university of groningen

faculty of mathematics and natural sciences

Bachelor Thesis in Mathematics and Physics

Thomas precession in very special relativity

Author:
Klaas Hakvoort

Supervisors
Dr. A.V. Kiselev
Dr. R.G.E. Timmermans

Abstract

In this text, I look at how Thomas precession looks like within the context of very special relativity, as proposed by Cohen and Glashow. I first derive the subgroup structure of the Lorentz group, before looking at the specific subgroups used by Cohen and Glashow and pointing out some properties of these particular subgroups. I then work inside the framework of these subgroups to derive the Thomas precession and look at differences between the precession in these subgroups versus classical Thomas precession. It is found that they behave quite differently, and thus differences could be found between special relativity and very special relativity.

July 21, 2014

Contents

1 Introduction

The group of all Lorentz transformations, often called the Lorentz Group, is a very important group in physics. The theory of special relativity is a direct consequence of this group, and many physical theories are derived from symmetries of this group. One example of this would be the Standard Model. The fact that the laws of physics in this model must be invariant under the Lorentz transformations leads to many of the particles in this model.
However, it is known that not all physical processes behave 'nice' under Lorentz transformations, most of these being due to the weak interaction. Thus in 2006 A.G. Cohen and S.L. Glashow [1] came up with the theory that it is only a subgroup of the Lorentz Group under which all of physics is invariant. In particular, they argue that one of the subgroups labelled $T(2)$, $E(2)$, $SIM(2)$ and $HOM(2)$ might be the actual symmetry group of nature.

In this thesis, I will first give a short introduction of the Lorentz Group before deriving its subgroup structure, by the guide of Toller [2]. I will then look into the subgroups that Cohen and Glashow propose and show some special properties of these. To conclude, I will look at Thomas precession, both in the classical sense, using the whole Lorentz group, and restricted to their subgroups $T(2)$ and $HOM(2)$. In particular, I will look at possible differences in the precession frequency to see if there is any validity in their ideas.

2 Lorentz Group

In order to discuss the subgroups of the Lorentz Group suggested by Cohen and Glashow [1], we first need to determine what the Lorentz Group is and what its subgroups are. The Lorentz Group consists of all real, linear space-time transformations that leave the length of space-time vectors invariant [3]. Space-time most commonly refers to Minkowski space-time $E(3,1)$. This is a vector space over the real numbers, with vectors $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ and metric tensor

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This allows us to define a distance $\boldsymbol{x}^2 = (\boldsymbol{x}, \boldsymbol{x}) = \eta_{\mu,\nu} x^\mu x^\nu$, where the Einstein-summation convention is used [3]. Let Λ denote a general Lorentz transformation, then its action on a vector can be described by

$$\boldsymbol{x}' = \Lambda \boldsymbol{x} \qquad \text{or} \qquad x'^\mu = \Lambda^\mu_\nu x^\nu.$$

Since vectors lengths are unchanged, $\eta_{\mu,\nu} x^\mu x^\nu = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho,\sigma} x^\mu x^\nu$, we have the following properties:

- $det(\Lambda) = \pm 1$
- $\eta_{\mu,\nu} = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho,\sigma}$

We denote the set of all Lorentz transformations by $\mathcal{L}$. It becomes the Lorentz Group if we take the rule of composition as our group law, let inverse transformations be inverse elements and choose ̀do nothingás our unit element. Since the Lorenz Group consists of linear transformations of 4-vectors, it can be represented by a sub-group of the group of all linear transformations on $\mathbb{R}^4$, $GL(4, \mathbb{R})$ [3].

Definition 1. *A representation D of dimension d of a group G is a homomorphism*

$$D : G \to GL(d, \mathbb{R}) \text{ s.t. } \forall g \in G \ det(D(g)) \neq 0$$

and $D(g_1 \circ g_2) = D(g_1)D(g_2)$ $\forall g_1, g_2 \in G$ [4].

Let us take a look at this representation. This representation forms a Lie group, which is a special kind of group, defined as follows:

Definition 2. *A Lie Group G is a group that is also a smooth manifold, such that the following mapping is smooth: $G \times G \to G : (\sigma, \tau) = \sigma\tau^{-1}$ [5].*

Since it is a manifold, at least locally it admits a parametrisation. This means that we can write elements of the group $g \in G$ as $g = g(\boldsymbol{\alpha})$ and we can choose our parametrisation such that $g(\mathbf{0}) = e$, the identity element of the group. For a representation $D(g(\boldsymbol{\alpha})) =: D(\boldsymbol{\alpha})$, we can make the following expansion in a neighbourhood close to the identity [4]:

$$D(\mathrm{d}\boldsymbol{\alpha}) = \mathbb{1}_{d\times d} + \mathrm{d}\alpha_a X^a.$$

The X^a are called generators of the group, and are calculated by $X^a := \frac{\partial}{\partial \alpha_a} D(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}}$. To move away from the identity, we can raise $D(\mathrm{d}\boldsymbol{\alpha})$ to some power k, and take the limit. Writing $\mathrm{d}\boldsymbol{\alpha}_a = \alpha_a/k$, we get [4]:

$$D(\boldsymbol{\alpha}) = \lim_{k\to\infty} \left(1 + \frac{\alpha_a X^a}{k}\right)^k = \exp(\alpha_a X^a). \tag{1}$$

This equation shows us how we can generate the group through its generators. These generators form a so called Lie algebra, which is a vector space g with the following properties:

Definition 3. *A vector space g is a Lie algebra if there is a Lie product on the space, $[\,\cdot\,,\,\cdot\,] : g \times g \to g$, with the following properties [4]*

1. $[a, b] \in g \ \forall a, b \in g$,
2. $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c] \ \forall a, b, c \in g; \forall \alpha, \beta \in \mathbb{R}$,
3. $[a, b] = -[b, a]$,
4. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \ \forall a, b, c \in g$.

