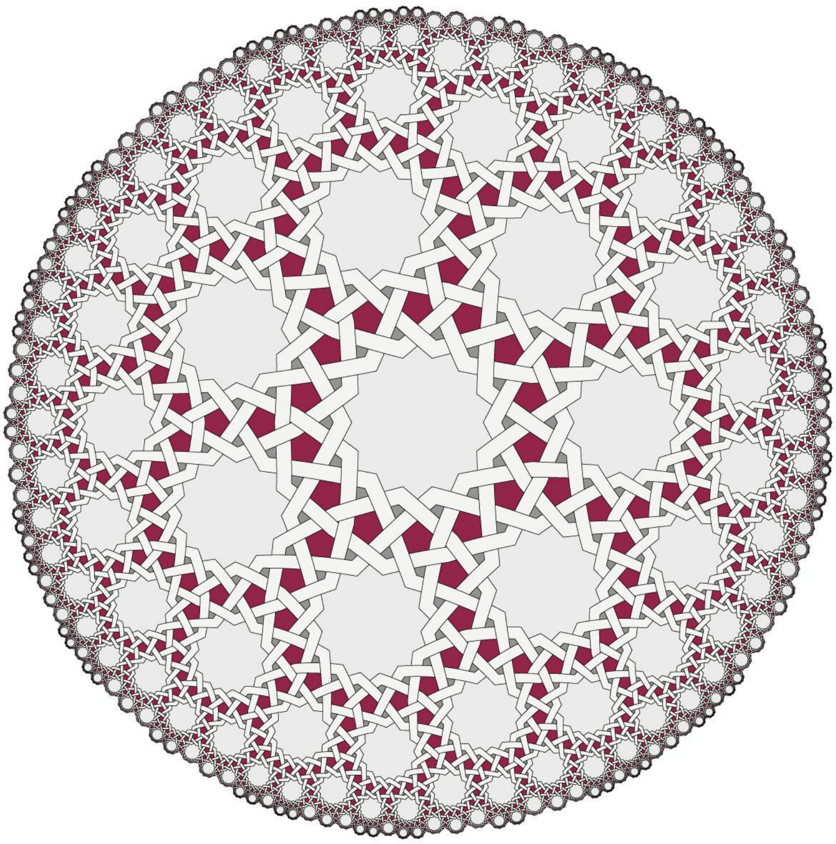
Hyperbolic geometry



Bachelor’s thesis in Mathematics

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Abstract

This paper concerns hyperbolic geometry as described by the Poincaré disk model. In addition three other models (Poincaré half-plane, Beltrami-Klein-Hilbert and Minkowski model) as well as their equivalences will be discussed.

Contents

Chapter 1

Introduction

Imagine your worst nightmare: you are walking to some edge but instead of getting closer it seems that every step takes more and more time. The best way out of such a nightmare is to understand that this cannot be. Unfortunately I am going to show you that this nightmare scenario can be realized. And it has a name: Hyperbolic geometry.

Hyperbolic geometry gives a different definition of straight lines, distances, areas and many other notions from common (Euclidean) geometry. Besides many differences, there are also some similarities between this geometry and Euclidean geometry, the geometry we all know and love, like the isosceles triangle theorem.

The essence of hyperbolic geometry is the difference in the fifth postulate of Euclidean geometry. Where we can find only one parallel to a line through a point not on that line in Euclidean geometry, we can find infinitely many in hyperbolic geometry.

The objective of this thesis is to get a better idea of hyperbolic geometry and of several models that describe it. This document begins with a literature study: an historical overview of this new geometry (chapter 2) and a specification of four models for this geometry (chapter 3 and 4). Subsequently there has been done some research to verify that the four models described are equivalent to each other (chapter 5).

M.C. Escher's 'Circle Limit III'

Chapter 2

Historical overview

In 300 BC. Euclid wrote his masterpiece 'Elements', consisting of thirteen books. In the first book he spoke about five postulates which form the basis of geometry. But instead of using all these five postulates, he only used the first four to prove the first twenty-eight propositions. This is why historians speculate that he had his doubts about the fifth one. A literal translation of these the postulates (with the exception of the fifth) is:

1. To draw a straight line from any point to any point
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to another.
5. (Parallel Postulate) Through a point outside of an infinitely long line there is only one infinitely long line that does not cut the first line.

This Parallel Postulate is known as Playfair's postulate after the Scottish physicist and mathematician John Playfair (1748-1819). It is the most used alternative for Euclides' real fifth postulate: That, if a straight line on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

Many years later Proclus (410-485) wrote a commentary on Euclid's first book. This was the first documented wrangling with the fifth postulate. Proclus stated that people were trying to prove the dependence of the postulate almost immediately after Eulclid's books. Like them, he also tried to prove that this postulate could be proven by the first four. So that he could prove that this postulate was not a postulate, but a proposition.

He tried it in many ways, but most important was the one in which he changed the definition of parallel lines. Instead of defining them as straight lines that do never meet each other, he stated that they were everywhere equidistant. If he wanted to use this definition, he first needed to prove that if two straight lines never meet, they are indeed equidistant. To do this, he unfortunately needed the fifth postulate. During the Middle Ages and the Renaissance many mathematicians tried to prove what Proclus failed to do but none of their proves turned out to be true. They always used some kind of assumption that was just as strong as the Parallel Postulate.

In 1733 Girolamo Saccheri (1667-1733) published his book 'Euclides ab omni naevo vindicatus' (Euclid freed of every flaw). He tried to prove the dependence of the fifth postulate by contradiction: he assumed it did not depend on the other ones and hoped to derive a contradiction. He tried this using a quadrilateral $ABCD$, with $\angle A$ and $\angle B$ having right angles and $AD = BC$. He proved that in this case $\angle C$ and $\angle D$ must be congruent to each other. Then there were three possibilities: $\angle C$ and $\angle D$ were right, obtuse or acute angles. He proved that if one of them is true for this quadrilateral, it is true for every quadrilateral. He tried to eliminate the hypothesizes of the angles being obtuse or acute. He found a contradiction for the angles to be obtuse, but later we will see that this contradiction is not always valid. Unfortunately he failed to find a contradiction for the angles to be acute.

Though he did not find a valid contradiction to prove the dependence of the fifth postulate, the conclusions he derived from the first four postulates and the negation of the fifth one turned out to become the basis of non-Euclidean geometry. If he had put less faith in the one thing he was trying to prove and had payed more attention to his reasonings, he would have discovered non-Euclidean years before it was invented.

Johann H. Lambert (1728-1777) followed Saccheri's method, but instead of using a quadrangle with two, he used one with three right angles. He likewise distinguished three possibilities: the fourth angle is right, obtuse or acute. Proving that the second case does not occur was not so hard, but he also had problems with the third one. However he was not able to prove this one, he came up with some other conclusions. Like Saccheri, he showed that the sum of the angles of a triangle is less than or equal to two right angles but the also showed that the difference between the sum of these angles and two right angles (the defect) is proportional to its area. His findings were published eleven years after his death in: 'Die theorie der Parallellinien'.

The first person who had serious doubts about the dependence of the Parallel Postulate was C.F. Gauss (1777-1855). He constructed a valid non-Euclidean geometry, but never published it. In a letter made public after his death he expressly asked his friends not to publish his findings while he was alive. He probably feared for his reputation, Gauss was considered as the greatest mathematician of his time and publishing something this progressive would be too shocking.

In 1829 N.L. Lobachevsky (1793-1856) published the article 'On the principals of geometry'. In here he spoke about a line, a point not on that line and infinitely many lines through that point that never cross the first line. He used this to develop hyperbolic geometry. This was the first time that this geometry was developed for its own sake instead of using it to prove the dependence of the fifth postulate. At the same time J. Bolyai (1802-1860) also discovered this geometry, independently from Lobachevsky. He published his findings in the article: 'Supplement Containing the Absolutely True Science of Space, Independent of the Truth or Falsity of Euclid's Axiom XI [5]' as an appendix to a survey of attempts to prove the Parallel Postulate written by his father, F. Bolyai.

Gauss read their work and was impressed by it. But he did not express this in public as strongly as he could have. Thus his immense prestige did not help them to make their work known by a larger audience. Bolyai became a little suspicious. He knew that Gauss and his father had corresponded for a long time and he had told his father about his findings many years before his article. So when Gauss said that he had been working the same subject, Bolyai thought that Gauss might want to gain credits for his work. Historical research has shown that these three man were ignorant of each others work.

B. Riemann (1826-1866) came with the next great breakthrough. He made a distinction between the infinitude of a line and its unboundedness. If you draw a line around a sphere that forms a circle, it is not of infinite length, but it is unbounded in the sense that one can continue around it without stopping. By this we can contradict the fifth postulate and the infinity of a line. If Saccheri and Lambert had noticed this fact, they could not have ended up with a contradiction for the existence of the situations with the obtuse angle, because they assumed the infinite length of a line. Riemann suggested a geometry without any parallel line and in which the sum of the angles of a triangle is greater than two right angles.

In 1868 E. Beltrami (1835-1900) gave the first complete proof of the consistency of the hyperbolic plane. He showed that hyperbolic geometry was just as consistent as Euclidean geometry. F. Klein (1849-1925) used projective geometry, also developed in the nineteenth century, to distinguish three types of geometry: parabolic; corresponding to Euclid's geometry, elliptic; to Riemanns', and hyperbolic geometry; to that of Gauss, Bolyai and Lobachevsky.

In 1868 Beltrami constructed a Riemannian geometry for the Poincaré half-plane and disk (both named after Herni Poincaré (1854-1912)) and another model. In 1871 Klein showed that the last model of Beltrami could be constructed by projective geometry. This model became the Beltrami-Klein or projective model. Later, in 1901, D. Hilbert (1862-1943) showed that there is no Euclidean space that could represent the whole geometry of the hyperbolic plane.

These self-consistent geometries prove that the fifth postulate indeed is a postulate and not a proposition. It does not depend on the first four postulates. At this point we have three versions of this postulate, each with its own geometry: the version with one parallel of Euclidean geometry, infinitely many for hyperbolic geometry and none for elliptic geometry.

Throughout time several non-Euclidean geometries have been developed, whose postulational basis contradicts some postulate of Euclidean geometry. Riemann described a whole class of non-Euclidean geometries which are very important up to present day. They are referred to as Riemannian geometries.

The graphic artist M. C. Escher (1898-1972), who loved to amaze his audience by deception of the eye, used this new geometry for some of his work. Although he did not think himself good enough to understand the mathematics behind his figures, he was fascinated by geometric figures that can fill a space endlessly and therefore used the help of some mathematicians like H.S.M. Coxeter (who studied his work) to be able to make regular tessellations in the hyperbolic plane, resulting in the figures below.

Angels and Demons.

Reducing Lizards.

Bibliographical notes

The content of this chapter is derived from several sources. Eves [3], Golos [4], and Stahl [12] each gave a general summary of the history of non-Euclidean geometry, where Milnor [8] gave a more technical outline. We further used a more or less literal translation of Euclid's 'Elements' of Heath [7]. For a biography of M.C. Escher we refer to Spaanstra-Polak [11]. A description of the interactions between Coxeter and M.C. Escher focusing on the construction of regular tessellations in the hyperbolic plane can be found in Coxeter [2].

Chapter 3

The Poincaré disk model

First we are going to discuss the Poincaré disk model. It is the most famous hyperbolic model, probably because of Escher's beautiful and fascinating 'Angels and Demons'. We will discuss its points and several similarities and differences with the Euclidean space.

