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The Henstock-Kurzweil integral

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Abstract

In this thesis we examine the Henstock-Kurzweil integral. First we look at the definition, which only differs slightly from the definition of the common Riemann integral; instead of using a constant δ, we use a strictly positive function γ. After proving some properties of the Henstock-Kurzweil integral, we look at a couple examples. At last we will see that the Henstock-Kurzweil integral has some nice benefits over the Riemann integral: we note that each derivative is Henstock-Kurzweil integrable when we look at the Fundamental theorem and we can prove some nice convergence theorems regarding Henstock-Kurzweil integrals.

Contents

1 Introduction

Historically integration was defined to be the inverse process of differentiating. So a function F was the integral of a function f if $F' = f$. Around 1850 a new approach occurred in the work of Cauchy and soon after in the work of Riemann. Their idea was to not look at integrals as the inverse of derivatives, but to come up with a definition of the integral independent of the derivative. They used the notion of the 'area under the curve' as a starting point for building a definition of the integral. Nowadays this is known as the Riemann integral. This integral has an intuitive approach and is usually discussed in calculus courses.

Suppose we have a function f on the interval $[a, b]$ and we want to find its Riemann integral. The idea is to divide the interval into small subintervals. In each subinterval $[x_{i-1}, x_i]$ we pick some point t_i. For simplicity the point t_i is often chosen to be one of the endpoints of the interval. The value $f(t_i)(x_i - x_{i-1})$ is then used to approximate the area

under the graph of f on $[x_{i-1}, x_i]$. The area of each rectangle that is obtained this way is $f(t_i)(x_i - x_{i-1})$. The total area under the curve on the interval $[a, b]$ is then approximated by $\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$. It is easily seen that the approximation improves as the rectangles get thinner, so if the length of the subintervals $[x_{i-1}, x_i]$ get smaller. We take the limit (if it exists) of this approximating sums as the length of those subintervals tends to zero. This limit is Riemann's definition of the integral $\int_a^b f$.
Riemann's definition of the integral turned out to have some limitations. For example, not every derivative can be integrated when considering Riemann's definition. To correct these deficiencies, Lebesgue came up with a new definition of integration. Lebesgue's method is a complex one and a considerably amount of measure theory is required even to define the integral. Around 1965 Kurzweil came up with a new definition for the integral which was further developed by Henstock. In their definition the intuitive approach of the Riemann integral is preserved. They just made a small adjustment to the standard ϵ,δ-definition of the Riemann integral; instead of a constant δ, Henstock and Kurzweil used a strictly positive function γ. By making this small adjustment, it turned out that their integral also correct some of the limitations of the Riemann integral.
The integral of Kurzweil and Henstock is known by various names; the Henstock-Kurzweil integral, the generalized Riemann integral, and just the Henstock or just the Kurzweil integral are examples. Because the strictly positive function γ used in the definition is called a *gauge*, it is also known as the gauge integral.
In this thesis we will look at the integral of Kurzweil and Henstock. The integral will go by the name *the Henstock-Kurzweil integral*. First we are going to define the Henstock-Kurzweil integral and compare it with the definition of the Riemann integral. In the next section, we will examine some properties and we will prove all those properties. Then we are going to look at some examples of Henstock-Kurzweil integrable function. At last we will investigate the fundamental theorem and some convergence theorems regarding Henstock-Kurzweil integration.

2 The Riemann and the Henstock-Kurzweil integral

In this first section we begin with giving the definition of the common *Riemann integral*. Then we expand this definition to the definition of the *Henstock-Kurzweil integral* in little steps. To get a better understanding of what the various definitions actually mean, we will also give some examples.

2.1 The Riemann integral

For Riemann and Henstock-Kurzweil integration we first need to divide the interval $[a, b]$ in subintervals. We call the collection of these subintervals a *partition* [2, 4].

Definition 2.1. *A* **partition** $P := \{[x_{i-1}, x_i] : 1 \leq i \leq n\}$ *of an interval* $[a, b]$*, is a finite collection of closed intervals whose union is* $[a, b]$ *and where the subintervals* $[x_{i-1}, x_i]$ *can only have the endpoints in common.*

If we have for example the interval $[0, 3]$, we could make a partition by dividing it into three subintervals of equal length: $P = \{[0, 1], [1, 2], [2, 3]\}$. We see that this is an finite collection of closed intervals. It is also clear that the union of these three subintervals is $[0, 3]$. Therefore this is indeed a partition. An other partition could be $P = \{[0, \frac{1}{2}], [\frac{1}{2}, 3]\}$. This is a partition because it is again a finite collection of closed subintervals, whose union is $[0, 3]$.
The next step is to take a point t_i in each subinterval. These points t_i will be the *tags*. These tags together with our partition will form a collection of ordered pairs which will be called a *tagged partition* [2, 4].

Definition 2.2. *A* **tagged partition** $P := \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ *of an interval* $[a, b]$*, is a finite set of ordered pairs where* $t_i \in [x_{i-1}, x_i]$ *and the intervals* $[x_{i-1}, x_i]$ *form a partition of* $[a, b]$*. The numbers* t_i *are called the corresponding* **tags**.

For simplicity the left or right endpoints of the subintervals $[x_{i-1}, x_i]$ are often chosen as tags. We take our partition $P = \{[0, 1], [1, 2], [2, 3]\}$ from the previous example and choose the tags t_i to be the left endpoints. The tagged partition we get by doing this is: $P = \{(0, [0, 1]), (1, [1, 2]), (2, [2, 3])\}$. We already knew that the subintervals form a partition of $[0, 3]$ and it is clear that $0 \in [0, 1]$, $1 \in [1, 2]$ and $2 \in [2, 3]$, so P is indeed a tagged partition of $[0, 3]$. We could also choose the points in the middle of the subintervals to be the tags. Then we get the tagged partition $P = \{(\frac{1}{2}, [0, 1]), (1\frac{1}{2}, [1, 2]), (2\frac{1}{2}, [2, 3])\}$. Here it is also obvious that each tag lies in the corresponding subinterval, so that this P is again a tagged partition of $[0, 3]$.
The next step that will lead us to the definition of the Riemann integral is to define the *Riemann sum* [2, 4].

Definition 2.3. *If* $P := \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ *is a tagged partition of* $[a, b]$ *and* $f : [a, b] \to \mathbb{R}$ *is a function, then the* **Riemann sum** $S(f, P)$ *of* f *corresponding to* P *is* $S(f, P) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$.

For example, let's take the function $f(x) = x$ on the interval $[0, 3]$ together with our tagged partition $P = \{(0, [0, 1]), (1, [1, 2]), (2, [2, 3])\}$ from the previous example. The Riemann sum will now be given by: $S(f, P) = f(0)(1 - 0) + f(1)(2 - 1) + f(2)(3 - 2) = 0 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 3$.
The last step is to take the limit of this Riemann sum when the lenght of the intervals tend to zero; so when $x_i - x_{i-1} \to 0$. In the following definition this is given in ϵ,δ-form [1].

Definition 2.4. *For a function* $f : [a, b] \to \mathbb{R}$*, the number* $A = \int_a^b f$ *is called the* **Riemann integral** *of* f *if for all* $\epsilon > 0$*, there exists a* $\delta_\epsilon > 0$ *such that if* $P := \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ *is any tagged partition of* $[a, b]$ *satisfying* $0 < x_i - x_{i-1} \leq \delta_\epsilon$ *for all* $i = 1, 2, \cdots, n$*, then* $|S(f, P) - A| \leq \epsilon$.

2.2 The Henstock-Kurzweil integral

We will now define the *Henstock-Kurzweil integral*. Instead of choosing δ to be a constant, we allow δ to be a function which we will call γ. This small change has great benefits as we will see later on. The only restriction to the function γ is that is has to be strictly positive. Such a function will be called a *gauge* [2].

Definition 2.5. *A function γ on an interval $[a, b]$ is called a* **gauge** *if $\gamma(x) > 0$, for all $x \in [a, b]$.*

We could come up with many examples of gauges. We could simply take $\gamma(x) = c$, where $c \in \mathbb{R}_{>0}$. We could also take the function $\gamma(x) = \sin x$. It is a gauge on the interval $[1, 3]$, because it is strictly positive on that interval.
In order to define the *Henstock-Kurzweil* integral we now want to use a gauge γ instead of a constant δ to compare the length of each subinterval $[x_{i-1}, x_i]$ [2].

Definition 2.6. *If γ is a gauge on the interval $[a, b]$ and $P := \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$, then P is* **γ-fine** *if $x_i - x_{i-1} \leq \gamma(t_i)$ for all $i = 1, 2, \cdots, n$.*

Suppose we have the function $\gamma(x) = x + 1$ on $[0, 3]$. This is a gauge on $[0, 3]$ because it is strictly greater than zero on that interval. If we look at our tagged partition $P := \{(0, [0, 1]), (1, [1, 2]), (2, [2, 3])\}$ of the interval $[0, 3]$ from our previous examples, then we see that P is γ-fine, because $1 - 0 = 1 \leq 1 = 0 + 1 = \gamma(0)$, $2 - 1 = 1 \leq 2 = 1 + 1 = \gamma(1)$ and $3 - 2 = 1 \leq 3 = 2 + 1 = \gamma(2)$. So a tagged partition is γ-fine if the length of each subinterval is less or equal than the value of γ at the corresponding tag.
It turns out that for every gauge γ on an interval $[a, b]$ there exists a γ-fine, tagged partition. This is the content of Cousin's lemma [8].

Lemma 2.1 - Cousin's lemma. *If γ is a gauge on the interval $[a, b]$, then there exists a tagged partition of $[a, b]$ that is γ-fine.*

Before we can prove Cousin's lemma, we need another little lemma.

Lemma 2.2. *Suppose γ is a gauge and I_1 and I_2 are intervals which have at most one point in common. If I_1 and I_2 are γ-fine then the union $I := I_1 \cup I_2$ is also γ-fine.*

Proof. If I_1 and I_2 are γ-fine, then there exist tagged partitions $P_1 := \{(t_i, [x_{i-1}, x_i] : 1 \leq i \leq n\}$ and $P_2 := \{t_j, [x_{j-1}, x_j] : 1 \leq j \leq m\}$ of I_1 respectively I_2 which are γ-fine. So $0 \leq x_i - x_{i-1} \leq \gamma(t_i)$ and $0 \leq x_j - x_{j-1} \leq \gamma(t_j)$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$. If we now take the union of our tagged partitions $P := P_1 \cup P_2$, then the length of each subinterval is still smaller than the value of γ in the corresponding tag, so P is γ-fine. It is clear that $P := P_1 \cup P_2$ is a tagged partition of $I := I_1 \cup I_2$. So for $I := I_1 \cup I_2$ there exists a tagged partition that is γ-fine and thus I is γ-fine. $\square$

We are now going to use this little lemma to prove Cousin's lemma.