If we are looking at a matrix representation, this Lie product is usually the common commutation relation between matrices $[\mathsf{A}, \mathsf{B}] = \mathsf{AB} - \mathsf{BA}$ [4].

Definition 4. *A sub-algebra is a sub-vector space h of some algebra g which is itself also an algebra. In particular, it is closed under the Lie product $[a, b] \in h \ \forall a, b \in g$ [5].*

We can now take another look at the Lorentz group and its algebra. The Lorentz algebra $\mathbb{L}$ is of dimension six [6], and is spanned by 3 rotations J_i and 3 Lorentz boosts K_i [3]. Let us define ϵ_{ijk} as the fully anti-symmetric tensor of rank three, then the generators satisfy the commutation relations (2) (3) (4) below [2].

$$[J_i, J_j] = \epsilon_{ijk} J^k, \tag{2}$$
$$[J_i, K_j] = \epsilon_{ijk} K^k, \tag{3}$$
$$[K_i, K_j] = -\epsilon_{ijk} J^k. \tag{4}$$

Since they span the algebra, we can write each element of the algebra as a linear combination of the generators. If $\boldsymbol{J} = (J_1, J_2, J_3)$ and $\boldsymbol{K} = (K_1, K_2, K_3)$, we can write $\forall X \in \mathbb{L}, \exists \alpha, \beta \in \mathbb{R}^3$ s.t. $X = \alpha \cdot \boldsymbol{J} + \beta \cdot \boldsymbol{K}$. By exponentiating this expression, all elements of the Lorentz group.

All elements of the Lorentz group? No actually, there are elements of the Lorentz group that are not in $e^{\mathbb{L}}$. This is due to the Lorentz group not being a connected group [3]. Instead, there are four connected subgroups. These are $L_+^\uparrow$, $L_+^\downarrow$, $L_-^\uparrow$ and $L_-^\downarrow$ [3]. It is only $L_+^\uparrow$ that is reached by exponentiating the Lorentz algebra, since it contains the identity element. The other subgroups can be obtained however by the discrete transformations of parity P and time reversal T [3]. For the rest of this text, I shall focus on the subgroup of the Lorentz group that is connected to the identity, and thus can be reached from the algebra, i.e. $L_+^\uparrow$.

3 Lorentz subgroups

In physics, the Lorentz group is considered the symmetry group of nature. This means that if we have a pair of inertial reference systems, that you can travel between using a transformation from the Lorentz Group, then in both systems the same laws of physics must hold. However, this might not be the case. It is known that the weak force does not hold under parity transformations, as was discovered by Wu and collaborators in 1957 [7]. They found that the emission of β-decay of Co^{60} does not abide this mirror symmetry. Thus, it might be that the true symmetry group of nature is a subgroup of the Lorentz group $L_+^\uparrow$. As such, it is worth the time to study these subgroups.

To study the subgroups, one first needs to know what these subgroups are. In order to look at the subgroups, I shall look at the subalgebras of the Lorentz algebra and from there we can generated all (closed) subgroups of the Lorentz group [5]. To do this, I shall introduce two bi-linear forms that are invariant under automorphisms. By definition, they will commute with all elements of the Lorentz group. I will then classify the elements of the Lorentz algebra using these forms. I shall then use these classes to find all 15 sub-algebras of the Lorentz algebra.
These bi-linear forms are called Casimir constants. If we use that a general element of the algebra is given as $X = \alpha \cdot \boldsymbol{J} + \beta \cdot \boldsymbol{K}$, then they are given as [2]

$$\begin{aligned} C_1 &= |\alpha|^2 - |\beta|^2. \\ C_2 &= \alpha \cdot \beta. \end{aligned}$$

With this, we can make the following classification of elements, up to conjugation [2]:

$$\begin{array}{rcl} A = \mu J_3 + \nu K_3 & \mu > 0, \nu \neq 0 & C_1 = \mu^2 - \nu^2, C_2 = \mu\nu \\ A = \mu J_3 & \mu > 0 & C_1 = \mu^2, C_2 = 0 \\ A = \nu K_3 & \nu > 0 & C_1 = -\nu^2, C_2 = 0 \\ A = J_1 + K_2 & & C_1 = C_2 = 0 \end{array}$$

This gives us the first 5 subalgebras:

1. 0-D: h_6, containing only the zero element 0.

2. 1-D: h_5^λ, generated by $J_3 + \lambda K_3$.

3. 1-D: h_5^0, generated by J_3.

4. 1-D: h_5^∞, generated by K_3.

5. 1-D: h_5^N, generated by $J_1 + K_2$.

Next, we want the subalgebra containing all elements with $C_1 = C_2 = 0$. We know that $J_1 + K_2$ is in this subalgebra, we can thus label a general element as $A = \mu(J_1 + K_2) + \alpha \cdot \boldsymbol{J} + \beta \cdot \boldsymbol{K}$. Applying the conditions $C_1 = C_2 = 0\ \forall \mu$ leads to the following conditions on the vectors α and β: $\beta_1 + \alpha_2 = 0$ & $\alpha_1 - \beta_2 = 0$. This also requires that $\alpha_3 = \beta_3 = 0$, and we thus find as sixth subalgebra:

6. 2-D: h_4^N, generated by $J_1 + K_2$ and $J_2 - K_1$, also called $T(2)$ [1].