3.1 D-points and D-lines

Definitions

The points in the Poincaré disk model, D-points, are the points of the open unit disk:

$$\mathcal{D} = \{z : |z| < 1\} = \{(x, y) : x^2 + y^2 < 1\}.$$

The boundary of the unit disk does not belong to the geometry. The boundary is denoted by:

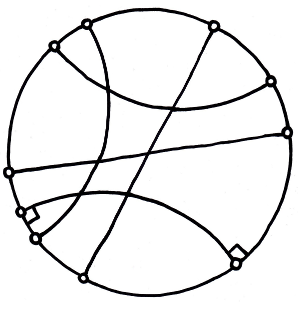
$$\mathcal{B} = \{z : |z| = 1\} = \{(x, y) : x^2 + y^2 = 1\}.$$

Before defining D-lines in the Poincaré disk model, we define a concept of Euclidean geometry:

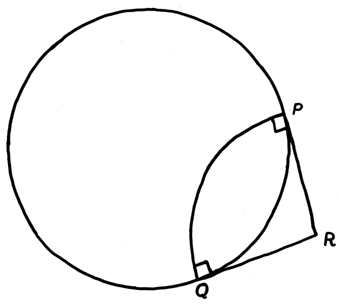
Definition 1. *A generalized circle is a Euclidean circle or a Euclidean line.*

In this model straight lines are not quite as we know them from Euclidean geometry. We call them D-lines and are defined as follows:

Definition 2. *A D-line is that part of a (Euclidean) generalized circle which meets $\mathcal{B}$ at right angles and which lies in $\mathcal{D}$.*



(a) $\mathcal{D}$, $\mathcal{B}$, D-lines



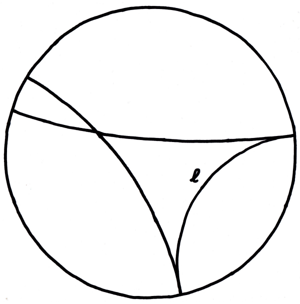
(b) Center of a generalized circle of a D-line

Figure 3.1: D-lines

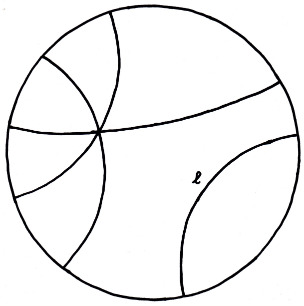
It is obvious that if a D-line is an Euclidean straight line, it must be a diameter of $\mathcal{D}$ to meet $\mathcal{B}$ at right angles. On the other hand, if any D-line is part of an Euclidean circle, it cannot pass through the origin. The Euclidean center of a D-line can be found by intersecting the tangents to $\mathcal{B}$ at the boundary points of the D-line, see figure 3.1b.

Like in Euclidean geometry we also define parallel lines:

Definition 3. *Two D-lines that do not meet in $\mathcal{D}$ are:* ***parallel*** *if the generalized circles of which they are part meet at a point on $\mathcal{B}$ and* ***ultra-parallel*** *if the generalized circles of which they are part do not meet at a point on $\mathcal{B}$.*



(a) Lines parallel to ℓ



(b) Lines ultra-parallel to ℓ

Figure 3.2: Parallel lines

Note that even parallel lines never meet, because $\mathcal{B}$ is not part of this geometry. With these definitions it can easily be seen that each D-line has two parallel D-lines an infinitely many ultra-parallel D-lines. So Euclid's fifth postulate does not hold in this setting.

Inversions and reflections

As is Euclidean geometry inversions and reflections play an important role in non-Euclidean geometry. It is obvious that reflection in a diameter of $\mathcal{D}$ maps $\mathcal{D}$ onto itself, because it works like normal Euclidean reflection. Fortunately it turns out that reflections in all D-lines map $\mathcal{D}$ onto itself. We begin by defining what reflection means in non-Euclidean geometry:

Definition 4. *Reflection in non-Euclidean geometry is Euclidean inversion in a D-line.*

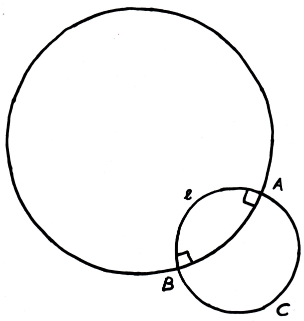


Figure 3.3: Inversion in a D-line ℓ

The formula for Euclidean inversion in a Euclidean circle is given in the proof of lemma 1. At this moment we will not use this formula and just provide ourselves with a visual interpretation.

Theorem 1. *Let ℓ be a D-line that is part of an Euclidean circle C. Then inversion in C maps $\mathcal{B}$ onto $\mathcal{B}$ and $\mathcal{D}$ onto $\mathcal{D}$.*

Proof. We call A and B the points where C meets $\mathcal{B}$, see figure 3.3. Under inversion in C, A and B will be mapped to themselves. Inversion preserves angles, so points of $\mathcal{B}$ must be mapped into some circle that meets C at right angles at the points A and B. The only circle that satisfies this condition is $\mathcal{B}$ itself. So inversion in C maps $\mathcal{B}$ onto itself.
Now points of $\mathcal{D}$ must be mapped into either the inside or outside of $\mathcal{B}$. Since ℓ is mapped onto itself, $\mathcal{D}$ can only be mapped onto itself. So inversion in C maps $\mathcal{D}$ onto itself. □

Non-Euclidean transformations are the composition of a finite number of non-Euclidean reflections. The set of all non-Euclidean transformations is called the non-Euclidean group $G_{\mathcal{D}}$.

Definition 5. *Non-Euclidean geometry consists of the unit disk, $\mathcal{D}$, together with the group $G_{\mathcal{D}}$ of non-Euclidean reflections.*

It can easily be seen that $G_{\mathcal{D}}$ forms a group: We have the identity operator (no reflection at all). For every reflection we have an inverse (reflection in the same line). The composition of two reflections also is an element of $G_{\mathcal{D}}$ and $(M \circ (N \circ P))(z) = ((M \circ N) \circ P)(z)$.

Next we define angles in the Poincaré disk model:

Definition 6. *The non-Euclidean angle between two D-lines through a given point A in $\mathcal{D}$ is the Euclidean angle between their tangents in A.*

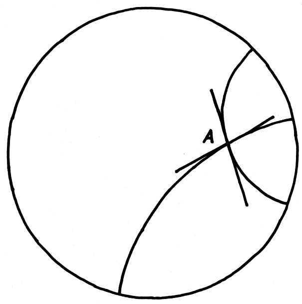


Figure 3.4: Angle in non-Euclidean geometry

It follows that non-Euclidean transformations preserve magnitudes of angles, just like in Euclidean geometry. Combining this with the fact that Euclidean inversions and reflections map generalized circles onto generalized circles gives us the following theorem:

Theorem 2. *Non-Euclidean transformations map D-lines onto D-lines and preserve the magnitudes of angles.*

Transformations

In Euclidean geometry we know we can draw infinitely many lines through a given point A. Could we also do this in non-Euclidean geometry? It is obvious that we can draw at least one line through such a given point, namely the Euclidean line through A and the origin. And we can draw infinitely many D-lines through the origin, namely the diameters of the unit disk. To prove that there are infinitely many D-lines through an arbitrary point A we use the following lemma:

Lemma 1. *Let A be a point of $\mathcal{D}$ other that the origin. Then there exists a D-line ℓ such that non-Euclidean reflection maps A to the origin O.*

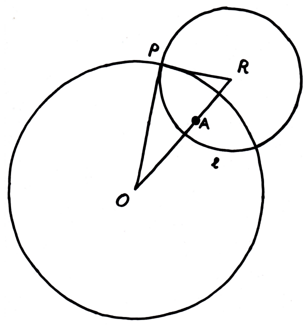


Figure 3.5: Reflecting A to the origin

Proof. The D-line we seek is part of an Euclidean circle C with center R. We call one of the two points in which this Euclidean circle meets the boundary of the unit disk P, then the radius of C is denoted by PR, see figure 3.5. To map A to O in the D-line ℓ we need the following condition:

$$RO \cdot RA = RP^2,$$

the formula for Euclidean inversion in a circle. Since C meets $\mathcal{B}$ at right angles, the triangle RPO is right angled. So we can apply Pythagoras' theorem:

$$RP^2 + PO^2 = RO^2.$$

And since $\mathcal{B}$ is a circle of radius 1, $PO = 1$. Wich gives us:

$$RP^2 = RO^2 - 1.$$

Combining the previous findings gives us the following condition:

$$RO \cdot RA = RO^2 - 1.$$

Which can be rewritten as:

$$RO(RO - RA) = 1.$$

A quick look at the figure gives us the following:

$$RO - RA = AO.$$

Thus we have:

$$RO \cdot AO = 1.$$

Now we know that we can construct a unique D-line in which we can reflect a point A to O. This D-line is part of the Euclidean circle with center on the line through O and A with Euclidean distance $\frac{1}{AO}$ from the origin and with radius $\sqrt{RO^2 - 1}$. □

Now we can answer our question: are there infinitely many lines through any point of $\mathcal{D}$?

Theorem 3. *Let A be a point of $\mathcal{D}$. Then there exist infinitely many D-lines through A.*

Proof. We already know that if A is the origin, there are infinitely many D-lines through A, namely the diameters of the unit disk.
Next we look at A not being the origin. By lemma 1 we can construct a non-Euclidean transformation t that maps A to O. Trough O are infinitely many D-lines. By theorem 2 the inverse transformation t^{-1} maps the D-lines of O onto D-lines of A. So A has infinitely many D-lines. □

Next we ask ourselves if there is a unique D-line through two points A and B inside the unit disk, just like in the Euclidean case. This turns out to be the case.

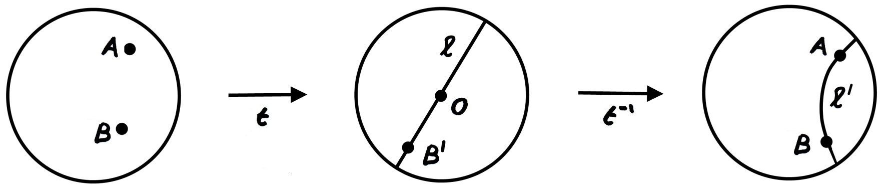


Figure 3.6: Constructing a unique D-line through A and B

Theorem 4. *Let A and B be any two distinct points of $\mathcal{D}$. Then there exists a unique D-line ℓ through A and B.*

Proof. First we will prove that this D-line exists, subsequently we will prove that this D-line is unique.
(Existence) By lemma 1 we know that there is a non-Euclidean transformation t that maps A to O. Let us call B' the image of B under t, see figure 3.6. We now have a unique D-line ℓ' that passes through O and B'. The inverse non-Euclidean transformation t^{-1} will map O to A, B' to B and by theorem 2 ℓ' will be mapped into ℓ, the D-line through A and B.
(Uniqueness) Suppose that there is another D-line ℓ_1 through A and B. Then $t(\ell_1)$ is a D-line that passes through O and B'. Since this D-line is unique, it follows that $t(\ell_1) = \ell'$. So ℓ_1 must be equal to ℓ, since $\ell_1 = t^{-1}(t(\ell_1)) = t^{-1}(\ell') = \ell$. □

3.2 Transformations

In this section we will look at explicit formulas for transformations of the Poincaré disk. We will look at inversion in a D-line ℓ that is part of a generalized circle C with center (a, b) and radius r, see figure 3.7. From now on we will write our coördinates as complex numbers and identify (a, b) with $\alpha = a + bi$. Since C makes right angles with $\mathcal{B}$ we can use Pythagoras' Theorem to obtain:

$$r^2 + 1^2 = a^2 + b^2.$$

Since $a^2 + b^2 = \alpha\overline{\alpha}$, we may rewrite this equation as:

$$r^2 + 1^2 = \alpha\overline{\alpha}.$$

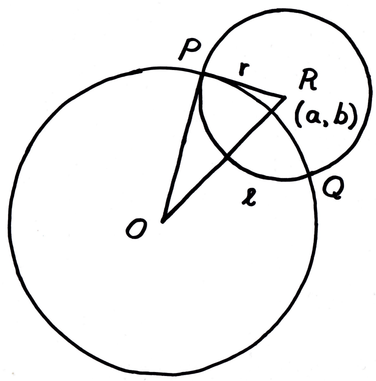


Figure 3.7: Reflection in ℓ

The general formula for inversion in the circle C is:

$$t(z) = \frac{r^2}{\overline{z - \alpha}} + \alpha \quad (z \in \mathbb{C} \setminus \{\alpha\}).$$

This, together with the formulas above, gives:

$$\begin{aligned} t(z) &= \frac{r^2}{\overline{z - \alpha}} + \alpha \\ &= \frac{r^2 + \alpha\overline{z} - \alpha\overline{\alpha}}{\overline{z} - \overline{\alpha}} \\ &= \frac{\alpha\overline{z} - 1}{\overline{z} - \overline{\alpha}} \quad (z \in \mathcal{D}). \end{aligned}$$

Now we have proven the following lemma:

Lemma 2. *The non-Euclidean reflection s in the D-line ℓ obtained from α is given by the non-Euclidean transformation*

$$s(z) = \frac{\alpha\overline{z} - 1}{\overline{z} - \overline{\alpha}} \quad (z \in \mathcal{D}).$$

One can see that this is the composition of the Möbius transformation $M(z) = \frac{\alpha z - 1}{z - \overline{\alpha}}$ and the complex conjugation $B(z) = \overline{z}$. The formula of lemma 2 can be used to give another prove of lemma 1 taking $\alpha = (\overline{z})^{-1}$.