Proof. Lemma 2.1 - Cousin's Lemma. Let S be the set of points $x \in (a, b]$ such that there exists a γ-fine tagged partition of $[a, x]$. We are going to show that there exists a γ-fine tagged partition of $[a, b]$ by showing that the supremum of set S exists and that it is equal to b.
Let x be a real number that satisfies $a < x < a + \gamma(a)$ and $x < b$. Now consider $\{[a, x]\}$ to be a partition of $[a, x]$ which consists of one subinterval. If we choose a to be a tag, then we get a tagged partition of $[a, x]$: $\{(a, [a, x])\}$. Because $x - a < a + \gamma(a) - a = \gamma(a)$, we know that the tagged partition $\{(a, [a, x])\}$ is γ-fine. Thus we know that $x \in S$ and therefore S is nonempty. The axioma of completeness states that every nonempty set of real numbers that is bounded above has a supremum. Since S is bounded above by b, we know that S has a supremum. Let β be the supremum of S.
Since β is the supremum of S, we have that $\beta \leq b$ and so the gauge γ is defined at β. Since $\gamma(\beta) > 0$ and since β is the supremum of S, there exists a point $y \in S$ such that $\beta - \gamma(\beta) < y < \beta$. Let P_1 be a γ-fine tagged partition of $[a, y]$; such a partition existst because $y \in S$. Now we consider $\{[y, \beta]\}$ to be a partition of $[y, \beta]$ which again consists of one subinterval. We choose β to be a tag and so we get the tagged partition $\{(\beta, [y, \beta])\}$. Because $\beta - y < \beta - (\beta - \gamma(\beta)) = \gamma(\beta)$ we know that $\{(\beta, [y, \beta])\}$ is γ-fine. Now we take the union: $\tilde{P_1} = P_1 \cup \{(\beta, [y, \beta])\}$ and by lemma 2.1 we know that this union $\tilde{P_1}$ is a tagged partition of $[a, \beta]$ that is γ-fine. Thus we can conclude that $\beta \in S$.
Now we need to demonstrate that $\beta = b$. To do this, we assume by way of contradiction that $\beta < b$. Since $\beta < b$ there exists a point $z \in (\beta, b)$ such that $z < \beta + \gamma(\beta)$. Let P_2 be a γ-fine tagged partition of $[a, \beta]$; such a partition exists because $\beta \in S$. We consider $\{[\beta, z]\}$ to be a partition of $[\beta, z]$ and by taking β as a tag we get the tagged partition $\{(\beta, [\beta, z])\}$. Because $z - \beta < \beta + \gamma(\beta) - \beta = \gamma(\beta)$ we know that $\{(\beta, [\beta, z])\}$ is a γ-fine tagged partition of $[\beta, z]$. Now we take the union $\tilde{P_2} = P_2 \cup \{(\beta, [\beta, z])\}$ and by applying lemma 2.1 again, we know that this $\tilde{P_2}$ is a γ-fine taggged partition of $[a, z]$. Therefore we conclude that $z \in S$. But this is a contradiction, because β is the supremum of S. So our assumption can not be true and thus $\beta = b$.
Finally, since $b = \beta$ is an element of S, we can conclude that there exists a γ-fine partition of $[a, b]$. □

With the definition of a γ-fine partition, we can define the *Henstock-Kurzweil integral* [2]. As said before, its definition differs only slightly from the definition of the Riemann integral.

Definition 2.7. *For a function $f : [a, b] \to \mathbb{R}$ the number $B = \int_a^b f$ is called the* **Henstock-Kurzweil integral** *of f if for all $\epsilon > 0$ there exists a gauge γ such that if $P := \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$ that is γ-fine, then $|S(f, P) - B| \leq \epsilon$.*

Instead of setting the length of each subinterval $[x_{i-1}, x_i]$ smaller than a constant δ as in the definition of the Riemann integral, we use a gauge γ to compare the lengths with. This little difference will have many benefits. In section 4 we will see some examples of Henstock-Kurzweil integrable functions.

3 Properties of the Henstock-Kurzweil integral

In this section we will look at some properties of the Henstock-Kurzweil integral. We will begin with comparing it with the Riemann integral. Next thing we will see is that the Henstock-Kurzweil integral is unique. Then some linearity properties will follow. After that we are going to look at the Cauchy criterion for Henstock-Kurzweil integrability and we are going to use this criterion to prove a theorem considering the integrability on subintervals of $[a, b]$. At last we are looking at the Henstock-Kurzweil integrals of functions that are zero, of two functions that are equal to each other and finally of a function that is less or equal than another Henstock-Kurzweil integrable function.
We saw in the previous section that the definitions of the Riemann integral and the Henstock-Kurzweil integral are very similar. Because the definitions are so similar we could wonder if a Riemann integrable function is also Henstock-Kurzweil integrable and the other way around.
It turns out that indeed every Riemann integrable function is Henstock-Kurzweil integrable. The idea behind proving this statement is to choose our gauge γ to be the constant δ from the definition of the Riemann integral. This statement is formally presented in the next theorem.

Theorem 3.1. *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ with $\int_a^b f = A$, then f is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b f = A$.*

Proof. Let $\epsilon > 0$. Since f is Riemann integrable, there exists a constant $\delta_\epsilon > 0$ such that if $P := \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$ such that $x_i - x_{i-1} \leq \delta_\epsilon$ for all $i = 1, 2, \cdots, n$ then $|S(f, P) - A| \leq \epsilon$. Now we choose our gauge to be $\gamma(x) = \delta_\epsilon$. Then we know that if P is a tagged partition of $[a, b]$ that is γ-fine then

$$|S(f, P) - A| \leq \epsilon.$$

Because our ϵ was arbitrary, we know this holds for all $\epsilon > 0$ and thus we can conclude that f is Henstock-Kurzweil integrable with $\int_a^b f = A$. □

The converse of this statement is false. Not every Henstock-Kurzweil integrable function is Riemann integrable. An example of a function that is not Riemann integrable, but is Henstock-Kurzweil integrable is Dirichlet's function. We will discuss this function in section 4.2.
The following theorem states that if a function f is Henstock-Kurzweil integrable, then its integral is unique [6].

Theorem 3.2 - Uniqueness of Henstock-Kurzweil integral. *If $f : [a, b] \to \mathbb{R}$ is Henstock-Kurzweil integrable on the interval $[a, b]$, then the Henstock-Kurzweil integral of f on $[a, b]$ is unique.*

Proof. We assume that there are two numbers $B_1, B_2 \in \mathbb{R}$ that are both the Henstock-Kurzweil integral of f on $[a, b]$. Let $\epsilon > 0$. Since B_1 is Henstock-Kurzweil integrable, there exists a gauge γ_1 on $[a, b]$ such that if P_1 is a γ_1-fine partition of $[a, b]$ then

$$|S(f, P_1) - B_1| \leq \frac{\epsilon}{2}.$$

Similarly, there exists a gauge γ_2 on $[a, b]$ such that if P_2 is a γ_2-fine partition of $[a, b]$ then

$$|(S(f, P_2) - B_2| \leq \frac{\epsilon}{2}.$$

We now define a new gauge: $\gamma := \min\{\gamma_1, \gamma_2\}$. Suppose now that P is a tagged partition of $[a, b]$ that is γ-fine. Because of the construction of γ, we know that this P is also γ_1- and γ_2-fine. Thus we have:

$$\begin{aligned} |B_1 - B_2| &= |B_1 - S(f, P) + S(f, P) - B_2| \\ &\leq |S(f, P) - B_1| + |S(f, P) - B_2| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Because our ϵ was arbitrary, we conclude that $B_1 = B_2$ and thus the Henstock-Kurzweil integral of f on $[a, b]$ is unique. □

Now we will look at some linearity properties of the Henstock-Kurzweil integral.

Theorem 3.3 - Linearity properties. *Let f and g be Henstock-Kurzweil integrable on $[a, b]$, then*

1. *kf is Henstock-Kurzweil integrable on $[a, b]$ for each $k \in \mathbb{R}$ with $\int_a^b kf = k \int_a^b f$;*
2. *$f + g$ is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.*

Proof. 1. Let $\epsilon > 0$. By assumption f is Henstock-Kurzweil integrable. So for $\frac{\epsilon}{|k|}$ there exists a gauge γ such that if P is a tagged partition of $[a, b]$ that is γ-fine, then

$$\left| S(f, P) - \int_a^b f \right| \leq \frac{\epsilon}{|k|}.$$

Now we have that

$$\begin{aligned} \left| S(kf, P) - k \int_a^b f \right| &= \left| \sum_{i=1}^{n} kf(t_i)(x_i - x_{i-1}) - k \int_a^b f \right| \\ &= |k| \left| S(f, P) - \int_a^b f \right| \\ &\leq |k| \frac{\epsilon}{|k|} = \epsilon. \end{aligned}$$

Because our ϵ was arbitrary, this holds for all $\epsilon > 0$ and therefore kf is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b kf = k \int_a^b f$.

2. Let $\epsilon > 0$. By assumption f and g are Henstock-Kurzweil integrable. So for $\frac{\epsilon}{2}$ there exist gauges γ_f and γ_g such that whenever P_f and P_g are tagged partitions of $[a, b]$ that are γ_f-fine respectively γ_g-fine, then

$$\left| S(f, P_f) - \int_a^b f \right| \leq \frac{\epsilon}{2}$$

and

$$\left| S(g, P_g) - \int_a^b g \right| \leq \frac{\epsilon}{2}.$$

Now we define $\gamma := \min\{\gamma_f, \gamma_g\}$. Furthermore we have:

$$\begin{aligned} S(f+g, P) &= \sum_{i=1}^{n} [(f+g)(t_i)(x_i - x_{i-1})] \\ &= \sum_{i=1}^{n} [f(t_i)(x_i - x_{i-1}) + g(t_i)(x_i - x_{i-1})] \\ &= \sum_{i=1}^{n} [f(t_i)(x_i - x_{i-1})] + \sum_{i=1}^{n} [g(t_i)(x_i - x_{i-1})] \\ &= S(f, P) + S(g, P). \end{aligned}$$

If we now have a partition P that is γ-fine, then because of the construction of γ this P is also γ_f- and γ_g-fine. Therefore we know that

$$\begin{aligned} \left| S(f+g, P) - (\int_a^b f + \int_a^b g) \right| &\leq \left| S(f, P) - \int_a^b f \right| + \left| S(g, P) - \int_a^b g \right| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since our ϵ was arbitrary, this holds for all $\epsilon > 0$ and therefore $f + g$ is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

□

The next thing we are going to look at is the Cauchy criterion for Henstock-Kurzweil integrals [4]. This theorem is a nice one, because with this theorem one can prove that a certain function is Henstock-Kurzweil integrable without knowing the value of the integral. It will be used in some proofs later on.