All other subalgebras contain at least one element for which $C_1 \neq 0$ and/or $C_2 \neq 0$. We can assume that the general form of this element is $\mu J_3 + \nu K_3$, by our classification earlier. For this element we can define the operator $\rho A = [\mu J_3 + \nu K_3, A]$, which we call the adjoint operator. This operator has a 6 dimensional eigenspace, since there are 6 eigenvectors. Toller [2] gives us the following eigenvectors in $\mathbb{C}$, which can easily be verified:

$$\begin{aligned}
\rho K_3 &= 0. \\
\rho J_3 &= 0. \\
\rho(J_1 + K_2 - i(J_2 - K_1)) &= (\nu + i\mu)(J_1 + K_2 - i(J_2 - K_1)). \\
\rho(J_1 + K_2 + i(J_2 - K_1)) &= (\nu - i\mu)(J_1 + K_2 + i(J_2 - K_1)). \\
\rho(J_1 - K_2 - i(J_2 + K_1)) &= (-\nu + i\mu)(J_1 - K_2 - i(J_2 + K_1)). \\
\rho(J_1 - K_2 + i(J_2 + K_1)) &= -(\nu + i\mu)(J_1 - K_2 + i(J_2 + K_1)).
\end{aligned}$$

Our eigenvectors and eigenvalues are not wholly real, as they include an imaginary element. We want real eigenspaces however, and thus we take the real and imaginary parts of each vector, by taking the sum and difference between conjugate vectors. In order for our eigenspace, and thus our subalgebra, to still be closed under the Lie bracket, our subalgebras must then contain both the eigenvector, and its conjugate.

We now first assume that both J_3 and K_3 are in our subalgebra, i.e. $\mu \neq 0 \neq \nu$. Then none of our eigenvectors are degenerate, and we can choose zero, one or two pairs of eigenvectors to form our subalgebra. This way, we find the following algebras [2]:

7. 2-D: h_4, generated by J_3 and K_3.

8. 4-D: h_2, generated by J_3, K_3, $J_1 + K_2$ and $J_2 - K_1$, called $SIM(2)$ [1].

9. 6-D: h_0, the entire Lorentz-algebra.

If we were to choose the second pair of eigenvectors for 8., we would obtain a subalgebra that is conjugate under time reversal. Now, assume that we have a specific choice of μ and ν, such that $\mu = 1$ and $\nu = \lambda \neq 0$. Again, we can choose zero, one or two pairs of eigenvectors. If we choose zero, we get h_5^λ and choosing both gives us the entire algebra h_0. Only if we choose one pair, do we get a new subalgebra. The choice is irrelevant, since they are conjugate under time reversal [2]:

10. 3-D: h_3^λ, generated by $J_3 + \lambda K_3$, $J_1 + K_2$ and $J_2 - K_1$.

Another option is that $K_3 \notin h$, i.e. $\mu = 1$ and $\nu = 0$. Now, the eigenvalues are degenerate. This allows us to choose eigenvectors of the form

$$\alpha(J_1 - iJ_2) + \beta(K_2 + iK_1).$$

However, not all α and β are suited. Otherwise, commutation of this vector with its complex conjugate leads to $K_3 \in h$. Proof:

$$[\alpha(J_1 - iJ_2) + \beta(K_2 + iK_1), \bar{\alpha}(J_1 - iJ_2) + \bar{\beta}(K_2 + iK_1)] = (\alpha\bar{\beta} - \beta\bar{\alpha})K_3. \quad (5)$$

Thus, we want $\alpha\bar{\beta} - \beta\bar{\alpha} = 0$ and we can assume that α & β are real [2]. Since we want our subalgebra to be Lorentz-invariant, i.e. conjugate under the group action, there are some additional restrictions: $\alpha = 0$, $\beta = 0$ or $\alpha = \beta$ [2]. For each case, we find a subalgebra. They are:

11. 3-D: h_3^+, generated by J_1, J_2 and J_3.

12. 3-D: h_3^-, generated by K_1, K_2 and J_3.

13. 3-D: h_3^0, generated by J_3, $J_1 + K_2$ and $J_2 - K_1$, called $E(2)$ [1].

The last option is $\mu = 0$, $\nu \neq 0$, i.e. $J_3 \notin h$. Proceeding as before, we find the following 2 additional subalgebras [2]:

14. 2-D: h_4^∞, generated by K_3, $J_1 + K_2$.

15. 3-D: h_3^∞, generated by K_3, $J_1 + K_2$ and $J_2 - K_1$, called $HOM(2)$ [1].

We can show how these subalgebras are related in figure 1 on the next page.

