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Swan's Theorem

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Abstract

In this thesis we examine Swan's theorem, which states that that given a compact Hausdorff space X, any finitely generated projective module over the continuous functions on X is isomorphic to the module of sections of a topological vector bundle over X. Conversely it states that any module of sections of a topological vector bundle over X is a finitely generated and projective module over the continuous functions on X. After covering some preliminaries concerning vector bundles and projective modules we examine vector bundles over compact Hausdorff spaces and subsequently give the proof of Swan's theorem. We also examine the scope of Swan's theorem and consider a generalization due to Vaserstein.

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Chapter 1

Introduction

The subject of this thesis is a theorem from the sixties, due to R.G. Swan, which states that given a compact Hausdroff space, any finitely generated projective module $C(X)$-module is isomorphic to the module of sections of a topological vector bundle over X, and conversely that any module of sections of a topological vector bundle over X is a finitely generated and projective $C(X)$-module. (Here $C(X)$ is the ring of continuous functions on X.)

For this theorem Swan was inspired by J.P. Serre's earlier work on algebraic vector bundles over affine varieties. Serre showed, in his famous article 'Faisceaux Algébriques Cohérents' [1], that there is exactly such a correspondence between algebraic vector bundles over affine varieties and finitely generated projective modules over its coordinate ring. As Swan notes in his article, after this observation made by Serre, the idea rose that an analogous statement concerning topological vector bundles could also be made. Swan thus eventually proved it in the article 'Vector bundles and Projective modules' [2].

Swan's theorem provides a nice link between topological objects, vector bundles over a compact Hausdorff space, and the algebraic notion of projective modules (over the ring of continuous functions over that same space). This enables us to perhaps approach topological problems from an algebraic point of view and vice-versa. Questions which might be very hard to answer on one side, might be much easier when approached from the other side. For example, Swan's theorem has been used to show that certain projective modules are in a sense non trivial by showing that the corresponding vector bundles are also non trivial in the same way. For some of these examples no purely algebraic proofs are known. However, our main interest lies not in the applications of the theorem. Rather, we are interested in its proof and its scope. That is, is the statement by Swan the final word on the subject or is there a more general statement to be made? One way to examine this is to relax the conditions on the topological space and try to come up with examples for which the conclusions of the theorem would no longer hold, or perhaps find examples for which

they *do* continue to hold. Such examples could shine some light on the matter.

Now, let's give a short overview of how this thesis is organized. In chapter 2 we will first introduce the basics concerning modules and then turn to the special case of projective modules and give several properties they exhibit. In the third chapter we will cover the subject of vector bundles: the definition, how they can be constructed, etc. The chapter will be concluded with some topology and special properties of vector bundles over compact Hausdorff spaces. In chapter 4 we will give the actual proof of the theorem. In chapter 5 we will examine the scope of Swan's theorem by looking at examples and finally comment on a generalization of Swan's theorem.

Chapter 2

Projective modules

In this chapter we cover the basics concerning modules and, more specifically, projective modules. We start off with an introduction about modules and some related concepts as submodules, homomorphisms, generators, direct sums and free modules. We will subsequently introduce projective modules. We will give some examples and also examine some of their properties.

2.1 Modules

Before we can discuss projective modules and their properties we must of course first pay some attention to modules in general and get some feeling for the concept. Most of what is discussed in this section can be found in [4]. Let us start off with the definition of a module:

Definition 1. *R-Module. Let R be a ring. An R-module is an abelian group M equiped with an action of a ring R, i.e. there is a map: $R \times M \to M$, $(a, m) \mapsto am$, such that for all $a, b \in R$ and all $m, n \in M$ the following holds:*
(1) $a(m + n) = am + an$
(2) $(a + b)m = am + bm$
(3) $a(bm) = (ab)m$
(4) $1m = m$

(This is actually the definition of a *left* R-module. A similar definition can be made for *right* R-modules. However, we will solely consider left R-modules and from now on when we say R-module we mean a left R-module.) In words the definition can be simply stated: given some commutative group M and a ring R, we define a multiplication of elements from the group M by elements of the ring R. Even though this might not immediately ring a bell, one has already encountered several such constructions in elementary mathematics courses. To see this, let us give some examples of modules that one has undoubtedly already seen before:

Example 1. *Let $R = \mathbb{K}$ be a field. The axioms of a $\mathbb{K}$-module are then exactly the axioms for a vector space over $\mathbb{K}$.*

Example 2. *Let $\mathbb{K}$ be a field and consider the ring $M(n, \mathbb{K})$ of $n \times n$ matrices with coefficients in $\mathbb{K}$. Then $\mathbb{K}^n$ can be viewed as an $M(n, \mathbb{K})$-module by the usual multiplication of a matrix with a vector.*

Example 3. *A ring R can also be viewed as an R-module by the standard ring multiplication.*

Example 4. *Any abelian group M can be thought of as a $\mathbb{Z}$-module. To see this define $0m = 0$, $-1m = -m$ (with $-m$ the inverse of m), $nm = m + m + .. + m$ (n times, $n \geq 1$) and $(-n)m = (-m) + (-m) + .. + (-m)$ (again n times, $n \geq 1$). It is easily verified that this turns M into a $\mathbb{Z}$-module.*

There is an obvious way to define submodules of a module:

Definition 2. *Submodule. A submodule N of an R-module M is a subgroup of M which is closed under the action of R. I.e. for all $a, b \in N$ and all $r \in R$ the following must hold:*
(1) $0 \in N$, $a - b \in N$
(2) $ra \in N$

A submodule is of course a module in its own right. Now for some examples:

Example 5. *Let $\mathbb{K}$ be a field. The submodules of a module over $\mathbb{K}$ are precisely the subspaces of the corresponding vector space over $\mathbb{K}$.*

Example 6. *View R as an R-module. One can easily verify that the submodules are precisely the ideals of R.*

Having introduced certain mathematical objecs, one would of course like to define mappings between them. In the case of modules, these are called R-module homomorphisms:

Definition 3. *Let M, N be R-modules. An R-module homomorphism is a map $f : M \to N$, that is a homomorphism of Abelian groups which is R-linear. I.e for all $x, y \in M$ and $r \in R$:*
(1) $f(x + y) = f(x) + f(y)$
(2) $f(ry) = rf(y)$

The kernel and image of such a morphism f are defined in the usual fashion and it is easily established that $\ker f$ is a submodule of M and $\operatorname{im} f$ is a submodule of N. An R-module isomorphism is an R-module homomorphism that is injective and surjective. Modules between which an isomorphism exist are called *isomorphic*.

Now for some further concepts we need to cover in order to be able to talk about finitely generated projective modules. We start off with the definition of a finitely generated module:

Definition 4. *Finitely generated. A module M is said to be finitely generated iff there exist $a_1, ..., a_n \in M$ such that for all $x \in M$, there exist $r_1, ..., r_n \in R$ with $x = r_1 a_1 + ... + r_n a_n$. The set $\{a_1, ..., a_n\}$ is called a generating set for M.*

This reminds us of the concept of a spanning set for vector spaces. However, there are some differences. In the case of vector spaces there is a well defined minimal value of n, say k, such that any set of k linearly independent elements span the vector space. Such a minimal spanning set is then called a basis. When dealing with modules and generating sets this is not the case. There is no minimal value of n (k), such that any set of k linearly independent elements would be a generating set of the module. However, we are not interested in the exact details, but simply in the existence of some finite generating set. Finitely generated modules are quite common, and of course the modules we will be dealing with are finitely generated. Let us give some examples:

Example 7. *The finitely generated $\mathbb{Z}$-modules precisely correspond to the finitely generated Abelian groups.*

Of course, not all modules are finitely generated. In fact, submodules of a finitely generated module need not even be finitely generated:

Example 8. *Consider the ring $R = \mathbb{Z}[X_1, X_2, ...]$ of all polynomials in countably many variables. Now view R as an R-module. It is of course finitely generated by the identity element 1. We already mentioned that its submodules are exactly its ideals. Now consider the ideal consisting of all polynomials whose constant term is zero. As every polynomial contains only finitely many nonzero terms, we conclude that the submodule is not finitely generated.*

Given a collection of R-modules we can construct new R-modules in various ways. One construction we will encounter very often is the *direct sum* of the modules. Let us give the definition:

Definition 5. *Direct sum. Let M_i be R-modules and let I be some index set. The direct sum M of the M_i is defined as:*

$$M = \oplus_{i \in I} M_i = \{(..., x_i, ...)_{i \in I} : x_i \in M_i \text{ and finitely many } x_i \neq 0\} \qquad (2.1)$$

where we define the R-module structure via:

$$(.., x_i, ..) + (.., y_i, ..) = (.., x_i + y_i, ..) \text{ for all } i \in I \qquad (2.2)$$

$$0 = (.., 0, ..) \text{ that is } x_i = 0 \text{ for all } i \in I \qquad (2.3)$$

$$r(.., x_i, ..) = (.., r x_i, ..) \text{ for all } i \in I \qquad (2.4)$$

Although thus also defined for a countable collection of modules, we will only encounter the case in which we consider merely two modules. Thus given two R-modules M and N we consider $M \oplus N = \{(m, n) | m \in M, n \in N\}$ with the componentwise action of R on the tuples. We can now introduce a very simple type of R-modules which are called *free*, and which are isomorphic to a direct sum of copies of R, i.e:

Definition 6. *Free module. A module F is called free if there is an index set I, such that if we consider copies of R counted by I (denoted R_i), there exists an isomorphism $f : F \to \oplus_{i \in I} R_i$, i.e. $F \simeq \oplus_{i \in I} R_i$.*

As we will see, free modules are closely related to projective modules: all free modules are projective. Given a free module, one can choose the standard generating set $e_i = (.., x_i, ..)$ where $x_i = 1$ and $x_j = 0$ if $i \neq j$. It is not hard to see that indeed $\{e_1, ..\}$ forms a generating set. Let us now consider the case $F \simeq \oplus_{i \in \{1, .., n\}} R_i$ (which we from now on denote as $F \simeq R^n$), i.e. the index set is finite. Then it is easily seen that F is finitely generated by $\{e_1, .., e_n\}$. An example of a free module:

Example 9. *Consider the polynomial ring $R[X]$ and view it as an R-module. Then one can readily verify that:*

$$f : R[X] \to \oplus_{i \in \mathbb{Z}_{\geq 0}} R, \quad \Sigma_{i=0}^n a_i X^i \to (a_0, .., a_n, 0, ..) \tag{2.5}$$

is an R-module isomorphism and thus $R[X]$ is free.

