Every knot is a billiard knot

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Abstract

In this article we study Daniel Pecker's proof of the theorem saying that for an ellipse E (which is not a circle) and an elliptic cylinder $D = E \times [0, 1]$ it can be stated that every knot/link is a billiard knot/link. To understand the proof first billiard trajectories in an ellipse, Jacobian elliptic functions and Poncelet polygons must be introduced. Then there are some statements preceding the main theorem, containing the famous Poncelet's closure theorem, the irregularity of Poncelet odd polygons and Kronecker's density theorem. In the final proof both the situation of knots and the situation of links will be considered.

Keywords: Knot theory, Billiard knots, Poncelet's closure theorem, Jacobian elliptic functions, Winding numbers

1 Introduction

In recent history knot theory became a very popular subject in mathematics. At first there were the mathematicians Vaughan Jones (winner of a fields medal in 1990) and Jozef H. Przytycki, who defined billiard knots as periodic billiard trajectories without self-intersections in a three-dimensional billiard. They proved that billiard knots in a cube are very special knots, called the Lissajous knots, named after the French mathematician Jules Antoine Lissajous (1822-1880). They also presented the conjecture that every knot is a billiard knot in some convex polyhedron [3]. After that Christoph Lamm proved that not all knots are billiard knots in a cylinder. Also he presented the conjecture that every non-circular elliptic cylinder contains all knots as billiard knots [5]. This actually implies the conjecture of Jones and Przytycki, because if K is a billiard knot in a convex set (i.e. a set in which all points, which lie on a line between two arbitrary points of the set, are part of the set), then it is also a billiard knot in the polyhedron delimited by the tangent planes. Patrick Dehornoy constructed a billiard which contains all knots, but this billiard is not convex [2]. In this article we study the work of Daniel Pecker who proved that every knot is a billiard knot under certain conditions [7].

Pecker considered an ellipse E (which is not a circle) and an elliptic cylinder $D = E \times [0, 1]$. Then he states that every knot/link is a billiard knot/link in D. He proves this in the following way. First he introduces the subjects of billiard trajectories and Poncelet polygons. Then he introduces Jacobian elliptic functions. He needs these to recall the Hermite-Laurent proof of Poncelet's closure theorem. The advantage of this proof is that it provides explicit computations of the Poncelet polygons, using Jacobian elliptic functions. Also he uses these functions to compute the coordinates of crossings and vertices. With this he can prove that among Poncelet polygons with an odd number of sides there are totally irregular ones. Combining this with Kronecker's density theorem, the proof of the main theorem follows. In this article we will discuss Pecker's proof in detail.

The first three sections will introduce some necessary tools. Section 2 deals with billiard trajectories, section 3 with Jacobian elliptic functions and section 4 with Poncelet polygons. Then in section 5 some results will be presented in which Jacobian elliptic functions are used on Poncelet polygons. Here the famous Poncelet's closure theorem and the irregularity of Poncelet odd polygons are discussed. In section 6 we take a look at the main theorem and Pecker's proof of it. And in the end, in section 7, we will draw some conclusions and do proposals for further research.

2 Billiard trajectories in an ellipse

This section will first provide some general information about billiard trajectories in an ellipse and after that some classical results will be presented. In 1927 this subject has been introduced by the American mathematician George David Birkhoff (1884-1944) [1].

2.1 General information

Billiard trajectories are formed by the free motion of a point particle in a domain with specular reflections at the boundary of the domain. This is comparable with a ball which keeps on rolling through the domain without friction for an infinite amount of time. But one must be careful with saying that, because a ball isn't quite the same as a point particle. Differences are that a ball does have friction and it also does have a moment of inertia.

In the upcoming text we will consider the special case of a billiard trajectory inside an ellipse. You can define an ellipse by two fixed points F_1 and F_2, which are called the foci of the ellipse. The segment between these two points we call the focal segment $[F_1F_2]$. An ellipse exists of all the points P, for which the sum of the distances to F_1 and F_2 equals $2a$, where a stands for the length of half the major axis. These distances PF_1 and PF_2 are called the focal radii of the point P.

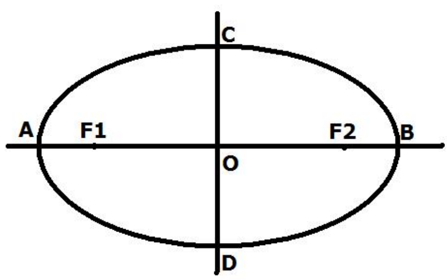

Figure 1: The foci and axes of an ellipse.

Now let us call O the middle point of the ellipse, the endpoints of the major axis A and B and the endpoints of the minor axis C and D. Then the length of the minor axis is equal to twice the distance between C and O. The length of the major axis is equal to twice the distance between C and F_2. The relation between the lengths then is $(2CF_2)^2 = (2CO)^2 + (2OF_2)^2$, where CF_2, CO and OF_2 are the distances between the two points. The circle can be seen as a special ellipse, where the distance between O and F_2 is equal to zero, such that the distances CF_2 and CO are equal to each other.

Useful to know for our subject is the fact that the angle of the line PF_1 with the tangent line to P is equal to the angle of the line PF_2 with the tangent line to P. If you take the normal to P, you will see that the angle of PF_1 with this normal is equal to the angle of PF_2 with this normal. Here the normal is the line that is orthogonal to the tangent.

Now let us return to our point particle. If the ball without friction or point particle passes through one of the foci, then it is sure that it will pass through the other focus after it has been reflected at the boundary of the domain. This is a consequence of the law of reflection, which states that the angle between the incoming path and the normal is equal to the angle between the outgoing path and the normal. An impression of this law can be seen in figure 2.

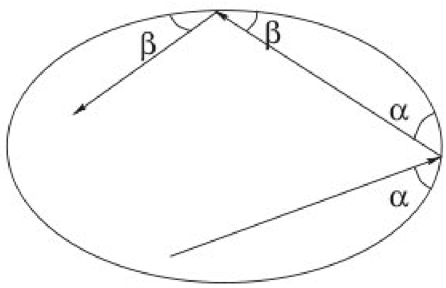

Figure 2: The law of reflection. [8]

In general a point particle doesn't hit one of the foci. Then there are two cases left. In the first case the point particle does not pass between the two foci. This will lead to the point particle never passing between the two foci, no matter how many times it is reflected. In the second case the point particle does pass between the two foci. This will lead to the point particle always passing between the two foci, no matter how many times it is reflected. Both statements are again a consequence of the law of reflection.

2.2 Classical results

Now we know some more about billiard trajectories in an ellipse, we can take a look at some classical results, surrounding this subject.

The following theorem will prove that if the initial segment of a billiard path in ellipse E avoids the focal segment $[F_1F_2]$ (this is the segment between the foci F_1 and F_2) of

E, then there exists an ellipse C called the caustic, such that the path is circumscribed about C. Caustic here stands for a curve that is tangent to each member of the family of segments, which are reflected at the inside of the boundary of the ellipse E.

