rijksuniversiteit groningen

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String theory & Generalized Geometry

Abstract

This thesis aims to show the role of Generalized Geometry in string theory. It is divided up into two parts. The first consists an introduction to bosonic and super-string theory and a brief discussion of type II superstring theory's low energy limit: the so-called supergravity theories. The second part deals with the problem of compactification in string theory and focusses on flux compactifications. It is in that part that Generalized Geometry is discussed and used to classify Minkowski compactifications.

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1 Introduction

The aim of this thesis is to offer an account on how a branch of geometry, called *Generalized Geometry*, ties in with string theory. To that end, this thesis is split up into two parts. The *first* part consists of an introduction in string theory.

String theory is based on the idea that particles are not point-like, but rather tiny loops (i.e. *closed* strings) or (*open*) pieces of string. As we will see, this assumption leads to the conclusion that the different vibrational modes of the strings represent different kinds of particles. This set is the main reason why string theory is interesting: it contains, among others, the graviton, different kinds of chiral fermions and Yang-Mills gauge fields. This means that string theory is a quantum theory that includes gravity, as well as roughly all the types of fields that exist in nature; it may just be the sought-after theory that consistently unifies all known particle fields and forces.

Still, string theory is not a very easy theory. It provides plenty of problems that one has to deal with. One of these is the central theme of this thesis' *second* part—in this part we will focus on the ten dimensions that string theoy predicts and how we can make the extra six disappear (almost, that is). The correct term for this is *compactification*: it involves "wrapping up" the six additional dimensions into a small, compact manifold—one that should be so small that it is no longer noticeable (in a sense) at 'long' distance scaler (low energies).

Generalized geometry will come in handy when we consider a particular way of compactifying that involves non-zero vacuum expectation values (or *vevs*) for some of the fields. It offers a very clean rephrasing of the conditions the compact manifold will have to satisfy.

In the following few sections we will introduce string theory.

2 Bosonic string theory

The first subject we will consider is *bosonic string theory.* It is a string theory that, as its name suggests, describes only a certain set of bosons. However far removed from reality it may be, it is the simplest of several string theories and serves, as such, as an introduction to string theory. *Superstring theory*, which does include fermions, is constructed in a way similar to the bosonic theory, and will be discussed after.

The way the theory will be developed in this section, is very much like the standard way quantum field theory is introduced (see, for example, the lecture notes on relativistic quantum mechanics by Mees de Roo). Firstly, the classical action and the equations of motion are studied; secondly, the solution to the equation of motion is expanded in modes and lastly, in canonical quantization, the modes' coefficients are promoted to creation and annihilation operators and the corresponding Fock space is constructed.

The current discussion is based on a number of sources, all of which serve as decent introductions to string theory, [1], [2], [3] and [4]. The lecture notes by David Tong, [3], though, focus solely on bosonic string theory.

2.1 The relativistic string

2.1.1 The classical action

We start our discussion of string theory with a simpler and more familiar object: the relativistic point particle. The action of a such a particle is given by:

$$S = m \int_{\tau_i}^{\tau_f} d\tau = m \int_{s_i}^{s_f} ds = m \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}, \tag{2.1}$$

where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is the "mostly plus" Minkowski metric, and $\dot{X}^\mu = \frac{dX^\mu}{d\lambda}(\lambda)$. It is proportional to the length of the particle's world-line, starting at some initial and ending at some final point. Note that we work in units where $\hbar = c = 1$.

The reason for this choice of action is that a classical point particle maximizes its proper time, which corresponds to a linear trajectory.

In the case of a string, we have a one dimensional object tracing out a two-dimensional "*world-sheet*". In analogy to the point particle action, we take the string's action to be proportional to the worldsheet's area. This defines the *Nambu-Goto action*, which is given by

$$S_{NG} = -T \int dA. \tag{2.2}$$

The constant T plays the same role as m does in the point particle action: it is there to make the action dimensionless. It has dimensions $[\text{length}]^{-2}$ or $[\text{mass}]^2$.

In order to calculate the world-sheet's surface area, we will need a metric. Let $\xi^i (i = 0, 1)$ denote the coordinates on the worldsheet and take $G_{\mu\nu}$ to be the metric of the d-dimensional space-time in which the string propagates[1]. Then $G_{\mu\nu}$ induces a metric on the worldsheet as follows:

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = G_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^i} \frac{\partial X^\nu}{\partial \xi^j} d\xi^i d\xi^j \equiv G_{ij} d\xi^i d\xi^j, \tag{2.3}$$

G_{ij} being the induced metric. In a flat Minkowski spacetime we have that $G_{\mu\nu} = \eta_{\mu\nu}$. The Nambu-Goto action then takes on the following form:

$$S_{NG} = -T \int \sqrt{-\det G_{ij}}\, d^2\xi = -T \int \sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X'^2)} d\tau d\sigma, \tag{2.4}$$

where we redefined the world-sheet coordinates as $\tau = \xi^0, \sigma = \xi^1$ and wrote $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$[2]. See for a simple image, figure 1.

[1] Initially, we take the number of spacetime dimensions, d, to be arbitrary. The reason for this will become clear further on

[2] Using coordinates $X^\mu = (t, \vec{x})$ and reparametrizing such that $\tau = t$, we find that in the rest frame of the string (i.e. $\frac{d\vec{x}}{dt} = 0$ and zero kinetic energy) $S = -T \int dt d\sigma |d\vec{x}/d\sigma|$. Identifying the Lagrangian as $L = E_{\text{kin}} - V$, with $E_{\text{kin}} = 0$, we find that $T = V/(\text{string length})$. The parameter T has dimensions energy per length and is therefore called the string's *tension*. As an unrelated note, it is often written as $T = \frac{1}{2\pi\alpha'}$. Note that is it only because of the quantum theory's zero point energy that a string does not minimize its potential energy by shrinking to zero length [3].

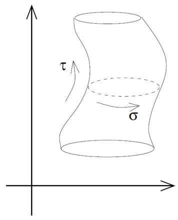

Figure 1: Part of the worldsheet [3].

2.1.2 The equations of motion and boundary conditions

The equations of motion follow from the Euler-Lagrange equations. They are

$$\partial_\tau \left(\frac{\delta L}{\delta \dot{X}^\mu} \right) + \partial_\sigma \left(\frac{\delta L}{\delta X'^\mu} \right) = 0. \tag{2.5}$$

The momentum conjugate to X^μ, Π^μ, is, as usual, given by:

$$\Pi^\mu \equiv \frac{\delta L}{\delta \dot{X}^\mu} = -T \frac{(\dot{X} \cdot X')X'^\mu - (X')^2 \dot{X}^\mu}{\sqrt{(X' \cdot \dot{X})^2 - (\dot{X})^2 (X')^2}}. \tag{2.6}$$

Using the conjugate momentum we obtain the following two constraints,

$$\Pi \cdot X' = 0, \qquad \Pi^2 + T^2 X'^2 = 0. \tag{2.7}$$

The dynamics of the string are fully governed by these equations as the Hamiltonian vanishes:

$$H = \int_0^{\bar{\sigma}} d\sigma (\dot{X} \cdot \Pi - L) = 0. \tag{2.8}$$

A similar situation arises in the case of the point particle, if we choose to describe it by the rightmost version of the action in (2.1)[3].

For the integral above, the integration domain for σ is usually taken to be $[0, \bar{\sigma}) = [0, 2\pi)$ for closed strings and $[0, \pi)$ for open strings. Apart from the range of σ we also need to specify the boundary conditions for the two types of string. In the closed string case the worldsheet is a tube and we will take the condition to be periodicity

$$X^\mu(\sigma + 2\pi) = X^\mu(\sigma). \tag{2.9}$$

For open strings there are two kinds of boundary conditions that are frequently used, the *Neumann* and *Dirichlet* conditions:

$$\left. \frac{\delta L}{\delta X'^\mu} \right|_{\sigma = 0, \pi} = 0 \qquad \text{(Neumann)}, \tag{2.10}$$

[3] The constraints mean that not all momenta are independent and that not all degrees of freedom (X^μ in the point particle case) are physical. In the point particle case the constraint is $\Pi^2 + m^2 = 0$ (the mass-shell relation) and the degree of freedom that is unphysical is X^0: a particle is forced to move in time in a way determined by its mass-shell condition. This becomes obvious when one considers the reparametrization (or "gauge") invariance $\lambda \to \tilde{\lambda}(\lambda)$. Gauge fixing then amounts to fixing the time parameter. The advantage of this action is, however, that it is Lorentz invariance in an obvious way.

$$\frac{\delta L}{\delta \dot{X}^\mu}\bigg|_{\sigma=0,\pi} = 0 \qquad \text{(Dirichlet)}. \tag{2.11}$$

The Neumann conditions imply that no momentum flows off the ends of the string while the Dirichlet conditions imply that the endpoints are fixed in space-time. If Dirichlet conditions are applied to a subset of the d indices of X^μ, for example to the indices $p+1, \ldots, d$, that would mean that the endpoints of the string are confined to move on a p-dimensional hyperplane. This hypersurface is called a *Dp-brane*, where p indicates its dimension and D stands for Dirichlet. It turns out that these objects should be considered to be dynamical as well. See, for an introduction to this particular subject, [5], or the short but insightful remarks in [3].

2.1.3 The Polyakov action

Although we now have an action for the relativistic string, the square-root makes it difficult to quantize. We can, however, get rid of it by introducing an additional field on the world-sheet: a fluctuating metric $g_{\alpha\beta}$. This allows us to write the *Polyakov action*, which will turn out to be classically equivalent to the Nambu-Goto action.

$$S_P = -\frac{T}{2} \int d^2\xi \sqrt{-\det g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.12}$$

Here, space-time is taken to be flat again. In the following we will write $\sqrt{-g} \equiv \sqrt{-\det g}$.

If we vary the action with respect to the metric $g_{\alpha\beta}$[4], we get, up to a factor, the stress-energy tensor:

$$T_{\alpha\beta} \equiv -\frac{2}{T\sqrt{-g}} \frac{\delta S_P}{\delta g^{\alpha\beta}} = \partial_\alpha X \cdot \partial_\beta X - \tfrac{1}{2} g_{\alpha\beta} g^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X. \tag{2.13}$$

Setting the variation of the action with respect to the metric equal to zero, we find for $g_{\alpha\beta}$ that

$$g_{\alpha\beta} = 2f(\sigma,\tau) \partial_\alpha X \cdot \partial_\beta X, \tag{2.14}$$

where

$$f^{-1}(\sigma,\tau) = g^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X. \tag{2.15}$$

Inserting this expression for the metric into the Polyakov action, yields the Nambu-Goto action, confirming that they are classically equivalent.

Varying the action with respect to X^μ yields the equations of motion,

$$\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X^\mu) = 0. \tag{2.16}$$

These equations correspond to d two-dimensional scalar fields coupled to a dynamical two-dimensional metric. That is to say, they describe a two-dimensional theory in which gravity is coupled to matter[5].

[4] We use the identity $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$.

[5] Actually, for this to be true, one should also add the Einstein-Hilbert action to the Polyakov action: $S = S_P - \frac{1}{2\kappa}\int d^2\xi \sqrt{-g}\, R$, where R is the Ricci scalar and $\kappa = 8\pi G$. As this action is a topological invariant (it yields the Euler number of the worldsheet), it does not influence the local dynamics of the string [2].

2.1.4 Symmetries of the Polyakov action

The Polyakov action in its current form is still relatively complicated. Luckily, the action has a number of symmetries that can be exploited to simplify it. The infinitessimal forms of these symmetries are listed below.

- *Poincaré invariance*, which is a global symmetry on the wordsheet:

$$\begin{aligned}\delta X^\mu &= \omega^\mu{}_\nu X^\nu + a^\mu, \\ \delta g_{\alpha\beta} &= 0,\end{aligned} \tag{2.17}$$

  where the $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are parameters defining an infinitesimal Lorentz transformation. The a^μ correspond to a translation.

- *Reparametrization* invariance, or invariance under *diffeomorphisms*[6] (or changes of coordinates ξ^i):

$$\begin{aligned}\delta X^\mu &= \xi^\alpha \partial_\alpha X^\mu, \\ \delta g_{\alpha\beta} &= \xi^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \xi^\gamma g_{\beta\gamma} + \partial_\beta \xi^\gamma g_{\alpha\gamma} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha, \\ \delta(\sqrt{-g}) &= \partial_\alpha(\xi^\alpha \sqrt{-g}).\end{aligned} \tag{2.18}$$

- Invariance under *Weyl rescaling*, or *conformal* invariance[7]:

$$\begin{aligned}\delta X^\mu &= 0, \\ \delta g_{\alpha\beta} &= 2\Lambda g_{\alpha\beta}.\end{aligned} \tag{2.19}$$

Above, Λ and ξ^α are arbitrary, infinitessimal functions of (σ, τ). For the non-infinitessimal versions of these transformations, see [3].

Using reparametrizations, the metric $g_{\alpha\beta}$ can be made conformally flat. This choice of gauge is called the *conformal gauge*:

$$g_{\alpha\beta} = e^{2\Lambda(\xi)} \eta_{\alpha\beta}. \tag{2.20}$$

Furthermore, using a Weyl transformation, the metric can be brought to the standard Minkowski form: $g_{\alpha\beta} = \eta_{\alpha\beta}$[8].

Using the flat metric, the Polyakov action becomes

$$S_P = -\frac{T}{2} \int d^2\xi \, \partial_\alpha X \cdot \partial^\alpha X, \tag{2.21}$$

while the equations of motion for the X^μ reduce to free wave equations,

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{2.22}$$

[6] This particular one is, of course, reminiscent of the point particle case.

[7] Non-infinitessimally this becomes: $g_{\alpha\beta} \to e^{2\Lambda(\xi)} g_{\alpha\beta}$. It corresponds to an angle-preserving, ξ^i-dependent rescaling [3].

[8] This trick of simplifying the metric is only possible for trivial topologies in two dimensions: reparametrization and Weyl invariance can be used to fix, respectively, d and 1 components of the metric; the sum of this is equal to the number of independent components of the metric, $d(d+1)/2$, only for $d = 2$, [2].

Although this looks particularly simple, there are still a number of constraints for X^μ that need to be taken into account. They result from the equation of motion for the metric: $T_{\alpha\beta} = 0$. Explicitly,

$$T_{10} = T_{01} = \dot{X} \cdot X' = 0, \tag{2.23}$$

$$T_{00} = T_{11} = \tfrac{1}{2}(\dot{X}^2 + X'^2) = 0. \tag{2.24}$$

They are known as the *Virasoro* constraints. They may be rewritten as

$$(\dot{X} \pm X')^2 = 0. \tag{2.25}$$

2.1.5 Mode expansions

In this section we expand the $X^\mu(\sigma, \tau)$ in Fourier modes. First, we switch to *lightcone* worldsheet coordinates:

$$\sigma^\pm = \tau \pm \sigma. \tag{2.26}$$

The equations of motion now read

$$\partial_+ \partial_- X^\mu = 0. \tag{2.27}$$

The most general solution is a sum of a left-moving (X_L^μ) and a right-moving wave (X_R^μ),

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-). \tag{2.28}$$

As these will need to satisfy the periodicity conditions mentioned in section 2.1.2, they can be expanded in Fourier modes. Here we consider only closed strings; the case of open strings (using Neumann boundary conditions) is quite similar.

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{x^\mu}{2} + \frac{p^\mu}{4\pi T}\sigma^+ + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{\tilde{a}_n^\mu}{n} e^{-in\sigma^+} \\ X_R^\mu(\sigma^-) &= \frac{x^\mu}{2} + \frac{p^\mu}{4\pi T}\sigma^- + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{a_n^\mu}{n} e^{-in\sigma^-} \end{aligned} \tag{2.29}$$

In the mode expansions above, x^μ and p^μ are, respectively, the string's centre of mass position and momentum. This may be checked by computing the integral over σ of X^μ or $\dot{X}^\mu$, including the appropriate factors of T and 2π. The reality of X^μ requires the Fourier coefficients to obey

$$a_n^\mu = (a_{-n}^\mu)^*, \qquad \tilde{a}_n^\mu = (\tilde{a}_{-n}^\mu)^*. \tag{2.30}$$

The Virasoro constraints become

$$(\partial_- X)^2 = (\partial_+ X)^2 = 0. \tag{2.31}$$

and may be expanded in terms of Fourier series as well. Their expansions can be found by inserting the expansion for X^μ in the formulas above. The Fourier coefficients, setting $\tilde{a}_0^\mu = a_0^\mu \equiv \frac{p^\mu}{\sqrt{4\pi T}}$, are

$$L_n = \tfrac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} \cdot a_m, \qquad \tilde{L}_n = \tfrac{1}{2} \sum_{m \in \mathbb{Z}} \tilde{a}_{n-m} \cdot \tilde{a}_m. \tag{2.32}$$

It follows that any classical solution of the string has to obey the constraints $L_n = \tilde{L}_n = 0, n \in \mathbb{Z}$. The L_0 and $\tilde{L}_0$ constraints are of special interest. As they contain the square of the centre of mass momentum p^μ, they determine the string's mass in terms of the excited oscillator modes:

$$m^2 = -p_\mu p^\mu = 8\pi T \sum_{n>0} a_n \cdot a_{-n} = 8\pi T \sum_{n>0} \tilde{a}_n \cdot \tilde{a}_{-n}. \tag{2.33}$$

The equality of the two expressions for the mass, one in terms of right-moving oscillators and one in terms of left-moving oscillators, is know as *level matching*. In the quantum theory level matching implies that the number and type of right-moving excitations of a string is equal to the number and type of its left-moving excitations. The Hamiltonian, in terms of the modes' coefficients, is simply:

$$H = L_0 + \tilde{L}_0. \tag{2.34}$$

The last step we take before we quantize the string consists of determining the equal-τ Poisson brackets that are of interest to the theory. In the Hamiltonian picture we have the following bracket for the dynamical variables X^μ and their conjugate momenta $(T\dot{X}^\nu)$:

$$\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{PB} = \tfrac{1}{T}\delta(\sigma-\sigma')\eta^{\mu\nu}. \tag{2.35}$$

The other brackets, $\{X, X\}$ and $\{\dot{X}, \dot{X}\}$, vanish. Using these, the Poisson brackets for the modes are easily found to be:

$$\begin{aligned}
&\{a_m^\mu, a_n^\nu\} = \{\tilde{a}_m^\mu, \tilde{a}_n^\nu\} = -im\delta_{m+n,0}\eta^{\mu\nu}, \\
&\{\tilde{a}_m^\mu, a_n^\nu\} = 0, \\
&\{x^\mu, p^\nu\} = \eta^{\mu\nu}.
\end{aligned} \tag{2.36}$$

These in turn determine the brackets that define the classical Virasoro algebra:

$$\begin{aligned}
&\{L_m, L_n\} = -i(m-n)L_{m+n}, \\
&\{\tilde{L}_m, \tilde{L}_n\} = -i(m-n)\tilde{L}_{m+n}, \\
&\{L_m, \tilde{L}_n\} = 0.
\end{aligned} \tag{2.37}$$

2.2 The quantized relativistic string

In the following sections we consider two different ways of quantizing the bosonic string: canonical quantization and lightcone quantization. For a discussion of a third way, that of path integral quantization, see [2] or [3]. The canonical and lightcone procedures are useful especially because they allow for an explicit construction of the Fock space.

We will conclude tha chapter on the bosonic string with a few remarks concerning the string's spectrum and the particles its oscillations represent.

2.2.1 Canonical quantization

The most straightforward way to quantize the string, and perhaps also the most traditional way to quantize any system, is to replace the fields, X^μ, by operators and to replace the Poisson brackets by commutator brackets:

$$\{\ ,\ \}_{PB} \to -i[\ ,\ ]. \tag{2.38}$$

This is just the canonical quantization procedure. It leads to the following brackets for the mode coefficients,

$$
\begin{aligned}
&[x^\mu, p^\nu] = i\eta^{\mu\nu} \\
&[a_m^\mu, a_n^\nu] = [\tilde{a}_m^\mu, \tilde{a}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}.
\end{aligned}
\tag{2.39}
$$

The second relation becomes, after a redefinition of the modes:

$$
a_n \equiv \frac{a_n}{\sqrt{n}}, \qquad a_n^\dagger \equiv \frac{a_{-n}}{\sqrt{n}}, \qquad \text{with } n > 0, \tag{2.40}
$$

$$
[a_m^\mu, a_n^{\nu\dagger}] = [\tilde{a}_m^\mu, \tilde{a}_n^{\nu\dagger}] = \delta_{m,n}\eta^{\mu\nu}. \tag{2.41}
$$

These are just the harmonic oscillator commutation relations for an infinite set of oscillators, labelled by $n \in \mathbb{N}$. The positive frequency mode coefficients a_n become annihilation (or lowering) operators while the negative frequency modes $a_n^\dagger$ creation (or raising) operators. The use of commutators implies that the theory's excitations comprise only bosons.

The Fock space of the theory is to be constructed from the vacuum state, which we will consider first. The vacuum state $|0\rangle$ is defined as the state that is annihilated by all annihilation operators:

$$
a_n|0\rangle = \tilde{a}_n|0\rangle = 0, \qquad \forall n > 0. \tag{2.42}
$$

This state needs to be defined still more precisely, since we have not yet considered the string's centre of mass variables, x^μ and p^μ. If we diagonalize p^μ, the string's vacuum state will be characterized by p. This state we denote by $|0;p\rangle$. The other states of the Fock space are then built by acting on $|0;p\rangle$ with the $a_n^\dagger$'s. Each different state in the Fock space then corresponds to a different excited state of the string.

There is one problem, though, that arises if one uses this procedure: it leads to negative norm states, dubbed *ghosts*,

$$
\|a_{-1}^0|0;p\rangle\|^2 = \langle 0;p|a_1^0 a_{-1}^0|0;p\rangle \sim -\delta^{d-1}(p-p). \tag{2.43}
$$

This is due to the Minkowski metric that appears in the commutation relations. Luckily, a no-ghost theorem can be proven stating that these ghosts decouple from the physical spectrum if we impose the Virasoro constraints, under the conditions that the number of spacetime dimensions d is 26 and that the normal ordering constant a (which will be discussed below) is 1.

In order to impose the Virasoro constraints, we have to prescribe a specific ordering of the oscillator operators they include, as because they are non-commuting, different ways of ordering lead to different constraints. The standard prescription is called *normal ordering*; it puts positive frequency modes to the right of negative frequency modes. The Virasoro operators are now defined as

$$
L_m = \tfrac{1}{2}\sum_{n\in\mathbb{Z}} : a_{m-n}\cdot a_n :, \tag{2.44}
$$

where the colons indicate that the operators are normal ordered. Of the L_m, only L_0 (and similarly $\tilde{L}_0$) is sensitive to normal ordering. It becomes

$$
L_0 = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_{-n}\cdot a_n. \tag{2.45}
$$

Since the commutator of two operators is a constant, and as we do not know what this constant should be in the case of L_0, we simply include an additional arbitrary constant a in the definition of L_0 ($L_0 \to L_0 - a$). This constant has some physical significance as it appears in the mass relation for the string:

$$m^2 = 8\pi T\left(-a + \sum_{n>0} a_n \cdot a_{-n}\right) = 8\pi T\left(-a + \sum_{n>0} \tilde{a}_n \cdot \tilde{a}_{-n}\right), \tag{2.46}$$

and thus directly affects the mass spectrum.

The Virasoro algebra for the quantized string is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\delta_{m+n,0}, \tag{2.47}$$

where d is the number of space-time dimensions. Looking at this equation, we see that we cannot impose the constraints as $L_m|\phi\rangle = 0$, as this leads to

$$0 = \langle\phi|[L_m, L_{-m}]|\phi\rangle = 2m\langle\phi|L_0|\phi\rangle + \frac{d}{12}m(m^2-1)\langle\phi|\phi\rangle \neq 0. \tag{2.48}$$

So instead we impose on physical states, as in the Gupta-Bleuler approach to quantizing QED,

$$L_{m>0}|\text{phys}\rangle = 0, \qquad (L_0 - a)|\text{phys}\rangle = 0. \tag{2.49}$$

In the case of the closed string, equivalent expressions apply for the $\tilde{L}$ operators. This definition of the constraints is consistent with the classical picture, since the expectation value of the operators vanishes: $\langle\text{phys}|L_n|\text{phys}\rangle = 0$.

2.2.2 Lightcone quantization

Lightcone quantization is similar to covariant quantization in that it also involves replacing the Poisson brackets by commutators and the fields by operators. The procedure differs because now we will not quantize first and impose the Virasoro constraints later, but we will do that in reverse order.

First, however, we consider the remaining gauge invariance. In the gauge where $g_{\alpha\beta} = \eta_{\alpha\beta}$ we are still allowed to reparametrize $\sigma^\pm$:

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+), \qquad \sigma^- \to \tilde{\sigma}^-(\sigma^-), \tag{2.50}$$

as this leads to a change of metric $g \to \Omega(\sigma,\tau)g$ which can be undone by a Weyl transformation[9]. To fix this gauge freedom, it is easiest to work in spacetime lightcone coordinates:

$$X^\pm \equiv \tfrac{1}{\sqrt{2}}(X^0 \pm X^{d-1}). \tag{2.51}$$

Although these coordinates do not look Lorentz invariant, as we picked out two particular directions, we will find further on that it still is.

[9]The existence of this remaining gauge freedom does not contradict the earlier discussion where we mentioned that we had to use all of our gauge invariances (1 Weyl, 2 reparametrizations) to fix the metric. This is because $\tilde{\sigma}^\pm$ are functions of only one variable each and form a set of measure zero among the original gauge transformations. This remaining freedom implies that of the $2(d-1)$ degrees of freedom of $X^\mu_{L/R}$ ($2d$ *minus* 2 because of the Virasoro contstraints) $2(d-2)$ are physical.

We fix a gauge by reparametrizing such that, $X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-)$,

$$X_L^+ = \tfrac{1}{2}x^+ + \tfrac{1}{2}\alpha' p^+ \sigma^+, \qquad X_R^+ = \tfrac{1}{2}x^+ + \tfrac{1}{2}\alpha' p^+ \sigma^-, \tag{2.52}$$

$$X^+ = x^+ + \alpha' p^+ \tau. \tag{2.53}$$

This choice of gauge is called the *lightcone gauge*.

Note that the wave equation for X^+ is satisfied, which confirms that we made a valid choice of coordinates X^μ. Further restrictions on X^+ cannot be made as the requirement that X^μ should be periodic in the transformed $\sigma^\pm$ does not leave us with enough reparametrization freedom.

A consequence of this choice of gauge is that X^- is determined, up to an integration constant, by the remaining $d-1$ coordinates. This becomes obvious when one rewrites for example the first Virasoro constraint (2.31) as:

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{d-2} \partial_+ X^i \partial_+ X^i, \tag{2.54}$$

and uses $X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-)$:

$$\begin{aligned} \partial_+ X_L^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{d-2} \partial_+ X^i \partial_+ X^i, \\ \partial_- X_R^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{d-2} \partial_- X^i \partial_- X^i. \end{aligned} \tag{2.55}$$

The resulting mode expanded solutions for $X^-_{L/R}$ are identical to expressions (2.29) with μ replaced by $-$; x^- plays the role of the undetermined integration constant. Because X^- is solved in terms of the other fields, its Fourier coefficients a_n^- can be expressed in terms of the $d-2$ independent modes a_n^i. Using $a_0^- = \tilde{a}_0^- = \sqrt{\alpha'/2}p^-$, we can rewrite the mass relation as:

$$m^2 = 2p^+p^- - \sum_{i=1}^{d-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{d-2} \sum_{n>0} a_{-n}^i a_n^i = \frac{4}{\alpha'} \sum_{i=1}^{d-2} \sum_{n>0} \tilde{a}_{-n}^i \tilde{a}_n^i. \tag{2.56}$$

In summary, we find that in the lightcone picture we end up with $2(d-2)$ independent oscillator modes a_n^i and $\tilde{a}_n^i$, which we will refer to as the *transverse* modes[10]. We were also left with the centre of mass and momentum variables x^i, p^i, p^+ and x^-. The other two, x^+ and p^-, are redundant: the former can be compensated for by a shift in τ, while the second is defined in terms op the p^i and a_n^i through a_0^-. $p-$ is interesting though since it generates shifts in x^+, or equivalently, in τ, and may for this reason be thought of as being proportional to the lightcone Hamiltonian.

In the quantized theory, the commutation relations turn out to be more or less as expected:

$$\begin{aligned} [x^i, p^i] &= i\delta^{ij}, \\ [x^-, p^+] &= [x^+, p^-] = -i, \\ [a_n^i, a_m^j] &= [\tilde{a}_n^i, \tilde{a}_m^j] = n\delta^{ij}\delta_{m+n,0}. \end{aligned} \tag{2.57}$$

[10]Although they are not necessarily physically transverse to $X^\pm$.

The Hilbert space of states is constructed in the same way as it is in covariant quantization. First, the vacuum is defined:

$$\hat{p}\mu|0;p\rangle = p^\mu|0\rangle, \quad a^i_n|0;p\rangle = \tilde{a}^i_n|0;p\rangle = 0, \quad n > 0. \tag{2.58}$$

The excited states are then created by acting on the vacuum with the raising operators a^i_n and $\tilde{a}^i_n$ with $n < 0$. The only constraint left in the quantum theory is the mass relation, which has to be imposed as p^- is not independent. The mass relation will again include a normal ordering constant a. In the following section we give a sketch of how to determine this constant, along with the required number of spacetime dimensions.

2.2.3 Lorentz invariance

Let us first take a look at the action for the free scalar fields X^μ before light-cone gauge fixing. This action is manifestly Poincaré invariant since Poincaré transformations appear as a global symmetry on the world-sheet:

$$X^\mu \to \Lambda^\mu_{\ \nu} X^\nu + c^\mu. \tag{2.59}$$

These transformations give rise to Noether currents and associated conserved charges. For the translations $X^\mu \to X^\mu + c^\mu$, we have the following current:

$$P^\alpha_\mu = T\partial^\alpha X_\mu. \tag{2.60}$$

This current is trivially conserved as $\partial_\alpha P^\alpha_\mu = 0$ is simply the equation of motion. The $d(d-1)/2$ currents associated to Lorentz transformations can be computed similarly. They are,

$$J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu. \tag{2.61}$$

The fact that these are conserved follows again from the equation of motion. The corresponding charges are given by $M_{\mu\nu} = \int d\sigma J^\tau_{\mu\nu}$, and become, after inserting the mode expansion for X^μ into their defining equation,

$$\begin{aligned} M^{\mu\nu} &= (p^\mu x^\nu - p^\nu x^\mu) - i\sum_{n=1}^{\infty}\frac{1}{n}(a^\nu_{-n}a^\mu_n - a^\mu_{-n}a^\nu_n) - i\sum_{n=1}^{\infty}\frac{1}{n}(\tilde{a}^\nu_{-n}\tilde{a}^\mu_n - \tilde{a}^\mu_{-n}\tilde{a}^\nu_n) \\ &\equiv l^{\mu\nu} + S^{\mu\nu} + \tilde{S}^{\mu\nu}, \end{aligned} \tag{2.62}$$

where $l^{\mu\nu}$ is the orbital angular momentum of the string, and $S^{\mu\nu}$ and $\tilde{S}^{\mu\nu}$ describe the angular momentum due to excited oscillator modes. Classically, these obey the Poisson brackets of the Lorentz algebra. In the covariantly quantized theory the corresponding operators obey the commutation relations of the Lorentz Lie algebra:

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}. \tag{2.63}$$

In the lightcone gauge, however, the Lorentz algebra is not generally reproduced by the generators $M^{\mu\nu}$, implying that the theory is not Lorentz invariant. Lorentz invariance requires, among other things,

$$[M^{i-}, M^{j-}] = 0. \tag{2.64}$$

This equality is the problematic one, as it is not generally satisfied due to the presence of p^- and a_n^-, which are defined in terms of the transverse oscillators. Now, if we first change $l^{\mu\nu}$ by replacing $x^\mu p^\nu$ by $\frac{1}{2}(x^\mu p^\nu + p^\nu x^\mu)$ such that it is Hermitian, we may go on and compute the commutator. As the computation is rather lengthy, we skip it and only quote the result:

$$\begin{aligned}[M^{i-}, M^{j-}] =& \frac{2}{(p^+)^2} \sum_{n>0} \left(\left[\frac{d-2}{24} - 1 \right] n + \frac{1}{n} \left[a - \frac{d-2}{24} \right] \right) (a_{-n}^i a_n^j - a_{-n}^j a_n^i) \\ &+ (a \leftrightarrow \tilde{a})\end{aligned} \tag{2.65}$$

The right-hand side of the equation only vanishes for $d = 26$ and $a = 1$, showing that the relativistic string can only be quantized properly in flat Minkowski space if we have 26 space-time dimensions [3], [4].