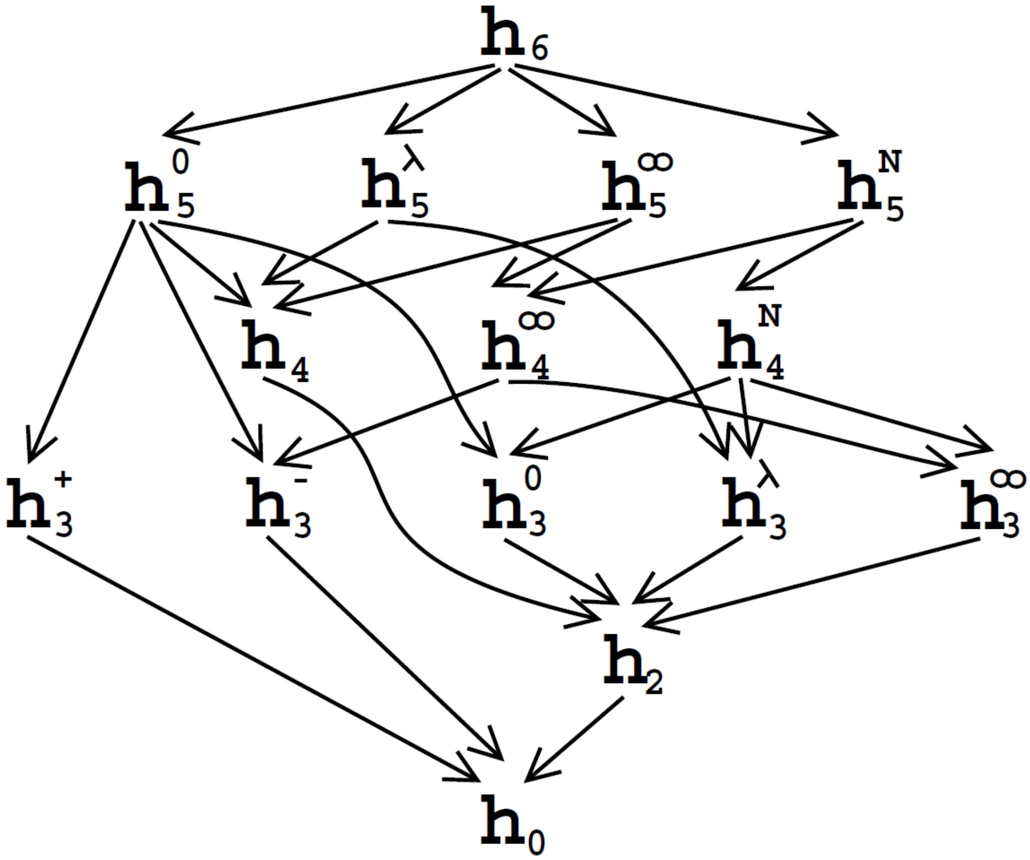

Figure 1: Subalgebras of the Lorentz algebra [2]. An arrow between two (sub)algebras means that the upper algebra is contained in the lower one.

We have thus found all subalgebras of the Lorentz algebra, and consequently all subgroups generated by these subalgebras [8]. We can now look at the subgroups generated by them, as we will do in the next section.

4 Glashow Groups

Cohen and Glashow [1] say that the true symmetry group of Nature is not the full Lorentz group, but rather is a subgroup. They introduce the idea that the true symmetry group of Nature is one of four subgroups they mention. These subgroups are generated by exponentiating certain subalgebras of the Lorentz algebra. These subalgebras are [1]:

$$T(2) = \text{span}(J_1 + K_2, J_2 - K_1) \tag{6}$$
$$E(2) = \text{span}(J_1 + K_2, J_2 - K_1, J_3) \tag{7}$$
$$HOM(2) = \text{span}(J_1 + K_2, J_2 - K_1, K_3) \tag{8}$$
$$SIM(2) = \text{span}(J_1 + K_2, J_2 - K_1, J_3, K_3) \tag{9}$$

They also postulates that, if you conjugate any of these subalgebras by time reversion T and parity reversal P, you will get the entire Lorentz algebra and group. We proof this for the smallest of these subalgebras, $T(2)$, since it is a subalgebra of the other ones. Thus, if the theorem holds for this subalgebra, it will also hold for the other three subalgebras.

Proposition 4.1. *Conjugating $T(2)$ by T and P, we get the entire Lorentz algebra $\mathbb{L}$.*

$$\mathbb{L} = T(2) \oplus T \oplus P$$

Proof. To prove the proposition, we will use that a subalgebra is closed under commutation, i.e. the Lie bracket. We first need to know the action of T and P on $T(2)$. Since J_i leaves time invariant, we have $TJ_iT^{-1} = J_i$. However, J_i is not invariant under parity, since parity affects the space axes. Thus $PJ_iP^{-1} = -J_i$. Since the Lorentz contraction in K_i does include the time axis, we have $TK_iT^{-1} = -K_i$. The boosts are also invariant under parity, $PK_iP^{-1} = K_i$.
Thus, the actions of T and P on $T(2)$ are given as

$$T(J_1 + K_2)T^{-1} = J_1 - K_2, \qquad P(J_1 + K_2)P^{-1} = -J_1 + K_2$$
$$T(J_2 - K_1)T^{-1} = J_2 + K_1, \qquad P(J_2 - K_1)P^{-1} = -J_2 - K_1$$

Thus, if we include T and P in our algebra, the right hand side of these equations must be included. Since a subalgebra is also a vector space, any linear combination of the right hand sides with the generators of $T(2)$ must also be inside the subalgebra. In particular

$$\frac{1}{2}(J_1 + K_2) + \frac{1}{2}(J_1 - K_2) = J_1, \tag{10}$$

$$\frac{1}{2}(J_1 + K_2) - \frac{1}{2}(J_1 - K_2) = K_2, \tag{11}$$

$$\frac{1}{2}(J_2 - K_1) + \frac{1}{2}(J_2 + K_1) = J_2, \tag{12}$$

$$-\frac{1}{2}(J_2 - K_1) + \frac{1}{2}(J_2 + K_1) = K_1, \tag{13}$$

are all included in our subalgebra. Now, the subalgebra must be closed under commutation. Using the first and third equation, we thus get

$$[J_1, J_2] = J_3,$$

while the first and second relations yield us

$$[J_1, K_2] = K_3.$$

This last set of six equations has thus given us all six generators of the Lorentz algebra and thus the (connected) Lorentz group. Further conjugation with T and P will give us the rest of the Lorentz group [1] [3]. □

Now, in processes for which T and P symmetries hold, we can thus not distinguish between the full Lorentz group and any of the subgroups Cohen and Glashow suggest. However, for processes that violate T and P, a difference is possible and may be measurable. This is however not within the scope of this thesis, and thus I shall not go into this any further.
However, I will take a little detour to show another special property of their subalgebras: they are all solvable.