Theorem 1. *Suppose that some segment of a billiard trajectory in an ellipse does not intersect the focal segment $[F_1F_2]$. Then the billiard trajectory remains forever tangent to a fixed confocal ellipse called the caustic.*

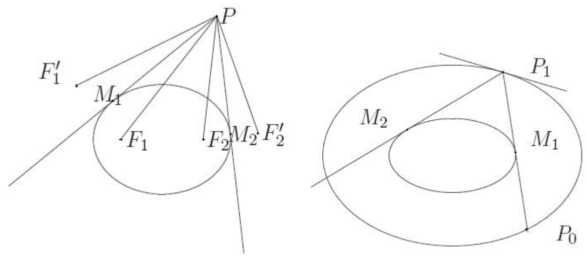

Figure 3: Reflecting F_1 and F_2 in P_0P_1 and the existence of the caustic. [7]

Proof: Let P_0P_1 be the segment of a billiard trajectory in an ellipse E, that does not intersect the focal segment $[F_1F_2]$. Consider ellipse $C = \{M \in \mathbb{R}^2 \mid MF_1 + MF_2 = F_2F_1'\}$, where F_1' can be found by reflecting F_1 in P_0P_1, like shown in Figure 3. Then we see that $M_1 = F_2F_1' \cap P_0P_1$ (i.e. M_1 is the point where these two lines intersect) belongs to C. Since P_0P_1 divides the angle $\widehat{F_1M_1F_2}$ in two equal parts, we can call P_0P_1 the bisector of $\widehat{F_1M_1F_2}$. Hence P_0P_1 is tangent to C at M_1. Now draw P_1M_2 as the second tangent to C. Then, because P_1M_1 and P_1M_2 are both tangent to C at respectively M_1 and M_2, it follows that angles $\widehat{M_1P_1M_2}$ and $\widehat{F_1M_1F_2}$ have the same bisectors. You can see this in the following way. Reflect F_2 in P_1M_2 to get F_2', just as we did earlier to get F_1'. Because P_1M_1 is a bisector of $\widehat{F_1M_1F_2}$, it follows that F_1', M_1 and F_2 are collinear, so $F_1'F_2 = M_1F_1 + M_1F_2$, which is the major axis of C. Furthermore, $F_1'F_2 = F_1F_2'$ and the triangles $F_1'P_1F_2$ and $F_1P_1F_2'$ are congruent, so $\widehat{F_1'P_1F_2} = \widehat{F_1P_1F_2'}$. Hence $\widehat{F_1'P_1F_1} = \widehat{F_1'P_1F_2} - \widehat{F_1P_1F_2} = \widehat{F_2'P_1F_1} - \widehat{F_1P_1F_2} = \widehat{F_2'P_1F_2}$, which gives us the desired result that angles $\widehat{M_1P_1M_2}$ and $\widehat{F_1M_1F_2}$ have the same bisectors. From this statement it is clear that P_1M_2 is the second segment of the billiard trajectory and it is tangent to C. With induction follows the rest. $\square$

The preceding theorem assumed that some segment does not intersect the focal segment. But what happens if some segment does contain only one focus or intersects the interior of the focal segment? In the first case this would mean that every segment contains a focus and there would not be a caustic. In the second case there is a caustic, which is a hyperbola with foci F_1 and F_2. With this in mind, let us take a look at when we can speak of a periodic billiard trajectory.

Corollary 2. *Let $P_0, P_1, \ldots, P_{n-1}, P_n = P_0$ be a billiard trajectory in an ellipse E such that P_0P_1 does not intersect the focal segment $[F_1F_2]$. Then it is a periodic billiard trajectory inscribed in E and circumscribed about a confocal ellipse C.*

Proof: Since $P_{n-1}P_0$ does not intersect the focal segment, its reflection at P_0 cannot be P_0P_{n-1}. Since it is another tangent to C through P_0, it must be P_0P_1, from which it follows that $P_{n+1} = P_1$. Hence we have proven that we have a periodic billiard trajectory here. □

The next proposition will prove that if $n \geq 2k+1$, then there exist n-periodic trajectories travelling k times around the caustic. The number of times these trajectories travel around the caustic will be denoted by the winding number p. Examples of billiard trajectories with period n and winding number p can be found in section 4.

Proposition 3. *Let P_0 be a point on the ellipse E and let n, p be coprime integers such that $n \geq 2p+1$. Coprime means that the only common divisor of n and p is 1. Then there exists a billiard trajectory in E of period n and winding number p, which starts at P_0.*

Proof: Let ellipse E be defined by the foci F_1 and F_2 and let the major axis of this ellipse have length $2a$. Consider a 1-parameter family of caustics defined by $C_\delta = \{M \in \mathbb{R}^2 \mid MF_1 + MF_2 = 2\delta, \delta \in \mathbb{R}\}$. C_δ shrinks down to the focal segment $[F_1F_2]$ when $2\delta = F_1F_2$ and expands to E if $\delta = a$. When C_δ is the focal segment, then the Poncelet trajectory $P_0, P_1, \ldots, P_n$ passes alternately through one focus and the other. Consequently the winding number ω of this trajectory is greater than $n/2$. When δ varies from $F_1F_2/2$ to a, then the winding number varies continuously from ω to 0. Hence the desired (integral) winding number is achieved. By the preceding corollary we can then say that this Poncelet trajectory is a periodic one. Since p and n are coprime, its exact period is n. □

It is even possible to say that the caustics $C_{n,p}$ do not depend on the initial point P_0, but that hasn't been proven yet by the Proposition 3. More about this can be found in Section 5.

3 Jacobian elliptic functions

This section will specifically take a look at the Jacobian elliptic functions. These are named after the German mathematician Carl Gustav Jacob Jacobi (1804-1851). Jacobian elliptic functions occur to solve equations of the pendulum and rigid body motion, where the usual trigonometric functions are used to solve equations of harmonic oscillators and free rotators. For more information on elliptic functions in general we refer to Appendix A.

3.1 Defining sn z, cn z and dn z

To define the Jacobian elliptic function we first must find a way to describe the Jacobian amplitude $\operatorname{am}(z)$. This can be done by inverting the elliptic integral

$$z = \int_0^\phi \frac{dt}{\sqrt{1 - k^2 \sin^2(t)}}. \tag{1}$$

Here k is called the elliptic modulus, for which you can choose values between 0 and 1. We find this elliptic integral by first seeing that the sine function can be defined as the inverse $x = x(u)$ of the relation $u = u(x)$, where

$$u(x) = \int_0^x \frac{dz}{\sqrt{1-z^2}} = \arcsin(x). \tag{2}$$

Here $u(x)$ can be seen as a path integral on the Riemann surface of the polynomial $w^2 = 1 - z^2$, whose critical point are at $z = +1$ and $z = -1$. The inverse $x = x(u)$ can be found in integral-form as $x = \sin(u)$ or in differential form as

$$du = \frac{dz}{\sqrt{1-z^2}}, \tag{3}$$

where $z =: \sin(u)$. Jacobi used this to define the sinus amplitudinis sn by using

$$du = \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \tag{4}$$

with $z =: \operatorname{sn}(u)$ or the equivalent integral form

$$u = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{5}$$

By the substitution $z = \sin(t)$ we obtain the relation $z = \sin(t) =: \sin(am(u)) = \operatorname{sn}(u)$, which gives the elliptic integral z.