The natural question to be asked at this moment is what happens if we compose two such reflections. It turns out that this composition is a Möbius transformation, which is really useful because this type of transformation preserves angles and maps generalized circles into generalized circles.

Theorem 5. *The composition of two non-Euclidean reflections $u(z)$ and $v(z)$ for $z \in \mathcal{D}$ given by:*

$$u(z) = \frac{\alpha\overline{z} - 1}{\overline{z} - \overline{\alpha}} \text{ and } v(z) = \frac{\beta\overline{z} - 1}{\overline{z} - \overline{\beta}}$$

is the non-Euclidean transformation

$$(v \circ u)(z) = \frac{(\overline{\alpha}\beta - 1)z + \alpha - \beta}{(\overline{\alpha} - \overline{\beta})z + \alpha\overline{\beta} - 1}.$$

The prove of this theorem is no more than writing out the composition of the two reflections and is given in appendix A.1.

Analogously it can be shown that every composition of an odd number of reflections is the composition of a Möbius transformation together with complex conjugation. Thus any non-Euclidean transformation, which is a composition of a finite number of non-Euclidean reflections, can be expressed in one of the to forms:

$$z \mapsto M(z) \text{ or } z \mapsto M(\overline{z}) \; z \in \mathcal{D},$$

where M is a Möbius transformation of the form

$$M(z) = \frac{az + b}{\overline{b}z + \overline{a}}, \; |z| < 1.$$

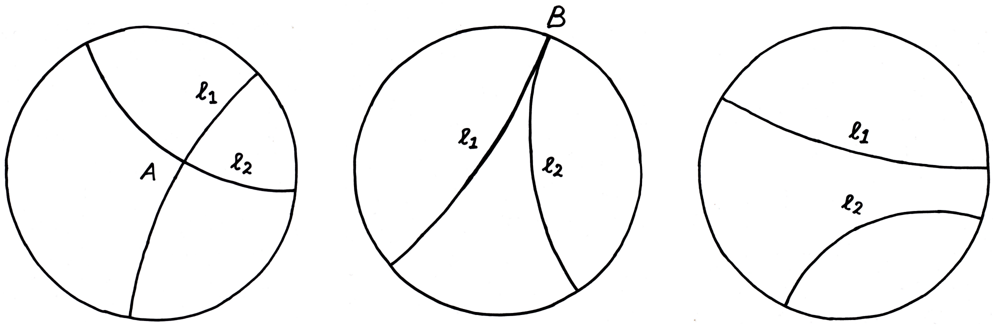
Since we want the origin to be mapped to a point of $\mathcal{D}$ and $M(0) = \frac{b}{\overline{a}}$ we add the condition $|b| < |a|$. This consideration gives us the following theorem. Since its proof is a little roundabout, the reader is referred to ([1], pp. 276).

Theorem 6. *Every Möbius transformation of the form $M(z) = \frac{az+b}{\overline{b}z+\overline{a}}$ with $z \in \mathcal{D}$ and $|b| < |a|$ is a composition of two non-Euclidean reflections and is for that reason a non-Euclidean transformation.*

Analogous to Euclidean reflections, an odd number of non-Euclidean reflections reverses orientation of angles between D-lines, while an even number preserves the orientation. Although we will not prove it here, it turns out that every non-Euclidean transformation can be written as the composition of at most three non-Euclidean reflections. These two observations give us the following:

A direct non-Euclidean transformation leaves orientation unchanged and can therefore be written as the composition of at most two non-Euclidean reflections.

A indirect non-Euclidean transformation reserves orientation and can therefore be written as the composition of at most three non-Euclidean reflections.



(a) Rotation (b) Limit rotation (c) Translation

Figure 3.8: Direct non-Euclidean transformations

In Euclidean geometry direct transformations can be divided in translations and rotations. In non-Euclidean geometry we have three families of direct transformations:

(Non-Euclidean) rotation. Consider a point $A \in \mathcal{D}$ and two D-lines ℓ_1 and ℓ_2 which intersect in A, see figure 3.8a. (There is no other point in which these lines intersect by theorem 4.) Then reflecting a point in $\mathcal{D}$ in the line ℓ_1 followed by reflecting it in the line ℓ_2 leaves the point A fixed and rotates any other point in $\mathcal{D}$ around A.

(Non-Euclidean) limit rotation Consider two parallel D-lines ℓ_1 and ℓ_2 which are tangent at some point $B \in \mathcal{B}$, see figure 3.8b. Reflecting a point in $\mathcal{D}$ in the line ℓ_1 followed by reflecting it in the line ℓ_2 rotates this point in $\mathcal{D}$ around B and leaves B fixed. But since B is not a point of $\mathcal{D}$, this composition of reflections has no fixed points. But it does leave parallel lines with B as their common point as parallel lines ending at B.

(Non-Euclidean) translation The last possibility is ℓ_1 and ℓ_2 being ultra-parallel. These lines have no points in common in $\mathcal{D}$ or $\mathcal{B}$ and therefore do not intersect or collide in $\mathcal{D}$ or $\mathcal{B}$, see figure 3.8c. This means that there is no fixed point in the Poincaré disk or its boundary. Reflecting a point in $\mathcal{D}$ in the line ℓ_1 followed by reflection in the line ℓ_2 moves points of $\mathcal{D}$ in one general direction.

It would be very useful if we could have a formula for non-Euclidean transformations that immediately shows us which point will be mapped to the origin. Fortunately this formula can easily be deduced from the formula in theorem 6.

Theorem 7. *Any direct non-Euclidean transformation M can be written in the form*

$$M(z) = K\frac{z-m}{1-\overline{m}z},$$

where K and m are complex numbers and $|K| = 1$ and $|m| < 1$.

Proof. Dividing both the nominator and denominator of the formula of theorem 6 by $\overline{a}$ gives:

$$M(z) = \frac{(az+b)/\overline{a}}{(\overline{b}z+\overline{a})/\overline{a}} = \frac{\frac{a}{\overline{a}}z + \frac{b}{\overline{a}}}{1+\frac{\overline{b}}{\overline{a}}z} = \frac{a}{\overline{a}}\frac{z - \frac{-b}{a}}{1-\frac{-\overline{b}}{\overline{a}}z}.$$

This is the required form with $K = a/\overline{a}$ and $m = -b/a$. Since a and $\overline{a}$ are of equal length, $|K| = 1$ and since $|b| < |a|$ we have $|m| < 1$, as desired. $\square$

Defined as such, M maps m to the origin, since $M(m) = 0$. Having this theorem, it is easy to find a non-Euclidean transformation that maps a point p to a point q, see figure 3.9.

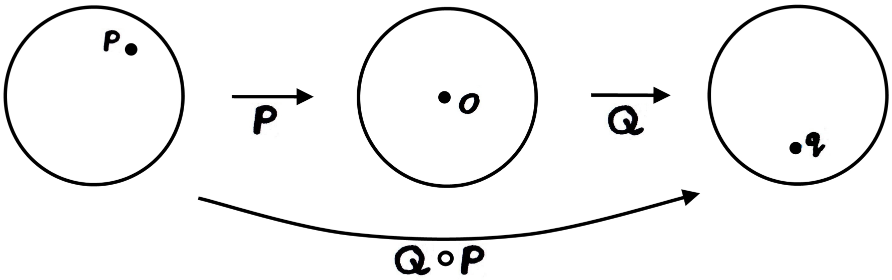


Figure 3.9: Mapping p to q

Step 1 Find a non-Euclidean transformation that maps p to the origin. This transformation is given by the formula:

$$P(z) = K\frac{z-p}{1-\overline{p}z}.$$

Step 2 Find a non-Euclidean transformation that maps the origin to the point q. This transformation is given by:

$$Q(z) = \frac{z+q}{1+\overline{q}z}.$$

Step 3 The non-Euclidean transformation mapping p to q is given by:

$$(Q \circ P)(z) = \frac{(K-\overline{p}q)z - pK + q}{(\overline{q}K-\overline{p})z - \overline{q}pK + 1},$$

with $|K| = 1$.

The last subject of this section will be finding the equations of D-lines. It is easy to construct an equation of a diameter, this is just: $ay = bx$ with $a, b \in \mathbb{R}$. Finding the corresponding equation for a D-line that is part of a circle requires more calculations. From lemma 2 we know that reflecting D-points in a D-line ℓ that is part of a circle with center α is given by the equation:

$$s(z) = \frac{\alpha\overline{z} - 1}{\overline{z} - \overline{\alpha}}.$$

This reflection leaves points of ℓ fixed. Thus for points d of ℓ we have $s(d) = d$. Solving this equation with $\alpha = a + bi$ and $d = x + iy$ gives the following equation for the D-line ℓ:

$$x^2 + y^2 - 2ax - 2by + 1 = 0 \text{ with } a^2 + b^2 > 1.$$

3.3 Distance

The explicit formula

In Euclidean geometry there are three properties concerning distance:

1. $d(z_1, z_2) \geq 0$ for all z_1 and z_2; $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$.

2. $d(z_1, z_2) = d(z_2, z_1)$ for all z_1 and z_2.

3. $d(z_1, z_3) + d(z_3, z_2) \geq d(z_1, z_2)$ for all z_1, z_2 and z_3.

These common properties also are valid in non-Euclidean geometry. In addition to our non-Euclidean geometry we need some extra properties to make the set complete:

4. $d(z_1, z_3) + d(z_3, z_2) = d(z_1, z_2)$ if and only if z_1, z_3 and z_2 lie in this order on a D-line.

5. $d(z_1, z_2) = d(\overline{z_1}, \overline{z_2})$ for all z_1 and z_2 in $\mathcal{D}$.

6. $d(z_1, z_2) = d(M(z_1), M(z_2))$ for all z_1 and z_2 in $\mathcal{D}$ and all direct non-Euclidean transformations M in $G_{\mathcal{D}}$.

The fourth property claims that distance is additive along a D-line. The fifth and sixth property claim that non-Euclidean transformations do not change distances between points.

Next we would like to find some expression for $d(z_1, z_2)$. We do this using the direct non-Euclidean transformation

$$M : z \mapsto \frac{z - z_1}{1 - \overline{z_1} z}, \text{ where } |z_1| < 1.$$

This transformations maps z_1 to O and z_2 to $\frac{z_2 - z_1}{1 - \overline{z_1} z_2}$. Next the rotation R sends $\frac{z_2 - z_1}{1 - \overline{z_1} z_2}$ to $\left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right|$. This gives us the direct transformation $R \circ M$.
By property 6 it must follow that

$$d(z_1, z_2) = d\left(0, \left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right|\right), \text{ for all } z_1,\ z_2 \in \mathcal{D}.$$

Thus, d should depend on $\left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right|$ alone. From ([1], pp. 285, 296-297) we know that the function that satisfies this condition together with the six properties for distance is given by:

Definition 7. *The non-Euclidean distance $d(z_1, z_2)$ between the points z_1 and z_2 in $\mathcal{D}$ is defined by:*

$$d(z_1, z_2) = C \tanh^{-1}\left(\left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right|\right),$$

with C a positive number.