2.2.4 The string's spectrum

With d and a fixed, we may analyse the (closed) string's spectrum. In this section we will only consider the ground state and the first excited states, as these will turn out to be the most relevant ones. As remarked before, all excitations are bosonic since the mode operators satisfy commutator relations.

The ground state of the string, $|0; p\rangle$, is a *tachyon*. Its mass is given by

$$m^2 = -\frac{4}{\alpha'}. \tag{2.66}$$

The presence of a tachyon indicates that the ground state is unstable[11]. Another, stable ground state might still exist, but it is unknown if there is any. In superstring theory the tachyon disappears after imposing the so-called *GSO-projection.*

The first excited states are of the form

$$a_{-1}^i \tilde{a}_{-1}^j |0; p\rangle. \tag{2.67}$$

These $(d-2)^2 = 24^2$ particles are massless. They are invariant under transformations of the little group of the Lorentz group: $SO(24)$, which is the transverse rotation group. The states can be decomposed into irreducible representations of that group in the following manner

$$\begin{aligned}a_{-1}^i \tilde{a}_{-1}^j |0; p\rangle =& a_{-1}^{[i} \tilde{a}_{-1}^{j]} |0; p\rangle + \left[a_{-1}^{\{i} \tilde{a}_{-1}^{j\}} - \tfrac{1}{24} \delta^{ij} a_{-1}^k \tilde{a}_{-1}^k \right] |0; p\rangle \\ &+ \tfrac{1}{24} \delta^{ij} a_{-1}^k \tilde{a}_{-1}^k |0; p\rangle.\end{aligned} \tag{2.68}$$

This corresponds to a decomposition into an anti-symmetric part, a symmetric and traceless part and a trace. To each of these representations we associate a field in spacetime, such that a string's oscillation is identified with a quantum of these fields. The anti-symmetric part corresponds to an anti-symmetric tensor

[11] In quantum field theory, the mass squared of a field $T(X)$ is determined by the potential part of the Lagrangian, $V(T)$. It is given by $m_T^2 = \left.\frac{\partial^2 V(T)}{\partial T^2}\right|_{T=0}$. If at $T = 0$, which is our reference value (i.e. the one that defines the ground state), $m_T^2 < 0$, T is a tachyon. This happens if the potential V has a local maximum at this value for T. In our case that means that our ground state is unstable. Another example of a tachyon is the Standard Model Higgs boson H at $H = 0$ [3].

field $B_{\mu\nu}$[12]. The trace part is invariant under $SO(24)$ and is therefore a massless scalar. It is called Φ, the dilaton. The symmetric and traceless part corresponds to a ditto tensor field $G_{\mu\nu}$. It is a spin 2 particle and it can be shown that it should be identified to the graviton or, equivalently, to the spacetime metric. This is why string theory includes general relativity.

Up until now, we have more or less ignored the open string. In the open string case we do not have the $\tilde{a}_n^i$ oscillator modes, if we choose Neumann boundary conditions.

The open string's ground state is tachyonic, while its first excited states are given by

$$a_{-1}^i|0;p\rangle, \tag{2.69}$$

which is the massless vector representation of $SO(24)$.

Remark 2.1. In this section we looked at the string spectrum only after we determined d and a. A different argument leading to the same values for d and a as we found, starts out by noting that the open string's massless states should transform under $SO(d-2)$, while its massive states should transform under $SO(d-1)$ for the theory to be Lorentz invariant [3]. This is because these groups are the relevant little groups of the Lorentz group $SO(d-1,1)$. Now, the first excited states of the open string form a $d-2$ dimensional vector of the transverse rotation group $SO(d-2)$ and should thus be massless. As its mass is given by $\alpha' m^2 = (1-a)$, which can be found by acting on the state with the mass operator $L_0 - a$, this implies $a = 1$.

The constant a came about as the result of normal ordering the expression

$$\begin{aligned}\sum_{n\neq 0} a_{-n}^i a_n^i &= \sum_{n\neq 0} :a_{-n}^i a_n^i: + (d-2)\sum_{n=1}^{\infty} n \\ &= 2\left\{\sum_{n=1}^{\infty} a_{-n}^i a_n^i + \frac{d-2}{2}\sum_{n=1}^{\infty} n\right\}.\end{aligned} \tag{2.70}$$

Above, the last sum on the right-hand side is the Riemann zeta function, $\xi(s) = \sum_{n=1}^{\infty} n^{-s}$, for $s = -1$. Although the function is converges only for $s > 1$, it does have a unique analytic continuation at $s = -1$, namely $\xi(-1) = -\frac{1}{12}$. This identification leads to $a = \frac{d-2}{24}$, implying $d = 26$.

In fact, it turns out that only for $d = 26$ and $a = 1$ do all the tachyonic, massless and massive excitations fall neatly into representations of $SO(25)$, $SO(24)$ and $SO(25)$, respectively. Similar arguments apply for the closed string and they lead to the same values for a and d.

Remark 2.2. So far, we have not yet talked about how string-string interactions arise in string theory. As one can see from the Polyakov action, there are no non-linear interaction terms present as there would be in a normal interacting quantum field theory, so string interaction must come about differently. They can in fact be introduced by allowing the *topology* of a worldsheet to change, so that strings may split or fuse as in figure 2.

Another thing that may happen is that open strings close, so that open-string string theories necessarily contain closed strings as well. For most closed string

[12] Or, in other terms, a two-form field with components $B_{\mu\nu}$.

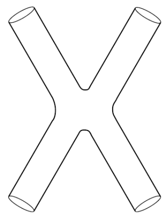

Figure 2: A collision of two strings [4].

theories, such as the type II superstring theories we will focus on hereafter, there exist arguments that show that open strings are required too.

In this thesis we will however not discuss these subjects further; for references on this (and the related subject of conformal field theory) see the lecture notes mentioned in the bibliography.

3 Superstring theory

In this section we extend the Polyakov action of the bosonic string to include also fermionic fields. In other words, we will introduce fermions on the worldsheet. The resulting string spectrum will turn out to also include fermions, which means that the superstring is already one step closer to offering a realistic model for particle physics than the bosonic string is.

The superstring version of string theory is constructed along the same lines as the bosonic theory: first we consider the classical action, the solutions to the equations of motion and their mode expansions and then we apply the canonical and lightcone quantization procedures. Like in the bosonic case, we will focus our attention on a particular kind of string: the *oriented*, closed string with a *supersymmetric* action.

Unorientedness is the symmetry of a string under interchange of left- and right-moving excitations. Oriented string are those that do not have this symmetry. *Supersymmetry* on the other hand is a property of the action: a supersymmetric action is invariant under a certain transformation that transforms fermions into bosons and vice versa. It implies that the number of fermionic and bosonic degrees of freedom are equal. Supersymmetry and the GSO-projection that was mentioned earlier together result in a supersymmetric, tachyonless spectrum. The usefulness of having a supersymmetric spectrum is discussed at the end of this section.

In concluding this section we will, as we did in the bosonic case, write a few words about the theory's low energy limit, which is physically perhaps the most interesting. The most useful references are [1], [2] and [4]. A more comprehensive one would be volume II of [5].

3.1 The relativistic string

3.1.1 Spinors on manifolds

Before we can actually write the superstring action, we need to know how to include fermions on a general, possibly curved, manifold. The mathematics required to do this we discuss below [4], [12]. In this discussion the dimension d of the manifold under consideration will be arbitrary.

Spinors, on a d-dimensional flat Minkowski manifold $\mathbb{R}^{d-1,1}$, have $2^{\left[\frac{d}{2}\right]}$ complex components[13], and transform under the spinor representations of the Lorentz group $SO(d-1,1)$.

Now, at every point p on a general d-dimensional curved Minkowski manifold M one has a tangent space T_pM which corresponds to flat Minkowski space. In this tangent space Lorentz transformations are well defined. To bridge the gap between flat and curved Minkowski space, we define an orthonormal basis each point p for the corresponding tangent space: $e^a_{\ \alpha}(x),\ a = 1,\ldots,d$. Orthonormality means:

$$\langle e^a, e^b \rangle \equiv g^{\alpha\beta} e^a_{\ \alpha} e^b_{\ \beta} = \eta^{ab}, \tag{3.1}$$

or, equivalently $g_{\alpha\beta} = e^a_{\ \alpha} e^b_{\ \beta} \eta_{ab}$. Here, η^{ab} is the Minkowski metric and $e^a_{\ \alpha}$ is a so-called *vielbein*. Its index α is called *curved* as it transforms under general coordinate transformations (diffeomorphisms) as a vector index, while its index a is called *flat* and transforms under local (x-dependent) Lorentz transformations. Its inverse is denoted $e^\alpha_{\ b}$ and obeys $e^a_{\ \alpha} e^\alpha_{\ b} = \delta^a_b$. Due to its two different indices, the vielbein allows one to couple gravity (that is, the curvature of space) to spinors.

The definition of the vielbein above is local, as it is defined on coordinate patches $U \subset M$. In order for spinors to be well defined globally, the vielbein should satisfy the following relation on the intersection of two patches, $U_\alpha \cap U_\beta$,

$$e_{(\alpha)}(x) = \Lambda_{(\alpha\beta)}(x) e_{(\beta)}(x), \tag{3.2}$$

where Λ is a local Lorentz transformation. In this context these are also referred to as the *transition functions*. In a region of triple intersection, $U_\alpha \cap U_\beta \cap U_\gamma$, these functions should satisfy the compatibility condition

$$\Lambda_{(\alpha\beta)} \Lambda_{(\beta\gamma)} \Lambda_{(\gamma\alpha)} = 1. \tag{3.3}$$

A spinor field ψ should then, on an intersection of two patches, transform as

$$\psi_{(\alpha)} = \rho(\Lambda_{(\alpha\beta)}) \psi_{(\beta)}, \tag{3.4}$$

where the $\rho(\Lambda)$ is the spinor representation of the Lorentz transformation. On regions of triple intersection we should have

$$\rho(\Lambda_{(\alpha\beta)}) \rho(\Lambda_{(\beta\gamma)}) \rho(\Lambda_{(\gamma\alpha)}) = \pm 1, \tag{3.5}$$

allowing one to use both $\rho(\Lambda)$ and $-\rho(\Lambda)$. We allow for the sign ambiguity as the spinor representation is double valued[14]. Manifolds for which the above relations

[13] The brackets indicate that only the integer part of the enclosed number is used.

[14] E.g. in the case of ordinary rotations in three dimensions the spinor representation is $SU(2)$ instead of $SO(3)$, for which we have that $SO(3) \cong SU(2)/\mathbb{Z}_2$.

holds, are called *spin manifolds* and are said to allow for *spin structures* (see also the section on G-structures, section 5). It has been proved that any oriented two- or three-dimensional manifold is a spin manifold. The two-dimensional case that is of our interest is of course the superstring's worldsheet.

The next thing to do is to define the Dirac action on M. With respect to the frame $e^a_{\ \alpha}$, the Dirac gamma matrices are $\gamma^a = e^a_{\ \alpha}\gamma^\alpha$. They satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. The object we need is a covariant derivative ∇_a, which should be a local Lorentz vector and transform as a spinor when applied to a spinor:

$$\nabla_a \psi \to \rho(\Lambda)\Lambda_a^{\ b}\nabla_b\psi. \tag{3.6}$$

The Lorentz invariant Lagrangian density will, with such a derivative, be given by

$$\mathcal{L} = \bar{\psi}(i\gamma^a\nabla_a + m)\psi. \tag{3.7}$$

We know that $e_a^{\ \alpha}\partial_\alpha\psi$ transforms as

$$e_a^{\ \alpha}\partial_\alpha\psi \to \Lambda_a^{\ b}e_b^{\ \alpha}\partial_\alpha\rho(\Lambda)\psi = \Lambda_a^{\ b}e_b^{\ \alpha}(\rho(\Lambda)\partial_\alpha\psi + \partial_\alpha\rho(\Lambda)\psi). \tag{3.8}$$

This seems to suggest us to take a derivative of the form

$$\nabla_a\psi = e_a^{\ \alpha}(\partial_\alpha + \Omega_\alpha)\psi. \tag{3.9}$$

The connection Ω then satisfies $\Omega_\alpha \to \rho(\Lambda)\Omega_\alpha\rho(\Lambda)^{-1} - \partial_\alpha\rho(\Lambda)\rho(\Lambda)^{-1}$. To find its explicit form, we consider infinitesimal Lorentz transformations $\Lambda_a^{\ b} = \delta_a^{\ b} + \epsilon_a^{\ b}(x)$. Then $\rho(\Lambda) \equiv \exp(\frac{1}{2}\epsilon^{ab}\gamma_{ab}) \simeq 1 + \frac{1}{2}i\epsilon^{ab}\gamma_{ab}$. (Note: $\gamma_{ab} \equiv \frac{i}{4}[\gamma_a, \gamma_b]$.) The γ_{ab}, being the Lorentz group generators of the spinor representation, satisfy the commutation relations of the Lie algebra $so(d-1,1)$:

$$i[\gamma_{ab}, \gamma_{cd}] = \eta_{cb}\gamma_{ad} - \eta_{ca}\gamma_{bd} + \eta_{db}\gamma_{ac} - \eta_{da}\gamma_{cb}. \tag{3.10}$$

Under the same infinitesimal transformation as before, Ω should transform as

$$\Omega_\alpha \to \Omega_\alpha + \tfrac{1}{2}i\epsilon^{ab}[\gamma_{ab}, \Omega_\alpha] - \tfrac{1}{2}i\partial_\alpha\epsilon^{ab}\gamma_{ab}. \tag{3.11}$$

Now, if we define the connection coefficients for vectors in terms of the vielbein: $\nabla_a\hat{e}_b \equiv \nabla_{\hat{e}_a}\hat{e}_b = \Gamma^c_{ab}\hat{e}_c$, where $\hat{e}_a \equiv e_a^{\ \alpha}e_\alpha$, with e_α the coordinate basis; we find

$$e_a^{\ \alpha}(\partial_\alpha e_b^{\ \beta} + e_b^{\ \lambda}\Gamma^\beta_{\alpha\lambda})e_\beta = \Gamma^c_{ab}e_c^{\ \beta}e_\beta, \tag{3.12}$$

or

$$\Gamma^c_{ab} = e^c_{\ \beta}e_a^{\ \alpha}(\partial_\alpha e_b^{\ \beta} + e_b^{\ \lambda}\Gamma^\beta_{\alpha\lambda}) = e^c_{\ \beta}e_a^{\ \alpha}\nabla_\alpha e_b^{\ \beta}, \tag{3.13}$$

which, in a slightly modified form, transforms under infinitesimal transformations as

$$\Gamma^a_{\alpha b} \to \Gamma^a_{\alpha b} + \epsilon^a_{\ c}\Gamma^c_{\alpha b} - \Gamma^a_{\alpha c}\epsilon^c_{\ b} - \partial_\alpha\epsilon^a_{\ b}. \tag{3.14}$$

Combining these with the γ_{ab} above as follows, we find an Ω with the required transformation properties:

$$\Omega_\alpha \equiv \tfrac{1}{2}i\Gamma^{a\ b}_{\ \alpha}\gamma_{ab} = \tfrac{1}{2}ie^a_{\ \beta}\nabla_\alpha e^{b\beta}\gamma_{ab}. \tag{3.15}$$

With it, the Dirac Lagrangian becomes

$$\mathcal{L} = \bar{\psi}(i\gamma^a e_a^{\ \alpha}(\partial_\alpha + \tfrac{1}{2}i\Gamma^{b\ c}_{\ \alpha}\gamma_{bc}) + m)\psi, \tag{3.16}$$

while the action becomes

$$S = \int_M d^dx\sqrt{-g}\,\mathcal{L}. \tag{3.17}$$

By adding certain total derivatives to this action, we obtain a Hermitian version:

$$S = \tfrac{1}{2}\int_M d^dx\sqrt{-g}\,\bar{\psi}(i\gamma^\alpha\overleftrightarrow{\partial}_\alpha + \tfrac{1}{2}i\Gamma^{b\ c}_{\ \alpha}\{i\gamma^\alpha, \gamma_{bc}\} + m)\psi. \tag{3.18}$$

In $d = 2$ dimensions, the connection term vanishes (in both forms of the action), which can be seen from the equation directly above. The only non-zero γ_{ab} are $\gamma_{01} \propto \gamma_3$ (the two-dimensional analogue of γ_5) and $\{\gamma^\alpha, \gamma_3\} = 0$, so that the connection term drops out.

Here, for later use, we state some further facts about gamma-matrices and spinors. The algebra for two-dimensional Dirac matrices $\rho^a,\ a = 0, 1$ is given by

$$\{\rho^a, \rho^b\} = 2\eta^{ab}, \tag{3.19}$$

where $\eta^{ab} = \mathrm{diag}(-1, +1)$. A particular set of matrices satisfying the algebra is:

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.20}$$

The charge conjugation matrix can be taken to be $C = \rho^0$. Majorana spinors satisfy $\bar{\psi} \equiv \psi^\dagger\rho^0 = \psi^T C = \psi^T\rho^0$. Thus $\psi^* = \psi$; the spinor has real components. Now, using the zweibein (the vielbein in two dimensions) we can define curved ρ-matrices as follows

$$\rho^\alpha = e^{\alpha}_{\ a}\rho^a. \tag{3.21}$$

These satisfy a modified algebra: $\{\rho^\alpha, \rho^\beta\} = 2g^{\alpha\beta}$. The next two identities, the first of which is valid for anti-commuting Majorana spinors and the second for general anti-commuting spinors, will be useful in showing the string action's invariance under supersymmetry.

- The *spin-flip* identity:

$$\bar{\psi}_1\rho^{\alpha_1}\ldots\rho^{\alpha_n}\psi_2 = (-1)^n\bar{\psi}_2\rho^{\alpha_n}\ldots\rho^{\alpha_1}\psi_1. \tag{3.22}$$

- The *Fierz* identity:

$$(\bar{\psi}\lambda)(\bar{\phi}\chi) = -\tfrac{1}{2}\left\{(\bar{\psi}\chi)(\bar{\phi}\lambda) + (\bar{\psi}\bar{\rho}\chi)(\bar{\phi}\bar{\rho}\lambda) + (\bar{\psi}\rho^\alpha\chi)(\bar{\phi}\rho_\alpha\lambda)\right\}, \tag{3.23}$$

  where $\bar{\rho} \equiv \rho^0\rho^1$.

Remark 3.1. The d-dimensional Clifford algebra is reducible, as there is a matrix that commutes with all generators:

$$\gamma_{d+1} \equiv e^{-i\pi(d-2)/4}\gamma^0 \dots \gamma^{d-1}, \tag{3.24}$$

for d even, while for d odd $\gamma_{d+1} \propto 1$ ($\gamma_{d+1}^2 = 1$, also $[\gamma_{d+1}, \gamma_{ab}] = 0$). This matrix is used to define the inequivalent and irreducible *Weyl representations*: given a Dirac spinor ψ, the left- and right-handed Weyl spinors are

$$\psi_L = \tfrac{1}{2}(1-\gamma_{d+1})\psi, \qquad \psi_R = \tfrac{1}{2}(1+\gamma_{d+1})\psi. \tag{3.25}$$

Weyl and Dirac spinors are complex, but in some cases the *Majorana condition*, which has been referred to before, can be imposed: $\psi^* = B\psi$, with a matrix such that $BB^* = 1$ is consistent with the Lorentz transformations $\delta\psi = \frac{1}{2}i\epsilon^{ab}\gamma_{ab}\psi$, i.e. $\gamma_{ab}^* = -B\gamma_{ab}B^{-1}$. The Majorana condition is allowed only in $d = 0, 1, 2, 3, 4 \mod 8$ and Majorana-Weyl spinors can be shown to only exist in $d = 2 \mod 8$.

For a useful reference on spinors, see [5] or [15].

3.1.2 The superstring action

As mentioned before, the superstring action is an extension of the Polyakov action. It is based on two *multiplets* that are supersymmetric in two dimensions. The first multiplet consists of d matter multiplets, labelled by μ:

$$(X^\mu, \psi^\mu, F^\mu)\,. \tag{3.26}$$

It consists of the d bosonic fields X^μ of the Polyakov string, in addition to d two-dimensional Majorana spinors ψ^μ, and d real scalar fields F^μ, guaranteeing the equality of the fermionic and bosonic degrees of freedom off-shell. On-shell, however, after the equations of motion for F^μ have been used to simplify the action, X^μ and ψ^μ suffice.

The second multiplet is called the supergravity multiplet:

$$(e^a_{\ \alpha}, \chi_\alpha, A)\,. \tag{3.27}$$

The second field, χ_α, is a Majorana "spinor-vector", called the *gravitino*; it has $2^{[\frac{d}{2}]}d$ independent components. The other two are A, a real scalar field, and $e^a_{\ \alpha}$, the vielbein. In two dimensions the vielbein and the scalar each have one degree of freedom, while the gravitino has four. If, however, we have one supersymmetry parameter (corresponding to one specific supersymmetry transformation), that is a Majorana spinor, it would have $2^{[\frac{d}{2}]}$ independent components and reduce the number of fermionic degrees of freedom by that number. The number of fermionic degrees of freedom then matches the number of bosonic degrees of freedom again off-shell.

The on-shell action takes on the following form:

$$\begin{aligned} S = -\frac{1}{8\pi}\int d^2\sigma\, e\, \big(& g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \\ & -i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \left(\partial_\beta X_\mu - \tfrac{i}{4}\bar{\chi}_\beta \psi_\mu\right)\big)\,. \end{aligned} \tag{3.28}$$

Here, e is defined as $e \equiv |\det e^a_{\ \alpha}| = \sqrt{-g}$. Note that the auxiliary fields A and F^μ do not appear. That the kinetic term for the gravitino, which is given by

$\bar{\chi}_\alpha \gamma^{\alpha\beta\gamma} D_\beta \chi_\gamma$, does not appear either is due to the fact that we only have two ρ matrices: the completely anti-symmetrized product of three Dirac matrices $\gamma^{\alpha\beta\gamma}$ requires at least three to exist for it to be non-zero.

3.1.3 Symmetries of the action

The superstring action is invariant under five local symmetries. The infinitessimal versions are listed below (σ and τ are again the coordinates on the world-sheet; $\sigma \in [0, 2\pi)$).

- Local *supersymmetry*; $\epsilon(\sigma, \tau)$ is a Majorana spinor parametrizing the supersymmetry transformations.

$$\begin{aligned} \delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu, & \delta_\epsilon \psi^\mu &= \tfrac{1}{2}\rho^\alpha(\partial_\alpha X^\mu - \tfrac{i}{2}\bar{\chi}_\alpha \psi^\mu)\epsilon, \\ \delta_\epsilon e^a_{\ \alpha} &= \tfrac{i}{2}\bar{\epsilon}\rho^a \chi_\alpha, & \delta_\epsilon \chi_\alpha &= 2D_\alpha \epsilon, \end{aligned} \tag{3.29}$$

  where $D_\alpha \epsilon = \partial_\alpha \epsilon - \frac{1}{2}\omega_\alpha \bar{\rho}\epsilon$ and ω_α is the connection with torsion $\omega_\alpha = \omega_\alpha(e) + \frac{i}{4}\bar{\chi}_\alpha \bar{\rho}\rho^\beta \chi_\beta$; $\omega_\alpha(e)$ is the spin connection: $\omega_\alpha(e) = -\frac{1}{e} e_{\alpha a}\epsilon^{\beta\gamma}\nabla_\beta e^a_{\ \gamma}$, as mentioned in section 3.1.1.

- *Weyl* invariance. The parameter in this case is the function $\Lambda(\sigma, \tau)$.

$$\begin{aligned} \delta_\Lambda X^\mu &= 0, & \delta_\Lambda \psi^\mu &= -\tfrac{1}{2}\Lambda \psi^\mu, \\ \delta_\Lambda e^a_{\ \alpha} &= \Lambda e^a_{\ \alpha}, & \delta_\Lambda \chi_\alpha &= \tfrac{1}{2}\Lambda \chi_\alpha. \end{aligned} \tag{3.30}$$

- *Super Weyl* invariance. The parameter, η, is a Majorana spinor. Only the gravitino is affected by the super Weyl transformations; invariance follows from the identity $\rho^\alpha \rho_\beta \rho_\alpha = 0$.

$$\delta_\eta \chi_\alpha = \rho_\alpha \eta. \tag{3.31}$$

- Two-dimensional *Lorentz* symmetry. The parameter $l(\sigma, \tau)$ is a function.

$$\begin{aligned} \delta_l X^\mu &= 0, & \delta_l \psi^\mu &= \tfrac{1}{2} l \bar{\rho}\psi^\mu, \\ \delta_l e^a_{\ \alpha} &= l \epsilon^a_{\ b} e^b_{\ \alpha}, & \delta_l \chi_\alpha &= \tfrac{1}{2} l \bar{\rho}\chi_\alpha. \end{aligned}$$

- *Reparametrizations* (or diffeomorphisms); $\xi^\alpha(\sigma, \tau)$ is a vector parameter.

$$\begin{aligned} \delta_\xi X^\mu &= \xi^\beta \partial_\beta X^\mu, & \delta_\xi e^a_{\ \alpha} &= \xi^\beta \partial_\beta e^a_{\ \alpha} + e^a_{\ \beta}\partial_\alpha \xi^\beta, \\ \delta_\xi \psi^\mu &= \xi^\beta \partial_\beta \psi^\mu, & \delta_\xi \chi_\alpha &= \xi^\beta \partial_\beta \chi_\alpha + \chi_\beta \partial_\alpha \xi^\beta. \end{aligned} \tag{3.32}$$

As in the bosonic case, we simplify the action by using its symmetries. First we will use local supersymmetry, Lorentz invariance and reparametrizations to fix two degrees of freedom of the zweibein and of the gravitino.

The gravitino may be decomposed as

$$\begin{aligned} \chi_\alpha &= (g^\beta_\alpha - \tfrac{1}{2}\rho_\alpha \rho^\beta)\chi_\beta + \tfrac{1}{2}\rho_\alpha \rho^\beta \chi_\alpha \\ &= \tfrac{1}{2}\rho^\beta \rho_\alpha \chi_\beta + \tfrac{1}{2}\rho_\alpha \rho^\beta \chi_\alpha \\ &\equiv \tilde{\chi}_\alpha + \rho_\alpha \lambda. \end{aligned} \tag{3.33}$$

The supersymmetry transformation for the gravitino may be decomposed similarly

$$\begin{aligned}\delta_\epsilon \chi_\alpha &= 2D_\alpha \epsilon \\ &\equiv 2(\Pi\epsilon)_\alpha + \rho_\alpha \rho^\beta D_\beta \epsilon,\end{aligned} \tag{3.34}$$

where we defined $(\Pi\epsilon)_\alpha = (g_\alpha^{\ \beta} - \frac{1}{2}\rho_\alpha\rho^\beta)D_\beta\epsilon = \frac{1}{2}\rho^\beta\rho_\alpha D_\beta\epsilon$. Now, one can show that locally there always exists a spinor κ, such that $\tilde{\chi}_\alpha = \rho^\beta\rho_\alpha D_\beta\kappa$. We then find that, looking at the supersymmetry variation for the gravitino, κ can always be made to vanish using a supersymmetry transformation.

To simplify things further we use reparametrizations and local Lorentz transformations. The zweibein can be transformed into $e_\alpha^a = e^\phi \delta_\alpha^a$ (this is just the result of a transformation that makes $g_{\alpha\beta}$ conformally flat). Under these tranformations, the decomposition of the gravitino ($\chi_\alpha = \rho_\alpha\lambda$) does not change. The gauge choice we have made is called the *superconformal* gauge, in analogy to the conformal gauge in the bosonic theory. Using a Weyl transformation and a super-Weyl transformation it may be further simplified to

$$\begin{aligned}e_\alpha^a &= \delta_\alpha^a, \\ \chi_\alpha &= 0.\end{aligned} \tag{3.35}$$

An important issue that we did not consider in the bosonic case, and that we will but mention here, is that the (super-)conformal gauge was derived locally. For it to hold globally, there need to be a globally defined spinor κ and a globally defined vector field ξ^α, both of which are non-trivial conditions on the worldsheet.

Finally, the action in superconformal gauge is given by

$$S = -\frac{1}{8\pi}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right), \tag{3.36}$$

where we rescaled the fermions by $e^{\phi/2}\psi \to \psi$. The spin connection is now $\omega_\alpha = \epsilon_\alpha^{\ \beta}\partial_\beta\phi$ and indices are raised and lowered using the flat metric $\eta_{\alpha\beta}$ and $\rho^\alpha = \delta_a^\alpha \rho^a$.

As in the bosonic case, there is still some gauge invariance left. Reparametrizations and supersymmetry transformations satisfying $P(\xi^\alpha) = 0$ and $\Pi\epsilon = 0$ respectively[15], do not change the action; though for the zweibein we need an accompanying Lorentz transformation with parameter $l = \frac{i}{2}\bar{\epsilon}\bar{\rho}\lambda$. The Weyl degree of freedom ϕ then changes by $\delta\phi = \frac{i}{2}\bar{\epsilon}\lambda$. The supersymmetry transformations can be made to be of the form

$$\begin{aligned}\delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu, \\ \delta_\epsilon \psi^\mu &= \frac{1}{2}\rho^\alpha \partial_\alpha X^\mu \epsilon,\end{aligned} \tag{3.37}$$

if we redefine $e^{\phi/2}\psi$ and $e^{-\phi/2}\epsilon$ as ψ and ϵ, respectively. The condition $\Pi\epsilon = 0$ then becomes $\rho^\beta\rho_\alpha\partial_\beta\epsilon = 0$.

[15] $P(\xi)$ is defined as follows: $(P\xi)_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha - (\nabla_\gamma\xi^\gamma)g_{\alpha\beta}$. It gives the traceless part of the metric variation under a reparametrization.

The equations of motion derived from the simplified action are:

$$
\begin{aligned}
\partial_\alpha \partial^\alpha X^\mu &= 0, \\
\rho^\alpha \partial_\alpha \psi^\mu &= 0.
\end{aligned}
\tag{3.38}
$$

These still need to be accompanied by boundary conditions. For the X^μ they are the same as in the bosonic theory. For the fermions the boundary conditions will be given somewhat further on. We also have two sets of constraints on the system, that come from the equations of motion for the zweibein and the gravitino:

- The stress tensor, which, for theories with fermions, is defined as:

$$
T_{\alpha\beta} \equiv -\frac{2\pi}{e} \frac{\delta S}{\delta e_a^\beta} e_{\alpha a} = 0,
\tag{3.39}
$$

- and an object related to the stress tensor by supersymmetry, which we will denote as T_F. It is the response of the action to variations in the gravitino:

$$
T_{F\alpha} \equiv \frac{2\pi}{e} \frac{\delta S}{i\delta \bar{\chi}^\alpha} = 0.
\tag{3.40}
$$

Both of these are conserved currents, as $\partial^\alpha T_{\alpha\beta} = \partial^\alpha T_{F\alpha} = 0$.

For the rest of the discussion we will again use lightcone coordinates $\sigma^\pm$. The action becomes

$$
S = \frac{1}{2\pi} \int d^2\sigma \left(\partial_+ X \cdot \partial_- X + i(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-) \right),
\tag{3.41}
$$

where we used the notation $\psi^\mu = (\psi_+^\mu, \psi_-^\mu)$ for the components of the spinors. The equations of motion are given by

$$
\begin{aligned}
\partial_+ \partial_- X^\mu &= 0, \\
\partial_- \psi_+^\mu = \partial_+ \psi_-^\mu &= 0.
\end{aligned}
\tag{3.42}
$$

We see that the X^μ can again be split up into a left-moving and a right-moving part, while for the fermions we have $\psi_\pm = \psi_\pm(\sigma^\pm)$. Using the same notation for the supersymmetry parameter, the conditions on the allowed reparametrizations and supersymmetry transformations become

$$
\begin{aligned}
\partial_+ \xi^- = \partial_- \xi^+ &= 0, \\
\partial_+ \epsilon^- = \partial_- \epsilon^+ &= 0.
\end{aligned}
\tag{3.43}
$$

Note that the indices $\pm$ for the vector ξ denote the vector components in lightcone coordinates. To determine the aforementioned boundary conditions for the fermions we simply require that the boundary term arising from the variation of the action with respect to the fermions vanishes. Thus we impose

$$
\int d\sigma \partial_\sigma \left(\psi_+ \delta \psi_+ - \psi_- \delta \psi_- \right) = 0,
\tag{3.44}
$$

or equivalently (for closed strings only)

$$\psi_+(\sigma) = \pm\psi_+(\sigma + 2\pi),$$
$$\psi_-(\sigma) = \pm\psi_-(\sigma + 2\pi). \tag{3.45}$$

As the conditions for the two spinor components can be chosen independently, they lead to four different combinations (R,R), (R,NS), (NS,R) and (NS,NS). The R is used to denote the periodic boundary condition, which are usually referred to as *Ramond* boundary conditions. Similarly the NS stands for *Neveu-Schwarz* or anti-periodic boundary conditions.