Let us also note that in our case we assume the ring R to be commutative. In that case we can make the following assertion: $R^n \simeq R^m \Rightarrow n = m$. (We will use this in an example.)

2.2 Projective Modules

Having dealt with the preliminaries concerning modules in general, we are ready to turn our attention to projective modules, which are one of the two main objects of interest throughout this thesis. In addition to [4] we refer to [5]. There are several equivalent ways in which one could define projective modules, which we will now give:

Definition 7. *Projective module. Let P be an R-module. It is said to be projective if any of the following equivalent properties hold:*
(1) For any surjective R-module homomorphism $f : M \to N$ and any R-module homomorphism $h : P \to N$, there exists a R-module homomorphism $\tilde{h} : P \to M$ such that $f\tilde{h} = h$.
(2) Any short exact sequence $0 \to N \to M \xrightarrow{f} P \to 0$ of R-modules splits and $M \simeq N \oplus P$.
(3) There exists an R-module Q such that $P \oplus Q$ is a free R-module.
(4) If $\pi : M \to N$ is a surjection, then the natural map $Hom(P, M) \to Hom(P, N)$ given by $\Phi \to \pi \circ \Phi$ is surjective.
(5) The functor $Hom(P, \ )$ is exact.

The last two properties are of little interest to us. The first three are more relevant and let us at least convince ourselves that the first three properties are indeed equivalent:

Proof. (1)$\Rightarrow$ (2): Consider the exact sequence $0 \to N \xrightarrow{g} M \xrightarrow{f} P \to 0$. Then $f : M \to P$ is surjective, $g : N \to M$ is injective and $\ker f = \mathrm{im} g$. Now in addition consider the identity map on P, i.e. $1|_P : P \to P$. Then there exists a homomorphism $\tilde{h} : P \to M$ such that $f \circ \tilde{h} = 1|_P$. Hence $\tilde{h}$ splits the sequence. Now the map $\tilde{h} \circ f : M \to M$ is a projection map since $(\tilde{h} \circ f) \circ (\tilde{h} \circ f) = \tilde{h} \circ (f \circ \tilde{h}) \circ f = \tilde{h} \circ (1_P) \circ f = \tilde{h} \circ f$. Therefor $M = \mathrm{im}(\tilde{h} \circ f) \oplus \mathrm{im}(\tilde{h} \circ f)$. Now, $\tilde{h} : P \to \mathrm{im} \tilde{h}$ has an inverse given by $f|_{\mathrm{im} \tilde{h}}$, i.e. $P \simeq \mathrm{im} \tilde{h}$. Also $\ker(\tilde{h} \circ f) = \ker f = \mathrm{im} g$ and since g is injective $N \simeq \mathrm{im} g$. We thus conclude that $M \simeq N \oplus P$.

(2)$\Rightarrow$ (3): Choose a set of generators $\{p_i\}$ for P. Let F be the free module given with generating set $\{e_i\}$. We can then construct a surjective homomorphism $f : F \to P$ via $f(e_i) = p_i$. This gives rise to the short exact sequence $0 \to \ker f \to F \to P \to 0$. Since it splits we get $F \simeq \ker f \oplus P$.

(3)$\Rightarrow$ (1): Let $F \simeq Q \oplus P$ be a free module. Consider the surjective map $f : M \to N$ and the map $h : P \to N$. Also consider $g : F \to N$ such that $g|_P = h$. Now consider the free generating set $\{e_i\}$ for F and let $n_i = g(e_i)$. Now consider m_i such that $f(m_i) = n_i$ which is possible due to the surjectivity of f. Now consider $\tilde{g} : F \to M$ with $\Sigma x_i e_i \to x_i n_i$. One easily verifies that this is an R-module homomorphism and $f \tilde{g} = g$. The restriction $\tilde{h} = \tilde{g}|_P : P \to M$ precisely gives the morphism such that $f \tilde{h} = h$. $\square$

Let us stress the following which we implicitly just proved above ((3)$\Rightarrow$(2)):

Corollary 1. *Every free module R-module F is a projective R-module.*

The reverse however, is certainly not the case. Let us give some examples of projective modules which are not free:

Example 10. *Let $\mathbb{K}$ be a field and let $R = M(2, \mathbb{K})$. Consider the R-module $P = \mathbb{K}^2$. By looking at the dimensions of the corresponding vector spaces we see that P is not free (dim $P = 2 < 4 = $dim R). Let us now show that $P \oplus P \simeq R$ by explicitly constructing an isomorphism. Consider therefor:*

$$\phi : P \oplus P \to R, \quad (\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}) \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{2.6}$$

Clearly ϕ is a homomorphism of Abelian groups. Now to verify that it is also R-linear consider $r = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} \in R$ and $x = (\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}) \in P$, then:

$$\phi(rx) = \phi(r(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix})) = \phi((r \begin{pmatrix} a \\ b \end{pmatrix}, r \begin{pmatrix} c \\ d \end{pmatrix}) \tag{2.7}$$

$$= \phi(\begin{pmatrix} r_1 a + r_2 b \\ r_3 a + r_4 b \end{pmatrix}, \begin{pmatrix} r_1 c + r_2 d \\ r_3 c + r_4 d \end{pmatrix}) \tag{2.8}$$

$$= \begin{pmatrix} r_1 a + r_2 b & r_1 c + r_2 d \\ r_3 a + r_4 b & r_3 c + r_4 d \end{pmatrix} \tag{2.9}$$

$$= \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{2.10}$$

$$= r\phi(x) \tag{2.11}$$

Injectivity and surjectivity of ϕ are immediate and we conclude that ϕ is an R-module isomorphism. Thus P is projective and not free.

Or more general:

Example 11. *Let $\mathbb{K}$ be a field and let $R = M(n, \mathbb{K})$. Consider the R-module $P = \mathbb{K}^n$. Again it is not difficult to convince oneself that $P \oplus Q \simeq R$, where $Q = \oplus_{i=1}^{n-1} P_i$ ($P_i = P$). Thus P is in fact projective. Again by examining the dimensions of the corresponding vector spaces it follows that P is not free.*

A more interesting example for our purposes is the Möbius module. It is a rather simple example which we will also encounter in the chapter about vector bundles. This because it is a finitely generated nonfree projective $C(X)$-module where $X = \mathbb{S}^1$.

Example 12. *Möbius module. Consider the ring:*

$$R = \{f \in C(\mathbb{R}) | f(x + 2\pi) = f(x)\} \tag{2.12}$$

i.e. the ring of continuous 2π-periodic real valued functions. Also consider the Abelian group of continuous 2π-odd real valued functions:

$$M = \{m \in C(\mathbb{R}) | m(x + 2\pi) = -m(x)\} \tag{2.13}$$

This is an R-module via:

$$(fm)(x) = f(x)m(x) \tag{2.14}$$

We will now show that $M \oplus M \simeq R \oplus R$, i.e. M is projective. Consider $\cos(\frac{x}{2})$ and $\sin(\frac{x}{2})$. We can use them to construct the following homomorphism:

$$\psi : R \oplus R \to M \oplus M \tag{2.15}$$

$$\psi(f, g) = (\cos(\frac{x}{2})f + \sin(\frac{x}{2})g, -\sin(\frac{x}{2})f + \cos(\frac{x}{2})g) \tag{2.16}$$

One can then easily verify that it has an inverse given by:

$$\psi^{-1} : M \oplus M \to R \oplus R \tag{2.17}$$

$$\psi^{-1}(m, n) = (\cos(\frac{x}{2})m - \sin(\frac{x}{2})n, \sin(\frac{x}{2})m + \cos(\frac{x}{2})n) \tag{2.18}$$

Therefor, ψ gives an isomorphism $M \oplus M \simeq R \oplus R$.

Let us now show that M is not free. To do this, we first assume M is free, i.e. $M \simeq R^n$ for some $n \in \mathbb{N}$. Then $R^2 \simeq M \oplus M \simeq R^n \oplus R^n = R^{2n}$. As R is commutative we then conclude that $n = 1$, i.e. $M \simeq R$. Therefor there exists an isomorphism of modules $\phi : R \to M$. Then in general $\phi(f) = \phi(f1) = f\phi(1)$. Now consider $g = \phi(1) \in M$. Since it is continuous and $g(0) = -g(2\pi)$ we know that it has at least one zero on $[0, 2\pi]$. Call it a. Now since ϕ is an isomorphism

we know that any $m \in M$ is the image of some $f \in R$, i.e: $m = \phi(f)$. From this we conclude: $m(a) = \phi(f)(a) = f(a)\phi(1)(a) = f(a)g(a) = 0$. Thus any $m \in M$ has a zero at a. However, consider $\cos(\frac{x-a}{2})$. This is an element of M but $\cos(\frac{a-a}{2}) = 1$. Thus we stumble upon a contradiction and we conclude that M and R are not isomorphic. And thus that there is no $n \in \mathbb{N}$ such that $M \simeq R$, i.e. M is not free.

A recurring question concerning projective modules is thus: under what conditions is a projective module free? In general, this is not an easy question. However, many special cases are known. Let us state a few (without proof):

Theorem 1. *Let R be a field. Any projective R-module is free.*

Theorem 2. *Let R be a principal ideal domain (a domain whose ideals are all principal ideals). Any finitely generated projective R-module is free.*

There is also the following, rather famous, theorem due to Quillen en Susslin:

Theorem 3. *Let P be a projective module over the polynomial ring over a field. Then P is free.*

This celebrated result contributed to Quillen receiving the Fields Medal. For an exposition of the theorem and the proof the reader is referred to [6].