From the integral (1) we see that $\operatorname{am}(u + 2nK) = \operatorname{am}(u) + n\pi$, where $\operatorname{am}(K) = \pi/2$ and $n \in \mathbb{Z}$. Then the Jacobian elliptic functions for real z are $\operatorname{sn}(z) = \sin(\operatorname{am}(z))$, $\operatorname{cn}(z) = \cos(\operatorname{am}(z))$ and $\operatorname{dn}(z) = \sqrt{1 - k^2 \operatorname{sn}^2(z)}$. These can be extended to meromorphic functions on $\mathbb{C}$. For $k = 0$ these functions are similar to the sine and cosine, but the difference is that elliptic functions are doubly periodic as function on $\mathbb{C}$ with periods $4K \in \mathbb{R}$ and $4iK' \in i\mathbb{R}$, where K' stands for K on the imaginary axis, and they also have poles. For example, the poles of $\operatorname{sn}(z)$ are congruent to $iK' \mod (2K, 2iK')$, the zeros of $\operatorname{sn}(z)$ are points congruent to $0 \mod (2K, 2iK')$ and the exact periods of $\operatorname{sn}(z)$ are $4K$, $2iK'$. The zeros of $\operatorname{cn}(z)$ are the points congruent to $K \mod (2K, 2iK')$). What follows here is that $\operatorname{sn}(z + 2K) = -\operatorname{sn}(z)$, $\operatorname{cn}(z + 2K) = -\operatorname{cn}(z)$ and $\operatorname{sn}(K + iK') = k^{-1}$, which implies that the zeros of $\operatorname{dn}(z)$ are the points congruent to $K + iK' \mod (2K, 2iK')$.

3.2 Useful formulas

In Section 5 several formulas based on Jacobian elliptic functions will be useful to prove some main facts. At first there are the addition formulas

$$\operatorname{sn}(x+y) = \frac{\operatorname{sn}(x)\operatorname{cn}(y)\operatorname{dn}(y) + \operatorname{sn}(y)\operatorname{cn}(x)\operatorname{dn}(x)}{1 - k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)}, \tag{6}$$

$$\operatorname{cn}(x+y) = \frac{\operatorname{cn}(x)\operatorname{cn}(y) - \operatorname{sn}(x)\operatorname{sn}(y)\operatorname{dn}(x)\operatorname{dn}(y)}{1 - k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)}. \tag{7}$$

Also there is a special formula due to Jacobi, which is of great importance in upcoming proofs:

$$\sin(\operatorname{am}(u+v) + \operatorname{am}(u-v)) = \frac{2\operatorname{sn}(u)\operatorname{cn}(u)\operatorname{dn}(v)}{1 - k^2\operatorname{sn}^2(u)\operatorname{sn}^2(v)}. \tag{8}$$

In Section 5 can be seen how these formulas will be used.

4 Poncelet polygons

This section will first introduce the notions of knots and links. Then we summarize several facts about (quasi)toric braids. Using a theorem on (quasi)toric braids due to Christoph Lamm and Vassily Manturov, we deduce that every knot has a planar projection, which is a billiard path in an ellipse. For some background information on braid theory we refer to Appendix B. After this we will take a look at some examples of Poncelet polygons. These are named after the famous French mathematician Jean-Victor Poncelet (1788-1867), who is one of the 72 French people whose name is on the Eiffel Tower.

4.1 Knots and links

A knot is a smooth embedding (i.e. a mathematical structure contained within another) of a circle in the three-dimensional Euclidean space $\mathbb{R}^3$. A mathematical knot differs from a usual knot in the sense that the ends of a mathematical knot are joined together, so that it cannot be undone. Two knots are equivalent or ambient isotopic if one can be transformed to the other through a deformation (i.e. a family of analytic spaces depending on parameters) of $\mathbb{R}^3$.

A link is a collection of knots which do not intersect. This means that a link is an embedding of the disjoint union of several circles in $\mathbb{R}^3$. To compare both terms, take a submanifold M of a manifold N and a non-trivial embedding of M in N. The latter means that both embeddings M and N are not isotopic to each other. If M is disconnected (i.e. it could be represented as the union of two or more disjoint nonempty open sets), the embedding is called a link. If M is connected, the embedding is called a knot.

A billiard knot is defined as the trajectory inside a domain, which leaves the boundary at rational angles with respect to the natural frame, and travels in a straight line except for reflecting perfectly off the boundaries. Because in general it will miss the "corners" and "edges" of the domain, the trajectory will form a (billiard) knot.

4.2 Toric braids and quasitoric braids

We will now look at a standardly embedded torus knot, which is a special kind of knot that lies on the surface of an unknotted torus in $\mathbb{R}^3$. Each torus knot is specified by a pair of coprime integers p and q. A toric braid is a braid corresponding to the closed braid obtained by projecting the torus knot into the xy-plane. It has the form $\tau_{p,n} = (\sigma_1\sigma_2\cdots\sigma_{p-1})^n$, where $\sigma_1,\ldots,\sigma_{p-1}$ are the standard generators of the full braid group B_p. By changing some crossings in the toric braid $\tau_{p,n}$ we can obtain a quasitoric braid. The Lamm-Manturov theorem tells us that every knot/link is realized as the closure of a quasitoric braid [6]. More precisely, every μ-component link can be realized as the closure of a quasitoric braid of type $B(p\mu, n\mu)$ where $(p,n) = 1$, p even and n odd. Quasitoric braids form a subgroup of the full braid group, hence there exists trivial quasitoric braids of arbitrary length. Consequently, we can suppose $n \geq 2p+1$ in the Lamm-Manturov theorem, just like we did in Proposition 3.

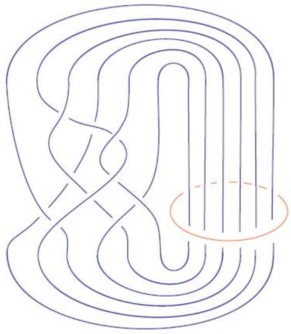

Figure 4: The closure of a braid.

4.3 Examples of Poncelet polygons

Let E and C be nested ellipses, such that the billiard trajectory $P_0, P_1, \ldots, P_{n-1}, P_n = P_0$, which is periodic, is inscribed in E and circumscribed by C. Then you can call this a Poncelet polygon. Every Poncelet polygon is the projection of a torus knot. Moreover, if we cut the elliptic annulus delimited by E and C (i.e. the remaining "donut" after removing ellipse C from ellipse E) along a half-tangent, then such a polygon is ambient isotopic (i.e. equivalent) to the projection of the closure of the toric braid $\tau_{p,n}$. Consequence of this all is, that it is even ambient isotopic to the star polygon $\binom{n}{p}$. Here a star polygon is a non-convex polygon, which has the shape of a star.

Figure 5 will show some examples of Poncelet polygons. For even more examples on periodic orbit representations like these we refer to Appendix C.

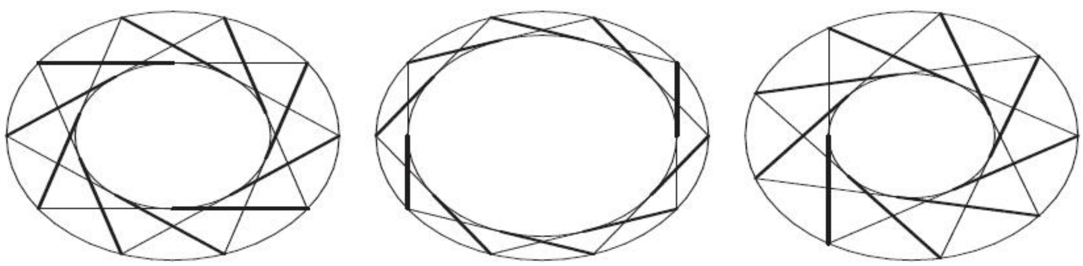

Figure 5: These Poncelet polygons (or unions of Poncelet polygons) in nested ellipses are projections of the toric braids $\tau_{3,10}$, $\tau_{2,10}$ and $\tau_{3,9}$, denoted by $\binom{10}{3}$, $\binom{10}{2}$ and $\binom{9}{3}$. [7]

5 Results combining Jacobian elliptic functions with Poncelet polygons

In this section we will first consider Poncelet's Closure Theorem, for which we need Jacobi's Lemma. After this we will take an extensive look at several statements concerning the irregularity of Poncelet odd polygons. These statements we need to prove the main theorem in the next section.