In lightcone coordinates the stress tensor $T_{\alpha\beta}$ and supercurrent $T_{F\alpha}$ become

$$\begin{aligned} T_{++} &= \tfrac{1}{2}\partial_+ X \cdot \partial_+ X + \tfrac{i}{2}\psi_+ \cdot \partial_+ \psi_+, & T_{F++} &\equiv T_{F+} = \tfrac{1}{2}\psi_+ \cdot \partial_+ X, \\ T_{--} &= \frac{1}{2}\partial_- X \cdot \partial_- X + \tfrac{i}{2}\psi_- \cdot \partial_- \psi_-, & T_{F--} &\equiv T_{F-} = \tfrac{1}{2}\psi_- \cdot \partial_- X, \\ T_{+-} &= T_{-+} = 0, & T_{F+-} &= T_{F-+} = 0. \end{aligned} \tag{3.46}$$

Note that for T_F the first index is a spinor index, while the second is a vector index.

3.1.4 Mode expansions

As in bosonic string theory we consider the mode expansions for the fields. These will again yield the constraints and creation and annihilation operators of the quantum theory. For the bosonic X^μ we have expansions identical to those of section 2.1.5. The general solution for the equations of motion for the fermions is given by

$$\psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+\phi} \bar{b}_r^\mu e^{-ir(\tau+\sigma)},$$
$$\psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+\phi} b_r^\mu e^{-ir(\tau-\sigma)}, \tag{3.47}$$

where $\phi = 0$ when Ramond conditions apply and $\phi = \frac{1}{2}$ when Neveu-Schwarz conditions apply. Reality of the Majorana spinors implies

$$b_r^\mu = (b_{-r}^\mu)^*, \qquad \tilde{b}_r^\mu = (\tilde{b}_{-r}^\mu)^*. \tag{3.48}$$

The Dirac part of the simplified superstring action implies the following brackets for the fermions

$$\{\psi_\pm^\mu(\sigma,\tau), \psi_\pm^\nu(\sigma',\tau)\} = -2\pi i\delta(\sigma-\sigma')\eta^{\mu\nu}. \tag{3.49}$$

These are, however, not regular Poisson brackets, but Dirac brackets as the Poisson brackets would vanish trivially. Some more words on this issue may be found in [1]. In terms of the fermionic oscillator modes we have

$$\{b_r^\mu, b_s^\nu\} = -i\eta^{\mu\nu}\delta_{r+s,0},$$
$$\{\bar{b}_r^\mu, \bar{b}_s^\nu\} = -i\eta^{\mu\nu}\delta_{r+s,0},$$
$$\{b_r, \bar{b}_s\} = 0. \tag{3.50}$$

The expansions of the constraints will again be of importance. For the right-moving sector the modes are defined by:

$$L_m = \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{-im\sigma} T_{--},$$
$$G_r = \frac{1}{\pi}\int_0^{2\pi} d\sigma e^{-ir\sigma} T_{F-}, \tag{3.51}$$

and are given by

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_{-n}\cdot a_{m+n} + \frac{1}{2}\sum_r \left(r + \frac{m}{2}\right) b_{-r}\cdot b_{m+r},$$
$$G_r = \sum_r a_{-n}\cdot b_{r+n}. \tag{3.52}$$

To find the analogous expressions for the right-moving sector, simply but bars over every b and a. The modes satisfy the classical *super-Virasoro* algebra

$$\{L_m, L_n\} = -i(m-n)L_{m+n},$$
$$\{L_m, G_r\} = -i\left(\frac{m}{2} - n\right)G_{m+r},$$
$$\{G_r, G_s\} = -2iL_{r+s}. \tag{3.53}$$

3.2 The quantized relativistic string

In what follows, we only briefly consider canonical and lightcone quantization of the superstring, as the story to be told is very similar to the bosonic version.

3.2.1 Canonical quantization

For the fields X^μ we replace the Poisson brackets by commutators, as is done in section 2.2.1. For the fermions we make the following replacement: $\{\ ,\ \} \to -i\{\ ,\ \}$, where the second bracket denotes the anti-commutator. The anti-commutators for the fermionic modes become

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}. \tag{3.54}$$

The oscillator mode coefficients with positive mode number become annihilation operators, while those with negative mode number become creation operators.

The super-Virasoro operators still need to be normal ordered, which again affects only L_0. As before, we allow for this by including a normal ordering constant a in all expressions containing L_0. The super-Virasoro algebra is then given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{8}m(m^2 - 2a)\delta_{m+n,0},$$
$$[L_m, G_r] = \left(\frac{m}{2} - n\right)G_{m+r},$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{d}{2}\left(r^2 - \frac{a}{2}\right)\delta_{r+s,0}. \tag{3.55}$$

The super-Virasoro constraints are to be imposed on physical states in a way similar to the Virasoro constraints in the bosonic case.

The ground state for the quantum theory is defined by, ignoring for the moment the dependence on the string's centre of mass momentum,

$$a_m^\mu|0\rangle = b_r^\mu|0\rangle = 0, \qquad m,r > 0. \tag{3.56}$$

The other states are constructed by acting on the ground state with the creation operators.

The mass operator of the theory is given by

$$m^2 = \frac{2}{\alpha'}\left(\sum_{m=1}^{\infty} a_{-m}\cdot a_m + \sum_{r>0}^{\infty} r b_{-r}\cdot b_r - a\right). \tag{3.57}$$

It implies that, in a Ramond sector, $[m^2, b_0^\mu] = 0$. That means that the states $|0\rangle$ and $b_0^\mu|0\rangle$ are degenerate in mass. For $n, r < 0$ we have that a_n^μ and b_r^μ increase $\alpha' m^2$ by $2n$ and $2r$ respectively, implying that in the NS sector we do have a unique ground state, which must therefore have spin zero[16].

The Ramond sector ground state is degenerate. Since the b_0^μ satisfy the Dirac algebra $\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$, we see that the Ramond $|0\rangle$ is acted upon by what are effectively gamma matrices. It must thus be a spinor of $SO(d-1,1)$. Furthermore, we can conclude that all states in the NS sector are spacetime bosons, while all states in the R sector are spacetime fermions—this is because the creation operators cannot change fermions into bosons or vice versa, as they carry a spacetime vector index μ. We denote the R ground state by $|a\rangle$, where a is the spinor index. The b_0^μ act on this state as

$$b_0^\mu|a\rangle = \tfrac{1}{\sqrt{2}}(\Gamma^\mu)^a_{\ b}|b\rangle, \tag{3.58}$$

where the Γ's are d dimensional Dirac matrices.

As in the bosonic case, the definition of states given above leads to the existence of negative norm states. In bosonic string theory we had a theorem stating that these ghosts decouple from the physical spectrum in 26 dimensions when $a = 1$. In superstring theory a similar theorem exists. It states that the decoupling will take place if we have 10 spacetime dimensions, and if $a = 0$ for the R sector and $a = \frac{1}{2}$ for the NS sector [4].

3.2.2 Lightcone quantization

In section , the remaining reparametrization invariance of the bosonic theory was fixed by choosing

$$X^+ = \alpha' p^+ \tau, \tag{3.59}$$

this choice is also possible in superstring theory. The residual supersymmetry can be used to transform the analogue of X^+, ψ^+ away, so that is in lightcone quantization we have

$$\psi^+ = 0, \tag{3.60}$$

[16] A non-zero spin of the ground state would imply the existence of a, say, "preferred" direction; it would break the ground state's rotational invariance.

where $\psi^\pm$ is defined as $\psi^\pm = \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^{d-1})$.

If we rewrite the Virasoro constraints using the newly defined lightcone components, we find not only that X^- can be defined (up to an integration constant) in terms of transverse bosonic and fermionic components, but also that ψ^- can be defined in such a way. The only physical components are thus the transverse ones.

The mass operator becomes

$$m^2 = m_L^2 + m_R^2, \tag{3.61}$$

where m_R^2 is the mass due to right-moving oscillator modes; it is given by

$$m_R^2 = \frac{2}{\alpha'}\left(\sum_{n>0} a_{-n}^i a_n^i + \sum_{r>0} r b_{-r}^i b_r^i - a\right). \tag{3.62}$$

A similar expression applies for m_L^2, the mass due to left-moving modes. For physical states, we require $m_R^2 = m_L^2$. This level-matching condition derives from the condition $(L_0 - \bar{L}_0)|\text{phys}\rangle = 0$, which expresses the fact that no point on a closed string is distinct[17].

3.2.3 The string spectrum

We can now proceed to analyse the superstring's mass spectrum. We will again focus mostly on the massless states. First we consider the discuss the spectrum of the NS and R sectors separately and last the spectra of the different possible combinations, (R,R), (R,NS), (NS,R) and (NS,NS).

The NS-sector. The ground state is the oscillator vacuum $|0\rangle$, it has mass $m^2 = -a/\alpha'$ and is tachyonic. The first excited state is $b_{-1/2}^i|0\rangle$, which is a vector of $SO(d-2)$. By the same argument as in the bosonic case, we are lead to the conclusion that the latter state must be massless. For the NS-sector we then have $a_{NS} = \frac{1}{2}$. As the formal expression for this normal ordering constant is given by

$$a = -\frac{d-2}{2}\left(\sum_{n=0}^{\infty} n - \sum_{r=1/2}^{\infty} r\right), \tag{3.63}$$

we find, using again ζ-function regularization, that $\sum_{n\geq 0}(n+a) = \zeta(-1,a) = -\frac{1}{12}(6a^2 - 6a + 1)$ and that $a = \frac{d-2}{2}(\frac{1}{12} + \frac{1}{24}) = \frac{d-2}{16}$. Or, finally, that $d = 10$. In the superstring case it can be shown that higher, massive, excited states fall into representations of $SO(9)$, as required.

The R-sector. We already found that the R ground state is a spinor of $SO(9,1)$. A Dirac spinor in ten spacetime dimensions would have 2^5 independent complex or 64 independent real components. On-shell this number is reduced to 32 as the Dirac equation $\Gamma^\mu \partial_\mu \psi = 0$ relates half of the components to the other half. Furthermore, in ten dimensions we may impose both a Majorana and a Weyl condition, each of which divides in half the number of independent

[17] This interpretation stems from the fact that $L_0 - \bar{L}_0$ generates translations in σ.

components, leaving only 8. These may then be interpreted as components of a massless Majorana-Weyl spinor of $SO(8)$. We can choose the ground state to be either of the two possible chiral Weyl spinors, which we denote by $|a\rangle$ and $|\dot{a}\rangle$. The first excited states are $a^i_{-1}|a\rangle$ and $b^i_{-1}|a\rangle$, along with their chiral partners. They have mass $m^2 = 1/\alpha'$ and can be assembled uniquely into representations of $SO(9)$.

Though we now have a string theory with both fermions and bosons, which is what we were after, it turns out that in this form it is inconsistent. And, on top of that, we still have not managed to get rid of the tachyon. These somewhat troubling problems may, luckily, be solved by considering only a specific part of the string spectrum. The selection procedure that we use is called the *GSO projection*, after Gliozzi, Scherk and Olive. Apart from solving the inconsistencies, it succeeds in removing the tachyon, and in providing spacetime supersymmetry in the resulting spectrum.

Looking at the massless part of the NS sector we see that it consists of a vector of $SO(8)$, and that the massless part of the R sector consists of two spinors of $SO(8)$. If we project out one of the two spinors, we will end up with a supersymmetric theory as the NS and R degrees of freedom will match in number. Mathematically, we define the projection as follows. First we define an operator G, which, for the NS sector, is given by

$$G = (-1)^F, \qquad F = \sum_{r=1/2}^{\infty} b^i_{-r} b^i_r - 1, \tag{3.64}$$

and we require that all states satisfy $G|\Phi\rangle = 1|\Phi\rangle$. In particular, this ensures that the tachyon is projected out. In the R sector G is defined as follows

$$G = (-1)^F = b_0^1 \cdots b_0^8 (-1)^{\sum_{n=1}^{\infty} b^i_{-n} b^i_n}, \tag{3.65}$$

which is a generalized chirality operator (the γ^5 in $d = 4$), as $b_0^1 \cdots b_0^8$ is the chirality operator in the transverse dimensions. The sum is the worldsheet fermion number. For the ground states we have that the eigenvalue of G that applies, depends solely on their chirality, and we define $G|a\rangle = 1$ and $G|\dot{a}\rangle = -1$. Its eigenvalues for a general state are then again ± 1, since $\{G, \psi^\mu\} = 0$. For the R sector we have two possible options, as we may demand that all states satisfy either $G = 1$ or $G = -1$.

The closed superstring has, as mentioned earlier, a left-moving and a right-moving sector, each of which may be either an NS or an R sector. This leads to the four combinations named in the first few remarks of this section. The superstring states are then tensor products of left and right-moving states (such that $m_L^2 = m_R^2$). The GSO projection is carried out for the left and right-movers separately, leading to two different supersymmetric theories. It turns out that the (R,R) and (NS,NS) sector combinations lead to spacetime bosons, while the (R,NS) and (NS,R) combinations lead to spacetime fermions.

First we consider what is called *type IIB string theory*. It is the result of demanding that $G = \bar{G} = 1$ in both the R and NS sectors (the bar on G indicates that it operates on the left-moving sector). The allowed massless states are then

$$
\begin{aligned}
\bar{b}^i_{-1/2}|0\rangle_L \times b^j_{-1/2}|0\rangle_R, &\qquad \text{in the (NS,NS) part,}\\
|\dot{a}\rangle_L \times |b\rangle_R, &\qquad \text{in the (R,R) part,}\\
|a\rangle_L \times b^i_{-1/2}|0\rangle_R, &\qquad \text{in the (R,NS) part and}\\
\bar{b}^i_{-1/2}|0\rangle_L \times |a\rangle_R, &\qquad \text{in the (NS,R) part.}
\end{aligned} \tag{3.66}
$$

These may be decomposed into irreducible representations of the little group. For the bosons this results in a graviton, two antisymmetric tensor fields of rank two (or two-forms), one antisymmetric tensor of rank four (which is in fact four-form with self-dual field strength, see section 3.3) and two real scalars. For fermions the decomposition results in two gravitinos of spin $\frac{3}{2}$ and two spin $\frac{1}{2}$ fermions. As both gravitinos have the same chirality, the theory is chiral. Furthermore, the presence of two gravitinos implies that the theory has $\mathcal{N} = 2$ supersymmetry, i.e. that there are two different supersymmetry transformations.

The second type of theory we consider is called *type IIA string theory.* It puts the same requirement on the NS sectors, but demands for the R sectors $G = -\bar{G} = 1$. The allowed massless states are now

$$
\begin{aligned}
\bar{b}^i_{-1/2}|0\rangle_L \times b^j_{-1/2}|0\rangle_R, &\qquad \text{in the (NS,NS) part,}\\
|\dot{a}\rangle_L \times |\dot{b}\rangle_R, &\qquad \text{in the (R,R) part,}\\
|\dot{a}\rangle_L \times b^i_{-1/2}|0\rangle_R, &\qquad \text{in the (R,NS) part and}\\
\bar{b}^i_{-1/2}|0\rangle_L \times |\bar{a}\rangle_R, &\qquad \text{in the (NS,R) part.}
\end{aligned} \tag{3.67}
$$

The decomposition into irreducible representations results in, for the bosonic part, a graviton, one real scalar (called the dilaton), one antisymmetric rank two tensor, one vector (or one-form) and one antisymmetric rank three tensor (or three-form). The fermionic part consists of two spin $\frac{3}{2}$ gravitinos and two spin $\frac{1}{2}$ fermions called dilatinos. In this case we have, however, that each pair of fermions contains one of each kind of chirality, leading to a non-chiral, $\mathcal{N} = 2$ supersymmetric theory.

3.3 The type IIA and IIB supergravities

In the previous section we wrote about the massless part of the type IIA and type IIB theories' $\mathcal{N} = 2$ supersymmetric spectra. We know however that these theories contain an infinite number of bosonic and fermionic particles, with masses $m^2 = n/\alpha'$, with $n \in \mathbb{N}$. We would like to somehow be able to ignore them when we consider for example collisions between particles that involve energies up to several TeV. This is because the Standard Model, which we know works quite well at those energy scales, does not contain an infinite number of particles.

The reason why these particles effectively disappear at 'low' energies is in fact due to the fact that they are massive. It turns out that if we choose to identify the symmetric tensor field $G_{\mu\nu}$ (henceforth referred to by a lower case g) with the graviton, we need to relate the constant α' to the Planck scale:

$\alpha' \sim G_N = M_P^{-p}$, where p is some positive integer related to the number of dimensions, G_N Newton's constant in ten dimensions and M_P the Planck mass that is about $10^{19} GeV/c^2$ [3]. This implies that all of string theory's massive particles are much too heavy to be produced in any modern-day experiment. This also explains our focus on the massless particles of the different string spectra we wrote about; they are physically interesting if one tries to make string theory tie up with our current knowledge of 'low'-energy physics.

In the following we will describe the actions (which are highly constrained by supersymmetry conditions [5]) that determine the behaviour of the massless particles. They are the so-called *supergravity actions*. They naturally fall apart in three pieces [3]:

$$S_{\text{supergravity}} = S_1 + S_2 + S_{\text{fermi}}. \tag{3.68}$$

The first two parts contain only bosons, while the third contains all the interactions and kinetic terms of the fermionic fields. The latter we will not describe here. S_{fermi} may be obtained by performing a supersymmetry variation of the bosonic fields (using the fact that both type II supergravities should be $\mathcal{N} = 2$ supersymmetric[18]) and adding in the appropriate kinetic terms. The supersymmetry parameters are sixteen-dimensional[19] Majorana-Weyl spinors; they are of the same chirality in type IIB and opposite chirality in type IIA [9]. The supersymmetry transformations are described at the end of this section.

The S_1 part is common to both IIA and IIB and contains the fields coming from their NS-NS sectors. It is given by

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g}\, e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} H^2 \right). \tag{3.69}$$

Above, we almost find the standard Einstein-Hilbert action; it's standard form may be obtained by a Weyl rescaling of the metric. The resulting κ'^2 is then the gravitational coupling in ten dimensions, $\kappa'^2 = 8\pi G$, where G is Newton's constant in ten dimensions. The fields S_1 includes are: the graviton $g_{\mu\nu}$, a scalar field Φ called the *dilaton* and H, the field strength of $B_{\mu\nu}$, $H = dB$.

The part S_2 differs in IIA and IIB and contains the remaining (R-R sector) bosonic fields. For type IIA these fields are a one-form, C_1, and a three-form, C_3. The IIA S_2 is, in form notation,

$$S_2 = -\frac{1}{4\kappa^2} \int B \wedge F_4 \wedge F_4 - \frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \left(|F_2|^2 + |\tilde{F}_4|^2 \right). \tag{3.70}$$

Here, F_2 is the field strength of C_1, $F_2 = dC_1$; $\tilde{F}_4$ is defined by $\tilde{F}_4 = F_4 - C_1 \wedge H$, where $F_4 = dC_3$ is the field strength of C_3.

The remaining fields in IIB are: a scalar that is called the *axion*, C_0, a two-form C_2 and a four-form C_4 with self-dual field strength, i.e. $F_5 = dC_4 = *F_5$. The self-duality condition must be implemented separately as it cannot be enforced by some type of Lagrangian. S_2 is given by

$$S_2 = -\frac{1}{4\kappa^2} \int C_4 \wedge H \wedge F_3 - \frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right). \tag{3.71}$$

[18] Hence the name "type II".

[19] Sixteen-dimensional as $2^{\left[\frac{d}{2}\right]} = 16$ for $d = 10$.

Now, $F_1 = dC_0$, $F_3 = dC_2$, $F_5 = dC_4$, while $\tilde{F}_3 = F_3 - C_0 \wedge H$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H + \frac{1}{2}B \wedge F_3$. In the case of IIB, however, the entire bosonic sector may be rewritten completely as the four form part may be set to zero [2]. Defining $\tau = C_0 + ie^{-\Phi}$, one finds

$$S_1 + S_2 = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\tau^2} - \frac{1}{12}\frac{|F_3 + \tau H|^2}{\tau} \right). \tag{3.72}$$

Remark 3.2. As a side note, we remark that the free field action for a p-form gauge field C_p with field strength $F_{p+1} = dC_p$ is generally given by, [12],

$$S_p = -\frac{1}{2(p+1)!} \int_{\mathcal{M}} F_{p+1} \wedge *F_{p+1} \equiv -\frac{1}{2(p+1)!} \int d^d x \sqrt{-g}\, |F_{p+1}|^2. \tag{3.73}$$

There is a gauge invariance of the form $C_p \to C_p + d\Lambda_{p-1}$ which may be fixed by imposing $d^\dagger C_p \equiv (-1)^{d(p+1)} * d * C_p = 0$ (d is the dimension of the manifold; the Lorentz condition of electromagnetism is of this form). The equations of motion are, after gauge fixing

$$\Delta C_p \equiv (d + d^\dagger)^2 C_p = (dd^\dagger + d^\dagger d) C_p = 0. \tag{3.74}$$

Δ is the Laplacian. Lastly, the "$*$" above is the Hodge star, which is given in coordinate basis dx^m, for an p-form ω, by

$$\begin{aligned} \omega &= \frac{1}{p!}\, \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}, \\ *\omega &= \frac{\sqrt{-g}}{p!(d-p)!}\, \omega_{i_1 \dots i_p} \epsilon^{i_1 \dots i_p}{}_{j_{p+1} \dots j_d} dx^{j_{p+1}} \wedge \dots \wedge dx^{j_d}. \end{aligned} \tag{3.75}$$

$\epsilon^{i_1 \dots i_d} = g^{i_1 j_1} \cdots g^{i_d j_d} \epsilon_{j_1 \dots j_d}$ and

$$\epsilon_{j_1 \dots j_d} = \begin{cases} +1 & \text{if } (j_1 \dots j_d) \text{ is an even permutation of } (12 \dots d), \\ -1 & \text{if } (j_1 \dots j_d) \text{ is an odd permutation of } (12 \dots d), \\ 0 & \text{otherwise.} \end{cases} \tag{3.76}$$

Note that $*1$ is the invariant volume element $\sqrt{-g}\, dx^1 \wedge \dots \wedge dx^d$, and that $**\omega = (-1)^{1+p(d-p)}\omega$ for a Lorentzian manifold $\mathcal{M}$ (like $\mathbb{R}^{d-1,1}$; an extra factor -1 is to be added for a Riemannian manifold).

3.3.1 Remarks on supersymmetry

Throughout the whole section on the superstring we have insisted on keeping supersymmetry around, though we never explained why. Its importance is related to the view that the Standard Model is, as more or less remarked in the previous section, a low energy limit of some underlying, more "fundamental" theory; much like the old Fermi theory of the weak interactions is a low energy limit of the later Glashow-Salam-Weinberg model of the electroweak interactions. In this thesis, the assumed underlying theory is string theory.

Whether the "true" theory turns out to be string theory or not, we will in any case need some sort of extension of the standard model that will be valid up to higher energies and that needs to come up with solutions to a number of problems the standard model suffers from. One of these is lack of candidates for dark matter, which is supposed to make up about 30% the energy content of our universe. Another problem is dark energy ($\sim 70\%$ of the energy content of the universe): if it is interpreted as the energy of the vacuum, i.e. as the cosmological constant Λ, the standard model predicts a value that is a hundred orders of magnitude or so too large. A third problem is the so-called *hierarchy problem*: the standard models Higgs boson's mass receives highly divergent contributions from loop-corrections, in order to cancel these an "unnaturally" precise cancelling would have to take place by using counter terms.

Supersymmetric extensions of the standard model are interesting as some offer solutions to at least some of these problems[20]. The minimal supersymmetric extension of the standard model (called the *Minimal Supersymmetric Standard Model*) for example solves the hierarchy problem by providing loop-corrections that cancel each other. Furthermore, if the model's supersymmetry is broken at energies of about one TeV, it provides a weakly interacting interacting massive particle that would be a candidate dark matter particle, and it would allow for the unification of the standard model gauge groups in a larger group[21].

Although it is not (yet) known whether the idea of supersymmetry is correct, it seems to be a rather useful model and so we choose to keep it around as part of our string theory based model. Hoping, of course, that one day a four-dimensional low energy limit of string theory that is compatible with the Standard Model will be found.

References. A good introduction of the Standard Model would be provided by *An Introduction to Quantum Field Theory* (1995, Westview Press) by Michael E. Peskin and Daniel V. Schroeder. For an introduction to supersymmetry (and also to string theory), see the book by Michael Dine: *Supersymmetry and String Theory* (2006, Cambridge University Press). A slightly more mathematical alternative to these books is the three volume book written by Steven Weinberg, *The Quantum Theory of Fields* (2000, Cambridge University Press).

3.3.2 Supersymmetry variations

From this point onwards, we will define the R-R fields somewhat differently [8], [9]. Before, they could be described as follows; the $p-$form fields (field strengths or fluxes) were F_p with $p = 0, 2, 4$ for type IIA and $p = 1, 3, 5$ for type IIB.

The F_0 field of type IIA does not have propagating degrees of freedom and corresponds to a constant $m = F_0$, called the Romans mass. The F_5 of type IIB satisfies a self-duality condition.

In what follows we will however work with the so-called *democratic formulation*, in which the number of R-R fields is doubled: $p = 0, 2, 4, 6, 8, 10$ for type IIA and $p = 1, 3, 5, 7, 9$ for type IIB. In order to undo the doubling of degrees of freedom in the democratic formulation, we impose duality relations for all R-R

[20] A problem these theories do not address is the inclusion of gravity; this is done in string theory.

[21] Results from the LHC experiments have, however, shown that this particular model is incorrect. There are, of course, a host of other models left. For the published results of the ATLAS and CMS experiments, see `https://twiki.cern.ch/twiki/bin/view/AtlasPublic` and `https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResultsSUS`.

fields. They read

$$F_p = (-1)^{\frac{(n-1)(n-2)}{2}} * F_{10-p}, \tag{3.77}$$

and should be imposed by hand after deriving the equations of motion from the action. The action is thus only a pseudo-action as it does not imply the duality relations.

Usually, we will work with the *polyform* $F = \sum_p F_p$. The duality condition can then be written as

$$F = *\sigma(F), \tag{3.78}$$

where σ is the operator defined in definition 6.2.

The R-R potentials C_{p-1} are also doubled in the democratic formulation, and will be collectively denoted as $C = \sum_p C_{p-1}$. Now, using the fact that $dH = 0$, we define the nilpotent H-twisted exterior derivative for polyforms as

$$d_H = d + H\wedge, \qquad d_H^2 = 0. \tag{3.79}$$

The relation between the potentials C_{p-1} and the field strengths F_p can then be conveniently written as H-twisted Bianchi identities:

$$\begin{aligned} F &= d_H C && \text{for type IIB,} \\ F &= d_H C + me^{-B} && \text{for type IIA.} \end{aligned} \tag{3.80}$$

Furthermore, the field polyform F satisfies the Bianchi identity

$$d_H F = 0, \tag{3.81}$$

again in the absence of sources.

The action, in the string frame, of the democratic formulation is given by

$$S = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-\mathrm{det}g}\left(e^{-2\Phi}\left(R + 4(d\phi)^2 - \tfrac{1}{2}H^2\right) - \tfrac{1}{4}F^2\right) + S_{\mathrm{loc}}, \tag{3.82}$$

where S_{loc} is the action corresponding to localised sources, which may include a variety of objects, such as D-branes, orientifolds, NS5-branes, fundamental strings and KK-monopoles [9].

The fermionic superpartners of the bosonic fields discussed above are two Majorana-Weyl gravitinos, ψ_M^A, $A = 1, 2$ and $M = 1, \ldots, 10$, and two Majorana-Weyl dilatinos λ^A. As mentioned before, the gravitinos are of opposite chirality in type IIA and of the same chirality in type IIB; the dilatinos have chiralities that are opposite to the gravitinos'. Again, both theories have $\mathcal{N} = 2$ supersymmetry with two Majorana-Weyl supersymmetry parameters, ϵ^A, with chiralities equal to those of the gravitinos.

Finally, using the democratic formulation and working in the string frame[22], the supersymmetry transformations for the supergravity theories, of the grav-

[22]The string frame is what has been used in this section, and refers to the form of the supergravity actions. The Einstein frame versions of the actions are obtained by a conformal rescaling of the metric, such that the usual form of the Einstein-Hilbert action appears ($S = \int d^{10}x\,\sqrt{-\det g}\,R$).

itino and dilatino doublets, become

$$
\begin{aligned}
\delta\psi^1_M &= (D_M\epsilon)^1 \equiv \Big(\nabla_M + \tfrac{1}{4}\underline{H_M}\Big)\epsilon^1 + \tfrac{1}{16}e^{\Phi}\underline{F}\,\Gamma_M\Gamma_{11}\epsilon^2,\\
\delta\psi^2_M &= (D_M\epsilon)^2 \equiv \Big(\nabla_M - \tfrac{1}{4}\underline{H_M}\Big)\epsilon^2 - \tfrac{1}{16}e^{\Phi}\sigma(\underline{F})\Gamma_M\Gamma_{11}\epsilon^1,\\
\delta\lambda^1 &= \Big(\underline{\partial}\Phi + \tfrac{1}{2}\underline{H}\Big)\epsilon^1 + \tfrac{1}{16}e^{\Phi}\Gamma^M\underline{F}\,\Gamma_M\Gamma_{11}\epsilon^2,\\
\delta\lambda^2 &= \Big(\underline{\partial}\Phi - \tfrac{1}{2}\underline{H}\Big)\epsilon^2 - \tfrac{1}{16}e^{\Phi}\Gamma^M\sigma(\underline{F})\Gamma_M\Gamma_{11}\epsilon^1.
\end{aligned}
\tag{3.83}
$$

Remark 3.3. In the equations above Γ_M denote the gamma matrices in 10 dimensions; Γ_{11} is the chirality operator. The fields that are underlined are defined using the Clifford map:

$$
\phi \to \underline{\phi} = \sum_l \frac{1}{l!}\phi_{i_1\ldots i_l}\Gamma^{i_1\ldots i_l}, \tag{3.84}
$$

where ϕ is a (poly-)form and, as usual, $\Gamma^{i_1\cdots i_n} \equiv \Gamma^{[i_1}\ldots\Gamma^{i_n]}$, where the brackets indicate a complete anti-symmetrization. Note that for H in $\delta\psi^A$ the first index is not contracted with gamma matrices through the Clifford map.

These equations are central to the discussion of compactification, that forms the second part of this thesis. Before studying them in greater detail, however, we first discuss compactification in more general terms.

4 Compactification

This section marks the beginning of what we referred to in the introduction as being the *second* part of this thesis. In it we will discuss the compactification of superstring theory. As remarked earlier, compactification is the “wrapping up” of 6 of the 10 dimensions (see section 3.2) into a small, compact manifold such that only a $d = 4$ Lorentzian (i.e. locally Minkowski, meaning that we have only one time dimension) manifold remains large. The latter part should correspond to, at distance scales large compared to the $d = 6$ manifold, the four-dimensional spacetime we experience daily.

We will study the subject in terms of field theory, not in terms of string theory[23], and we will mostly pay attention to the massless supergravity fields of the preceding section.

Useful references are [6], [7] and [8]. Several chapters in [5] are also dedicated to the subject.

4.1 Compactified space

Essentially, the foregoing paragraph says this: we want to replace $\mathcal{M}_{10}$, our ten-dimensional manifold, by a product of manifolds: $\mathcal{M}_4 \times K_6$, where the former is our spacetime and the latter is something compact, Riemannian and ‘small’. That this could be possible is however not evident; it is after all the

[23]When studying strings on compactified spaces, one might wonder about the possibility of strings ‘winding’ around the compact space etc.. Such questions do not arise in a field theory treatment.

dynamical metric field g that determines the geometry of $\mathcal{M}_{10}$. The sought-after product form (of the *vacuum*) is possible whenever the metric field has a vacuum expectation value of the form:

$$\langle 0|g|0\rangle = e^{2A(y)}\tilde{g}_{\mu\nu}(x)dx^\mu \otimes dx^\nu + g_{mn}(y)dy^m \otimes dy^n. \tag{4.1}$$

Here x^μ, $\mu = 0, \ldots, 3$ are the coordinates of $\mathcal{M}_4$ and y^m, $m = 1 \ldots 6$ the coordinates of K_6. $A(y)$ is a *warp factor*, it depends only on the $y-$coordinates and does not change the symmetry group associated to the spacetime vacuum (e.g. the Poincaré invariance of a Minkowski vacuum). The manifold $\mathcal{M}_4$ with metric $e^{2A(y)}\tilde{g}_{\mu\nu}(x)$ is usually referred to as the *external* manifold, while K_6 with metric $g_{mn}(y)$ is referred to as the *internal* manifold.