Let us now mention an important property of projective modules which we will use throughout the proof of Swan's theorem, which has to do with the relation between projective modules and projections (or idempotent endomorphisms). Consider therefore the following theorem:

Theorem 4. *Every projective module is isomorphic to the image of an idempotent endomorphism $g : F \to F$ where F is a free module.*

Proof. Let P be a projective module, then it is the direct summand of a free module F. Now consider the surjection $f : F \to P$. Now consider the exact sequence: $0 \to \ker f \to F \to P \to 0$. Since P is projective, this sequence splits. I.e. there is some $\sigma : P \to F$ such that $f \circ \sigma = 1_P$. We can now construct an idempotent endomorphism $g : F \to F$, via $g = \sigma \circ f$, since $g^2 = (\sigma \circ f) \circ (\sigma \circ f) = \sigma \circ (f \circ \sigma) \circ f = \sigma \circ 1_P \circ f = \sigma \circ f = g$. This results in the following decomposition $F = \ker g \oplus \operatorname{im} g = \operatorname{im}(1 - g) \oplus \operatorname{im} g$. Also $\operatorname{im} g = \operatorname{im} \sigma$ and $\sigma : P \to \operatorname{im} \sigma$ is an isomorphism with inverse $f|_{\operatorname{im}\sigma}$. We thus conclude that $P \simeq \operatorname{im} g$. $\square$

Let us conclude this chapter by noting that projectivity is a property that in general is difficult to check. Thus this might form a problem later on when we would like to try to check for explicit examples whether they are projective or not.

Chapter 3

Vector bundles

In this chapter we will introduce the necessary prerequisites as far as vector bundles go. We will start off with the formal definition and some examples. After that we turn to the construction of vector bundles via glueing data. We will then introduce sections and some of their properties, in particular the fact that the set of sections can be given a $C(X)$-module structure. Also we introduce subbundles and other bundle constructions. We then define bundle homomorphisms and examine their kernels and images. We end this chapter with some notions from topology such as (para)compactness, Hausdorffness and normality of a topological space and also some preliminary results for vector bundles over compact Hausdorff spaces. From now on, whenever we write $\mathbb{K}$ we either mean $\mathbb{R}$ or $\mathbb{C}$. (One could also let $\mathbb{K} = \mathbb{H}$, however due to its noncommutativity one would have to be a bit more precise and make slight modifications. We will not pay attention to this case.) Most of what we will discuss can be found in [7] and [8]. For some topology we refer to [9] and [10]. For the part about vector bundles over compact Hausdorff spaces we refer to [2].

3.1 Definition

Let us just start off with the formal definition of a vector bundle and work from there:

Definition 8. *A rank n $\mathbb{K}$-vector bundle ξ consists of a topological space E called the total space, a topological space X called the base space, and a continuous map $\pi : E \to X$ which is onto, and is called the projection, subject to the following conditions:*
(1) For any $x \in X$, $\pi^{-1}(x) = F_x(\xi)$ is an n-dimensional $\mathbb{K}$-vector space, called the fibre at point x.
(2) For each $x \in X$ there exists an open neighbourhood U of x and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{K}^n$ called the local trivialization *such that $\pi = \pi_1 \circ \Phi$, where π_1 is the projection map from $U \times \mathbb{K}^n$ to U. For each $y \in U$ the restriction of Φ to $\pi^{-1}(y)$ must be a linear isomorphism between $\pi^{-1}(y)$ and $\mathbb{K}^n$.*

At first it might not be clear what this definition exactly entails. One way to get a feeling for the concept is by drawing a picture. So let us do just that:
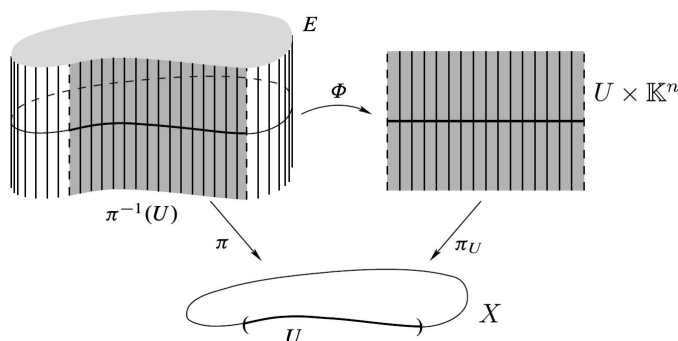

Figure 3.1: An illustration of a vector bundle. Taken from [8].

Let us make notion of the analogy with manifolds. Manifolds locally look like some open subset of $\mathbb{F}^n$. The different subsets are then glued together in some fashion resulting in the manifold which most certainly does not have to look like $\mathbb{F}^n$ globally. In much the same way a vector bundle locally looks like the direct product of fibres $\mathbb{F}^n$ over an open subset of some topological space. These are then suitably glued together to form a general vector bundle. (We will make this more precise in the next section.) Thus although locally they look like a direct product, globally they might have some kind of twist depending on how the different trivializations are glued together.

Some notes about notation: a vector bundle thus comes with several objects and there are several ways to denote a vector bundle, for example: ξ, $\pi : E \to X$, or let ξ be a vector bundle over X, etc. We will mostly use the first, ξ, but we will also use the alternative notations depending on what is more convenient. Let us also note that one could have chosen a definition without the constant rank condition we imposed by letting all fibres be of the same dimension. However, one can easily see that in any case the dimension of the fibres is locally constant. So if we would like a more general setup we would be automatically led to nonconnected base spaces. In this thesis all the statements would very easily be extended to nonconnected base spaces and therefore we will solely consider connected base spaces. Explicitly considering nonconnected base spaces would just clutter the discussion.

Now let's look at some examples. Vector bundles which *do* globally look like such a direct product, i.e. bundles for which a global trivialization exist, are called trivial bundles. This gives our first example of a vector bundle:

Example 13. *Direct product bundle. Let X be some topological space. Construct the direct product vector bundle of dimension n as $X \times \mathbb{K}^n$. The projection*

map is the usual map $\pi(x, v) = x$. The global trivialization is given by the identity.

Of course trivial bundles are not very interesting, so we would like to give some examples of nontrivial bundles, i.e. bundles with some kind of twist. However, first let us construct the one-dimensional trivial real bundle over S^1, i.e. the (infinite) cylinder. We do this to contrast with our first nontrivial bundle we will construct: the Möbius bundle.

Example 14. *Cylinder. Let $X = S^1$. Consider the trivial bundle $S^1 \times \mathbb{R}$. This bundle can be visualised as an infinite cylinder.*

Apart from the cylinder we can construct another one-dimensional real bundle over S^1, which is *not* trivial. To see this, let's cut the infinite cylinder of the previous example along one of the fibres, say at F_0. Now let's apply a twist, and glue it back together with one side inverted. What we now get is an infinite Möbius band which actually forms a vector bundle which is not trivial. To make things a bit more precise:

Example 15. *Möbius bundle. Consider the space $E = I \times \mathbb{R}/ \sim$ with $(0, y) \sim (1, -y)$ (i.e. we let $S^1 = [0, 1]/(0 \sim 1)$). Let us denote the corresponding equivalences in E and S^1 by $[(x, y)]$ respectively $[x]$. Then $\pi : E \to S^1$ given by $\pi([(x, y)]) = [x]$ defines a vector bundle. To see this, consider the open cover $U = S^1 \backslash \{[0]\}$, $V = S^1 \backslash \{[\frac{1}{2}]\}$. Now consider:*

$$\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}, \quad \Phi([(x, y)]) = (x, y) \tag{3.1}$$

$$\Phi_V : \pi^{-1}(V) \to V \times \mathbb{R}, \quad \Phi([(x, y)]) = \begin{cases} (x, y), & \frac{1}{2} < x \leq 1 \\ (x, -y), & 0 \leq x < \frac{1}{2} \end{cases} \tag{3.2}$$

Φ_U has an inverse defined via $\Phi_U^{-1}(x, y) = [(x, y)]$. Now Φ_U and Φ_U^{-1} are seen to be continuous by noting that if $W \subset U \times \mathbb{R}$ is open, then $\Phi_U^{-1}(W)$ is open by the quotient topology on E. Similarly one sees that if $W \subset \pi^{-1}(U)$ is open, then $\Phi_U(W)$ is open. Also it is direct that $\pi_1 \circ \Phi_U = \pi$ and that the restriction of Φ_U to $\pi^{-1}(x)$ with $x \in U$ gives a linear isomorphism from $\pi^{-1}(x)$ to $\mathbb{R}$. Similar reasoning holds for Φ_V. Therefor $\pi : E \to S^1$ is a vector bundle.

Let us consider one last example one often comes across:

Example 16. *(Co)tangent bundles. Let X be a manifold and consider the tangent space TX_x at each $x \in X$. Glueing all these spaces together one ends up with what is called the tangent bundle denoted TX. Analogously one can consider the corresponding cotangent spaces T^*X_x and glueing them into what we call the cotangent bundle T^*X_x.*

3.2 Bundles via glueing data

Later on in this thesis we will be interested in constructing various vector bundles. However, the definition given in the previous section is not the most suitable starting point from which to actually construct a vector bundle. Luckily,
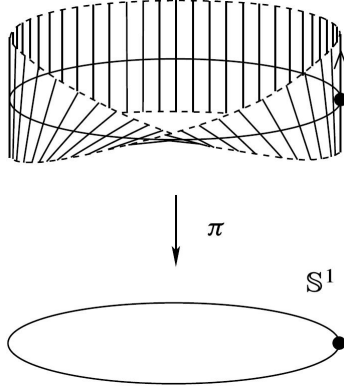
Figure 3.2: An illustration of the (rescaled) Möbius bundle. Taken from [8].

there is an equivalent formulation of vector bundles which enables us to explicitly construct vector bundles in a rather direct fashion. We have already said something about glueing the local trivial parts together. It is time to make that a bit more precise. At the basis lie the so called *transition maps* which come with every vector bundle. To see this, consider a vector bundle ξ with open cover $\{U_i\}$ and local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^n$. Now consider U_i and U_j in the cover and define the following map: $\psi_{ij} = \phi_j \circ \phi_i^{-1} : U_i \cap U_j \times \mathbb{K}^n \to U_i \cap U_j \times \mathbb{K}^n$. It can be easily seen that for $x \in U_i \cap U_j$ and $v \in \mathbb{K}^n$ the map takes the following form: $\psi_{ij}(x, v) = (x, g_{ij}(x)v)$ where $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{K})$ is continuous. The functions g_{ij} are called the transition maps, which tells us how exactly we patch together $U_i \times \mathbb{K}^n$ and $U_j \times \mathbb{K}^n$ (and thus $\pi^{-1}(U_i)$ and $\pi^{-1}(U_j)$).