5.1 Poncelet's Closure Theorem

We will present here the Hermite-Laurent proof of Poncelet's Closure Theorem. Pecker has chosen this speific proof, because it has the advantage of providing explicit computations of the Poncelet polygons. For other proofs we refer to [4].

Before we can prove Poncelet's Closure Theorem we need a lemma, which is a variant of Jacobi's uniformisation of the Poncelet problem.

Lemma 4. *Let E and C be ellipses defined by $E = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, $a > b > 1$, $C = \{x^2 + y^2 = 1\}$. Parameterize E and C respectively by $P(\psi) = (a\, cn(\psi), b\, sn(\psi))$ and $M(\phi) = (\, cn(\phi),\, sn(\phi))$. Then the tangent to C at $M(\phi)$ intersects E at $P(\phi - \beta)$ and $P(\phi + \beta)$. Here k is the elliptic modulus defined by $k^2(a^2 - 1) = (a^2 - b^2)$ and β is a real number such that $cn(\beta) = 1/a$.*

Proof: Using $\operatorname{dn}(z) = \sqrt{1 - k^2 \operatorname{sn}^2(z)}$ with $z = \beta$, it follows that $\operatorname{dn}^2(\beta) = 1 - k^2 \operatorname{sn}^2(\beta) = b^2/a^2$, hence $\operatorname{dn}(\beta) = b/a$. Let us take the equation of the tangent to C at $M(\phi)$ as $x \operatorname{cn}(\phi) + y \operatorname{sn}(\phi) = 1$. Then $S = a \operatorname{cn}(\phi + \beta) \operatorname{cn}(\phi) + b \operatorname{sn}(\phi + \beta) \operatorname{sn}(\phi)$. The addition formulas (6) and (7) at the end of Section 3 tell us that

$$\begin{aligned} S(1 - k^2 \operatorname{sn}^2(\phi) \operatorname{sn}^2(\beta)) &= \operatorname{cn}^2(\phi) + \operatorname{sn}^2(\phi)(1 - k^2 \operatorname{sn}^2(\beta)) && (9) \\ &= 1 - k^2 \operatorname{sn}^2(\phi) \operatorname{sn}^2(\beta) && (10) \end{aligned}$$

Hence $S = 1$ and we see that $P(\phi + \beta)$ belongs to the tangent to C at $M(\phi)$. In the same way you can see that $P(\phi - \beta)$ belongs also to this tangent by changing β to $-\beta$. □

With this lemma, called Jacobi's Lemma, in our hands we can now prove Poncelet's Closure Theorem.

Theorem 5. *Let C and E be two ellipses. If there is one n-sided polygon inscribed in E and circumscribed about C, then there are infinitely many of them. Furthermore, every point on E and every tangent to C belongs to such a polygon.*

Proof: Let $P_0, P_1, \ldots, P_{n-1}, P_n = P_0$ be a Poncelet polygon inscribed in E and circumscribed about C. Let $P_j P_{j+1}$ be a tangent to C at M_j. Then we can use the Jacobi parametrizations of E and C, like they have been introduced in Lemma 4. So parameterize E and C respectively by $P(\psi) = (a \operatorname{cn}(\psi), b \operatorname{sn}(\psi))$ and $M(\phi) = (\operatorname{cn}(\phi), \operatorname{sn}(\phi))$, where k is the elliptic modulus defined by $k^2(a^2 - 1) = (a^2 - b^2)$. Assuming $M_0 = M(\phi)$ we see that $P_1 = P(\phi+\beta)$. By induction this will lead from $M_1 = M(\phi+2\beta)$ to $M_j = M(\phi+2j\beta)$. Since the polygon closes after n steps, we have $M_n = M_0$, or $M(\phi + 2n\beta) = M(\phi)$. Using the properties of the Jacobi amplitude we get $\operatorname{am}(\phi+2n\beta) = \operatorname{am}(\phi)+2q\pi = \operatorname{am}(\phi+4qK)$.

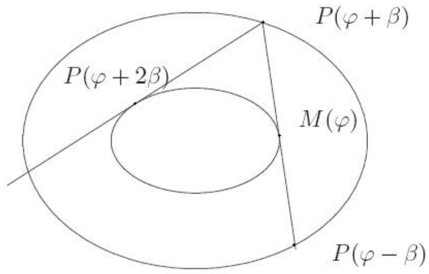

Figure 6: First two lines of the Poncelet polygon. [7]

Both equalities follow from $\mathrm{am}(u+2nK) = \mathrm{am}(u) + n\pi$ with $u = \phi$ and $n = 2q$. As a consequence we obtain $2n\beta = 4qK$, hence $\beta = 2qK/n$. Now consider a Poncelet polygon, which starts from an arbitrary point $M_0' = M(\phi')$ of C. Lemma 4 makes it possible to derive the equalities $M_n' = M(\phi' + 2n\beta) = M(\phi' + 4qK) = M(\phi') = M_0'$. Taking this all together we see that every Poncelet polygonal line closes after n steps. $\square$

Combining Poncelet's closure theorem with the Lamm-Manturov theorem and Proposition 3 we obtain the following result, which will be very useful in proving the main theorem. For this we let E be an ellipse. Then we can say that every μ-component link has a projection which is the union of μ billiard trajectories in E with the same odd period, and with the same caustic C.

5.2 Irregularity of Poncelet odd polygons

Now we have arrived at a key step in Pecker's strategy to prove the main theorem. First the coordinates of crossings and vertices will be computed, using Jacobian elliptic functions. Then we will prove that if the number of sides of a Poncelet polygon is odd, then there exists a Poncelet polygon which is totally irregular. This means that if we start at some initial point and we use arc length to produce some polygon, then we get a sequence of distances $t_1, t_2, \ldots$ between the initial point and the following crossing point, where the numbers $1, t_1, t_2, \ldots$ are linearly independent over $\mathbb{Q}$. But first we go back to computing the arc lengths of the crossings and vertices of Poncelet polygons.