The meaning of the earlier phrase "... [$\mathcal{M}_4$] should correspond to, at distance scales large compared to the $d = 6$ manifold, the four-dimensional spacetime we experience daily." is, in more precise terms, that in the limit of low energies (or, equivalently, long wavelengths and long distance scales), the compactified ten-dimensional theory should yield an *effective* four-dimensional theory. Furthermore, at energies of up to 1 TeV this effective theory should correspond to the Standard Model.

In the following we will not try to determine whether the equations of motion explicitly allow for product form solutions. Instead we will simply assume the form of the metric vev given above. The solutions that are of interest are those that preserve some supersymmetry, for reasons that were mentioned in section 3.3.1.

4.1.1 A simple example: dimensional reduction

The example we describe in this section illustrates how to determine the effective theory after compactification, [6]. Furthermore, it gives a feeling for what the word 'small', as used before, really means.

Suppose we have a real, massless scalar field φ, living in a five dimensional Minkowski space $\mathbb{R}^{4,1}$. Its action is given by

$$S = -\frac{1}{2}\int d^5x\, \partial_m\varphi\partial^m\varphi, \tag{4.2}$$

where $\eta^{nm} = \mathrm{diag}(-, +, \ldots, +)$ and $n = 0, \ldots, 4$.

One way to compactify the fourth spatial dimension is to replace it by a circle; that is, we replace $\mathbb{R}^{4,1}$ by $\mathbb{R}^{3,1} \times S^1$, where $\mathbb{R}^{3,1}$ is the standard four-dimensional Minkowski spacetime and S^1 a circle of radius R. The coordinates on this new manifold become $x^m = (x^\mu, y)$, with $\mu = 0, \ldots, 3$ and $y \in [0, 2\pi)$.

The field φ's equation of motion is,

$$\partial_m\partial^m\varphi = \partial_\mu\partial^\mu\varphi + \partial_y^2\varphi = 0. \tag{4.3}$$

We require φ to be single-valued, which means that it should be 2π-periodic in y. This implies that φ can be partially Fourier expanded:

$$\varphi(x, y) = \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{n=\infty}\varphi_n(x)e^{iny/R}. \tag{4.4}$$

The $Y_n(y) \equiv \frac{1}{\sqrt{2\pi R}} e^{iny/R}$ are the orthonormalized eigenfunctions of ∂_y^2 on S^1. The equation of motion now turns into

$$\partial_\mu \partial^\mu \varphi_n - \frac{n^2}{R^2} \varphi_n = 0, \tag{4.5}$$

and implies that the $\varphi_n(x)$ are four-dimensional scalar fields with masses n/R. This also shows in the action, which, after substituting the Fourier series and integrating over y, becomes

$$S = - \sum_{n=-\infty}^{\infty} \frac{1}{2} \int d^4x \left(\partial_\mu \varphi_n^* \partial^\mu \varphi_n + \frac{n^2}{R} \varphi_n^* \varphi_n \right). \tag{4.6}$$

The limit we are interested in is that of $R \to 0$, i.e. that of 'small' R[24]. In this limit all fields but the zero mode φ_0 acquire very large masses and share the same fate as string theory's massive excitations discussed earlier: they may be left out of the (effective, low energy) theory.

In this particular case, the only interesting mode, $\varphi_0(x)$, is independent of the internal coordinate y; this is called *dimensional reduction.* The most general case in which it occurs is that of compactification on a $d-$dimensional torus T^d.

In general, the determination of the effective theory proceeds more or less along the lines described above: a particular product $\mathcal{M}_4 \times K_6$ is chosen, (the fields are expanded around their vevs,) the equations of motion and the relevant eigenfunctions on K_6 (the analogues of the $Y_n(y)$ of ∂_y^2 on S^1) are studied and, whenever that is possible, the resulting massive modes or particles are ignored.

4.1.2 Decomposing fields

On any manifold, the fields that live on it transform under certain representations of the *structure group* of that manifold. The structure group G (or $G-$*structure*) is, roughly, the group that is required to consistently patch the tangent bundle together. See, for a more careful and precise definition, section 5.

On a ten-dimensional Lorentzian manifold $\mathcal{M}_{10}$ this group is the group of local changes of basis (of vectors in $T_p\mathcal{M}_{10}$, $p \in U \subset \mathcal{M}_{10}$ where U is a small patch—or equivalently, changes of coordinates—) that leave the metric invariant (which, locally, can be made equal to the Minkowski η). This is $O(1,9)$. For a Riemannian manifold, the local metric to be left invariant is δ_{ij} and so the stucture group is $O(d)$. If these manifolds are also orientable, which means that there exists a globally defined volume form, these groups further reduce to $SO(1,9)$ and $SO(d)$, respectively.

In our case, we start out with fields that transform under $SO(1,9)$, the structure group of $\mathcal{M}_{10}$. After compactification, however, we should end up with fields that transform under $SO(1,3) \times SO(6)$, the structure group associated to $\mathcal{M}_4 \times K_6$, so we need to know how the original fields decompose into representations of that group. Below, we list a number of examples.

[24]Smallness means, for a more serious compactification, that the internal space's characteristic size (which for the circle this is determined by R) should be smaller than, say, 10^{-18}, which is roughly the de Boglie wavelength for a light particle with an energy of 1 TeV. If this were not the case, the presence of an internal space (and the associated massive modes like $\varphi_n(x)$, $n \neq 0$) should have been noticed by the particle accelerators exploring particle collisions at these energies (such as the Tevatron in the US: `http://www.fnal.gov/pub/tevatron/`).

Example 4.1. A vector A_M, that transforms under the defining representation $\mathbf{10}$, decomposes as $\mathbf{10} \to (\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{6})$. This means that the vector splits up into two parts, A_μ and A_m ($\mu = 0, \ldots, 3$ and $m = 4, \ldots, 9$), the former being a vector of $SO(1,3)$ and a scalar under the internal $SO(6)$, and vice versa for the latter. As the A_m, for each m, are singlets under $SO(1,3)$, they appear as scalars on $\mathcal{M}_4$.

Example 4.2. The antisymmetric tensor B_{MN} decomposes into $B_{\mu\nu}$, $B_{\mu m}$ and B_{mn}—an antisymmetric tensor and a number of vectors and scalars on $\mathcal{M}_4$. The case of the graviton is similar.

In the decomposition of, for example, B_{MN} above, each part is an $n-$form on the internal manifold: a 0, 1 or 2$-$form respectively. After decomposing B, what remains to be done is determining the number of massless $n-$form eigenfunctions on K_6. Those numbers are defined by the topology of K_6, and are equal to the number of tensors, vectors and scalars we end up with on $\mathcal{M}_4$. The $g_{\mu\nu}$ and $B_{\mu\nu}$ will be unique, but vectors and scalars may be multiple[25].

Example 4.3. The case of spinors is somewhat different. As was mentioned before, spinors have $2^{[d/2]}$ components in d components, the d Dirac matrices Γ^M being $2^{[d/2]} \times 2^{[d/2]}$-matrices. The Dirac matrices can be decomposed as below, into Γ^μ and Γ^m; these can then be used to construct the generators of $SO(1, D-1)$ and $SO(d-D)$ respectively (when compactifying to $\mathcal{M}_D \times K_{d-D}$). As these are still $2^{[d/2]} \times 2^{[d/2]}$-matrices, they act on all spinor components, so that an $SO(1, d-1)$ spinor should transform as a spinor under both $SO(1, D-1)$ and $SO(d-D)$.

The $\mathbf{16}$ Majorana-Weyl representation of $SO(1,9)$, for example, decomposes under $SO(1,3) \times SO(6)$ as $\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}})$, where $\mathbf{4}$ and $\bar{\mathbf{4}}$ are Weyl spinors of $SO(6)$ and $\mathbf{2}$, $\bar{\mathbf{2}}$ are a Weyl spinors of $SO(1,3)$ that are conjugate to each other [5].

Remark 4.4. A standard decomposition in $D = 10$ of the Dirac matrices $\Gamma^M = (\Gamma^\mu, \Gamma^m)$ is given by [8]

$$\Gamma^\mu = \gamma^\mu \otimes 1,\ \mu = 0, 1, 2, 3, \qquad \Gamma^M = \gamma_5 \otimes \gamma^m,\ m = 1, \ldots, 6. \tag{4.7}$$

And,

$$\gamma_5 = \tfrac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}, \quad \gamma_7 = -\tfrac{i}{6!}\epsilon_{mnpqrs}\gamma^{mnpqrs}, \quad \Gamma_{11} = \gamma_5 \otimes \gamma_7. \tag{4.8}$$

The γ^m and γ^μ, except for γ^0, are Hermitian.

[25]The case of the g_{mn} part is interesting in that it determines the geometry of the internal space. Expanded around its vev: $g_{mn} = \langle g_{mn} \rangle + h_{mn}$; we find that h may vary as a function of x^μ as long as it leaves the internal space Ricci flat (i.e. $R_{mn}(\langle g \rangle + h) = R_{mn}(\langle g \rangle) = 0$; a condition that results from demanding the $d = 4$ theory to be supersymmetric). The h that satisfy this condition are called *moduli*, they will appear again a little later on. Their variation implies that the internal geometry changes (in terms of shape and size, not in terms of topology) as a function on spacetime, changing physics from one place to the next. In the circle example we had something similar, as the value of R was small but otherwise arbitrary. This means that R shows up as a massless scalar field, with vanishing potential, in the spacetime theory [6].

4.1.3 The vacuum revisited

Earlier we stated that we were interested in a product form manifold $\mathcal{M}_4 \times K_6$ with a metric vev of the form

$$\langle g \rangle = e^{2A(y)} \tilde{g}_{\mu\nu}(x) dx^\mu \otimes dx^\nu + g_{mn}(y) dy^m \otimes dy^n. \tag{4.9}$$

The case we will study further is that of $\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu}$, that of a Minkowski spacetime vacuum. This case is of course physically interesting as the universe, or at least the part we inhabit, is approximately Minkowski. Variations of the metric will be treated as perturbations with respect to this *background* metric.

The other property we wish the vacuum to have is maximal Poincaré symmetry. This means that the vacuum expectation values of the other fields should be constants on spacetime (as we want the vacuum to be translation invariant) and that they should transform as scalars—that is, trivially—under $SO(1,9)$. So, again, for $\langle g \rangle|_{\mathcal{M}_4}$ only something of the form $e^{2A(y)}\eta_{\mu\nu}$ is allowed, while for a $d-$form field ($d \leq 4$) only $\epsilon_{\mu_1 \ldots \mu_d}$ is allowed[26].

All fermionic vevs should be zero: $\langle \psi^A_M \rangle = \langle \lambda^A \rangle = 0$, since fermions transform non-trivially under the Lorentz group (, a non-zero fermion vev would thus reduce the symmetry of the vacuum). Also, as we want our theory to be supersymmetric, the vevs of the supersymmetry transformations of the fermionic fields, which are dependent on the bosonic fields, should equal zero: $\langle \delta\psi^A_M \rangle = \langle \delta\lambda^A \rangle = 0$. The susy variations of the fermions were given in equations (3.83).

On the internal manifold, the story is different, as the symmetry condition on the spacetime vacuum does not affect the internal vacuum. The internal parts of the bosonic fields may therefore take on arbitrary $y-$dependent vevs. Explicitly, the field strength polyform F appearing in the susy variations should decompose as follows:

$$F = \hat{F} + \mathrm{vol}_4 \wedge e^{4A} \tilde{F}, \tag{4.10}$$

if it is to preserve maximal spacetime Poincaré symmetry after compactification. Here vol_4 is the four-dimensional Minkowski volume element, which, in components, is given by the aforementioned tensor $\epsilon_{\mu\nu\rho\sigma}$. $\hat{F}$ and $\tilde{F}$ denote, respectively, the so-called "magnetic" and "electric" components of F.

Hodge duality implies the following relation between the two

$$\tilde{F} = *_6 \sigma(\hat{F}), \tag{4.11}$$

where $*_6$ is the six-dimensional (internal) Hodge star. This relation, written out in forms, becomes

$$\tilde{F}_{n-4} = (-1)^{\frac{(n-1)(n-2)}{2}} *_6 \hat{F}_{10-n}, \tag{4.12}$$

and we find that we can relate external components to internal ones, allowing us to rewrite the susy transformation in terms of the internal fields only ($F_n,\ n = 0, \ldots, 6$, by exchanging for example an F_4 with μ-type indices with an 'internal' F_6 with m-type indices, [8]).

In the following sections we will discuss two cases separately:

[26] Ignoring a possible $y-$dependent factor.

- the *fluxless* case, where the bosonic fields vevs are all zero: $\langle F_p \rangle = 0$, for all $p-$form fields (, which are often called *fluxes*),
- the case of *flux compactification*, in which not all $\langle F_p \rangle$ are zero.

4.2 Fluxless compactification

The fluxless case is obviously the simplest to work out as the susy variations simplify enormously. Initially, we will assume a general spacetime metric and show that in fact, only a (warped) Minkowski spacetime is possible.

In the following equations we will no longer write the brackets $\langle , \rangle$ to indicate that we are talking about vevs. When we are, it will be remarked. A useful reference

In the absence of fluxes, $\langle F \rangle = \langle H \rangle = 0$, the vev of the gravitino susy variation reduces to

$$\delta\psi_M^A = \nabla_M \epsilon^A. \tag{4.13}$$

If this is to be zero, there needs to exist a covariantly constant spinor on the ten-dimensional manifold: $\nabla_M \epsilon = 0$.

The space-time components of this equation are

$$\nabla_\mu \epsilon + \tfrac{1}{2}(\gamma_\mu \gamma_5 \otimes \gamma^m \nabla_m A)\epsilon = 0. \tag{4.14}$$

Here the gamma matrices have been decomposed as described in remark 4.4. The γ^μ are defined with respect to $\tilde{g}_{\mu\nu}$ and the γ^m with respect to g_{mn}.

From this equation we derive an integrability condition for the spacetime manifold. First, we have

$$[\nabla_\mu, \nabla_\nu]\epsilon = -\tfrac{1}{2}(\nabla_m A \nabla^m A)\gamma_{\mu\nu}\epsilon, \tag{4.15}$$

and, from the definition of the Riemann curvature tensor, we know that

$$[\nabla_\mu, \nabla_\nu]\epsilon = \tfrac{1}{4}R_{\mu\nu\lambda\rho}\gamma^{\lambda\rho}\epsilon = \tfrac{k}{2}\gamma_{\mu\nu}\epsilon. \tag{4.16}$$

Above we used that for a maximally symmetric space, $R_{\mu\nu\lambda\rho}$ is given by $R_{\mu\nu\lambda\rho} = k(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$. For a Minkowski spacetime, $k = 0$, while for the anti-de Sitter and de Sitter spacetimes, $k < 0$ and $k > 0$, respectively. Combining the two equations, and using the fact that $\gamma_{\mu\nu}$ is invertible, we find

$$k + \nabla_m A \nabla^m A = 0. \tag{4.17}$$

On a compact manifold, the only possible constant value for $(\nabla A)^2$ is zero. This implies that the four-dimensional spacetime can only be Minkowski.

To analyse the susy variation equation in more detail, we split the susy parameters ϵ^A up into two parts: an external and an internal spinor. For the internal spinor we use only one Weyl spinor. The reasons for this will become clear later. For the type IIA supergravity the decomposition reads

$$\begin{aligned}\epsilon^1 &= \xi_+^1 \otimes \eta_+ + \xi_-^1 \otimes \eta_-, \\ \epsilon^2 &= \xi_+^2 \otimes \eta_- + \xi_-^2 \otimes \eta_+,\end{aligned} \tag{4.18}$$

where $\gamma_{11}\epsilon^1 = \epsilon^1$ and $\gamma_{11}\epsilon^2 = -\epsilon^2$ (they have opposite chirality). The external and internal spinors obey, respectively, $\xi_-^A = (\xi_+^A)^*$ and $\eta_- = (\eta_+)^*$; the ϵ^A are real. The IIB decomposition is different, as the ϵ^A have the same chirality:

$$\epsilon^A = \xi_+^A \otimes \eta_+ + \xi_-^A \otimes \eta_-, \quad A = 1, 2. \tag{4.19}$$

These decompositions describe explicitly what we wrote in section 4.1.2 as

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}}); \tag{4.20}$$

a sixteen-dimensional Majorana-Weyl spinor is replaced by (real) tensor products of four- and eight-dimensional Weyl spinors.

If we insert these decompositions into the gravitino variation, we find

$$\nabla_m \eta_\pm = 0, \tag{4.21}$$

which says that the internal manifold should have a covariantly constant spinor on it. Supposing we have only one such spinor, we also find from the internal gravitino equation that we have two external supersymmetry parameters, ξ^1 and ξ^2, yielding $\mathcal{N} = 2$ spacetime supersymmetry.

Requiring the internal space to allow for a globally defined, covariantly constant spinor strongly constrains its topological properties. The condition implies that the manifold has reduced *holonomy*, or a reduced *holonomy group*. To understand how this comes about, let us first mention the definition of the holonomy group (see also appendix A.4).

Definition 4.5. Let (M, g) be a Riemannian manifold, with metric g and affine connection ∇. Let $p \in M$. We consider the set of closed loops at p: $\{c(t)|0 \leq t \leq 1, c(0) = c(1) = p\}$. Now, take a vector $X \in T_pM$ and parallel transport it around a curve $c(t)$. After this trip, we end up with a new vector $X_c \in T_pM$. Thus we see that the loop $c(t)$ and the connection ∇ induce a linear transformation

$$P_c : T_pM \to T_pM. \tag{4.22}$$

The set of these transformations is denoted by $H(p)$ and is called the *holonomy group*[27]. (Spinors, when transported around a loop, transform under a spinor representation of $H(p)$.)

It has been proven that a manifold has a torsion free $G-$structure if and only if there exists a torsion free covariant derivative so that the associated holonomy group is a subgroup of the structure group, [14]. Standard examples are Riemannian manifolds ($O(d)$-structure and holonomy, torsion free Levi-Civita derivative) and Kähler manifolds, which are described in appendix A.4 ($U(d)$-structure and holonomy).

Earlier we stated that fields, and so also spinor fields, transform under the structure group; on K_6 they transform as the $\mathbf{4}$ under the spinor representation of $SO(6)$: $Spin(6) \cong SU(4)$. If K_6's structure group is reduced to $SU(3)$, the $\mathbf{4}$ decomposes into irreducible representations of $SU(3)$ as $\mathbf{4} \to \mathbf{3} + \mathbf{1}$. The $\mathbf{1}$ is the sought-after globally defined, internal spinor. If it is also covariantly

[27]The holonomy group is, in fact, base point (p-) independent, but not connection independent [14].

constant, it forces the holonomy group to be contained in $SU(3)$[28]. Conversely, it is known that a manifold with $SU(3)$ holonomy admits a globally defined, covariantly constant spinor, [14].

We arrive to the conclusion that K_6 should be a manifold with $SU(3)$-holonomy. Manifolds of this type (with $SU(d)$ holonomy) are called Calabi-Yau. A different, but equivalent definition is given in appendix A.4, and a rephrasing of this definition's conditions is provided for by theorem 5.14.

In summary, a fluxless compactification can only be the product of a four-dimensional Minkowski spacetime with a six-dimensional Calabi-Yau manifold, having $\mathcal{N} = 2$ spacetime supersymmetry.

In the presence of fluxes, the biggest change with respect to the fluxless case is that the internal spinor no longer needs to be covariantly constant. The only remaining requirement is that a globally defined spinor exists, which, as we have seen, is the case if the internal manifold has a $SU(3)$-*structure*. The degree of spacetime supersymmetry remains unchanged.

The subject of $G-$structures is discussed in much more detail than was provided above, in section 5.

Remark 4.6. Above, we found that the resulting degree of spacetime supersymmetry is $\mathcal{N} = 2$: there are two spacetime susy parameters ξ^A. The extensions of the Standard Model that are of particular interest, however, have only one susy parameter (and thus $\mathcal{N} = 1$ supersymmetry). In order to achieve this degree of supersymmetry we simply take the ξ^A to be proportional to each other. The proportionality factor used is in general a function of the internal coordinates (see apppendix B).

4.3 Flux compactification

In this section we briefly discuss flux compactification, which is in a sense the subject of the rest of this thesis. One of the goals of the remaining sections on $G-$structures and Generalized Geometry is to provide a mathematical framework in which it is easier to work with the messy equations that arise in flux compactifications. The central equations are again the susy variations of the spinor fields.

In this section however, we keep the mathematics to a minimum and explain why flux compactification is interesting and possibly useful. A good reference at the introductory level is [7].

4.3.1 Moduli

Perhaps the main reason why flux compactification is useful has to do with *moduli*. They were mentioned before in one of the footnotes of section 4.1.2.

In the previous section we found that in fluxless compactifications we have Calabi-Yau internal manifolds; these are Ricci flat (i.e. the Ricci tensor vanishes, $R_{mn}(\langle g \rangle) = 0$). The variations of the internal metric, at a certain spacetime point, with respect to its vev $g_{mn} = \langle g_{mn} \rangle + h_{mn}$ that are allowed are those that preserve the topology of the internal space. They are those that preserve

[28] This argument is repeated in slightly more careful wording in section 5.1.

its Ricci flatness. It is these h_{mn} that are called *moduli*. If they also vary as a function of x^μ, which is perfectly allowable, they show up as massless scalar fields on the spacetime manifold.

The space of moduli h_{mn} is called the *moduli space* and is a manifold.

One explicit example of a modulus is a rescaling factor multiplying the internal metric: λg_{mn}, $\lambda \in \mathbb{R}$; it parametrizes a family of metrics (if g is Ricci flat, then λg is too).

The problem with moduli is the (interacting) massless fields that they yield after going through the whole compactification procedure: physically, no such particles are found to exist. Furthermore, they pose an arbitrariness problem inherent to the theory as no particular set of moduli values is preferred. Flux compactification might solve this problem. By introducing non-zero flux vevs on the internal space, one introduces through the flux's action a metric-, and thus moduli-dependent potential. In trying to find a vacuum for the compactified theory, we minimize this potential energy and we may in this way come up with a (unique and) stable set of moduli values if the potential has a (global) minimum. A slightly more mathematical sketch of this argument is given below.

The fluxes we are considering are the aforementioned F_p and H.

4.3.2 A sketch

First, let us consider a classical example, [7]. In the Maxwell theory, we have a one form potential A_1, with field strength $F_2 = dA_1$. In the presence of a magnetic monopole, we may compute its magnetic charge by taking the integral of the magnetic flux over a two-sphere surrounding the monopole:

$$g = \int_{S^2} F_2. \tag{4.23}$$

This charge has to satisfy the Dirac quantization condition, [12]: $e \cdot g = 2\pi$ (in units such that $\hbar = 1$, e is the electron charge).

In this case, the field strength of a magnetic monopole placed at the origin of $\mathbb{R}^3$, restricted to a sphere centred on the origin, $B(\theta, \phi) = g \sin\theta \, d\theta \, d\phi$, satisfies the Maxwell equations restricted to that sphere. So we find that, if we start out with Maxwell's theory in six dimensions and compactify two on the two-sphere, we may do so with a non-zero background field B on S^2. This leads to a potential energy term associated to S^2 that is proportional to B^2, the square of the magnetic field.

In string theory compactifications we may proceed similarly. Suppose we compactify on a manifold with a non-trivial p cycle Σ_p, meaning that the corresponding homology group $H_p(K)$ should be non-trivial, then we may consider a field configuration on Σ_p (with or without sources) with a non-zero total flux[29]:

$$\frac{1}{(2\pi\sqrt{\alpha'})^{p-1}} \int_{\Sigma_p} F_p = n \neq 0. \tag{4.24}$$

As in the monopole case, this flux should be quantized: $n \in \mathbb{Z}$. If the homology group contains multiple basis elements, we can turn on an independent flux on each of them.

[29]Note, we are considering fields that satisfy standard Bianchi identities: $dF_p = 0$, instead of H-twisted ones.

The potential energy associated to an internal flux is given by

$$\begin{aligned} V &\propto \int_{K_6} F_p \wedge *F_p \\ &= \int_{K_6} d^6 y \sqrt{-\det g}\, g^{ij} \cdots g^{kl} (F_p)_{i\ldots k} (F_p)_{j\ldots l}. \end{aligned} \tag{4.25}$$

The metric dependence enters in the definition of the six-dimensional Hodge star $*$. A simple, explicit, though somewhat sketchy example may be found in [7].

5 Manifolds and G-structures

In the preceding section on compactification, we encountered the notion of G-structures and saw its relevance: the structure group determines in part the fields that the final theory contains, and it needs to be (contained in) $SU(3)$ if we wish to end up with some degree of spacetime supersymmetry.

In this section we define the structure group and its context more carefully. Furthermore, we examine a number of invariant tensor fields and the structures they define. Our focus will lie on the $SU(3)$ structure. We also consider equivalent descriptions in terms of forms and spinors of several structures.

A second theme of this section are the particular allowed kinds of internal manifolds that lead to $\mathcal{N} = 1$ supersymmetry. In fluxless compactifications they were necessarily Calabi-Yau and had $\mathcal{N} = 2$ supersymmetry; in flux compactifications the essential demand was $SU(3)$-structure, implying $\mathcal{N} = 2$ susy as well. So in the sub-sub-sections of this section we simply put $\mathcal{N} = 1$, and study the consequences in terms of so-called *torsion classes*; these will determine the type of manifold the internal space will be.

The discussion of structures is geared towards a subsequent generalization in Generalized Geometry. The reason why that branch of geometry is interesting (or useful) will be explained at the beginning of the corresponding section.

The most relevant reference is [9] (although [8], [12], and [14] occasionally come in handy as well).

5.1 Basic definitions

First we give the definition of fibre bundles, [12]. It is in the context of fibre bundles that G-structures are defined.

Definition 5.1. A *differentiable fibre bundle* (E, π, M, F, G) consists of the following elements:

1. A differentiable manifold E, which is called the *total space*,
2. a differentiable manifold M, which is called the *base space* (e.g. our K_6),
3. a differentiable manifold F, which is called the *fibre*.
4. A surjective *projection map* $\pi : E \to M$. The inverse image $\pi^{-1}(p) = F_p \cong F$ is called the *fibre at p*.
5. A Lie group G that is called the *structure group*. It acts on F on the left.

6. An open covering $\{U_i\}$ of M with diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The maps ϕ_i are called *local trivializations* as they map parts of the fibre bundle E, $\pi^{-1}(U_i)$, onto the direct product $U_i \times F$.

7. If we write $\phi_i(p, f) = \phi_{i,p}(f)$ we have a diffeomorphism $\phi_{i,p} : F \to F_p$. On $U_i \cap U_j \neq \emptyset$, we require that $t_{ij} \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ be an element of G. Then ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f). \tag{5.1}$$

   The maps t_{ij} are called *transition functions*.

In a nutshell, this definition says that a fibre bundle E is something that is locally diffeomorphic to $U \times F$, $U \subset M$. The transition functions serve to smoothly glue these pieces together. As this needs to be done consistently, the transition functions should satisfy:

$$\begin{aligned} t_{ii}(p) &= \text{the identity map} \quad (p \in U_i), \\ t_{ij}(p) &= t_{ji}(p)^{-1} \qquad (p \in U_i \cap U_j), \\ t_{ij}(p) \cdot t_{jk}(p) &= t_{ik}(p) \qquad (p \in U_i \cap U_j \cap U_k). \end{aligned} \tag{5.2}$$

If, for example, all transition functions may be taken to be the identity, the fibre bundle is called *trivial* and is simply a product $M \times F$. Two simple examples of fibre bundles over S^1 are: a) a cylinder, which is trivial as it is the direct product of S^1 with an interval; and b) the Möbius strip, which locally looks like a part of S^1 times an interval but is non-trivial.

Remark 5.2. Actually, the definition above defines a *coordinate bundle* $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$. Two coordinate bundles, $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$ and $(E, \pi, M, F, G, \{V_j\}, \{\psi_j\})$, are said to be equivalent if $(E, \pi, M, F, G, \{U_i\} \cup \{V_j\}, \{\phi_i\} \cup \{\psi_j\})$ is again a coordinate bundle. A fibre bundle is then defined as an equivalence class of coordinate bundles. In physical applications, however, one usually considers a certain definite covering and forgets about the distinction.

Sections of a fibre bundle E are smooth maps $s : M \to E$ sucht that $\pi \circ s = \mathrm{id}_M$. The space of sections of E is denoted by $\Gamma(E)$. In a sense, they are nothing but F-valued functions on M.

The type of fibre bundles that are of interest to us are *vector bundles*: fibre bundles of which the fibre is a vector space. Two important examples are the *tangent bundle*, the collection of all tangent spaces, and the *cotangent bundle*, the collection of all cotangent spaces:

$$TM \equiv \bigcup_{p \in M} T_p M, \qquad T^*M \equiv \bigcup_{p \in M} T_p^* M. \tag{5.3}$$

The sections of the former are vector fields, while the sections of the latter are one-forms. If M has d real dimensions, the structure group of TM is $GL(d, \mathbb{R})$. A related fibre bundle is,

Definition 5.3. The *tangent frame bundle* FM, associated to the tangent bundle TM, is the bundle over M with as its fibre at each point $p \in M$ the set of ordered bases of the tangent space $T_p M$.

Given a patch $U_\alpha \subset M$ with coordinates x^m, the fibre bundle elements can be projected to the product of base and fibre using a local trivialization: (p, e_a), $p \in U_i$, where the basis e_a may be expanded in the standard basis $\{\partial/\partial x^i\}$ of T_pM as $e_a = e^i{}_a \frac{\partial}{\partial x^i}$. The a-indices are naturally acted upon by elements of $GL(d, \mathbb{R})$. Furthermore, given two local trivializations (p, e_a) and (p, e'_a) on overlapping patches U_α, U_β, we find the following relation, inherited from the associated tangent bundle

$$e'^i{}_a = \frac{\partial x'^i}{\partial x^j} e^j{}_a. \tag{5.4}$$

This relation may be translated to one of the form $e'^i{}_a = e^j{}_b (t_{\beta\alpha}(p))^b{}_a$, with $t_{\beta\alpha}(p) \in Gl(d, \mathbb{R})$, yielding the transition functions. As the elements of the fibre bundle may be considered to be the $d \times d$ matrices $e^i{}_a \in GL(d, \mathbb{R})$, the frame bundle is an example of a *principal fibre bundle*: a bundle of which the fibre is the same as the structure group.

The cases that are of interest are those whereby through a particular choice of local frame on each patch we may introduce a *reduced* tangent frame bundle such that its structure group is a proper subgroup $G \subset GL(d, \mathbb{R})$. The topological properties of M determine whether this is possible or not.

Definition 5.4. A manifold M is said to have a *G-structure* with $G \subset GL(d, \mathbb{R})$ if it is possible to reduce the tangent frame bundle such that it has structure group G. In other words, a G-structure corresponds to a principal sub-bundle of FM with fibre G.

In the extreme case (e.g. $M = \mathbb{R}^d$), where a global frame may be defined, the manifold is said to be *parallelizable*, as through changes of local frames the global section may be made to be constant. Then $G = \{e\}$.

One way to describe G-structures is via one or more G-invariant tensors (or spinors) that are globally defined on M and non-degenerate. For in this case, frames e_a may be chosen on each patch such that these tensors take on the same form on all patches. It then follows that only those transition functions are allowed that leave these tensors invariant, reducing the structure group to G or a subgroup of G.

Typically, the set of G-invariant tensors is not unique, leading to several descriptions of the same G-structure. Also, sometimes multiple tensors are needed to specify a structure exactly. One example is the $SO(d)$-structure defined by a metric and a volume form (see example 5.6).

The G-invariant tensors may be found using representation theory: one starts by considering the representation of $GL(d, \mathbb{R})$ in which the tensor transforms and then one decomposes it into irreducible representations of G; if one (or more) of these is an invariant (i.e. a representation transforming as a scalar under G, such as the volume form in the case of $SO(d)$), that one will be the structure defining, G-invariant tensor we were after.

In the case that the manifold has an $SO(d)$-structure and if its transition functions may be lifted to the double cover $Spin(d)$ in a globally consistent way (as mentioned before, the manifold is then said to be a *spin manifold*), we may also consider spinor bundles. Invariant spinors on such bundles are of our interest since they appeared in the construction of susy parameters. As explained earlier, and as is repeated below, one globally defined spinor reduces an $SO(6)$- (or $Spin(6)$-) structure to an $SU(3)$-structure.

Example 5.5. (*Spinors and $SU(3)$-structures.*) We are considering an orientable, six-dimensional, Riemannian manifold, K_6. As has been remarked several times before, it has the structure group $SO(6)$. The standard four-component spinor representation of the double covering $Spin(6) \cong SU(4)$ is denoted by $\mathbf{4}$. The $\mathbf{4}$ can be decomposed in $SU(3)$ representations as $\mathbf{4} \to \mathbf{3}+\mathbf{1}$, [5]. The $\mathbf{1}$ is a singlet of $SU(3)$ and depends trivially on the tangent bundle. This means that, on a six-dimensional manifold with structure group $SU(3)$, we have a globally defined, non-vanishing spinor. The converse of this statement is true as well.