In this section we will use $C = 1$, in section 3.5 we will use $C = 2$. This choice of C will just rescale the distance function and will not have an important impact on the topics described in this thesis. It is obvious that these functions satisfy properties 1, 2 and 5. For the other properties the reader is referred to ([1], pp. 292-293, 295-297). Replacing z_1 by 0 and z_2 by z gives the following definition:

Corollary 1. *The non-Euclidean distance between 0 and a point z in $\mathcal{D}$ is defined by:*

$$d(0, z) = \tanh^{-1}(|z|).$$

This definition may seem redundant, but it is quite useful if we want to get some idea of how distance works in the Poincaé disk. Looking at figure 3.10 we see that near the origin, the non-Euclidean distance is approximately equal to the Euclidean distance. If we move to the boundary of the disk, the distance to the origin goes to infinitely. This looks a little bit like the nightmare described in the introduction. Beginning from the origin, one gains less and less with every non-Euclidean step measured in a Euclidean manner; the Euclidean distance to the boundary hardly decreases.

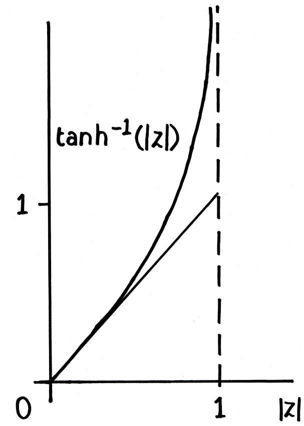


Figure 3.10: Graph of $\tanh^{-1}(|z|)$

Stahl [12] gives another formula for the distance between two points using cross-ratios. Instead of just looking at the two points we want to know the distance between, we also look at where their unique D-line intersects the unit circle. This new distance function is defined by:

Definition 8. *Let p and q be two points of the unit disk that are joined by a D-line that meets the unit circle at p' and q'. If these points are labeled such that their counterclockwise order on the D-line is p', p, q, q', then the distance between p and q is given by:*

$$d(p,q) = \ln\left(\frac{(p'-q)(q'-p)}{(p'-p)(q'-q)}\right).$$

By properties 5 and 6 we only have to verify that definitions 7 and 8 give the same result for the distance between the origin and any real point A that lies between 0 and 1, see figure 3.11. Plugging in these values in the formula of definition 8 yields:

$$\begin{aligned} d(0,A) &= \ln\left(\frac{(1-0)(-1-A)}{(1-A)(-1-0)}\right) \\ &= \ln\left(\frac{1+A}{1-A}\right) = 2\tanh^{-1}(A), \end{aligned}$$

the same as in definition 7 with $C = 2$.

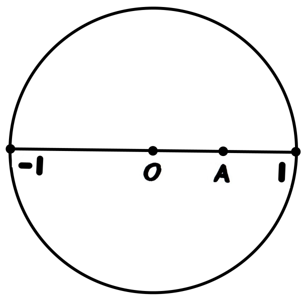


Figure 3.11: Distance using cross-ratios

Midpoints and circles

Knowing how to calculate the distance between a point z in $\mathcal{D}$ and the origin, we can find the midpoint of two points p and q in $\mathcal{D}$. Below it is shown how to do this when p and q lie on the same diameter of the unit disk. If p and q do not lie on the same diameter, we use a Möbius transformation M to map p to 0 and q to $M(q)$. We are allowed to do this by the sixth property of distance.
Finding a midpoint begins with defining which condition a midpoint should satisfy:

Definition 9. *A point m is the non-Euclidean midpoint of two points p and q in $\mathcal{D}$ if*

$$d(p,m) = d(q,m) = \frac{1}{2}d(p,q).$$

This means that m should lie on the unique D-line trough p and q with the same non-Euclidean distance from p as from q.

Knowing this, we can calculate where m lies:

1. Look which of the two points has the smallest Euclidean distance from the the origin and call this point q. Call the other one p.

2. Calculate $d(0,p)$ and $d(0,q)$.
3. If p and q lie on opposite sides of the origin, then calculate:

$$d = \frac{1}{2}(d(0,p) - d(0,q)).$$

   If p and q lie on the same side of the origin, then calculate:

$$d = \frac{1}{2}(d(0,p) + d(0,q)).$$

4. The non-Euclidean midpoint m lies on the line-segment Op at a Euclidean distance $\tanh(d)$ from the origin.

The last remarkable statement of this section concerns circles. In Euclidean geometry a circle is defined as the set of points z with equal distance r from a center c in $\mathcal{D}$. This is the same in non-Euclidean geometry, except for the fact that the distance is measured in some other way.

You may wonder what such a non-Euclidean circle may look like. It turns out that non-Euclidean circles are just the same as Euclidean, except for a difference in the place of its center:

Theorem 8. *Every non-Euclidean circle is a Euclidean circle and visa versa.*

Proof. The proof is based on the fact that we already know that non-Euclidean circles with centers at the origin are Euclidean circles. We also know that Möbius transformations map generalized circles into generalized circles and a non-Euclidean circle must be some closed loop. Using a Möbius transformation M that maps the non-Euclidean center to the origin and the non-Euclidean circle to a Euclidean circle centered at the origin, one can prove that every non-Euclidean circle is a Euclidean circle an visa versa. □

3.4 Geometrical theorems

The angle-sum

Like in Euclidean geometry, there is a huge number of geometrical theorems that are valid in non-Euclidean geometry. Many results for Euclidean geometry still hold in a non-Euclidean setting, while some are replaced by others which often are clearly similar. In this section we will discuss some of the most famous Euclidean theorems in their non-Euclidean version. We begin with a basic result: in Euclidean geometry the sum of the angles of a triangle is always equal to π, in non-Euclidean geometry we have to replace the equality by a 'less than' sign:

Theorem 9. *The sum of the angles of a D-triangle is less than π.*

Proof. Proving this theorem turns out to be quite easy using a theorem and a lemma we have proven before. Let us call the corners of an arbitrary D-triangle A, B and C. By theorem 2 and lemma 1 we know that there is a non-Euclidean transformation that maps A to O and B, C to B' and C', see figure 3.12. Since non-Euclidean transformations preserve angles, the sum of this new D-triangle $OB'C'$ is the same as that of the D-triangle ABC.

It is obvious that OB' and OC' are parts of Euclidean lines. Since the third line $B'C'$ is not a Euclidean but a non-Euclidean D-line bending towards the origin, the sum of the angles of the D-triangle $OB'C'$ must be strictly less than the sum of the angles of the Euclidean triangle $OB'C'$, which is π. □

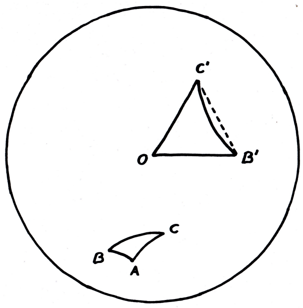


Figure 3.12: D-triangle ABC and its reflection $OB'C'$

You may wonder that if we have 'equal to' and 'less than', we should also have a geometry in which this theorem has 'greater than'. This geometry indeed exists, it is called spherical geometry and is alluded in the second chapter.
Besides D-triangles which lie entirely in $\mathcal{D}$, there are also D-triangles with one ore more of its vertices lying on the boundary of $\mathcal{D}$. We call them asymptotic triangles and distinguish them in three categories:

Definition 10. *A D-triangle is called a* ***simply asymptotic triangle*** *if only one vertex lies on* $\mathcal{B}$*, a* ***doubly asymptotic triangle*** *if two vertices lie on* $\mathcal{B}$ *and a* ***trebly asymptotic triangle*** *if all of its vertices lie on* $\mathcal{B}$*.*

These triangles are essentially different from 'ordinary' D-triangles, because there is no non-Euclidean transformation that maps points of $\mathcal{B}$ into points of $\mathcal{D}$. Apart from this difficulty it turns out that the result above also holds for asymptotic triangles:

Theorem 10. *The sum of the angles of an asymptotic triangle is less than* π*. The sum of the angles of an trebly asymptotic triangle is zero.*

Even though we will not discuss its prove in this essay, it is quite similar to that of the ordinary D-triangles.

Other results

In non-Euclidean geometry we have a same theorem for isosceles triangles as in Euclidean geometry:

Theorem 11. *Let ABC be a D-triangle in which $\angle ABC = \angle ACB$. Then the sides AB and AC are of equal length.*

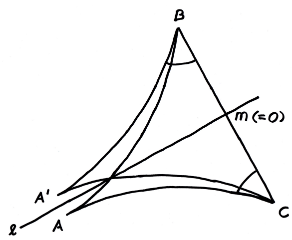


Figure 3.13: D-triangle ABC with $\angle ABC = \angle ACB$

Proof. Let M be the midpoint of the line segment BC. For simplicity we assume that M coincides with the origin, see figure 3.13. We may do this using lemma 1. Let ℓ be the perpendicular bisector of BC (the diameter of $\mathcal{D}$ perpendicular to BC). Next we reflect the triangle ABC in the D-line ℓ. Reflections preserve length and M is the midpoint of BC, so B and C interchange places.

Suppose that A reflects to some point A'. Reflections also preserve angles, so $\angle A'BC = \angle ACB$. And since $\angle ABC = \angle ACB$, we have that $\angle ABC = \angle A'BC$. This can only happen if A' lies on the (unique) D-line through A and B. Similarly, $\angle A'CB = \angle ABC = \angle ACB$. Therefore A' must also lie on the (unique) line through A and C.

Since there is only one D-line trough two given points in $\mathcal{D}$ and B and C are not the same point, AB and AC intersect at only one point: A. So A' must coincide with A. Thus the D-line AB reflects into the D-line AC, and visa versa. Reflections preserve length, so AB and AC must be of equal length. □

One of the most used theorems in Euclidean geometry will be 'Pythagoras' Theorem'. Almost everyone, mathematician or not, will know the famous '$a^2 + b^2 = c^2$'. In non-Euclidean geometry we also have a version of 'Pythagoras' Theorem', which differs just a little bit from the Euclidean version:

Theorem 12. *Let ABC be a D-triangle with right angle at C. And let a, b and c be the non-Euclidean length of BC, AC and AB, respectively. Then*

$$\cosh 2c = \cosh 2a \times \cosh 2b.$$

Although the prove is not that difficult, we will not discuss it at the moment. The interested reader will be referred to ([1], pp. 310-312). The following theorems also will not be proven, but are too interesting not to mention. We begin with a trebly asymptotic D-triangle. Looking at the Euclidean case, having your vertices at points infinitely far away means that the area of your triangle will be infinitely large. A trebly asymptotic triangle has its vertices on $\mathcal{B}$, which lie infinitely far away from any point in $\mathcal{D}$. Surprisingly its area turns out to be finite:

Theorem 13. *The area of a trebly asymptotic d-triangle is finite*

Another interesting result is that the area of a D-triangle does not depend on the length of its D-lines and its angles, but only on the sum of its angles:

Theorem 14. *The area of a D-triangle with angles α, β and γ is given by $K(\pi - (\alpha + \beta + \gamma))$, where K is the same for all D-triangles.*

3.5 Metric tensor

In each two-dimensional Riemannian geometry an inner product is defined by the triple E, F, G. The inner product of two vectors u and v in the tangent space of a point (x, y) is given by:

$$\langle u, v\rangle = Eu_1v_1 + F(u_1v_2 + u_2v_1) + Gu_2v_2,$$

where E, F and G are chosen such that the inner product is positive definite ($\langle u, u\rangle \geq 0$, $\langle u, u\rangle = 0 \iff u = 0\ \forall u \in T_pM$).

We often use the symbolic notation: $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$. For the Poincaré disk we have:

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

Sometimes this metric is given by $ds^2 = \frac{dx^2+dy^2}{(1-(x^2+y^2))^2}$. As we have seen in defining a distance function, this is a consequence of rescaling and has no important impact on the topics discussed in this chapter.

This metric tensor is useful when we would like to find a formula for the distance between two points. By properties 5 and 6 of the distance function we only have to find a function for the distance between the origin and a given point $|z|$. Since distances are measured along D-lines, we have to parametrize the unique D-line between the origin and $|z|$, which is just a Euclidean straight line between these points. It is given by: $\alpha(t) = (t|z|, 0)$, $t \in [0, 1]$.

