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Approximately optimal control via discrete abstractions

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Abstract

In this thesis we use a symbolic approach to provide a methodology to approximately solve optimal control problems. This is done by considering approximate (alternating) simulation relations to formalize the approximation of infinite-state systems by finite-state systems. We show how such relations enable the transfer of optimality information between systems. Since optimal control problems can be solved automatically on finite-state systems by relying on standard computer science tools, optimal control problems can be solved approximately on infinite-state control systems by working with a finite-state system which is a discrete approximation of the infinite-state control system. We show how the costs computed on the approximated finite-state system provide lower and upper bounds for the optimal achievable cost of the infinite-state control system. Moreover, the approach provides also a method to automatically synthesize controllers respecting these bounds.

Keywords: Symbolic Approach, Simulation relations, Optimal control.

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Chapter 1

Introduction

Traditionally, in computer science finite-state machines have been studied extensively while in control theory researchers have been focusing primarily on differential equations. The field of hybrid systems, in which smooth dynamics of differential/difference equations are combined with discrete-finite dynamics of finite-state machines, has greatly grown in the last two decades. This is motivated by the increasing amount of software and hardware placed in interaction with the physical world. To emphasize the combination of the cyber world (i.e. computing entities modeled by finite-state machines) with the physical world, hybrid systems have been renamed cyber-physical systems in the more engineering applications.

Cyber-physical systems are in general hard to analyze and even harder to control since the finite-state machines from the cyber world evolve on finite-state sets while the differential/difference equations from the physical world evolve on infinite state sets. Symbolic abstractions of control systems are simpler descriptions of control systems, with typically finitely many states, where each state represents infinitely many states of the control system. If symbolic abstractions can be constructed to represent cyber-physical systems, then the symbolic abstractions are of the same nature as the software and hardware models in computer science. Hence, there is an unified language to study problems of control in which software and hardware interact with the physical world provided by the symbolic abstractions. The use of the symbolic abstractions now enables one to leverage tools from computer science and game theory for automated controller synthesis purposes.

If the symbolic abstraction is taken such that some (approximate) equivalence relation is retained between the cyber-physical system and the symbolic abstraction, then both analysis and design made on the abstracted system can be transferred back to the cyber-physical system. It has been shown in [1,2,3] that under mild assumptions one can construct symbolic abstractions for cyber-physical system retaining the afore mentioned (approximate) equivalence relations.

In control problems we are interested in modifying a system's behavior such that the design conforms to some provided specifications. Qualitative specifications, i.e. specifications in which the trajectories of the plant are divided into bad trajectories (those that need to be avoided) and good trajectories, are easily captured by finite-state machines, i.e. symbolic abstractions of control systems. Additionally, in many applications the optimization of quantitative measures of the trajectories retained by the controller, as specified by a cost or utility function, is also required. Hence, the control design problem is required to remove the undesirable trajectories (qualitative specification) and to select the minimum cost or maximum utility trajectory (quantitative measure).

In the paper 'Symbolic approximate time-optimal control' by Mazo Jr. and Tabuada [4] a first step towards the objective of synthesizing controllers enforcing qualitative and quantitative objectives is taken by considering the synthesis of time-optimal controllers for reachability specifications. Our objective is to extend the analysis of Mazo Jr. and Tabuada in [4] to more general optimal controllers for reachability specifications. We want to know what the best achievable costs are for an optimal control problem by working with symbolic abstractions which are related to the control system by some (approximate) equivalence relation. By determining the best achievable costs for the symbolic abstractions, we provide bounds for the best achievable costs in the control system. As a side benefit, the approach provides also a method to automatically synthesize controllers for the optimal control problem respecting those bounds.

The thesis is organized as follows: in Chapter 2 we review the basic material on the notion of systems, relationships between systems and composition of systems. Chapter 2 also contains some illustrative examples about these notions. In Chapter 3 first the formalization of a meaningful instantaneous cost function is considered. Then the notions of reachability, system entry time and final cost are formalized and some illustrative examples are also given about these notions. Thereafter it is shown that when a system S_a is related to a system S_b by an approximate (alternating) simulation relation, optimality information can be transferred from S_a to S_b. Using this result, it is explained how lower and upper bounds for the cost in system S_b can be computed on system S_a. These lower and upper bounds provide a performance guarantee for the optimal control problem of system S_b. Chapter 4 concludes this thesis with a conclusion and future work discussions.

Chapter 2

Preliminaries

2.1 Notation

To be able to define our problem properly in Chapter 3, we start by introducing some notation and background.

We denote by $\mathbb{N}$ the natural numbers including zero and by $\mathbb{N}^+$ the strictly positive natural numbers. With $\mathbb{R}_0^+$ we denote the positive real numbers including zero.

For a set Z we denote by $1_Z : Z \to Z$ the identity map on Z defined by $1_Z(z) = z$ for every $z \in Z$. The map $\pi_X : X_a \times X_b \times U_a \times U_b \to X_a \times X_b$ is called the projection map and maps $(x_a, x_b, u_a, u_b) \in X_a \times X_b \times U_a \times U_b$ onto $(x_a, x_b) \in X_a \times X_b$.

The pre-image of a set $W \subseteq Y$ under a map $H : X \to Y$ is denoted by $H^{-1}(W)$ and defined as the set $H^{-1}(W) = \{x \in X \mid H(x) \in W\}$.

A normed vector space V is a vector space equipped with a norm $\|\cdot\|$. Now, $\forall x, y \in V$ this norm $\|\cdot\|$ induces the metric $d(x, y) = \|x - y\|$ on V, where $d : V \times V \to \mathbb{R}_0^+$. The norm $\|\cdot\|$ denotes the infinity norm given by $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, $\forall x \in \mathbb{R}^n$, where $|x_i|$ denotes the absolute value of the i-th component of x ($i \in \{1, 2, \ldots, n\}$).

We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(A)$ if $(a, b) \in R$. The set 2^B denotes the power set of B, i.e. the set of all the subsets of B. For a set $S \subseteq A$, the set $R(S)$ is definded as $R(S) = \{b \in B \mid \exists a \in S \,\text{such that}\, (a, b) \in R\}$. The inverse relation R^{-1} is defined as $R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}$.

2.2 Systems

In the following the word *system* refers to a mathematical model of a dynamical phenomenon. A versatile notion of system is required since we want to describe different kinds of mathematical models with finite-state sets, infinite-state sets and combinations thereof. Such a notion of system can then be used to describe finite-state machines, i.e. models with finitely many states, infinite-state systems described by differential or difference equations with solutions evolving in infinite-state sets and hybrid systems, i.e. systems combining aspects of finite-state and infinite-state systems. The mathematical notion of system, as described in [5], satisfies this requirement and is formalized as follows:

Definition 2.1. (System [5]). A system S is a sextuple $(X, X_0, U, \rightarrow, Y, H)$ consisting of:

- a set of states X;
- a set of initial states $X_0 \subseteq X$;
- a set of inputs U;
- a transition relation $\rightarrow \subseteq X \times U \times X$;
- a set of outputs Y;
- an output map $H : X \rightarrow Y$.

A system is said to be *metric* if the output set Y is equipped with a metric $d : Y \times Y \rightarrow \mathbb{R}_0^+$.

The transition relation in definition (2.1) captures the evolution of the system and the notation $x \xrightarrow{u} x'$ is used to denote a transition $(x, u, x') \in \rightarrow$. For such a transition, state x' is called an u-successor (or simply successor) of state x. The set of u-successors of a state $x \in X$ is denoted by $\text{Post}_u(x)$. Since $\text{Post}_u(x)$ may be empty, the set $U(x)$ denotes the set of inputs $u \in U$ for which $\text{Post}_u(x)$ is nonempty.

If there exists a state x in the system with no u-successors, i.e. $\forall u \in U : \text{Post}_u(x) = \emptyset$, then the system is called *blocking*. The system is called *nonblocking* if the set of u-successors of every $x \in X$ is nonempty, i.e. $\forall x \in X, \exists u \in U : \text{Post}_u(x) \neq \emptyset$.

Additionally, the system is called *deterministic* if for any state $x \in X$ and any input $u \in U$, there exists at most one u-successor, i.e. for any $x \in X$ and for any $u \in U$, $x \xrightarrow{u} x'$ and $x \xrightarrow{u} x''$ implies $x' = x''$. The system is called *non-deterministic* if the system is not determistic which means that the system has at least one state with two (or more) distinct u-successors, i.e. $\exists x \in X$ and $\exists u \in U : \text{Post}_u(x) > 1$.

Every non-deterministic system S_a can be associated with a deterministic system $S_{d(a)}$ by extending the set of inputs. This result will be used in the main theorem of this thesis and is formalized as follows:

Definition 2.2. (Associated Deterministic System [4]). The deterministic system $S_{d(a)} = (X_a, X_{a0}, U_{d(a)}, \underset{d(a)}{\longrightarrow}, Y_a, H_a)$ associated with a given non-deterministic system $S_a = (X_a, X_{a0}, U_a, \underset{a}{\rightarrow}, Y_a, H_a)$ is defined by:

- $U_{d(a)} = U_a \times X_a$;
- $x_a \xrightarrow[d(a)]{(u_a, x'_a)} x'_a$ if there exists $x_a \xrightarrow[a]{u_a} x'_a$ in S_a.

So, the set of inputs $U_{d(a)}$ of system $S_{d(a)}$ consists of pairs consisting of one input and one state (u_a, x'_a) where $u_a \in U_a$ and $x'_a \in X_a$. With this set of inputs the transition relation $\underset{d(a)}{\longrightarrow} \subseteq X_a \times U_{d(a)} \times X_a$ in system $S_{d(a)}$ defines the set of u_a-successors as $\text{Post}_{(u_a, x'_a)}(x_a) = \{x'_a\}$. In other words, the input $(u_a, x'_a) \in U_a \times X_a$ can only take the state $x_a \in X_a$ in system $S_{d(a)}$ to the state $x'_a \in X_a$ (provided that the transistion is allowed in system S_a), while the input $u_a \in U_a$ can take the state $x_a \in X_a$ in system S_a to multiple states (given that this transition is one of the non-deterministic transtions in S_a). Hence, the system $S_{d(a)}$ is deterministic while the associated system S_a is non-deterministic.

As we will see later on, the notions of *internal and external behaviors* of systems turn out to be useful. Internal and external behaviors, which are the possible sequences of states and outputs generated by the system respectively, are formalized as follows:

Definition 2.3. (Behavior [5]). For a system S and given any state $x \in X$, an *finite internal behavior* generated from x is a finite sequence of transitions:

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \dots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n \tag{2.1}$$

such that $x_0 = x$ and for all $0 < i \leq n : (x_{i-1}, u_{i-1}, x_i) \in \rightarrow$. An *infinite internal behavior* generated from x is an infinite sequence of transitions:

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} x_3 \xrightarrow{u_3} \dots \tag{2.2}$$

such that $x_0 = x$ and for all $i \in \mathbb{N}^+ : (x_{i-1}, u_{i-1}, x_i) \in \rightarrow$.
Through the output map H, every finite internal behavior as in (2.1) defines a *finite external behavior*:

$$y_0 \rightarrow y_1 \rightarrow y_2 \rightarrow \dots \rightarrow y_{n-1} \rightarrow y_n \tag{2.3}$$

with $H(x_i) = y_i \in Y$ for all $0 \leq i \leq n$. Similarly, every infinite internal behavior as in (2.2) defines an *infinite external behavior*:

$$y_0 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow \dots \tag{2.4}$$

with $H(x_i) = y_i \in Y$ for all $i \in \mathbb{N}$.

The set of all finite external behaviors that are defined by finite internal behaviors generated from state x, as defined above, is denoted by $\mathcal{B}_x(S)$ and is called the *finite external behavior* from state x. Similarly, the set of all infinite external behaviors generated from x, also defined above, is denoted by $\mathcal{B}^\omega_x(S)$ and is called the *infinite external behavior* from state x.

More concise notations $\mathbf{x} = x_0x_1x_2 \dots x_n$ and $\mathbf{x} = x_0x_1x_2x_3 \dots$ are sometimes used to denote the internal behaviors (2.1) and (2.2), respectively. Similarly, the notations $\mathbf{y} = y_0y_1y_2 \dots y_n$ and $\mathbf{y} = y_0y_1y_2y_3 \dots$ are sometimes used to denote the external behaviors (2.3) and (2.4), respectively. The distinction between a finite and an infinite internal behavior, which can both be referred to as $\mathbf{x}$, follows always from the context. The same holds for the distinction between a finite and an infinte external behavior, which can both be referred to as $\mathbf{y}$. Additionally, $\mathbf{x}(k)$ denotes the k-th state of an internal behavior $\mathbf{x}$, i.e. x_k, and $\mathbf{y}(k)$ denotes the corresponding k-th output of the external behavior $\mathbf{y}$, i.e. y_k.

A behavior $\mathbf{y}$ is said to be *maximal* if there is no other behavior containing $\mathbf{y}$ as a prefix. In other words, $\mathbf{y} \in \mathcal{B}(S) \cup \mathcal{B}^\omega(S)$ is said to be maximal if $\mathbf{y} \in \mathcal{B}^\omega(S)$ (i.e. $\mathbf{y}$ is an infinite external behavior) or if $\mathbf{y} = y_0y_1y_2 \dots y_k \in \mathcal{B}(S)$ with $k \in \mathbb{N}^+$ (i.e. $\mathbf{y}$ is a finite external behavior) and there exists no external behavior $\mathbf{w} = w_0w_1w_2 \dots w_kw_{k+1} \dots w_l \in \mathcal{B}(S) \cup \mathcal{B}^\omega(S)$ with some $k < l \in \mathbb{N}^+$ satisfying $y_i = w_i$ for $i = 0, 1, 2, \dots, k$.

If an internal behavior is generated from a state $x \in X_0$, then we call the internal behavior *initialized*. An external behavior corresponding to an initialized internal behavior is called an *initialized external behavior*.

2.3 Systems relations

The results that are proved in this paper are built upon certain equivalence relations that can be established between systems. These equivalence relations between systems are called simulation relations and are formulated as follows.

In the definition of *Approximate Simulation Relation* it is formalized how one system can approximate another system.

Definition 2.4. (Approximate Simulation Relation [5]). Consider two metric systems S_a and S_b with $Y_a = Y_b$ normed vector spaces, and let $\varepsilon \in \mathbb{R}_0^+$. A relation $R \subseteq X_a \times X_b$ is an *ε-approximate simulation relation* from S_a to S_b if the following three conditions are satisfied:

1. for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
2. for every $(x_a, x_b) \in R$ we have $d(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
3. for every $(x_a, x_b) \in R$ we have that: $x_a \underset{a}{\xrightarrow{u_a}} x_a'$ in S_a implies the existence of $x_b \underset{b}{\xrightarrow{u_b}} x_b'$ in S_b satisfying $(x_a', x_b') \in R$.