The above example shows that an $SU(3)$-structure is indeed what we need to be able to properly decompose the susy parameters when compactifying. The following examples show how an number of related structures can be defined in terms of tensors.

Example 5.6. (*Metric and orientation.*) Our first example here is that of a metric tensor g: a globally defined, positive-definite (and thus non-degenerate) symmetric two-tensor $g \in \Gamma(S^2(T^*M))$. The structure group can in this case be reduced to $G = O(d, \mathbb{R})$, by defining local orthonormal frames such that locally (on each patch) $g_{ij} = \delta_{ij}$. The manifold M is now called a Riemannian manifold.

If there is also a globally defined volume-form vol_d associated to g the manifold is orientable and the structure group reduces to $G = SO(d, \mathbb{R})$. This is because the determinant of the $O(d)$-transformations can be fixed to plus or minus one using vol_d.

Example 5.7. (*Almost complex structure.*) A third example is a globally defined almost complex structure J with $J_p : T_pM \to T_pM$, $J_p^2 = -\mathrm{id}_{T_pM}$. It is described in more detail in appendices A.2.3 and A.5. In this case the structure group reduces to $G = GL(d/2, \mathbb{C})$.

As mentioned in the appendices, when we consider such a structure we need to work with the complexification of the tangent bundle: $TM \otimes \mathbb{C}$ (which has structure group $GL(d, \mathbb{C})$).

The $\pm i$ eigenspaces of J_p are denoted $T_pM^{\pm}$. Their bases can be extended locally, for J is supposed to be smooth, to local bases of smooth vector fields (that are elements of $\mathcal{X}^{\pm}(U_\alpha)$). Between patches, the transition functions of G may mix these basis vectors, but as they leave J invariant, they will also leave the decomposition into $TM^{\pm}$ ($\equiv \cup_p T_pM^{\pm}$) invariant.

Furthermore, as J is real, $X \in TM^+$ implies that $\bar{X} \in TM^-$, which in turn implies that the dimensions of the sub-bundles $TM^{\pm} \subset TM \otimes \mathbb{C}$ are equal (and equal to $d/2$). These facts, taken together, imply that the structure group is reduced to $GL(d/2, \mathbb{C})$.

Example 5.8. (*Pre-symplectic structure.*) In the presence of a globally defined non-degenerate two-form $\omega \in \Gamma(\Lambda^2 T^*M)$, the structure group can be reduced to $G = Sp(d, \mathbb{R})$. Note that ω is non-degenerate if and only if $w^{d/2} \neq 0$.

Example 5.9. (*Hermitian metric.*) The last example we mention here is that of the Hermitian metric, which, as described in appendix A.3, satisfies (in index notation) $J^i{}_k g_{ij} J^j{}_l = g_{kl}$, where g is the metric and J an almost complex structure.

In this case the structure group can be reduced to $G = U(d/2) \subset GL(d/2, \mathbb{C})$. Furthermore,

$$\omega_{ij} = g_{ik} J^k_{\ j}, \tag{5.5}$$

is anti-symmetric (this is just the Kähler form, see appendix A.4) so that we have a pre-symplectic structure as well.

Conversely, given a pre-symplectic structure ω and an almost complex structure J that satisfy the compatibility condition

$$J^i_{\ k} \omega_{ij} J^j_{\ l} = \omega_{kl}; \tag{5.6}$$

one may define a metric by

$$g_{ij} = -\omega_{ik} J^k_{\ j}. \tag{5.7}$$

Lastly, given a metric and a pre-symplectic structure, one can define an almost complex structure, so that two out of three imply the third.

An important differential condition on is *integrability*, it is defined in appendix A.5 for almost complex structures (and for distributions in general). In short, in an almost complex structure is integrable, the manifold is complex.

A pre-symplectic structure is integrable if $d\omega = 0$ and is then called a *symplectic structure*. We will not provide a description in terms of distributions for it, but we will see in section 6 that both the pre-symplectic and the almost complex structure can be described in essentially the same way.

5.2 Forms

The spaces of real and complex forms we denote as follows:

$$\Gamma(\Lambda^l T^* M) \equiv \Omega^l(M, \mathbb{R}), \qquad \Gamma(\Lambda^l T^* M) \otimes \mathbb{C} \equiv \Omega^l(M, \mathbb{C}). \tag{5.8}$$

The goal of this section is to describe the G-structures mentioned in the previous section in terms of forms. The first example we will discuss is that of the almost complex structure.

Example 5.10. (*Almost complex structure.*) Earlier we saw that an almost complex structure J divides the complexified tangent bundle $TM \otimes \mathbb{C}$ up into two sub-bundles $TM^{\pm}$. Similarly, it splits the cotangent bundle into two sub-bundles of dimensionality $d/2$,

$$\Lambda^1 T^* M \otimes \mathbb{C} = \Lambda^{1,0} T^* M \oplus \Lambda^{0,1} T^* M, \tag{5.9}$$

where $\theta \in \Lambda^{1,0} T^* M$ if and only if $\theta(X) = 0, \forall X \in TM^+$, analogously for $\Lambda^{0,1} T^* M$ (see definition A.5). This leads to the decomposition of forms into forms of bidegrees (p, q). The sets of smooth (p, q)-forms are denoted $\Omega^{p,q}(M) = \Gamma(\Lambda^{p,q} T^* M)$.

Now, we may define a local frame of $d/2$ independent $(1, 0)$-forms θ^i, and a corresponding local section $\Omega \in \Lambda^{d/2,0} T^* M$, as:

$$\Omega = \theta^1 \wedge \cdots \wedge \theta^{d/2}. \tag{5.10}$$

So, in the presence of an almost complex structure, we can define this particular form Ω. Conversely, using Ω we can construct TM^- and thus TM^+ and J. TM^- can be defined as[30]

$$TM^- = \{X \in TM | \imath_X \Omega = 0\}. \tag{5.11}$$

Note however that the θ^a are determined only up to a $GL(d/2, \mathbb{C})$-transformation, by an almost complex structure. This means that Ω is determined up to a complex function. Between patches, the $GL(d/2, \mathbb{C})$ transition functions may change this complex factor, which implies that an almost complex structure does not require the existence of a globally defined $(d/2, 0)$-form. Requiring a globally defined Ω reduces the structure group to $SL(d/2, \mathbb{C})$.

The form Ω has to be *decomposable* (or *simple*): locally it should be possible to write it as a wedge product of one-forms. In $d = 6$[31] this condition is easily verified by constructing it as follows. We take a real three-form ρ and set

$$\tilde{J}^i_{\ j} = \epsilon^{ii_1 \ldots i_5} \rho_{j i_1 i_2} \rho_{i_3 i_4 i_5}. \tag{5.12}$$

The Hitchin function

$$H(\rho) \equiv \sqrt{-q(\rho)} \equiv \left(-\tfrac{1}{6} \operatorname{tr} \tilde{J}^2\right)^{\frac{1}{2}}, \tag{5.13}$$

should be non-zero (ρ is then called *stable*). In the case $q(\rho) < 0$ a complex decomposable form may be constructed, and in the case $q(\rho) > 0$ this form will be real. Supposing the former, we can define the almost complex structure by

$$J = \pm \tilde{J}/H(\rho), \tag{5.14}$$

and the complex decomposable form by

$$\operatorname{Re}\Omega = \rho, \qquad \operatorname{Im}\Omega = \hat{\rho} = \tfrac{1}{6} J^i_{\ j} (\imath_i \wedge dx^j - dx^j \wedge \imath_i)\rho. \tag{5.15}$$

We see that J and $\operatorname{Im}\Omega$ are determined (up to sign) by the real part of Ω ($= \rho$).

In appendix A.5 the integrability of J was defined by $\forall X, Y \in \mathcal{X}(M)^{\pm} \Rightarrow [X, Y] \in \mathcal{X}(M)^{\pm}$. As $TM^- = \{X \in TM | \imath_X \Omega = 0\}$, we have the equivalent condition $\forall X, Y \in \mathcal{X}(M)^- \Rightarrow \imath_{[X,Y]}\Omega = 0$. That is, J is integrable if and only if[32]

$$\imath_{[X,Y]}\Omega = \imath_Y \imath_X d\Omega = 0, \ \forall X, Y \in \mathcal{X}(M)^-. \tag{5.16}$$

This means that $d\Omega$ should be of type $(3,1)$ ($d\Omega = \theta \wedge \Omega$, for some one-form θ).

Example 5.11. (*Hermitian pre-symplectic or $U(d/2)$-structure.*) As discussed before in example 5.9, this structure is partially defined by a pre-symplectic form $\omega \in \Omega^2(M, \mathbb{R})$. Actually, $\omega \in \Omega^{1,1}(M)$, if ω satisfies the condition of equation (5.6). As ω is real, this may be reformulated as

$$\omega \wedge \Omega = 0. \tag{5.17}$$

---

[30]Note: $\imath_X : \Omega^l(M) \to \Omega^{l-1}(M)$ is called the *interior product* and is defined by $\imath_X \phi(Y_1, \ldots, Y_{l-1}) = \phi(X, Y_1, \ldots, Y_{l-1})$.

[31]In general dimension this construction does not necessarily work for every real form ρ with $q(\rho) < 0$, [9].

[32]The *Lie derivative* of forms, $\mathcal{L}_X : \Omega^l(M) \to \Omega^l(M)$ is given by $\mathcal{L}_X = \{\imath_X, d\} = \imath_X\, d + d\, \imath_X$. We then have $\imath_{[X,Y]} = [\mathcal{L}_X, \imath_Y]$.

Example 5.12. (*$SU(d/2)$-structure.*) An almost complex structure and a compatible pre-symplectic structure reduce the structure group to $U(d/2)$ (example 5.9). A globally defined $(d/2, 0)$-form Ω associated to the almost complex structure further reduces it to $SU(d/2)$. Expressed in terms of forms these statements become: if there is a globally defined, decomposable, complex $d/2$-form Ω that is non-degenerate everywhere:

$$\Omega \wedge \bar{\Omega} \neq 0, \tag{5.18}$$

and a compatible, non-degenerate two-form ω such that $\omega \wedge \Omega = 0$ and such that the associated (Hermitian) metric is positive definite, the structure group reduces to $SU(d/2)$.

Ω is usually normalized such that

$$\Omega \wedge \bar{\Omega} = (-1)^{\frac{d(d-2)}{8}} \frac{(2i\omega)^{d/2}}{(d/2)!}. \tag{5.19}$$

Using this normalization, a local basis of $(1,0)$-forms θ^a can be found so that J and Ω take on the standard form

$$J = \tfrac{i}{2} \sum_a \theta^a \wedge \bar{\theta}^a, \qquad \Omega = \theta^1 \wedge \cdots \wedge \theta^{d/2}. \tag{5.20}$$

5.2.1 $SU(3)$-structure and torsion classes

Remark 5.13. As a reminder, let us again consider the decomposition of 1- and l-forms in the presence of an almost complex structure (appendix A.2.4):

$$\begin{aligned} \Lambda^1 T^*M &= \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M, \\ \Lambda^l T^*M &= \bigoplus_{0 \leq p \leq l} \Lambda^{p,l-p}T^*M. \end{aligned}$$

This, along with the definition of the exterior derivative d implies, that for a (p,q)-forms $\phi \in \Omega^{p,q}(M)$ (see proposition A.6 and its footnotes):

$$d\phi^{p,q} \in \Omega^{p+2,q-1}(M) \cup \Omega^{p+1,q}(M) \cup \Omega^{p,q+1}(M) \cup \Omega^{p-1,q+2}(M). \tag{5.21}$$

This collapses to $d\phi^{p,q} \in \Omega^{p+1,q}(M) \cup \Omega^{p,q+1}(M)$ for complex manifolds (i.e. for integrable almost complex structures). We will use this decomposition in what follows.

In example 5.12 we found the necessary form ingredients for an $SU(3)$-structure in $d = 6$. The exterior derivatives of these forms may be decomposed in $SU(3)$-representations as follows, using the so-called *torsion classes* W_i.

$$\begin{aligned} d\omega &= -\tfrac{3}{2}\mathrm{Im}(\bar{W}_1 \Omega) + W_4 \wedge \omega + W_3, \\ d\Omega &= W_1 \omega^2 + W_2 \wedge \omega + \bar{W}_5 \wedge \Omega. \end{aligned} \tag{5.22}$$

Here, W_1 is a complex scalar, W_2 a complex primitive $(1,1)$-form, W_3 a real primitive $(2,1)+(1,2)$-form, W_4 a real one-form, and W_5 a complex $(1,0)$-form.

In order to illustrate why this decomposition makes sense, consider the decomposition of $d\omega$: it consists of a $(3,0) + (0,3)$-part and a $(2,1) + (1,2)$-part.

The former is described by W_1 and transforms in the $\mathbf{1}+\mathbf{1}$ of $SU(3)$. As for the latter: a $(2,1)$-form transforms under $SU(3)$ as

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{6}+\bar{\mathbf{3}}, \tag{5.23}$$

and a $(2,1)+(1,2)$-form as $\mathbf{6}+\bar{\mathbf{6}}+\mathbf{3}+\bar{\mathbf{3}}$; the $(\mathbf{3}+\bar{\mathbf{3}})$-part is described by the real one-form W_4 and the $(\mathbf{6}+\bar{\mathbf{6}})$-part by W_3, which must satisfy the primitiveness condition $W_3\wedge\omega=0$ in order to remove its $(\mathbf{3}+\bar{\mathbf{3}})$-component.

Similarly, $d\Omega$ consists of a $(3,1)$-part and a $(2,2)$-part. The first of these is described by W_5 which transforms as $\bar{\mathbf{3}}$. The second transforms as $\bar{\mathbf{3}}\otimes\mathbf{3}=\mathbf{8}+\mathbf{1}$ and is described by the primitive W_2 ($W_2\wedge\omega\wedge\omega=0$) and W_1 respectively. As the W_i appearing in $d\Omega$ are complex, the representations count twice.

The conclusions we can draw from the decompositions of $d\omega$ and $d\Omega$ are that (see appendix A):

- if $W_1=W_2=0$, the manifold is complex, as the almost complex structure is integrable in this case,
- if $W_1=W_3=W_4=0$, then $d\omega=0$ and the manifold is symplectic,
- if $W_1=W_2=W_3=W_4=0$, the manifold is both complex and symplectic and thus Kähler, ω is then called the Kähler form and the manifold has $U(3)$-holonomy,
- if all W_i are zero, the holonomy group reduces to $SU(3)$ and the manifold is Calabi-Yau. The holonomy does not change under the transformation $(\omega,\Omega)\to(e^A\omega,e^A\Omega)$; the transformed forms define what is called a *conformal* Calabi-Yau manifold. For the latter class of manifolds: $W_1=W_2=W_3=0$ and $\frac{1}{2}W_4=\frac{1}{3}W_5=-dA$.

The list above shows why the torsion classes are useful: they allow for a clear classification of different types of (complex) manifolds. For Calabi-Yau manifolds we have the following theorem[33].

Theorem 5.14. (Calabi-Yau.) On a compact Kähler manifold M of dimension d with Kähler form $\tilde{\omega}$ and complex structure J, for which there exists a globally defined, nowhere-vanishing $(d/2,0)$-form Ω, there is a unique metric with Kähler form ω in the same Kähler class as $\tilde{\omega}$ (i.e. $\tilde{\omega}=\omega+d\alpha$, for some function α) such that $(\omega,f\Omega)$, for some appropriate normalisation function f, is Calabi-Yau.

This theorem states that metrics exist for Calabi-Yau manifolds; these are, however, except in the case of tori, not analytically known, [9].

One thing we did not yet mention is the reason why the W_i are called *torsion classes*; they have that name because they derive from the torsion part of covariant derivatives. We will say a little more on this in section 5.3.1. In that and the subsequent section we will also briefly discuss their role in flux compactifications.

[33] The existence of a globally defined, nowhere-vanishing $(d/2,0)$-form is equivalent to the vanishing of the first Chern class $c_1(M)$. It is in terms of the vanishing of $c_1(M)$ that Calabi-Yau manifolds are defined in appendix A.4.

5.3 Spinors

The last way we define an $SU(d/2)$-structure is in terms of invariant spinors. $GL(d,\mathbb{R})$ does not have a spinor representation, so we will need to work with spin manifolds. As remarked earlier, these manifolds have a structure group $SO(d,\mathbb{R})$ that can be consistently lifted to its double cover $Spin(d,\mathbb{R})$.

As a slight variation of the discussion in section 3.1.1, we introduce the vielbein as follows

$$g_{ij} = \delta_{ab} e^a_{\ i} e^b_{\ j}. \tag{5.24}$$

A spinor field $\eta(x)$ is then a field which transforms under an infinitesimal rotation, $\omega \in so(d,\mathbb{R})$, as

$$\delta\eta = \tfrac{1}{4}\omega^{ab}\gamma_{ab}\eta, \tag{5.25}$$

where for the gamma matrices γ_a: $\{\gamma_a,\gamma_b\} = 2\delta_{ab}$ and $\gamma_{ab} = [\gamma_a,\gamma_b]$; the latter form a basis for the Lie algebra $spin(d) \cong so(d)$. The curved gamma matrices are again defined as $\gamma_i = e^a_{\ i}\gamma_a$. Contractions of a form ϕ involving gamma matrices we denote as we did in section 3.3.2:

$$\underline{\phi} \equiv \frac{1}{l!}\phi_{i_1\ldots i_l}\gamma^{i_1\ldots i_l} = \frac{1}{l!}\phi_{i_1\ldots i_l} e^{i_1}_{\ a_1}\cdots e^{i_l}_{\ a_l}\gamma^{a_1\ldots a_l}. \tag{5.26}$$

For manifolds of even dimension, a globally defined, invariant, pure[34] spinor η (, along with its complex conjugate $\eta' \equiv C\eta^*$, where C is the charge conjugation matrix, $\gamma_a^* = -C^{-1}\gamma_a C$), reduces the structure group to $SU(d/2)$. For the case $d=6$ we saw this explicitly in example 5.5.

The forms ω and Ω can be constructed using the invariant spinor and its complex conjugate as follows

$$\omega_{ij} = i\eta^\dagger\gamma_{ij}\eta, \qquad \Omega_{i_1\ldots i_{d/2}} = \eta'^\dagger\gamma_{i_1\ldots i_{d/2}}\eta. \tag{5.27}$$

For $d=6$ we have that for the chirality operator $\gamma_7 = -C^{-1}\gamma_7 C$ so that η and η' have different chirality; for this reason we label them as $\eta = \eta_+$ and $\eta' = \eta_-$.

5.3.1 The torsion classes revisited

The class of manifolds we are considering (but also more generally) always admit a covariant derivative with torsion, ∇_T, that satisfies

$$\begin{aligned} \nabla_T g &= 0,\\ \nabla_T\eta_+ &= \nabla_{LC}\eta_+ + T\eta_+ = 0, \end{aligned} \tag{5.28}$$

where ∇_{LC} is the Levi-Civita covariant derivative[35]. The difference between the two covariant derivatives, T, is called the *contorsion tensor.* In Riemannian geometry, there is a one-to-one correspondence between the contorsion tensor T

[34]A spinor η is *pure* if $\gamma_a\eta = 0$ for half of the $\{\gamma_a\}$, which is the maximal amount. In $d=6$ any Weyl spinor is pure.

[35]The Levi-Civita covariant derivative is the unique one for which $\nabla_{LC}g = 0$ and which is torsion-free (i.e. $T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y] = 0$ for all vector fields X, Y).

and the torsion of ∇_T[36]. The contorsion tensor decomposes as follows in $SU(3)$ representations:

$$T_{ab}{}^c \in (su(3) + su(3)^\perp) \otimes V. \tag{5.29}$$

The indices ab span the space of anti-symmetric matrices, which is isomorphic to $so(6)$, the adjoint representation of which splits up into $su(3) \oplus su(3)^\perp$. The upper index transforms as a vector (indicated by V). Acting on an $SU(3)$-invariant form, the $su(3)$ part drops out. The remaining piece contains

$$\begin{aligned}(su(3)^\perp) \otimes V &= (\mathbf{1} + \mathbf{3} + \bar{\mathbf{3}}) \otimes (\mathbf{3} + \bar{\mathbf{3}}) \\ &= (\mathbf{1} + \mathbf{1}) + (\mathbf{8} + \mathbf{8}) + (\mathbf{6} + \bar{\mathbf{6}}) + 2(\mathbf{3} + \bar{\mathbf{3}}), \\ &\quad\; W_1 \qquad\quad W_2 \qquad\quad W_3 \qquad\quad W_4, W_5.\end{aligned} \tag{5.30}$$

The $SU(3)$ representations above correspond to the five torsion classes W_i that appear in the decompositions of $d\omega$ and $d\Omega$. The exact relations between the contorsion components and the torsion classes may be found using equations (5.22), (5.27), and (5.28).

5.3.2 $\mathcal{N} = 1$ supersymmetry

At the end of section 4.2 we concluded that using the following decomposition of the susy parameters ϵ^A,

$$\begin{aligned}\epsilon^1 &= \xi^1_+ \otimes \eta_+ + \xi^1_- \otimes \eta_-, \\ \epsilon^2 &= \xi^2_+ \otimes \eta_- + \xi^2_- \otimes \eta_+, \qquad \text{(IIA)} \\ \epsilon^A &= \xi^A_+ \otimes \eta_+ + \xi^A_- \otimes \eta_-, \qquad \text{(IIB)}\end{aligned}$$

leads to $\mathcal{N} = 2$ spacetime supersymmetry, with parameters ξ^A. As we are interested in the case $\mathcal{N} = 1$ we take the ξ^A to be proportional[37] to each other and use instead, [8]:

$$\begin{aligned}\epsilon^1 &= \xi_+ \otimes (a\eta_+) + \xi_- \otimes (\bar{a}\eta_-), \\ \epsilon^2 &= \xi_+ \otimes (\bar{b}\eta_-) + \xi_- \otimes (b\eta_+),\end{aligned} \tag{5.31}$$

for IIA; for IIB

$$\begin{aligned}\epsilon^1 &= \xi_+ \otimes (a\eta_+) + \xi_- \otimes (\bar{a}\eta_-), \\ \epsilon^2 &= \xi_+ \otimes (b\eta_+) + \xi_- \otimes (\bar{b}\eta_-),\end{aligned} \tag{5.32}$$

where a and b are complex functions of the internal coordinates y^m.

In the susy variations, the spinors $\xi_\pm$ can be factored out. The other parts should then equal zero (as $\langle\psi\rangle = \langle\lambda\rangle = 0$); they are, rewritten using the basis $\{\eta_+, \gamma^m\eta_-, \gamma^m\eta_+, \eta_-\}$, schematically given by

$$\begin{aligned}\delta\psi_\mu : &\quad P\eta_+ + P_m\gamma^m\eta_- = 0, \\ \delta\psi_m : &\quad Q_m\eta_+ Q_{mn}\gamma^n\eta_- = 0, \\ \delta\lambda : &\quad R\eta_+ + R_m\gamma^m\eta_- = 0.\end{aligned} \tag{5.33}$$

[36] In fact, the connection coefficients Γ^k_{ij}, when separated into a symmetric, $\Gamma^k_{(ij)}$, and an anti-symmetric part $\Gamma^k_{[ij]}$, correspond to the usual Christoffel symbols and the contorsion tensor, respectively.

[37] Maximal four-dimensional Poincaré symmetry leaves us with no other option.

Here, P, Q, and R contain contributions from the *torsion* (as a covariant derivative appears in the susy variations), the NS and R-R fluxes, the warp factor and derivatives of the functions a and b. We used that η_+ is a $Spin(6)$ vacuum: it is annihilated by the γ^m, $\gamma^m \eta_+ = 0$.

Through these equations, the functions a and b, the torsion classes W_i, and the fluxes are linked. Their solutions are quoted in appendix B and are given in terms of the conditions on the fluxes and the torsion classes for given values of a and b. The conditions on the torsion classes W_i reveal what kinds of internal manifolds are allowed in $\mathcal{N} = 1$ flux compactifications.

6 Generalized Complex Geometry

The formalism of Generalized Complex Geometry was introduced by N. Hitchin and M. Gualtieri. In [10], Hitchin states that their research in this area was part of a programme "for characterizing special geometry in low dimensions by means of invariant functionals of differential forms." As it turned out, the formalism proved useful for describing background geometries in string theory. P. Koerber names two applications at the beginning of his lecture notes, [9]: "the description of supersymmetric flux compactifications and the supersymmetric embedding of D-branes". The former of the two is the one we will try to describe in some detail.

In section 5 and appendix B we found that in flux compactifications, the internal space can be either complex or symplectic or both (i.e. Kähler or Calabi-Yau). Previously, these structures were described in different ways: a symplectic manifold had a two-form ω living on it, while a complex manifold had a complex structure, J. In what follows we will see that both are really just special cases of *generalized complex structures.* We will end our discussion of generalized geometry with a discussion of the susy variations, rewritten in terms of *pure spinors*, and their (generalized) implications.

Useful references are the lecture notes of Koerber, [9], the papers of Hitchin and Gualtieri, and the review paper on flux compactifications by M. Graña, [8].

6.1 The basics

In this first section we describe the basic building blocks of generalized geometry. We more or less adopt the notation of [9], but otherwise the treatment is similar to that of any introduction to the subject (see, e.g. [10] or [11]).

6.1.1 Metric and structure

The name generalized geometry carries the adjective "generalized" perhaps because it does not consider TM, the tangent bundle, but rather $TM \oplus T^*M$, the sum of the tangent and cotangent bundles (or its complexification). This object is referred to as the *generalized tangent bundle.*

On it we naturally have a metric $\mathcal{G}$, which is defined as follows:

$$\mathcal{G}(\mathbf{X}, \mathbf{Y}) \equiv \tfrac{1}{2}\left(\xi(Y) + \eta(X)\right), \tag{6.1}$$

where $\mathbf{X} = X + \xi$ and $\mathbf{Y} = Y + \eta$ are generalized tangent vectors (to avoid saying "generalized" all the time, we simply refer to them as vectors from now on—they will be easily distinguishable from normal vectors as we continue to print them in boldface letters), with $X, Y \in TM$ and $\xi, \eta \in T^*M$. Note that it has signature (d, d), so that the structure group reduces to $O(d, d)$. A naturally associated volume form, $\mathrm{vol}_{\mathcal{G}} \in \Gamma(\Lambda^{2d}(TM \oplus T^*M))$, may be defined by

$$\mathrm{vol}_{\mathcal{G}} = \frac{1}{(d!)^2} \epsilon^{i_1 \dots i_d} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_d}} \wedge \epsilon_{j_1 \dots j_d} dx^{j_1} \wedge \dots \wedge dx^{j_d}. \tag{6.2}$$

As the fully antisymmetric tensor ϵ appears twice, this form does not depend on the choice of orientation on M. This reduces the structure group further, to $SO(d, d)$.

The structure group is generated by elements of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \qquad e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \qquad e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}. \tag{6.3}$$

Here, $A \in GL(d, \mathbb{R})$, the structure group of the tangent bundle; B corresponds to a two-form and β to an anti-symmetric "two-vector". The action of the B- and β-*transforms*, e^B and e^β, is given by

$$\begin{aligned} e^B &: \mathbf{X} = X + \xi \to X + (\xi - \imath_X B), \\ e^\beta &: \mathbf{X} \to (X - \imath_\xi \beta) + \xi. \end{aligned} \tag{6.4}$$

They obviously leave the metric invariant.

For vector fields we define a substitute for the Lie bracket.

Definition 6.1. For two vector fields $\mathbf{X} = X + \xi$, $\mathbf{Y} = Y + \eta \in \Gamma(TM \oplus T^*M)$ the *Courant bracket* is given by[38]

$$[\mathbf{X}, \mathbf{Y}]_C \equiv [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \tfrac{1}{2} d(\imath_X \eta - \imath_Y \xi). \tag{6.5}$$

Note that the Courant bracket is not a Lie bracket as it does not satisfy the Jacobi identity. It does, however, reduce to the Lie bracket if the one-form components of the fields $\mathbf{X}, \mathbf{Y}$ are zero. Furthermore, just as the Lie bracket, it is invariant under diffeomorphisms (the endomorphisms of TM, i.e. under the action of A as defined in (6.3)). The Courant bracket is also invariant under B-transforms with closed B ($dB = 0$).

6.1.2 Spinors

In generalized geometry, the role of differential forms changes. They become a *Clifford module* for the Clifford algebra defined by $TM \oplus T^*M$ and $\mathcal{G}$, [10]. This follows directly if we define the action of a vector $\mathbf{X} = X + \xi$ on a (*poly-*) form $\varphi \in \Omega^\bullet(M)$ (a polyform is defined as a sum of forms of different degrees) as follows

$$\mathbf{X} \cdot \phi = \imath_X \varphi + \xi \wedge \varphi. \tag{6.6}$$

This action leads to:

$$\{\mathbf{X}, \mathbf{Y}\} \cdot \varphi = (\mathbf{X} \cdot \mathbf{Y} + \mathbf{Y} \cdot \mathbf{X}) \cdot \varphi = 2\mathcal{G}(\mathbf{X}, \mathbf{Y})\varphi. \tag{6.7}$$

And so we find that the forms are acted on by a Clifford algebra, i.e. $\Omega^\bullet(M)$ is a module for the Clifford algebra[39].

[38] A useful formula is the Cartan formula for the Lie derivative of a differential form α: $\mathcal{L}_X \alpha = d(\imath_X \alpha) + \imath_X d\alpha$, [11]. It was given earlier in footnote 32 of section 5.2.

[39] Any set of orthonormal vectors that form a basis with respect to $\mathcal{G}$ (i.e. a basis such that $\mathcal{G}$ is diagonal with ± 1's on the diagonal) correspond to a set of gamma matrices.

Polyforms do, however, not quite correspond to actual spinors. Spinors transform under $Spin(d,d)$, the lift of the structure group $SO(d,d)$ (this lift is possible whenever M is oriented). The spin representation is given by

$$S = \Omega^{\bullet}(M) \otimes (\Lambda^n T^* M)^{\frac{1}{2}}. \tag{6.8}$$

This means that there is an isomorphism between spinors and polyforms defined by a particular choice of volume form $\epsilon \in \Lambda^n T^* M$:

$$\varphi = \epsilon^{\frac{1}{2}} \varphi_s, \tag{6.9}$$

where φ_s is a spinor. It also means that there is an invariant bilinear form on $\Omega^{\bullet}(M)$ taking values in the line bundle $\Lambda^n T^* M$. This form is called the *Mukai pairing* and is defined below.

The irreducible representations of $Spin(d,d)$ are Majorana-Weyl; the Majorana condition defines the spinors to be real, while the Weyl condition restricts polyforms to be either of only odd or even degrees (corresponding to negative or positive chirality). They are

$$\begin{aligned} S^{+} &= \Omega^{even}(M) \otimes (\Lambda^n T^* M)^{\frac{1}{2}}, \\ S^{-} &= \Omega^{odd}(M) \otimes (\Lambda^n T^* M)^{\frac{1}{2}}. \end{aligned} \tag{6.10}$$

Definition 6.2. The *Mukai pairing* of two polyforms φ_1, φ_2 is given by

$$\langle \varphi_1, \varphi_2 \rangle = \sum_j (-1)^j \left(\varphi_1^{2j} \wedge \varphi_2^{d-2j} + \varphi_1^{2j+1} \wedge \varphi_2^{d-2j-1} \right), \tag{6.11}$$

where the superscript p denotes the p-form component of the form. Defining $\sigma : \Omega^{ev/odd}(M) \to \Omega^{ev/odd}(M)$ by

$$\sigma(\varphi^{2m}) = (-1)^m \varphi^{2m} \qquad \sigma(\varphi^{2m+1}) = (-1)^m \varphi^{2m+1}, \tag{6.12}$$

the bracket becomes

$$\langle \varphi_1, \varphi_2 \rangle = (\sigma(\varphi_1) \wedge \varphi_2)^d. \tag{6.13}$$

One can check that the result of a pairing transforms as a top($/d$)-form under $Spin(d,d)$. Furthermore, the pairing is invariant under B-transforms:

$$\langle e^B \varphi_1, e^B \varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle. \tag{6.14}$$

In fact, the Mukai pairing is the map of the usual bilinear form of spinors, [9]:

$$\varphi_{s1}^T C \varphi_{s2}, \tag{6.15}$$

under the isomorphism of equation (6.9) (C is the charge conjugation matrix). This implies the pairing's property

$$C = (-1)^{\frac{d(d-1)}{2}} C^T \Leftrightarrow \langle \varphi_1, \varphi_2 \rangle = (-1)^{\frac{d(d-1)}{2}} \langle \varphi_2, \varphi_1 \rangle, \tag{6.16}$$

so that for $d = 6$ the pairing is anti-symmetric. Two other properties are:

$$\begin{aligned} \langle \mathbf{X} \cdot \varphi_1, \varphi_2 \rangle &= -(-1)^d \langle \varphi_1, \mathbf{X} \cdot \varphi_2 \rangle, \\ \int_M \langle d_H \varphi_1, \varphi_2 \rangle &= (-1)^d \int_M \langle \varphi_1, d_H \varphi_2 \rangle, \end{aligned} \tag{6.17}$$

where d_H is *the twisted exterior derivative*: $d_H \phi \equiv d\phi + H \wedge \phi$.