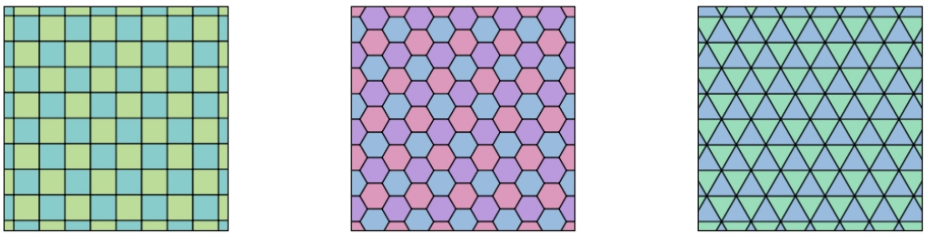
The length s of this finite D-line is given by:

$$s = \int_0^1 \|\alpha'(t)\| dt = \int_0^1 \sqrt{\langle \alpha'(t), \alpha'(t) \rangle} dt.$$

Straightforward integration yields: $s = 2\tanh^{-1}(|z|)$, which corresponds to the function given in definition 7. The calculations are given in appendix A.2.

3.6 Tessellations

In Euclidean geometry it only is possible to make a regular tessellation using regular squares, hexagons and triangles. One will end up with gaps if they use any other polygon. Take for instance a pentagon. Its corners have an angle of 108°. When we put the corners of four pentagons together, they will overlap because they make an angle of $4 \times 108 = 432°$. But we will end up with a gap if we try to put three of these corners together, since $3 \times 108 = 324°$.



Regular squares Regular hexagons Regular triangles

In non-Euclidean geometry it is possible to make regular tilings with any regular n-gon with angle $(n-2)\pi/k$ for each pair of integers $k > n > 2$. The reason for this is that there are infinitely many noncongruent n-gons in non-Euclidean geometry for any $n \geq 3$. M.C. Escher used this principle in his 'Angels and Demons'. He began with a hexagon centered at the origin and reflected it in its edges. As you can see in figure 3.14, beginning with a hexagon does not define the angles of its corners. Making the radius of each Euclidean circle of which the edges are part larger, makes the angles of the corners larger and visa versa. Thus there are infinitely many hexagons which are noncongruent.

Since non-Euclidean reflections preserve angles just like common Euclidean reflections, the sums of the angles of all hexagons in figure 3.14 are the same. They therefore all have the same area, by theorem 14, since each n-gon can be build up by triangles. With our Euclidean eye we may think that the angels and demons become smaller and smaller as they are being reflected to the edge of the unit disk, but in this geometry this is not the case.

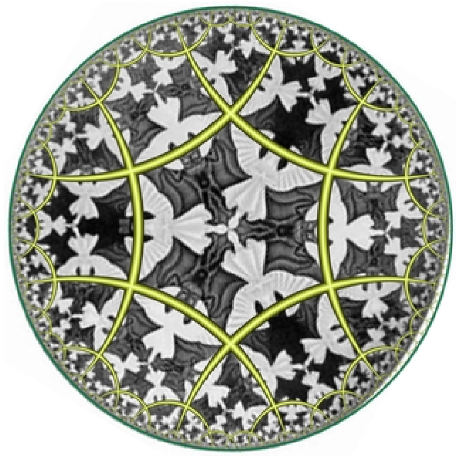


Figure 3.14: Tessellation of M.C. Escher's 'Angels and Demons'

Bibliographical notes

We follow the outline of Brannan, Esplen and Gray ([1], chapter 6). For a general introduction to Riemannian geometry in low dimensions we refer to O'Neill [10]. Stahl ([12], section 6.2 and 13.6), Brannan, Esplen and Gray ([1], section 6.5) and Goodmann-Strauss [5] explain how the Poincaré disk model can be used to make regular tessellations in the hyperbolic plane.

Chapter 4

Other models

4.1 The Poincaré half-plane

H-points, H-lines and the metric tensor

The points in the Poincaré half-plane model, H-points, are the points of the xy-plane that lie above the x-axis:

$$\mathcal{H} = \{(x, y) : y > 0\}.$$

The boundary is denoted by:

$$\mathcal{B} = \{(x, y) : y = 0\}.$$

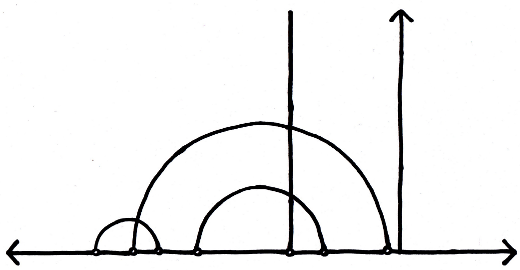


Figure 4.1: H-lines

The metric tensor of this model is given by:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

H-lines of this model are Euclidean lines and circles which make right angles with the x-axis, see figure 4.1. Their parameterizations can easily be found and are given by:

- $s(t) = (a, t)$ with $a \in \mathbb{R}$ and $t > 0$,
- $p(t) = (a + r\cos(t/r), r\sin(t/r))$ with $a, r \in \mathbb{R}$ and $0 < t < \pi r$.

We can prove that these lines and circles are H-lines using ([10], theorem 7.4.2 and lemma 7.4.5). Since s and t have unit speed and $s'(t)$ and $p'(t)$ are never orthogonal to E_2 (the vector pointing in the y-direction), we only need to show that:

$$h_1'' + \frac{1}{2E}(E_{h_1}h_1'^2 + 2E_{h_2}h_1'h_2' - G_{h_1}h_2'^2) = 0, \tag{4.1}$$

for $h(t) = (h_1(t), h_2(t)) = s(t)$ and $h(t) = (h_1(t), h_2(t)) = p(t)$. In the Poincaré half-plane $E = G = \frac{1}{h_2^2}$. Substituting this in equation 4.1 gives:

$$h_1'' + \frac{h_2^2}{2} \cdot \frac{-2h_1' h_2'}{h_2^3} = h_1" - \frac{1}{h_2} h_1' h_2' = 0.$$

Thus we have to solve the partial differential equation $h_2 h_1'' = h_1' h_2'$. The only solutions are given by:

- $h_1 = t$, $h_2 = a$ with a some constant and $t \in \mathbb{R}$,
- $h_1 = a + r\cos t$, $h_2 = r \sin t$ with a and r some constants and $t \in \mathbb{R}$.

Deleting the parts on and under the x-axis indeed gives us the given H-lines parameterized by $s(t)$ and $p(t)$.

Tessellations

Like in section 3.6 this model can be used to make beautiful regular tessellations. For example, M.C. Escher used this model for his 'Reducing Lizards'. The closer we get to the origin the smaller the lizards seem to be. But this is just an optical illusion since we are only capable to look with our Euclidean eyes.

Figure 4.2: Reducing Lizards.

4.2 The Minkowski model

M-points and M-lines

The Minkowski model is relatively young compared to the other models. It is closely related to the Poincaré disk and Beltrami-Klein model (next section) as we will see in the next chapter. There even are indications that Poincaré used this model implicitly in his personal notes before W. Killing (1847-1923) published an article explicitly describing the model. The Minkowski model can be used in the theory of relativity.

The Minkowski model is the top shell of the hyperboloid with equation $-t^2 + x^2 + y^2 = 1$ and is for this reason also called the Hyperbolic model. Since this is just a plane in $\mathbb{R}^3$ it is a two dimensional space and therefore the M-points can be parametrized by two variables. We will use the following parametrization:

$$\begin{pmatrix} t \\ x \\ y \end{pmatrix} = \begin{pmatrix} \cosh\chi \\ \sinh\chi\cos\theta \\ \sinh\chi\sin\theta \end{pmatrix}.$$

It can easily be seen that this parametrizes the top shell of the given hyperboloid. The next we would like to know is what the M-lines of this space are. They are the intersections with planes through the point $(0,0,0)$, see figure 4.3.

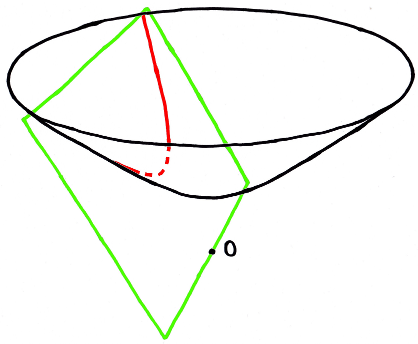


Figure 4.3: M-lines

Metric tensor

The metric tensor in the ambient space is given by: $ds^2 = -dt^2 + dx^2 + dy^2$. The isometric embedding induces the metric on the hyperboloid:

$$ds^2 = d\chi^2 + \sinh^2\chi \; d\theta^2.$$

The calculations are given in appendix A.3.

4.3 The Beltrami-Klein-Hilbert model

B-points, B-lines and the metric tensor

The Beltrami-Klein-Hilbert model (BKH model) looks a lot like the Poincaré disk model. B-points of the BKH model are the same as in the disk model:

$$\mathcal{K} = \{z : |z| < 1\} = \{(x,y) : x^2 + y^2 < 1\}.$$

The boundary of the unit disk does not belong to the geometry. The boundary is:

$$\mathcal{B} = \{z : |z| = 1\} = \{(x,y) : x^2 + y^2 = 1\}.$$

Its difference lies in the fact that B-lines are Euclidean straight lines. In fact, B-lines are the cords joining the endpoints of D-lines, see figure 4.4. Since these two models have a different definition of 'straight lines', while their space is the same, we must have a different metric for the BKH model. This is because the distance between two points measured along the unique 'straight line' joining them must be the smallest of all measurements along curves joining these points.

Unfortunately, the metric tensor of this model is not as pretty as the other models. In this model F is not equal to zero, which gives us as extra factor $2Fdxdy$. The metric tensor is given by:

$$ds^2 = \frac{(1-y^2)dx^2 + 2xy\; dxdy + (1-x^2)dy^2}{(1-(x^2+y^2))^2}.$$

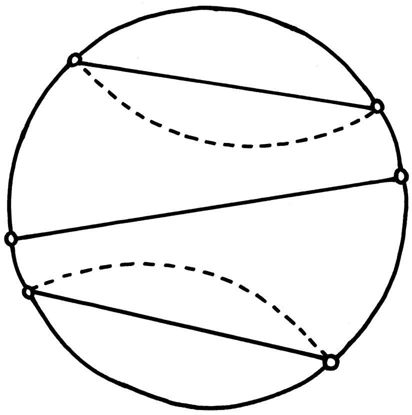


Figure 4.4: B-lines

Bibliographical notes

We made use of the very accessible explanation of the Poincaré half-plane by Stahl [12]. Misner, Thorne and Wheeler [9] and Thurston [13] each briefly discussed the Minkowski model. The Beltrami-Klein-Hilbert model has been described in Stahl ([12], chapter 14) and Brannan, Esplen and Gray ([1], section 8.3).

Chapter 5

Equivalence of the models

To show that these four models are equivalent, we must show that there are bijective functions between them who map the space of the first model into the other and transform the metric of one model into that of the other. In this manner one could also deduce a metric of a model knowing the metric of another model and a bijective function between the first and second model.

We will prove that the Poincaré disk and Beltrami-Klein-Hilbert model are equivalent to the Minkowski model and that the Poincaré disk model is equivalent to the Poincaré half-plane model. Once this is done, we know that all models are equivalent to each other, because the equivalence of models A and B and the equivalence of models B and C imply the equivalence of models A and C.

5.1 The Minkowski and the Poincaré disk model

Starting from the Minkowski model, one will end up at the space of the Poincaré disk model by intersecting the line through a point of the Minkowski model and the point $(-1, 0, 0)$ with the plane $t = 0$, see figure 5.1. This gives us the function: $f(t, m_1, m_2) = (\frac{m_1}{t+1}, \frac{m_2}{t+1})$. Details are given in appendix B.1.

On the other hand, getting from a point of the Poincaré disk to its corresponding point on the Minkowski model means that one has to intersect the line through the point $(-1, 0, 0)$ and the given point in the disk (at hight $t = 0$) with the upper half sheet of the hyperboloid. This method defines a bijective function between the two spaces.