Theorem 3.4 - Cauchy criterion. *A function $f : [a, b] \to \mathbb{R}$ is Henstock-Kurzweil integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a gauge γ on $[a, b]$ such that if P_1 and P_2 are tagged partitions of $[a, b]$ that are γ-fine, then $|S(f, P_1) - S(f, P_2)| \leq \epsilon$.*

Proof. First we assume that f is Henstock-Kurzweil integrable on $[a,b]$. Let $\epsilon > 0$. We know that for $\frac{\epsilon}{2}$, there exists a gauge γ on $[a,b]$ such that if P_1 and P_2 are tagged partitions of $[a,b]$ that are γ-fine then,

$$\left| S(f,P_1) - \int_a^b f \right| \leq \frac{\epsilon}{2}$$

and

$$\left| S(f,P_2) - \int_a^b f \right| \leq \frac{\epsilon}{2}.$$

Now it follows that

$$\begin{aligned} |S(f,P_1) - S(f,P_2)| &\leq \left| S(f,P_1) - \int_a^b f \right| + \left| \int_a^b f - S(f,P_2) \right| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Because our ϵ was arbitrary, we know that this holds for all $\epsilon > 0$.
Conversely, we assume that for every $\epsilon > 0$ there exists a gauge γ on $[a,b]$ such that if P_1 and P_2 are tagged partitions of $[a,b]$ that are γ-fine, then $|S(f,P_1) - S(f,P_2)| \leq \epsilon$. For each $n \in \mathbb{N}$, choose a gauge γ_n such that

$$|S(f,P_1) - S(f,P_2)| \leq \frac{1}{n}$$

whenever P_1 and P_2 are γ_n-fine. We may assume that the sequence $\{\gamma_n\}$ is non-increasing. For each n, let P_n be a tagged partition of $[a,b]$ that is γ_n-fine. If $m > n \geq N$, then $\gamma_N \geq \gamma_n \geq \gamma_m$. Now P_n is γ_n-fine and thus also γ_N-fine. The same holds for P_m: P_m is γ_m-fine and thus γ_N-fine. Now we have that

$$|S(f,P_n) - S(f,P_m)| \leq \frac{1}{N}$$

for $m > n \geq N$. So the sequence $\{S(f,P_n)\}$ is a Cauchy sequence. Now let A be the limit of this sequence and let $\epsilon > 0$. Choose an integer $N \in \mathbb{N}$ such that $\frac{1}{N} \leq \frac{\epsilon}{2}$ and

$$|S(f,P_n) - A| \leq \frac{\epsilon}{2}$$

for all $n \geq N$. Now let P be a partition of $[a,b]$ that is γ_N-fine, then

$$\begin{aligned} |S(f,P) - A| &\leq |S(f,P) - S(f,P_N)| + |S(f,P_N) - A| \\ &\leq \frac{1}{N} + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Because our ϵ was arbitrary, we conclude that f is Henstock-Kurzweil integrable on $[a,b]$. $\square$

If we have a function f that is Henstock-Kurzweil integrable on an interval $[a, b]$, then intuitively we suppose that it is also Henstock-Kurzweil integrable on subintervals of $[a, b]$. We also suppose by intuition that if a function is Henstock-Kurzweil integrable on $[a, c]$ and on $[c, b]$, then it is also Henstock-Kurzweil integrable on $[a, b]$. It turns out that these two statements are indeed true. This is the content of the following theorem. We are going to use the previous theorem, the Cauchy criterion, to prove this. [4].

Theorem 3.5. *Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$.*

1. *If f is Henstock-Kurzweil integrable on $[a, c]$ and on $[c, b]$, then f is Henstock-Kurzweil integrable on $[a, b]$.*

2. *If f is Henstock-Kurzweil integrable on $[a, b]$, then f is Henstock-Kurzweil integrable on every subinterval $[\alpha, \beta] \subseteq [a, b]$.*

Proof. 1. Let $\epsilon > 0$. Because f is Henstock-Kurzweil integrable on $[a, c]$ we know that for $\frac{\epsilon}{2}$ there exists a gauge γ_a such that if P_1 and P_2 are tagged partitions of $[a, c]$ that are γ_a-fine, then

$$|S(f, P_1) - S(f, P_2)| \leq \frac{\epsilon}{2}.$$

We also apply the Cauchy criterion to the interval $[c, b]$. We know that for $\frac{\epsilon}{2}$ there exists a gauge γ_b such that

$$\left|S(f, \tilde{P}_1) - S(f, \tilde{P}_2)\right| \leq \frac{\epsilon}{2}$$

whenever $\tilde{P}_1$ and $\tilde{P}_2$ are tagged partitions of $[c, b]$ that are γ_b-fine. Now we define a new gauge: $\gamma := \max\{\gamma_a, \gamma_b\}$. If we now have a partition that is γ_a-fine or γ_b-fine, then it is also γ-fine. Now we take $P_a := P_1 \cup \tilde{P}_1$ and $P_b := P_2 \cup \tilde{P}_2$; these P_a and P_b are tagged partitions of $[a, b]$ that are γ-fine. Now we have:

$$\begin{aligned}
|S(f, P_a) - S(f, P_b)| &= \left|S(f, P_1) + S(f, \tilde{P}_1) - [S(f, P_2) + S(f, \tilde{P}_2)]\right| \\
&\leq \left|S(f, P_1) - S(f, P_2)\right| + \left|S(f, \tilde{P}_1) + S(f, \tilde{P}_2)\right| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Because our ϵ was arbitrary, this holds for all $\epsilon > 0$ and therefore we conclude by the Cauchy criterion that f is Henstock-Kurzweil integrable on $[a, b]$

2. Let $\epsilon > 0$. Choose a gauge γ such that

$$|S(f, P_1) - S(f, P_2)| \leq \epsilon$$

whenever P_1 and P_2 are γ-fine. We can do this because of the Cauchy criterion. Now fix partitions P_α of $[a, \alpha]$ and P_β of $[\beta, b]$ that are γ-fine. Let $\tilde{P}_1$ and $\tilde{P}_2$ be tagged

partitions of $[\alpha, \beta]$ that are γ-fine. Such partitions exist because of Cousin's lemma (lemma 2.1). Now define $P_1 := P_\alpha \cup \tilde{P}_1 \cup P_\beta$ and $P_2 := P_\alpha \cup \tilde{P}_2 \cup P_\beta$. P_1 and P_2 are now tagged partitions of $[a, b]$ that are γ-fine. So we get:

$$\begin{aligned}\left|S(f, \tilde{P}_1) - S(f, \tilde{P}_2)\right| &= \left|S(f, P_\alpha) + S(f, \tilde{P}_1) + S(f, P_\beta) - [S(f, P_\alpha) + S(f, \tilde{P}_2) + S(f, P_\beta)]\right| \\ &= |S(f, P_1) - S(f, P_2)| \\ &\leq \epsilon.\end{aligned}$$

Because our ϵ was arbitrary, we have that for every $\epsilon > 0$ there exists a gauge γ on $[\alpha, \beta]$ such that $\left|S(f, \tilde{P}_1) - S(f, \tilde{P}_2)\right| \leq \epsilon$ when $\tilde{P}_1$ and $\tilde{P}_2$ are γ-fine tagged partitions of $[\alpha, \beta]$. By using the Cauchy criterion again, we conclude that f is Henstock-Kurzweil integrable on $[\alpha, \beta]$.

□

We are now going to look at functions that are zero almost everywhere on an interval $[a, b]$. It turns out that such functions are Henstock-Kurzweil integrable and that their integrals are, as expected, equal to zero [4].

Theorem 3.6. *Let $f : [a, b] \to \mathbb{R}$. If $f = 0$ except on a countable number of points on $[a, b]$, then f is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b f = 0$.*

Proof. Let $\epsilon > 0$. First we define the set where $f(x) \neq 0$: $A := \{a_n : n \in \mathbb{Z}_{>0}\} = \{x \in [a, b] : f(x) \neq 0\}$. Now we are going to define an appropriate gauge on $[a, b]$:

$$\gamma(x) := \begin{cases} 1 & \text{if } x \in [a, b] \text{ and } x \notin A \\ \frac{\epsilon 2^{-n}}{|f(a_n)|} & \text{if } x \in A \end{cases}.$$

Suppose that $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$ that is γ-fine. So $x_i - x_{i-1} \leq \gamma(t_i)$ for all $1 \leq i \leq n$. Now let π be the set of all indices i such that the tags $t_i \in A$. Choose n_i such that $t_i = a_{n_i}$. Let σ be the set of indices i such that $t_i \notin A$. We know that if we have such a tag, then $f(t_i) = 0$ and thus $f(t_i)(x_i - x_{i-1}) = 0$. So tags

that are not in A do not contribute to $S(f,P)$. We have

$$\begin{aligned}
|S(f,P)| &= \left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})\right| \\
&\leq \left|\sum_{i\in\pi} f(t_i)(x_i - x_{i-1})\right| + \left|\sum_{i\in\sigma} f(t_i)(x_i - x_{i-1})\right| \\
&\leq \sum_{i\in\pi} |f(a_{n_i})| \frac{\epsilon 2^{-n_i}}{|f(a_{n_i})|} \\
&= \sum_{i\in\pi} \epsilon 2^{-n_i} \\
&\leq \epsilon \sum_{i=1}^{\infty} 2^{-n_i} = \epsilon
\end{aligned}$$

and because our ϵ was arbitrary, this holds for all $\epsilon > 0$. Thus we can conclude that f is Henstock-Kurzweil integrable on $[a,b]$ with $\int_a^b f = 0$. □

With the previous theorem we can prove that if we have two functions that are the same almost everywhere on an interval $[a,b]$ and if one of them is Henstock-Kurzweil integrable on that interval, then the other is also Henstock-Kurzweil integrable and their integrals are the same. This is the content of the next theorem [4].

Corollary 3.1. *Let $f : [a,b] \to \mathbb{R}$ be Henstock-Kurzweil integrable on $[a,b]$ and let $g : [a,b] \to \mathbb{R}$. If $f = g$ except on a countable number of points on $[a,b]$, then g is Henstock-Kurzweil integrable on $[a,b]$ with $\int_a^b g = \int_a^b f$.*

Proof. We define a new function: $h := g - f$ on $[a,b]$. We have $h = 0$ except on a countable number of points on $[a,b]$. So from theorem 3.6 we conclude that h is Henstock-Kurzweil integrable on $[a,b]$ with $\int_a^b h = 0$. Since f and h are both Henstock-Kurzweil integrable on $[a,b]$, from theorem 3.3 we know that $g = f + h$ is Henstock-Kurzweil integrable on $[a,b]$ with $\int_a^b g = \int_a^b f + \int_a^b h = \int_a^b f$. □

The last property we are going to look at considers two Henstock-Kurzweil integrable functions on an interval $[a,b]$. If one of them is less or equal than the other almost everywhere on the interval $[a,b]$, then the statement is that the integral of the smallest function is less or equal than the biggest [8]. We will also use this theorem when we are proving convergence theorems in section 6.