Lemma 6. *Let n and p be coprime integers with n odd. Also, define for every integer j the function $f_j(z) = sn^2(z + j\theta) + r^2$, where $r^2 > 0$ and $\theta = 4pK/n$. Then, if $h \neq j \mod (n)$, the functions $f_j(z)$ and $f_h(z)$ do not possess any common zero.*

Proof: To find the zeros of the function $f_j(z)$ we will first take a look at the somewhat simpler function $g(z) = \mathrm{sn}(z) + ir$, $r > 0$. So the goal is to find the solution of the equation $\mathrm{sn}(z) = -ir$. For this equation the poles lie on the imaginary axis, so if we look at the imaginary part of $\mathrm{sn}(iy)$, then the solution is real. Restricting the function in this way to the y-axis we discover that that there does exist a purely imaginary α such that $g(\alpha) = 0$. From the periodicity of elliptic functions we see that $\mathrm{sn}(2K - \alpha) = \mathrm{sn}(\alpha)$, so $2K - \alpha$ is also a zero of g(z). Hence the zeros of $g(z)$ are all the points congruent to $\alpha, 2K + \alpha \mod (4K, 2iK')$. But then the zeros of $f_j(z) = \mathrm{sn}^2(z + j\theta) + r^2$ are numbers that are congruent with $+\alpha - j\theta, -\alpha - j\theta, 2K + \alpha - j\theta, 2K - \alpha - j\theta$. Now let's take a look at one of those zeros, for example $\alpha - j\theta$. Let's assume that this zero is equal to $\alpha - h\theta$, one of the zeros of $f_h(z)$. This is the same as saying that $\alpha - j\theta = 2K + \alpha - h\theta \mod (4K, 2iK')$, from which we can deduce that $(h-j)\theta \equiv 0 \mod (2K)$. But this implies that $2(h-j)p/n$ is an integer, which is impossible, because n is odd, $(n,p) = 1$ and $h \neq j \mod (n)$. The only possibility for $2(h-j)p/n \in \mathbb{Z}$ is that $n = 2q$, but then n would be even. Hence $\alpha - j\theta \neq \alpha - h\theta$. Another candidate zero of $f_h(z)$ that could be equal to $\alpha - j\theta$ is $-\alpha - h\theta$. This tells us that $\alpha - j\theta \equiv 2K - \alpha - h\theta$, from which we can deduce that $2\alpha \equiv ((j-h)\theta \mod (2K, 2iK')$. Taking the real parts of both sides, we get $(j-h)\theta \equiv 0 \mod (2K)$, which is impossible for the same reason as we mentioned earlier. Everytime we try to make a zero of $f_j(z)$ equal to a zero of $f_h(z)$, we will arrive at an expression like $(h-j)\theta \equiv 0 \mod (2K)$ or $(j-h)\theta \equiv 0 \mod (2K)$. Continuing this way of reasoning will lead to the conclusion that all the zeros of $f_j(z)$ cannot be a zero of $f_h(z)$. Hence the functions $f_j(z)$ and $f_h(z)$ do not possess any common zero if $h \neq j \mod (n)$. $\square$

Lemma 7. *Let n and p be coprime integers. For $j \neq 0 \mod (n)$, define the functions $D_j(z) = sn(z+j\theta)\, cn(z) - cn(z+j\theta)\, sn(z)$ and $F_j(z) = \frac{sn(z+j\theta) - sn(z)}{D_j(z)}$, where $\theta = 4pK/n$. Then for every integer j, there exists a complex number α_j such that $F(\alpha_j) = \infty$, and $F(\alpha_j) \neq \infty$ for $h \neq j \mod (n)$.*

Proof: Using $\operatorname{sn}(z) = \sin(\operatorname{am}(z))$ and $\operatorname{cn}(z) = \cos(\operatorname{am}(z))$ we get $D_j(z) = \sin(\operatorname{am}(z + j\theta) - \operatorname{am}(z)) = \sin(\operatorname{am}(z + j\theta) + \operatorname{am}(-z))$. Now let $u = j\beta$ and $v = z + j\beta$, then $u + v = z + 2j\beta = z + j\theta$ and $u - v = z$. Filling this in gives

$$
\begin{aligned}
D_j(z) &= \sin(\operatorname{am}(u+v) + \operatorname{am}(u-v)) && (11)\\
&= \frac{2\operatorname{sn}(u)\operatorname{cn}(u)\operatorname{dn}(v)}{1 - k^2\operatorname{sn}^2(u)\operatorname{sn}^2(v)} && (12)\\
&= \frac{2\operatorname{sn}(j\beta)\operatorname{cn}(j\beta)\operatorname{dn}(z+j\beta)}{1 - k^2\operatorname{sn}^2(j\beta)\operatorname{sn}^2(z+j\beta)}, && (13)
\end{aligned}
$$

where for the second equality Jacobi's formula (8) at the end of Section 3 is used. Here $\beta = \theta/2$. If we now let $\alpha_j = -j\beta + K + iK'$, then $\operatorname{dn}(\alpha_j + j\beta) = \operatorname{dn}(K + iK') = 0$. But since $\operatorname{dn}^2(z) + k^2\operatorname{sn}^2(z) = 1$, it follows from $\operatorname{dn}^2(z) = 1 - k^2\operatorname{sn}^2(z) = 0$ that $\operatorname{sn}^2(\alpha_j + j\beta) = 1/k^2$. Hence $D_j(\alpha_j) = 0$, because $\operatorname{dn}(\alpha_j + j\beta) = 0$. If we call $N(\alpha_j)$ the numerator of $F_j(\alpha_j)$, then $N(\alpha_j) = \operatorname{sn}(\alpha_j + j\theta) - \operatorname{sn}(\alpha_j) = \operatorname{sn}(K + iK' + j\beta) - (K + iK' - j\beta)$. Now we can use the addition formula (6) for

$$
\operatorname{sn}(x+y) = \frac{\operatorname{sn}(x)\operatorname{cn}(y)\operatorname{dn}(y) + \operatorname{sn}(y)\operatorname{cn}(x)\operatorname{dn}(x)}{1 - k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)} \tag{14}
$$

with $x = K + iK'$ and $y = j\beta$, where $\operatorname{dn}(x) = \operatorname{dn}(K + iK') = 0$ to obtain

$$
\begin{aligned}
N(\alpha_j) &= \frac{\operatorname{sn}(K+iK')\operatorname{cn}(j\beta)\operatorname{dn}(j\beta)}{1 - k^2\operatorname{sn}^2(K+iK')\operatorname{sn}^2(j\beta)} - \frac{\operatorname{sn}(K+iK')\operatorname{cn}(-j\beta)\operatorname{dn}(-j\beta)}{1 - k^2\operatorname{sn}^2(K+iK')\operatorname{sn}^2(-j\beta)} && (15)\\
&= 2\frac{\operatorname{sn}(K+iK')\operatorname{cn}(j\beta)\operatorname{dn}(j\beta)}{1 - k^2\operatorname{sn}^2(K+iK')\operatorname{sn}^2(j\beta)}. && (16)
\end{aligned}
$$

We know that $\operatorname{sn}(K+iK') = \sqrt{\operatorname{sn}^2(\alpha_j + j\beta)} = \sqrt{1/k^2} = 1/k$. So

$$
\begin{aligned}
N(\alpha_j) &= 2\frac{(1/k)\operatorname{cn}(j\beta)\operatorname{dn}(j\beta)}{1-k^2(1/k^2)\operatorname{sn}^2(j\beta)} && (17)\\
&= 2k^{-1}\frac{\operatorname{cn}(j\beta)\operatorname{dn}(j\beta)}{\operatorname{cn}^2(j\beta)} && (18)\\
&= 2k^{-1}\frac{\operatorname{dn}(j\beta)}{\operatorname{cn}(j\beta)} && (19)
\end{aligned}
$$