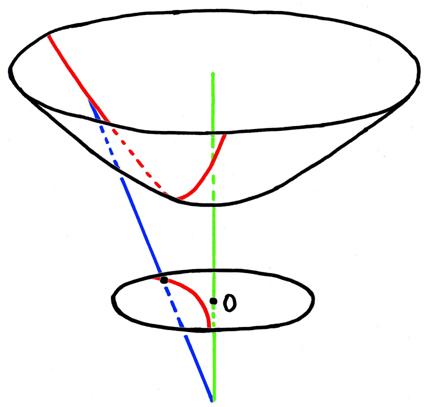


Figure 5.1: Mapping from the Minkowski model to the Poincaré disk

Proving that these two models are equivalent means showing that the function f transforms the metric of the Minkowski model into that of the Poincaré disk model. Thus we have to show that:

$$4\frac{dp_1^2+dp_2^2}{(1-(p_1^2+p_2^2))^2}=4\frac{(d(\frac{m_1}{t+1}))^2+(d(\frac{m_2}{t+1}))^2}{(1-((\frac{m_1}{t+1})^2+(\frac{m_2}{t+1})^2))^2}=-dt^2+dm_1^2+dm_2^2|_{\{(t,m_1,m_2):-t^2+m_1^2+m_2^2=1\}}.$$

With (p_1,p_2) points on the Poincaré disk and (t,m_1,m_2) points on the Minkowski space. We do this using the transformation given in section 4.2, the calculations are given in appendix C.1.

5.2 The Minkowski and the Beltrami-Klein-Hilbert model

The isometry between the Minkowski and the Beltrami-Klein-Hilbert model looks a lot like the one we have seen in the previous section. In here we intersect the line through a point on the Minkowski model and the point $(0,0,0)$ with the plane $t=1$, see figure 5.2. Since the M-lines of the Minkowski model are the intersections with planes through $(0,0,0)$, M-lines will, with this transformation, correspond to straight lines on a unit disk. These lines indeed correspond to the B-lines of the BKH-model. This gives us the following function: $g(t,m_1,m_2)=(\frac{m_1}{t},\frac{m_2}{t})$. Details are given in appendix B.2.

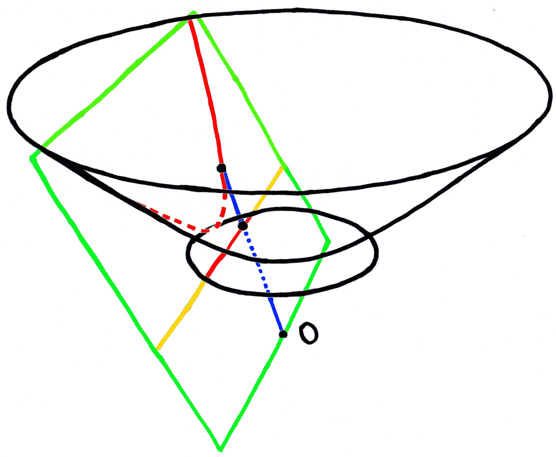


Figure 5.2: Mapping from the Minkowski model to Beltrami-Klein-Hilbert disk

Likewise one can define a function that maps a point of the BKH-space to a point of the Minkowski model by intersecting the line through the point $(0,0,0)$ and the given point in the disk (at hight $t=1$) with the upper half sheet of the hyperboloid. In appendix C.2 it will be shown that this function indeed transforms the metric of the Minkowski model into that of the BKH-model. This gives us the isometry between the Minkowski and the Beltrami-Klein-Hilbert model.

5.3 The Poincaré disk and Poincaré half-plane

The isometry between the Poincaré disk and the Poincaré half-plane is given by a Möbius transformation:

$$U : PD \to PH,\ z \mapsto \frac{iz - 1}{-z + i}\ z \in \mathbb{C}.$$

One can easily see that $U(-i) = 0$, $U(1) = 1$ and $U(-1) = -1$. Thus these three points of the unit circle are mapped to three points of the x-axis. We know that a Möbius transformation maps generalized circles into generalized circles. Thus the unit circle is mapped into the x-axis. Since U is continuous, the interior of the unit disk is mapped into either the upper or lower half-plane. We see that $U(0) = i$, thus U maps the interior of the unit disk into the upper half-plane, see figure 5.3. As shown in this figure, the D-lines of the unit disk are mapped into the H-lines of the upper half-plane.

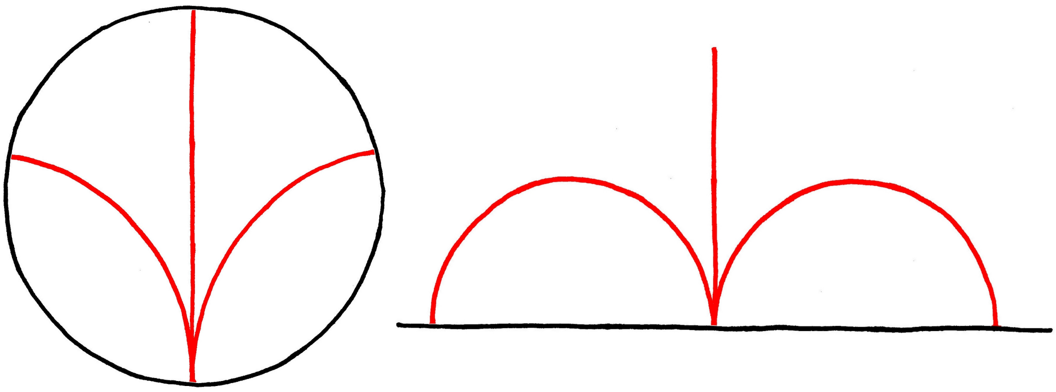


Figure 5.3: Mapping from the Poincaré disk to the Poincaré half-plane

One can easily verify that its inverse is given by: $V(z) = \frac{iz+1}{z+i}$. We will use U to demonstrate that these models are equivalent. This will be done showing that under $U(z)$

$$\frac{dh_1^2 + dh_2^2}{h_2^2} = 4\frac{dp_1^2 + dp_2^2}{(1 - p_1^2 - p_2^2)^2}.$$

With (h_1, h_2) points of the Poincaré half-plane and (p_1, p_2) points of the Poincaré disk. Details are given in appendix C.3.

In Stahl ([12], pp. 182) it has been proven that Möbius transformations preserve cross-ratios. Since the mapping between the Poincaré disk and half-plane model is given by such a Möbius transformation, the formula for the distance between two points of the half-plane model lying on a line not being a Euclidean straight line is given by that of definition 8.

Bibliographical notes

We have relied on Thurston [13] and Hege [6] for the mapping between the Beltrami-Klein-Hilbert model and the Minkowski model. Hege [6] also provided us with some information with regard to the mapping between the Poincaré disk and Minkowski model. The Möbius transformation between the Poincaré disk and half-plane can be found in for example Stahl [12].

Chapter 6

Conclusion

In 300 BC. Euclid wrote his masterpiece 'Elements'. In the first of these thirteen books he spoke about 5 postulates which should, together with a collection of definitions, form the basis of geometry. Since the fifth had a different structure than the other four and he only used the first four for the first twenty-eight propositions, people speculated that this postulate should be a proposition. If this is the case, it must be possible to prove this 'Parallel Postulate' using the first four postulates.

Over two-thousand years mathematicians tried to find such a prove. Until C.F. Gauss, N.L. Lobachevsky and J. Bolyaj independently invented hyperbolic geometry. This geometry turned out to be completely consistent, but differs significantly from Euclidean geometry, the only geometry known till then. The first four postulates equal those of Euclidean geometry, but instead of having exactly one parallel line, each line has infinitely many parallel lines. This proves that the Parallel Postulate is independent of the first four and therefore is a postulate and not a proposition.

M.C. Escher used the Poincaré disk model in his 'Angels and Demons'. This is one of the models for hyperbolic geometry. Its space is given by the open unit disk (D-points) and lines (D-lines) are generalized circles which make right angles with the unit circle. In this model reflection is defined by Euclidean inversion in these D-lines. These reflections map D-points to D-points and the unit circle onto the unit circle. Angles between D-lines are measured using their tangents and do not differ from Euclidean angles.

Another model that we have discussed is: The Poincaré half-plane model. Its H-points are points of the xy-plane which lie above the x-axis. H-lines are Euclidean straight lines and circles which make right angles with the x-axis. M.C. Escher used this model for his 'Reducing Lizards'. We also discussed the Minkowski model or Hyperbolic model. This model has M-points embedded in the Minkowski space on the top shell of the hyperboloid with equation $-t^2 + m_1^2 + m_2^2 = 1$. M-lines are the intersections with planes through the point $(0, 0, 0)$. The last model discussed is the Beltrami-Klein-Hilbert model, which looks a lot like the disk model. The main difference lies in the fact that its B-lines are Euclidean straight lines.

The last subject discussed is the equivalence of these models. This has been proven by showing that there are bijective mappings between each of these models that transform the metric of one model into that of the other. Knowing that these models are equivalent, we know that all the theorems we have proven for the Poincaré disk model are valid in all the models discussed in this thesis.

Chapter 7

Thanks

First of all I would like to thank Prof.Dr. G. Vegter and M.H.M.J. Wintraecken MSc. for suggesting this interesting subject and for helping me with all the problems I had to face making this thesis. Without them this thesis would not be what it is at this moment an I would not have enjoyed studying this subject as much as I have in the last few months.

But always remember that ‘studying’ partly is ‘dying’ and therefore I want to thank my family and friends and specially my parents for giving me the moral support I needed. Not just for this thesis but during my whole study.

Last but not least I would like to thank Prof. Dr. H. W. Broer for being my second supervisor.

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Appendix A

Miscellaneous

A.1 Proof of theorem 5

The composition of to non-Euclidean reflections $u(z)$ and $v(z)$ for $z \in \mathcal{D}$ given by:

$$u(z) = \frac{\alpha\overline{z} - 1}{\overline{z} - \overline{\alpha}} \text{ and } v(z) = \frac{\beta\overline{z} - 1}{\overline{z} - \overline{\beta}}$$

is the non-Euclidean transformation

$$(v \circ u)(z) = \frac{(\overline{\alpha}\beta - 1)z + \alpha - \beta}{(\overline{\alpha} - \overline{\beta})z + \alpha\overline{\beta} - 1}.$$

Proof. The composition of the to reflections $u(z) = \frac{\alpha\overline{z}-1}{\overline{z}-\overline{\alpha}}$ and $v(z) = \frac{\beta\overline{z}-1}{\overline{z}-\overline{\beta}}$ can be calculated using:

$$\begin{aligned}(v \circ u)(z) =& \frac{\beta\overline{\left(\frac{\alpha\overline{z}-1}{\overline{z}-\overline{\alpha}}\right)} - 1}{\overline{\left(\frac{\alpha\overline{z}-1}{\overline{z}-\overline{\alpha}}\right)} - \overline{\beta}} = \frac{\beta\frac{\overline{\alpha}z-1}{z-\alpha} - 1}{\frac{\overline{\alpha}z-1}{z-\alpha} - \overline{\beta}} = \frac{\frac{\beta\overline{\alpha}z-\beta}{z-\alpha} - 1}{\frac{\overline{\alpha}z-1}{z-\alpha} - \overline{\beta}} = \frac{\frac{\beta\overline{\alpha}z-\beta-z+\alpha}{z-\alpha}}{\frac{\overline{\alpha}z-1-\overline{\beta}z+\alpha\overline{\beta}}{z-\alpha}} \\ =& \frac{\beta\overline{\alpha}z - \beta - z + \alpha}{\overline{\alpha}z - 1 - \overline{\beta}z + \alpha\overline{\beta}} = \frac{(\overline{\alpha}\beta - 1)z + \alpha - \beta}{(\overline{\alpha} - \overline{\beta})z + \alpha\overline{\beta} - 1}.\end{aligned}$$

□

A.2 Distance function in the Poincaré disk model

We would like to find a function for the distance between the origin and a point $|z|$. Since distances are measured along D-lines, we have to parametrize the unique D-line between the origin and $|z|$. This is given by: $\alpha(t) = (t|z|, 0)$, $t \in [0, 1]$. The length s of this finite D-line is given by:

$$s = \int_0^1 \|\alpha'(t)\| dt.$$

In this section we will calculate this integral:

$$\begin{aligned}s =& \int_0^1 \|\alpha'(t)\| dt = \int_0^1 \langle\alpha'(t), \alpha'(t)\rangle \, dt = \int_0^1 \langle(|z|, 0), (|z|, 0)\rangle \, dt \\ =& \int_0^1 \sqrt{4\frac{|z|^2 + 0^2}{(1 - (|z|t)^2 - 0^2)^2}} dt = 2\int_0^1 \frac{|z|}{1 - (|z|t)^2} dt = 2\tanh^{-1}(|z|).\end{aligned}$$

A.3 Modification of the Minkowski metric

We need to prove that the parametrization for the Minkowski model:

$$\begin{pmatrix} t \\ x \\ y \end{pmatrix} = \begin{pmatrix} \cosh\chi \\ \sinh\chi\cos\theta \\ \sinh\chi\sin\theta \end{pmatrix}.$$

Gives us the metric tensor $ds^2 = d\chi^2 + \sinh^2\chi\ d\theta^2$.