From now on, polyforms will be often referred to as spinors, assuming implicitly the isomorphism defined earlier.

6.1.3 Twisted structures

One reason why generalized geometry turned out to be interesting for string theory, is that it manages to incorporate the B-field in a natural way. To achieve this, we introduce the *H-twisted Courant bracket*:

$$[\mathbf{X}, \mathbf{Y}]_H = [\mathbf{X}, \mathbf{Y}]_C + \imath_X \imath_Y H, \tag{6.18}$$

for a closed three-form H that lives on our manifold (as is the case in string theory with the NS-NS form). This bracket now satisfies

$$[e^B \mathbf{X}, e^B \mathbf{Y}]_{H-dB} = e^B [\mathbf{X}, \mathbf{Y}]_H. \tag{6.19}$$

From this property we see that we have a choice to either describe our geometry ($TM \oplus T^*M$) in terms of the twisted bracket, or to locally untwist it by twisting our geometry, i.e. by locally B-transforming our vectors.

Locally untwisting means finding a two-form B_α on each patch U_α such that $H|_\alpha = dB_\alpha$ and B-transforming the vectors of $TU_\alpha \oplus T^*U_\alpha$. As B functions as a potential it may be shifted by a gauge transform $B_\alpha \to B_\alpha + d\Lambda_\alpha$ without changing the physical field H. Also, since B is not necessarily globally defined we need to allow for a gauge transformation on the overlap of two patches, $U_\alpha \cap U_\beta$,

$$B_\alpha = B_\beta + d\Lambda_{\alpha\beta}. \tag{6.20}$$

Consistency requires that on $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = d\Lambda_{\alpha\beta\gamma}. \tag{6.21}$$

In the following, we will not try to twist the geometry and untwist the H-twisted bracket. The theorems and definitions of the following sections are the same for both pictures, except that in them, the bundle $TM \oplus T^*M$ should be replaced by the twisted bundle E if one chooses to work in the untwisted-bracket-picture.

From the preceding discussion we conclude that the structure group of E is not $SO(d,d)$, as the β-transforms are not symmetries of the Courant bracket. The full structure group is

$$GL(d, \mathbb{R}) \rtimes \Omega^2(M)_{closed}, \tag{6.22}$$

the semi-direct product of the group of diffeomorphisms and the group of B-transforms with B closed. It is this group that leaves both the metric and the Courant bracket invariant.

Remark 6.3. In [11] it is shown that a B-field acts on spinors as follows

$$\varphi \to e^{-B\wedge} \varphi. \tag{6.23}$$

Now, on E, as on $TM \oplus T^*M$ earlier, we have a spin bundle S, since E has an orthogonal structure. A global section ψ of S is given by local forms $\varphi_\alpha, \varphi_\beta$ such that

$$\varphi_\alpha = e^{-d\Lambda_{\alpha\beta}} \varphi_\beta, \tag{6.24}$$

or

$$\psi = e^{-B_\alpha}\varphi_\alpha. \tag{6.25}$$

The operator d on the forms $\psi \in S$ is then equivalent to the *H-twisted operator* d_H, which we encountered at the end of the previous section, on the original untwisted forms. This follows from

$$d\psi = d(e^{-B_\alpha}\varphi_\alpha) = -H \wedge \psi + e^{-B_\alpha} d\varphi_\alpha, \tag{6.26}$$

and it is the operator we need to use in the twisted-bracket-picture. Note that $d_H^2 = 0$ as H is closed.

6.2 Generalized complex structures

In this section we extend the concept of (almost) complex structures to generalized geometry.

6.2.1 Generalized (almost) complex structures

Definition 6.4. A *generalized almost complex structure* is a map

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M, \tag{6.27}$$

that respects the bundle structure: $\pi(\mathcal{J}\mathbf{X}) = \pi(\mathbf{X})$, and that satisfies the following two conditions; first, its square should be minus the identity:

$$\mathcal{J}^2 = -1, \tag{6.28}$$

and second, the canonical metric $\mathcal{G}$ should be Hermitian:

$$\mathcal{G}(\mathcal{J}\mathbf{X}, \mathcal{J}\mathbf{Y}) = \mathcal{G}(\mathbf{X}, \mathbf{Y}). \tag{6.29}$$

Such a structure reduces the structure group from $SO(d,d)$ to $U(d/2, d/2)$. Furthermore, it defines, as in the case of the normal almost complex structure, two sub-bundles $L^\pm \subset (TM \oplus T^*M) \otimes \mathbb{C}$ with fibres the $(\pm i)$-eigenspaces of $\mathcal{J}$ at each point.

Definition 6.5. A sub-bundle L is *isotropic* if

$$\mathcal{G}(\mathbf{X}, \mathbf{Y}) = 0, \quad \text{for all } \mathbf{X}, \mathbf{Y} \in L. \tag{6.30}$$

It is *maximally isotropic* if its rank is half of the rank of $TM \oplus T^*M$ (i.e. d), which is the maximal rank an isotropic sub-bundle can have for a non-degenerate $\mathcal{G}$ [11].

Obviously, the $L^\pm$ introduced above, are maximally isotropic. The specification of a maximally isotropic sub-bundle L satisfying $L \cap \bar{L} = 0$ is in fact equivalent to the introduction of a generalized almost complex structure $\mathcal{J}$. The correspondence being, of course, $L = L^+$.

The integrability of $\mathcal{J}$ is defined as follows.

Definition 6.6. $\mathcal{J}$ is *H-integrable*, making it into a *generalized complex structure*, if L^+ is involutive under the H-twisted Courant bracket:

$$[\mathbf{X}, \mathbf{Y}]_H \in \Gamma(L^+), \quad \text{for all } \mathbf{X}, \mathbf{Y} \in \Gamma(L^+). \tag{6.31}$$

In fact, there does exist a theorem analogous to the Frobenius theorem, stating that if a generalized complex structure is involutive, adapted coordinates can be found. We will however not go into this further, other than remarking the following theorem.

Theorem 6.7. (The generalized Darboux theorem.) Any regular point in a manifold with an H-twisted integrable generalized complex structure has a neighbourhood that is equivalent, via a diffeomorphism and a B-transform for which $dB = H$, to the product of an open set in $\mathbb{C}^k$, described by complex coordinates, and an open set in the standard symplectic space $(\mathbb{R}^{d-2k}, \omega_0)$, described by Darboux coordinates. The number k is the type of the associated pure spinor, a number we define in the following section.

This theorem basically states that, in adapted coordinates, a generalized complex structure interpolates between a complex structure and a symplectic structure. The latter two are just special cases of generalized complex structures.

Example 6.8. (*Almost complex structure.*) Using an ordinary almost complex structure we can construct a generalized version as follows

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix}. \tag{6.32}$$

It follows that $\mathcal{J}$ is H-integrable if and only if J is integrable and H is of type $(2,1) \oplus (1,2)$.

Example 6.9. (*Symplectic structure.*) Similarly, using a symplectic structure ω, we may construct a generalised complex structure, defining

$$\mathcal{J} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \tag{6.33}$$

$\mathcal{J}$ is H-integrable if and only if $d\omega = 0$ and $H = 0$.

6.2.2 Pure spinors

The definition of generalized almost complex structures can be easily recast in terms of so-called *pure spinors*. To do so, we first define a spinor's *null space*.

Definition 6.10. The *null space* of a spinor φ is defined to be the sub-bundle L_φ consisting of all annihilators of φ:

$$L_\varphi = \{\mathbf{X} | \mathbf{X} \cdot \varphi = 0\}. \tag{6.34}$$

The null space is an isotropic bundle, since

$$2\mathcal{G}(\mathbf{X}, \mathbf{Y})\varphi = (\mathbf{X} \cdot \mathbf{Y} + \mathbf{Y} \cdot \mathbf{X}) \cdot \varphi = 0 \Rightarrow \mathcal{G}(\mathbf{X}, \mathbf{Y}) = 0, \forall \mathbf{X}, \mathbf{Y} \in L_\phi. \tag{6.35}$$

Definition 6.11. A spinor φ is called *pure* if its null space L_φ is maximally isotropic.

In summary, a (complex) pure spinor defines a maximally isotropic bundle, its null space $L_\varphi \subset (TM \oplus T^*M) \otimes \mathbb{C}$, which may be identified as the L^+ of the previous section. This definition determines the complex pure spinor up to a multiplicative factor, which may vary from patch to patch.

This conclusion is similar to that of section 5.3, where we stated that pure spinors of $Spin(d)$ could be related to almost complex structures.

The integrability condition on a pure spinor is given in the following theorem.

Theorem 6.12. A generalized almost complex structure $\mathcal{J}$ is integrable if and only if its associated pure spinor φ satisfies

$$\langle\varphi,\bar{\varphi}\rangle \neq 0, \qquad d_H\varphi = \mathbf{X}\cdot\varphi, \tag{6.36}$$

for some $\mathbf{X} \in (TM \oplus T^*M)\otimes\mathbb{C}$.

The first condition is equivalent to $L^+ \cap L^- = 0$.

Proof. Since d_H is closed, we have that

$$[\mathbf{X},\mathbf{Y}]_H\cdot\varphi = [\{\mathbf{X},d_H\},\mathbf{Y}]\cdot\varphi - d(\mathcal{G}(\mathbf{X},\mathbf{Y}))\wedge\varphi, \tag{6.37}$$

for all sections $\mathbf{X},\mathbf{Y}$ and all polyforms φ. Restricting the discussion to the isotropic null space of a pure spinor, the last term above drops out. For all $\mathbf{X},\mathbf{Y}\in\Gamma(L_\varphi)$, we have:

$$\begin{aligned}[\mathbf{X},\mathbf{Y}]_H\cdot\varphi &= [\{\mathbf{X},d_H\},\mathbf{Y}]\cdot\varphi - d(\mathcal{G}(\mathbf{X},\mathbf{Y}))\wedge\varphi\\ &= \mathbf{X}\cdot\mathbf{Y}\cdot d_H\varphi.\end{aligned} \tag{6.38}$$

This implies that L_φ is involutive if and only if the bottom line above yields zero, which means that $d_H\varphi = \mathbf{Z}\cdot\varphi$ for some $\mathbf{Z}$. (It also means that $d_H\varphi\in\Gamma(U_{d/2-1})$, which will be defined in the following section.) □

If a φ satisfies the conditions stated in the theorem, then $\varphi' = f\varphi$ does so as well for $\mathbf{X}' = \mathbf{X} + df$. We find that the undetermined overall factor does not influence the integrability condition. If the factor may be fixed, for a globally defined, complex pure spinor, the structure group of the generalized tangent bundle reduces to $SU(d/2,d/2)$.

It is natural to consider a somewhat stronger constraint: $\mathbf{X} = 0$.

Definition 6.13. A complex pure spinor φ is *generalized Calabi-Yau à la Hitchin* ([10]) if it satisfies

$$d_H\varphi = 0. \tag{6.39}$$

This condition is, however, not a suitable generalization of the standard Calabi-Yau geometry, as this will involve two generalized almost complex structures. The proper generalization will be discussed further on.

Remark 6.14. (*The general form of pure spinors.*) The origins of the generalized Darboux theorem mentioned in section 6.2.1 become clear when one considers a theorem proved by Gualtieri in his PhD thesis (see the references in [9]). The theorem states that every non-degenerate, pure spinor φ can be written as

$$\varphi = \Omega_k\wedge e^{i\omega+B}, \tag{6.40}$$

where ω and B are real two-forms and Ω_k a decomposable k-form such that

$$\langle\varphi,\bar{\varphi}\rangle = (-1)^{\frac{k(k-1)}{2}}\frac{2^{d/2-k}}{(d/2-k)!}\Omega_k\wedge\bar{\Omega}_k\wedge\omega^{d/2-k}\neq 0. \tag{6.41}$$

The number k is called the *type* of the pure spinor.

Definition 6.15. The *type* of a pure spinor is the lowest form-dimension appearing.

Example 6.16. The pure spinor associated to the generalized almost complex structure $\mathcal{J}$ of example 6.8 is

$$\varphi = f\Omega, \tag{6.42}$$

where f is a nowhere-vanishing complex function and Ω the complex, decomposable three-form associated to the almost complex structure J (see example 5.10). Its null space is given by $\bar{L}_\varphi \oplus \Lambda^{(0,1)}T^*M$, which is also the $(+i)$-eigenspace of $\mathcal{J}$. The spinor's type is 3. The integrability conditions of theorem 6.12 become

$$d\Omega = \bar{\mathcal{W}}_5 \wedge \Omega, \qquad H \wedge \Omega = 0. \tag{6.43}$$

Example 6.17. The pure spinor associated to the $\mathcal{J}$ of example 6.9 is

$$\varphi = fe^{i\omega}, \tag{6.44}$$

for some function f; ω defines a pre-symplectic structure. Its null space is given by $\{X + \xi | \xi = -i\imath_X\omega\}$, which is also the $(+i)$-eigenspace of $\mathcal{J}$. Its type is 0. The integrability conditions become

$$d\omega = 0, \qquad H = 0. \tag{6.45}$$

6.2.3 The Lie algebroid L_φ

It has been shown (see the references in [9]) that the Courant bracket satisfies the Jacobi identity on an integrable and isotropic sub-bundle, so that for the aforementioned L_φ, $(L_\varphi, [\ ,\ ]_H)$ is a Lie algebra. The projection map $\pi_{TM} : TM \oplus T^*M \to TM$ restricted to L_φ is a Lie algebra homomorphism: $\pi_{TM} : L_\varphi \to TM$. The Lie algebra $(L_\varphi, [\ ,\ ]_H)$ is then what is called a *Lie algebroid*. On the forms of L_φ one can now define a so-called Lie algebroid derivative d_{L_φ} in way similar to the usual exterior derivative on forms.

Definition 6.18. Given a Lie algebroid $(L, \pi_{TM}, [\ ,\ ]_H)$, the *Lie algebroid derivative* $d_L : \Gamma(\Lambda^l L^*) \to \Gamma(\Lambda^{l+1} L^*)$ acts on a l-form α and produces a $(l+1)$-form $d_L\alpha$ as follows:

$$\begin{aligned} d_L\alpha(\mathbf{Y}_0, \ldots, \mathbf{Y}_l) = &\sum_{0\le a\le l} (-1)^a \pi_{TM}(\mathbf{Y}_a)\left[\alpha(\mathbf{Y}_0, \ldots, \hat{\mathbf{Y}}_a, \ldots, \mathbf{Y}_l)\right] \\ &+ \sum_{0\le a<b\le l} (-1)^{a+b}\alpha([\mathbf{Y}_a, \mathbf{Y}_b]_H, \mathbf{Y}_0, \ldots, \hat{\mathbf{Y}}_a, \ldots, \hat{\mathbf{Y}}_b, \ldots, \mathbf{Y}_l), \end{aligned} \tag{6.46}$$

for all $\mathbf{Y}_0, \ldots, \mathbf{Y}_l \in \Gamma(L)$.

This derivative is related to the derivative d_H that was defined earlier. How exactly, we find at the end of this subsection.

Now, a complex pure spinor φ and its associated isotropic null space L_φ induce a decomposition of the space of polyforms

$$\Lambda^\bullet T^*M \otimes \mathbb{C} = \bigoplus_{-d/2\le k\le d/2} U_k, \tag{6.47}$$

where

$$U_k = (\Lambda^{d/2-k}\bar{L}_\varphi) \cdot \varphi. \tag{6.48}$$

The last line says that U_k is the sub-bundle of polyforms one gets by acting with an anti-symmetric product of $d/2-k$ generalized vectors $\mathbf{X}_i \in \bar{L}_\varphi$ on φ. In effect, φ provides an isomorphism between

$$\Gamma(\Lambda^{d/2-k}\bar{L}_\varphi) \to U_k \,:\, \alpha \to \alpha \cdot \varphi. \tag{6.49}$$

The type of decomposition is called a *filtration.* One can consider the decomposition as building up the spinor representation by acting with anti-holomorphic gamma-matrices—taking the role of creation operators—on a "null state", [9].

Remark 6.19. An alternative definition of the U_k consists of taking it to be the ik-eigenbundle of $\mathcal{J}$, acting in the spinor representation on forms. In detail, given a local frame $\mathbf{X}_a$, $a = 1, \ldots, 2d$ of the generalised tangent bundle, the action of $\mathcal{J}$ in the spinor representation of an arbitrary spinor ψ is given by

$$\mathcal{J} \cdot \psi = \tfrac{1}{2}\mathcal{G}^{ab}\mathcal{J}(\mathbf{X}_a) \cdot \mathbf{X}_b \cdot \psi, \tag{6.50}$$

where $\mathcal{G}^{ab} \equiv (\mathcal{G}_{ab})^{-1}$, $\mathcal{G}_{ab} \equiv \mathcal{G}(\mathbf{X}_a, \mathbf{X}_b)$.

Note that the original spinor φ we have $\varphi \in U^{d/2}$. For a polyform $\varphi' \in \Gamma(U_k)$ we have $\bar{\varphi}' \in \Gamma(U_{-k})$. The decomposition of polyforms is compatible with the Mukai pairing, for $\varphi_1 \in \Gamma(U_k)$

$$\langle \varphi_1, \varphi_2 \rangle = 0, \quad \text{if} \quad \varphi_2|_{U_{-k}} = 0, \tag{6.51}$$

where $\varphi_2|_{U_{-k}}$ is the projection on U_{-k}.

For the exterior derivative d_H one can prove that

$$d_H : \Gamma(U_k) \to \Gamma(U_{k-3}) \oplus \Gamma(U_{k-1}) \oplus \Gamma(U_{k+1}) \oplus \Gamma(U_{k+3}), \tag{6.52}$$

the right-hand part of which reduces to $\Gamma(U_{k-1}) \oplus \Gamma(U_{k+1})$ in the case that L_φ is integrable. In the latter case d_H splits as

$$d_H = \partial_H + \bar{\partial}_H, \tag{6.53}$$

where both parts behave exactly similar to the Dolbeault operators defined in section A.2.4.

The relation between d_L and d_H follows from the aforementioned isomorphism provided by φ: it maps d_L into $\bar{\partial}_H$,

$$(d_L\alpha) \cdot \varphi = \bar{\partial}_H(\alpha \cdot \varphi). \tag{6.54}$$

6.3 Generalized Kähler structures

In the preceding sections we saw that a globally defined, invariant spinor—a prerequisite for spacetime supersymmetry—puts an $SU(d/2)$-structure on the internal manifold. We have also seen that such a structure corresponds to having both a pre-symplectic and an almost complex structure that satisfy a certain compatibility condition. Both of these turned out to be special cases of generalized almost complex structures. In this section we generalize the notion of an $SU(3)$-structure, which will be defined by two (in some sense) *compatible* generalized almost complex structures, or equivalently by two compatible pure spinors.

6.3.1 Generalized complex structures

First we define the notion of generalized Kähler structures. Ordinary Kähler structures were defined by a complex structure J and symplectic structure ω. These needed to be compatible: ω ought to be of bidegree $(1,1)$ with respect to J and the Hermitian metric they define ought to be positive-definite. Using the generalized complex structures of examples 6.8 and 6.9, these conditions translate to those of the following definition, [11].

Definition 6.20. A $U(d/2)\times U(d/2)$-structure consists of two commuting generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$: $[\mathcal{J}_1, \mathcal{J}_2] = 0$; that define a (symmetric) positive-definite metric G by $G = -\mathcal{G}\mathcal{J}_1\mathcal{J}_2$. This metric is *bihermitian.*

Definition 6.21. A $U(d/2)\times U(d/2)$-structure is an *H-twisted generalized generalized Kähler structure* if both $\mathcal{J}_1$ and $\mathcal{J}_2$ are H-integrable.

Using this 'generalized' definition, the class of generalized Kähler structures is larger than that of ordinary Kähler ones. It does, of course, include the ordinary $U(d/2)$ Kähler structures.

Example 6.22. (*Kähler structure.*) The ordinary Kähler structure defined by ω and J is now defined by the generalized structures of examples 6.8 and 6.9; these structures we refer to as $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. They satisfy the compatibility condition $[\mathcal{J}_1, \mathcal{J}_2] = 0$ and define the metric

$$G = \begin{pmatrix} -\omega J & 0 \\ 0 & J^T\omega^{-1} \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \tag{6.55}$$

where g is the ordinary Kähler metric defined by ω and J.

The metric G defined above satisfies $(\mathcal{G}^{-1}G)^2 = (-\mathcal{J}_1\mathcal{J}_2)^2 = 1$, so that we have orthogonal ±1-eigenspaces $C_\pm$. Splitting up vectors as $\mathbf{X} = \mathbf{X}_+ + \mathbf{X}_-$, $\mathbf{X}_\pm \in C_\pm$ we find

$$G(\,\cdot\,,\,\cdot\,) = \mathcal{G}(\,\cdot\,,\,\cdot\,)|_{C_+} - \mathcal{G}(\,\cdot\,,\,\cdot\,)|_{C_-}. \tag{6.56}$$

As the signature of $\mathcal{G}$ is (d,d), we find that $\dim C_+ = \dim C_- = d$. Now, taking into account that G is positive-definite, we find that the structure group of $(TM \oplus T^*M)$ is reduced to $O(d)\times O(d)$.

In short, the description of the metric G as in its definition is equivalent to the specification of a sub-bundle C_+ on which $\mathcal{G}$ is positive-definite. Once we have defined the C_+ associated to a given $\mathcal{G}^{-1}G$, we can further reduce the structure group to $U(d/2)\times U(d/2)$ by adding a generalized almost complex structure $\mathcal{J}_1$ that commutes with $\mathcal{G}^{-1}G$.

Since for a $U(d/2)\times U(d/2)$-structure, $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, they may be diagonalized simultaneously. Then we can define

$$\begin{aligned} L_1^+ &= L_1 \cap L_2, \\ L_1^- &= L_1 \cap \bar{L}_2, \end{aligned} \tag{6.57}$$

where L_i is the isotropic sub-bundle associated to $\mathcal{J}_i$. From the positive-definiteness of G we have that $L_1^\pm$ both have rank $d/2$. Furthermore, the bundle

C_+ has as fibres the generalized vectors with equal eigenvalues of $\mathcal{J}_1$ and $\mathcal{J}_2$, while for C_- the eigenvalues are opposite, i.e.

$$\begin{aligned} C_+ \otimes \mathbb{C} &= L_1^+ \oplus \bar{L}_1^+, \\ C_- \otimes \mathbb{C} &= L_1^- \oplus \bar{L}_1^-. \end{aligned} \tag{6.58}$$

Remark 6.23. It has been shown that, in general, $\mathcal{G}^{-1}G$ is a B-transform of the metric appearing in the ordinary Kähler example:

$$\mathcal{G}^{-1}G = e^B \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-B}. \tag{6.59}$$

The easiest way to find these B and g is to use the fact that vectors $\mathbf{X}_\pm = X+\xi \in C_\pm$ correspond to ± 1-eigenvectors of $\mathcal{G}^{-1}G$ and thus take on the general form

$$\mathbf{X}_\pm = (X, \xi) = e^B(X, \pm gX) = (X, (\pm g + B)X). \tag{6.60}$$

We find, for $\mathbf{X}_\pm, \mathbf{Y}_\pm \in C_\pm$,

$$\begin{aligned} \mathcal{G}(\mathbf{X}_\pm, \mathbf{Y}_\pm) &= \pm g(X, Y), \\ \mathcal{A}(\mathbf{X}_\pm, \mathbf{Y}_\pm) &= B(X, Y), \end{aligned} \tag{6.61}$$

where $\mathcal{A}(\mathbf{X}, \mathbf{Y}) = \frac{1}{2}(\eta(X) - \xi(Y))$ is the canonical antisymmetric bilinear product.

6.3.2 Pure spinors

In terms of pure spinors, the $U(d/2)\times U(d/2)$-structure is defined in the following theorem.

Theorem 6.24. The generalized tangent bundle has a $U(d/2)\times U(d/2)$-structure if and only if there exist pure spinor line bundles[40] ψ_1 and ψ_2, that satisfy the compatibility condition

$$\psi_2 \in \Gamma(U_0), \tag{6.62}$$

where U_i is the filtration associated to ψ_1, and that are such that the metric associated to their generalized almost complex structures is positive-definite.

Remark 6.25. The compatibility condition may, equivalently, be written as

$$\psi_1 \in \Gamma(V_0), \tag{6.63}$$

where V_i is the filtration associated to ψ_2. It expresses the condition that $\mathcal{J}_1$ and $\mathcal{J}_2$ should commute. For $d = 6$ it may be formulated as

$$\langle \psi_1, \mathbf{X} \cdot \psi_2 \rangle = \langle \psi_1, \mathbf{X} \cdot \bar{\psi}_2 \rangle = 0. \tag{6.64}$$

The latter equation generalizes the compatibility condition for forms in the ordinary $U(d/2)$-structure case.

[40] This is to say that we have two globally defined spinors that are defined up to a scalar function.

Fixing the overall factor of the spinors, such that they are globally defined and non-degenerate, results in an $SU(d/2) \times SU(d/2)$. Subsequently, the spinors may be normalized.

Definition 6.26. The generalized tangent bundle has an $SU(d/2) \times SU(d/2)$-structure if there exist globally defined pure spinors ψ_1 and ψ_2, that satisfy the normalisation condition

$$\langle \psi_1, \bar{\psi}_1 \rangle = \langle \psi_2, \bar{\psi}_2 \rangle \neq 0, \tag{6.65}$$

and the compatibility condition

$$\psi_2 \in \Gamma(U_0), \tag{6.66}$$

where U_i is again the filtration associated to ψ_1, and are such that the metric associated to their generalized almost complex structures is positive-definite.

A stronger condition than H-integrability of both $\mathcal{J}_i$ that applies to the case of an $SU(d/2) \times SU(d/2)$-structure appears in the following definition.

Definition 6.27. A *generalized Calabi-Yau geometry à la Gualtieri* is an $SU(d/2) \times SU(d/2)$-structure such that

$$d_H \psi_1 = 0, \qquad d_H \psi_2 = 0. \tag{6.67}$$

From example 6.22 and the remarks in section 5.2.1 we find that an ordinary Calabi-Yau geometry is an example of a generalized Calabi-Yau geometry.

6.3.3 Spinor bilinears

In this section we consider the relation between spinors of $Spin(d,d)$ and the ordinary ones of $Spin(d)$, [8], [9].

The generalized metric $G(g,B)$ allows us to define the Clifford map isomorphism between polyforms and operators acting on spinors; it is given by

$$\phi' = e^B \wedge \phi \leftrightarrow \underline{\phi} = \sum_l \frac{1}{l!} \phi_{i_1 \ldots i_l} \gamma^{i_1 \ldots i_l}, \tag{6.68}$$

where the gamma matrices on the left-hand side are defined using the vielbein associated to g. In the following we will take $B = 0$ and work with untwisted ϕ's, etc..

Now, we can define an $SU(d/2) \times SU(d/2)$-structure using two (not necessarily everywhere independent) spinors η^1 and η^2 that define two (again not necessarily everywhere independent) $SU(d/2)$-structures. The spinors should be normalized such that $\eta^{1\dagger}\eta^1 = \eta^{2\dagger}\eta^2 = 1$.

Using the Clifford map, we first construct two polyforms ψ_1 and ψ_2 from the following spinor bilinears

$$\underline{\psi_1} = \dim(S)\, \eta^1 \eta^{2\dagger}, \qquad \underline{\psi_2} = \dim(S)\, \eta^1 \eta'^{2\dagger}, \tag{6.69}$$

where $\eta'^2 = C\eta^{2*}$ and $\dim(S) = 2^{[d/2]}$, the dimension of the spinor representation. Using Fierz identities results in

$$\begin{aligned} \sigma(\psi_1)_{i_1 \ldots i_l} &= \eta^{2\dagger} \gamma_{i_1 \ldots i_l} \eta^1, \\ \sigma(\psi_2)_{i_1 \ldots i_l} &= \eta'^{2\dagger} \gamma_{i_1 \ldots i_l} \eta^1. \end{aligned} \tag{6.70}$$

The action of elements $\mathbf{X}_\pm = (X, \pm gX) \in \Gamma(C_\pm)$ on polyforms is given by

$$\underline{\mathbf{X}_+ \cdot \phi} = X^i \gamma_i \underline{\phi}, \qquad \underline{\mathbf{X}_- \cdot \phi} = -P \underline{\phi} \gamma_i X^i, \tag{6.71}$$

where $P = \pm 1$ is the parity of ϕ. Elements of C_+ thus act as gamma-matrices from the left and those of C_- as gamma-matrices from the right.

This action reveals that if η^1 and η^2 are pure spinors of $Spin(d)$ (a trivial property of Weyl spinors in $d = 6$), then the polyforms ψ_1 and ψ_2 as defined above will be pure spinors of $Spin(d, d)$. This is because the null spaces of ψ_1 and ψ_2 have dimension d at every point: they consist of $d/2$ basis vectors of C_+, constructed from the annihilators of η^1, and $d/2$ basis vectors of C_-, constructed from the annihilators of η^2 (or η'^2, in the case of ψ_2).

Furthermore, the conditions of definition 6.26 are also fulfilled for ψ_1 and ψ_2, which means that they do indeed define an $SU(d/2) \times SU(d/2)$-structure.

Example 6.28. (*The case* $d = 6$.) We take $\eta^{1,2} \equiv \eta_+^{1,2}$ to have positive chirality and $\eta'^{1,2} \equiv \eta_-^{1,2}$ to have negative chirality.

The most general relation between η^1 and η^2 is given by

$$\eta_+^2 = c\, \eta_+^1 + \tfrac{1}{2} V^i \gamma_i \eta_-^1. \tag{6.72}$$

This relation we rewrite using two mutually orthogonal spinors[41], η_+ and χ_+, where $\chi_+ = \frac{1}{2} v^i \gamma_i \eta_-$ (and $|v|^2 = 2$):

$$\eta_+^1 = e^{i\vartheta/2} \eta_+, \qquad \eta_+^2 = e^{-i\vartheta/2} (\cos\varphi\, \eta_+ + \sin\varphi\, \chi_+). \tag{6.73}$$

Above, $0 \le \varphi \le \pi/2$ indicates the angle between η^1 and η^2, which may vary over the manifold M. The relation between both descriptions is given by: $c = e^{-i\vartheta} \cos\varphi$, $V^i = v^i \sin\varphi$.

In points where $\sin\varphi = 0$, χ_+ need not be defined. In other points, η_+ and χ_+ define a local $SU(2)$-structure, which is described by the three forms:

$$\begin{aligned} v^i &= \eta_-^\dagger \gamma^i \chi_+, \\ \omega_{ij} &= \tfrac{i}{2} \eta_+^\dagger \gamma_{ij} \eta_+ - \tfrac{i}{2} \chi_+^\dagger \gamma_{ij} \chi_+, \\ \Omega_{ij} &= \chi_+^\dagger \gamma_{ij} \eta_+, \end{aligned} \tag{6.74}$$

that satisfy

$$\begin{aligned} &\Omega \wedge \omega = \Omega \wedge \Omega = 0, \\ &\imath_v \omega = \imath_v \Omega = \imath_v \bar{\Omega} = 0, \\ &\Omega \wedge \bar{\Omega} = 2\,\omega^2. \end{aligned} \tag{6.75}$$

The pure spinors become

$$\begin{aligned} \psi_1 &= e^{i\vartheta}\, e^{\frac{1}{2} v \wedge \bar{v}} \left(\cos\varphi\, e^{i\omega} - \sin\varphi\, \Omega \right), \\ \psi_2 &= -v \wedge \left(\cos\varphi\, \Omega + \sin\varphi\, e^{i\omega} \right). \end{aligned} \tag{6.76}$$

In the case that $\sin\varphi = 0$ everywhere we have what is called a *strict* $SU(3)$-*structure*. The types of (ψ_1, ψ_2) are $(0, 3)$. If $\cos\varphi = 0$ everywhere one speaks of a *static* $SU(2)$-*structure* (with types $(2, 1)$), while if both $\sin\varphi \neq 0$ and $\cos\varphi \neq 0$ one speaks of an *intermediate* $SU(2)$-*structure* (with types $(0, 1)$). In the latter case, if φ varies over M, the structure is called dynamic and may involve *type-changing* of the two ψ_i.