An ε-approximate simulation relation $R \subseteq X_a \times X_b$ relates the states of system S_b to the states of system S_a as follows: if $(x_a, x_b) \in R$, then state x_b of system S_b is ε-related to state x_a of system S_a. Following this intuitive idea, the first condition in Definition 2.4 actually states that every initial state of system S_a must have an equivalent initial state in system S_b. So the ε-approximate simulation relation R must respect *initial states*. The second condition states that if state x_b is the equivalent of state x_a, then the output generated by state x_b should be the same as the output generated by state x_a up to an error ε. So, the ε-approximate simulation relation R must respect *observations*. The third condition in Definition 2.4 states that all transitions $x_a \underset{a}{\xrightarrow{u_a}} x_a'$ in system S_a can be matched to a transition $x_b \underset{b}{\xrightarrow{u_b}} x_b'$ in system S_b, where matching means that the successor states remain in the ε-approximate simulation relation, i.e. $(x_a', x_b') \in R$. So, the ε-approximate simulation relation R must also respect *transitions*.

We say that system S_a is ε-approximately simulated by system S_b or that system S_b ε-approximately simulates system S_a, denoted by $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b.

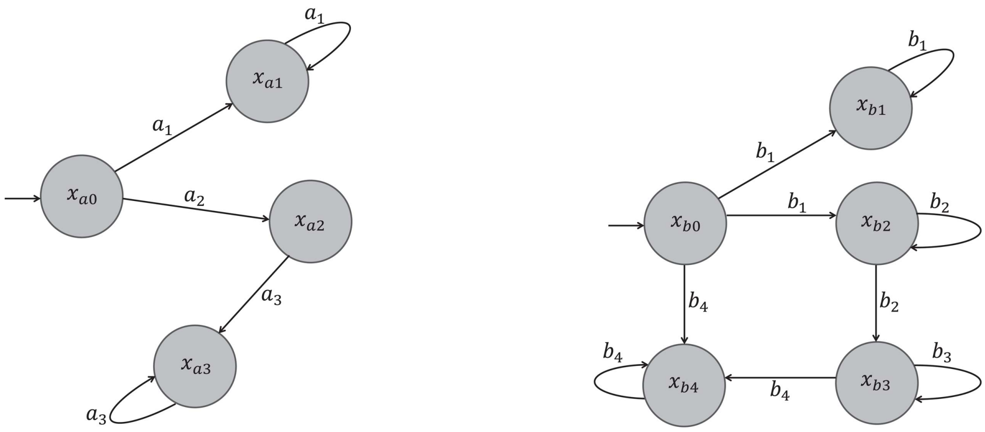
If $\varepsilon = 0$, then the second condition of Definition 2.4 (the observation condition) implies that the output generated by state x_b should be exactly the same as the output generated by state x_a, i.e. $(x_a, x_b) \in R$ implies $H_a(x_a) = H_b(x_b)$, and we say that the relation R is an *exact* simulation relation.

The following example shows how it can be verified that a given relation R is an ε-approximate simulation relation between two given discrete-state systems.

Example 2.5. (Simulation Relation). *Consider the two metric systems S_a and S_b with $Y_a = Y_b$ normed vector spaces displayed in Figure 2.1. Let $\varepsilon \in \mathbb{R}_0^+$ and assume that we are given: $d(H_a(x_{ai}), H_b(x_{bi})) \leq \varepsilon$ for $i = 0, 1, 2, 3$. The relation*

$$R = \{(x_{a0}, x_{b0}),\ (x_{a1}, x_{b1}),\ (x_{a2}, x_{b2}),\ (x_{a3}, x_{b3})\}$$

is an ε-approximate simulation relation from S_a to S_b. Note that the state x_{b4} of system S_b is not covered by R. That R is an ε-approximate simulation relation from S_a to S_b



(a) Deterministic system S_a (b) Non-deterministic system S_b

Figure 2.1: Two discrete-state systems.

can be verified by checking the conditions of Definition 2.4:

1. *$X_{a0} = \{x_{a0}\}$ and for x_{a0} there exists $x_{b0} \in X_{b0}$ satisfying $(x_{a0}, x_{b0}) \in R$.*
2. *For $(x_{a0}, x_{b0}) \in R : d(H_a(x_{a0}), H_b(x_{b0})) \leq \varepsilon$,*
   for $(x_{a1}, x_{b1}) \in R : d(H_a(x_{a1}), H_b(x_{b1})) \leq \varepsilon$,
   for $(x_{a2}, x_{b2}) \in R : d(H_a(x_{a2}), H_b(x_{b2})) \leq \varepsilon$
   and for $(x_{a3}, x_{b3}) \in R : d(H_a(x_{a3}), H_b(x_{b3})) \leq \varepsilon$.
3. *(i) $(x_{a0}, x_{b0}) \in R$: for the transition $x_{a0} \xrightarrow[a]{a_1} x_{a1}$ in S_a there exists the matching transition $x_{b0} \xrightarrow[b]{b_1} x_{b1}$ in S_b satisfying $(x_{a1}, x_{b1}) \in R$, and for the transition $x_{a0} \xrightarrow[a]{a_2} x_{a2}$ in S_a there exists the matching transition $x_{b0} \xrightarrow[b]{b_1} x_{b2}$ in S_b satisfying $(x_{a2}, x_{b2}) \in R$.*

   (ii) $(x_{a1}, x_{b1}) \in R$: for the transition $x_{a1} \xrightarrow[a]{a_1} x_{a1}$ in S_a there exists the matching transition $x_{b1} \xrightarrow[b]{b_1} x_{b1}$ in S_b satisfying $(x_{a1}, x_{b1}) \in R$.

   (iii) $(x_{a2}, x_{b2}) \in R$: for the transition $x_{a2} \xrightarrow[a]{a_3} x_{a3}$ in S_a there exists the matching transition $x_{b2} \xrightarrow[b]{b_2} x_{b3}$ in S_b satisfying $(x_{a3}, x_{b3}) \in R$.

   (iv) $(x_{a3}, x_{b3}) \in R$: for the transition $x_{a3} \xrightarrow[a]{a_3} x_{a3}$ in S_a there exists the matching transition $x_{b3} \xrightarrow[b]{b_3} x_{b3}$ in S_b satisfying $(x_{a3}, x_{b3}) \in R$.

Since R satisfies all conditions of Definition 2.4 we conclude that R is an ε-approximate simulation relation from S_a to S_b.

If system S_a in Definition 2.4 is non-deterministic, then the notion of simulation can be modified to explicitely account for non-determinism.

Definition 2.6. (Approximate Alternating Simulation Relation [5]). Let S_a and S_b be two metric systems with $Y_a = Y_b$ normed vector spaces, and let $\varepsilon \in \mathbb{R}_0^+$. A relation $R \subseteq X_a \times X_b$ is an *ε-approximate alternating simulation relation* from S_a to S_b if the following three conditions are satisfied:

1. for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
2. for every $(x_a, x_b) \in R$ we have $d(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
3. for every $(x_a, x_b) \in R$ and for every $u_a \in U_a(x_a)$ there exists $u_b \in U_b(x_b)$ such that for every $x'_b \in \mathrm{Post}_{u_b}(x_b)$ there exists $x'_a \in \mathrm{Post}_{u_a}(x_a)$ satisfying $(x'_a, x'_b) \in R$.

We say that system S_a is ε-approximately alternatingly simulated by system S_b or that system S_b ε-approximately alternatingly simulates system S_a, denoted by $S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b$, if there exists an ε-approximate alternating simulation relation from S_a to S_b.

As in the case of exact simulation relations, we say that the alternating simulation relation is *exact* if $\varepsilon = 0$.

Note 2.7. In the case of determistic systems, the notion of alternating simulation coincides with the notion of simulation. This can easily be seen as follows: determinism implies $|\mathrm{Post}_{u_b}(x_b)| \leq 1$ and $|\mathrm{Post}_{u_a}(x_a)| \leq 1$ for every $u_b \in U_b(x_b)$ and $u_a \in U_a(x_a)$.

Note 2.8. Any non-deterministic system S_a and its deterministic counterpart $S_{d(a)}$ (as described in Definition 2.2) satisfy $S_a \preceq^{0}_{\mathcal{AS}} S_{d(a)}$. The exact alternating simulation relation $R_{d(a)}$ from S_a to $S_{d(a)}$ is given by $R_{d(a)} = \{(x_a, x_{d(a)}) \in X_a \times X_a \,|\, x_{d(a)} = x_a\}$. This can be verified by checking the conditions of Definition 2.6. The first two conditions follow directly by the definition of $S_{d(a)}$ and $R_{d(a)}$. The third condition is less obvious and is therefore explained now: for every $(x_a, x_{d(a)}) \in R_{d(a)}$ (i.e. $x_a = x_{d(a)}$) and for every $u_a \in U_a(x_a)$ there exists $u_{d(a)} \in U_{d(a)}(x_a)$, namely $u_{d(a)} = (u_a, x'_a)$, such that for $x'_a = Post_{(u_a, x'_a)}(x_a)$ in $S_{d(a)}$ there exists $x'_a \in Post_{u_a}(x_a)$ in S_a satisfying $(x'_a, x'_a) \in R_{d(a)}$.

Example 2.9 shows that for non-determinism the ε-approximate simulation relation from Example 2.5 does not satsify the ε-approximate *alternating* simulation relation definition (Definition 2.6). Furthermore, another relation R' is given that does satisfy the ε-approximate alternating simulation relation definition.

Example 2.9. (Alternating Simulation Relation). *From Example 2.5 we already know that the relation*

$$R = \{(x_{a0}, x_{b0}),\, (x_{a1}, x_{b1}),\, (x_{a2}, x_{b2}),\, (x_{a3}, x_{b3})\}$$

is an ε-approximate simulation relation from S_a to S_b (depicted in Figure 2.1).

However, the relation R is not an ε-approximate alternating simulation relation from S_a to S_b since the third condition of Definition 2.6 is violated: Consider the element $(x_{a2}, x_{b2}) \in R$. If R would be an ε-approximate alternating simulation relation from S_a to S_b, then for every input in $U_a(x_{a2})$ there would exist an input in $U_b(x_{b2})$ such that for all successor states resulting from applying the specific input from $U_b(x_{b2})$ to x_{b2}, there exists an successor state in the set resulting from applying the specific input from $U_a(x_{a2})$ to x_{a2} such that the combination of the successor states in S_a and S_b are again in R. But, if we consider $a_3 \in U_a(x_{a2})$, then we have to consider the input $b_2 \in U_b(x_{b2})$ (since b_2 is the only input to be applied to x_{b2} in system S_b, i.e. $U_b(x_{b2}) = b_2$) resulting in

$Post_{b_2}(x_{b2}) = \{x_{b2}, x_{b3}\}$. Now for both successor states in $Post_{b_2}(x_{b2})$, there should exist a matching state in $Post_{a_3}(x_{a2})$. But, since $Post_{a_3}(x_{a2}) = \{x_{a3}\}$ and $(x_{a3}, x_{b2}) \notin R$ (while $(x_{a3}, x_{b3}) \in R$), the third condition is not satisfied here.
The relation

$$R' = \{(x_{a0}, x_{b0}),\ (x_{a1}, x_{b1}),\ (x_{a1}, x_{b2}),\ (x_{a1}, x_{b3}),\ (x_{a2}, x_{b4}),\ (x_{a3}, x_{b4})\}$$

is an ε-approximate alternating simulation relation from S_a to S_b if we assume that

$$\begin{aligned} d(H_a(x_{a1}), H_b(x_{bj})) &\leq \varepsilon \qquad\qquad \textit{for } j = 2,3 \\ d(H_a(x_{a2}), H_b(x_{b4})) &\leq \varepsilon \\ d(H_a(x_{a3}), H_b(x_{b4})) &\leq \varepsilon. \end{aligned}$$

That the relaton R' is an ε-approximate alternating simulation relation from S_a to S_b under the above given assumptions can be verified by checking the conditions of Definition 2.6, which is not done here.

2.4 Composition of systems

The feedback composition of a controller S_c with a plant S_a describes the concurrent evolution of systems S_c and S_a subject to the synchronization prescribed by the interconnection relation R^e. In this thesis, by R^e we denote the extended alternating simulation relation associated with alternating simulation relation R, formalized as follows:

Definition 2.10. (Extended alternating simulation relation [5]). Let R be an alternating simulation relation from system S_a to system S_b. The *extended alternating simulation relation* $R^e \subseteq X_a \times X_b \times U_a \times U_b$ associated with R is defined by all the quadruples $(x_a, x_b, u_a, u_b) \in X_a \times X_b \times U_a \times U_b$ for which the following three conditions hold:

1. $(x_a, x_b) \in R$;
2. $u_a \in U_a(x_a)$;
3. $u_b \in U_b(x_b)$ and for every $x'_b \in \mathrm{Post}_{u_b}(x_b)$ there exists $x'_a \in \mathrm{Post}_{u_a}(x_a)$ satisfying $(x'_a, x'_b) \in R$.

The extended alternating simulation relation relates not only the states of system S_a and system S_b, but also the inputs (see [5]).

A composition operation describes mathematically how larger systems are constructed by interconnecting smaller systems. This composition operation is based on an interconnection relation which describes how two systems interact with each other. For control, the interconnection relation is in this paper required to be the extended relation R^e of an ε-approximate alternating simulation relation R from the controller system S_c to the system S_a. The definition of *Approximate Feedback Composition* is formalized as follows:

Definition 2.11. (Approximate Feedback Composition [5]). Let $S_a = (X_a, X_{a0}, U_a, \underset{a}{\rightarrow}, Y_a, H_a)$ and $S_c = (X_c, X_{c0}, U_c, \underset{c}{\rightarrow}, Y_c, H_c)$ be two metric systems with the same output set $Y_a = Y_c$, normed vector spaces, and let R be an ε-approximate alternating simulation relation from S_c to S_a. The feedback composition of S_c and S_a with interconnection relation $\mathcal{F} = R^e$, denoted by $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$, is the system $(X_{\mathcal{F}}, X_{\mathcal{F}0}, U_{\mathcal{F}}, \underset{\mathcal{F}}{\longrightarrow}, Y_{\mathcal{F}}, H_{\mathcal{F}})$ consisting of:

- $X_{\mathcal{F}} = \pi_X(\mathcal{F}) = R$;
- $X_{\mathcal{F}0} = X_{\mathcal{F}} \cap (X_{c0} \times X_{a0})$;
- $U_{\mathcal{F}} = U_c \times U_a$;
- $(x_c, x_a) \xrightarrow[\mathcal{F}]{(u_c,u_a)} (x'_c, x'_a)$ if the following three conditions hold:
  1. $(x_c, u_c, x'_c) \in \underset{c}{\longrightarrow}$;
  2. $(x_a, u_a, x'_a) \in \underset{a}{\longrightarrow}$;
  3. $(x_c, x_a, u_c, u_a) \in \mathcal{F}$;
- $Y_{\mathcal{F}} = Y_c = Y_a$;
- $H_{\mathcal{F}}(x_c, x_a) = \frac{1}{2}(H_c(x_c) + H_a(x_a))$.