We thus see that each vector bundle comes with these transition maps which tell us how all the different trivializations are patched together. In fact, a vector bundle is completely determined by its transition maps. To see this consider the following theorem:

Theorem 5. *Bundle construction. Let X be a topological space. Let $\{U_i\}$ be an open cover of X and consider continuous maps $g_{ij} : U_i \cap U_j \to GL(\mathbb{F})$ such that $g_{ij} \circ g_{jk} = g_{ik}$. Then there is a unique (up to isomorphism) vector bundle giving rise to them.*

Proof. We will give a sketch. Consider a topological space X with open cover $\{U_i\}$. And assume we are also given a collection of continuous maps $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{K})$ such that $g_{ij}g_{jk} = g_{ik}$. Now consider $E = (\amalg_i i \times U_i \times \mathbb{K}^n)/ \sim$, with $(j, x, v) \sim (i, x, g_{ij}(x)v)$. It can be easily checked that $\sim$ indeed is an equivalence relation. One then consider:

$$q : \amalg_i i \times U_i \times \mathbb{K}^n \to E \qquad (3.3)$$

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and the natural projection map:

$$\pi : E \to X, \quad [i, x, v] \to x \tag{3.4}$$

One can verify that both maps are continuous and by considering the restrictions $q_{\alpha \times U_{\alpha} \times \mathbb{F}^n}$ and also $\pi_1 = \pi \circ q_{\alpha}$ one can check that all properties of a vector bundle are satisfied.

Uniqueness can be rather straightforwardly shown by constructing an explicit vector bundle isomorphism between bundles with the same glueing data. (We still have to explain what we mean by vector bundle homomorphisms though.)
□

Let us see how we can put it in practice by considering, again, the Möbius bundle.

Example 17. *Möbius revisited. Let $X = S^1$ and again consider the open cover: $U = S^1 \backslash \{[0]\}$, $V = S^1 \backslash \left\{ \left[\frac{1}{2} \right] \right\}$. Now consider $U \cap V$ with the transition function $g : U \cap V \to GL(1, \mathbb{R})$ given by:*

$$g(x) = \begin{cases} 1 & x < \frac{1}{2} \\ -1 & x > \frac{1}{2} \end{cases} \tag{3.5}$$

One can verify that this indeed describes the Möbius bundle.

3.3 Sections

The last concept we have to introduce is that of *sections* of a vector bundle. These are the objects that will be at the core of Swan's theorem. Let's start off with the definition:

Definition 9. *A section. A (global) section of a vector bundle is a continuous map $s : X \to E$ such that $\pi \circ s = id_X$. A map $s : U \to E$ with $U \subset X$, is called a local* local.

We thus see, from $\pi \circ s = id_X$, that a section assigns to each x an element of the corresponding fibre $\pi^{-1}(x)$. The set of all global sections, $\Gamma(\xi)$, forms a vector space under:

$$(c_1 s_1 + c_2 s_2)(x) = c_1 s_1(x) + c_2 s_2(x), \quad c_1, c_2 \in \mathbb{K} \tag{3.6}$$

Actually, there is a stronger notion which we will be using throughout this thesis and which lies at the basis of our entire discussion. It is not difficult to see that we can multiply continuous sections with elements from $C(X)$ turning $\Gamma(\xi)$ into a $C(X)$-*module*:

Theorem 6. *Let ξ be a vector bundle over a topological space X and consider the set of continuous sections $\Gamma(\xi)$. Then $\Gamma(\xi)$ is a $C(X)$-module.*
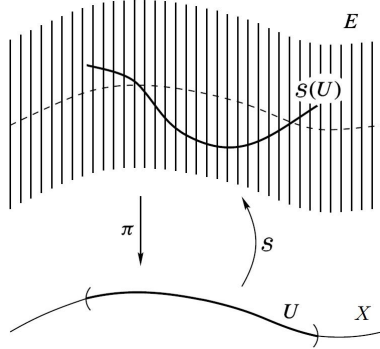
Figure 3.3: An illustration of a (local) section. Taken from [8].

Proof. Consider $f \in C(X)$ and $s \in \Gamma(\xi)$ and define:

$$(fs)(x) = f(x)s(x) \tag{3.7}$$

The map $fs : X \to E$ defined in this manner is a section. Since $\Gamma(\xi)$ forms a vector space, it is of course an additive group. Let $f, g, 1 \in C(X)$ with 1 identity, and $s, t \in \Gamma(\xi)$. Then:

$$(f(s+t))(x) = f(x)(s(x) + t(x)) = f(x)s(x) + f(x)t(x) \tag{3.8}$$
$$= (fs)(x) + (ft)(x) \tag{3.9}$$
$$((f+g)(s))(x) = (f+g)(x)s(x) = (f(x) + g(x))s(x) \tag{3.10}$$
$$= f(x)s(x) + g(x)s(x) = (fs)(x) + (gs)(x) \tag{3.11}$$
$$((fg)(s))(x) = (fg)(x)s(x) = (f(x)g(x))s(x) \tag{3.12}$$
$$= f(x)(g(x)s(x)) = (f(gs))(x) \tag{3.13}$$
$$(1s)(x) = 1(x)s(x) = s(x) \tag{3.14}$$

and we conclude that $\Gamma(\xi)$ is a $C(X)$-module. $\qquad\square$

We are now finally able to fully understand all the objects about which Swan's theorem makes a statement. It claims that if X is compact Hausdorff then $\Gamma(\xi)$ is in fact a finitely generated projective module over $C(X)$. The other way around, the theorem tells us that if we happen to have a finitely generated projective $C(X)$-module for some compact Hausdorff space X, then we know that it can be viewed as being the module of sections of some vector bundle over X. From now on we will view $\Gamma(\xi)$ as a $C(X)$-module.

Before introducing some more concepts let us look at some examples:

Example 18. *Consider the trivial bundle $X \times \mathbb{K}^n$. The sections are precisely the continuous functions $C(X, \mathbb{K}^n)$.*

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Example 19. *Vectorfields/covectorfields. Let X be a manifold and consider the tangent and cotangent bundles. The sections of the tangent bundle are precisely the vector fields over X. The sections of the cotangent bundle correspond to 1-forms on X.*

Example 20. *Consider the Möbius bundle. Its module of sections is precisely the Möbius module discussed in chapter 2. (This is an example of Swan's theorem at work: we have already seen that the Möbius module is projective.)*

Now for some further definitions concerning sections:

Definition 10. *Independent, span, basis. Local sections of E over $U \subset X$, $s_1, ..., s_k$ are said to be* independent *if their values $s_1(x), ..., s_k(x)$ are linearly independent elements of $\pi^{-1}(x)$ for each $x \in U$. If the values span the spaces $\pi^{-1}(x)$, they are said to* span *E. If they are both independent and span the space, they are said to form a* local basis *for E. If $U = X$ they form a* global basis.

Given a vector bundle, one can say a lot about it by examining its sections, in particular its global sections. Due to possible twists of the vector bundle there may be severe restrictions on the possible global sections. If the bundle is trivial however, the case is quite simple (we define bundle (iso)morphisms in section 3.5.):

Theorem 7. *Let ξ be a vector bundle. ξ is isomorphic to a trivial bundle iff ξ has a global basis.*

Proof. ($\Rightarrow$) Just take e_i, with e_i the section which assigns to x the vector $(0, .., 1, ..., 0)$ with the 1 on the i-the entry.
($\Leftarrow$) Consider such a global frame $\{s_1, .., s_n\}$. Then consider the bundle map $f : X \times \mathbb{K}^n \to E$ given by: $f(x, (v_1, .., v_n)) = \Sigma v_i s_i(x)$. This map is a linear isomorphism on each fibre. Also since the composition of this map with a trivialization is continuous, it must be continuous. Hence we have constructed a bundle isomorphism. $\square$

Let us use this theorem to prove the non-triviality of the Möbius bundle:

Example 21. *Möbius bundle. We know that every continuous section of the Möbius bundle vanishes at some point. Thus at that given point the section does not form a basis for the corresponding fibre. Hence the Möbius bundle is not trivial.*

Let us also make the following, due to the local triviality of vector bundles, very natural observation regarding the existence of local bases:

Theorem 8. *Let ξ be a vector bundle over X. For any $x \in X$ there exists a local basis defined on some neighbourhood of x.*

Proof. For any $x \in X$ there is a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{F}^n$ with $x \in U \subset X$. As $U \times \mathbb{F}^n$ is trivial, we can construct a global basis on it, let it be $\{e_1, .., e_n\}$. Now consider $s_i = \Phi^{-1}(e_i)$. Due to composition of continuous functions s_i is continuous. It follows from the fact that Φ restricted to any fibre gives a linear isomorphism $F_x \simeq \mathbb{F}^n$ and that $\{e_1, .., e_n\}$ is a basis for $U \times \mathbb{F}^n$, that $\{s_1, .., s_n\}$ forms basis for each $\pi^{-1}(x)$. From this and the continuity of each s_i we conclude that they form a basis for $\pi^{-1}(U)$. $\qquad\square$

We shall make use of this fact several times later on. The last thing we will say about sections for now is stated in the following theorem:

Theorem 9. *Let $s_1, ..., s_k$ be a collection of sections on some neighbourhood U of x such that $s_1(x), ..., s_k(x)$ are linearly independent. Then, there is a neighbourhood V of x such that $s_1(y), ..., s_k(y)$ are linearly independent for all $y \in V$.*

Proof. Let $t_1, .., t_n$ be a local basis at x. Let $s_i(y) = \Sigma a_{ij}(y) t_j(y)$ with $a_{ij} \in C(X)$. Since the t_i are linearly independent in x, we know that some $k \times k$ submatrix of $(a_{ij}(x))$ must be nonsingular. Therefore, by the continuity of the determinant we conclude that the same submatrix must be nonsingular for all y sufficiently close to x. $\qquad\square$

3.4   Subbundles and other constructions

We would also like to define subbundles as these will play a role in Swan's theorem. The definition is a straightforward one:

Definition 11. *Subbundle. Let ξ be a vector bundle over X. A subbundle of ξ is a subset $E' \subset E(\xi)$ such that $E' \cap F_x$ is a $\mathbb{K}$-subspace of F_x for each x and such that E' with the projection $\pi|_{E'}$ and the $\mathbb{K}$-structure on its fibres induced by that of E forms a $\mathbb{K}$-vector bundle over X.*

There is a convenient test to check whether a given subset of E is a subbundle. We state it without proof:

Theorem 10. *Let ξ be a rank n $\mathbb{K}$-vector bundle over X. Consider a collection of k-dimensional linear subspaces $F'_x \subset F_x$, then $E' = F'_x \subset E$ is a subbundle of E if and only if for each $x \in X$ there is a neighbourhood U of x on which there is a local frame.*