Corollary 3.2. *If f and g are Henstock-Kurzweil integrable on $[a,b]$ and $f(x) \leq g(x)$ except on a countable number of points on $[a,b]$, then $\int_a^b f \leq \int_a^b g$.*

Proof. Define two new functions:

$$\tilde{f}(x) = \begin{cases} f(x) & \forall x \text{ such that } f(x) \leq g(x) \\ 0 & \forall x \text{ such that } f(x) > g(x) \end{cases}$$

and

$$\tilde{g}(x) = \begin{cases} g(x) & \forall x \text{ such that } f(x) \leq g(x) \\ 0 & \forall x \text{ such that } f(x) > g(x) \end{cases}.$$

Because $f(x) = \tilde{f}(x)$ and $g(x) = \tilde{g}(x)$ almost everywhere on $[a, b]$, we know from corollary 3.1 that $\int_a^b f = \int_a^b \tilde{f}$ and $\int_a^b g = \int_a^b \tilde{g}$. Because $\tilde{g} \geq \tilde{f}$ we know $\tilde{g} - \tilde{f} \geq 0$ and thus it follows that $\int_a^b (\tilde{g} - \tilde{f}) \geq 0$. Now we have $\int_a^b g - \int_a^b f = \int_a^b \tilde{g} - \int_a^b \tilde{f} = \int_a^b (\tilde{g} - \tilde{f}) \geq 0$, so it follows that $\int_a^b g \geq \int_a^b f$. □

4 Examples

Now we have seen al those nice properties of the Henstock-Kurzweil integral in the previous section, we are going to look at some explicit examples of Henstock-Kurzweil integrable functions. For all the examples that we treat, we will give a proof that they are Henstock-Kurzweil integrable mostly by defining an appropriate gauge.
We will start easy: first we are going to look at linear functions. Secondly we are going to prove that Dirichlet's function is not Riemann integrable, but that it is Henstock-Kurzweil integrable as said before. Then we will modify Dirichlet's function and see what happens with the integrability. At last we are going to investigate Thomae's function.

4.1 Linear functions

We begin here with investigating the easiest linear function. We want to find out if the function $f(x) = c$ with $c \in \mathbb{R}$ is Henstock-Kurzweil integrable on the interval $[a, b]$. We know that it is Riemann integrable with $\int_a^b f = c(b - a)$. So we know by theorem 3.1 that it is also Henstock-Kurzweil integrable with $\int_a^b f = c(b - a)$. In the following proposition we are going to prove this by looking at the definition.

Proposition 4.1. *The function $f(x) = c$ with $c \in \mathbb{R}$ is Henstock-Kurzweil integrable on the interval $[a, b]$ with $\int_a^b f = c(b - a)$.*

Proof. Let $\epsilon > 0$. We know that $f(x) = c$, $\forall x \in [a, b]$. So for any tagged partition P of $[a, b]$, we have

$$S(f, P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} c(x_i - x_{i-1}).$$

This is a telescoping sum, so $\sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b-a)$. Therefore we have that

$$\begin{aligned} |S(f,P) - c(b-a)| &= |c(b-a) - c(b-a)| \\ &= 0 < \epsilon \end{aligned}$$

for any tagged partition P. So this certainly holds if P is a tagged partition that is γ-fine for any gauge γ. Since our ϵ is arbitrary, it holds for all $\epsilon > 0$ and thus $f(x) = c$ is Henstock-Kurzweil integrable on $[a,b]$ with $\int_a^b f = c(b-a)$. $\square$

We will now look at the function $f(x) = x$. We know that this function is Riemann integrable and therefore Henstock-Kurzweil integrable. Here we are going to prove it by finding an appropriate gauge.

Proposition 4.2. *The function $f(x) = x$ is Henstock-Kurzweil integrable on the interval $[a,b]$ with $\int_a^b f = \frac{1}{2}(b^2 - a^2)$.*

Proof. Let $\epsilon > 0$. We take $\gamma(x) := \sqrt{\frac{2\epsilon}{n}}$ to be our gauge, where n is the number of tags and thus the number of subintervals in the tagged partition. Suppose P is a tagged partition of $[a,b]$ that is γ-fine. So $x_i - x_{i-1} \leq \gamma(t_i) = \sqrt{\frac{2\epsilon}{n}}$ for all $1 \leq i \leq n$. We can write $\frac{1}{2}(b^2 - a^2)$ as a telescoping sum:

$$\frac{1}{2}(b^2 - a^2) = \frac{1}{2}\sum_{i=1}^{n}[x_i^2 - x_{i-1}^2].$$

If we look at the distance from a tag t_i to the middle of a subinterval $[x_{i-1}, x_i]$, we know it is always less or equal to half the length of the subinterval. So:

$$\left| t_i - \frac{x_i + x_{i-1}}{2} \right| \leq \frac{1}{2} \left| x_i - x_{i-1} \right|.$$

Now we have:

$$\begin{aligned}
\left|S(f,P)-\frac{1}{2}(b^2-a^2)\right| &= \left|\sum_{i=1}^{n}[t_i(x_i-x_{i-1})]-\frac{1}{2}\sum_{i=1}^{n}[x_i^2-x_{i-1}^2]\right| \\
&= \left|\sum_{i=1}^{n}[t_i(x_i-x_{i-1})-\frac{1}{2}(x_i-x_{i-1})(x_i+x_{i-1})]\right| \\
&= \left|\sum_{i=1}^{n}[(x_i-x_{i-1})(t_i-\frac{x_i+x_{i-1}}{2})]\right| \\
&\leq \sum_{i=1}^{n}|x_i-x_{i-1}|\left|t_i-\frac{x_i+x_{i-1}}{2}\right| \\
&\leq \sum_{i=1}^{n}\frac{1}{2}|x_i-x_{i-1}|^2 \\
&\leq \sum_{i=1}^{n}\frac{1}{2}\sqrt{\frac{2\epsilon}{n}}^2 \\
&= \sum_{i=1}^{n}\frac{\epsilon}{n}=\epsilon.
\end{aligned}$$

Because our ϵ was arbitrary, we know that this holds for all $\epsilon > 0$ and thus $f(x) = x$ is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b f = \frac{1}{2}(b^2 - a^2)$. □

Because we know now that the functions $f(x) = c$ with $c \in \mathbb{R}$ and $f(x) = x$ are Henstock-Kurzweil integrable on an arbitrary interval $[a, b]$, we know that all linear functions are Henstock-Kurzweil integrable on an interval $[a, b]$. This follows directly from theorem 3.3 and propositions 4.1 and 4.2 and the fact that a linear function is always of the form $f(x) = dx + c$ with $c, d \in \mathbb{R}$.

4.2 Dirichlet's function

Dirichlet's function $h : [a, b] \to \mathbb{R}$ is the characteristic function of the rational numbers. It is a function which is discontinuous everywhere on the interval $[a, b]$. Dirichlet's function is given by:

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

This function is bounded, but not Riemann integrable. We will prove that now.

Proposition 4.3. *Dirchlet's function $h(x)$ is not Riemann integrable on $[a, b]$.*

Proof. Recall from section 2 that the number A is the Riemann integral of $h(x)$ on $[a,b]$ if for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that if P is any tagged partition of $[a,b]$ satisfying $x_i - x_{i-1} \leq \delta_\epsilon$ for all $i = 1, 2, \cdots, n$ then $|S(h,P) - A| \leq \epsilon$. Suppose that $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a,b]$ that satisfies $x_i - x_{i-1} \leq \delta_\epsilon$ for all $i = 1, 2, \cdots, n$. Because the rationals and irrationals are both dense in $\mathbb{R}$, each subinterval contains a point that is rational and a point that is irrational. We distinguish between two cases: the first case is when $A \neq 0$ and the second is when $A = 0$.
First, assume $A \neq 0$. We choose all our tags to be irrational, so $t_i \notin \mathbb{Q}$. If we do this we get:

$$\begin{aligned} |S(h,P) - A| &= \left| \sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) - A \right| \\ &= \left| \sum_{i=1}^{n} 0(x_i - x_{i-1}) - A \right| \\ &= |A| \,. \end{aligned}$$

So we can not always make the expression $|S(h,P) - A|$ smaller than ϵ for any tagged partition P that satisfies $x_i - x_{i-1} \leq \delta_\epsilon$. Therefore we can conclude that $h(x)$ is not Riemann integrable on $[a,b]$.
Now we assume that $A = 0$. Now we choose all our tags to be rational, so $t_i \in \mathbb{Q}$. Now we get:

$$\begin{aligned} |S(h,P)| &= \left| \sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) \right| \\ &= \left| \sum_{i=1}^{n} 1(x_i - x_{i-1}) \right| \\ &= |b - a| \,. \end{aligned}$$

So again, we can not always make the expression $|S(h,P) - A|$ smaller than ϵ for any tagged partition P that satisfies $x_i - x_{i-1} \leq \delta_\epsilon$. Therefore we can conclude that $h(x)$ is not Riemann integrable on $[a,b]$. □

We have now shown that Dirichlet's function is not Riemann integrable on an interval $[a,b]$. Dirichlet's function is actually Henstock-Kurzweil integrable. We will prove that statement now [2].

Proposition 4.4. *Dirichlet's function $h : [a,b] \to \mathbb{R}$ is Henstock-Kurzweil integrable with $\int_a^b h = 0$.*

Proof. Let $\epsilon > 0$. First we enumerate the rational numbers in $[a,b]$ as $\{r_1, r_2, \cdots\}$. Now we define our gauge γ_ϵ:

$$\gamma_\epsilon(x) := \begin{cases} \frac{\epsilon}{2^i} & \text{if } x = r_i \\ 1 & \text{if } x \notin \{r_1, r_2, \cdots\} \end{cases} .$$

Note that our γ_ϵ is not a constant value. Suppose $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$ that is γ_ϵ-fine. If $t_i \in \{r_1, r_2, \cdots\}$ is a tag, then $h(t_i) = 1$ and $x_i - x_{i-1} \leq \gamma_\epsilon(t_i) = \frac{\epsilon}{2^i}$ where $[x_{i-1}, x_i]$ is the subinterval corresponding to the tag t_i. Thus we have: $h(t_i)(x_i - x_{i-1}) \leq \frac{\epsilon}{2^i}$ for tags $t_i \in \{r_1, r_2, \cdots\}$. If $t_i \notin \{r_1, r_2, \cdots\}$ is a tag, then $h(t_i) = 0$ and $x_i - x_{i-1} \leq \gamma_\epsilon(t_i) = 1$ where $[x_{i-1}, x_i]$ is the subinterval corresponding to the tag t_i. Therefore: $h(t_i)(x_i - x_{i-1}) = 0$. So the subintervals with tags $t_i \notin \{r_1, r_2, \cdots\}$ do not contribute to $S(h, P)$. Let π be the set of indices i such that $t_i \in \{r_1, r_2, \cdots\}$ and let σ be the set of indices i such that $t_i \in \{r_1, r_2, \cdots\}$. We conclude that:

$$\begin{aligned}
|S(h, P)| &= \left| \sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) \right| \\
&\leq \left| \sum_{i \in \pi} h(t_i)(x_i - x_{i-1}) \right| + \left| \sum_{i \in \sigma} h(t_i)(x_i - x_{i-1}) \right| \\
&\leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \\
&= \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = \epsilon.
\end{aligned}$$

Since our ϵ was arbitrary, we know that this holds for all ϵ and thus Dirichlet's function h is Henstock-Kurzweil integrable with $\int_a^b h = 0$. □

In section 3 we saw that every Riemann integrable function is Henstock-Kurzweil integrable. Here we have shown that the converse statement is false by giving an example of a function that is Henstock-Kurzweil integrable, but not Riemann integrable.