The term *feedback composition* is justified by the following intepretation of $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$ (from [5]): Assume that $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$ is at the state $(x_c, x_a) \in R$. Then the controller system S_c offers to execute any of the inputs $u_c \in U_c(x_c)$. System S_a responds on this action by selecting any input $u_a \in U_a(x_a)$ satisfying $(x_c, x_a, u_c, u_a) \in \mathcal{F}$ and by taking any transition $x_a \xrightarrow[a]{u_a} x'_a$ labeled by the chosen input u_a. This transition in system S_a triggers a matching transition in the controller system S_c, i.e. S_c measures the new state x'_a of S_a and takes a transition $x_c \xrightarrow[c]{u_c} x'_c$ satisfying $(x'_a, x'_c) \in R$.

Note 2.12. The existence of the matching transition is guaranteed by the fact that R is an ε-approximate alternating simulation relation from S_c to S_a.

The internal behavior of $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$ can thus be interpret as being the result of a feedback process during which the controller S_c offers a set of inputs, measures the state of S_a, updates its own state, offers again a new set of inputs based on its own updated state, and so on.

The apparently arbitrary choice of output map in Definition 2.11 is justified by the following three important properties of approximate composition (from [5]):

- $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$ is commutative, i.e. $S_c \times_{\mathcal{F}}^{\varepsilon} S_a \cong_{\mathcal{S}} S_a \times_{\mathcal{F}}^{\varepsilon} S_c$;
- $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$ generalizes exact composition, i.e. if $\varepsilon = 0$, then $S_c \times_{\mathcal{F}}^{0} S_a = S_c \times_{\mathcal{F}} S_a$;
- $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$ satisfies Proposition 2.13.

Proposition 2.13. (from [5]). *Let S_c and S_a be metric systems with $Y_c = Y_a$ normed vector spaces with the same norm-induced metric. Furthermore, let $\mathcal{F} = R^e$ be an interconnection relation where R is an ε-approximate alternating simulation relation from S_c to S_a, Then, the following holds:*

- $S_c \times_{\mathcal{F}}^{\varepsilon} S_a \preceq_{\mathcal{S}}^{\frac{1}{2}\varepsilon} S_a$;
- $S_c \times_{\mathcal{F}}^{\varepsilon} S_a \preceq_{\mathcal{S}}^{\frac{1}{2}\varepsilon} S_c$.

Proof. The proof consists of checking that the relations

$$\{ ((x_c, x_a), x'_a) \in X_{\mathcal{F}} \times X_a \,|\, x_a = x'_a \}$$
$$\{ ((x_c, x_a), x'_c) \in X_{\mathcal{F}} \times X_c \,|\, x_c = x'_c \}$$

are $\frac{1}{2}\varepsilon$-approximate simulation relations from $S_c \times^{\varepsilon}_{\mathcal{F}} S_a$ to S_a and from $S_c \times^{\varepsilon}_{\mathcal{F}} S_a$ to S_c, respectively. We will only show $S_c \times^{\varepsilon}_{\mathcal{F}} S_a \preceq^{\frac{1}{2}\varepsilon}_{\mathcal{S}} S_c$ since the case $S_c \times^{\varepsilon}_{\mathcal{F}} S_a \preceq^{\frac{1}{2}\varepsilon}_{\mathcal{S}} S_a$ can be proved similarly.
Verifying that the relation

$$R_c = \{ ((x_c, x_a), x'_c) \in X_{\mathcal{F}} \times X_c \,|\, x_c = x'_c \}$$

is an $\frac{1}{2}\varepsilon$-approximate simulation relation from $S_c \times^{\varepsilon}_{\mathcal{F}} S_a$ to S_c is done by checking the conditions of Definition 2.4:

1. For all initial states in R_c, i.e. $((x_{c0}, x_{a0}), x'_{c0}) \in R_c \cap (X_{\mathcal{F}0} \times X_{c0})$, we know $x'_{c0} = x_{c0} \in X_{c0}$ and $(x_{c0}, x_{a0}) \in X_{\mathcal{F}0} = X_{\mathcal{F}} \cap (X_{c0} \times X_{a0})$.
$\Rightarrow \forall (x_{c0}, x_{a0}) \in X_{\mathcal{F}0}, \exists x'_{c0} \in X_{c0}$ (namely $x'_{c0} = x_{c0}$) $: ((x_{c0}, x_{a0}), x'_{c0}) \in R_c$.
2. Since $\mathcal{F} = R^e$ is an interconnection relation where R is an ε-approximate alternating simulation relation from S_c to S_a, the following holds:

$$\forall (x_c, x_a) \in X_{\mathcal{F}} = R \,:\, d(H_c(x_c), H_a(x_a)) \leq \varepsilon. \tag{2.5}$$

Hence, $\forall ((x_c, x_a), x'_c) \in R_c$ we have:

$$\begin{aligned}
d(H_{\mathcal{F}}(x_c, x_a), H_c(x'_c)) &= d(H_{\mathcal{F}}(x_c, x_a), H_c(x_c)) && (x'_c = x_c) \\
&= \|H_{\mathcal{F}}(x_c, x_a) - H_c(x_c)\| && (\text{def. metric } d) \\
&= \|\frac{1}{2}(H_c(x_c) + H_a(x_a)) - H_c(x_c)\| && (\text{def. output map } H_{\mathcal{F}}) \\
&= \|-\frac{1}{2}H_c(x_c)) + \frac{1}{2}H_a(x_a)\| && \\
&= \frac{1}{2}d(H_c(x_c), H_a(x_a)) && (\text{def. metric } d) \\
&\leq \frac{1}{2}\varepsilon. && (\text{by } 2.5)
\end{aligned}$$

3. $\forall ((x_c, x_a), x'_c) \in R_c$ we know $(x_c, x_a) \in X_{\mathcal{F}}$. If a transition $(x_c, x_a) \xrightarrow[\mathcal{F}]{(u_c, u_a)} (x''_c, x''_a)$ exists in $S_c \times^{\varepsilon}_{\mathcal{F}} S_a$, then by the interconnection relation $\mathcal{F}$ we know that the transition $x_c \xrightarrow[c]{u_c} x''_c$ exists in S_c. Since $x'_c = x_c$ for all $((x_c, x_a), x'_c) \in R_c$ we know that the transition $x'_c \xrightarrow[c]{u_c} x''_c$ exists in S_c. Note that $(x''_c, x''_a) \in R = X_{\mathcal{F}}$ and $x''_c \in X_c$.
$\Rightarrow \forall ((x_c, x_a), x'_c) \in R_c$ we have that $(x_c, x_a) \xrightarrow[\mathcal{F}]{(u_c, u_a)} (x''_c, x''_a)$ in $S_c \times^{\varepsilon}_{\mathcal{F}} S_a$ implies the existence of $x'_c \xrightarrow[c]{u_c} x''_c$ in X_c satisfying $((x''_c, x''_a), x''_c) \in R_c$.

$\square$

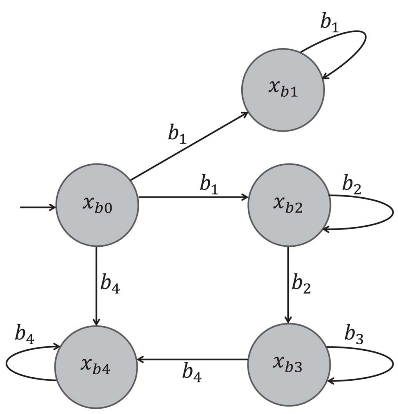
Note 2.14. The proofs of the first two important properties of approximate composition are omitted here. The third property (i.e. Proposition 2.13) is proven since we need this result later on in the proof of Lemma 3.9.

In the following example a relation R''' is verified to be an ε-approximate alternating simulation relation from a controller system S_c to the system S_b from example 2.5. This is done to compose the controller system S_c with the system S_b using Definition 2.11.

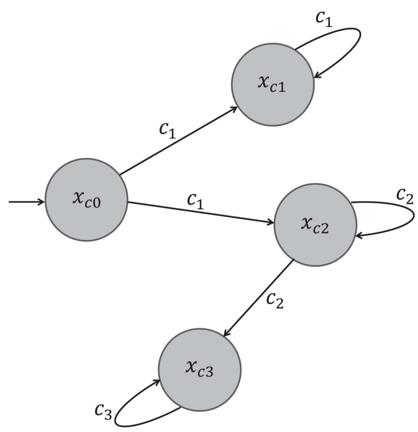
Example 2.15. (Feedback composition). *Consider the system S_b from Example 2.5 (also depicted in Figure 2.2(a)) and assume that we want to eliminate all the internal behaviors containing transitions of the form:*

$$x_{b0} \xrightarrow[b]{b_4} x_{b4},$$
$$x_{b3} \xrightarrow[b]{b_4} x_{b4},$$
$$x_{b4} \xrightarrow[b]{b_4} x_{b4}.$$

A reason for doing this can be that state x_{b4} is an unsafe state which one never wants to reach. Therefore, metric system S_b needs to be controlled to achieve this objective.



(a) Non-deterministic system S_b



(b) Non-deterministic controller system S_c

Figure 2.2: Two discrete-state systems.

To do so, we consider the metric controller system S_c depicted in Figure 2.2(b) (which is not the same as system S_a depicted in Figure 2.1(a)) and the relation

$$R''' = \{(x_{c0}, x_{b0}), (x_{c1}, x_{b1}), (x_{c2}, x_{b2}), (x_{c3}, x_{b3})\}.$$

We will show first that this relation R''' is an ε-approximate alternating simulation relation from S_c to S_b using Definition 2.6. Secondly, we will feedback compose system S_b with the controller system S_c using the interconnection relation $\mathcal{F} = (R''')^e$ from Definition 2.11.

Verifying that R''' is an ε-approximate alternating simulation relation from S_c to S_b is done by checking the conditions of Definition 2.6. To satisfy all the requirements in the definition we assume that $Y_c = Y_b$ normed vector spaces and $d(H_c(x_{ci}), H_b(x_{bi})) \leq \varepsilon$ for $i = 0, 1, 2, 3$.

1. *$X_{c0} = \{x_{c0}\}$ and for x_{c0} there exists $x_{b0} \in X_{b0}$ satisfying $(x_{c0}, x_{b0}) \in R'''$.*

2. *For* $(x_{c0}, x_{b0}) \in R''' : d(H_c(x_{ca0}), H_b(x_{b0})) \leq \varepsilon$,
for $(x_{c1}, x_{b1}) \in R''' : d(H_c(x_{c1}), H_b(x_{b1})) \leq \varepsilon$,
for $(x_{c2}, x_{b2}) \in R''' : d(H_c(x_{c2}), H_b(x_{b2})) \leq \varepsilon$
and for $(x_{c3}, x_{b3}) \in R''' : d(H_c(x_{c3}), H_b(x_{b3})) \leq \varepsilon$.

3. (i) *Consider* $(x_{c0}, x_{b0}) \in R'''$.
$U_c(x_{c0}) = \{c_1\}$. *For* $c_1 \in U_c(x_{c0})$, *there exists* $b_1 \in U_b(x_{b0})$ *(resulting in* $Post_{b_1}(x_{b0}) = \{x_{b1}, x_{b2}\}$*) such that for* $x_{b1} \in Post_{b_1}(x_{b0})$ *there exists* $x_{c1} \in Post_{c_1}(x_{c0})$ *satisfying* $(x_{c1}, x_{b1}) \in R'''$ *and for* $x_{b2} \in Post_{b_1}(x_{b0})$ *there exists* $x_{c2} \in Post_{c_1}(x_{c0})$ *satisfying* $(x_{c2}, x_{b2}) \in R'''$.

   (ii) *Consider* $(x_{c1}, x_{b1}) \in R'''$.
$U_c(x_{c1}) = \{c_1\}$. *For* $c_1 \in U_c(x_{c1})$, *there exists* $b_1 \in U_b(x_{b1})$ *(resulting in* $Post_{b_1}(x_{b1}) = \{x_{b1}\}$*) such that for* $x_{b1} \in Post_{b_1}(x_{b1})$ *there exists* $x_{c1} \in Post_{c_1}(x_{c1})$ *satisfying* $(x_{c1}, x_{b1}) \in R'''$.

   (iii) *Consider* $(x_{c2}, x_{b2}) \in R'''$.
$U_c(x_{c2}) = \{c_2\}$. *For* $c_2 \in U_c(x_{c2})$, *there exists* $b_2 \in U_b(x_{b2})$ *(resulting in* $Post_{b_2}(x_{b2}) = \{x_{b2}, x_{b3}\}$*) such that for* $x_{b2} \in Post_{b_2}(x_{b2})$ *there exists* $x_{c2} \in Post_{c_2}(x_{c2})$ *satisfying* $(x_{c2}, x_{b2}) \in R'''$ *and for* $x_{b3} \in Post_{b_2}(x_{b2})$ *there exists* $x_{c3} \in Post_{c_2}(x_{c2})$ *satisfying* $(x_{c3}, x_{b3}) \in R'''$.

   (iv) *Consider* $(x_{c3}, x_{b3}) \in R'''$.
$U_c(x_{c3}) = \{c_3\}$. *For* $c_3 \in U_c(x_{c3})$, *there exists* $b_3 \in U_b(x_{b3})$ *(resulting in* $Post_{b_3}(x_{b3}) = \{x_{b3}\}$*) such that for* $x_{b3} \in Post_{b_3}(x_{b3})$ *there exists* $x_{c3} \in Post_{c_3}(x_{c3})$ *satisfying* $(x_{c3}, x_{b3}) \in R'''$.

Since R''' *satisfies all conditions of Definition 2.6 we conclude that* R''' *is an* ε*-approximate alternating simulation relation from* S_c *to* S_b.