Definition 5 (Solvability of an algebra)**.** *Let for a Lie algebra g and $n \in \mathbb{N}$, $g_0 = g$ and $g_i = [g_{i-1}, g_{i-1}]$. Then this Lie algebra is called solvable if $\exists n \in \mathbb{N}$ s.t. $g_n = 0$, i.e. if g_n is the trivial subalgebra.*

We can also define the stronger property of nilpotency as follows:

Definition 6 (Nilpotence of an algebra)**.** *Let for a Lie algebra g and $n \in \mathbb{N}$, $h_0 = g$ and $h_i = [h_{i-1}, g]$. Then this Lie algebra is called nilpotent if $\exists n \in \mathbb{N}$ s.t. $h_n = 0$, i.e. if h_n is the trivial subalgebra.*

Theorem 4.2 (Solvability of Cohen and Glashow's algebras)**.** *The following Lorentz subalgebras are all solvable:*

$$\begin{aligned} T(2) &= \text{span}(J_1 + K_2, J_2 - K_1) \\ E(2) &= \text{span}(J_1 + K_2, J_2 - K_1, J_3) \\ HOM(2) &= \text{span}(J_1 + K_2, J_2 - K_1, K_3) \\ SIM(2) &= \text{span}(J_1 + K_2, J_2 - K_1, J_3, K_3) \end{aligned}$$

Now for our proof, I will restrict myself to the largest of subalgebras, $SIM(2)$, since the other algebras are contained therein. Thus, if it is solvable, so are all its subalgebras.

Proof. To show that $SIM(2)$ is solvable, we need to look at the Lie bracket operation on the algebra:

$$[J_1 + K_2, J_2 - K_1] = 0 \tag{14}$$
$$[J_1 + K_2, J_3] = -J_2 + K_1 \tag{15}$$
$$[J_2 - K_1, J_3] = J_1 + K_2 \tag{16}$$
$$[J_1 + K_2, K_3] = -K_2 - J_1 \tag{17}$$
$$[J_2 - K_1, K_3] = K_1 - J_2 \tag{18}$$

Thus, we can see that $g_1 =$ span$(J_1 + K_2, J_2 - K_1)$. The next step is to compute $g_2 = [g_1, g_1]$. We can see from equation (14) that $g_2 = 0$, and thus take $n = 2$ in our definition and find that $SIM(2)$ is solvable. □

From this proof, we can also see that $T(2)$ itself is actually nilpotent, since (14) is the commutator of the two basis elements in this algebra.

So why is it important that these algebras are solvable? Because they are linked to differential equations [9]. The algebras are vector spaces on a differentiable manifold [5], and thus can be expressed as differential operators. We can then look at the symmetries of such a differential equation F. If F is first order, then the symmetries are all differential operators V such that $[V, F] = 0$. If these symmetries form a solvable algebra of dimension m, then a partial differential equation F in n variables can be transformed into a equation in $n - m$ variables [9] [10].

It is also possible to do the reverse, and to find the differential equation from the algebra. For example, if one defines vectorfields $M_{i,j} = \epsilon_{i,j,k} J_k$ and $M_{0,i} = K_i$ anti-symmetrically, i.e. $M_{\mu,\nu} = -M_{\nu,\mu}$, then the set $\bar{M}_{\mu,\nu} = M_{\mu,\nu} + S_{\mu,\nu}$, where $S_{\mu,\nu} = 1/4[\gamma_\mu, \gamma_\nu]$ is the commutator of the gamma-matrices, are fields that leave the Dirac-equation invariant [10]. As such, they can be used to derive this equation.

5 Thomas precession

One of the classic tests for relativity is Thomas precession. In 1926 Thomas used relativity to solve a problem that appeared in atomic spectroscopy. The different band lines in hydrogen were twice as far apart as the theory predicted, however Thomas explained the difference between theory and practice by using a relativistic effect [11].

In this section, I will first derive this Thomas factor using the full Lorentz group. I will then focus our view on the Cohen and Glashow subgroups and try to see if any Thomas-like precession arises. I will look mainly at the smaller group generated by $T(2)$ and this group adjoined with K_3, i.e. $HOM(3)$.

5.1 Thomas' half

We start with normal Thomas precession. During the 1920's, when researchers such as Pauli and Landé were trying to explain the different frequency bands found in atomic spectroscopy, they realized that these bands were caused by magnetic fields [11]. In order to calculate the magnetic field at the electron however, instead of doing this from the lab frame, in which the electron was orbiting the core, they did the calculation from the electron rest frame. In this frame, the electron is at rest and the core is orbiting around it. Since the magnetic field at the electron is of interest, this makes the calculation easier. Researchers then directly applied the result they found in the electron frame to the lab frame [11]. This would be fine, if the electron was moving in an inertial frame. However, the electron is not in such a frame and is instead accelerated at each instant, so as to orbit the core. These constant accelerations lead to a relativistic effect and thus to a relativistic correction to the theory, known as Thomas' half [11].