which makes $F_j(\alpha_j) = N(\alpha_j)/D_j(\alpha_j) = \infty$. From Lemma 6 we know that if $h \neq j \mod (n)$, then $\alpha_j + h\beta = K + iK' + 2(h-j)pK/n$. Because $\alpha_j + j\beta = K + iK'$, we see that $\alpha_j + h\beta \neq K + iK' \mod (2K, 2iK')$, which implies that $\operatorname{dn}(\alpha_j + h\beta) \neq 0$. Also $\alpha_j + h\beta \neq iK' \mod (2K, 2iK')$, which implies that $\operatorname{sn}(\alpha_j + h\beta) \neq \infty)$. Hence we can conclude that $D_h(\alpha_j) \neq 0$. Let us now take a look at $\operatorname{sn}(z) = \infty$. Since the functions $\operatorname{sn}(z)$ and $\operatorname{sn}(z+h\theta)$ do not have common poles, it follows that $\operatorname{sn}(z+h\theta) \neq \infty$. Because $\operatorname{sn}^2(z) + \operatorname{cn}^2(z) = 1$, we obtain $\frac{\operatorname{cn}^2(z)}{\operatorname{sn}^2(z)} = -1$, such that

$$
\begin{aligned}
F_h(z) &= \frac{\operatorname{sn}(z+h\theta) - \operatorname{sn}(z)}{D_j(z)} && (20)\\
&= \frac{\operatorname{sn}(z+h\theta) - \operatorname{sn}(z)}{\operatorname{sn}(z+h\theta)\operatorname{cn}(z) - \operatorname{cn}(z+h\theta)\operatorname{sn}(z)} && (21)\\
&= \frac{\frac{\operatorname{sn}(z+h\theta)}{\operatorname{sn}(z)} - \frac{\operatorname{sn}(z)}{\operatorname{sn}(z)}}{\operatorname{sn}(z+h\theta)\frac{\operatorname{cn}(z)}{\operatorname{sn}(z)} - \operatorname{cn}(z+h\theta)\frac{\operatorname{sn}(z)}{\operatorname{sn}(z)}} && (22)\\
&= \frac{-1}{\operatorname{sn}(z+h\theta)\frac{\operatorname{cn}(z)}{\operatorname{sn}(z)} - \operatorname{cn}(z+h\theta)}. && (23)
\end{aligned}
$$

If we assume that $F_h(z) = \infty$, then $\operatorname{sn}(z+h\theta)\frac{\operatorname{cn}(z)}{\operatorname{sn}(z)} = \operatorname{cn}(z+h\theta)$. But then $-\operatorname{cn}^2(z+h\theta) = \operatorname{sn}^2(z+h\theta) \neq \infty$ and $\operatorname{cn}^2(z+h\theta) + \operatorname{sn}^2(z+h\theta) = 0$, which is impossible. Hence $F_h(z) \neq \infty$. Taking it all together we have $D_h(\alpha_j) \neq 0$, $\operatorname{sn}(\alpha_j) \neq \infty$ and $\operatorname{sn}(\alpha_j + h\theta) \neq \infty$. Hence we have proven that

$$
F_h(\alpha_j) = \frac{\operatorname{sn}(\alpha_j + h\theta) - \operatorname{sn}(\alpha_j)}{D_h(\alpha_j)} \neq \infty. \tag{24}
$$

□

Proposition 8. *Let E and C be confocal ellipses such that there exists a Poncelet polygon $\mathcal{P}$ inscribed in E and circumscribed about C. If we suppose that the number of sides of $\mathcal{P}$ is odd, then there exists a Poncelet polygon, which satisfies the following condition: "If the arc lengths t_i of the vertices and crossing are measured from a vertex P_0, then the numbers 1 and t_i for $i \neq 0$ are linearly independent over $\mathbb{Q}$."*

Proof: We start with two confocal ellipses (i.e. two ellipses with the same foci) E and C, such that there exists a Poncelet polygon inscribed in E and circumscribed in C. We define these two ellipses by $E = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, $a > b > 1$, $C = \{x^2 + \frac{y^2}{c^2} = 1\}$, $c < 1$. Let us then consider the Jacobi parametrizations $P(\psi) = (a\operatorname{cn}(\psi), b\operatorname{sn}(\psi))$ and $M(\phi) = (\operatorname{cn}(\phi), c\operatorname{sn}(\phi))$ of E and C. For each real number ϕ there is a Poncelet polygon P_ϕ through $M(\phi) = (\operatorname{cn}(\phi), c\operatorname{sn}(\phi))$. Let us denote $\phi_j = \phi + j\theta$, where $\theta = 4pK/n$, and $M_j = M(\phi_j)$.

We call l_j the tangent to C at M_j, which has the equation $x\,\mathrm{cn}(\phi_j) + \frac{y}{c}\,\mathrm{sn}(\phi_j) = 1$. If we let $Q_{h,j} = l_h \cap l_{h+j}$, where $j \neq 0 \mod (n)$, then we can find the abcissa (i.e. element of an ordered pair which is plotted on the horizontal x-axis) $x_{h,j}$ of $Q_{h,j}$. So

$$x_{h,j} = \frac{-\,\mathrm{sn}(\phi_h) + \,\mathrm{sn}(\phi_h + j\theta)}{\mathrm{sn}(\phi_h + j\theta)\,\mathrm{cn}(\phi_h) - \,\mathrm{cn}(\phi_h + j\theta)\,\mathrm{sn}(\phi_h)} = F_j(\phi_h) \tag{25}$$

where F_j is the function which we defined in Lemma 7. The abscissa of $P_h = Q_{h,-1} = Q_{h-1,-1}$ will also be denoted by $x_h = x_{h.-1}$. Then the distance $P_hQ_{h,j}$ is$|d_{h,j}|$, where

$$d_{h,j} = d_{h,j}(\phi) = \frac{\sqrt{1-c^2}}{\mathrm{sn}(\phi_h)}\sqrt{\mathrm{sn}^2(\phi_h) + \frac{c^2}{1-c^2}}\,(x_h - x_{h,j})\,. \tag{26}$$

Since $c^2/(1-c^2) > 1$, the functions $d_{h,j}(\phi)$ are meromorphic (i.e. complex differentiable at all points, except some isolated poles) in a neighborhood of the real axis. Let $\lambda_{h,j}$ and λ be complex numbers such that $\sum_{h=1}^{n}\sum_{j=1}^{n-2}\lambda_{h,j}d_{h,j} = \lambda$, or

$$\sum_{h=1}^{n}\frac{\sqrt{1-c^2}}{\mathrm{sn}(\phi_h)}\sqrt{\mathrm{sn}^2(\phi_h) + \frac{c^2}{1-c^2}}\left(\sum_{j=1}^{n-2}\lambda_{h,j}(x_h - x_{h,j})\right) = \lambda \tag{27}$$