By the existence of the ε*-approximate alternating simulation relation* R''' *between metric controller system* S_c *and metric system* S_b*, we can now feedback compose* S_b *with* S_c *using Definition 2.11 with the interconnection relation* $\mathcal{F} = (R''')^e$*:*

The feedback composed system $S_c \times^{\varepsilon}_{\mathcal{F}} S_b$ *is according to Definition 2.11 the system* $(X_{\mathcal{F}}, X_{\mathcal{F}0}, U_{\mathcal{F}}, \underset{\mathcal{F}}{\longrightarrow}, Y_{\mathcal{F}}, H_{\mathcal{F}})$ *consisting of:*

- $X_{\mathcal{F}} = R''' = \{(x_{c0}, x_{b0}), (x_{c1}, x_{b1}), (x_{c2}, x_{b2}), (x_{c3}, x_{b3})\}$;
- $X_{\mathcal{F}0} = X_{\mathcal{F}} \cap (X_{c0} \times X_{b0}) = \{(x_{c0}, x_{b0})\}$;
- $U_{\mathcal{F}} = U_c \times U_b$;
- $(x_c, x_b) \xrightarrow[\mathcal{F}]{(u_c, u_b)} (x'_c, x'_b)$ *if the following three conditions hold:*

  1. $(x_c, u_c, x'_c) \in \{(x_{c0}, c_1, x_{c1}), (x_{c0}, c_1, x_{c2}), (x_{c1}, c_1, x_{c1}), (x_{c2}, c_2, x_{c2}), (x_{c2}, c_2, x_{c3}), (x_{c3}, c_3, x_{c3})\}$
  2. $(x_b, u_b, x'_b) \in \{(x_{b0}, b_1, x_{b1}), (x_{b0}, b_1, x_{b2}), (x_{b0}, b_4, x_{b4}), (x_{b1}, b_1, x_{b1}), (x_{b2}, b_2, x_{b2}), (x_{b2}, b_2, x_{b3}), (x_{b3}, b_3, x_{b3}), (x_{b3}, b_4, x_{b4}), (x_{b4}, b_4, x_{b4})\}$
  3. $(x_c, x_b, u_c, u_b) \in X_c \times X_b \times U_c \times U_b$ *satisfying:*

     (i) $(x_c, x_b) \in R''' = \{(x_{c0}, x_{b0}), (x_{c1}, x_{b1}), (x_{c2}, x_{b2}), (x_{c3}, x_{b3})\}$;

     (ii) $u_c \in U_c(x_c) = \begin{cases} c_1 & \textit{if } x_c = x_{c0} \\ c_1 & \textit{if } x_c = x_{c1} \\ c_2 & \textit{if } x_c = x_{c2} \\ c_3 & \textit{if } x_c = x_{c3} \end{cases}$

$$(iii)\ \ u_b \in U_b(x_b) = \begin{cases} b_1 \vee b_4 & \text{if } \ x_b = x_{b0} \\ b_1 & \text{if } \ x_b = x_{b1} \\ b_2 & \text{if } \ x_b = x_{b2} \\ b_3 \vee b_4 & \text{if } \ x_b = x_{b3} \\ b_4 & \text{if } \ x_b = x_{b4} \end{cases}$$

and $\forall x_b' \in Post_{u_b}(x_b), \exists x_c' \in Post_{u_c}(x_c)$ *satisfying* $(x_c', x_b') \in R'''$*;*

- $Y_{\mathcal{F}} = Y_c = Y_b$;
- $H_{\mathcal{F}} = \frac{1}{2}(H_c(x_c) + H_b(x_b))$;

Hence, the feedback composed system $S_c \times_{\mathcal{F}}^{\varepsilon} S_b$ *is depicted in Figure 2.3.*

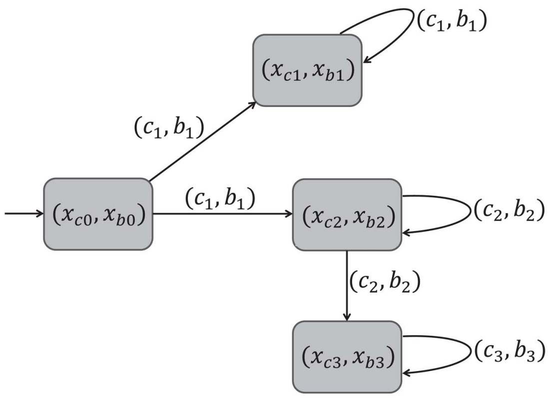


Figure 2.3: Feedback composed system $S_c \times_{\mathcal{F}}^{\varepsilon} S_b$.

We now can see that, up to a relabeling of states and inputs, $S_c \times_{\mathcal{F}}^{\varepsilon} S_b$ *is equal to the controller* S_c*. Therefore, the controller* S_c *enforces the desired requirements (i.e. avoid the unsafe state* x_{b4}*) on the system* S_b*.*

Chapter 3

Optimal control and simulation relations

The first section of this chapter contains the problem formulation of this thesis consisting of a definition of optimal cost for a generalized optimal control problem, and how approximate simulation relations can be related to this. Section 3.2 contains the main theorem of this thesis.

3.1 Problem definition

To simplify the discussion, we consider only systems S_a in which $Y_a = X_a$ and $H_a = 1_{X_a}$ (without loss of generality). The apparently more general problems of this chapter when $Y_a \neq X_a$ and $H_a \neq 1_{X_a}$, can be reduced to the problems studied in this chapter. It suffices to consider a new target set given by the pre-image of the given target set, i.e. $W' \subseteq X_a$ defined by $W' = H_a^{-1}(W)$ where $W \subseteq Y_a$, and to apply the results to system $(X_a, X_{a0}, U_a, \xrightarrow{a}, X_a, 1_{X_a})$ and specification set W'.

In this chapter we consider systems S_a and their instantaneous cost function f_a, where the instantaneous cost function $f_a : X_a \to \mathbb{R}_0^+$ describes the cost incurred by visiting a certain state. The combination of system and instantaneous cost function is denoted by (S_a, f_a) and referred to as system-cost pair. Additionally, we assume that for all possible controllers S_c for a system S_a, the controllers have no influence on the instantaneous cost function of the interconnected system $S = S_c \times^{\varepsilon}_{R^e} S_a$, i.e.

$$\forall x_a \in X_a \text{ and } \forall x_c \in R^{-1}(x_a) \subseteq X_c \quad : \quad f_{ca}(x_c, x_a) = f_a(x_a), \tag{3.1}$$

where R is an ε-approximate alternating simulation relation from S_c to S_a. Note that this assumption is exactly what we expect from controllers in practical terms.

The reachability problem is now regarded which involves a system S_a and a controller system S_c such that there exists an ε-approximate alternating simulation relation R from S_c to S_a. The controller system S_c and the approximate feedback composition relation $\mathcal{F} = R^e$ solve the reachability problem if for at least one initial state of the composed system $S_c \times^{\varepsilon}_{\mathcal{F}} S_a$, every maximal external behavior generated from that initial state enters the target set W in $k \in \mathbb{N}$ (depending on the initial state) steps.

Problem 3.1. (Reachability [4]). Let S_a be a metric system with $Y_a = X_a$ and $H_a = 1_{X_a}$, and let S_c be a metric controller system with $X_c = X_a$. Furthermore, let $W \subseteq X_a$ be a subset of the set of outputs of S_a and R an ε-approximate alternating simulation relation from S_c to S_a for some $\varepsilon \in \mathbb{R}_0^+$. The controller-interconnection pair $(S_c, \mathcal{F})$, with $\mathcal{F} = R^e$, is said to solve the *reachability problem* if

$$\begin{array}{lll} \exists x_0 \in X_{\mathcal{F}0} \quad \text{s.t.} & \forall \quad \mathbf{y}^{ca} \in \mathcal{B}_{x_0}(S_c \times_{\mathcal{F}}^{\varepsilon} S_a) \cup \mathcal{B}_{x_0}^{\omega}(S_c \times_{\mathcal{F}}^{\varepsilon} S_a) \quad \text{maximal}, \\ & \exists \quad k(x_0) \in \mathbb{N} \quad : \quad \mathbf{y}^{ca}(k(x_0)) = y_{cak(x_0)} \in W. \end{array}$$

So, $(S_c, \mathcal{F})$ solves the reachability problem if there exists at least one initial state x_0 in the composed system $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$ such that for every maximal behavior $\mathbf{y}^{ca}$ (i.e. there is no other behavior containing $\mathbf{y}^{ca}$ as a prefix) in that composed system, there exists a finite natural number k depending on the initial state x_0 such that the $k(x_0)$-th output of the maximal behavior $\mathbf{y}^{ca}$, i.e. $y_{cak(x_0)}$, belongs to the target set W.

The target set W is called *reachable* from $\overline{X_{\mathcal{F}0}} \subseteq X_{\mathcal{F}0}$ for system S_a if there exists a controller-interconnection pair $(S_c, \mathcal{F})$ solving the reachability problem for all $x_0 \in \overline{X_{\mathcal{F}0}}$. The target set W is called "*reachable from all initial states*" for a system S_a if it is reachable from $\overline{X_{\mathcal{F}0}} = X_{\mathcal{F}0}$.

Furthermore, the controller-interconnection pairs $(S_c, \mathcal{F})$ that solve the reachability problem for system S_a with target set W as specification and from initial states $\overline{X_{\mathcal{F}0}} \subseteq X_{\mathcal{F}0}$, are denoted by $\mathcal{R}_{\overline{X_{\mathcal{F}0}}}(S_a, W)$. The controller-interconnection pairs that solve this reachability problem for all initial states, i.e. $\overline{X_{\mathcal{F}0}} = X_{\mathcal{F}0}$, are denoted by $\mathcal{R}(S_a, W)$.

Let $S = S_c \times_{\mathcal{F}}^{\varepsilon} S_a$, i.e. system S is the result of the approximate feedback composition of the metric system S_a and a metric controller system S_c (where the output set satisfies: $X = X_c = X_a$) with interconnection relation $\mathcal{F} = R^e$ where R is an ε-approximate alternating simulation relation from S_c to S_a. So, system S is a controlled system.

After interconnecting the metric system S_a with a controller system S_c, we regard the controlled system S as a system having no inputs, i.e. the transitions in system S are not labeled. Systems that do not have inputs are called *unlabeled systems*. Unlabeled systems can be non-deterministic, but non-determinism causes no problems since the instantaneous cost function is defined on the states and not on the transitions of the system.

Now, for controlled unlabeled system S the minimum (natural) number of steps it takes for all maximal behaviors (finite or infinite), for some initial state, to enter the target set W is called the system entry time. This is formalized as follows:

Definition 3.2. ((System) Entry time [4]). Let S be an unlabeled system and let $W \subseteq X$ be a subset of the set of outputs of S. The *entry time* of an initialized maximal external behavior $\mathbf{y}$ in S to reach the target set W, denoted by $\tau(S, W, \mathbf{y})$, is given by:

$$\tau(S, W, \mathbf{y}) = \min_{k \in \mathbb{N}} \{ \mathbf{y}(k) = y_k \in W \mid \mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}_{x_0}^{\omega}(S) \text{ maximal},\ x_0 \in X_0 \}.$$

The *system entry time* of the unlabeled system S reaching the target set W from $x_0 \in X_0$, denoted by $\overline{\tau}(S, W, x_0)$, is then given by:

$$\overline{\tau}(S, W, x_0) = \max_{\mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}_{x_0}^{\omega}(S)} \tau(S, W, \mathbf{y}).$$

If the target set W is not reachable for an initialized maximal behavior $\mathbf{y}$ in S, then we define the entry time $\tau(S, W, \mathbf{y}) = \infty$ and consequentely the system entry time $\overline{\tau}(S, W, x_0) = \infty$.

Note 3.3. Since S might be non-deterministic, there might be more than one behavior contained in $\mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S)$. Therefore Definition 3.2 selects the smallest entry time for which all maximal behaviors in S generated from x_0 have reached W.

Using the notion of entry time, the final cost of reaching the target set W for an unlabeled system S within or at the system entry time is a twofold problem. Firstly, the costs to reach the target set W for all possible behaviors from x_0 have to be determined. Secondly, the final cost for the system S to reach the target set W from x_0 is determined by taking the maximum cost of all possible behaviors from x_0. This is formalized as follows:

Definition 3.4. ((Final) Cost). Let (S, f) be a system-cost pair and let $W \subseteq X$ be a subset of the set of outputs of the unlabeled system S. The *cost* to reach the target set W for a maximal behavior $\mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S)$ at the corresponding entry time $\tau(S, W, \mathbf{y})$, denoted by $\mathcal{C}(\mathbf{y})$, is given by:

$$\mathcal{C}(\mathbf{y}) = \sum_{i=0}^{\tau(S,W,\mathbf{y})-1} f(x_i).$$

The *final cost* to reach the target set W within or at the system entry time $\overline{\tau}(S, W, x_0)$ for unlabeled system S from x_0, denoted by $J(S, W, x_0, f)$, is then given by

$$J(S, W, x_0, f) = \max_{\mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S)} \mathcal{C}(\mathbf{y}). \tag{3.2}$$

Since S might be a non-deterministic system, there might be more than one behavior contained in $\mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S)$ as we stated before. Therefore the cost to reach the target set for an unlabeled system S from some initial condition x_0, as defined in Definition 3.4, is taken to be the maximum cost of all the possible behaviors from x_0. So no matter which initialized maximal external behavior $\mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S)$ is considered to reach W, the costs $\mathcal{C}(\mathbf{y})$ are always smaller than or equal to the cost $J(S, W, x_0, f)$.
If $\overline{\tau}(S, W, x_0) = \infty$, then $J(S, W, x_0, f) = \infty$.

Note 3.5. If for a system-cost pair (S, f) the instantaneous cost function f is equal to one at each state, i.e $\forall x \in X : f(x) = 1$, then the following holds for all $x_0 \in X_0$:

$$\forall \mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S) \text{ maximal} \quad : \quad \mathcal{C}(\mathbf{y}) \; = \; \sum_{i=0}^{\tau(S,W,\mathbf{y})-1} f(x_i) \; = \; \tau(S, W, \mathbf{y})$$

and consequentely

$$\begin{aligned} J(S, W, x_0, f) &= \max_{\mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S)} \mathcal{C}(\mathbf{y}) \\ &= \max_{\mathbf{y} \in \mathcal{B}_{x_0}(S) \cup \mathcal{B}^{\omega}_{x_0}(S)} \tau(S, W, \mathbf{y}) \\ &= \overline{\tau}(S, W, x_0). \end{aligned}$$

In other words: for all maximal behaviors of unlabeled system S, the cost to reach the target set within or at entry time of the considered behavior is equal to that entry time. Consequentely, the final cost of reaching the target set for an unlabeled system S within or at the system entry time is equal to the system entry time.

Hence, section 3 of [4] can be seen as a special case of this section.

Definition 3.6. (Minimum cost over all possible initial controller states [4]). For the system-cost pair (S, f), where the unlabeled system S is the result of the approximate feedback composition of a system S_a and a controller system S_c, the minimum cost over all possible initial states of the controller related to x_{a0}, denoted by $\tilde{J}(S_c, \mathcal{F}, S_a, W, x_{a0}, f)$, is given by:

$$\begin{aligned}\tilde{J}(S_c, \mathcal{F}, S_a, W, x_{a0}, f) &= \min_{x_{c0} \in X_{c0}} \{J(S, W, x_0, f) \mid x_0 \in \mathcal{X}_{\mathcal{F}0}\} \\ &= \min_{x_{c0} \in X_{c0}} \{J(S_c \times^{\varepsilon}_{\mathcal{F}} S_a, W, (x_{c0}, x_{a0}), f) \mid (x_{c0}, x_{a0}) \in \mathcal{X}_{\mathcal{F}0}\}.\end{aligned}$$

The optimal control problem now asks to select the minimal cost behavior for every $x_0 = (x_{c0}, x_{a0}) \in \mathcal{X}_{\mathcal{F}0}$ for which $\overline{\tau}(S, W, x_0)$ and $J(S, W, x_0, f)$ are both finite.

Problem 3.7. (Optimal reachability [4]). Let (S_a, f_a) be a system-cost pair where S_a is a metric unlabeled system with $Y_a = X_a$ and $H_a = 1_{X_a}$, and let $W \subseteq X_a$ be a subset of the set of outputs S_a. The *optimal reachability problem* asks to find the controller-interconnection pair $(S_c^*, \mathcal{F}^*) \in \mathcal{R}(S_a, W)$ such that for any other pair $(S_c, \mathcal{F}) \in \mathcal{R}(S_a, W)$ the following is satisfied:

$$\forall x_{a0} \in X_{a0} \quad : \quad \tilde{J}(S_c, \mathcal{F}, S_a, W, x_{a0}, f) \geq \tilde{J}(S_c^*, \mathcal{F}^*, S_a, W, x_{a0}, f).$$

So, Problem 3.7 selects of all the possible controller-interconnection pairs $(S_c, \mathcal{F}) \in \mathcal{R}(S_a, W)$ the controller-interconnection pair $(S_c^*, \mathcal{F}^*)$ such that the approximate feedback composed system (i.e. the controlled system $S = S_c^* \times^{\varepsilon}_{\mathcal{F}^*} S_a$) has the lowest cost to reach the target set W within or at the entry time $\overline{\tau}(S, W, x_0)$ from x_0, for all $x_0 \in \mathcal{X}_{\mathcal{F}0}$, compared to all the other possible controller-interconnection pairs.