4.3 Modified Dirichlet function

We are now going to look what happens if we adjust Dirichlet's function a little bit. We define a new function g by

$$g(x) := \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We are now going to investigate the Riemann and the Henstock-Kurzweil integrability of this function. It turns out that this little modification does not affect the integrability: the function g is still not Riemann integrable but it is Henstock-Kurzweil integrable. To prove these two statements, we can follow the proofs of Dirichlet's function and we only need to make a few adjustments.

Proposition 4.5. *The modified Dirichlet function $g(x)$ is not Riemann integrable on $[a, b]$.*

Proof. Recall again that the number A is the Riemann integral of $g(x)$ on $[a,b]$ if for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that if P is any tagged partition of $[a,b]$ satisfying $0 \leq x_i - x_{i-1} \leq \delta_\epsilon$ for all $i = 1, 2, \cdots, n$ then $|S(g,P) - A| \leq \epsilon$. Suppose that $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a,b]$ that satisfies $0 \leq x_i - x_{i-1} \leq \delta_\epsilon$ for all $i = 1, 2, \cdots, n$. Because the rationals and irrationals are both dense in $\mathbb{R}$, each subinterval contains a point that is rational and a point that is irrational. Again, we distinguish between two cases: $A \neq 0$ and $A = 0$.
The proof of the first case, when $A \neq 0$, is identical to the proof of that of Dirichlet's function, so we will skip that part here.
If $A = 0$, we choose our $x_i \in \mathbb{Q}$. Next we choose our tags to be the midpoints of the subintervals: $t_i = \frac{x_{i-1}+x_i}{2} \in \mathbb{Q}$. It could be that $a = x_0$ and $b = x_n$ are no rational numbers. Now we get

$$\begin{aligned}
|S(g,P)| &= \left| \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) \right| \\
&= \left| g(t_1)(x_1 - x_0) + \sum_{i=2}^{n-1} g(t_i)(x_i - x_{i-1}) + g(t_n)(x_n - x_{n-1}) \right| \\
&= \left| g(t_1)(x_1 - a) + g(t_n)(b - x_{n-1}) + \sum_{i=2}^{n-1} \frac{x_{i-1} + x_i}{2}(x_i - x_{i-1}) \right| \\
&= \left| g(t_1)(x_1 - a) + g(t_n)(b - x_{n-1}) + \sum_{i=2}^{n-1} \frac{1}{2}(x_i^2 - x_{i-1}^2) \right| \\
&= \left| g(t_1)(x_1 - a) + g(t_n)(b - x_{n-1}) + \frac{1}{2}(x_{n-1}^2 - x_1^2) \right|
\end{aligned}$$

We can not make the term $\frac{1}{2}(x_{n-1}^2 - x_1^2)$ arbitrarily small by refining our partition. So we can not make the whole expression $|S(g,P) - A|$ smaller than ϵ for any tagged partition P that satisfies $x_i - x_{i-1} \leq \delta_\epsilon$. Therefore we can conclude that $g(x)$ is not Riemann integrable on $[a,b]$. $\square$

Proposition 4.6. *The modified Dirichlet function $g : [a,b] \to \mathbb{R}$ is Henstock-Kurzweil integrable with $\int_a^b g = 0$.*

Proof. Let $\epsilon > 0$. We start again with enumerating the rationals in $[a,b]$ as $\{r_1, r_2, \cdots\}$. Now we define our gauge γ_ϵ:

$$\gamma_\epsilon(x) := \begin{cases} \left|\frac{\epsilon}{x2^i}\right| & \text{if } x = r_i \\ 1 & \text{if } x \notin \{r_1, r_2, \cdots\} \end{cases}.$$

Suppose $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a,b]$ that is γ_ϵ-fine. If $t_i \in \{r_1, r_2, \cdots\}$ is a tag, then $g(t_i) = t_i$ and $x_i - x_{i-1} \leq \gamma_\epsilon(t_i) = \left|\frac{\epsilon}{t_i 2^i}\right|$ where $[x_{i-1}, x_i]$

is the subinterval corresponding to the tag t_i. If $t_i \notin \{r_1, r_2, \cdots\}$ is a tag, then $g(t_i) = 0$ and $x_i - x_{i-1} \leq \gamma_\epsilon(t_i) = 1$ where $[x_{i-1}, x_i]$ is the subinterval corresponding to the tag t_i. Therefore: $g(t_i)(x_i - x_{i-1}) = 0$. So the subintervals with tags $t_i \notin \{r_1, r_2, \cdots\}$ do not contribute to $S(h, P)$. Let π be the set of integers i such that $t_i \in \{r_1, r_2, \cdots\}$ and let σ be the set of integers i such that $t_i \in \{r_1, r_2, \cdots\}$. We can conclude that:

$$\begin{aligned}
|S(g, P)| &= \left| \sum_{i \in \pi} g(t_i)(x_i - x_{i-1}) \right| + \left| \sum_{i \in \sigma} g(t_i)(x_i - x_{i-1}) \right| \\
&\leq \sum_{i \in \pi} |t_i| \left| \frac{\epsilon}{t_i 2^i} \right| \\
&\leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.
\end{aligned}$$

Since our ϵ was arbitrary, we know this holds for all $\epsilon > 0$ and thus is the modified Dirichlet function g Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b g = 0$. □

4.4 Thomae's function

The next function we will look at is Thomae's function. It is given by:

$$T(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

This function is a special one, because it has the property that it is continuous at every irrational point and discontinuous at every rational point. To show this we look at sequences [1].

Proposition 4.7. *Thomae's function T is continuous at every irrational point and discontinuous at every rational point.*

Proof. We will first look at the rational points; so we look at $c \in \mathbb{Q}$. We know that $T(c) > 0$. We can find a sequence $(y_n) \notin \mathbb{Q}$ which converges to c, because the irrationals are dense in $\mathbb{R}$. Because $(y_n) \notin \mathbb{Q}$, we have $T(y_n) = 0$. So while $(y_n) \to c$, $T(y_n) \to 0 \neq T(c)$. Because c was an arbitrary rational number, we conclude that $T(x)$ is not continuous at all rational points.
Now we are going to look at the irrational points; so we look at $d \notin \mathbb{Q}$. Because the rationals are also dense in $\mathbb{R}$ we can find a sequence $(z_n) \in \mathbb{Q}$ which converges to d. The closer a rational number is to a fixed irrational number, the larger its denominator must necessarily be. If the denominator is large, then its reciprocal is close to zero. So if $(z_n) \to d$ it follows that $T(z_n) \to 0 = T(d)$. Because d was an arbitrary irrational number, we can conclude that $T(x)$ is continuous at all irrational points. □

Now we want to investigate if Thomae's function is Henstock-Kurzweil integrable. It turns out that it is. We will prove that statement now. We are doing this again by finding an appropriate gauge. This gauge will look like the gauges we used for Dirichlet's function and the modified version of it: we distinguish for the different cases just like in Thomae's function itself.

Proposition 4.8. *Thomae's function T is Henstock-Kurzweil integrable on the interval $[a, b]$ with $\int_a^b T = 0$.*

Proof. Let $\epsilon > 0$. Suppose that $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$. Let π be the set of indices i such that the tags t_i are rational; so $\pi := \{i : t_i = \frac{m_i}{n_i} \in \mathbb{Q} \setminus \{0\}$ is in lowest terms, with $n_i > 0\}$. Let ρ be the set of indices i such that the tags t_i are irrational; so $\rho := \{i : t_i \notin \mathbb{Q}\}$. Let σ be the set of indices i such that the tags t_i are zero; so $\sigma := \{i : t_i = 0\}$. The set σ contains at most two elements, because there can be at most be two intervals that have the same tag. This tag should be the right endpoint of one of the subintervals and the left endpoint of the other subinterval. Now we define our gauge γ_ϵ as follows:

$$\gamma_\epsilon(x) = \begin{cases} \frac{\epsilon}{4} & \text{if } x = 0 \\ \frac{n\epsilon}{2q} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

where q is the number of elements of the set π. Now suppose our tagged partition P is γ_ϵ-fine. Then:

$$\begin{aligned} |S(T, P)| &= \left| \sum_{i=1}^{n} T(t_i)(x_i - x_{i-1}) \right| \\ &\leq \left| \sum_{i \in \pi} T(t_i)(x_i - x_{i-1}) \right| + \left| \sum_{i \in \rho} T(t_i)(x_i - x_{i-1}) \right| + \left| \sum_{i \in \sigma} T(t_i)(x_i - x_{i-1}) \right| \\ &= \left| \sum_{i \in \pi} \frac{1}{n_i}(x_i - x_{i-1}) \right| + \left| \sum_{i \in \sigma} (x_i - x_{i-1}) \right| \\ &\leq \left| \sum_{i \in \pi} \frac{1}{n_i} \frac{n_i \epsilon}{2q} \right| + \left| \sum_{i \in \sigma} \frac{\epsilon}{4} \right| \\ &\leq \left| \sum_{i \in \pi} \frac{\epsilon}{2q} \right| + \left| \frac{\epsilon}{2} \right| \\ &= \left| \frac{\epsilon}{2} \right| + \left| \frac{\epsilon}{2} \right| = \epsilon \end{aligned}$$

Because our ϵ was arbitrary, this holds for all $\epsilon > 0$ and therefore we conclude that Thomae's function $T(x)$ is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b T = 0$. $\square$

5 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus states that if F is differentiable on $[a, b]$ and $F' = f$, then $\int_a^b f = F(b) - F(a)$. In the case of Riemann integration it is necessary to add the additional condition that f must be integrable. This condition is necessary, because not every derivative turns out to be Riemann integrable. This is one of the shortcomings of the Riemann integral. We are going to investigate if this problem also occurs when we are working with the Henstock-Kurzweil integral.
To show that not every derivative is Riemann integrable, we will now present an example. We consider the following function:

$$F(x) := \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

We claim that this function is continuous and differentiable everywhere. For $x \neq 0$ this is easy to see: F is a product of a composition of continuous, differentiable functions. For $x = 0$ we have that $\lim_{x\to 0} F(x) = 0$ and $F(0) = 0$, so $F(x)$ is continuous at $x = 0$ and because $\lim_{x\to 0} \frac{F(x)-F(0)}{x-0} = \lim_{x\to 0} \frac{F(x)}{x} = \lim_{x\to 0} x \sin(\frac{1}{x^2}) = 0$, $F(x)$ is also differentiable at $x = 0$. By using the basic rules of differentiation we get:

$$F'(x) := \begin{cases} 2x \sin(\frac{1}{x^2}) - \frac{2}{x} \cos(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Lebesgue's theorem states that a function is Riemann integrable if and only if it is bounded and continuous almost everywhere [8]. We are going to show that $F'(x)$ is not Riemann integrable on an interval $[0, b]$ (where $b \in \mathbb{R}^+$) by showing that it is unbounded [8].