Since $c^2/(1-c^2) > 0$, Lemma 6 tells us that the functions $f_h(\phi) = \sqrt{\mathrm{sn}^2(\phi_h) + c^2/(1-c^2)}$ do not possess any common zero. Hence, in the neighborhood of a zero of $f_h(\phi)$ this function is not meromorphic, while the others are. This implies that for every $h = 1, \ldots, n$ we have $\sum_{j=1}^{n-2}\lambda_{h,j}(x_h - x_{h,j}) = 0$, from which it follows that $\lambda = 0$. Using $x_{h,j} = F_j(\phi_h)$ and $x_h = x_{h,-1}$ we obtain $\sum_{j=1}^{n-2}\lambda_{h,j}(F_{-1} - F_j) = 0$, which can be seen as a relation between meromorphic functions. Lemma 7 tells us that for every integer $j \neq 0$ there exists a number α_j such that $F_j(\alpha_j) = \infty$ and $F_h(\alpha_j) \neq \infty$ if $h \neq j \mod (n)$. If we let $z = \alpha_j$, then we obtain $\lambda_{h,j}$. Hence we have proven that the functions 1 and $d_{h,j}(\phi)$ with $j \neq -1 \mod (n)$ are linearly independent over $\mathbb{C}$. Now, for every nonzero collection of rational numbers $\Lambda = (\lambda, \lambda_{h,j})$, let us define the function F_Λ by $F_\Lambda(\phi) = \lambda - \sum_{h,j}\lambda_{h,j}d_{h,j}(\phi)$. We can see that this function is not identically zero, and it is meromorphic in a neighborhood of $\mathbb{R}$. This makes the set of its real zeros countable, because these zeros form a (discrete) set of isolated points. Consequently, the set of all real numbers ϕ, such that 1 and the numbers $d_{h,j}(\phi)$ are linearly independent over $\mathbb{Q}$, is countable. Cardinality then tells us that the complementary set is not countable. But this set must also be nonempty. Hence there exists a real ϕ, such that 1 and the numbers $|d_{h,j}(\phi)|$ are linearly independent over $\mathbb{Q}$. Now its possible to parameterize the Poncelet polygon by arc length $t_{h,j}$ of $Q_{h,j}$, starting from P_0 for $t_0 \in \mathbb{Q}$. So

$$\begin{aligned} t_{h,j} &= t_0 + d(P_0,P_1) + d(P_1,P_2) + \cdots + d(P_{h-1},P_h) + d(P_h,Q_{h,j}) && (28)\\ &= t_0 + |d_{0,1}| + |d_{1,1}| + |d_{2,1}| + \cdots + |d_{h-1,1}| + |d_{h,j}| && (29)\end{aligned}$$

Because the numbers 1 and $|d_{h,j}|$ are independent, we see that indeed the numbers 1 and t_i, $t_i \neq 0$, are linearly independent over $\mathbb{Q}$. $\square$

The following proposition is an analogous result of the previous proposition, but this time it focuses on links. Remember that links are a collection of knots which do not intersect, but which may be linked or knotted together. The preceding proposition showed that if ellipses C and E possess a Poncelet polygon with an odd number of sides, then there exists a totally irregular Poncelet polygon. The next proposition will show something equivalent, but then for unions of finitely many Poncelet polygons.

Proposition 9. *Let E and C be confocal ellipses such that there exists a polygon of an odd number of sides inscribed in E and circumscribed about C. For any integer μ, there exist μ Poncelet polygons $\mathcal{P}(0), \mathcal{P}(1), , \mathcal{P}(\mu-1)$, which satisfy the following condition: For each such polygon, if t_i are the arc lengths corresponding to its vertices, crossings and intersections with the other polygons, then the numbers 1 and t_i for $i \neq 0$ are linearly independent over $\mathbb{Q}$.*

Proof: Let l_h the tangent to C at $M_h = M(\phi + h\tau) \in C$, where $\tau = \frac{\theta}{\mu}$. We consider the Poncelet polygons $\mathcal{P} = \mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(\mu-1)}$ through the points $M_0, M_1, \ldots, M_{\mu-1}$. Here the polygon $\mathcal{P}$ is tangent to C at the points $M_0, M_\mu, M_{2\mu}, \ldots, M_{(n-1)\mu}$. The vertices and crossings of $\mathcal{P}$ are the points $Q_{h,j} = l_h \cap l_{h+j}$, where $h = 0 \mod (\mu)$. With the same way of reasoning as in the proof of Proposition 8 we see that the distances 1 and $|d_{h,j}(\phi)|$, $h = 0$, $j \neq 0$, $j \neq -1(\mod (\mu))$ are linearly independent over $\mathbb{Q}$, except for a countable set of numbers ϕ. Consequently, the number 1 and the arc lengths t_i, $i \neq 0$ of the crossings and vertices of $\mathcal{P}$ are linearly independent over $\mathbb{Q}$, except on a countable set of numbers ϕ. Using cardinality we can suppose that this is true for each polygon $\mathcal{P}^{(j)}$, $j = 0, \ldots, \mu - 1$. Hence we have proven the statement. $\square$

6 The final proof

In this section the proof of Daniel Pecker for the main theorem "every knot is a billiard knot under certain condition" will be presented. The proof makes use of several results that we mentioned earlier. One result was that for an ellipse E we can say that every μ-component link has a projection which is the union of μ billiard trajectories in E with the same odd period, and with the same caustic C. The other two results that are quite important are Proposition 8 and Proposition 9 at the end of Section 5.

In the upcoming theorem we will consider a three-dimensional elliptic cylinder, in contrast with the two-dimensional ellipse of the previous lemma's and propositions. As a consequence we have to introduce the sawtoothfunction $z(t) = 2|(mt + \phi) - 1/2|$, where m is the slope.

Before examining the proof, first the famous density theorem of the German mathematician Leopold Kronecker (1823-1891) must be introduced. This theorem states that if $\theta_1, \ldots, \theta_k, 1$ are linearly independent over $\mathbb{Q}$, then the set of points $((m\theta_1), \ldots, (m\theta_k))$ is dense in the unit cube, when m varies over $\mathbb{N}$. Here (x) denotes the fractional part of x. Now we have all the tools to discuss the proof.

Theorem 10. *Let E be an ellipse, which is not a circle, and let D an elliptic cylinder, where $D = E \times [0, 1]$. Then every knot/link is a billiard knot/link.*

Proof: First we consider knots. Let us call the desired knot K. Using the theory around braids we can say there exists a knot isotopic to K, whose projection on the xy-plane is a billiard trajectory of odd period in the ellipse E. So what we get here is a copy of K whose planar projection is a totally irregular polygon P. Using Proposition 8 we can say that if $t_0, t_1, \ldots, t_k$ are the arc lengths corresponding to the vertices and crossings of this polygon, then $t_1, \ldots, t_k, 1$ are linearly independent over $\mathbb{Q}$. Rescaling can make the total length of the trajectory equal to 1. Keep the horizontal component the same and let the slope m variate. By adjusting the slope it's possible to find m such that at the pre-images of crossings, the heights of P_m are arbitrarily close to some specified list. It's even possible to obtain over-and-under crossings of P_m to match those of K. Even if P_m bounces up and down a huge number of times, there will still be a match. Now lets consider a polygonal curve defined by $(x(t), y(t), z(t))$, where $z(t) = 2|(mt + \phi) - 1/2|$ is the sawtoothfunction depending on $m \in \mathbb{Z}$ and $\phi \in \mathbb{R}$. If the heights $z(P_j)$ of the vertices are such that $z(P_j) \neq 0$ and $z(P_j) \neq 1$, then it is a periodic billiard trajectory in the elliptic cylinder $D = E \times [0, 1]$. Finally, set $\phi = 1/2 + z_0/2$ with $z_0 \in (0, 1)$, then $z(0) = z_0$. Using Kronecker's theorem we see that there is an $m \in \mathbb{Z}$ such that the numbers $z(t_i)$ are arbitrary close to any specified collection of heights. This concludes the proof for knots. The case of links follows the same pattern. First you can find a diagram by the theorem around braids, which is the union of μ Poncelet polygons with the same odd number of sides. Combining Proposition 9 and Kronecker's theorem it's possible to parameterize each component such that the heights of the vertices and crossings are close to any specified list. $\square$

7 Conclusions

Daniel Pecker started his article with some classical results on billiard trajectories, where he also shortly mentioned winding numbers. After this he took a long road through Jacobian elliptic functions. Looking from a different perspective you might want to ask yourself why this is necessary. Daniel Pecker makes no use of the Liouville integratibility in an elliptic cylinder. This integrability implies a foliation of the system's phase space by invariant tori. It seems that the result of Pecker can be understood more easily from studying the winding numbers associated with these tori and the projection of the periodic orbits to the elliptic cylinder. More research is needed to substantiate this feeling with mathematical proofs.