In the following example the notions of this section are illustrated in the case of discrete-state systems. To show how these notions apply to different systems, we constructed three different controller systems for a system S_a such that the interconnected systems have different behaviors. For convience, we only consider exact alternating simulations relations between the systems.

Example 3.8. (Reachability, entry time and cost). *Consider the system-cost pair* (S_a, f_a) *where* S_a *is a metric system with* $Y_a = X_a$ *and* $H_a = 1_{X_a}$*, depicted in Figure 3.1. The instantaneous cost function* $f_a : X_a \to \mathbb{R}_0^+$ *is given in the little grey boxes displayed at the states, see Figure 3.1. For this system* S_a *we define the target set as* $W = \{x_{a5}, x_{a6}\} \subseteq X_a = \{x_{a0}, x_{a1}, x_{a2}, x_{a3}, x_{a4}, x_{a5}, x_{a6}\}$.

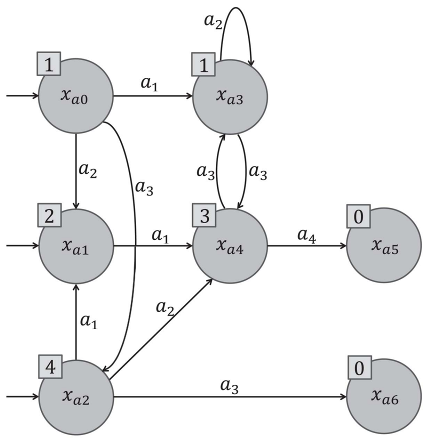


Figure 3.1: Discrete-state system S_a

1. *Consider the controller system* S_c *with* $X_c = X_a$*, i.e.* $x_{ci} = x_{ai}$ *for* $i = 0, \ldots, 6$*, depicted in Figure 3.2. Note: there is no instantaneous cost function given for the controller system* S_c *since after interconnecting the controller system with another system, the controller has no influence on the instantaneous cost function of the interconnected system, i.e.* S_c *satisfies Equation 3.1.*

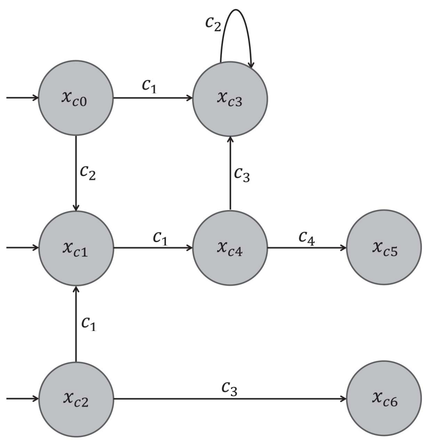


Figure 3.2: Discrete-state system S_c

The relation

$$R = \{(x_{c0}, x_{a0}), (x_{c1}, x_{a1}), (x_{c2}, x_{a2}), (x_{c3}, x_{a3}), (x_{c4}, x_{a4}), (x_{c5}, x_{a5}), (x_{c6}, x_{a6})\}$$

is an exact alternating simulation relation from S_c to S_a. This can be verified by checking the conditions of Definition 2.6 (which is not done here). By the existence of the exact alternating simulation relation R between controller system S_c and system S_a, S_a can now be feedback composed with S_c using Definition 2.11 with the interconnection relation $\mathcal{F} = (R)^e$. The feedback composed system $S_c \times_{\mathcal{F}} S_a$ is depicted in Figure 3.3. The instantaneous cost function $f : X_{\mathcal{F}} \to \mathbb{R}_0^+$ is again given in the little grey boxes displayed at the states, see Figure 3.3. Note: system $S_c \times_{\mathcal{F}} S_a$ is an unlabeled system, since after composing S_c and S_a with $\mathcal{F}$ the transitions are not labeled with inputs.

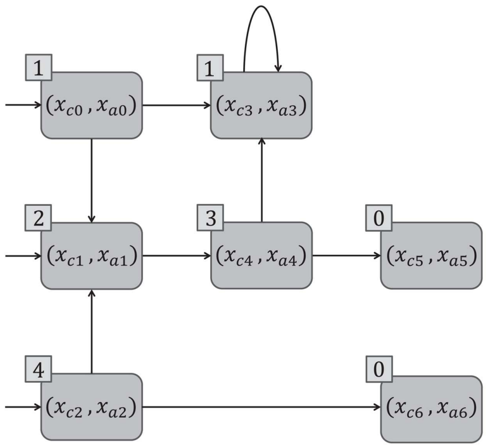


Figure 3.3: Unlabeled feedback composed system $S_c \times_{\mathcal{F}} S_a$.

We are now going to investigate if $(S_c, \mathcal{F})$ solves the reachability problem 3.1 for system S_a and target set W. So, does there exists an intial state in the composed system $S_c \times_{\mathcal{F}} S_a$ such that for every initialized maximal external behavior $\mathbf{y}^{ca}$ in the composed system, there exists finite natural number k depending on the initial state such that the k-th output of $\mathbf{y}^{ca}$ belongs to the target set W.
The set of initial states of system $S_c \times_{\mathcal{F}} S_a$ is given by (see Figure 3.3):

$$X_{\mathcal{F}0} = \{(x_{c0}, x_{a0}), (x_{c1}, x_{a1}), (x_{c2}, x_{a2})\}.$$

Note: since $X = X_a = X_c$ by definition of feedback composition, we have $X \ni x_i = (x_{ci}, x_{ai}) = x_{ai} = x_{ci}$ for $i = 0, \dots, 6$.

- *All maximal behaviors of $S_c \times_{\mathcal{F}} S_a$ generated from $x_0 = (x_{c0}, x_{a0})$ are given by:*

$$\begin{aligned} \mathbf{y}_1^{ca} &= x_0 x_3 x_3 x_3 \dots \\ \mathbf{y}_2^{ca} &= x_0 x_1 x_4 x_3 x_3 x_3 \dots \\ \mathbf{y}_3^{ca} &= x_0 x_1 x_4 x_5. \end{aligned}$$

  Since $x_3 \notin W$, there exists no finite number k (depending on x_0) such that the $k(x_0)$-th output of the maximal behavior $\mathbf{y}_1^{ca}$ belongs to the target set W. Also for $\mathbf{y}_2^{ca}$ such a finite number k (depending on x_0) does not exist.

- *All maximal behaviors of $S_c \times_{\mathcal{F}} S_a$ generated from $x_1 = (x_{c1}, x_{a1})$ are given by:*

$$\begin{aligned} \mathbf{y}_4^{ca} &= x_1 x_4 x_3 x_3 x_3 \ldots \\ \mathbf{y}_5^{ca} &= x_1 x_4 x_5. \end{aligned}$$

  Since $x_3 \notin W$, there exists no finite number k (depending on x_1) such that the $k(x_1)$-th output of the maximal behavior $\mathbf{y}_4^{ca}$ belongs to the target set W.

- *All maximal behaviors of $S_c \times_{\mathcal{F}} S_a$ generated from $x_2 = (x_{c2}, x_{a2})$ are given by:*

$$\begin{aligned} \mathbf{y}_6^{ca} &= x_2 x_1 x_4 x_3 x_3 x_3 \ldots \\ \mathbf{y}_7^{ca} &= x_2 x_1 x_4 x_5 \\ \mathbf{y}_8^{ca} &= x_2 x_6. \end{aligned}$$

  Since $x_3 \notin W$, there exists no finite number k (depending on x_2) such that the $k(x_2)$-th output of the maximal behavior $\mathbf{y}_6^{ca}$ belongs to the target set W.

Hence, there exists no initial state in $S_c \times_{\mathcal{F}} S_a$ such that $(S_c, \mathcal{F})$ solves the reachability problem for system S_a and target set W.
Consequentely, the system entry times and the final costs are given by:

$$\begin{aligned} \overline{\tau}(S_c \times_{\mathcal{F}} S_a,\, W,\, (x_{c0}, x_{a0})) &= \infty \;\Rightarrow\; J(S_c \times_{\mathcal{F}} S_a,\, W,\, (x_{c0}, x_{a0}),\, f) = \infty \\ \overline{\tau}(S_c \times_{\mathcal{F}} S_a,\, W,\, (x_{c1}, x_{a1})) &= \infty \;\Rightarrow\; J(S_c \times_{\mathcal{F}} S_a,\, W,\, (x_{c1}, x_{a1}),\, f) = \infty \\ \overline{\tau}(S_c \times_{\mathcal{F}} S_a,\, W,\, (x_{c2}, x_{a2})) &= \infty \;\Rightarrow\; J(S_c \times_{\mathcal{F}} S_a,\, W,\, (x_{c2}, x_{a2}),\, f) = \infty \end{aligned}$$

2. *Now, consider the controller system S_{ca} with $X_{ca} = X_a$, i.e. $x_{cai} = x_{ai}$ for $i = 0, \ldots, 6$, depicted in Figure 3.4. Note: as in 1., there is no instantaneous cost function given for the controller system.*

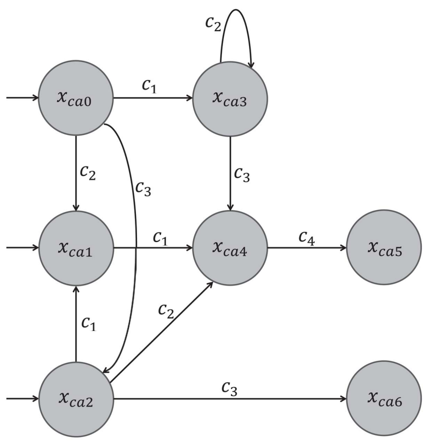


Figure 3.4: Discrete-state system S_{ca}

The relation

$$R_a = \{(x_{ca0}, x_{a0}), (x_{ca1}, x_{a1}), (x_{ca2}, x_{a2}), (x_{ca3}, x_{a3}),$$
$$(x_{ca4}, x_{a4}), (x_{ca5}, x_{a5}), (x_{ca6}, x_{a6})\}$$

is an exact alternating simulation relation from S_{ca} *to* S_a. *This can be verified by checking the conditions of Definition 2.6 (which is not done here). By the existence of the exact alternating simulation relation* R_a *between controller system* S_{ca} *and system* S_a, S_a *can now be feedback composed with* S_{ca} *using Definition 2.11 with the interconnection relation* $\mathcal{F}_a = (R_a)^e$. *The feedback composed system* $S_{ca} \times_{\mathcal{F}_a} S_a$ *is depicted in Figure 3.5. The instantaneous cost function* $f : X_{\mathcal{F}_a} \to \mathbb{R}_0^+$ *is again given in the little grey boxes displayed at the states, see Figure 3.5. Note: as the composed system in 1., system* $S_{ca} \times_{\mathcal{F}_a} S_a$ *is an unlabeled system.*

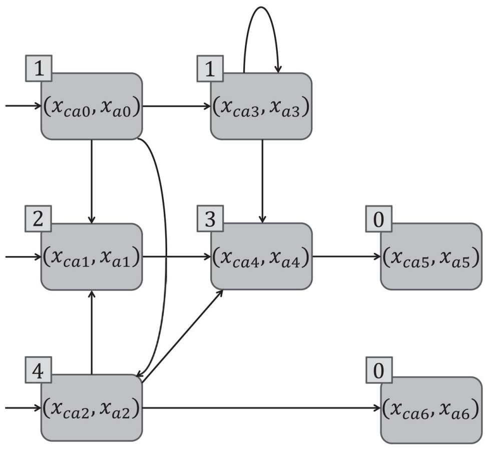


Figure 3.5: Unlabeled feedback composed system $S_{ca} \times_{\mathcal{F}_a} S_a$.

We are now going to investigate if $(S_{ca}, \mathcal{F}_a)$ *solves the reachability problem 3.1 for system* S_a *and target set* W.
The set of initial states of system $S_{ca} \times_{\mathcal{F}_a} S_a$ *is given by (see Figure 3.5):*

$$X_{\mathcal{F}_a 0} = \{(x_{ca0}, x_{a0}), (x_{ca1}, x_{a1}), (x_{ca2}, x_{a2})\}.$$

Note: as in 1., we have $X_{\mathcal{F}_a} \ni x_i = (x_{cai}, x_{ai}) = x_{ai} = x_{ci}$ *for* $i = 0, \ldots, 6$.

- *All maximal behaviors of* $S_{ca} \times_{\mathcal{F}_a} S_a$ *generated from* $x_0 = (x_{ca0}, x_{a0})$ *are given by:*

$$\begin{aligned}
\mathbf{y}_1^{ca} &= x_0 x_3 x_3 x_3 \ldots \\
\mathbf{y}_2^{ca} &= x_0 x_3 \ldots x_3 x_4 x_5 \qquad \textit{(where } x_3 \textit{ is visited } k < \infty \textit{ times, } k \in \mathbb{N}^+) \\
\mathbf{y}_3^{ca} &= x_0 x_1 x_4 x_5 \\
\mathbf{y}_4^{ca} &= x_0 x_2 x_1 x_4 x_5 \\
\mathbf{y}_5^{ca} &= x_0 x_2 x_4 x_5 \\
\mathbf{y}_6^{ca} &= x_0 x_2 x_6.
\end{aligned}$$

  Since $x_3 \notin W$, *there exists no finite number* k *(depending on* x_0*) such that the* $k(x_0)$*-th output of the maximal behaviors* $\mathbf{y}_1^{ca}$ *belongs to the target set* W.