Proof. We begin by calculating dt, dx and dy and insert this in: $ds^2 = -dt^2 + dx^2 + dy^2$.

$$\begin{aligned} dt &= d(\cosh\chi) & dx &= d(\sinh\chi\cos\theta) & dy &= d(\sinh\chi\sin\theta) \\ &= \sinh\chi\ d\chi, & &= \cosh\chi\cos\theta\ d\chi - \sinh\chi\sin\theta\ d\theta, & &= \cosh\chi\sin\theta\ d\chi + \sinh\chi\cos\theta\ d\theta. \end{aligned}$$

$$\begin{aligned} ds^2 =& - dt^2 + dx^2 + dy^2 \\ =& - \sinh^2\chi\ d\chi^2 + (\cosh\chi\cos\theta\ d\chi - \sinh\chi\sin\theta\ d\theta)^2 + (\cosh\chi\sin\theta\ d\chi + \sinh\chi\cos\theta\ d\theta)^2 \\ =& - \sinh^2\chi\ d\chi^2 \\ &+ \cosh^2\chi\cos^2\theta\ d\chi^2 + \sinh^2\chi\sin^2\theta\ d\theta^2 - 2\sinh\chi\cosh\chi\sin\theta\cos\theta\ d\chi d\theta \\ &+ \cosh^2\chi\sin^2\theta\ d\chi^2 + \sinh^2\chi\cos^2\theta\ d\theta^2 + 2\sinh\chi\cosh\chi\sin\theta\cos\theta\ d\chi d\theta \\ =& - \sinh^2\chi\ d\chi^2 + \cosh^2\chi(\sin^2\theta + \cos^2\theta)\ d\chi^2 + \sinh^2\chi(\sin^2\theta + \cos^2\theta) d\theta^2 \\ =& (\cosh^2\chi - \sinh^2\chi)\ d\chi^2 + \sinh^2\chi d\theta^2 \\ =& d\chi^2 + \sinh^2\chi d\theta^2. \end{aligned}$$

□

Appendix B

Mappings

B.1 From the Minkowski model to the Poincaré disk

Starting from the Minkowski model, one will end up at the space of the Poincaré disk model by intersecting the line through a point of the Minkowski model and the point $(-1, 0, 0)$ with the plane $t = 0$, see figure B.1.

We will parameterize this line by: $(t, m_1, m_2) + p(-1 - t, -m_1, -m_2) = (t - p(t + 1), m_1(1 - p), m_2(1 - p))$. We would like to know for which p this line intersects with the plane $t = 0$, thus we have to solve: $t - p(t+1) = 0$. It is not to hard to see that the solution of this equation is: $p = \frac{t}{t+1}$. Substituting this in the equation of the line gives us the point: $(0, m_1(1 - \frac{t}{t+1}), m_2(1 - \frac{t}{t+1}) = (0, \frac{m_1}{t+1}, \frac{m_2}{t+1})$. Now we have found the required equation:

$$f(t, m_1, m_2) = \left(\frac{m_1}{t+1}, \frac{m_2}{t+1} \right).$$

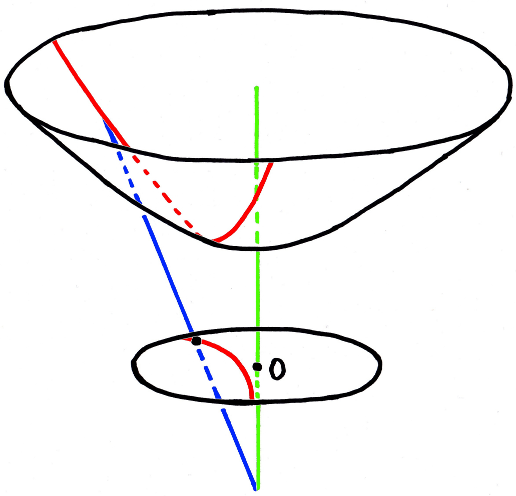

Figure B.1: Mapping from the Minkowski model to the Poincaré disk

B.2 From the Minkowski model to the Beltrami-Klein-Hilbert disk

Starting from the Minkowski model, one will end up at the space of the Beltrami-Klein-Hilbert model by intersecting the line through a point of the Minkowski model and the point $(0,0,0)$ with the plane $t = 1$, see figure B.2.

We will parameterize this line by: $(t, m_1, m_2)+p(-t, -m_1, -m_2) = (t-pt, m_1(1-p), m_2(1-p))$. We would like to know for which p this line intersects with the plane $t = 1$, thus we have to solve: $t-pt = 1$. It is not to hard to see that the solution of this equation is: $p = \frac{t-1}{t}$. Substituting this in the equation of the line gives us the point: $(0, m_1(1-\frac{t-1}{t}), m_2(1-\frac{t-1}{t}) = (0, \frac{m_1}{t}, \frac{m_2}{t})$. Now we have found the required equation:

$$f(t, m_1, m_2) = \left(\frac{m_1}{t}, \frac{m_2}{t}\right).$$

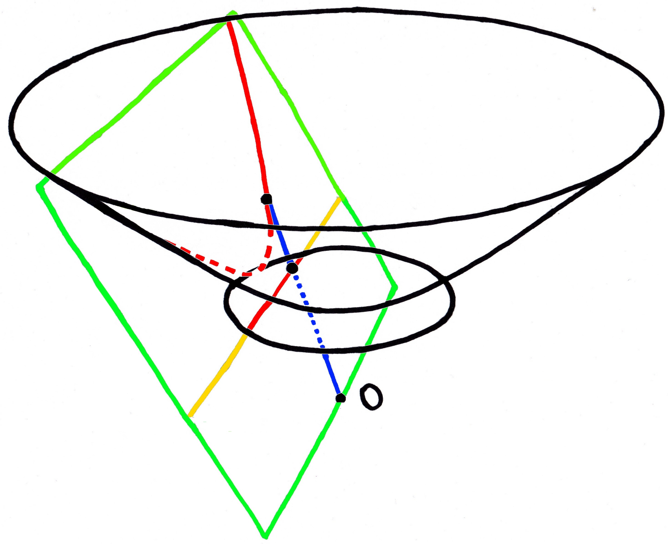

Figure B.2: Mapping from the Minkowski model to Beltrami-Klein-Hilbert disk

Appendix C

Equivalences of the models

C.1 Equivalence of the Minkowski and the Poincaré disk model

The function between the Minkowski model and the unit disk is given by: $f(t, m_1, m_2) = (\frac{m_1}{t+1}, \frac{m_2}{t+1})$. Using the parametrization of the Minkowski model gives us:

$$g(\chi, \theta) = \left(\frac{\sinh\chi\cos\theta}{1+\cosh\chi}, \frac{\sinh\chi\sin\theta}{1+\cosh\chi} \right).$$

We will show that this function is the required isometry between the two models by verifying that

$$4\frac{dp_1^2 + dp_2^2}{(1-(p_1^2+p_2^2))^2} = d\chi^2 + \sinh\chi^2 \ d\theta^2.$$

Like in section A.3 we will start by calculating dp_1 and dp_2:

$$\begin{aligned}
dp_1 =& d\left(\frac{\sinh\chi\cos\theta}{1+\cosh\chi}\right) \\
=& \frac{(1+\cosh\chi)(\cosh\chi\cos\theta) - (\sinh\chi\cos\theta)(\sinh\chi)}{(1+\cosh\chi)^2} d\chi - \frac{\sinh\chi\sin\theta}{1+\cosh\chi} d\theta \\
=& \frac{\cosh\chi\cos\theta + \cos\theta}{(1+\cosh\chi)^2} d\chi - \frac{\sinh\chi\sin\theta}{1+\cosh\chi} d\theta \\
=& \frac{\cos\theta}{1+\cosh\chi} d\chi - \frac{\sinh\chi\sin\theta}{1+\cosh\chi} d\theta.
\end{aligned}$$

$$\begin{aligned}
dp_2 =& d\left(\frac{\sinh\chi\sin\theta}{1+\cosh\chi}\right) \\
=& \frac{(1+\cosh\chi)(\cosh\chi\sin\theta) - (\sinh\chi\sin\theta)(\sinh\chi)}{(1+\cosh\chi)^2} d\chi + \frac{\sinh\chi\cos\theta}{1+\cosh\chi} d\theta \\
=& \frac{\cosh\chi\sin\theta + \sin\theta}{(1+\cosh\chi)^2} d\chi + \frac{\sinh\chi\cos\theta}{1+\cosh\chi} d\theta \\
=& \frac{\sin\theta}{1+\cosh\chi} d\chi + \frac{\sinh\chi\cos\theta}{1+\cosh\chi} d\theta.
\end{aligned}$$

Next we will calculate their squares:

$$dp_1^2 = \frac{\cos^2\theta}{(1+\cosh\chi)^2}d\chi^2 + \frac{\sinh^2\chi\sin^2\theta}{(1+\cosh\chi)^2}d\theta^2 - 2\frac{\sinh\chi\sin\theta\cos\theta}{(1+\cosh\chi)^2}d\chi d\theta.$$

$$dp_2^2 = \frac{\sin^2\theta}{(1+\cosh\chi)^2}d\chi^2 + \frac{\sinh^2\chi\cos^2\theta}{(1+\cosh\chi)^2}d\theta^2 + 2\frac{\sinh\chi\sin\theta\cos\theta}{(1+\cosh\chi)^2}d\chi d\theta.$$

Now we calculate $1-(p_1^2+p_2^2)$:

$$\begin{aligned}1-(p_1^2+p_2^2) =& 1 - \frac{\sinh^2\chi}{(1+\cosh\chi)^2} = \frac{(1+\cosh\chi)^2-\sinh^2\chi}{(1+\cosh\chi)^2}\\ =& \frac{1+\cosh^2\chi+2\cosh\chi-\sinh^2\chi}{(1+\cosh\chi)^2} = \frac{2(1+\cosh\chi)}{(1+\cosh\chi)^2}\\ =& \frac{2}{1+\cosh\chi}.\end{aligned}$$

Putting it all together gives:

$$\begin{aligned}4\frac{dp_1^2+dp_2^2}{(1-(p_1^2+p_2^2))^2} =& \frac{4}{4/(1+\cosh\chi)^2}\left(\frac{\cos^2\theta+\sin^2\theta}{(1+\cosh\chi)^2}d\chi^2 + \frac{\sinh^2\chi(\sin^2\theta+\cos^2\theta)}{(1+\cosh\chi)^2}d\theta^2\right)\\ =& (1+\cosh\chi)^2\left(\frac{1}{(1+\cosh\chi)^2}d\chi^2 + \frac{\sinh^2\chi}{(1+\cosh\chi)^2}d\theta^2\right)\\ =& d\chi^2 + \sinh^2\chi\ d\theta^2,\end{aligned}$$

as required. This proves that the Minkowski and the Poincaré disk model are equivalent.