[41] Here, spinor orthogonality is defined as $\eta_+^\dagger \eta_+ = \chi_+^\dagger \chi_+ = 1$, $\chi_+^\dagger \eta_+ = 0$.

6.4 Minkowski compactification

In this last section we once again consider compactification. It is here that the usefulness of generalized geometry, as referred to in the introduction of section 6, becomes clear. The compactifications we are interested in, are of the form $\mathbb{R}^{3,1} \times K_6$, where K_6 is the six-dimensional internal space as before; $\mathbb{R}^{3,1}$ denotes the flat Minkowski space.

We consider vacuum solutions of type II supergravities and use the following compactification ansatz for the susy parameters ϵ^i, $i = 1, 2$:

$$\begin{aligned}\epsilon^1 &= \xi_+ \otimes \eta^1_+ + \text{c.c.},\\ \epsilon^2 &= \xi_+ \otimes \eta^2_\mp + \text{c.c.},\end{aligned} \tag{6.77}$$

where the upper signs correspond to type IIA and the lower signs to type IIB. Note that we use two internal spinors η^i, instead of one as in section 5.3.2.

In the equations above, ξ_+ is a constant Weyl spinor on the flat Minkowski space, with four real components, so that we have $\mathcal{N} = 1$ spacetime supersymmetry.

Had we chosen two different ξ^i_+ spinors the result would have been $\mathcal{N} = 2$ supersymmetry. In this case, however, the susy variations of section 3.3.2 relate ϵ^1 and ϵ^2, often leading to $\mathcal{N} = 1$ supersymmetric solutions. The effective theory on the other hand, will have a different degree of supersymmetry as the two ϵ^i are not related off-shell. To achieve an off-shell relation and thus a $\mathcal{N} = 1$ effective theory, the presence of orientifolds is sufficient—the introduction of these objects, that serve as negative-tension sources, is necessary to find Minkowski solutions with non-zero fluxes, according to the no-go theorem of Maldacena-Núñez[42] [9].

The two internal chiral spinors η^1_+ and $\eta^2_\mp$ are fixed and characterize the background geometry. They (see section 6.3.3) reduce the structure of $TM \oplus T^*M$ from $SO(6,6)$ to $SU(3) \times SU(3)$. Supersymmetry thus imposes a topological constraint on the internal manifold.

In the following the η^i are taken to have the same norm: $|a|$. A more general compactification decomposition of the parameters is discussed in section 5.3.2 appendix and B. The pure spinors ψ_1 and ψ_2 may then be defined in a way similar to that of equation (6.69) (using only a different normalization convention):

$$\underline{\psi_1} = \underline{\psi}^\mp = -\frac{8i}{|a|^2}\eta^1_+\eta^{2\dagger}_\mp, \qquad \underline{\psi_2} = \underline{\psi}^\pm = -\frac{8i}{|a|^2}\eta^1_+\eta^{2\dagger}_\pm, \tag{6.78}$$

for types IIA and IIB. The difference between types IIA and IIB is that the positive-chirality and negative-chirality spinors are interchanged.

The susy transformations, (3.83), become:

$$\begin{aligned}d_H(e^{4A-\Phi}\mathrm{Re}\,\psi_1) &= e^{4A}\tilde{F},\\ d_H(e^{3A-\Phi}\psi_2) &= 0,\\ d_H(e^{2A-\Phi}\mathrm{Im}\,\psi_1) &= 0.\end{aligned} \tag{6.79}$$

[42] The no-go theorem basically states that compactification to a Minkowski (or a de Sitter) space times a compact internal manifold in the presence of fluxes is impossible unless sources with negative tension are introduced [8], [9].

The calculation is given in the references of [9].

Looking back at the different generalized stucture classifications we discussed in the preceding sections, we see that the susy variations imply that ψ_2 is a generalised Calabi-Yau structure à la Hitchin, and that its associated generalised complex structure, $\mathcal{J}_2$, is integrable. We also find that the generalized almost complex structure $\mathcal{J}_1$, associated to ψ_1, is not integrable, due to the presence of the R-R fluxes. Thus, the two ψ_i do, although they define an $SU(3) \times SU(3)$-structure, not form a generalized Calabi-Yau structure à la Gualtieri.

Example 6.29. An example of a solution is given in [9], it is an F-theory (for a non-constant dilaton Φ) or a ($SU(3)$-structured) warped Calabi-Yau solution (for a constant dilaton) of a type IIB flux compactification. In this case the pure spinors are

$$\psi_1 = e^{i\omega}, \qquad \psi_2 = \Omega. \tag{6.80}$$

The two internal spinors satisfy $\eta_+^2 = -i\eta_+^1$. The equations we can derive from the susy variations are:

$$\begin{aligned} d(e^{3A-\Phi}\Omega) &= 0, & *_6 G_3 &= iG_3, \\ d(e^{2A-\Phi}\omega) &= 0, & \bar{\partial}\tau &= 0, \\ H \wedge \Omega = H \wedge \omega &= 0, & 4dA - d\Phi &= e^{\Phi} *_6 F_5, \end{aligned} \tag{6.81}$$

where $G_3 = F_3 + ie^{-\Phi}H$.

7 Summary

In this thesis, we have discussed a number of aspects of string theory. Our starting point was bosonic string theory (2), which we used to point out some of string theory's essential features, such as the particle spectrum it predicts and the number of dimensions it needs to exist in a consistent way. We then repeated this discussion for the case of superstring theory (3) and finished by describing its massless sector (and low-energy limit): the (type II) supergravities (3.3).

The subsequent subject was compactification (4); we discussed the effects concerning the particle spectrum and the conditions on the nature of the internal space. We found that in fluxless compactifications the internal space is necessarily Calabi-Yau, if we wished to have some remaining, unbroken spacetime supersymmetry (4.2). For flux compactifications the condition on the internal space turned out to be that of $SU(3)$-structure, something we carefully defined in the subsequent sections (5). In these sections, we also found, by studying the supergravities' susy variations, the kinds of manifolds that can be used for flux compactifications (5.3.2, B and A).

In the last section (6) we turned to generalized geometry, a theory that offers a unified description of complex and symplectic manifolds. In the concluding paragraphs of this section, we found from the susy variations the generalized conditions the internal manifolds has to satisfy (6.4). The conciseness and apparent naturalness of the formulation of these conditions illustrated the (possible) usefulness of generalized geometry in a field theory approach to string theory compactifications.

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A Symplectic and complex manifolds

This appendix serves as a short introduction to the different kinds of manifolds that appear in (flux) compactifications.

A.1 Symplectic manifolds

The first class we consider is that of symplectic manifolds (for a short, but much more in depth introduction, see [13]).

Definition A.1. A *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω a non-degenerate two-form (, which is called the symplectic form).

The standard example is the phase space of classical mechanics, which is defined as a $2n$-dimensional Euclidean space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic form:

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i. \tag{A.1}$$

A basic result is the Darboux theorem.

Theorem A.2. (Darboux.) Every symplectic manifold (M, ω) is locally diffeomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

A.2 Complex manifolds

This section will be notably longer than the previous one, as we took the time to consider multiple kinds of complex manifolds—all of which will be of some relevance to this thesis. The discussion here is mostly based on parts of [12], though for the section on almost complex manifolds [9] was also used.

A.2.1 Definitions

Definition A.3. M is a *complex manifold* if it is a manifold whose charts are open subsets of $\mathbb{C}^n$ and of which the transition functions between charts are holomorphic.

The number n above is the complex dimension of M, denoted $\dim_{\mathbb{C}} M = n$. Interpreting the n complex coordinates of a chart, $(z^\mu = x^\mu + iy^\mu)$, $\mu = 1, \ldots, n$, as $2n$ real ones: (x^μ, y^μ), we find that a complex manifold is also a differentiable manifold with real dimension $2n$. This is denoted $\dim_{\mathbb{R}} M = 2n$. The analyticity of the coordinate transformation functions ensures that the differentiability of the real coordinates.

Definition A.4. Given complex manifolds N and M with $\dim_{\mathbb{C}} M = m$ and $\dim_{\mathbb{C}} N = n$. A map $f : M \to N$ is called a *holomorphic map* if, for charts (U, φ) of M (with $p \in U$) and (V, ψ) of N (such that $f(p) \in V$), all component functions of $\psi \circ f \circ \varphi^{-1} : \mathbb{C}^m \to \mathbb{C}^n$ are holomorphic functions.

A holomorphic map $f : M \to N$ that is also a diffeomorphism is called a *biholomorphism* (as f^{-1} is then also holomorphic), and M is said to be *biholomorphic* to N.

A *holomorphic function* is a holomorphic map $f : M \to \mathbb{C}$. Note that for compact M, f is constant.

A.2.2 Complexifications

Given the set of smooth functions $\mathcal{F}(M)$ on a differentiable manifold M with $\dim_{\mathbb{R}} M = m$. Its complexification $\mathcal{F}(M)^{\mathbb{C}}$ is defined as the set of functions $f : M \to \mathbb{C}$ of the form $f = g + ih$, $g, h \in \mathcal{F}(M)$. Note that these do not necessarily satisfy the Cauchy-Riemann equations. The complex conjugate of f is $\bar{f} \equiv g - ih$; f is real if and only if $f = \bar{f}$ or, equivalently, $f \in \mathcal{F}(M)$.

Similarly, the complexification $V^{\mathbb{C}}$ of a vector space V consists of elements of the form $X + iY$, where $X, Y \in V$. This set becomes a vector space if we define addition and scalar multiplication (with scalars $a + ib \in \mathbb{C}$) as follows

$$\begin{aligned}(X_1 + iY_1) + (X_2 + iY_2) &= (X_1 + X_2) + i(Y_1 + Y_2),\\ (a + ib)(X + iY) &= (aX - bY) + i(bX + aY).\end{aligned} \tag{A.2}$$

The complex conjugate of $Z = X + iY$ is $\bar{Z} = X - iY$. Again, Z is real iff $Z = \bar{Z}$ or, equivalently, $Z \in V \subset V^{\mathbb{C}}$.

A linear operator A on V may be extended to act on $V^{\mathbb{C}}$ by defining

$$A(X + iY) = A(X) + iA(Y). \tag{A.3}$$

A linear function $A \in V^*$ is thus extended to a linear function $A : V^{\mathbb{C}} \to \mathbb{C}$. More generally, any tensor defined on V and V^* can be extended so that it is defined on $V^{\mathbb{C}}$ and $(V^*)^{\mathbb{C}}$. And, similar to the vector case, an extended tensor may be complexified as $t = t_1 + it_2$, where t_1 and t_2 are of the same type.

Given a basis e_k of V, these vectors form, if they are regarded as complex vectors, a basis for $V^{\mathbb{C}}$. To see this, take $X = X^k e_k, Y = Y^k e_k \in V$, then $Z = X + iY$ is uniquely expressed as $(X^k + iY^k)e_k$. Then also $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$.

The complexification of the tangent space T_pM is $T_pM^{\mathbb{C}}$. Its vectors Z act on functions $f \in \mathcal{F}(M)^{\mathbb{C}}$ as

$$\begin{aligned}Z[f] &= X[f_1 + if_2] + iY[f_1 + if_2]\\ &= X[f_1] - Y[f_2] + i(X[f_2] + Y[f_1]).\end{aligned} \tag{A.4}$$

The complexified dual space $(T_p^*M)^{\mathbb{C}}$ consists of elements $\xi = \eta + i\omega$ with $\omega, \eta \in T_p^*M$. (In fact, $(T_p^*M)^{\mathbb{C}} = (T_pM^{\mathbb{C}})^* \equiv T_p^*M^{\mathbb{C}}$.) Any tensor defined on T_pM and T_pM^* is first extended to the complexified spaces and subsequently complexified itself.

The complexification of the set of smooth vector fields $\mathcal{X}(M)$ is denoted $\mathcal{X}(M)^{\mathbb{C}}$. The Lie bracket of $X + iY, U + iV \in \mathcal{X}(M)^{\mathbb{C}}$ is

$$[X + iY, U + iV] = ([X, U] - [Y, V]) + i([X, V] + [Y, U]). \tag{A.5}$$

A.2.3 Almost complex structure

We again consider a complex manifold M with complex dimension m. Its tangent space at p, T_pM is spanned by $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}\}$, where $z^\mu = x^\mu + iy^\mu$ are the coordinates of p in a certain chart. T_p^*M is spanned by $\{dx^1, \ldots, dx^m, dy^1, \ldots, dy^m\}$.

The $2m$ vectors

$$\frac{\partial}{\partial z^\mu} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right), \qquad \frac{\partial}{\partial \bar{z}^\mu} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\right), \tag{A.6}$$

form a basis for the $2m$-dimensional $T_pM^{\mathbb{C}}$. Note that $\overline{\frac{\partial}{\partial z^\mu}} = \frac{\partial}{\partial \bar{z}^\mu}$. The dual one-forms are

$$dz^\mu \equiv dx^\mu + i dy^\mu, \qquad d\bar{z}^\mu \equiv dx^\mu - i dy^\mu. \tag{A.7}$$

They span $T_p^*M^{\mathbb{C}}$ and satisfy

$$\begin{aligned} \langle dz^\mu, \partial/\partial\bar{z}^\nu\rangle &= \langle d\bar{z}^\mu, \partial/\partial z^\nu\rangle = 0, \\ \langle dz^\mu, \partial/\partial z^\nu\rangle &= \langle d\bar{z}^\mu, \partial/\partial\bar{z}^\nu\rangle = \delta^\mu_\nu. \end{aligned} \tag{A.8}$$

The linear map $J_p : T_pM \to T_pM$

$$J_p\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial y^\mu}, \qquad J_p\left(\frac{\partial}{\partial y^\mu}\right) = -\frac{\partial}{\partial x^\mu}, \tag{A.9}$$

defines a real tensor of type $(1,1)$ satisfying $J_p^2 = -\mathrm{id}_{T_pM}$. Its definition is coordinate independent, because of the analyticity of the transition functions. J_p is given by, in components, $J_p = \left(\begin{smallmatrix} 0 & -1_m \\ 1_m & 0 \end{smallmatrix}\right)$, where 1_m is the $m \times m$ unit matrix.

Such a map J (for which $J^2 = -1_{2m}$) may be defined, locally, on any $2m$-dimensional manifold. However, only on a complex manifold may the local J be patched together across charts to define a globally defined smooth tensor field J. This J is called the *almost complex structure* of a complex manifold M.

J_p may be extended to $T_pM^{\mathbb{C}}$ and can then be expressed in (anti-)holomorphic bases as

$$J_p = i\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i\, d\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu}, \tag{A.10}$$

which is, in components, $J_p = \left(\begin{smallmatrix} i1_m & 0 \\ 0 & -i1_m \end{smallmatrix}\right)$.

J_p is used to define the projectors

$$\mathcal{P}^\pm \equiv \frac{1}{2}(1_{2m} \mp iJ_p). \tag{A.11}$$

For any $Z \in T_pM^{\mathbb{C}}$, we find $J_p\mathcal{P}^\pm Z = \pm i\mathcal{P}^\pm Z$. Thus, $Z^\pm \equiv \mathcal{P}^\pm Z \in T_pM^\pm$, where $T_pM^\pm \equiv \{Z \in T_pM^{\mathbb{C}} | J_pZ = \pm iZ\}$. Evidently, T_pM^+ is spanned by $\{\partial/\partial z^\mu\}$ and T_pM^- by $\{\partial/\partial\bar{z}^\mu\}$.

Because of the properties of $\mathcal{P}^\pm$[43], every vector $Z \in T_pM^{\mathbb{C}}$ may be uniquely decomposed as $Z = Z^+ + Z^-$, implying a decomposition of $T_pM^{\mathbb{C}}$ into two disjoint vector spaces:

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-. \tag{A.13}$$

[43] The projectors satisfy

$$\begin{aligned} \mathcal{P}^+ + \mathcal{P}^- &= 1_{2m}, \\ \mathcal{P}^+\mathcal{P}^- = \mathcal{P}^-\mathcal{P}^+ &= 0, \\ (\mathcal{P}^\pm)^2 &= \mathcal{P}^\pm. \end{aligned} \tag{A.12}$$

$Z \in T_pM^{(-)+}$ is called a(n) (anti-)holomorphic vector. We have $T_pM^+ = \overline{T_pM^-}$ and therefore $\dim_{\mathbb{C}} T_pM^+ = \dim_{\mathbb{C}} T_pM^- = \frac{1}{2}\dim_{\mathbb{C}} T_pM^{\mathbb{C}} = \frac{1}{2}\dim_{\mathbb{C}} M$.

Similarly, complexified vector fields $Z \in \mathcal{X}(M)^{\mathbb{C}}$ may be decomposed using the field J: $Z = Z^+ + Z^-$, $Z^{\pm} = \mathcal{P}^{\pm}Z$, where $JZ|_p = J_p \cdot Z|_p$. $Z^+(Z^-)$ is called a(n) holomorphic (antiholomorphic) vector field. So, given a J, $\mathcal{X}(M)^{\mathbb{C}}$ decomposes as $\mathcal{X}(M)^{\mathbb{C}} = \mathcal{X}(M)^+ \oplus \mathcal{X}(M)^-$. Both $\mathcal{X}(M)^{\pm}$ are closed under the Lie bracket[44].

A.2.4 Complex differential forms

Let M be a differential manifold. At $p \in M$, complex q-forms are defined as $\zeta \in \Omega_p^q(M)^{\mathbb{C}} : \zeta = \omega + i\eta$, where $\omega, \eta \in \Omega_p^q(M)$. A complex q-form $\zeta \in \Omega^q(M)^{\mathbb{C}}$ defined on M is a smooth assignment of an element of $\Omega_p^q(M)^{\mathbb{C}}$, it may be uniquely decomposed as $\zeta = \omega + i\eta$, $\omega, \eta \in \Omega^q(M)$.

The exterior product of $\zeta = \omega + i\eta$ and $\xi = \phi + i\psi$ is defined by

$$\begin{aligned}\zeta \wedge \xi &= (\omega + i\eta) \wedge (\phi + i\psi) \\ &= (\omega \wedge \phi - \eta \wedge \psi) + i(\omega \wedge \psi + \eta \wedge \phi).\end{aligned} \tag{A.14}$$

And the exterior derivative d acts as

$$d\zeta = d\omega + i\, d\eta. \tag{A.15}$$

It is a real operator: $\overline{d\zeta} = d\omega - i\, d\eta = d\bar{\zeta}$.

Next, we restrict ourselves to complex manifolds M with $\dim_{\mathbb{C}} M = m$, on which we have the decompositions $T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-$ and $\mathcal{X}(M)^{\mathbb{C}} = \mathcal{X}(M)^+ \oplus \mathcal{X}(M)^-$.

Definition A.5. Let $\omega \in \Omega_p^q(M)^{\mathbb{C}}$ with $q \leq 2m$ and let $r, s \in \{0, 1, \ldots, 2m\}$ with $r{+}s = q$. Let V_i be q vectors be in either T_pM^+ or T_pM^-. If $\omega(V_1, \ldots, V_q) = 0$ unless r of the $V_i \in T_pM^+$ and s of the $V_i \in T_pM^-$, ω is said to be of *bidegree* (r, s) or simply an (r, s)-form. The set of such forms is denoted $\Omega_p^{r,s}(M)$. A smoothly assigned, to each point of M, (r, s)-form defines an (r, s)-form over M. The set of such forms is denoted by $\Omega^{r,s}(M)$.

From the definitions of $\{dz^{\mu}\}$ and $\{d\bar{z}^{\mu}\}$, we see that they correspond to $(1, 0)$- and $(0, 1)$-forms at p respectively. Using these bases, an (r, s)-form is written as

$$\omega = \frac{1}{r!s!}\omega_{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_s}. \tag{A.16}$$

The set $\{dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_s}\}$ forms a basis for $\Omega_p^{r,s}(M)$. Using this, one may prove the following.

Proposition A.6. Let ω and ξ be complex differential forms on M as defined above.

- If $\omega \in \Omega^{r,s}(M)$, then $\bar{\omega} \in \Omega^{s,r}(M)$.
- If $\omega \in \Omega^{r,s}(M)$ and $\xi \in \Omega^{r',s'}(M)$, then $\omega \wedge \xi \in \Omega^{r+r',s+s'}(M)$.

[44] This means that $X, Y \in \mathcal{X}(M)^+$ implies that $[X, Y] \in \mathcal{X}(M)^+$, and similarly for $\mathcal{X}(M)^-$.

- A complex q-form ω is uniquely written as

$$\omega = \sum_{r+s=q} \omega^{(r,s)}, \tag{A.17}$$

where $\omega^{(r,s)} \in \Omega^{r,s}(M)$. We then also have

$$\Omega^q(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{r,s}(M). \tag{A.18}$$

Next, considering the action of the exterior derivative d on an (r,s)-form ω[45]:

$$\begin{aligned} d\omega =& \frac{1}{r!s!}\left(\frac{\partial}{\partial z^\lambda}\omega_{\mu_1\dots\mu_r\nu_1\dots\nu_s}dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1\dots\mu_r\nu_1\dots\nu_s}d\bar{z}^\lambda\right)\wedge \\ & dz^{\mu_1}\wedge\cdots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\cdots\wedge d\bar{z}^{\nu_s}. \end{aligned} \tag{A.19}$$

we find that $d\omega$ is the sum of an $(r+1,s)$- and an $(r,s+1)$-form. We define the *Dolbeault operators* ∂ and $\bar{\partial}$ such that $d = \partial + \bar{\partial}$, $\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M)$ and $\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M)$. Explicitly, for $\omega = \omega_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu$[46] we have

$$\begin{aligned} \partial\omega &= \frac{\partial\omega_{\mu\bar{\nu}}}{\partial z^\lambda}dz^\lambda\wedge dz^\mu\wedge d\bar{z}^\nu \\ \bar{\partial}\omega &= \frac{\partial\omega_{\mu\bar{\nu}}}{\partial \bar{z}^\lambda}d\bar{z}^\lambda\wedge dz^\mu\wedge d\bar{z}^\nu = -\frac{\partial\omega_{\mu\bar{\nu}}}{\partial \bar{z}^\lambda}dz^\mu\wedge d\bar{z}^\lambda\wedge d\bar{z}^\nu. \end{aligned} \tag{A.20}$$

On a general q-form the actions of the Dolbeault operators are defined as

$$\partial\omega = \sum_{r+s=q}\partial\omega^{(r,s)}, \qquad \bar{\partial}\omega = \sum_{r+s=q}\bar{\partial}\omega^{(r,s)}. \tag{A.21}$$

Theorem A.7. The Dolbeault operators have the following properties. $\omega \in \Omega^q(M)^{\mathbb{C}}$ and $\xi \in \Omega^p(M)^{\mathbb{C}}$.

$$\begin{aligned} \partial\partial\omega = (\partial\bar{\partial}+\bar{\partial}\partial)\omega = \bar{\partial}\bar{\partial}\omega = 0, \\ \partial\bar{\omega} = \overline{\bar{\partial}\omega},\ \bar{\partial}\bar{\omega} = \overline{\partial\omega}, \\ \partial(\omega\wedge\xi) = \partial\omega\wedge\xi + (-1)^q\omega\wedge\partial\xi, \\ \bar{\partial}(\omega\wedge\xi) = \bar{\partial}\omega\wedge\xi + (-1)^q\xi\wedge\bar{\partial}\xi. \end{aligned}$$

All properties follow from the identities $d = \partial + \bar{\partial}$, $\bar{d} = d$ and $d^2 = 0$.

Definition A.8. A *holomorphic r-form* is an $\omega \in \Omega^{r,0}(M)$ such that $\bar{\partial}\omega = 0$.

Holomorphic 0-forms are just holomorphic functions and holomorphic r-forms $\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r}dz^{\mu_1}\wedge\cdots\wedge dz^{\mu_r}$ have holomorphic functions $\omega_{\mu_1\dots\mu_r}$.

[45] The exterior derivative $d\omega \in \Omega^{l+1}(M)$ of $\omega \in \Omega^l(M)$ is equivalently defined as

$$\begin{aligned} d\omega(Y_0,\dots,Y_l) = &\sum_{0\le a\le l}(-1)^a Y_a[\omega(Y_0,\dots,\hat{Y}_a,\dots,Y_l')] \\ &+\sum_{0\le a<b\le l}(-1)^{a+b}\omega([Y_a,Y_b],Y_0,\dots,\hat{Y}_a,\dots,\hat{Y}_b,\dots,Y_l), \end{aligned}$$

where 'hatted' vector fields are missing.

[46] The bar over the ν is used to indicate that the index is to be contracted with an anti-holomorphic $(0,1)$-form

A.3 Hermitian manifolds

Let M be a complex manifold with $\dim_{\mathbb{C}} M = m$ and g a Riemannian metric of M as a differentiable manifold. Taking $Z = X + iY$, $W = U + iV \in T_pM^{\mathbb{C}}$, we extend g so that

$$g_p(Z, W) = g_p(X, U) - g_p(Y, V) + i\{g_p(X, V) + g_p(Y, U)\}. \tag{A.22}$$

The components of g with respect to the (anti-)holomorphic basis are

$$\begin{aligned} g_{\mu\nu}(p) &= g_p(\partial/\partial z^\mu, \partial/\partial z^\nu), \\ g_{\mu\bar{\nu}}(p) &= g_p(\partial/\partial z^\mu, \partial/\partial \bar{z}^\nu), \\ g_{\bar{\mu}\nu}(p) &= g_p(\partial/\partial \bar{z}^\mu, \partial/\partial z^\nu), \\ g_{\bar{\mu}\bar{\nu}}(p) &= g_p(\partial/\partial \bar{z}^\mu, \partial/\partial \bar{z}^\nu), \end{aligned} \tag{A.23}$$

satisfying

$$g_{\mu\nu} = g_{\nu\mu}, \quad g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}}, \quad g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}}, \quad \overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu}, \quad \overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}}. \tag{A.24}$$

Definition A.9. Let M and g be as above. The pair (M, g) is called a *Hermitian manifold* and g a *Hermitian metric* if at each point $p \in M$, for arbitrary vectors $X, Y \in T_pM$, $g_p(J_pX, J_pY) = g_p(X, Y)$.

Note, the vector J_pX is orthogonal to X for a Hermitian metric, as $g_p(J_pX, X) = g_p(J_p^2X, J_pX) = -g_p(J_pX, X) = 0$.

Theorem A.10. A complex manifold always admits a Hermitian metric.

Proof. Let g be any Riemannian metric of a complex manifold M. Then the metric $\hat{g}_p(X, Y) \equiv \frac{1}{2}\{g_p(X, Y) + g_p(J_pX, J_pY)\}$ is Hermitian. □

From the definition of a Hermitian metric g we find $g_{\mu\nu} = -g_{\mu\nu} = 0$ and $g_{\bar{\mu}\bar{\nu}} = -g_{\bar{\mu}\bar{\nu}} = 0$, so that

$$g = g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\nu} d\bar{z}^\mu \otimes dz^\nu. \tag{A.25}$$

Definition A.11. The *Kähler form* Ω of a Hermitian metric g is defined as

$$\Omega_p(X, Y) = g_p(J_pX, Y), \qquad X, Y \in T_pM. \tag{A.26}$$

It is anti-symmetric since $\Omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(JY, X) = -\Omega(Y, X)$. It is also invariant under the action of J.

If its domain is extended to $T_pM^{\mathbb{C}}$, Ω becomes a real ($\bar{\Omega} = \Omega$) two-form of bidegree $(1, 1)$ with components

$$\Omega_{\mu\nu} = ig_{\mu\nu} = 0, \quad \Omega_{\bar{\mu}\bar{\nu}} = -ig_{\bar{\mu}\bar{\nu}} = 0, \quad \Omega_{\mu\bar{\nu}} = ig_{\mu\bar{\nu}} = -\Omega_{\bar{\nu}\mu}. \tag{A.27}$$

That is,

$$\Omega = ig_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu - ig_{\bar{\nu}\mu} d\bar{z}^\nu \otimes dz^\mu = ig_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu. \tag{A.28}$$

Equivalently,

$$\Omega = -J_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu, \tag{A.29}$$

where $J_{\mu\bar{\nu}} = g_{\mu\bar{\lambda}} J^{\bar{\lambda}}{}_{\bar{\nu}} = -ig_{\mu\bar{\nu}}$.

On a Hermitian manifold, we may choose an orthonormal basis of the form $\{\hat{e}_1, J\hat{e}_1, \ldots, \hat{e}_m, J\hat{e}_m\}$. If $g(\hat{e}_1, \hat{e}_1) = 1$, then $J\hat{e}_1$ is normalized and orthogonal to $\hat{e}_1$. Now, taking a normalized $\hat{e}_2$ orthogonal to $\{\hat{e}_1, J\hat{e}_1\}$, adding $J\hat{e}_2$ and repeating this procedure will yield the aforementioned basis. Using it we can prove the following.

Lemma A.12. Let Ω be the Kähler form of a Hermitian manifold with $\dim_{\mathbb{C}} M = m$. Then

$$\underbrace{\Omega \wedge \cdots \wedge \Omega}_{m} \tag{A.30}$$

is a nowhere vanishing $2m$-form.

Proof. For the orthonormal basis discussed above, we have

$$\Omega(\hat{e}_i, J\hat{e}_j) = g(J\hat{e}_i, J\hat{e}_j) = \delta_{ij}, \qquad \Omega(\hat{e}_i, \hat{e}_j) = \Omega(J\hat{e}_i, J\hat{e}_j) = 0. \tag{A.31}$$

Then

$$\begin{aligned}
&\underbrace{\Omega \wedge \cdots \wedge \Omega}_{m}(\hat{e}_1, J\hat{e}_1, \ldots, \hat{e}_m, J\hat{e}_m) \\
&\qquad = \sum_P \Omega(\hat{e}_{P(1)}, J\hat{e}_{P(1)}) \ldots \Omega(\hat{e}_{P(m)}, J\hat{e}_{P(m)}) \\
&\qquad = m!\Omega(\hat{e}_1, J\hat{e}_1) \ldots \Omega(\hat{e}_m, J\hat{e}_m) = m!,
\end{aligned} \tag{A.32}$$

where P is an element of the permutation group of m objects. This shows that $\Omega \wedge \cdots \wedge \Omega$ does not vanish at any point. □

Now that we have a real, nowhere vanishing $2m$-form, we have, in fact, a volume element. This implies the following.

Theorem A.13. A complex manifold is orientable.