We will now look at some other constructions one typically sees when dealing with vector bundles. These are methods which takes two bundles and construct a new vector bundle out of them. The first one we will consider, is the product bundle:

Definition 12. *Product bundle. Let $\pi_1 : E_1 \to X_1$ and $\pi_2 : E_2 \to X_2$ be vector bundles with local trivilizations $\phi_\alpha : \pi_1^{-1}(U_\alpha) \to U_\alpha \times F^n$ and $\phi_\beta : \pi_2^{-1}(U_\beta) \to U_\beta \times F^n$. Consider $\pi_1 \times \pi_2 : E_1 \times E_2 \to X_1 \times X_2$. This is also a vector bundle with fibres $\pi_1^{-1}(x_1) \times \pi_2^{-1}(x_2)$ and local trivilizations $\phi_\alpha \times \phi_\beta$.*

Another way to construct a new bundle is considering two bundles with the same base space and construct the bundle whose fibres are the direct sum of the fibres of the original bundles, i.e:

Definition 13. *Direct sum bundle. Let η and ξ be vector bundles with the same base space. Then we can construct what we call the direct sum bundle $\zeta = \eta \oplus \xi$ with the same base space by $\eta \oplus \xi = \{(u,v) \in \eta \times \xi | \pi_1(u) = \pi_2(v)\}$*

The direct sum construction will play an important role in the proof of Swan's theorem. An important observation, which can be checked rather straightforwardly, is that if $\zeta = \xi \oplus \eta$ then $\Gamma(\zeta) = \Gamma(\xi) \oplus \Gamma(\eta)$.

3.5 Vector bundle morphisms

To last thing we have to introduce are bundle morphisms. These are defined as follows:

Definition 14. *Vector bundle homomorphism. A vector bundle homomorphism is a pair of continuous maps $f : E_1 \to E_2$ and $g : X_1 \to X_2$ such that $g \circ \pi_1 = \pi_2 \circ f$ and for every x_1 the map $\pi_1(x) \to \pi_2(g(x))$ induced by f is a linear map between vector spaces. If the vector bundles have the same base space X we let $g = 1_X$. Another way to put it is by demanding that the following diagrams should commute:*

$$
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
X_1 & \xrightarrow{g} & X_2
\end{array}
\qquad
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
& \pi_1 \searrow \quad \swarrow \pi_2 & \\
& X &
\end{array}
\tag{3.15}
$$

Note that any vector bundle homomorphism $f : \xi \to \eta$ induces a $C(X)$-module homomorphism, denoted $\Gamma(f) : \Gamma(\xi) \to \Gamma(\eta)$. A vectorbundle isomorphism is a bundle homomorphism which has an inverse which is also a bundle homomorphism. Two bundles which are related via an isomorphism are called isomorphic. Unlike for example in the case of modules where all images and kernels of module homomorphisms are again modules, the image and the kernel of a bundle morphism are not always vector bundles. To see this consider the following example:

Example 22. *Let $X = [0,1]$ and let E be the product bundle $[0,1] \times \mathbb{K}$, with the projection $\pi(x,y) = x$. Consider the following bundle map: $f : E \to E$ given by $f(x,y) = (x,xy)$. The image of f has a fibre of dimension 0 at $x = 0$ and fibres of dimension 1 elsewhere. As the base space is connected we conclude that the image of f cannot be a vector bundle. Similarly we see that the kernel of f has a fibre of dimension 1 at $x = 0$ and fibres of dimension 0 elsewhere.*

However, one can show that this is the only way in which the kernel and image can fail to be bundles; that is by the failure of the dimensions of the fibres to be locally constant. Consider therefore the following theorem:

Theorem 11. *Let $f : \xi \to \eta$ be a map of vector bundles. Then the following statements are equivalent:*
(1) $\mathrm{im}f$ is a subbundle of η
(2) $\ker f$ is a subbundle of ξ
(3) the dimensions of the fibres of $\mathrm{im}f$ are locally constant
(4) the dimensions of the fibres of $\ker f$ are locally constant

Proof. $(1)\Rightarrow(3)$, $(2)\Rightarrow(4)$ are readily established as the dimensions of the fibres of a vector bundle are always locally constant by the local triviality.
$(3)\Leftrightarrow(4)$ is easily established by noting that fibrewise (f_x is the restriction of f to $(x, F_x(\xi))$): $\dim(\mathrm{im}f_x) + \dim(\ker f_x) = \mathrm{rank}(F_x(\xi))$ which is constant.
$(3)\Rightarrow(1)$ Let $x \in X$, and let $s_1, .., s_m$ be a local basis for ξ at x, and let $t_1, ..., t_n$ be a local basis for η at x. Let k be the dimension of $\mathrm{im}f$ at x. Without loss of generality we can assume that $fs_1(x), .., fs_k(x)$ span $F_x(imf)$ and are thus linearly independent. Again without loss of generality we can assume that $fs_1(x), .., fs_k(x), t_{k+1}(x), .., t_n(x)$ are linearly independent. They form a basis for η at x and thus also a local basis at x for η since η is a bundle. This implies that $fs_1, .., fs_k$ are linearly independent on some neighbourhood of x, and hence form a local basis for $\mathrm{im}f$ at x. Thus by the subbundle test we conclude that $\mathrm{im}f$ is a subbundle of η.
$(3)\Rightarrow(2)$ Let $s_1, .., s_m$ be a local basis for ξ at x and again let k be the dimension of $\mathrm{im}f$ at x. We can write $fs_i(y) = \Sigma_1^k a_{ij}(y)fs_j(y)$ for $i > k$ and all y near x. Now consider $s'(y) = s(y) - \Sigma_i^k a_{ij}(y)s_j(y)$, then $s'_{k+1}, .., s'_m$ are local sections of $\ker f$ and are linearly independent. As there are $m - k$ of them, they form a local basis of $\ker f$. Thus by the subbundle test we conclude that $\ker f$ is a subbundle of ξ. $\qquad\square$

Let us also note that without any hypothesis we can make the following statement: if $\dim_k F_x(\mathrm{im}f) = n$, then $\dim_k F_y(\mathrm{im}f) \geq n$ for all y in some neighbourhood of x. We will use this as a last step in proving Swan's theorem.

3.6   Compact Hausdorff spaces

As Swan's theorem deals with vector bundles over compact Hausdorff spaces we will give a short summary of compactness and Hausdorfness and some other concepts related to them.

3.6.1   Compact and Hausdorff

The Hausdorff condition is a separation condition on a topological space. It basically states that two distinct points should be separable by two disjoint open sets:

Definition 15. *Hausdorff space. A topological space X is called Hausdorff if for any two distinct points $x,y \in X$ there exist disjoint open set $U,V \subset X$ such that $x \in U$, $y \in V$*

Most 'nice' topological spaces one encounters satisfy the Hausdorff condition:

Example 23. *Every metric space is Hausdorff.*

Example 24. *Every subspace of a Hausdorff space is Hausdorff.*

Examples of non Hausdorff spaces would be:

Example 25. *Let $X = \{x_1, x_2\}$ with the topology $Top = \{\emptyset, \{x_1\}, X\}$*

Example 26. *The Zariski topology used in algebraic geometry is in general non Hausdorff.*

Now let us recall the definition of compactness:

Definition 16. *Compact space. A topological space X is called compact if any open cover of X has a finite subcover.*

Example 27. *Every finite space is compact.*

Example 28. *$\mathbb{R}^n$ is not compact. A subspace of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Actually we will barely be using the direct consequences of compactness and Hausdorfness separately, but rather some other properties which are implied by the combination of the two.

3.6.2   Paracompact and normal

The first such implication is reflected in the following theorem, which we state without proof:

Theorem 12. *Every compact Hausdorff space is normal.*

Normality is a separation conditions which mimics the Hausdorff condition, with distinct points replaced by disjoint closed sets:

Definition 17. *Normal space. A topological space X is called normal if for any two disjoint closed sets $Y, Z \subset X$ there exist disjoint open sets $U,V \subset X$ such that $Y \subset U, Z \subset V$.*

Example 29. *Any metric space is normal.*

A property of normal spaces we will use when examining vector bundles over compact Hausdorff spaces is given by the following theorem:

Theorem 13. *Urysohn's lemma. A topological space is normal iff any two disjoint closed sets can be separated by a continuous function.*

Now what does seperation by a continuous function entail? Two sets X and Y are said to be separable by a continuous function if there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in X$ and $f(y) = 1$ for $y \in Y$. This is all we need to know about normal spaces.

The next property to be discussed is paracompactness. Although we are not really interested in paracompactness on its own, but rather in combination with Hausdorffness, we will nevertheless give the definition of a paracompact space:

Definition 18. *Paracompact space. A space is paracompact if every open cover has a refinement that is locally finite. (An open cover is locally finite if any point has an open neighborhood which only intersects finitely many sets in the open cover.)*

We will look at some examples:

Example 30. *Any compact space is paracompact. Also any metric space is paracompact.*

Now we will look at the actual reason we find paracompactness interesting:

Theorem 14. *Let X be a topological space. X is paracompact and Hausdorff iff every open cover of X has a partition of unity subordinate to it.*

To make sense of this statement we recall the following definition:

Definition 19. *Partition of unity. A partition of unity is a collection of continuous functions $\{f_n : X \to [0,1]\}$ such that for each $x \in X$ only finitely many f_n are unequal to zero and together they sum to 1.*

However, partitions of unity only become interesting after imposing an extra condition. Consider therefore the following definition:

Definition 20. *Partition of unity subordinate to cover. Given an open cover $\{U_\alpha\}$ and a collection of continuous functions $\{f_\alpha\}$ counted by the same index set. Now let $\{f_\alpha\}$ be a partition of unity such that for all α we have $\operatorname{supp} f_\alpha \subset U_\alpha$. We call $\{f_\alpha\}$ a partition of unity subordinate to the cover $\{U_\alpha\}$.*

These partitions *are* particularly interesting. They can serve as a tool to pass from something local to something global. It is not difficult to see that due to the local triviality of vector bundles this is a nice property to have. This could possibly enable us to promote some nice local properties to global ones.

3.7 Vector bundles over compact Hausdorff spaces

Now that we have covered some basic topology we need in order to describe vector bundles over compact Hausdorff spaces, let us give some properties of these bundles which will prove to be important in proving Swan's Theorem.