Proposition 5.1. *The function $F'(x)$ as given above is unbounded on the interval $[0, b]$.*

Proof. Let n be a positive integer and let $x = \frac{n^2}{2\pi}$. The Archimedean property states that for every real number x, there exists a natural number N such that $N > x$. Our $x = \frac{n^2}{2\pi}$ is a real number, so there exists $N_1 \in \mathbb{N}$ such that $N_1 > x$. This is equivalent to the statement that $n < \sqrt{2\pi N_1}$. Now take $y = \frac{1}{2\pi b^2}$. This y is again a real number, so there exists $N_2 \in \mathbb{N}$ such that $N_2 > y$ and this is equivalent to $\frac{1}{\sqrt{2\pi N_2}} < b$. Now we take $N = \max\{N_1, N_2\}$. Then $n < \sqrt{2\pi N}$ and $0 < \frac{1}{\sqrt{2\pi N}} < b$. So we have that $\frac{1}{\sqrt{2\pi N}} \in [0, b]$ and we are going to evaluate $F'(x)$ at this value:

$$\begin{aligned} \left| F'(\frac{1}{\sqrt{2\pi N}}) \right| &= \left| \frac{2}{\sqrt{2\pi N}} \sin(2\pi N) - 2\sqrt{2\pi N} \cos(2\pi N) \right| \\ &= 2\sqrt{2\pi N} > n. \end{aligned}$$

Because our n was an arbitrary integer, we conclude that $F'(x)$ is unbounded on the interval $[0, b]$. $\square$

So we have found a function that is a derivative and that is unbounded. Therefore this derivative is not Riemann integrable. So the condition that $f = F'$ is Riemann integrable is necessary in the Fundamental Theorem.
Now we are going to show that we do not need this extra condition when we are considering Henstock-Kurzweil integration, because it turns out that every derivative is Henstock-Kurzweil integrable. We will see this while we are proving the Fundamental Theorem considering Henstock-Kurzweil integration [2, 7].

Theorem 5.1 - The Fundamental Theorem. *If $F : [a, b] \to \mathbb{R}$ is differentiable at every point of $[a, b]$ then $f = F'$ is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b f = F(b) - F(a)$.*

Proof. Let $\epsilon > 0$. Suppose $t \in [a, b]$. Because $f(t) = F'(t)$ exists, we know from the definition of differentiability that for $\frac{\epsilon}{b-a}$ there exists a δ_t such that if $0 < |z - t| \leq \delta_t$ for $z \in [a, b]$ then

$$\left| \frac{F(z) - F(t)}{z - t} - f(t) \right| \leq \frac{\epsilon}{b - a}.$$

Now we let these δ_t's form our gauge:

$$\gamma(t) := \delta_t.$$

Note that this gauge γ is not a constant, because δ_t is not the same for all t; δ_t depends on t. Now if $|z - t| \leq \gamma(t)$ for $z, t \in [a, b]$ then

$$|F(z) - F(t) - f(t)(z - t)| \leq \frac{\epsilon}{b - a} |z - t|.$$

If $a \leq \alpha \leq t \leq \beta \leq b$ and $0 < |\beta - \alpha| \leq \gamma(t)$ then by the triangle inequality, we get

$$\begin{aligned} |F(\beta) - F(\alpha) - f(t)(\beta - \alpha)| &\leq |F(\beta) - F(t) - f(t)(\beta - t)| + |F(t) - F(\alpha) - f(t)(t - \alpha)| \\ &\leq \frac{\epsilon}{b - a} |\beta - t| + \frac{\epsilon}{b - a} |t - \alpha| \\ &= \frac{\epsilon}{b - a} |\beta - \alpha|. \end{aligned}$$

Let $P := \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$ that is γ-fine. We can write $F(b) - F(a)$ as a telescoping sum: $F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))$. Now we

have:

$$\begin{aligned}
|S(f,P) - (F(b) - F(a))| &= |F(b) - F(a) - S(f,P)| \\
&= \left| \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \right| \\
&= \left| \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})) \right| \\
&\leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \\
&\leq \sum_{i=1}^{n} \frac{\epsilon}{b-a} |x_i - x_{i-1}| \\
&= \frac{\epsilon}{b-a}(b-a) = \epsilon.
\end{aligned}$$

Because our ϵ was arbitrary, this holds for all $\epsilon > 0$ and thus we conclude that $f = F'$ is Henstock-Kurzweil integrable on $[a,b]$ with $\int_a^b f = F(b) - F(a)$. $\square$

We see that every derivative is Henstock-Kurzweil integrable, because we could take $\gamma(t) := \delta_t$ as gauge, where δ_t stems from the definition of differentiability. So the shortcoming that not every derivative is Riemann integrable does not occur when we are considering Henstock-Kurzweil integrability and the Fundamental theorem does not need an extra condition.

6 Convergence theorems

Another limitation of the Riemann integral arises when we consider limits of sequences of functions. To illustrate this we look at Dirichlet's function again. Recall from subsection 4.2 that Dirichlet's function is the characteristic function of the rationals and is given by:

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Now we focus on the interval $[0,1]$. Let $\{r_1, r_2, ...\}$ be an enumeration of the rational points in this interval. Now define $h_1(x) = 1$ if $x = r_1$ and $h_1(x) = 0$ otherwise. Next, define $h_2(x) = 1$ if $x = r_1$ or $x = r_2$ and $h_2(x) = 0$ otherwise. By continuing this, we get:

$$h_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \cdots, r_n\} \\ 0 & \text{if } x \notin \{r_1, r_2, \cdots, r_n\} \end{cases}.$$

Each h_n has a finite number of discontinuities and is therefore Riemann integrable with $\int_a^b h_n = 0$. We also have that $h_n \to h$ pointwise. We saw in subsection 4.2 that Dirichlet's function is not Riemann integrable. Thus we have that

$$\lim_{n\to\infty} \int_0^1 h_n = \int_0^1 h$$

does not hold, because the value on the right-hand side does not exist [1].
In this section we are going to look at convergence theorems for the Henstock-Kurzweil integral. We want to investigate if we can overcome shortcomings as above while considering Henstock-Kurzweil integration instead of Riemann integration.
Before we begin our investigation regarding convergence and Henstock-Kurzweil integration, we define *uniformly Henstock-Kurzweil integrability* [4]. Then we continue with our first convergence theorem which will involve uniformly Henstock-Kurzweil integrals. The next theorem will be about uniform convergence and Henstock-Kurzweil integrability and at last we are going to look at pointwise convergence.

Definition 6.1. *Let $\{f_n\}$ be a sequence of Henstock-Kurzweil integrable functions on $[a,b]$. The sequence $\{f_n\}$ is* **uniformly Henstock-Kurzweil integrable** *on $[a,b]$ if for all $\epsilon > 0$ there exist a gauge γ on $[a,b]$ such that if P is a tagged partition of $[a,b]$ that is γ-fine then $\left|S(f_n,P) - \int_a^b f_n\right| \leq \epsilon$ for all n.*

What this definition actually means is that if we can find one gauge γ that works for all f_n, then we call the sequence $\{f_n\}$ uniformly Henstock-Kurzweil integrable.
Now we are going to prove our first convergence-theorem [4].

Theorem 6.1. *Suppose $\{f_n\}$ is a sequence of Henstock-Kurzweil integrable functions on $[a,b]$ and that $\{f_n\}$ converges pointwise to f. If $\{f_n\}$ is uniformly Henstock-Kurzweil integrable on $[a,b]$, then f is Henstock-Kurzweil integrable on $[a,b]$ with $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

Proof. Let $\epsilon > 0$. Since $\{f_n\}$ is uniformly Henstock-Kurzweil integrable there exists a gauge γ on $[a,b]$ such that if $\tilde{P}$ is a tagged partition of $[a,b]$ that is γ-fine, then

$$\left|S(f_n,\tilde{P}) - \int_a^b f_n\right| \leq \frac{\epsilon}{3}$$

for all n. Since $\{f_n\}$ converges pointwise on $[a,b]$, there exists an integer N such that

$$\left|S(f_n,\tilde{P}) - S(f_m,\tilde{P})\right| \leq \frac{\epsilon}{3}$$

for all $m,n \geq N$. Now it follows that:

$$\begin{aligned}\left|\int_a^b f_n - \int_a^b f_m\right| &\leq \left|\int_a^b f_n - S(f_n,\tilde{P})\right| + \left|S(f_n,\tilde{P}) - S(f_m,\tilde{P})\right| + \left|S(f_m,\tilde{P}) - \int_a^b f_m\right| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

for all $m, n \geq N$. This means that $\{\int_a^b f_n\}$ is a Cauchy sequence. Let A be the limit of this sequence. Our claim is now that $\int_a^b f = A$.
Let $\epsilon > 0$. Since $\{\int_a^b f_n\}$ is a Cauchy sequence, we can choose an integer N such that

$$\left| \int_a^b f_n - A \right| \leq \frac{\epsilon}{3}$$

for all $n \geq N$. Since $\{f_n\}$ is uniformly Henstock-Kurzweil integrable, there exists a gauge γ such that if P is a tagged partition of $[a, b]$ that is γ-fine, then

$$\left| S(f_n, P) - \int_a^b f_n \right| \leq \frac{\epsilon}{3}$$

for all n. Now suppose P is a tagged partition of $[a, b]$ that is γ-fine. Since $\{f_n\}$ converges pointwise to f there exist $k \geq N$ such that

$$|S(f, P) - S(f_k, P)| \leq \frac{\epsilon}{3}.$$

Thus, it follows that:

$$\begin{aligned} |S(f, P) - A| &\leq |S(f, P) - S(f_k, P)| + \left| S(f_k, P) - \int_a^b f_k \right| + \left| \int_a^b f_k - A \right| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Since our ϵ was arbitrary, we conclude that f is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b f = A = \lim_{n\to\infty} f_n$. □

The next convergence theorem we are going to look at is about uniform convergence in combination with Henstock-Kurzweil integrability.

Theorem 6.2. *If $\{f_k\}$ is a sequence of Henstock-Kurzweil integrable functions on $[a, b]$ that converges uniformly to f on $[a, b]$, then it follows that f is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b f = \lim_{k\to\infty} \int_a^b f_k$.*

Before we can prove this theorem, we need to prove a lemma.