Thomas conjectured that, although the electron is not in an inertial frame, the electron frame at each instant is an inertial frame and the electron's orbit comprises of many 'jumps' between these frames. There is one rule however, we are not entirely free in the choice of coordinate systems in these inertial frames, in order to provide a consistent theory [11]. The coordinate systems we are allowed to use were called proper axes by Thomas. These systems were related, in that the systems at $t = t_0$ and $t = t_0 + \Delta t$ are called *quasi-parallel*.

Definition 7. *Let I and I' be two inertial reference frames, and S and S' be coordinate systems on these frames. Then a vector $\boldsymbol{a}$ in I and a vector $\boldsymbol{a'}$ in I' are called quasi-parallel if $a_x : a_y : a_z = a'_x : a'_y : a'_z$.*

Definition 8. *Let I and I' be two inertial reference frames, and S and S' be coordinate systems on these frames. If in these frames, the basis vectors $\mathbf{e_x}$, $\mathbf{e_y}, \mathbf{e_z}$ are quasi-parallel to respectively $\mathbf{e}'_{\boldsymbol{x}}$, $\mathbf{e}'_{\boldsymbol{y}}$, $\mathbf{e}'_{\boldsymbol{z}}$, these coordinate systems are called quasi-parallel.*

Let's assume that we have two inertial frames I and I', with coordinate systems S and S'. If we look at the motion of the origin of S' from S, then for the velocity, we get the vector $\boldsymbol{w}_{O'}$. Vice versa, the velocity of the origin of S as measured in S' is given by $\boldsymbol{w}'_O$. If these velocity vectors satisfy

$$\boldsymbol{w}_{O'} = -\boldsymbol{w}_O, \tag{19}$$

then the systems S and S' are quasi-parallel and there is a special Lorentz transform, a pure boost, relating these two systems. If instead $|\boldsymbol{w}_{O'}| = |\boldsymbol{w}'_O|$, then one frame is rotated w.r.t the other one. I.e. $\exists \bar{S}$, a coordinate system in I, such that $\bar{S}$ and S' are quasi-parallel and in order to reach $\bar{S}$ from S, the coordinate axes need to be rotated by some angles about certain axis. I omit the proof of this for brevity, but the interested reader can find it in 'The history of spin' by Sin-itiro Tomonaga [11], page 190 and further.

We now proceed with our calculation of the Thomas factor. Let I be the lab inertial frame. A particle is moving through this frame with velocity v. We define our coordinate system S such that the particle is moving in the x direction at $t = 0$, with the particle located at the origin of S. The y and z directions are arbitrary, but orthogonal to x. We define the particle rest frame at this instant as I' with coordinate system S'. Its origin is located at the particle at $t = t' = 0$, and S and S' are thus related by a boost in the x direction. If we define the rapidity $\alpha = \tanh^{-1}(\frac{v}{c})$, then points in the different coordinate systems are related by [12], page 98, formula 3.27

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{20}$$

Throughout this text, instead of time t, I use $x_0 = c \cdot t$ to indicate the time axis. We now look into a third inertial frame I'', with coordinate system S''. It is related to S' by a boost in the y' direction with rapidity β. Thus the coordinate transformation from I' to I'' is given by [12]

$$\begin{pmatrix} x''_0 \\ x''_1 \\ x''_2 \\ x''_3 \end{pmatrix} = \begin{pmatrix} \cosh\beta & 0 & \sinh\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\beta & 0 & \cosh\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}. \tag{21}$$

If we now look at how the origin of S is moving in the S'' system, we can combine (20) and (21) to find

$$\begin{pmatrix} x_0'' \\ x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}_O'' = \begin{pmatrix} \cosh\beta & 0 & \sinh\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\beta & 0 & \cosh\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_O$$

$$= x_0 \begin{pmatrix} \cosh\alpha\cosh\beta \\ \sinh\alpha \\ \cosh\alpha\sinh\beta \\ 0 \end{pmatrix}_O'' = \frac{x_0''}{\cosh\alpha\cosh\beta} \begin{pmatrix} \cosh\alpha\cosh\beta \\ \sinh\alpha \\ \cosh\alpha\sinh\beta \\ 0 \end{pmatrix}_O'' = \begin{pmatrix} x_0'' \\ \frac{x_0''\tanh\alpha}{\cosh\beta} \\ x_0''\tanh\beta \\ 0 \end{pmatrix}_O'' .$$

We can equally compute the movement of the origin of S'', O'', as seen from S, by taking the inverses of (20) and (21). Thus we get

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}_{O''} = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cosh\beta & 0 & \sinh\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\beta & 0 & \cosh\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_0'' \\ 0 \\ 0 \\ 0 \end{pmatrix}_O$$

$$= x_0'' \begin{pmatrix} \cosh\alpha\cosh\beta \\ -\cosh\beta\sinh\alpha \\ -\sinh\beta \\ 0 \end{pmatrix}_{O''} = \frac{x_0}{\cosh\alpha\cosh\beta} \begin{pmatrix} \cosh\alpha\cosh\beta \\ -\cosh\beta\sinh\alpha \\ -\sinh\beta \\ 0 \end{pmatrix}_{O''} = \begin{pmatrix} x_0 \\ -x_0\tanh\alpha \\ -\frac{x_0\tanh\beta}{\cosh\alpha} \\ 0 \end{pmatrix}_{O''} .$$

Therefore the velocity of O, as observed in S'' is given by

$$\boldsymbol{w}_O'' = c \cdot \begin{pmatrix} \frac{\tanh\alpha}{\cosh\beta} \\ \tanh\beta \\ 0 \end{pmatrix} . \tag{22}$$