3.7.1 Inner products

A first property of vector bundles over compact Hausdorff spaces is the existence of an inner product. Let us explain what we mean by an inner product on a vector bundle:

Definition 21. *Inner product. An inner product on a vector bundle ξ is a continuous map $g : E \oplus E \to \mathbb{F}$ whose restriction to each fibre is an inner product on it.*

On each fibre of a vector bundle it is of course trivial to construct an inner product via the standard inner product on $\mathbb{K}^n$. The question would be whether we can assign to each fibre an inner product such that they vary continuously with the base point and thus taken together constitute a continuous inner product on the entire bundle. A first step in trying to create a global inner product, is to define an inner product on each trivialization $p^{-1}(U_\alpha)$. This is always possible due to the trivial nature and is achieved by simply pulling back the inner product of $U \times \mathbb{K}^n$. Our hope would then be that somehow, when glueing all the trivializations together, the inner products would turn out to nicely patch together in a continuous manner. In general, due to the possible nontrivial global structure of the bundle, this is not possible.
As noted before, one often used concept to turn something which is local, like our inner product on a trivialization, into something global is that of partitions of unity subordinate to an open cover. As we have seen in the previous section, paracompact Hausdorff spaces are precisely the spaces for which such subordinate partitions of unity exist. Let us see how such a partition may be used to construct a global inner product by considering the proof of the following theorem:

Theorem 15. *Let X be paracompact Hausdorff and let ξ be a vector bundle over X. Then ξ has an inner product.*

Proof. First consider a locally finite cover $\{U_\alpha\}$, such that $p^{-1}(U_\alpha) = U_\alpha \times \mathbb{K}^n$. Now consider a real partition of unity on X, $\{\omega_\alpha\}$, subordinate to the covering $\{U_\alpha\}$. We can define an inner product on each $p^{-1}(U_\alpha)$, call it $(,)_{\alpha,x}$. Now define $(e_1, e_2)_x = \sum_\alpha \omega_\alpha(x)(e_1, e_2)_{\alpha,x}$. One can easily verify that this defines a global inner product. $\square$

Thus, all vector bundles over compact Hausdorff spaces (as they are paracompact Hausdorff) come with an inner product. This fact enables us to define certain projection maps onto subbundles and their direct complements, which will turn out to be very useful.

3.7.2 Global and local sections

Now we will look at the second property of bundles over compact Hausdorff spaces, which has to do with going from local sections to global sections. Of course, for any $x \in X$ we can always find a local section with some desireable

property because the bundle is locally trivial. For example, given any $e \in E$ with $p(e) = x$, we can always find a local section s such that $s(x) = e$. However, there is nothing that ensures the existence of a global section with the same property. Luckily, due to the nice properties of compact Hausdorff spaces, we *can* make such a statement for vector bundles over such spaces. In fact, all one needs is the base space to be normal:

Theorem 16. *Let X be normal. Let U be a neighbourhood of x, and let s be a section of a vector bundle ξ over U. Then there is a section s' of ξ over X such that s' and s agree in some neighbourhood of x.*

Proof. Let V, W be neighbourhoods of x such that $\bar{V} \subset U$, $\bar{W} \subset V$. Let ω be a real-valued function on X such that $\omega|\bar{W} = 1$, $\omega|X - V = 0$. Such a function exists due to Urysohn's lemma. The wanted section is now given by: $s'(y) = \omega(y)s(y)$ if $y \in U$ and $s'(y) = 0$ if $y \notin U$. □

An immediate consequence is the following theorem:

Theorem 17. *For any $x \in X$ there are global sections which form a local basis at x.*

Proof. As we know there exist local sections which form a local basis at x. Now from the previous theorem we see that these can be promoted to global sections agreeing in some neighbourhood of x. □

Thus in the case of bundles over compact Hausdorff spaces, local sections with desireable properties can be promoted to global sections which much the same properties. This will turn out to be an important fact which will be used in the proof of Swan's Theorem.

Chapter 4

Proof of Swan's theorem

Now that we have covered the basics concerning vector bundles and projective modules we are ready to start examining the proof of Swan's theorem. We will closely follow Swan's original article [2] adding minor details every now and then. Let us restate the theorem for convenience:

Theorem 18. *Let X be compact Hausdorff. Then a $C(X)$-module P is isomorphic to a module of the form $\Gamma(\zeta)$ if and only if P is finitely generated and projective.*

The first step towards proving it, is the observation that an analogous statement holds concerning trivial bundles and free modules:

Theorem 19. *Let X be a topological space. Then a $C(X)$-module F is isomorphic to a module of the form $\Gamma(\zeta)$ for some trivial vector bundle ζ over X i.f.f. F is finitely generated and free.*

Proof. Let ζ be a trivial bundle, then $\zeta = X \times \mathbb{K}^n$ for some n. Now consider $s \in \Gamma(\zeta)$. As we have already seen $\{e_1, .., e_n\}$, with $e_i = (0, .., 1, .., 0)$ with the 1 on the i-th entry, is a global free basis for $\Gamma(\zeta)$. Thus $\Gamma(\zeta)$ is free. Conversely, let F be a finitely generated free $C(X)$-module. Then F is generated by the free basis $\{e_1, ..., e_n\}$ for some n. It then follows that $F \simeq \Gamma(\zeta)$ with $\zeta = X \times K^n$. $\square$

(Note that in this case there are no restrictions on the topological space X.)

Let us give a short overview of the line of reasoning which is to follow. In the ($\Rightarrow$)-direction the general idea is that we will prove that any vectorbundle over a compact Hausdorff space is a direct summand of a trivial bundle, i.e. $\zeta = \eta \oplus \xi$. As we have already noted in the previous chapter this results in a similar decomposition for the corresponding modules of sections, i.e. $\Gamma(\zeta) = \Gamma(\eta) \oplus \Gamma(\xi)$. We have just seen that if ζ is trivial, $\Gamma(\zeta)$ is a finitely generated free module. We would thus conclude that $\Gamma\xi$ is direct summand of a finitely generated free module and hence a finitely generated projective module.

In the ($\Leftarrow$)-direction we will make use of the fact that any projective module is isomorphic to the image of some idempotent endomorphism and that if X is normal, all maps between modules of sections are induced by maps between their corresponding bundles. (This is actually also needed in the ($\Rightarrow$)-direction.) The next section is dedicated to proving this statement.

4.1 All module morphisms are induced by bundle morphisms

Let us give a more precise formulation of the statement we are set out to prove in this section:

Theorem 20. *Let X be normal. Given any $C(X)$-map $F : \Gamma(\xi) \to \Gamma(\eta)$, there is a unique K-bundle map $f : \xi \to \eta$ such that $F = \Gamma(f)$.*

The reason this theorem holds can be traced back to the fact that if our base space is normal we can extend local sections to global sections (which we have seen in the previous chapter). Before we will prove the existence of such a bundle morphism, we will first show that if it were to exist, it would be unique:

Theorem 21. *Let X be normal. If $f, g : \xi \to \eta$ and $\Gamma(f) = \Gamma(g) : \Gamma(\xi) \to \Gamma(\eta)$, then $f = g$.*

Proof. Let $e \in E(\xi)$ with $\pi(e) = x$, there is a section s over a neighborhood U of x with $s(x) = e$. Since X is normal there is a section $s' \in \Gamma(\xi)$ with $s'(x) = e$. Now assume we are given the bundle morphisms f, g such that $\Gamma(f) = \Gamma(g)$. Then $f(e) = fs'(x) = (\Gamma(f)s')(x) = (\Gamma(g)s')(x) = gs'(x) = g(e)$ and we thus conclude that $f = g$. $\qquad\square$

Now, to be able to construct such a bundle morphism when presented with a morphism between the modules of sections of two bundles, we must first make some preliminary observations. Consider the following two theorems:

Theorem 22. *Let X be normal. Let $s \in \Gamma(\xi)$ and $s(x) = 0$. Then there are elements $s_1, .., s_k \in \Gamma(\xi)$, $a_1, .., a_k \in C(X)$ such that $a_i(x) = 0$ and $s = \Sigma a_i s_i$,*

Proof. Let $s \in \Gamma(\xi)$ and $s(x) = 0$. Let $s_1, .., s_n \in \Gamma(\xi)$ be a local base at x. Let $s(y) = \Sigma b_i(y) s_i(y)$ near x, $b_i(y) \in \mathbb{K}$. Now let $a_i(x) \in C(X)$ such that a_i and b_i agree on some neighborhood of x. Then $s' = s - \Sigma a_i s_i$ vanishes on some neighborhood U of x. Now let V be a neighborhood such that $\bar{V} \subset U$. Let $a \in C(X)$ be zero at x and 1 on $X - V$ (this is possible due to Urysohn's lemma). Then $s = as' + \Sigma a_i s_i$. But $a(x) = 0$ and we thus conclude that $a_i(x) = b_i(x) = 0$ as the s_i form a basis at x. Thus $s = \Sigma a_i s_i$ with $a_i(x) = 0$. $\qquad\square$

This theorem enables us to identify a given fibre of the vector bundle with the set of all equivalence classes of sections under the equivalence relation $s_1 \sim s_2$ if $s_1(x) = s_2(x)$:

Theorem 23. *Let I_x be the ideal of C(X) consisting of all $a \in C(X)$ with $a(x) = 0$. Then the map $s \to s(x)$ gives the isomorphism $\Gamma(\xi)/I_x\Gamma(\xi) \simeq F_x(\xi)$.*

Proof. It is clear that the map is a homomorphism. Rests us to check surjectivity and injectivity. To show surjectivity consider some $u \in F_x(\xi)$. There is a corresponding $e \in E(\xi)$. Since X is normal there is an $s \in \Gamma(\xi)$ with $s(x) = e$, thus the map is surjective. Now for injectivity: consider $s_1, s_2 \in \Gamma(\xi)/I_x\Gamma(\xi)$ such that $s_1(x) = s_2(x)$. Then $s(x) = s_1(x) - s_2(x) = 0$ and by the previous theorem we conclude that $s = \Sigma a_i s_i$ with $a_i(x) = 0$, i.e. $s \in I_x\Gamma(\xi)$. We thus conclude that that s_1 and s_2 lie in the same equivalence class. The map is thus also injective and we conclude that it is thus an isomorphism. $\qquad\square$