Lemma 6.1. *If $f : [a, b] \to \mathbb{R}$ is Henstock-Kurzweil integrable and $|f(x)| \leq M$ with $M \in \mathbb{R}$ for all $x \in [a, b]$ then $\left| \int_a^b f \right| \leq M(b - a)$.*

Proof. $|f(x)| \leq M$ means that $-M \leq f(x) \leq M$. In section 4.1 we saw that the function $f(x) = c$ with $c \in \mathbb{R}$ is Henstock-Kurzweil integrable on $[a, b]$ with $\int_a^b f = c(b - a)$. So we know that M and $-M$ are Henstock-Kurzweil integrable. Applying theorem 3.3, we get

$$\int_a^b M = M(b - a)$$

$$\int_a^b -M = -\int_a^b M = -M(b-a).$$

Corollary 3.2 states that if f and g are Henstock-Kurzweil integrable on $[a,b]$ and $f(x) \leq g(x)$ on $[a,b]$, then $\int_a^b f \leq \int_a^b g$. We apply this corollary to $-M \leq f(x)$ and to $f(x) \leq M$ and thus we get

$$\int_a^b -M = -M(b-a) \leq \int_a^b f$$

and

$$\int_a^b f \leq \int_a^b M = M(b-a).$$

We conclude that

$$\left| \int_a^b f \right| \leq M(b-a).$$

□

Now we have proven this lemma, we can prove theorem 6.2 [3].

Proof. Theorem 6.2. Let $\epsilon > 0$. Because $\{f_k\}$ converges uniformly to f, there exists a $K \in \mathbb{N}$ such that

$$|f_k - f| \leq \frac{\epsilon}{2(b-a)}$$

for all $x \in [a,b]$ whenever $k \geq K$. Consequently, if $h, k \geq K$ then we have

$$\begin{aligned} |f_k - f_h| &\leq |f_k - f| + |f - f_h| \\ &\leq \frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{(b-a)}. \end{aligned}$$

By assumption f_k and f_h are Henstock-Kurzweil integrable and thus follows from theorem 3.3 that $f_k - f_h$ is Henstock-Kurzweil integrable. Now we can apply lemma 6.1 to $f_k - f_h$:

$$\left| \int_a^b (f_k - f_h) \right| \leq \frac{\epsilon}{(b-a)}(b-a) = \epsilon.$$

Because of the linearity properties of the Henstock-Kurzweil integral, we know that $\left| \int_a^b (f_k - f_h) \right| = \left| \int_a^b f_k - \int_a^b f_h \right| \leq \epsilon$. Because our ϵ was arbitrary, this means that the sequence $\{\int_a^b f_k\}$ is a Cauchy sequence. Let A be the limit of this sequence. Our claim is again that $\int_a^b f = A$. Let $\epsilon > 0$ and let $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ be a tagged partition of $[a,b]$.
Since $\{\int_a^b f_k\}$ is a Cauchy sequence with limit A, there exist an integer $N \in \mathbb{N}$ such that for $k \geq N$, we have

$$\left| \int_a^b f_k - A \right| \leq \frac{\epsilon}{3}.$$

Since $\{f_k\}$ converges uniformly to f, there exists an $K \in \mathbb{N}$ such that if $k \geq K$, then

$$|f_k - f| \leq \frac{\epsilon}{3(b-a)}$$

for all $x \in [a, b]$. Because of this, it follows that

$$\begin{aligned}
|S(f_k, P) - S(f, P)| &= \left| \sum_{i=1}^{n} f_k(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \right| \\
&= \left| \sum_{i=1}^{n} [(f_k(t_i) - f(t_i))(x_i - x_{i-1})] \right| \\
&\leq \sum_{i=1}^{n} |(f_k(t_i) - f(t_i))(x_i - x_{i-1})| \\
&\leq \sum_{i=1}^{n} \frac{\epsilon}{3(b-a)}(x_i - x_{i-1}) \\
&= \frac{\epsilon}{3(b-a)}(b-a) = \frac{\epsilon}{3}.
\end{aligned}$$

All f_k are Henstock-Kurzweil integrable. So for each f_k there exists a gauge γ_k on $[a, b]$ such that whenever P is γ_k-fine we have

$$\left| S(f_k, P) - \int_a^b f_k \right| \leq \frac{\epsilon}{3}.$$

Now we choose a k such that $k \geq K$ and $k \geq N$. Now we combine all the previous and we get

$$\begin{aligned}
|S(f, P) - A| &\leq |S(f, P) - S(f_k, P)| + \left| S(f_k, P) - \int_a^b f_k \right| + \left| \int_a^b f_k - A \right| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Because our ϵ was arbitrary, this holds for all $\epsilon > 0$ and we conclude that f is Henstock-Kurzweil integrable with $\int_a^b f = \lim_{k\to\infty} \int_a^b f_k$. □

We saw here that uniform convergence guarantees the limit function to be Henstock-Kurzweil integrable as well. Now we are going to investigate if this also holds for pointwise convergence. As it turns out, supposing that each function in a sequence that converges pointwise is Riemann integrable is not enough to guarantee that the limit function is also Riemann integrable. We saw a counterexample of this at the beginning of this section. The monotone convergence theorem provides a set of conditions that guarantees that the limit of a pointwise convergent sequence of Henstock integrable functions is also Henstock integrable. This theorem does not hold for the Riemann integral; the sequence of functions

$h_n(x)$ at the beginning of this section turns out to be a counterexample again.
Before we can prove the monotone convergence theorem, we need to take a look at the *Saks-Henstock* lemma [3]. This lemma is about subpartitions. A subpartition of $[a, b]$ is a partition that does not need to cover the whole interval $[a, b]$. So the union of the subintervals of a subpartition does not have to be $[a, b]$.

Definition 6.2. *A* **subpartition** $\tilde{P} := \{[x_{j-1}, x_j] : 1 \leq j \leq m\}$ *of an interval* $[a, b]$ *is a finite collection of closed intervals such that* $[x_{j-1}, x_j] \subset [a, b]$ *and where the subintervals* $[x_{j-1}, x_j]$ *can only have the endpoints in common.*

Definition 6.3. *A* **tagged subpartition** $\tilde{P} := \{(t_j, [x_{j-1}, x_j]) : 1 \leq j \leq m\}$ *of an interval* $[a, b]$ *is a finite set of order pairs where* $t_j \in [x_{j-1}, x_j]$ *and the intervals* $[x_{j-1}, x_j]$ *form a subpartition of* $[a, b]$*. The numbers* t_j *are again called the* **tags**.

The Saks-Henstock lemma states that if we have a subpartition of the interval $[a, b]$, then we could make the difference between the Riemann sum and the integral over the subintervals of the subpartition also smaller than ϵ. It also says that if we first take absolute values and then summarize, this term will be smaller than 2ϵ.

Lemma 6.2 - Saks-Henstock lemma. *Let* f *be a function that is Henstock-Kurzweil integrable on* $[a, b]$ *and let for* $\epsilon > 0$ γ_ϵ *be a gauge on* $[a, b]$ *such that if* $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ *is a tagged partition of* $[a, b]$ *that is* γ_ϵ*-fine, then* $\left|S(f, P) - \int_a^b f\right| \leq \epsilon$*. If* $\tilde{P} = \{(t_j, [x_{j-1}, x_j]) : 1 \leq j \leq s\} = \{(t_j, J_j) : 1 \leq j \leq s\}$ *is any* γ_ϵ*-fine subpartition of* $[a, b]$*, then*

1. $\left|S(f, \tilde{P}) - [\int_{J_1} f + \cdots + \int_{J_s} f]\right| = \left|\sum_{j=1}^s [f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f]\right| \leq \epsilon$ *and*
2. $\sum_{j=1}^s \left|f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f\right| \leq 2\epsilon$.

Proof. 1. Let $K_1, \cdots, K_m$ be closed subintervals in $[a, b]$ such that $\{J_j\}_{j=1}^s \cup \{K_k\}_{k=1}^m$ forms a partition of $[a, b]$. Now let $\alpha > 0$ be arbitrary. Because each $K_k \subseteq [a, b]$, we know that f is Henstock-Kurzweil integrable on each K_k by theorem 3.5. So for each K_k, there exists a gauge $\gamma_{\alpha,k}$ such that if Q_k is a $\gamma_{\alpha,k}$-fine tagged partition of K_k then

$$\left|S(f, Q_k) - \int_{K_k} f\right| \leq \frac{\alpha}{m}.$$

If we have that $\gamma_{\alpha,k}(x) \geq \gamma_\epsilon(x)$ for some k, then we could add a positive constant to $\gamma_\epsilon(x)$ such that we get a new gauge that is greater than $\gamma_{\alpha,k}(x)$: $\tilde{\gamma}_\epsilon = \gamma_\epsilon + c$ with $c \in \mathbb{R}_{>0}$ such that $\gamma_{\alpha,k}(x) \leq \tilde{\gamma}_\epsilon(x)$ for all $x \in K_k$. A partition that is γ_ϵ-fine is then also $\tilde{\gamma}_\epsilon$-fine. So this $\tilde{\gamma}_\epsilon$ is also a gauge that satisfies the assumption in the statement of this theorem. So we may assume that $\gamma_{\alpha,k}(x) \leq \gamma_\epsilon(x)$ for all $x \in K_k$. Then we

know that if Q_k is $\gamma_{\alpha,k}$-fine, then it is also γ_ϵ-fine. Now let $P^* := \tilde{P} \cup Q_1 \cup \cdots \cup Q_m$. P^* is a tagged partition of $[a,b]$. If P^* is γ_ϵ-fine, then

$$\left| S(f, P^*) - \int_a^b f \right| \leq \epsilon.$$

Furthermore we have

$$\begin{aligned} S(f, P^*) &= S(f, \tilde{P}) + S(f, Q_1) + \cdots + S(f, Q_m) \\ \int_a^b f &= \int_{J_1} f + \cdots + \int_{J_s} f + \int_{K_1} f + \cdots + \int_{K_m} f. \end{aligned}$$

From this, it follows that

$$\begin{aligned} \left| S(f, \tilde{P}) - [\int_{J_1} f + \cdots + \int_{J_s} f] \right| &= \left| [S(f, P^*) - \sum_{k=1}^m S(f, Q_k)] - [\int_a^b f - \sum_{k=1}^m \int_{K_k} f] \right| \\ &\leq \left| S(f, P^*) - \int_a^b f - \sum_{k=1}^m [S(f, Q_k) - \int_{K_k} f] \right| \\ &\leq \left| S(f, P^*) - \int_a^b f \right| + \sum_{k=1}^m \left| S(f, Q_k) - \int_{K_k} f \right| \\ &\leq \epsilon + \sum_{k=1}^m \frac{\alpha}{m} \\ &= \epsilon + m\frac{\alpha}{m} = \epsilon + \alpha. \end{aligned}$$

Because α was arbitrary, we have $\left| S(f, P^*) - [\int_{J_1} f + \cdots + \int_{J_s} f] \right| \leq \epsilon$ as desired.