And the velocity of O'' is given by

$$\boldsymbol{w}_{O''} = c \cdot \begin{pmatrix} -\tanh\alpha \\ -\frac{\tanh\beta}{\cosh\alpha} \\ 0 \end{pmatrix} . \tag{23}$$

We now consider that the particle is at the origin of S'', as seen from the laboratory frame, at time $t = \Delta t$. Thus the system S'' is the rest system of the particle at $t = \Delta t$. Since the velocity of the particle was $(c * \tanh\alpha, 0, 0)$, we have a change in velocity of $(0, c * \frac{\tanh\beta}{\cosh\alpha}, 0)$. Now, the rapidity β is related to a velocity $\Delta u = c * \tanh\beta$ [11]. If we then take Δt infinitesimally small,

we have an acceleration $\boldsymbol{a} = (0, a, 0)$, with $a = \frac{1}{\cosh\alpha}\frac{\Delta u}{\Delta t} = \sqrt{1 - \frac{v^2}{c^2}}\frac{\Delta u}{\Delta t}$, using $\cosh\alpha = \gamma_\alpha = (\sqrt{1 - \frac{v^2}{c^2}})^{-1}$ [11] [12]. Also, since Δt is small, so is Δu and thus we can neglect $(\Delta v)^2$ to get $|\boldsymbol{w}_{O''}| = |\boldsymbol{w}''_O|$.
Since the transformation of S' to S'' was non-rotational, i.e. it was a proper boost, S'' has the proper coordinate axes for the particle [11]. The question now becomes, by how much have the axes of S'' rotated with respect to the axes of S? To compute this, we first note that there exists a coordinate frame $\bar{S}$ in I such that $\bar{S}$ and S'' are quasi-parallel. This relation can be seen in figure 2.

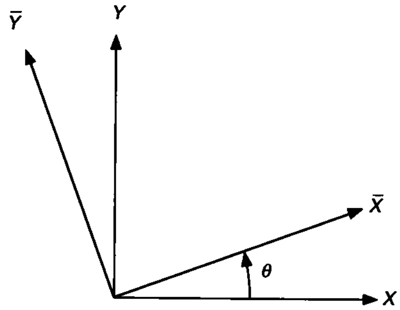

Figure 2: A diagram of the coordinate frames S and $\bar{S}$ [11].

In this system $\bar{S}$ the movement of O'' is given by

$$\bar{\boldsymbol{w}}_{O''} = -\boldsymbol{w}''_O = -c\begin{pmatrix} \frac{\tanh\alpha}{\cosh\beta} \\ \tanh\beta \\ 0 \end{pmatrix}. \tag{24}$$

We want the angle $\Delta\theta$ between these two velocity vectors, $\bar{\boldsymbol{w}}_{O''}$ and $\boldsymbol{w}_{O''}$. Since the z-axis is unchanged in these transformations, the rotation we are after is a rotation about this z-axis. We can express the velocity (24) in the components of (23) [11]:

$$\bar{w}_{O''_x} = w_{O''_x}\cos\Delta\theta + w_{O''_y}\sin\Delta\theta. \tag{25}$$

$$\bar{w}_{O''_y} = -w_{O''_x}\sin\Delta\theta + w_{O''_y}\cos\Delta\theta. \tag{26}$$

After some computation, which Tomonaga has done for us [11] on pages 196 to 199, we get the following equation for the angle between S and $\bar{S}$:

$$\Delta\theta = \frac{\Delta u}{v}(\sqrt{1 - \frac{v^2}{c^2}} - 1), \tag{27}$$

and an angular velocity of

$$\Omega = \frac{\Delta\theta}{\Delta t} = -(\gamma_\alpha - 1)\frac{a}{v}. \tag{28}$$

The electron travels at non-relativistic speeds inside the atom ($v \approx c/137$), thus we can take a Taylor expansion in order to simplify the above expression. Using

$$\gamma_\alpha = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + O(v^5) \tag{29}$$

and truncating it after the v^2 term, (28) reduces to

$$\Omega = -\frac{av}{2c^2}. \tag{30}$$

We have so far only looked at $t = 0$ and $t = \Delta t$, however, if we expand our view to $t = 2\Delta t, 3\Delta t, \ldots$, and assume that at each instant the particle gets accelerated by a, orthogonal to its current velocity, we get a precessing particle with frequency

$$\boldsymbol{\Omega}_T = -\frac{1}{2c^2}\boldsymbol{u} \times \boldsymbol{a}. \tag{31}$$

This then is the Thomas precession [11].
To now see where Thomas' half comes from, we consider a particle, with angular momentum S, precessing around a magnetic field. The torque of this electron is given by

$$S\boldsymbol{\Omega}_H = -M\boldsymbol{B}. \tag{32}$$

We split the magnetic moment M in a normal and abnormal part. The normal part is Dirac's theoretical value [11] and is given by

$$M_D = \frac{-e}{2mc} g_0 S.$$

In this equation, g_0 is the ratio between the magnetic moment and angular momentum. Also, the magnetic field comprises of an external magnetic field $\boldsymbol{B}_{ext}$ and an internal magnetic field, generated by the rotating electron, $\hat{\boldsymbol{B}}$. This is schematically drawn in figure 3.
Thus (32) becomes

$$S\boldsymbol{\Omega}_H = -M(\boldsymbol{B}_{ext} + \hat{\boldsymbol{B}}). \tag{33}$$

By now also going back to the lab frame, we observe a precession

$$S\boldsymbol{\Omega}_{lab} = -M(\boldsymbol{B}_{ext} + \hat{\boldsymbol{B}}) + S\boldsymbol{\Omega}_T. \tag{34}$$