A.3.1 Covariant derivatives

In this section we define a connection that is compatible with the complex structure, on a Hermitian manifold (M, g). We start out by requiring that a holomorphic vector $V \in T_pM^+$, parallel transported to a point q, be again holomorphic: $\tilde{V}(q) \in T_qM^+$. First, we define the derivative's action on the basis vector fields,

$$\nabla_\mu \frac{\partial}{\partial z^\nu} = \Gamma^\lambda{}_{\mu\nu}(z) \frac{\partial}{\partial z^\lambda}, \tag{A.33}$$

and for the conjugate vector fields,

$$\nabla_{\bar{\mu}} \frac{\partial}{\partial \bar{z}^\nu} = \Gamma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}}(z) \frac{\partial}{\partial \bar{z}^\lambda}, \tag{A.34}$$

where $\Gamma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}} = \overline{\Gamma^\lambda{}_{\mu\nu}}$, that are the only non-vanishing components of the connection coefficients. Then also $\nabla_\mu \partial/\partial \bar{z}^\nu = \nabla_{\bar{\mu}} \partial/\partial z^\nu = 0$. For the dual basis the non-vanishing covariant derivatives are

$$\nabla_\mu dz^\nu = -\Gamma^\nu{}_{\mu\lambda} dz^\lambda, \qquad \nabla_{\bar{\mu}} d\bar{z}^\nu = -\Gamma^{\bar{\nu}}{}_{\bar{\mu}\bar{\lambda}} d\bar{z}^\lambda. \tag{A.35}$$

The covariant derivative of $X^+ = X^\mu \partial/\partial z^\mu \in \mathcal{X}(M)^+$ is now given by

$$\nabla_\mu X^+ = (\partial_\mu X^\lambda + X^\nu \Gamma^\lambda{}_{\mu\nu}) \frac{\partial}{\partial z^\lambda}, \tag{A.36}$$

where $\partial_\mu \equiv \partial/\partial z^\mu$. For $X^- = X^{\bar{\mu}} \partial/\partial \bar{z}^\mu \in \mathcal{X}(M)^-$ we have

$$\nabla_\mu X^- = \partial_\mu X^{\bar{\lambda}} \frac{\partial}{\partial \bar{z}^\lambda}, \tag{A.37}$$

since $\Gamma^{\bar{\lambda}}{}_{\mu\nu} = \Gamma^{\bar{\lambda}}{}_{\mu\bar{\nu}} = 0$. Similarly,

$$\begin{aligned} \nabla_{\bar{\mu}} X^+ &= \partial_{\bar{\mu}} X^\lambda \frac{\partial}{\partial z^\lambda}, \\ \nabla_{\bar{\mu}} X^- &= (\partial_{\bar{\mu}} X^{\bar{\lambda}} + X^{\bar{\nu}} \Gamma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}}) \frac{\partial}{\partial \bar{z}^\lambda}. \end{aligned} \tag{A.38}$$

This may all be generalized to arbitrary tensor fields; for $t = t_{\mu\nu}{}^{\bar{\lambda}} dz^\mu \otimes dz^\nu \otimes \partial/\partial \bar{z}^\lambda$ for example, we have

$$\begin{aligned} (\nabla_\kappa t)_{\mu\nu}{}^{\bar{\lambda}} &= \partial_\kappa t_{\mu\nu}{}^{\bar{\lambda}} - t_{\xi\nu}{}^{\bar{\lambda}} \Gamma^\xi{}_{\kappa\mu} - t_{\mu\xi}{}^{\bar{\lambda}} \Gamma^\xi{}_{\kappa\nu}, \\ (\nabla_{\bar{\kappa}} t)_{\mu\nu}{}^{\bar{\lambda}} &= \partial_{\bar{\kappa}} t_{\mu\nu}{}^{\bar{\lambda}} + t_{\mu\nu}{}^{\bar{\xi}} \Gamma^{\bar{\lambda}}{}_{\bar{\kappa}\bar{\xi}}. \end{aligned} \tag{A.39}$$

We also demand *metric compatibility*, which in this case amounts to $\nabla_\kappa g_{\mu\bar{\nu}} = \nabla_{\bar{\kappa}} g_{\mu\bar{\nu}} = 0$. In components, this becomes

$$\partial_\kappa g_{\mu\bar{\nu}} - g_{\lambda\bar{\nu}} \Gamma^\lambda{}_{\kappa\mu} = 0, \qquad \partial_{\bar{\kappa}} g_{\mu\bar{\nu}} - g_{\mu\bar{\lambda}} \Gamma^{\bar{\lambda}}{}_{\bar{\kappa}\bar{\nu}} = 0. \tag{A.40}$$

The connection coefficients are thus $\Gamma^\lambda{}_{\kappa\mu} = g^{\bar{\nu}\lambda} \partial_\kappa g_{\mu\bar{\nu}}$ and $\Gamma^{\bar{\lambda}}{}_{\bar{\kappa}\bar{\nu}} = g^{\bar{\lambda}\mu} \partial_{\bar{\kappa}} g_{\mu\bar{\nu}}$, where $g^{\bar{\nu}\lambda}$ is the inverse matrix of $g_{\mu\bar{\nu}}$: $g_{\mu\bar{\nu}} g^{\bar{\nu}\lambda} = \delta_\mu{}^\lambda$, $g^{\bar{\nu}\lambda} g_{\lambda\bar{\mu}} = \delta^{\bar{\nu}}{}_{\bar{\mu}}$. A metric-compatible connection for which Γ(mixed indices) $= 0$ is called the *Hermitian connection*. By construction, it is unique and given above.

Theorem A.14. The almost complex structure J is covariantly constant with respect to the Hermitian connection,

$$(\nabla_\kappa J)_\nu{}^\mu = (\nabla_{\bar{\kappa}} J)_\nu{}^\mu = (\nabla_\kappa J)_{\bar{\nu}}{}^{\bar{\mu}} = (\nabla_{\bar{\kappa}} J)_{\bar{\nu}}{}^{\bar{\mu}} = 0. \tag{A.41}$$

Proof. We use the fact that in the basis that has been used in this section, we have $J = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}$. Then for the first equality we find (the others are proven similarly),

$$(\nabla_\kappa J)_\nu{}^\mu = \partial_\kappa i\delta_\nu{}^\mu - i\delta_\xi{}^\mu \Gamma^\xi{}_{\kappa\nu} + i\delta_\nu{}^\xi \Gamma^\mu{}_{\kappa\xi} = 0. \tag{A.42}$$

□

The torsion tensor T and the Riemann curvature tensor R are defined by

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y], \\ R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned} \tag{A.43}$$

For the torsion tensor we find that

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = \overline{T^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}}}, \tag{A.44}$$

all other components being zero. For the Riemann tensor all components of the form $R^{\kappa}{}_{\bar{\lambda}AB}$, $R^{\bar{\kappa}}{}_{\lambda AB}$, $R^{A}{}_{B\kappa\lambda}$ and $R^{A}{}_{B\bar{\kappa}\bar{\lambda}}$ are zero. Taking into account the trivial symmetry $R^{\kappa}{}_{\lambda\bar{\mu}\nu} = -R^{\kappa}{}_{\lambda\nu\bar{\mu}}$, the independent components are reduced to $R^{\kappa}{}_{\lambda\bar{\mu}\nu}$ and $R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = \overline{R^{\kappa}{}_{\lambda\bar{\mu}\nu}}$; for these we have

$$\begin{aligned} R^{\kappa}{}_{\lambda\bar{\mu}\nu} &= \partial_{\bar{\mu}}\Gamma^{\kappa}{}_{\nu\lambda} = \partial_{\bar{\mu}}(g^{\bar{\xi}\kappa}\partial_{\nu}g_{\lambda\bar{\xi}}), \\ R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} &= \partial_{\mu}\Gamma^{\bar{\kappa}}{}_{\bar{\nu}\bar{\lambda}} = \partial_{\mu}(g^{\bar{\kappa}\xi}\partial_{\bar{\nu}}g_{\xi\bar{\lambda}}). \end{aligned} \tag{A.45}$$

These satisfy certain other symmetries as well, which we will however not prove here. Two of R's indices may be contracted as follows,

$$\mathcal{R}_{\mu\bar{\nu}} \equiv R^{\kappa}{}_{\kappa\mu\bar{\nu}} = -\partial_{\bar{\nu}}(g^{\kappa\bar{\xi}})\partial_{\mu}g_{\kappa\bar{\xi}} = -\partial_{\mu}\partial_{\bar{\nu}}\log G, \tag{A.46}$$

where $G \equiv \det(g_{\mu\bar{\nu}}) = \sqrt{g}$. For the last equality the identity $\delta G = Gg^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ was used. Now, the *Ricci form* is defined by

$$\mathcal{R} \equiv \mathcal{R}_{\mu\bar{\nu}}dz^{\mu}\wedge d\bar{z}^{\nu} = i\partial\bar{\partial}\log G. \tag{A.47}$$

$\mathcal{R}$ is a real closed form, real since $\bar{\mathcal{R}} = -i\overline{\partial\bar{\partial}}\log G = -i\bar{\partial}\partial\log G = \mathcal{R}$ and closed since $d\partial\bar{\partial} = d(-\frac{1}{2}d(\partial-\bar{\partial})) = -\frac{1}{2}d^2(\partial-\bar{\partial}) = 0$. $\mathcal{R}$ is however not exact as G is not a scalar and $(\partial-\bar{\partial})\log G$ is not defined globally. $\mathcal{R}$ thus defines a non-trivial element $c_1(M) \equiv [\mathcal{R}/2\pi] \in H^2(M,\mathbb{R})$, called the *first Chern class*. Here, $H^r(M,\mathbb{R}) \equiv Z^r(M,\mathbb{R})/B^r(M,\mathbb{R})$, is the rth *de Rham cohomology group*, where $Z^r(M,\mathbb{R})$ is the set of all closed r-forms and $B^r(M,\mathbb{R})$ is the set of all exact r-forms.

Proposition A.15. The first Chern class $c_1(M)$ is invariant under a smooth change of the metric $g \to g+\delta g$.

Proof. We have $\delta\log G = g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$. Then $\delta\mathcal{R} = \delta i\partial\bar{\partial}\log G = i\partial\bar{\partial}g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}} = -\frac{1}{2}d(\partial-\bar{\partial})ig^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$. As $g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a scalar, $\omega \equiv -\frac{1}{2}d(\partial-\bar{\partial})ig^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a well-defined one-form on M. Then $\delta\mathcal{R} = d\omega$ is an exact two-form and $[\mathcal{R}] = [\mathcal{R}+\delta\mathcal{R}]$, that is, $c_1(M)$ is left invariant under $g \to g+\delta g$. □

A.4 Kähler manifolds

Definition A.16. A *Kähler manifold* is a Hermitian manifold (M,g) whose Kähler form Ω is closed: $d\Omega = 0$; g is then called the *Kähler metric* of M.

Theorem A.17. A Hermitian manifold (M,g) is a Kähler manifold if and only if the almost complex structure J satisfies $\nabla_{\mu}J = 0$, where ∇_{μ} is the Levi-Civita connection associated to g.

Proof. First we note that for any r-form ω

$$d\omega = \nabla\omega \equiv \frac{1}{r!}\nabla_{\mu}\omega_{\nu_1\ldots\nu_r}dx^{\mu}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_r}, \tag{A.48}$$

where the first equality is a result of the symmetry $\Gamma^{\kappa}{}_{\mu\nu} = \Gamma^{\kappa}{}_{\nu\mu}$. Secondly, using $\nabla_Z g = 0$, we have the following equalities,

$$\begin{aligned} (\nabla_Z\Omega)(X,Y) &= \nabla_Z\{\Omega(X,Y)\} - \Omega(\nabla_Z X,Y) - \Omega(X,\nabla_Z Y) \\ &= \nabla_Z\{g(JX,Y)\} - g(J\nabla_Z X,Y) - g(JX,\nabla_Z Y) \\ &= (\nabla_Z g)(JX,Y) + g(\nabla_Z JX,Y) - g(J\nabla_Z X,Y) \\ &= g(\nabla_Z JX - J\nabla_Z X,Y) = g((\nabla_Z J)X,Y), \end{aligned} \tag{A.49}$$

for any X,Y,Z. Thus we have that $\nabla_Z\Omega = 0$ if and only if $\nabla_Z J = 0$. □

Let g be a Kähler metric, from $d\Omega = 0$ we have

$$\begin{aligned}
&(\partial + \bar{\partial}) i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu \\
&\quad = i\partial_\lambda g_{\mu\bar{\nu}} dz^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu + i\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} d\bar{z}^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu \\
&\quad = \tfrac{1}{2} i(\partial_\lambda g_{\mu\bar{\nu}} - \partial_\mu g_{\lambda\bar{\nu}}) dz^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu \\
&\qquad + \tfrac{1}{2} i(\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} - \partial_{\bar{\nu}} g_{\mu\bar{\lambda}}) d\bar{z}^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu = 0,
\end{aligned} \tag{A.50}$$

that is,

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial z^\lambda} = \frac{\partial g_{\lambda\bar{\nu}}}{\partial z^\mu}, \qquad \frac{\partial g_{\mu\bar{\nu}}}{\partial \bar{z}^\lambda} = \frac{\partial g_{\mu\bar{\lambda}}}{\partial \bar{z}^\nu}, \tag{A.51}$$

implying that the metric is, in fact, torsion free:

$$\begin{aligned}
T^\lambda{}_{\mu\nu} &= g^{\bar{\xi}\lambda}(\partial_\mu g_{\nu\bar{\xi}} - \partial_\nu g_{\mu\bar{\xi}}) = 0, \\
T^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}} &= g^{\bar{\lambda}\xi}(\partial_{\bar{\mu}} g_{\bar{\nu}\xi} - \partial_{\bar{\nu}} g_{\bar{\mu}\xi}) = 0.
\end{aligned} \tag{A.52}$$

In this sense, the Kähler metric defines a connection which is very similar to the Levi-Civita connection. The Riemann tensor has, for this metric, an extra symmetry

$$R^\kappa{}_{\lambda\mu\bar{\nu}} = -\partial_{\bar{\nu}}(g^{\bar{\xi}\kappa}\partial_\mu g_{\lambda\bar{\xi}}) = -\partial_{\bar{\nu}}(g^{\bar{\xi}\kappa}\partial_\lambda g_{\mu\bar{\xi}}) = R^\kappa{}_{\mu\lambda\bar{\nu}}. \tag{A.53}$$

This means that $\mathcal{R}_{\mu\bar{\nu}} = R^\kappa{}_{\kappa\mu\bar{\nu}} = R^\kappa{}_{\mu\kappa\bar{\nu}} = R_{\mu\bar{\nu}}$, i.e. the components of the Ricci form are the same as the components of the Ricci tensor. If $\mathcal{R}_{\mu\bar{\nu}} = R_{\mu\bar{\nu}} = 0$, the Kähler metric is said to be Ricci flat.

Theorem A.18. Let (M, g) be a Kähler manifold. If M admits a Ricci flat metric h, its first Chern must vanish.

Proof. By assumption $\mathcal{R}(h) = 0$. As was shown before, $\mathcal{R}(g) - \mathcal{R}(h) = \mathcal{R}(g) = d\omega$. Hence, $c_1(M)$ computed from g agrees with that computed from h and hence vanishes. □

Compact Kähler manifolds with vanishing first Chern classes are called *Calabi-Yau manifolds*. Calabi conjectured that if $c_1(M) = 0$, the Kähler manifold M admits a Ricci-flat metric, a conjecture that was later proven by Yau.

The last remarks here consider the *holonomy groups* of Kähler manifolds.

Definition A.19. Let (M, g) be a Riemannian manifold with an affine connection ∇ and let $p \in M$. We consider the set of closed loops at p: $\{c(t) | 0 \leq t \leq 1, c(0) = c(1) = p\}$. Now, take a vector $X \in T_pM$ and parallel transport it around a curve $c(t)$. After this trip, we end up with a new vector $X_c \in T_pM$. Thus we see that the loop $c(t)$ and the connection ∇ induce a linear transformation

$$P_c : T_pM \to T_pM. \tag{A.54}$$

The set of these transformations is denoted by $H(p)$ and is called the *holonomy group*[47].

[47] The holonomy group is, in fact, base point (p-) independent, but not connection independent [14].

(M, g) is taken to be a Hermitian manifold with $\dim_{\mathbb{C}} M = m$. Now, if we parallel transport a vector $X \in T_pM^+$ around a loop c at p, we end up with a vector $X' \in T_pM^+$ for which $X'^\mu = X^\nu h_\nu{}^\mu$. As ∇ does not mix holomorphic and anti-holomorphic indices, X' has no components in T_pM^-, and as ∇ preserves the length of a vector, we must have $h_\nu{}^\mu(c) \in U(m) \subset O(2m)$.

Theorem A.20. If g is the Ricci-flat metric of an m-dimensional Calabi-Yau manifold M, the holonomy group is contained in $SU(m)$.

Proof. The proof will be sketchy. We parallel transport a vector $X \in T_pM^+$ around a small parallelogram $pqrs$, the vertices of which have coordinates z^μ, $z^\mu + \epsilon^\mu$, $z^\mu + \epsilon^\mu + \bar{\delta}^\mu$ and $z^\mu + \bar{\delta}^\mu$ respectively. For parallel transport from p to q we have (up to first order) $X''^\mu(q) = X^\mu(p) - X^\mu(p)\Gamma^\kappa{}_{\nu\kappa}(p)\epsilon^\nu$, etcetera. Around the full loop this becomes

$$X'^\mu = X^\mu + X^\nu R^\mu{}_{\nu\kappa\bar{\lambda}}\epsilon^\kappa\bar{\delta}^\lambda, \tag{A.55}$$

and so

$$h_\mu{}^\nu = \delta_\mu{}^\nu + R^\nu{}_{\mu\kappa\bar{\lambda}}\epsilon^\kappa\bar{\delta}^\lambda. \tag{A.56}$$

$U(m)$ decomposes as $U(m) = SU(m) \times U(1)$ in the vicinity of the unit element. In particular the Lie algebra $u(m) = T_e(U(m))$ is separated into $u(m) = su(m) \oplus u(1)$, where $su(m)$ is the traceless part of $u(m)$ and $u(1)$ contains the trace. Now, since the metric is Ricci flat, the trace part vanishes

$$R^\kappa{}_{\kappa\mu\bar{\nu}}\epsilon^\mu\bar{\delta}^\nu = \mathcal{R}_{\mu\bar{\nu}}\epsilon^\mu\bar{\delta}^\nu = 0. \tag{A.57}$$

This shows that the holonomy group (or, strictly speaking, only the restricted holonomy group[48]—which remains true even when M is multiply connected) is contained in $SU(m)$. $\square$

A.5 Almost complex manifolds

Definition A.21. Let M be a differentiable manifold. The pair (M, J), is called an *almost complex manifold* if there exists a tensor field J of type $(1, 1)$ such that at each point $p \in M$, $J_p^2 = -\mathrm{id}_{T_pM}$. J is called the *almost complex structure.*

Since $J_p^2 = -\mathrm{id}_{T_pM}$, J_p has eigenvalues $\pm i$. If it has m that are $+i$, it must have also m that are $-i$, making J_p into a $2m \times 2m$ matrix. This means that M must be even-dimensional. Complex manifolds are always almost complex as well. The converse, however, is not true in general.

Next, we complexify the tangent space at a point p of an almost complex manifold M so that the complex eigenvalues of J_p may appear explicitly. Once extended to a $\mathbb{C}$-linear map on $T_pM\mathbb{C}$, the identity $J_p^2 = -1_{2m}$ is of course still valid. $T_pM^{\mathbb{C}}$ may be split up into two disjoint vector subspaces, according to the eigenvalue of J_p:

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-, \tag{A.58}$$

[48] The restricted holonomy group considers only closed curves that can be contracted continuously to a point.

where $T_pM^\pm = \{Z \in T_pM^\mathbb{C} | J_pZ = \pm iZ\}$.

Any vector $V \in T_pM^\mathbb{C}$ is then written as $V = W_1 + \bar{W}_2$, where $W_1, W_2 \in T_pM^+$. We also separate $\mathcal{X}(M)^\mathbb{C}$ into $\mathcal{X}(M)^\pm$, the treatment differing only in the less strict condition J_p is required to satisfy. A consequence is that there does not necessarily exist a basis of T_pM^+ of the form $\{\partial/\partial z^\mu\}$. We may still define the projectors

$$\mathcal{P}^\pm \equiv \tfrac{1}{2}(\mathrm{id}_{T_pM} \mp iJ_p) : T_pM^\mathbb{C} \to T_pM^\pm. \tag{A.59}$$

A vector in $T_pM^{+(-)}$ is again called a(n) (anti-)holomorphic vector and a vector field in $\mathcal{X}(M)^{+(-)}$ a(n) (anti-)holomorphic vector field.

Definition A.22. Let (M, J) be an almost complex manifold. If the Lie bracket of any holomorphic vector fields $X, Y \in \mathcal{X}(M)^+$ is again a holomorphic vector field, $[X, Y] \in \mathcal{X}(M)^+$, the almost complex structure is said to be *integrable*.

Remark A.23. Actually, the definition above is a something of a shortcut. It is the consequence of the Frobenius theorem, which states the following.

Theorem A.24. (Frobenius.) A distribution L is integrable if and only if it is involutive.

To make sense out of this statement we need a few definitions.

Definition A.25. Sub-bundles L of the tangent bundle TM ($TM \equiv \cup_p T_pM$) that are locally spanned by smooth vector fields are called *distributions*.

Note that if J is globally defined and smooth, the sub-bundles $TM^\pm \equiv \cup_p T_pM^\pm \subset TM \otimes \mathbb{C}$ will be distributions. Note also that the local bases are not necessarily globally defined, this follows from the fact that the group of transition functions (the structure group) preserves J from patch to patch, also preserving the decomposition in $TM^\pm$, but mixes up the individual basis vectors.

Definition A.26. A distribution L is called *involutive* if for any two vector fields $X, Y \in \Gamma(L)$ ($\Gamma(L)$ is the space of sections of L),

$$X, Y \in \Gamma(L) \Rightarrow [X, Y] \in \Gamma(L). \tag{A.60}$$

The last definition we need is the following.

Definition A.27. A distribution L is *integrable* if through every point $P \in M$ there exists (a vector-valued funtion x that is) a solution $x(\lambda^1, \ldots, \lambda^{\mathrm{rank}(L)}; p)$ in a neighbourhood of p of the following equation:

$$\frac{\partial x}{\partial \lambda^a} = X_a, \tag{A.61}$$

where the vector fields X_a, $a = 1, \ldots, \dim(L)$, locally span L.

As a side-note, the solutions of the equation above defines a new system of coordinates (called *adapted coordinates*) such that L is locally spanned by $\{\partial/\partial x^i | i = 1, \ldots, \mathrm{rank}(L)\}$.

Also, on such a manifold M, we define the *Nijenhuis tensor field* $N : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ by

$$N(X, Y) \equiv [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \tag{A.62}$$

$N(X, Y)$ is linear in X and Y as J and the Lie bracket are linear. If J is a complex structures, N trivially vanishes.

Theorem A.28. An almost complex structure J on a manifold M is integrable if and only if $N(A, B) = 0$ for all $A, B \in \mathcal{X}(M)$.

Proof. Let $Z = X + iY, W = U + iV \in \mathcal{X}(M)^{\mathbb{C}}$. The Nijenhuis tensor field may then be extended to act on $\mathcal{X}(M)^{\mathbb{C}}$ as follows

$$\begin{aligned} N(Z, W) &= [Z, W] + J[JZ, W] + J[Z, JW] - [JZ, JW] \\ &= \{N(X, U) - N(Y, V)\} + i\{N(X, V) + N(Y, U)\}. \end{aligned} \tag{A.63}$$

Suppose $N(A, B) = 0$ for all $A, B \in \mathcal{X}(M)$. The equation above then implies that $N(Z, W) = 0$ for $Z, W \in \mathcal{X}(M)^{\mathbb{C}}$. In particular, for $Z, W \in \mathcal{X}(M)^+$ we have, since $JZ = iZ$ and $JW = iW$, $N(Z, W) = 2\{[Z, W] + iJ[Z, W]\}$. $N(Z, W) = 0$ then implies $J[Z, W] = i[Z, W]$ and $[Z, W] \in \mathcal{X}(M)^+$. The almost complex structure is integrable.

Conversely, suppose that J is integrable. Since $\mathcal{X}(M)^{\mathbb{C}} = \mathcal{X}(M)^+ \oplus \mathcal{X}(M)^-$, we can separate Z and W into $Z = Z^+ + Z^-$ and $W = W^+ + W^-$. Then

$$N(Z, W) = N(Z^+, W^+) + N(Z^+, W^-) + N(Z^-, W^+) + N(Z^-, W^-) \tag{A.64}$$

and it is easy to compute that each of the contributions on the right hand side yield zero. Thus $N(Z, W) = 0$ for all $Z, W \in \mathcal{X}(M)^{\mathbb{C}}$ and, in particular, for $Z, W \in \mathcal{X}(M)$. $\square$

We had already found that for a complex structure J on a complex manifold the Nijenhuis tensor field vanishes. For the converse we have the following theorem.

Theorem A.29. Let (M, J) be a $2m$-dimensional almost complex manifold. If J is integrable, the manifold M is a complex manifold with the almost complex structure J.

B Type II $\mathcal{N} = 1$ flux vacua

The solutions of the susy variation equations of section 5.3.2, for IIA and IIB Minkowksi flux compactifications, are shown in tables 1 and 2.

The last column in table 2 corresponds to so-called intermediate "ABC" solutions, satisfying the following two sets of equations. The first set, corresponding to $(*)$, is

$$\begin{aligned} 2abW_3 &= e^{\Phi}(a^2 + b^2) *_6 F_3^{(6)}, \\ (a^2 - b^2)W_3 &= -(a^2 + b^2) *_6 H^{(6)}, \\ 2abH^{(6)} &= -e^{\Phi}(a^2 - b^2)F_3^{(6)}. \end{aligned} \tag{B.1}$$

<table>
<tr><th>IIA</th><th>$a=0$ or $b=0$ (A)</th><th colspan="2">$a=b\,e^{i\beta}$ (BC)</th></tr>
<tr><td rowspan="2">1</td><td colspan="3">$W_1=H^{(1)}=0$</td></tr>
<tr><td>$F_0^{(1)}=\mp F_2^{(1)}=$
$F_4^{(1)}=\mp F_6^{(1)}$</td><td colspan="2">$F_{2n}^{(1)}=0$</td></tr>
<tr><td rowspan="2">8</td><td rowspan="2">$W_2=F_2^{(8)}=F_4^{(8)}=0$</td><td>generic β</td><td>$\beta=0$</td></tr>
<tr><td>$W_2^+=e^\Phi F_2^{(8)}$
$W_2^-=0$</td><td>$W_2^+=e^\Phi F_2^{(8)}+e^\Phi F_4^{(8)}$
$W_2^-=0$</td></tr>
<tr><td>6</td><td>$W_3=\mp *_6\,H^{(6)}$</td><td colspan="2">$W_3=H^{(6)}=0$</td></tr>
<tr><td>3</td><td>$\bar W_5=2W_4=$
$\mp 2iH^{(\bar 3)}=\bar\partial\Phi$
$\bar\partial A=\bar\partial a=0$</td><td colspan="2">$F_2^{(\bar 3)}=2i\bar W_5=-2i\bar\partial A=\frac{2}{3}i\bar\partial\Phi$, $W_4=0$</td></tr>
</table>

Table 1: Possible $\mathcal{N}=1$ vacua in IIA [8].

<table>
<tr><th>IIB</th><th>$a=0$ or $b=0$ (A)</th><th colspan="2">$a=\pm ib$ (B)</th><th>$a=\pm b$ (C)</th><th>(ABC)</th></tr>
<tr><td>1</td><td colspan="5">$W_1=F_3^{(1)}=H^{(1)}=0$</td></tr>
<tr><td>8</td><td colspan="5">$W_2=0$</td></tr>
<tr><td>6</td><td>$F_3^{(6)}=0$
$W_3=\pm *H^{(6)}$</td><td colspan="2">$W_3=0$
$e^\phi F_3^{(6)}=\mp *H_3^{(6)}$</td><td>$H_3^{(6)}=0$
$W_3=\pm e^\phi *F_3^{(6)}$</td><td>$(*)$</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">$\bar W_5=2W_4=$
$\mp 2iH^{(\bar 3)}=2\bar\partial\Phi$
$\bar\partial A=\bar\partial a=0$</td><td colspan="2">$e^\Phi F_5^{(\bar 3)}=\frac{2}{3}i\bar W_5=iW_4=$
$-2i\bar\partial A=-4i\bar\partial\log a$
$\bar\partial\Phi=0$</td><td rowspan="2">$\pm e^\Phi F_3^{(\bar 3)}=2i\bar W_5=$
$-2i\bar\partial A=-4i\bar\partial\log a=$
$-i\bar\partial\Phi$</td><td rowspan="2">$(**)$</td></tr>
<tr><td>F</td><td>$e^\Phi F_1^{(\bar 3)}=2e^\Phi F_5^{(\bar 3)}=$
$i\bar W_5=iW_4=i\bar\partial\Phi$</td></tr>
</table>

Table 2: Possible $\mathcal{N}=1$ vacua in IIB [8].

Note that the upper '(n)' indicate the representation under consideration. $*_6$ is a six-dimensional Hodge-star. The second set, corresponding to $(**)$, is

$$
\begin{aligned}
e^\Phi F_3^{(\bar 3)} &= \frac{-4i\,ab(a^2+b^2)}{a^4-2ia^3b+2iab^3+b^4}\,\bar\partial a\,, \\
e^\Phi F_5^{(\bar 3)} &= \frac{-4\,ab(a^2-b^2)}{a^4-2ia^3b+2iab^3+b^4}\,\bar\partial a\,, \\
H^{(\bar 3)} &= \frac{-2i(a^2+b^2)(a^2-b^2)}{a^4-2ia^3b+2iab^3+b^4}\,\bar\partial a\,,
\end{aligned}
\qquad
\begin{aligned}
W_4 &= \frac{2(a^2-b^2)^2}{a^4-2ia^3b+2iab^3+b^4}\bar\partial a\,, \\
\bar W_5 &= \frac{2(a^4-4a^2b^2+b^4)}{a^4-2ia^3b+2iab^3+b^4}\bar\partial a\,, \\
\bar\partial A &= -\frac{4(ab)^2}{a^4-2ia^3b+2iab^3+b^4}\bar\partial a\,, \\
\bar\partial\Phi &= \frac{2(a^2+b^2)^2}{a^4-2ia^3b+2iab^3+b^4}\bar\partial a\,.
\end{aligned}
\tag{B.2}
$$

The complex functions a and b satisfy $|a|^2+|b|^2=e^A$, reducing their four degrees of freedom to three. A second constraint reducing that number to two follows from gauge fixing their phase gauge freedom. That freedom amounts to the transformation $\eta_+\to e^{i\psi}\eta_+$, or, equivalently $(a,b)\to e^{i\psi}(a,b)$. Then also $\Omega\to e^{2i\psi}\Omega$, while J is left invariant. Furthermore $(W_1,W_2)\to e^{2i\psi}(W_1,W_2)$ and $W_5\to W_5+2id\psi$. As in the supersymmetry transformations, the gauge transformation of (a,b) cancels that of W_5, the overall phase of ab may be fixed (by rotating W_2–table 1 is given in a fixed gauge), reducing the number of degrees of freedom by one. Because of this, all $\mathcal{N}=1$ vacua with internal $SU(3)$ structure may be parametrized by two angles, as follows

$$\begin{aligned} a &= e^{A/2}\cos\alpha\, e^{i\frac{\beta}{2}}, \\ b &= e^{A/2}\sin\alpha\, e^{-i\frac{\beta}{2}}. \end{aligned} \tag{B.3}$$

These angles parametrize a $U(1)_R$ subspace in the $SU(2)_R$ symmetry of the underlying $\mathcal{N}=2$ theory.

B.1 The susy variations in terms of pure spinors

In the previous section we discussed the supersymmetry variation equations for the $\mathcal{N}=1$ case. We mentioned that imposing $\delta_\epsilon\psi^i_M=\delta_\epsilon\lambda^i=0$ translates to certain differential conditions on the internal spinor η. These conditions then turn into differential conditions on the pure $Spin(6,6)$ spinors $\psi_\pm$. The result is the following set of equations for the spinors. They are the generalization of those of the section 6.4 for general a and b.

$$\begin{aligned} e^{-2A+\Phi}d_H(e^{2A-\Phi}\tilde{\psi}_+) &= 0, \\ e^{-2A+\Phi}d_H(e^{2A-\Phi}\tilde{\psi}_-) &= dA\wedge\bar{\tilde{\psi}}_- - \tfrac{1}{16}e^{\Phi}\left[(|a|^2-|b|^2)F_{A-}\right. \\ &\qquad \left. -i(|a|^2+|b|^2)*F_{A+}\right], \end{aligned} \tag{B.4}$$

for type IIA, while for type IIB

$$\begin{aligned} e^{-2A+\Phi}d_H(e^{2A-\Phi}\tilde{\psi}_+) &= dA\wedge\bar{\tilde{\psi}}_+ + \tfrac{1}{16}e^{\Phi}\left[(|a|^2-|b|^2)F_{B+}\right. \\ &\qquad \left. -i(|a|^2+|b|^2)*F_{B-}\right], \\ e^{-2A+\Phi}d_H(e^{2A-\Phi}\tilde{\psi}_-) &= 0. \end{aligned} \tag{B.5}$$

In these equations we used

$$F_{A\pm}=F_0\pm F_2+F_4\pm F_6, \qquad F_{B\pm}=F_1\pm F_3+F_5. \tag{B.6}$$

$\tilde{\psi}_\pm$ are non-normalized pure $Spin(6,6)$ spinors, they are constructed in the same way as the $\psi_\pm$ were earlier: $\psi_\pm=\eta_+\,\eta^\dagger_\pm$, but now out of non-normalized $\tilde{\eta}^{1,2}$, defined by

$$\tilde{\eta}^1_+=a\,\eta^1_+, \qquad \tilde{\eta}^2_+=b\,\eta^2_+. \tag{B.7}$$

These are the internal spinors that help form the $\mathcal{N}=1$ supersymmetry parameter (note that in tables 1 and 2 we had $\eta^1=\eta^1$). The non-normalized $\tilde{\psi}_\pm$ are related to the normalized versions by

$$\tilde{\psi}_+=a\bar{b}\,\psi_+, \qquad \tilde{\psi}_-=ab\,\psi_-. \tag{B.8}$$

$\mathcal{N}=1$ supersymmetry imposes the following relations on these norms (for both IIA and IIB)

$$d|a|^2=|b|^2dA, \qquad d|b|^2=|a|^2dA. \tag{B.9}$$

The conclusions of section 6.4 remain essentially unchanged. The R-R fluxes still prevent the integrability of one spinor, while the other is Calabi-Yau à la Hitchin.

In the case of the pure $SU(3)$ structure, one finds from the generalized complex structures corresponding to the $\psi_{\pm}$ that internal manifold is complex for type IIB and (twisted) symplectic for type IIA. In the general $SU(3,3)$ case, the $\mathcal{N} = 1$ vacua may be hybrid complex-symplectic manifolds with k complex and $6 - 2k$ (real) symplectic dimensions. Given the chiralities of the preserved $Spin(6,6)$ spinors, k must be even in IIA and odd in IIB.

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