For each fibre one can consider this isomorphism. By considering these maps for all different x we are able to construct a bundle map and prove the theorem:

Theorem 24. *Let X be normal. Given any C(X)-map $F : \Gamma(\xi) \to \Gamma(\eta)$, there is a unique K-bundle map $f : \xi \to \eta$ such that $F = \Gamma(f)$.*

Proof. The map F induces the maps $f_x : \Gamma(\xi)/I_x\Gamma(\xi) \to \Gamma(\eta)/I_x\Gamma(\eta)$ by letting $f_x s(x) = (F(s))(x)$. The totality of these maps yield a map $f : E(\xi) \to E(\eta)$ which is $\mathbb{K}$-linear on the fibres. Also $F = \Gamma(f)$ by construction: let $s \in \Gamma(\xi)$, then $(fs)(x) = f_x s(x) = (F(s))(x)$. What remains to be showns is the continuity of f. Let $s_1, .., s_n$ be a local base at x. Now consider $e \in E(\xi)$ such that $p(e)$ is near x. Then $e = \Sigma a_i(p(e))s_i(p(e))$ where $a_i \in C(X)$ and thus $f(e) = \Sigma a_i(p(e))fs_i(p(e))$. Since $fs_i = F(s_i)$, fs_i is a continuous section of η. All terms are thus continuous and we conclude that f is continuous. We have already proven its uniqueness. $\qquad\square$

4.2 Direct summands

The goal of this section is to show that any vector bundle over a compact Hausdorff space is a direct summand of a trivial bundle over X. The first step towards proving this is the following theorem:

Theorem 25. *Let X be compact Hausdorff and let ξ be a vector bundle. Then there is a trivial bundle ζ and an epimorphism $f : \zeta \to \xi$.*

Proof. For every $x \in X$ choose a set of global sections $s_{x,1}, ..., s_{x,k} \in \Gamma(\xi)$ that forms a local basis over some neighbourhood U_x. Since X is compact we know that a finite number of U_x cover X. Now consider this finite open cover and take the collection of their corresponding local bases. There are finitely many of them resulting in the set (after renumbering) $\{s_1, ..., s_n\}$ (where n is k times the number of open sets in the cover). This set spans $F_x(\xi)$ for any x. Now consider $\zeta = X \times \mathbb{K}^n$, then $\Gamma(\zeta)$ is a free $C(X)$-module with n generators $e_1, ..., e_n$. Now consider $F : \Gamma(\zeta) \to \Gamma(\xi)$, with $F(e_i) = s_i$. By the previous section we know this module morphism is induced by a bundle morphism $f : \zeta \to \xi$. Now $fe_i = s_i$ and therefor we see that $s_i \in \mathrm{im} f$. As the s_i span each fibre we conclude that f is an epimorphism. $\qquad\square$

Another interesting thing to note is the following theorem, which reminds us of a property of projective modules:

Theorem 26. *Sequence splits. Let ζ be a vector bundle with an inner product an let η be a subbundle. Any exact sequence $0 \to \eta \to \zeta \to \xi \xrightarrow{f} 0$ splits and $\zeta \simeq \xi \oplus \eta$.*

Proof. We can define an orthogonal projection $P : \zeta \to \eta$ with $\mathrm{im}P = \eta = \ker f$ via the inner product. The restriction of the map $f : \zeta \to \xi$ to $\ker P$ gives an isomorphism $\xi \simeq \mathrm{im}f$. The inverse gives the splitting of the sequence.
Since the sequence splits we have $\sigma : \xi \to \zeta$ such that $f \circ \sigma = 1$. Now $\sigma \circ f : \zeta \to \zeta$, is seen to be a projection and $\zeta = \mathrm{im}(\sigma \circ f) \oplus \ker(\sigma \circ f)$. Also $f|_{\mathrm{im}(\sigma \circ f)} : \mathrm{im}(\sigma \circ f) \to \mathrm{im}f = \xi$ gives an isomorphism $\mathrm{im}(\sigma \circ f) \simeq \xi$. Also we see that $\ker(\sigma \circ f) = \ker f$, since $\ker\sigma = \{0\}$. We thus conclude that $\zeta = \mathrm{im}(\sigma \circ f) \oplus \ker(\sigma \circ f) \simeq \mathrm{im}f \oplus \ker f \simeq \xi \oplus \eta$. $\qquad\square$

Combining the two theorems we immediatly get:

Theorem 27. *Let X be compact Hausdorff and let ξ be a vector bundle. Then there is a trivial bundle ζ and another bundle η, such that $\zeta \simeq \xi \oplus \eta$.*

Proof. Let ζ abe a trivial bundle then there exists an epimorphism $f : \zeta \to \xi$. We directly see that $\ker f = \eta$ is a subbundle since $\mathrm{im}f = \xi$ is a bundle (and thus a subbundle) and we conclude that $\zeta \simeq \mathrm{im}f \oplus \ker f \simeq \xi \oplus \eta$. $\qquad\square$

4.3 Putting everything together

We have now proven everything we need in order put everything together and prove Swan's theorem. We start off with the $(\Rightarrow)$-direction, which in essence we have already proven in the previous sections:

Theorem 28. *Let X be compact Hausdorff and let ξ be a vector bundle. Then $\Gamma(\xi)$ is a finitely generated projective module.*

Proof. Since $\zeta = \eta \oplus \xi$ where ζ is trivial, we have $\Gamma(\zeta) = \Gamma(\eta) \oplus \Gamma(\xi)$ where $\Gamma(\zeta)$ is free and finitely generated. $\Gamma(\xi)$ is thus projective and finitely generated. $\square$

The other direction requires a bit more reasoning still:

Theorem 29. *Let X be compact Hausdorff en let P be a finitely generated projective $C(X)$-module. Then $P \simeq \Gamma(\xi)$ for some topological vector bundle ξ over X.*

Proof. Let P be a finitely generated projective $C(X)$-module. This means that it is the direct summand of a finitely generated free $C(X)$-module F. This in turn implies that P is isomorphic to the image of some idempotent endomorphism $g : F \to F$, i.e. $P \simeq \mathrm{im}g$. Now $F = \Gamma(\zeta)$ where ζ is a trivial bundle since F is free and finitely generated. Now, $g : \Gamma(\zeta) \to \Gamma(\zeta)$, and thus g is induced by a bundle endomorphism $f : \zeta \to \zeta$ and $g = \Gamma(f)$. If we are now able to show

that $\xi = \mathrm{im} f$ is a subbundle of ζ we are done. To see this note that if it is a subbundle we can conclude that $P \simeq \mathrm{im} g \simeq \mathrm{im} \Gamma(f) = \Gamma(\mathrm{im} f) = \Gamma(\xi)$

To show that it is a subbundle first note that since $g^2 = g$, also $f^2 = f$. This the following standard results: $\eta = \ker f = \mathrm{im}(1 - f)$ and $F_x(\zeta) = F_x(\xi) \oplus F_x(\eta)$. Now suppose $\dim_K F_x(\xi) = h$ and $\dim_K F_x(\eta) = k$. We know that $\dim_K F_y(\xi) \geq h$ and $\dim_K F_y(\eta) \geq k$ for all y in some neighborhood of x. Also: $\dim_K F_y(\xi) + \dim_K F_y(\eta) = \dim_K F_x(\zeta) = h + k$ is constant. Thus we conclude that $\dim_K F_x(\xi)$ is constant, and hence ξ is a subbundle. This concludes the proof. $\qquad\square$

We have thus proven Swan's Theorem, and with that we conclude this chapter.

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Chapter 5

Swan and beyond

Now that we have gone through the proof, it is time to examine the scope of Swan's theorem. That is: is Swan's theorem the definitive and final statement about the relation between topological vector bundles and projective $C(X)$-modules? Or is there a meaningful generalization? That is: is Swan's proof the best proof, that is, are all the steps necessary or merely sufficient? We have tried to examine these questions by trying to come up with explicit 'counter examples', which fall into any of the following four categories:

(1) A vector bundle over a space which is not compact Hausdorff whose module of sections is not projective.
(2) A vector bundle over a space which is not compact Hausdorff whose module of sections *is* projective.
(3) A projective $C(X)$-module, where X is not compact Hausdorff, which is not isomorphic to the module of sections of some vector bundle over X.
(4) A projective $C(X)$-module, where X is not compact Hausdorff, which *is* isomorphic to the module of sections of some vector bundle over X.

We will see that this is not as easy as one might initially think. We will comment on a partial explanation of this by means of a theorem by Vaserstein.

5.1   'Counter examples'

In this section we will give an overview of some of the things we tried in coming up with suitable counter examples. We first consider the construction of projective $C(X)$-modules.

5.1.1   Constructing projective $C(X)$-modules

A first step in trying to come up with suitable $C(X)$-modules is of course examining the ring $C(X)$. The structure of the ring $C(X)$ is completely determined

by the space X and the topology on it. Unfortunately the rings $C(X)$ are in general very large which makes explicit analysis difficult. Thus somehow we would like to choose our space and topology in such a way that $C(X)$ becomes much smaller, but not too small. We give two examples we tried.

Finite base space

As a first try we considered finite spaces. However, the resulting rings $C(X)$ are too simple. To see this consider the finite space $X = \{x_1, .., x_n\}$. Let us construct $C(X)$ by examining what continuous functions $f : X \to \mathbb{R}$ look like. Any function on a finite space can be represented as $f = (a_1, .., a_n)$ where $a_i = f(x_i) \in \mathbb{R}$. Depending on the topology one considers there will be restrictions on such a function in terms of the a_i. By making use of the fact that a function is continuous i.f.f. the preimage of an open set is open we will now examine this. Let us first consider the case where we take the discrete topology, i.e. every subset of X is taken to be open. In this case there is no restriction and any function is continuous: $C(X) \simeq \mathbb{R}^n$. If we consider the indiscrete topology, i.e. only $\emptyset$ and X are taken to be open, we conclude that any continuous function should map each $x_i \in X$ to the same point. I.e. $C(X) \simeq \mathbb{R}$. Now it is not difficult to convince oneself of the fact that in general $C(X) \simeq \mathbb{R}^k$ with $k = 1, .., n$ where the value of k depends on the precise structure of the topology. To see this note that in general, for any $x_i \in X$ there is some smallest open set U which contains x_i. Now in order for a function to be continuous, the function should map all elements in U to the same point. Depending on the sizes of the different smallest open sets containing the different points and other specifics of the topology, one would end up with one or more (or no) restrictions, effectively equating some of the a_i resulting in $C(X) \simeq \mathbb{R}^k$. Unfortunately no nonfree projective $\mathbb{R}^k$-modules were found that did not correspond to modules of sections of vector bundles over X.