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A Elliptic functions in general

The research in this area started with Carl Friedrich Gauss (1777-1855) and Niels Hendrik Abel (1802-1829), who discovered that the inverse of elliptic integrals defines elliptic functions. Here elliptic functions are meant like doubly periodic functions from the complex plane to the complex plane. If you have a polynomial of degree 3 or 4, it will lead to an elliptic integral. Whether a (Jacobian) elliptic integral is of first, second or third kind depends on the analytic nature of the differential $du = R(z, w)dz$.

Now lets take a Riemann surface. To study the nature of these surfaces, we start with copies/sheets C_1 and C_2 of the Riemann z-sphere, each representing one of the two signs of w, where $w^2 = z$. Then we open these sheets along certain branch cuts and we glue them together in such a way that the function w changes everywhere smoothly as z varies. This is necessary, because if we had only one sheet the function w would get a smoothness-problem after walking a distance of 2π through the sheet. The branch cuts connects the critical points z_k, which are easily identified, because they are the zeros of the polynomial inside the integral.

If we take $\sin(t) = z$, then $\cos(t)dt = dz$ and $dt = \frac{dz}{cost} = \frac{dz}{\sqrt{1-\sin^2(t)}} = \frac{dz}{\sqrt{1-z^2}}$. Speaking of elliptic integrals, we have to add an extra factor k such that: $\int \frac{dt}{\sqrt{1-k^2\sin^2(t)}} = \int \frac{dz}{\sqrt{1-k^2z^2}\sqrt{1-z^2}}$. The zeros of the polynomial inside this integral are $z = 1$, $z = -1$, $z = 1/k$, $z = -1/k$. One branchcut connects two of these points, so there are two branchcuts in totals. Glueing these together there appears to be a hole in the middle. Hence in this case it isn't possible for a closed line to shrink to a point.

B Braid theory

Braid theory is an abstract geometric theory that studies the braid concept and its generalizations. The idea is to organize these braids into a group, in which the group operation is to do the first braid on a set of strands and then to do the second braid on the set of twisted strands. The connection to knot theory is that any knot can be represented as the closure of certain braids.

The braid group on n strands, denoted by B_n, is a group with an intuitive geometrical representation, which in a sense generalizes the symmetric group S_n. Here for $n > 1$, B_n is an infinite group. For each n-strand braid it is possible to define a permutation $\sigma \in S_n$. The standard generators of B_n are the elements $\sigma_i,\ i = 1, \ldots, n-1$. The braid group B_n is built on top of two cubic relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. These are called the braid relations. This all leads to the following definition: $B_n =< \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i >$.

Now let us say we have k strands. The operation “the first strand passes the second strand from above” differs from “the first strand passes the second strand from below”. On the opposite, the operation “the first strand passes the second strand from above” is the same as the operation “the first strand passes the second strand from above, then the third from below, the fourth from above, again the fourth from above and in the end the third from below”. Simply said, braids are considered the same if their beginning point and their endpoints are in the same position. See Figure 7 for a more clearer picture on this. Important to note here is that the strands always move from left to right, because else it isn't a braid. Hence this is what we take as our convention. It determines the order of the σ_i.

If the braid is closed, meaning that corresponding ends can be connected in pairs, then we can speak of links.

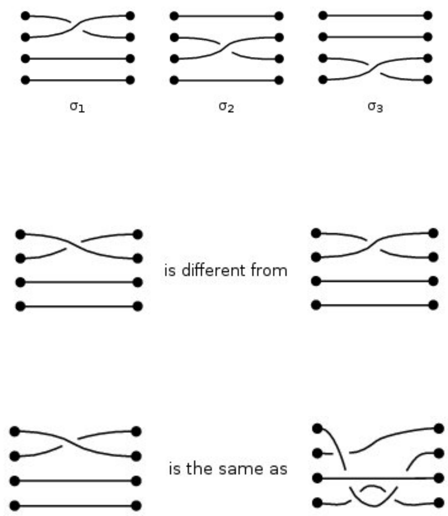

Figure 7: The braids of B_4 and some operations.

C Even more examples of Poncelet polygons

Figure 8 shows some more examples of periodic orbit representatives, such like the Poncelet polygons in Section 4.

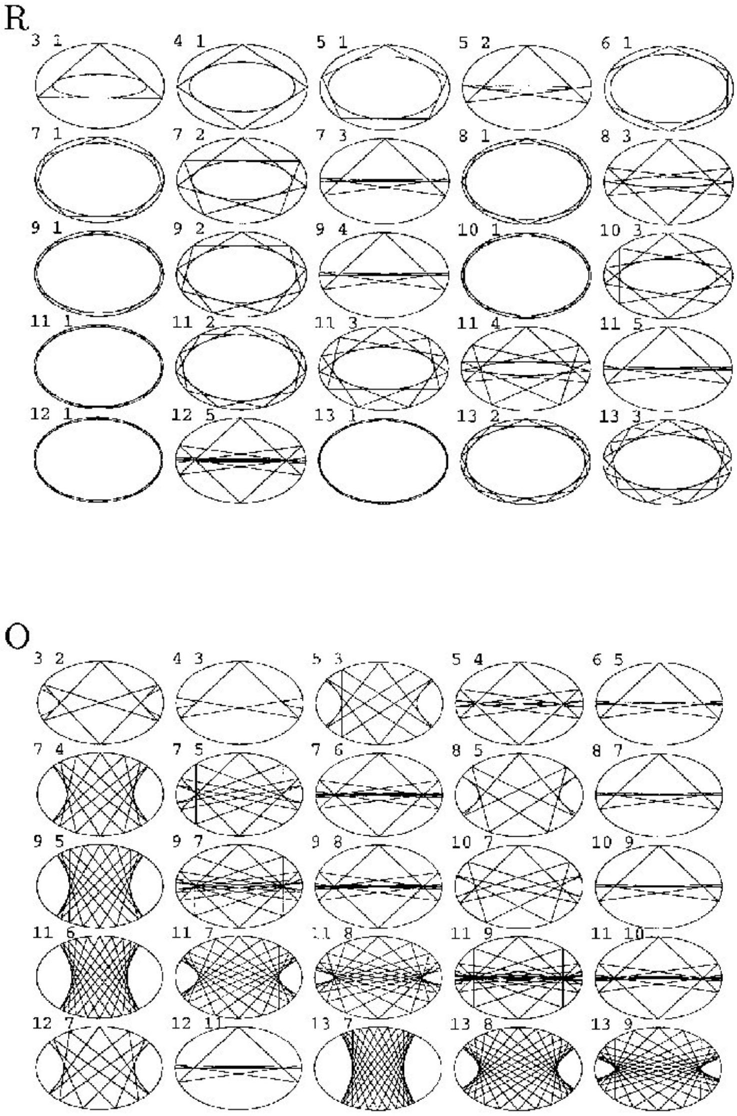

Figure 8: Periodic orbit representatives. [9]