- *All maximal behaviors of* $S_{ca} \times_{\mathcal{F}_a} S_a$ *generated from* $x_1 = (x_{ca1}, x_{a1})$ *are given by:*

$$\mathbf{y}_7^{ca} \quad = \quad x_1 x_4 x_5.$$

  Since $x_5 \in W$ *and maximal behavior* $\mathbf{y}_7^{ca}$ *reaches* x_5 *after the second transition, i.e.* $k(x_1) = 2$*, the target set* W *is reachable from* x_1 *for system* S_a*.*

- *All maximal behaviors of* $S_{ca} \times_{\mathcal{F}_a} S_a$ *generated from* $x_2 = (x_{ca2}, x_{a2})$ *are given by:*

$$\begin{aligned} \mathbf{y}_8^{ca} &= x_2 x_1 x_4 x_5 \\ \mathbf{y}_9^{ca} &= x_2 x_4 x_5 \\ \mathbf{y}_{10}^{ca} &= x_2 x_6. \end{aligned}$$

  Since $x_5, x_6 \in W$ *and maximal behaviors* $\mathbf{y}_8^{ca}$*,* $\mathbf{y}_9^{ca}$ *and* $\mathbf{y}_{10}^{ca}$ *reach* x_5 *or* x_6 *after the third, second and first transition respectively, i.e.* $k(x_2) = 3$*,* $k(x_2) = 2$ *and* $k(x_2) = 1$ *respectively, the target set* W *is reachable from* x_2 *for system* S_a*.*

Hence, $(S_{ca}, \mathcal{F}_a)$ *solves the reachability problem for system* S_a *and target set* W *from initial states* x_1 *and* x_2*, i.e.* $(S_{ca}, \mathcal{F}_a) \in \mathcal{R}_{\overline{X_{\mathcal{F}_a 0}}}(S_a, W)$ *with* $\overline{X_{\mathcal{F}_a 0}} = \{x_1, x_2\}$*.*
From the above made observations, we can immediately write down what the entry times of the initialized maximal external behaviors of system $S_{ca} \times_{\mathcal{F}_a} S_a$ *are by using Definition 3.2:*

$$\begin{aligned} \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_1^{ca}) &= \infty \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_2^{ca}) &= 2 + k \qquad (\textit{where } k < \infty, k \in \mathbb{N}^+) \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_3^{ca}) &= 3 \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_4^{ca}) &= 4 \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_5^{ca}) &= 3 \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_6^{ca}) &= 2 \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_7^{ca}) &= 2 \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_8^{ca}) &= 3 \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_9^{ca}) &= 2 \\ \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_{10}^{ca}) &= 1. \end{aligned}$$

Hence, the system entry time of $S_{ca} \times_{\mathcal{F}_a} S_a$ *from each initial conditions is then given by (using Definition 3.2 again):*

$$\begin{aligned} \overline{\tau}(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca0}, x_{a0})) &= \max_{i=1,\ldots,6} \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_i^{ca}) = \infty \\ \overline{\tau}(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca1}, x_{a1})) &= \max_{i=7} \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_i^{ca}) = 2 \\ \overline{\tau}(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca2}, x_{a2})) &= \max_{i=8,9,10} \tau(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, \mathbf{y}_i^{ca}) = 3. \end{aligned}$$

We can now determine the cost for the initialized external maximal behaviors by using the calculated entry times and the defined instantaneous cost function f *(see*

Figure 3.5) and apply Definition 3.4:

$$
\begin{aligned}
\mathcal{C}(\mathbf{y}_1^{ca}) &= f(x_0) + f(x_3) + f(x_3) + f(x_3) + \ldots \\
&= 1 + 1 + 1 + 1 + \ldots \;=\; \infty \\
\mathcal{C}(\mathbf{y}_2^{ca}) &= f(x_0) + f(x_3) + \ldots + f(x_3) + f(x_4) \\
&= 1 + 1 + \ldots + 1 + 3 \;=\; 4 + k \qquad\qquad (k < \infty, k \in \mathbb{N}^+) \\
\mathcal{C}(\mathbf{y}_3^{ca}) &= f(x_0) + f(x_1) + f(x_4) \\
&= 1 + 2 + 3 \;=\; 6 \\
\mathcal{C}(\mathbf{y}_4^{ca}) &= f(x_0) + f(x_2) + f(x_1) + f(x_4) \\
&= 1 + 4 + 2 + 3 \;=\; 10 \\
\mathcal{C}(\mathbf{y}_5^{ca}) &= f(x_0) + f(x_2) + f(x_4) \\
&= 1 + 4 + 3 \;=\; 8 \\
\mathcal{C}(\mathbf{y}_6^{ca}) &= f(x_0) + f(x_2) \\
&= 1 + 4 \;=\; 5 \\
\mathcal{C}(\mathbf{y}_7^{ca}) &= f(x_1) + f(x_4) \\
&= 2 + 3 \;=\; 5 \\
\mathcal{C}(\mathbf{y}_8^{ca}) &= f(x_2) + f(x_1) + f(x_4) \\
&= 4 + 2 + 3 \;=\; 9 \\
\mathcal{C}(\mathbf{y}_9^{ca}) &= f(x_2) + f(x_4) \\
&= 4 + 3 \;=\; 7 \\
\mathcal{C}(\mathbf{y}_{10}^{ca}) &= f(x_2) \\
&= 4.
\end{aligned}
$$

Hence, the final cost of $S_{ca} \times_{\mathcal{F}_a} S_a$ from each initial condition is then given by (using Definition 3.4 again):

$$
\begin{aligned}
J(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca0}, x_{a0}),\, f) &= \max_{i=1,\ldots,6} \mathcal{C}(\mathbf{y}_i^{ca}) = \infty \\
J(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca1}, x_{a1}),\, f) &= \max_{i=7} \mathcal{C}(\mathbf{y}_i^{ca}) = 5 \\
J(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca2}, x_{a2}),\, f) &= \max_{i=8,9,10} \mathcal{C}(\mathbf{y}_i^{ca}) = 9.
\end{aligned}
$$

Since the relation R_a relates every initial state of S_a to only one single initial state of S_{ca}, we have the following:

$$
\begin{aligned}
\tilde{J}(S_{ca},\, \mathcal{F}_a,\, S_a,\, W,\, x_{a0},\, f) &= J(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca0}, x_{a0}),\, f) = \infty \\
\tilde{J}(S_{ca},\, \mathcal{F}_a,\, S_a,\, W,\, x_{a1},\, f) &= J(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca1}, x_{a1}),\, f) = 5 \\
\tilde{J}(S_{ca},\, \mathcal{F}_a,\, S_a,\, W,\, x_{a2},\, f) &= J(S_{ca} \times_{\mathcal{F}_a} S_a,\, W,\, (x_{ca2}, x_{a2}),\, f) = 9.
\end{aligned}
$$

So, after regarding the reachability problem for system S_a and target set W using controller-interconnection pair $(S_{ca}, \mathcal{F}_a)$, we determined the system entry times and final costs to reach the target set W from each initial state to eventuelly determine the minimum costs over all initial states of the controller S_{ca} related to the initial states of S_a to reach the target set W.

3. *Now, consider the optimal controller system S^*_{ca} with $X^*_{ca} = X_a$, i.e. $x^*_{cai} = x_{ai}$ for $i = 0, \dots, 6$, depicted in Figure 3.6. Note: as in 1. and 2., there is no instantaneous cost function given for the controller system.*

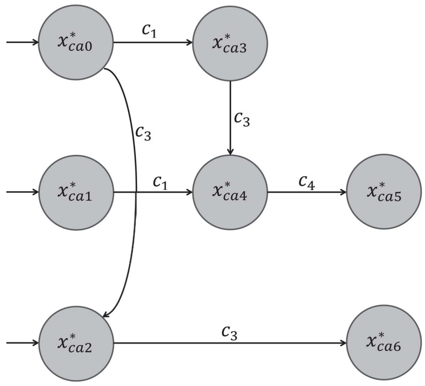


Figure 3.6: Discrete-state system S^*_{ca}

The relation

$$R^*_{ca} = \{(x^*_{ca0}, x_{a0}), (x^*_{ca1}, x_{a1}), (x^*_{ca2}, x_{a2}), (x^*_{ca3}, x_{a3}),\\ (x^*_{ca4}, x_{a4}), (x^*_{ca5}, x_{a5}), (x^*_{ca6}, x_{a6})\}$$

is an exact alternating simulation relation from S^*_{ca} to S_a. This can be verified by checking the conditions of Definition 2.6 (which is not done here). By the existence of the exact alternating simulation relation R^*_{ca} between controller system S^*_{ca} and system S_a, S_a can now be feedback composed with S^*_{ca} using Definition 2.11 with the interconnection relation $\mathcal{F}^*_a = (R^*_{ca})^e$. The feedback composed system $S^*_{ca} \times_{\mathcal{F}^*_a} S_a$ is depicted in Figure 3.7. The instantaneous cost function $f : X_{\mathcal{F}^*_a} \to \mathbb{R}^+_0$ is again given in the little grey boxes displayed at the states, see Figure 3.7. Note: as the composed systems in 1. and 2., system $S^*_{ca} \times_{\mathcal{F}^*_a} S_a$ is an unlabeled system.

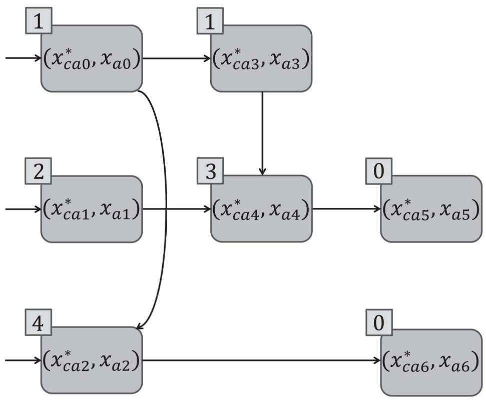


Figure 3.7: Unlabeled feedback composed system $S^*_{ca} \times_{\mathcal{F}^*_a} S_a$.

We are now going to investigate if $(S^*_{ca}, \mathcal{F}^*_a)$ solves the reachability problem 3.1 for system S_a and target set W.

The set of initial states of system $S_{ca}^* \times_{\mathcal{F}_a^*} S_a$ *is given by (see Figure 3.7):*

$$X_{\mathcal{F}_a^*0} = \{(x_{ca0}^*, x_{a0}), (x_{ca1}^*, x_{a1}), (x_{ca2}^*, x_{a2})\}.$$

Note: as in 1. and 2., we have $X_{\mathcal{F}_a^*} \ni x_i = (x_{cai}^*, x_{ai}) = x_{ai} = x_{cai}^*$ *for* $i = 0, \ldots, 6$.

- *All maximal behaviors of* $S_{ca}^* \times_{\mathcal{F}_a^*} S_a$ *generated from* $x_0 = (x_{ca0}^*, x_{a0})$ *are given by:*

  $$\begin{aligned} \mathbf{y}_1^{ca^*} &= x_0 x_3 x_4 x_5 \\ \mathbf{y}_2^{ca^*} &= x_0 x_2 x_6. \end{aligned}$$

  Since $x_5, x_6 \in W$ *and maximal behaviors* $\mathbf{y}_1^{ca}$ *and* $\mathbf{y}_2^{ca}$ *reach* x_5 *or* x_6 *after the third and second transition respectively, i.e.* $k(x_0) = 3$ *and* $k(x_0) = 2$ *respectively, the target set* W *is reachable from* x_0 *for system* S_a.

- *All maximal behaviors of* $S_{ca}^* \times_{\mathcal{F}_a^*} S_a$ *generated from* $x_1 = (x_{ca1}^*, x_{a1})$ *are given by:*

  $$\mathbf{y}_3^{ca^*} = x_1 x_4 x_5.$$

  Since $x_5 \in W$ *and maximal behavior* $\mathbf{y}_3^{ca^*}$ *reaches* x_5 *after the second transition, i.e.* $k(x_1) = 2$*, the target set* W *is reachable from* x_1 *for system* S_a.

- *All maximal behaviors of* $S_{ca}^* \times_{\mathcal{F}_a^*} S_a$ *generated from* $x_2 = (x_{ca2}^*, x_{a2})$ *are given by:*

  $$\mathbf{y}_4^{ca^*} = x_2 x_6.$$

  Since $x_6 \in W$ *and maximal behavior* $\mathbf{y}_4^{ca^*}$ *reaches* x_6 *after the first transition,* $k(x_2) = 1$*, the target set* W *is reachable from* x_2 *for system* S_a.

Hence, $(S_{ca}^*, \mathcal{F}_a^*)$ *solves the reachability problem for system* S_a *and target set* W *from all initial states, i.e.* $(S_{ca}^*, \mathcal{F}_a^*) \in \mathcal{R}(S_a, W)$.
From the above made observations, we can immediately write down what the entry times of the initialized maximal external behaviors of system $S_{ca}^* \times_{\mathcal{F}_a^*} S_a$ *are by using Definition 3.2:*

$$\begin{aligned} \tau(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, \mathbf{y}_1^{ca^*}) &= 3 \\ \tau(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, \mathbf{y}_2^{ca^*}) &= 2 \\ \tau(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, \mathbf{y}_3^{ca^*}) &= 2 \\ \tau(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, \mathbf{y}_4^{ca^*}) &= 1. \end{aligned}$$

Hence, the system entry time of $S_{ca}^* \times_{\mathcal{F}_a^*} S_a$ *from each initial conditions is then given by (using Definition 3.2 again):*

$$\begin{aligned} \overline{\tau}(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, (x_{ca0}^*, x_{a0})) &= \max_{i=1,2} \tau(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, \mathbf{y}_i^{ca^*}) = 3 \\ \overline{\tau}(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, (x_{ca1}^*, x_{a1})) &= \max_{i=3} \tau(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, \mathbf{y}_i^{ca^*}) = 2 \\ \overline{\tau}(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, (x_{ca2}^*, x_{a2})) &= \max_{i=4} \tau(S_{ca}^* \times_{\mathcal{F}_a^*} S_a, W, \mathbf{y}_i^{ca^*}) = 1. \end{aligned}$$

We can now determine the cost for the initialized external maximal behaviors by using the calculated entry times and the defined instantaneous cost function f (see Figure 3.7) and apply Definition 3.4:

$$\begin{aligned}
\mathcal{C}(\mathbf{y}_1^{ca^*}) &= f(x_0) + f(x_3) + f(x_4) \\
&= 1 + 1 + 3 = 5 \\
\mathcal{C}(\mathbf{y}_2^{ca^*}) &= f(x_0) + f(x_2) \\
&= 1 + 4 = 5 \\
\mathcal{C}(\mathbf{y}_3^{ca^*}) &= f(x_1) + f(x_4) \\
&= 2 + 3 = 5 \\
\mathcal{C}(\mathbf{y}_4^{ca^*}) &= f(x_2) \\
&= 4.
\end{aligned}$$

Hence, the final cost of $S_{ca}^* \times_{\mathcal{F}_a^*} S_a$ from each initial condition is then given by (using Definition 3.4 again):

$$\begin{aligned}
J(S_{ca}^* \times_{\mathcal{F}_a^*} S_a,\, W,\, (x_{ca0}^*, x_{a0}),\, f) &= \max_{i=1,2} \mathcal{C}(\mathbf{y}_i^{ca^*}) = 5 \\
J(S_{ca}^* \times_{\mathcal{F}_a^*} S_a,\, W,\, (x_{ca1}^*, x_{a1}),\, f) &= \max_{i=3} \mathcal{C}(\mathbf{y}_i^{ca^*}) = 5 \\
J(S_{ca}^* \times_{\mathcal{F}_a^*} S_a,\, W,\, (x_{ca2}^*, x_{a2}),\, f) &= \max_{i=4} \mathcal{C}(\mathbf{y}_i^{ca^*}) = 4.
\end{aligned}$$

Since the relation R_{ca}^* relates every initial state of S_a to only one single initial state of S_{ca}^*, we have the following:

$$\begin{aligned}
\tilde{J}(S_{ca}^*,\, \mathcal{F}_a^*,\, S_a,\, W,\, x_{a0},\, f) &= J(S_{ca}^* \times_{\mathcal{F}_a^*} S_a,\, W,\, (x_{ca0}^*, x_{a0}),\, f) = 5 \\
\tilde{J}(S_{ca}^*,\, \mathcal{F}_a^*,\, S_a,\, W,\, x_{a1},\, f) &= J(S_{ca}^* \times_{\mathcal{F}_a^*} S_a,\, W,\, (x_{ca1}^*, x_{a1}),\, f) = 5 \\
\tilde{J}(S_{ca}^*,\, \mathcal{F}_a^*,\, S_a,\, W,\, x_{a2},\, f) &= J(S_{ca}^* \times_{\mathcal{F}_a^*} S_a,\, W,\, (x_{ca2}^*, x_{a2}),\, f) = 4.
\end{aligned}$$

So, after regarding the reachability problem for system S_a and target set W using controller-interconnection pair $(S_{ca}^*, \mathcal{F}_a^*)$, we determined the system entry times and final costs to reach the target set W from each initial state to eventuelly determine the minimum costs over all initial states of the controller S_{ca}^* related to the initial states of S_a to reach the target set W.
Compared to 2., we immediately see that:

$$\text{for } i = 0, 1, 2 \quad : \quad \tilde{J}(S_{ca},\, \mathcal{F}_a,\, S_a,\, W,\, x_{ai},\, f) \geq \tilde{J}(S_{ca}^*,\, \mathcal{F}_a^*,\, S_a,\, W,\, x_{ai},\, f).$$

Actually, the controller-interconnection pair $(S_{ca}^*, \mathcal{F}_a^*)$ satisfies this equation for all other possible controller-interconnection pairs $(S_{ca}, \mathcal{F}_a) \in \mathcal{R}(S_a, W)$. Hence, the controller-interconnection pair $(S_{ca}^*, \mathcal{F}_a^*)$ solves the optimal reachability problem 3.7.