Countable base space

Another attempt would be to look at countable spaces, for example $\mathbb{N}$. In general a function could be represented by an infinite sequence a_n, where $a_n = f(x_n) \in \mathbb{R}$. This seems to be too large, so we would like to restrict it by choosing a suitable topology on $\mathbb{N}$. We considered the Zariski topology (which is non Hausdorff) in which the open sets are precisely the complements of finite sets. Again, after some thought and by making use of the preimage test for continuity we conclude that the ring $C(X)$ is in no way interesting: $C(X) \simeq \mathbb{R}$. Our topology is thus too restrictive.

Other attempts to find interesting results were also undertaken, but unfortunately also not successful. It thus seems hard to find suitable base spaces such that the resulting ring of continuous functions is not too big and not too small. Thus sadly, our attempt to construct suitable projective $C(X)$-modules already halts at the early stage of finding suitable function rings $C(X)$.

5.1.2 Constructing vector bundles

Let us now try to construct vector bundles which would be counter examples. We will give an overview of some of the bundles we considered. We first considered base spaces which are no longer compact. Our starting point was to look at for example $\mathbb{R}$ as a base space. However, $\mathbb{R}$ is contractible and the following theorem thus renders it useless [11]:

Theorem 30. *Every vector bundle over a contractible space is trivial.*

Proof. As the space is contractible it is homotopic to a point. This is also induced from the vector bundle point of view. That is: any vector bundle over the space is homotopic to a vector bundle over a point. Clearly any vector bundle over a point is trivial. □

In fact, this theorem rendered many of our attempts useless, due to the fact that we imbedded them in $\mathbb{R}^k$.

We also tried to let go of the Hausdorff condition, in particular when dealing with finite base spaces. However, we were not able to construct nontrivial vector bundles over such spaces.

5.1.3 Some further remarks

We thus see that finding explicit, interesting, counter examples is difficult due to several reasons. Therefore we, sadly, did not get a better grasp of the scope of Swan's Theorem. As noted there are a couple of reasons for this. First of all, in trying to construct projective modules, we had to deal with the fact that the rings $C(X)$ are generally quite large. We had to find a balance between constructing a ring $C(X)$ which is not too large, but also not too small. We were not able to come up with satisfactory results, perhaps due to an insufficient knowledge of topology. Even if we were able to construct nice rings $C(X)$ and projective modules, the question would still be if we would be able to determine whether they are isomorphic to the modules of sections of bundles.
As far as the construction of bundles is concerned we were also not successful, probably due to the fact that most ideas one naturally gets are too 'nice' for our case. Also, we tried several bases spaces embedded in $\mathbb{R}^k$, however its contractibility rendered our examples not very interesting. Also even if we were presented with suitable vector bundles, we would still be challenged to examine the projectivity of their modules of sections.
Extensive research in the known literature did not result in any examples, hinting at the difficulty of the task.

5.2 Beyond Swan

However, near the end of the research we came across a generalization of Swan's theorem due to Vaserstein. This theorem no longer puts any restrictions on the

underlying topological space, but rather on the type of vector bundle we consider over it, namely vector bundles of *finite type*. This generalization partially explains our difficulty in coming up with good examples. Let us first state the theorem [3]:

Theorem 31. *Let X be a topological space. A $C(X)$-module P is isomorphic to a module of the form $\Gamma(\zeta)$ where ζ is of finite type if and only if P is finitely generated and projective.*

The first thing that strikes us is that given a finitely generated projective $C(X)$-module over *any* topological space X, one can identify it with the module of sections of a vector bundle (though of finite type) over X. Thus any hopes we had in the previous section of finding a finitely generated projective $C(X)$-module for which the conclusion of Swan's Theorem would not hold, are now gone. So apart from the difficulty of constructing suitable $C(X)$-modules, let alone examining whether they have an interpretation as modules of sections of a bundle, it is now seen to be fundamentally impossible to find such modules. What it also tells us is that the proof used by Swan in the ($\Leftarrow$)-direction is certainly not the best one can do. We will further comment on this later on.

To examine the implications of this theorem in the other direction, we must first give some attention to these so called vector bundles of finite type:

Definition 22. *A vector bundle over a topological space X, is said to be of finite type if there is a finite partition S of 1 on X (i.e. a finite set S of nonnegative continuous functions on X whose sum is 1), such that the restriction of the bundle to the set $\{x \in X : f(x) \neq 0\}$ is trivial for each $f \in S$.*

It may not immediately be clear what this definition exactly entails, but let us try to find out. The definition basically has two major implications regarding the vector bundle. The first is that we can always find a finite number of open sets over which the bundle trivializes by means of the finite partition of unity. Secondly we thus see that effectively the finite partition of unity is subordinate to that finite open cover. Put like this, it is much more clear what the definition is all about. The fact that there is a finite trivialization very much resembles compactness of the underlying topological space. Most certainly if X is compact this would imply the existence of a finite trivialization. The existence of a partition of unity subordinate to that trivialization would then closely mimic the paracompact Hausdorff property of the underlying space. Thus really, the definition of a finite type vector bundle seems to be constructed in such a way that any bundle effectively behaves as if its base space is compact Hausdorff. Instead of demanding nice properties of the base space, one transfers these restrictions to the bundle itself.

It is not difficult to see that indeed this is a generalization of Swan's Theorem in the sense that it contains it: any vector bundle over a compact Hausdorff space is seen to be of finite type. Also the analogous theorem for free modules

and trivial vector bundles is included as a special case. To see this note that there is nothing in the definition of a vector bundle of finite type which says that given some open cover there should be a partition of unity subordinate to it. It is rather the other way around: there is a finite partition of unity which gives rise to a finite cover over which the bundle trivializes. Of course, for any topological space one can choose a trivial partition of unity by considering the unit function $f : X \to 1$. One thus sees that any trivial bundle is of finite type.

This theorem also partially explains why we had such a hard time in the previous section constructing bundles whose modules of sections are not finitely generated and projective. In fact, most bundles one encounters, even if the base space is not compact, are of finite type. One thus has to come up with rather exotic examples, which we unfortunately were not able to construct. Although they do exist, vector bundles which are not of finite type are not often come across and an extensive search through the literature also did not result in any examples.

However, the idea of what we would like to do is clear: find nonfinite type vector bundles and examine their modules of sections. There are a couple of difficulties one would probably encounter whilst trying to find such examples. The first is that it would in general be difficult to show that a bundle is not of finite type. For all we know, we are simply not smart enough to come up with a suitable partition of unity over which the bundle trivializes. Also, even if we are presented with a finite type vector bundle, we would still have to check whether its module of sections is projective or not, which, as we have already discussed, is not an easy task. The absence of such counter examples in the literature, also in the article by Vasertein himself, would also seem to hint at the difficulty.

5.2.1 Sketch of the proof

Let us conclude by giving a sketch of the proof, beginning in the ($\Leftarrow$)-direction. In this direction this theorem is certainly much stronger than Swan's theorem as it in no way puts any restrictions on the finitely generated projective $C(X)$-module. Let us see what differs with the proof given by Swan. Vaserstein uses the following, slightly stronger statement regarding projective modules: every projective $C(X)$-module P is isomorphic to the column space of a square hermitian idempotent matrix $e = e^2$ over $C(X)$. He then states that every such module gives rise to a vector bundle by taking the fibre at each point to be P evaluated at x, that is: $F_x = e(x)\mathbb{F}^n$, where e is the $n \times n$ matrix to whose column space P is isomorphic. To show that the bundle is of finite type he uses the fact that the set of matrices e, call it Y, is compact and he claims that one can find a finite partition of 1, S_Y, on Y with certain properties. Now the matrix e can be considered as a continuous map $X \to Y$. Therefore the finite partition S_Y on Y gives a finite partition of 1, S, on X. He then looks at the restriction of the column space of e to $U(f) = \{x \in X | f(x) \neq 0 \text{ with } f \in S\}$ and shows that it is a finitely generated free $C(U(f))$-module and hence the restriction of the bundle to $U(f)$ is trivial and he concludes that the bundle is of finite type.

In the other direction he uses the fact that for each $f \in S$ one can of course find a free basis $a_1(f), .., a_{n(f)}(f)$ for $\Gamma(\xi)|_{U(f)}$. Just as in Swans theorem he then extends these to global secions after suitable modifications. Taking these all together forms a generating set with $N = \sigma_{f \in S} n(f)$ elements for $\Gamma(\xi)$. He then constructs a surjection $g : F^N \to \Gamma(\xi)$ whose kernel can be shown to be a direct summand of F^N. It then follows that $\Gamma(\xi)$ is also a direct summand and thus projective. Vaserstein thus circumvents the part in Swan's theorem where Swan uses the result that every homomorphism between modules of sections are induced by bundle homomorphism, and directly examines the module of sections of ξ.

Chapter 6

Conclusions

In this thesis we set out to examine Swan's theorem which states that given a compact Hausdorff space, any finitely generated projective $C(X)$-module is isomorphic to the module of sections of a topological vector bundle over X, and conversely that any module of sections of a topological vector bundle over X is a finitely generated and projective $C(X)$-module. Before we could do so we had to cover some preliminaries regarding modules, more specifically projective modules, and topological vector bundles (over compact Hausdorff spaces). After these introductions we went through the proof in detail and found that it leans heavily on the property that certain local properties can be extended to global properties when dealing with vector bundles over compact Hausdorff spaces.

Finally we tried to examine the scope of the theorem by constructing explicit 'counter examples': examples to which Swan's theorem does not apply. This turned out to be a difficult task and unfortunately no satisfactory results were obtained. One reason is that in general $C(X)$ is very large and dificult to manage. Also projectivity of a module is a difficult property to check. Another partial explanation was found in a generalization of Swan's theorem due to Vaserstein. This theorem states that a projective $C(X)$-module over *any* topological space X can be viewed as the module of sections of some bundle over X. Conversely, any bundle of finite type has a projective module of sections. Upon examining finite type vector bundles we concluded that in fact almost all reasonable bundles are of finite type and no counter examples were found. It thus seems that Vaserstein's result gives a sensible generalization of Swan's theorem.

Chapter 7

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