C.2 Equivalence of the Minkowski and the Beltrami-Klein-Hilbert model

The function between the Minkowski model and the unit disk is given by: $f(t,m_1,m_2) = (\frac{m_1}{t},\frac{m_2}{t})$. Using the parametrization of the Minkowski model gives us:

$$g(\chi,\theta) = \left(\frac{\sinh\chi\cos\theta}{\cosh\chi}, \frac{\sinh\chi\sin\theta}{\cosh\chi}\right) = (\tanh\chi\cos\theta, \tanh\chi\sin\theta).$$

We will show that this function is the required isometry between the two models by verifying that

$$\frac{(1-b_2^2)db_1^2 + 2b_1b_2db_1db_2 + (1-b_1^2)db_2^2}{(1-(b_1^2+b_2^2))^2} = d\chi^2 + \sinh\chi^2\ d\theta^2.$$

Like in section A.3 we will begin by calculating db_1 and db_2:

$$\begin{aligned}db_1 =& d(\tanh\chi\cos\theta)\\ =& \frac{\cos\theta}{\cosh^2\chi}d\chi - \tanh\chi\sin\theta\ d\theta.\end{aligned}\qquad \begin{aligned}db_2 =& d(\tanh\chi\sin\theta)\\ =& \frac{\sin\theta}{\cosh^2\chi}d\chi + \tanh\chi\cos\theta\ d\theta.\end{aligned}$$

Next we will calculate their squares:

$$\begin{aligned}db_1^2 =& \frac{\cos^2\theta}{\cosh^4\chi}d\chi^2 + \tanh^2\chi\sin^2\theta\ d\theta^2 - 2\frac{\tanh\chi\sin\theta\cos\theta}{\cosh^2\theta}d\chi d\theta.\\ db_2^2 =& \frac{\sin^2\theta}{\cosh^4\chi}d\chi^2 + \tanh^2\chi\cos^2\theta\ d\theta^2 + 2\frac{\tanh\chi\sin\theta\cos\theta}{\cosh^2\chi}d\chi\ d\theta.\end{aligned}$$

Now we calculate $1-(b_1^2+b_2^2)$:

$$1-(b_1^2+b_2^2)=1-\tanh^2\chi=\frac{\cosh^2\chi-\sinh^2\chi}{\cosh^2\chi}=\frac{1}{\cosh^2\chi}.$$

Calculating $(1-b_2^2)db_1^2+2b_1b_2\ db_1db_2+(1-b_1^2)db_2^2$ gives:

$(1-b_2^2)db_1^2+2b_1b_2\ db_1db_2+(1-b_1^2)db_2^2$

$$\begin{aligned}
=&\frac{(1-\tanh^2\chi\sin^2\theta)\cos^2\theta}{\cosh^4\chi}d\chi^2+(1-\tanh^2\chi\sin^2\theta)\tanh^2\chi\sin^2\theta\ d\theta^2\\
&-2\frac{(1-\tanh^2\chi\sin^2\theta)\tanh\chi\cos\theta\sin\theta}{\cosh^2\chi}d\chi d\theta\\
&+2\frac{\tanh^2\chi\sin^2\theta\cos^2\theta}{\cosh^4\chi}d\chi^2-2\tanh^4\chi\sin^2\theta\cos^2\theta\ d\theta^2\\
&+\left(2\frac{\tanh^2\chi\cos^3\theta\sin\theta}{\cosh^2\chi}-2\frac{\tanh^2\chi\sin^3\theta\cos\theta}{\cosh^2\chi}\right)\ d\chi d\theta\\
&+\ \frac{(1-\tanh^2\chi\cos^2\theta)\sin^2\theta}{\cosh^4\chi}d\chi^2+(1-\tanh^2\chi\cos^2\theta)\tanh^2\chi\cos^2\theta\ d\theta^2\\
&+2\frac{(1-\tanh^2\chi\cos^2\theta)\tanh\chi\sin\theta\cos\theta}{\cosh^2\chi}d\chi\ d\theta\\
=&\frac{\cos^2\theta+\sin^2\theta}{\cosh^4\chi}d\chi^2+(\tanh^2\chi\sin^2\theta-\tanh^4\sin^4\theta-2\tanh^4\chi\sin^2\theta\cos^2\theta\\
&+\tanh^2\chi\cos^2\theta-\tanh^4\chi\cos^4\theta)d\theta^2\\
=&\frac{1}{\cosh^4\chi}d\chi^2+(\tanh^2\chi(\sin^2\theta+\cos^2\theta-\tanh^2(\sin^4\theta+2\sin^2\theta\cos^2\theta+\cos^4\theta)))d\theta^2\\
=&\frac{1}{\cosh^4\chi}d\chi^2+\tanh^2\chi(1-\tanh^2)d\theta^2=\frac{1}{\cosh^4\chi}d\chi^2+\tanh^2\chi\left(\frac{1}{\cosh^2\chi}\right)d\theta^2\\
=&\frac{1}{\cosh^4\chi}\left(d\chi^2+\sinh^2\chi\ d\theta^2\right).
\end{aligned}$$

Calculating the entire metric tensor gives:

$$\begin{aligned}
\frac{(1-b_2^2)db_1^2+2b_1b_2db_1db_2+(1-b_1^2)db_2^2}{(1-(b_1^2+b_2^2))^2}=&\cosh^4\chi\frac{1}{\cosh^4\chi}\left(d\chi^2+\sinh^2\chi\ d\theta^2\right)\\
=&d\chi^2+\sinh^2\chi\ d\theta^2,
\end{aligned}$$

as required. This proves that the Minkowski and the Beltrami-Klein-Hilbert model are equivalent.

C.3 Equivalence of the Poincaré disk and the Poincaré half-plane model

The mapping from the Poincaré disk to the Poincaré half-plane is given by: $U(z) = \frac{iz-1}{-z+i}$ with $|z| \leq 1$. At first we rewrite this function in coordinates using $z = p_1 + ip_2$:

$$\begin{aligned}
U(p_1+ip_2) =& \frac{i(p_1+ip_2)-1}{-(p_1+ip_2)+i} = \frac{ip_1-p_2-1}{-p_1-ip_2+i} \\
=& \frac{ip_1-(p_2+1)}{-p_1-i(p_2-1)} = \frac{ip_1-(p_2+1)}{-p_1-i(p_2-1)}\frac{-p_1+i(p_2-1)}{-p_1+i(p_2-1)} \\
=& \frac{-ip_1^2-p_1p_2+p_1+p_1p_2+p_1-ip_2^2+i}{p_1^2+(p_2-1)^2} \\
=& \frac{i(1-p_1^2-p_2^2)+2p_1}{p_1^2+(p_2-1)^2}.
\end{aligned}$$

Thus we have:

$$k(p_1,p_2) = \left(\frac{2p_1}{p_1^2+(p_2-1)^2}, \frac{1-p_2^2-p_2^2}{p_1^2+(p_2-1)^2}\right).$$

We will show that this function is the required isometry between the two models by verifying that

$$\frac{dh_1^2+dh_2^2}{h_2^2} = 4\frac{dp_1^2+dp_2^2}{(1-p_1^2-p_2^2)^2}.$$

As in section A.3 we first calculate dh_1 and dh_2:

$$\begin{aligned}
dh_1 =& \frac{2(p_1^2+(p_2-1)^2)-4p_1^2}{(p_1^2+(p_2-1)^2)^2}dp_1 + \frac{-4p_1(p_2-1)}{(p_1^2+(p_2-1)^2)^2}dp_2 \\
=& 2\frac{(p_2-1)^2-p_1^2}{(p_1^2+(p_2-1)^2)^2}dp_1 - 4\frac{p_1(p_2-1)}{(p_1^2+(p_2-1)^2)^2}dp_2
\end{aligned}$$

$$\begin{aligned}
dh_2 =& \frac{-2p_1(p_1^2+(p_2-1)^2)-2p_1(1-p_1^2-p_2^2)}{(p_1^2+(p_2-1)^2)^2}dp_1 + \frac{-2p_2(p_1^2+(p_2-1)^2)-2(p_2-1)(1-p_1^2-p_2^2)}{(p_1^2+(p_2-1)^2)^2}dp_2 \\
=& \frac{-2p_1(p_1^2+(p_2-1)^2+1-p_1^2-p_2^2)}{(p_1^2+(p_2-1)^2)^2}dp_1 + \frac{-2p_2(p_1^2+(p_2-1)^2+1-p_1^2-p_2^2)+2-2p_1^2-2p_2^2}{(p_1^2+(p_2-1)^2)^2}dp_2 \\
=& \frac{-2p_1(-2p_2+2)}{(p_1^2+(p_2-1)^2)^2}dp_1 + \frac{-2p_2(-2p_2+2)+2-2p_1^2-2p_2^2}{(p_1^2+(p_2-1)^2)^2}dp_2 \\
=& 4\frac{p_1p_2-p_1}{(p_1^2+(p_2-1)^2)^2}dp_1 + 2\frac{2p_2^2-2p_2+1-p_1^2-p_2^2}{(p_1^2+(p_2-1)^2)^2}dp_2 \\
=& 4\frac{p_1(p_2-1)}{(p_1^2+(p_2-1)^2)^2}dp_1 + 2\frac{p_2^2-2p_2+1-p_1^2}{(p_1^2+(p_2-1)^2)^2}dp_2 \\
=& 4\frac{p_1(p_2-1)}{(p_1^2+(p_2-1)^2)^2}dp_1 + 2\frac{(p_2-1)^2-p_1^2}{(p_1^2+(p_2-1)^2)^2}dp_2.
\end{aligned}$$

Hence we have that:

$$\begin{aligned}
dh_1^2 =& 4\frac{((p_2-1)^2-p_1^2)^2}{(p_1^2+(p_2-1)^2)^4}dp_1^2 + 16\frac{p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_2^2 \\
&- 16\frac{p_1(p_2-1)((p_2-1)^2-p_1^2)}{(p_1^2+(p_2-1)^2)^4}dp_1dp_2 \\
=& 4\frac{(p_2-1)^4+p_1^4-2p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_1^2 + 16\frac{p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_2^2 \\
&- 16\frac{p_1(p_2-1)((p_2-1)^2-p_1^2)}{(p_1^2+(p_2-1)^2)^4}dp_1dp_2 \\
dh_2^2 =& 16\frac{p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_1^2 + 4\frac{((p_2-1)^2-p_1^2)^2}{(p_1^2+(p_2-1)^2)^4}dp_2^2 \\
&+ 16\frac{p_1(p_2-1)((p_2-1)^2-p_1^2)}{(p_1^2+(p_2-1)^2)^4}dp_1dp_2 \\
=& 16\frac{p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_1^2 + 4\frac{(p_2-1)^4+p_1^4-2p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_2^2 \\
&+ 16\frac{p_1(p_2-1)((p_2-1)^2-p_1^2)}{(p_1^2+(p_2-1)^2)^4}dp_1dp_2.
\end{aligned}$$

From the previous we find:

$$\begin{aligned}
dh_1^2+dh_2^2 =& 4\frac{(p_2-1)^4+p_1^4-2p_1^2(p_2-1)^2+4p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_1^2 \\
&+ 4\frac{(p_2-1)^4+p_1^4-2p_1^2(p_2-1)^2+4p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_2^2 \\
=& 4\frac{(p_2-1)^4+p_1^4+2p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_1^2 \\
&+ 4\frac{(p_2-1)^4+p_1^4+2p_1^2(p_2-1)^2}{(p_1^2+(p_2-1)^2)^4}dp_2^2 \\
=& 4\frac{(p_1^2+(p_2-1)^2)^2}{(p_1^2+(p_2-1)^2)^4}dp_1^2 + 4\frac{(p_1^2+(p_2-1)^2)^2}{(p_1^2+(p_2-1)^2)^4}dp_2^2 \\
=& 4\frac{1}{(p_1^2+(p_2-1)^2)^2}dp_1^2 + 4\frac{1}{(p_1^2+(p_2-1)^2)^2}dp_2^2.
\end{aligned}$$

Putting it all together gives:

$$\begin{aligned}
\frac{dh_1^2+dh_2^2}{h_2^2} =& \frac{(p_1^2+(p_2-1)^2)^2}{(1-p_1^2-p_2^2)^2}4\frac{1}{(p_1^2+(p_2-1)^2)^2}(dp_1^2+dp_2^2) \\
=& 4\frac{dh_1^2+dh_2^2}{(1-p_1^2-p_2^2)^2},
\end{aligned}$$

as required. This proves that the Poincaré disk and the Poincaré half-plane model are equivalent.