3.2 Cost bounds

In this section it is shown how to obtain upper and lower bounds for the optimal cost in a system S_b by working with a related system S_a. To prove this main theoretical result, first it is proven how an upper bound can be obtained.

Lemma 3.9. (Upper bound for optimal cost). *Let S_a and S_b be two metric systems with $Y_a = X_a$, $H_a = 1_{X_a}$, $Y_b = X_b$ and $H_b = 1_{X_b}$ and let the output sets be equal* [1]*, i.e. $X_a = X_b$. Furthermore, let $W_a \subseteq X_a$ and $W_b \subseteq X_b$ be subsets of the output sets and let $\varepsilon \in \mathbb{R}_0^+$. If the following three conditions are satisfied:*

- *$S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b$ with the relation $R_\varepsilon \subseteq X_a \times X_b$;*
- *$R_\varepsilon(W_a) \subseteq W_b$;*
- *$\forall (x_a, x_b) \in R_\varepsilon \subseteq X_a \times X_b \,:\, f_b(x_b) \leq f_a(x_a)$,*

then the following holds:

$$\forall (x_{a0}, x_{b0}) \in R_\varepsilon \quad \Rightarrow \quad \tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W_b, x_{b0}, f_b) \leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, W_a, x_{a0}, f_a)$$

where $(S^*_{ca}, \mathcal{F}^*_a) \in \mathcal{R}(S_a, W_a)$ and $(S^*_{cb}, \mathcal{F}^*_b) \in \mathcal{R}(S_b, W_b)$ denote the optimal controller-interconnection pairs for their respective optimal control problems, and $x_{a0} \in X_{a0}$, $x_{b0} \in X_{b0}$.

Proof. In the case when $\tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, W_a, x_{a0}, f_a) = \infty$, the result is trivially true. Hence, we assume $\tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, W_a, x_{a0}, f_a) < \infty$. In this case, we prove the result by parts:

1. We start by defining a controller S_c and an interconnection relation $\mathcal{G}$ for the system S_b such that 2. holds.
2. We show that for every maximal behavior
$$\mathbf{y}^{cb} \in \mathcal{B}_{(x_{c0}, x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b) \cup \mathcal{B}^{\omega}_{(x_{c0}, x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b)$$
there exists a maximal behavior
$$\mathbf{y}^{ca} \in \mathcal{B}_{(x_{ca0}, x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a) \cup \mathcal{B}^{\omega}_{(x_{ca0}, x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a)$$
ε-related to $\mathbf{y}^{cb}$.
3. With the result from step 2. we show:
$$\tilde{J}(S_c, \mathcal{G}, S_b, W_b, x_{b0}, f_b) \leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, W_a, x_{a0}, f_a). \tag{3.3}$$
4. The proof is finalized by noting that for a controller-interconnection pair $(S^*_{cb}, \mathcal{F}^*_b)$ to be optimal, it has to satisfy:
$$\tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W_b, x_{b0}, f_b) \leq \tilde{J}(S_c, \mathcal{G}, S_b, W_b, x_{b0}, f_b).$$
Consequentely,
$$\tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W_b, x_{b0}, f_b) \leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, W_a, x_{a0}, f_a)$$
for all $x_{a0} \in X_{a0}$ and $x_{b0} \in X_{b0}$ such that $(x_{a0}, x_{b0}) \in R_\varepsilon$, hence proving the result.

[1] The output sets X_a and X_b do not necessarily have to be equal. It suffices that the output sets belong to the same normed vector space X such that it is possible to determine distances between outputs of X_a and outputs of X_b by the metric induced by the norm of X.

The proof of the above steps will now be given.

1. To define the controller-interconnection pair $(S_c, \mathcal{G})$ for the system S_b let R_a be the exact alternating simulation relation from S^*_{ca} to S_a, defining the interconnection relation $\mathcal{F}^*_a = R^e_a$. We define the interconnection relation $\mathcal{G} = R^e_G$ as an interconnection relation that allows us to use the system $S_c = S^*_{ca} \times_{\mathcal{F}^*_a} S_a$ as a controller for system S_b. The interconnection relation $\mathcal{G} = R^e_G$ is then determined by the relation:

$$R_G = \{((x^*_{ca}, x_a), x_b) \in (X^*_{ca} \times X_a) \times X_b \mid (x^*_{ca}, x_a) \in R_a \wedge (x_a, x_b) \in R_\varepsilon\}.$$

   Furthermore, from Proposition 2.13 it follows that

$$S_c \times^{\varepsilon}_{\mathcal{G}} S_b \preceq^{\frac{1}{2}\varepsilon}_{\mathcal{S}} S_c = S^*_{ca} \times_{\mathcal{F}^*_a} S_a \tag{3.4}$$

   with the relation $R_{cb} \subseteq X_{\mathcal{G}} \times X_c$:

$$R_{cb} = \{((x_c, x_b), x'_c) \in X_{\mathcal{G}} \times X_{\mathcal{F}^*_a} \mid x_c = x'_c\}.$$

2. In order to show that for every maximal behavior

$$\mathbf{y}^{cb} \in \mathcal{B}_{(x_{c0}, x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b) \cup \mathcal{B}^{\omega}_{(x_{c0}, x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b)$$

   there exists an ε-related maximal behavior

$$\mathbf{y}^{ca} \in \mathcal{B}_{(x_{ca0}, x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a) \cup \mathcal{B}^{\omega}_{(x_{ca0}, x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a),$$

   we distinguish two distinct cases: finite and infinite behaviors.
   From the previous step we already have the existence of the simulation relation (3.4) which implies that for every behavior $\mathbf{y}^{cb}$ there exists an $\frac{1}{2}\varepsilon$-related behavior $\mathbf{y}^{ca}$. Since $\frac{1}{2}\varepsilon < \varepsilon$, for every behavior $\mathbf{y}^{cb}$ there exists an ε-related behavior $\mathbf{y}^{ca}$. So what is left to proof is that this also holds for $\mathbf{y}^{cb}$ and $\mathbf{y}^{ca}$ maximal.

   - Since every infinite behavior is a maximal behavior, we immediately know that for every (maximal) infinite behavior $\mathbf{y}^{cb}$ there exists an ε-related (maximal) infinite behavior $\mathbf{y}^{ca}$.

   To discuss the finite behavior case, we first make the following remark:
   For any pair $(x_a, x_b) \in R_\varepsilon$, by the definition of alternating simulation relation:

$$U_b(x_b) = \emptyset \quad \Rightarrow \quad U_a(x_a) = \emptyset.$$

   From the definition of $\mathcal{G}$ it follows that $\forall((x^*_{ca}, x_a), x_b) \in X_{\mathcal{G}} : (x_a, x_b) \in R_\varepsilon$.
   Thus, for any pair of related states $(x_a, x_b) \in R_\varepsilon$, there exists $x_{\mathcal{G}} \in X_{\mathcal{G}}$, namely $x_{\mathcal{G}} = (x_c, x_b)$ with $x_c = (x^*_{ca}, x_a)$, so that

$$U_{\mathcal{G}}(x_{\mathcal{G}}) = \emptyset \quad \Rightarrow \quad U_c(x_c) = \emptyset.$$

   - Hence, if $\mathbf{y}^{cb}$ is a maximal finite behavior of length l, the set of inputs $U_{\mathcal{G}}(x_{\mathcal{G}l})$ is empty, where $x_{\mathcal{G}l} = (x_{cl}, x_{bl})$ is the state corresponding to the l-th output of the behavior $\mathbf{y}^{cb}$. As shown above, this implies that $U_c(x_{cl}) = \emptyset$, where $x_{cl} = (x^*_{cal}, x_{al})$ is the state corresponding to the l-th output of the behavior $\mathbf{y}^{ca}$. Hence $\mathbf{y}^{ca}$ is also a maximal finite behavior of length l, where $\mathbf{y}^{ca}$ is the corresponding behavior of $S^*_{ca} \times_{\mathcal{F}^*_a} S_a$ ε-related to $\mathbf{y}^{cb}$.

3. We now show that (3.3) holds.
The existence of the simulation relation (3.4) implies that for every maximal behavior
$$\mathbf{y}^{cb} \in \mathcal{B}_{(x_{c0},x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b) \cup \mathcal{B}^{\omega}_{(x_{c0},x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b)$$
there exists an ε-related maximal behavior
$$\mathbf{y}^{ca} \in \mathcal{B}_{(x_{ca0},x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a) \cup \mathcal{B}^{\omega}_{(x_{ca0},x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a),$$
and from the second assumption, i.e. $R_\varepsilon(W_a) \subseteq W_b$, we now know that the ε-related maximal behaviors $\mathbf{y}^{cb}$ and $\mathbf{y}^{ca}$ satisfy:
$$\begin{aligned} \tau(S_c \times^{\varepsilon}_{\mathcal{G}} S_b, W_b, \mathbf{y}^{cb}) &\leq \tau(S_c \times^{\varepsilon}_{\mathcal{G}} S_b, R_\varepsilon(W_a), \mathbf{y}^{cb}) \\ &= \tau(S^*_{ca} \times_{\mathcal{F}^*_a} S_a, W_a, \mathbf{y}^{ca}). \end{aligned} \tag{3.5}$$
Additionally, by remarking that the relation $R_{cb} \subseteq X_{\mathcal{G}} \times X_{\mathcal{F}^*_a}$ relies on the ε-approximate alternating simulation relation $R_\varepsilon \subseteq X_a \times X_b$ and using the third condition, i.e. $\forall (x_a, x_b) \in R_\varepsilon : f_b(x_b) \leq f_a(x_a)$, we have the following:
$$\begin{aligned} \text{If for all}\,(x_a,x_b) \in R_\varepsilon \ &: \quad & f_b(x_b) \leq f_a(x_a), \\ \text{then for all}\,(x_{cb},x_{ca}) \in R_{cb} \ &: \quad f(x_{cb}) \ = & f_b(x_b) \leq f_a(x_a) \ = \ f(x_{ca}), \end{aligned} \tag{3.6}$$
where $x_{ca} = (x^*_{ca}, x_a) \in X_{\mathcal{F}^*_a}$ and $x_{cb} = (x_c, x_b) \in X_{\mathcal{G}}$.

So, we now know that for every maximal behavior
$$\mathbf{y}^{cb} \in \mathcal{B}_{(x_{c0},x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b) \cup \mathcal{B}^{\omega}_{(x_{c0},x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b)$$
which is ε-related by R_{cb} to some maximal behavior
$$\mathbf{y}^{ca} \in \mathcal{B}_{(x_{ca0},x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a) \cup \mathcal{B}^{\omega}_{(x_{ca0},x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a),$$
the following holds:
$$\begin{aligned} \mathcal{C}(\mathbf{y}^{cb}) &= \sum_{i=0}^{\tau(S_c \times^{\varepsilon}_{\mathcal{G}} S_b, W_b, \mathbf{y}^{cb})-1} f(x_{cbi}) \\ &\leq \sum_{i=0}^{\tau(S^*_{ca} \times_{\mathcal{F}^*_a} S_a, W_a, \mathbf{y}^{ca})-1} f(x_{cai}) \qquad \text{(from (3.5) and (3.6))} \\ &= \mathcal{C}(\mathbf{y}^{ca}). \end{aligned}$$
Consequentely, by Definition 3.4 we have
$$\begin{aligned} \tilde{J}(S_c, \mathcal{G}, S_b, W_b, x_{b0}, f_b) &= \max_{\mathbf{y}^{cb} \in \mathcal{B}_{(x_{c0},x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b) \cup \mathcal{B}^{\omega}_{(x_{c0},x_{b0})}(S_c \times^{\varepsilon}_{\mathcal{G}} S_b)} \mathcal{C}(\mathbf{y}^{cb}) \\ &\leq \max_{\mathbf{y}^{ca} \in \mathcal{B}_{(x_{ca0},x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a) \cup \mathcal{B}^{\omega}_{(x_{ca0},x_{a0})}(S^*_{ca} \times_{\mathcal{F}^*_a} S_a)} \mathcal{C}(\mathbf{y}^{ca}) \\ &= \tilde{J}(S^*_{ca}, F^*_a, S_a, W_a, x_{a0}, f_a) \end{aligned}$$
for all $x_{a0} \in X_{a0}$ and $x_{b0} \in X_{b0}$ such that $(x_{a0}, x_{b0}) \in R_\varepsilon$.

4. Note that for a controller-interconnection pair $(S^*_{cb}, \mathcal{F}^*_b)$ to be optimal, it has to satisfy:
$$\tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W_b, x_{b0}, f_b) \leq \tilde{J}(S_{cb}, \mathcal{F}, S_b, W_b, x_{b0}, f_b)$$
for any pair $(S_{cb}, \mathcal{F}) \in \mathcal{R}(S_b, W_b)$ and for all $x_{b0} \in X_{b0}$. In particular this inequality holds for $(S_c, \mathcal{G}) \in \mathcal{R}(S_b, W_b)$ and for all $x_{b0} \in X_{b0}$ such that $\exists x_{a0} \in X_{a0}$ satisfying $(x_{a0}, x_{b0}) \in R_\varepsilon$. Hence,
$$\begin{aligned}\tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W_b, x_{b0}, f_b) &\leq \tilde{J}(S_c, \mathcal{G}, S_b, W_b, x_{b0}, f_b)\\ &\leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, W_a, x_{a0}, f_a)\end{aligned}$$
for all $x_{a0} \in X_{a0}$ and $x_{b0} \in X_{b0}$ such that $(x_{a0}, x_{b0}) \in R_\varepsilon$.