2. Let $\tilde{P}^+$ be the pairs $(t_j, [x_{j-1}, x_j])$ of $\tilde{P}$ such that $f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f \geq 0$ and let $\tilde{P}^-$ be the pairs $(t_j, [x_{j-1}, x_j])$ of $\tilde{P}$ such that $f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f < 0$. $\tilde{P}^+$ and $\tilde{P}^-$ are again subpartitions of $[a,b]$ which are γ_ϵ-fine so we are going to apply

the first part of this lemma to both $\tilde{P}^+$ and $\tilde{P}^-$:

$$\begin{aligned}
\sum_{\tilde{P}^+} \left| f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f \right| &= \sum_{\tilde{P}^+} [f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f] \\
&= \left| \sum_{\tilde{P}^+} [f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f] \right| \\
&\leq \epsilon \\
\sum_{\tilde{P}^-} \left| f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f \right| &= -\sum_{\tilde{P}^-} [f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f] \\
&= \left| \sum_{\tilde{P}^-} [f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f] \right| \\
&\leq \epsilon.
\end{aligned}$$

Now we combine this and we get

$$\begin{aligned}
\sum_{j=1}^{s} \left| f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f \right| &= \sum_{\tilde{P}^+} \left| f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f \right| \\
&\quad + \sum_{\tilde{P}^-} \left| f(t_j)(x_j - x_{j-1}) - \int_{x_{j-1}}^{x_j} f \right| \\
&\leq \epsilon + \epsilon = 2\epsilon
\end{aligned}$$

as desired.

□

Now we can use the Saks-Henstock lemma to prove the monotone convergence theorem [8].

Theorem 6.3 - Monotone convergence theorem. *Let $\{f_n\}$ be a monotone sequence of Henstock-Kurzweil integrable functions on $[a, b]$ that converges pointwise to a limit function f. If $\lim_{n\to\infty} \int_a^b f_n$ exists, then f is Henstock-Kurzweil integrable and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

Proof. We suppose $\{f_n\}$ is increasing. Let $\epsilon > 0$ and let $A = \lim_{n\to\infty} \int_a^b f_n$. Then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left| \int_a^b f_n - A \right| \leq \frac{\epsilon}{3}.$$

Since f_n is Henstock-Kurzweil integrable for each n, there exists a gauge γ_n on $[a,b]$ for each n such that

$$\left| S(f_n, P) - \int_a^b f_n \right| \leq \frac{\epsilon}{3 \cdot 2^n}$$

if P is γ_n-fine. Without loss of generality, we suppose that $\gamma_n \geq \gamma_{n+1}$. Since $\{f_n\}$ converges pointwise to f on $[a,b]$, we choose for each $x \in [a,b]$ an integer $M_x \geq N$ such that

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{3(b-a)}$$

for $n \geq M_x$. Now we define our gauge: $\gamma(x) := \gamma_{M_x}(x)$ and we let $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq m\}$ be a γ-fine tagged partition of $[a,b]$. Note that our γ is not a constant. Now we divide the set of tagged subintervals of P into classes based on the length of subintervals. P_m is the collection of $\gamma_{M_{t_m}}$-fine tagged partitions and let for each $m-1 \geq i \geq 1$ P_i be the collection of $\gamma_{M_{t_i}}$-fine partitions of $P - \{\bigcup_{j=i+1}^m P_j\}$. The set of P_i for $1 \leq i \leq m$ is disjoint and the union of elements in this set is P. Let for each $1 \leq i \leq m$, K_i be the collection of intervals used in the collection of tagged intervals of P_i.
By the triangle inequality we get:

$$|S(f,P) - A| \leq \left| S(f,P) - \sum_{i=1}^m S(f_{M_{t_i}}, P_i) \right| + \left| \sum_{i=1}^m S(f_{M_{t_i}}, P_i) - \sum_{i=1}^m \int_{K_i} f_{M_{t_i}} \right| + \left| \sum_{i=1}^m \int_{K_i} f_{M_{t_i}} - A \right|. \quad (1)$$

We are now going to look at those three terms seperately.
We can use the fact that $\{f_n\}$ converges pointwise to f to estimate the first term. $M_{t_i} \geq N$ by definition, so $\left| f_{M_{t_i}} - f \right| \leq \frac{\epsilon}{3(b-a)}$. Using this, we get:

$$\begin{aligned}
\left| S(f,P) - \sum_{i=i}^m S(f_{M_{t_i}}, P_i) \right| &= \left| \sum_{i=1}^m (f(t_i) - f_{M_{t_i}}(t_i))(x_i - x_{i-1}) \right| \\
&\leq \sum_{i=1}^m \left| f(t_i) - f_{M_{t_i}}(t_i) \right| (x_i - x_{i-1}) \\
&\leq \sum_{i=1}^m \frac{\epsilon}{3(b-a)} (x_i - x_{i-1}) \\
&= \frac{\epsilon}{3(b-a)} (b-a) = \frac{\epsilon}{3}.
\end{aligned}$$

Now we are going to look at the second term of (1). Here we are going to use Henstock's lemma. The collection of intervals $P_{M_{t_i}}$ is by definition $\gamma_{M_{t_i}}$-fine for each $1 \leq i \leq m$. Because for each n f_n is Henstock-Kurzweil integrable, $f_{M_{t_i}}$ is Henstock Kurzweil integrable and thus

$$\left| S(f_{M_{t_i}}, P) - \int_a^b f_{M_{t_i}} \right| \leq \frac{\epsilon}{3 \cdot 2^{M_{t_i}}}$$

if P is a $\gamma_{M_{t_i}}$-fine partition of $[a,b]$. Using Henstock's lemma, we get:

$$\left| S(f_{M_{t_i}}, P_i) - \int_{K_i} f_{M_{t_i}} \right| \leq \frac{\epsilon}{3 \cdot 2^{M_{t_i}}}$$

for each $1 \leq i \leq m$. Now we use this for the second term:

$$\begin{aligned} \left| \sum_{i=1}^{m} S(f_{M_{t_i}}, P_i) - \sum_{i=1}^{m} \int_{K_i} f_{M_{t_i}} \right| &\leq \sum_{i=1}^{m} \left| S(f_{M_{t_i}}, P_i) - \int_{K_i} f_{M_{t_i}} \right| \\ &\leq \sum_{i=1}^{m} \frac{\epsilon}{3 \cdot 2^{M_{t_i}}} \leq \frac{\epsilon}{3}. \end{aligned}$$

Now we are going to look at the last term of (1). We know that $\{f_n\}$ is an increasing sequence of functions. So $f_{M_{t_i}} \geq f_N$ because $M_{t_i} \geq N$. Because $\{f_n\}$ converges pointwise to f from below, we have that $\{\int_a^b f_n\}$ converges to A from below and we have

$$\left| \int_{K_i} f_{M_{t_i}} - A \right| \leq \left| \int_{K_i} f_N - A \right|$$

for each $1 \leq i \leq m$. Therefore we have

$$\begin{aligned} \left| \sum_{i=1}^{m} \int_{K_i} f_{M_{t_i}} - A \right| &\leq \left| \sum_{i=1}^{m} \int_{K_i} f_N - A \right| \\ &= \left| \int_a^b f_N - A \right| \\ &\leq \frac{\epsilon}{3}. \end{aligned}$$

Now we are combining the three terms again and we get:

$$\begin{aligned} |S(f,P) - A| &\leq \left| S(f,P) - \sum_{i=1}^{m} S(f_{M_{t_i}}, P_i) \right| + \left| \sum_{i=1}^{m} S(f_{M_{t_i}}, P_i) - \sum_{i=1}^{m} \int_{K_i} f_{M_{t_i}} \right| + \left| \sum_{i=1}^{m} \int_{K_i} f_{M_{t_i}} - A \right| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Because our ϵ was arbitrary, we conclude that f is Henstock-Kurzweil integrable with $\int_a^b f = A = \lim_{n\to\infty} \int_a^b f_n$. $\square$

Of course, a similar result holds when the sequence $\{f_n\}$ is decreasing. The proof is similar, and hence omitted.

The example in the beginning of this section shows that the monotone convergence theorem does not hold for Riemann integrability. Recall the sequence of functions defined by

$$h_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \cdots, r_n\} \\ 0 & \text{if } x \notin \{r_1, r_2, \cdots, r_n\} \end{cases}$$

on the interval $[0, 1]$. For this sequence of functions it is clear that $h_n \leq h_{n+1}$, but as we saw the limit function is Dirichlet's function, which is not Riemann integrable.
The dominated convergence theorem provides an other set of conditions that also guarantees the pointwise limit f of a sequence $\{f_n\}$ to be Henstock-Kurzweil integrable. Again a similar theorem for the Riemann integral does not exist. We do not prove this theorem here. For those who would like to see a proof we refer to Gordon's book *The Integrals of Lebesgue, Denjoy, Perron, and Henstock* [4].

Theorem 6.4 - Dominated convergence theorem. *Suppose $\{f_k\}$ is a sequence of Henstock-Kurzweil integrable functions on $[a, b]$ that converges pointwise to $f(x)$. If $g(x)$ and $h(x)$ are Henstock-Kurzweil integrable functions on $[a, b]$ such that $g(x) \leq f_k(x) \leq h(x)$ for all k, then $f(x)$ is Henstock-Kurzweil integrable with $\int_a^b f = \lim_{k\to\infty} \int_a^b f_k$.*

To show that the dominated convergence theorem does not hold for Riemann integrable functions, we could again take a look at the sequence of functions $h_n(x)$. These functions do only take the values 0 and 1. So lets take $r(x) = c$ with $c \in \mathbb{R}$ and $c \leq 0$ and $s(x) = d$ with $d \in \mathbb{R}$ and $d \geq 1$. Then we have that $r(x) \leq h_n(x) \leq s(x)$ for all n and we know that $r(x)$ and $s(x)$ are Riemann integrable. But still Dirichlet's function, the limit function of h_n, is not Riemann integrable.

7 Conclusion and discussion

We have seen that making a small change in the definition of the Riemann integral resulted in a more general integral: the Henstock-Kurzweil integral. We saw that using the gauge γ instead of a constant δ led to certain benefits over the Riemann integral. First we saw examples of functions that are Henstock-Kurzweil integrable, but not Riemann integrable. So by using the Henstock-Kurzweil definition of integrals, we enlarged the range of integrable functions. Next thing we saw is that not every derivative is Riemann integrable. So in the fundamental theorem we have to include the extra condition that $f = F'$ is integrable. When we examined the fundamental theorem regarding Henstock-Kurzweil integrals, we concluded that we do not need this extra condition; it turned out that every derivative is Henstock-Kurzweil integrable. In the last section we saw that for the Henstock-Kurzweil integral there exist some nice convergence theorems. For example we examined the monotone convergence theorem and the dominated convergence theorem. These theorems turned out to work for Henstock-Kurzweil integrals, but not for Riemann integrals.

When we do take measure theory in account, we could also compare the Henstock-Kurzweil integral with Lebesgue's integral. Furthermore we could use the monotone convergence theorem and the dominated convergence theorem for Lebesgue's integral to prove the monotone convergence theorem and the dominated convergence theorem for the Henstock-Kurzweil integral. This is the way R.A. Gordon proved the convergence theorems in his book *"The integrals of Lebesgue, Denjoy, Perron, and Henstock"*. He also shows equivalence between various integrals. For simplicity's sake we did not take the measure theory in account. We focused on the easy and intuitive approach of the Henstock-Kurzweil integral and its similarity with the Riemann integral. We wanted to focus on the ϵ,γ-definition of the Henstock-Kurzweil integral and investigate the advantages it has above the ϵ,δ-definition of the Riemann integral.

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