□

Note 3.10. The proof of steps 1. and 2. of the above lemma are direct results of the main lemma in [4]. Steps 3. and 4. are not direct results but more 'in the sense of' the main lemma in [4].

Note 3.11. The second assumption of Lemma 3.9 requires the target sets W_a and W_b to be related by the ε-approximate alternating simulation R_ε. This assumption can always be satisfied by appropriately enlarging or shrinking the target set W_b.

Definition 3.12. (Resizing the target set [5]). For any relation $R_\varepsilon \subseteq X_a \times X_b$ and any set $W \subseteq X_b$, the sets $\lfloor W \rfloor_{R_\varepsilon}, \lceil W \rceil_{R_\varepsilon}$ are given by:
$$\begin{aligned}\lfloor W \rfloor_{R_\varepsilon} &= \{x_a \in X_a \,|\, R_\varepsilon(x_a) \subseteq W\},\\ \lceil W \rceil_{R_\varepsilon} &= \{x_a \in X_a \,|\, R_\varepsilon(X_a) \cap W \neq \emptyset\}.\end{aligned}$$

So, by taking a more careful look at the formal definition of the sets $\lfloor W \rfloor_{R_\varepsilon}$ and $\lceil W \rceil_{R_\varepsilon}$, we see that we actually have:
$$\begin{aligned}\lfloor W \rfloor_{R_\varepsilon} &= \{x_a \in X_a \,|\, \forall (x_a, x_b) \in R_\varepsilon : x_b \in W\} \subseteq X_a,\\ \lceil W \rceil_{R_\varepsilon} &= \{x_a \in X_a \,|\, \exists x_b \in W : (x_a, x_b) \in R_\varepsilon\} \subseteq X_a.\end{aligned}$$
If we now apply the relation R_ε on both the sets, then we get:
$$R_\varepsilon(\lfloor W \rfloor_{R_\varepsilon}) \subseteq W \tag{3.7}$$
$$R_\varepsilon(\lceil W \rceil_{R_\varepsilon}) \supseteq W \quad \Rightarrow \quad \lceil W \rceil_{R_\varepsilon} \supseteq R_\varepsilon^{-1}(W). \tag{3.8}$$
Equations 3.7 and 3.8 can be used in Lemma 3.9 to satisfy the second assumption at all times to obtain an upper bound and a lower bound for the optimal cost respectively.

Note 3.13. The third assumption of Lemma 3.9 requires the instantaneous cost functions f_a and f_b to satisfy an inequality for all the states in the ε-approximate alternating simulation R_ε. This assumption can always be satisfied by appropriately labeling the cost in system S_a.

Definition 3.14. (Labeling the cost). For any ε-approximate alternating simulation relation $R_\varepsilon \subseteq X_a \times X_b$ between system S_a and deterministic system S_b satisfying that R_ε^{-1} is an ε-approximate simulation relation from S_b to S_a and any instantaneous cost function $f : X_b \to \mathbb{R}_0^+$, the functions $\lceil f \rceil_{R_\varepsilon} : X_a \to \mathbb{R}_0^+$ and $\lfloor f \rfloor_{R_\varepsilon} : X_a \to \mathbb{R}_0^+$ are for all $(x_a, x_b) \in R_\varepsilon$ defined by:

$$\begin{aligned} \lceil f \rceil_{R_\varepsilon}(x_a) &= \max_{x_b \in R_\varepsilon(x_a)} f(x_b) \\ \lfloor f \rfloor_{R_\varepsilon}(x_a) &= \min_{x_b \in R_\varepsilon(x_a)} f(x_b). \end{aligned}$$

Hence, we now always have for all $(x_a, x_b) \in R_\varepsilon$:

$$\lceil f \rceil_{R_\varepsilon}(x_a) \geq f(x_b) \tag{3.9}$$

$$\text{and} \quad \lfloor f \rfloor_{R_\varepsilon}(x_a) \leq f(x_b). \tag{3.10}$$

Based on Lemma 3.9 and Definitions 3.12 and 3.14, the main theorem explains now how to obtain both an upper and a lower bound for the optimal cost in a system S_b by working with a related system S_a.

Theorem 3.1. (Approximately optimal control). *Let S_a and S_b be two metric systems with $Y_a = X_a$, $H_a = 1_{X_a}$, $Y_b = X_b$ and $H_b = 1_{X_b}$ and let the output sets be equal*[1]*, i.e. $X_a = X_b$. Furthermore, let $f : X_b \to \mathbb{R}_0^+$ be the instantaneous cost function for system S_b. If S_b is deterministic and there exists an ε-approximate alternating simulation relation R_ε from S_a to S_b such that R_ε^{-1} is an ε-approximate simulation relation from S_b to S_a, i.e.*

$$S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b \preceq^{\varepsilon}_{\mathcal{S}} S_a,$$

then the following holds for any $W \subseteq X_b$ and $(x_{a0}, x_{b0}) \in R_\varepsilon$:

$$\begin{aligned} \tilde{J}(S^*_{cd(a)}, \mathcal{F}^*_{d(a)}, S_{d(a)}, \lceil W \rceil_{R_\varepsilon}, x_{a0}, \lfloor f \rfloor_{R_\varepsilon}) &\leq \tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W, x_{b0}, f) \\ &\leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, \lfloor W \rfloor_{R_\varepsilon}, x_{a0}, \lceil f \rceil_{R_\varepsilon}) \end{aligned}$$

where the interconnection-controller pairs $(S^*_{cd(a)}, F^*_{d(a)}) \in \mathcal{R}(S_{d(a)}, \lceil W \rceil_{R_\varepsilon}), (S^*_{cb}, F^*_b) \in \mathcal{R}(S_b, W)$ and $(S^*_{ca}, F^*_a) \in \mathcal{R}(S_a, \lfloor W \rfloor_{R_\varepsilon})$ are optimal for their respective optimal control problems.

Proof. The validation of this theorem follows immediately by Lemma 3.9.
We are given two metric systems S_a and S_b with $Y_a = X_a$, $H_a = 1_{X_a}$, $Y_b = X_b$, $H_b = 1_{X_b}$ and $X_a = X_b$. If S_b is deterministic and there exists an ε-approximate alternating simulation relation R_ε from S_a to S_b such that R_ε^{-1} is an ε-approximate simulation relation from S_b to S_a, then we have the following:

1. *Upper bound for the optimal cost*:
   System S_a and S_b satisfy the following three conditions:

   - $S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b$ with the relation $R_\varepsilon \subseteq X_a \times X_b$;
   - $R_\varepsilon(\lfloor W \rfloor_{R_\varepsilon}) \subseteq W$ where $\lfloor W \rfloor_{R_\varepsilon} \subseteq X_a$ and $W \subseteq X_b$; (see equation 3.7)
   - $\forall (x_a, x_b) \in R_\varepsilon : f(x_b) \leq \lceil f \rceil_{R_\varepsilon}(x_a)$. (see equation 3.9)

According to Lemma 3.9 the following now holds for all $(x_{a0}, x_{b0}) \in R_\varepsilon$:

$$\tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W, x_{b0}, f) \leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, \lfloor W \rfloor_{R_\varepsilon}, x_{a0}, \lceil f \rceil_{R_\varepsilon})$$

where the interconnection-controller pairs $(S^*_{cb}, F^*_b) \in \mathcal{R}(S_b, W)$ and $(S^*_{ca}, F^*_a) \in \mathcal{R}(S_a, \lfloor W \rfloor_{R_\varepsilon})$ are optimal for their respective optimal control problems, and $x_{a0} \in X_{a0}$, $x_{b0} \in X_{b0}$.

2. *Lower bound for the optimal cost*:
   System S_a and S_b also satisfy the following three conditions:

   - $S_b \preceq^\varepsilon_{\mathcal{S}} S_a$ with the relation $R_\varepsilon^{-1} \subseteq X_b \times X_a$.
     From Note 2.8 we know $S_a \preceq^0_{\mathcal{AS}} S_{d(a)}$ with the relation $R_{d(a)} = \{(x_a, x_{d(a)}) \in X_a \times X_a \,|\, x_{d(a)} = x_a\}$. Consequentely, we have $S_b \preceq^\varepsilon_{\mathcal{S}} S_{d(a)}$ and $S_b \preceq^\varepsilon_{\mathcal{AS}} S_{d(a)}$ (since S_b and $S_{d(a)}$ are both deterministic) with the relation $R_\varepsilon^{-1} \subseteq X_b \times X_a$;
   - $R_\varepsilon^{-1}(W) \subseteq \lceil W \rceil_{R_\varepsilon}$ where $W \subseteq X_b$ and $\lceil W \rceil_{R_\varepsilon} \subseteq X_a$; (see equation 3.8)
   - $\forall (x_b, x_a) \in R_\varepsilon^{-1} : \lfloor f \rfloor_{R_\varepsilon}(x_a) \leq f(x_b)$. (see equation 3.10)

   According to Lemma 3.9 the following now holds for all $(x_{b0}, x_{a0}) \in R_\varepsilon^{-1}$:

$$\tilde{J}(S^*_{cd(a)}, \mathcal{F}^*_{d(a)}, S_{d(a)}, \lceil W \rceil_{R_\varepsilon}, x_{a0}, \lfloor f \rfloor_{R_\varepsilon}) \leq \tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W, x_{b0}, f)$$

   where the interconnection-controller pairs $(S^*_{cd(a)}, F^*_{d(a)}) \in \mathcal{R}(S_{d(a)}, \lceil W \rceil_{R_\varepsilon})$ and $(S^*_{cb}, F^*_b) \in \mathcal{R}(S_b, W)$ are optimal for their respective optimal control problems, and $x_{a0} \in X_{a0}$, $x_{b0} \in X_{b0}$.

Hence, for system S_a and S_b the following holds for any $W \subseteq X_b$ and $(x_{a0}, x_{b0}) \in R_\varepsilon$:

$$\begin{aligned} \tilde{J}(S^*_{cd(a)}, \mathcal{F}^*_{d(a)}, S_{d(a)}, \lceil W \rceil_{R_\varepsilon}, x_{a0}, \lfloor f \rfloor_{R_\varepsilon}) &\leq \tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W, x_{b0}, f) \\ &\leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, \lfloor W \rfloor_{R_\varepsilon}, x_{a0}, \lceil f \rceil_{R_\varepsilon}) \end{aligned}$$

where the interconnection-controller pairs $(S^*_{cd(a)}, F^*_{d(a)}) \in \mathcal{R}(S_{d(a)}, \lceil W \rceil_{R_\varepsilon}), (S^*_{cb}, F^*_b) \in \mathcal{R}(S_b, W)$ and $(S^*_{ca}, F^*_a) \in \mathcal{R}(S_a, \lfloor W \rfloor_{R_\varepsilon})$ are optimal for their respective optimal control problems. □

Remark 3.15. If the system S_b is not deterministic in Theorem 3.1, then the inequality

$$\tilde{J}(S^*_{cb}, \mathcal{F}^*_b, S_b, W, x_{b0}, f) \leq \tilde{J}(S^*_{ca}, \mathcal{F}^*_a, S_a, \lfloor W \rfloor_{R_\varepsilon}, x_{a0}, \lceil f \rceil_{R_\varepsilon})$$

still holds by Lemma 3.9.

Theorem 3.1 explains how upper and lower bounds for the cost in system S_b can be computed on system S_a, where the optimality considerations are decoupled by using specific algorithms to compute the abstractions. When S_a is a much simpler system than system S_b, this possibility is of great value.

Chapter 4

Conclusions and future work

The results proved in Chapter 3 provide a methodology to solve optimal control problems for control systems reconciling both qualitative and quantitative specifications using the symbolic approach from [5]. The methodology is started by considering an infinite-state (deterministic) control system S_b and a finite-state symbolic abstraction S_a of S_b which satisfy $S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b \preceq^{\varepsilon}_{\mathcal{S}} S_a$. Since S_a is a finite-state system, system entry times and final costs for S_a can be computed efficiently using theory from computer science such as dynamic programming or Dijkstra's algorithm [6,7]. By Theorem 3.1 it then follows that the final costs computed on S_a (and a deterministic version of S_a) immediately provide bounds for the optimal cost in S_b to reach a target set W. Furthermore, along the process of computing the optimal costs for the abstracted system S_a, an optimal controller for S_a is provided which can be refined to an approximately optimal controller for the control system S_b. Composing the refined controller with the control system S_b is guaranteed to perform within the performance cost-bounds provided by the controlled abstracted system.

The result in which it is explained how upper and lower bounds for the cost in the control system S_b can be computed on the symbolic abstraction S_a, requires the existence of an ε-approximate alternating simulation relation R_ε from S_a to S_b where its inverse relation is an ε-approximate simulation relation from S_b to S_a (provided that S_b is deterministic). When a time-discretized cyber-physical system S represents the control system S_b, it has been shown in [1,2,3] that one can construct (under mild assumptions) symbolic abstractions S_{abs} in the form of finite-state systems satisfying:

$$S_{abs} \preceq^{\varepsilon}_{\mathcal{AS}} S \preceq^{\varepsilon}_{\mathcal{S}} S_{abs} \tag{4.1}$$

with arbitrary precision ε, as required. Note that the time-discretization, i.e. a time-triggered sampled version (see [5]), of the cyber-physical system is considered in equation (4.1) since this is the appropriate model for the synthesis of optimal controllers to be implemented on digital platforms.

While in general no improvement is guaranteed, in practice one normally improves the performance guarantees for the control system by improving the precision in equation (4.1). So, finding a symbolic abstraction of a control system having a higher precision ε' in equation (4.1) compared to another symbolic abstraction with precision ε, i.e. $\varepsilon' < \varepsilon$, then the symbolic abstraction with precision ε' will provide at least equally good cost-bounds in Theorem 3.1 as the symbolic abstraction with precision ε and possibly better.

As future work it should be explored how to automatically abstract the instantaneous cost functions $\lfloor f \rfloor_{R_\varepsilon}$ and $\lceil f \rceil_{R_\varepsilon}$ for the symbolic abstraction system S_a from the instantaneous cost function f of the control system S, as needed in Theorem 3.1. By making some well considered assumptions, this should be possible by considering the inverse relation of the approximate alternating simulation relation between the symbolic abstraction and the control system. Assuming the instantaneous cost function f of the control system to be continuously differentiable, the output space X to be a compact space and the output map H to be continuously differentiable as well, could be sufficient to solve the problem.

Other future work on this topic is the compositional approach. Here it has to be determined how a symbolic abstraction and specification system can be decomposed into smaller pieces such that after solving optimal control problems on the decomposed pieces of the symbolic abstractions and specification systems, the optimal controllers need to composed in such a way that the resulting controller is an optimal controller for the symbolic abstraction satisfying the specification system. This is convenient for large complex control problems which can be decomposed in smaller less complex control problems.

On the practical side, the results obtained in this thesis should be implemented in a toolbox, such as the Matlab toolbox Pessoa (see [8]), to make it possible to solve concrete optimal control problems.

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