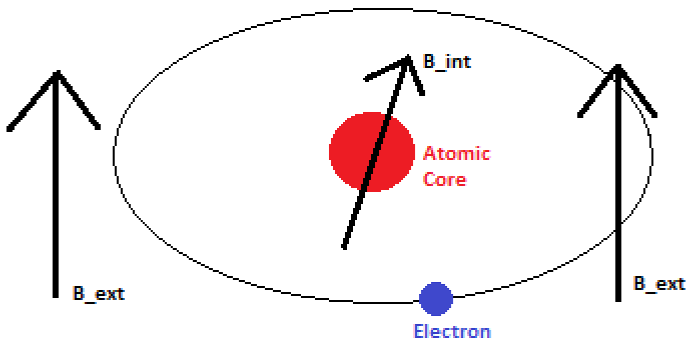

Figure 3: Diagram of the magnetic fields inside an atom. Present are the internal magnetic field B_int, generated by the rotating electron, and an external magnetic field B_ext, caused by outside influences (e.g. a big magnet).

Now, the particle is moving in an electric field, and therefore the acceleration is given by $\boldsymbol{a} = \frac{-e}{m}\boldsymbol{E}$, while the internal magnetic field is given by $\hat{\boldsymbol{B}} = \frac{1}{c}\boldsymbol{E}\times\boldsymbol{v}$. This allows us to rewrite the equation for the Thomas precession into

$$\boldsymbol{\Omega}_T = \frac{-e}{2mc}\hat{\boldsymbol{B}} = \frac{1}{2S}M_D\hat{\boldsymbol{B}}.$$

Substituting this into (34) then gives

$$S\boldsymbol{\Omega}_{lab} = -M\boldsymbol{B} - (M - \frac{1}{2}M_D)\hat{\boldsymbol{B}}. \tag{35}$$

For an electron, for which $M = M_D$, this gives

$$S\boldsymbol{\Omega}_{lab} = -M_D\boldsymbol{B} - \frac{1}{2}M_D\hat{\boldsymbol{B}}. \tag{36}$$

This then shows the Thomas factor, since the effect of the internal field is weighted by a factor 1/2 less than the effect of an external field.

5.2 The subgroup T(2)

In the last section we have seen that the combination of two non-collinear boosts leads to a rotation. In particular, the combination of a boost in the x-direction and one in the y-direction leads to a rotation about the z-axis. If we look at the commutator of the generators of these two boosts, we get $[K_x, K_y] = -J_z$, which is the generator for rotations about the z-axis. In this subgroup, we have elements of the form $\exp(\alpha(J_x + K_y) + \beta(J_y - K_x))$.

We are interested to see if first applying $\exp(\alpha(J_x + K_y))$ and then applying $\exp(\beta(J_y - K_x))$ gives another relativistic correction.
Again, we start in an inertial frame I, which we call the lab frame. We apply a coordinate system such that our particle is at the origin at $t = 0$ and such that the transformation $\exp(\alpha(J_x + K_y))$ transforms the system into the particle rest system at $t = 0$. In order to compute this transformation, we first need to know the matrix representations of these transformations. The generators J_x and J_y are given by [12]

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{37}$$

$$J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{38}$$

Meanwhile, the generators K_x and K_y are given by [12]

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{39}$$

$$K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

Thus the generators $J_x + K_y$ and $J_y - K_x$ are given by

$$J_x + K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{41}$$

$$J_y - K_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{42}$$

From this we can compute how the actual group elements look like:

$$\exp(\alpha(J_x+K_y)) = \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix}, \tag{43}$$

$$\exp(\beta(J_y-K_x)) = \begin{pmatrix} 1+\beta^2/2 & \beta & 0 & -\beta^2/2 \\ \beta & 1 & 0 & -\beta \\ 0 & 0 & 1 & 0 \\ \beta^2/2 & \beta & 0 & 1-\beta^2/2 \end{pmatrix}. \tag{44}$$

In these matrices α and β are two variables that determine the 'size' of the transformation. Knowing our transformation matrices, we can compute the transformation from I to I'. Again, we consider the motion of the origin and we are especially interested in its velocity.

$$\begin{aligned} \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix}_O' &= \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= x_0 \cdot \begin{pmatrix} 1+\alpha^2/2 \\ 0 \\ \alpha \\ \alpha^2/2 \end{pmatrix} = \frac{x_0'}{1+\alpha^2/2} \begin{pmatrix} 1+\alpha^2/2 \\ 0 \\ \alpha \\ \alpha^2/2 \end{pmatrix} = \begin{pmatrix} x_0' \\ 0 \\ \frac{x_0'\alpha}{1+\alpha^2/2} \\ \frac{-x_0'\alpha^2/2}{1+\alpha^2/2} \end{pmatrix}_O . \end{aligned}$$

Thus, the velocity of the origin of the lab frame measured in the particle frame is $\boldsymbol{w}_O' = \frac{1}{1+\alpha^2/2}(0, \alpha, \alpha^2/2)$. One thing we need to determine however, is whether S' is quasi parallel to S. To do this, we do the inverse calculation of what we just did, and calculate the velocity of the particle I' as seen from the lab frame I. We find $\boldsymbol{w}_{O'} = \frac{1}{1+\alpha^2/2}(0, -\alpha, \alpha^2/2) \neq -\boldsymbol{w}_O'$. Instead, the quasi parallel system seems to be rotated by π about the y-axis. Thus, in any further calculations, we need to take this rotation into account. Let us now consider a third inertial frame I'' with coordinate system S''. It is related to I' by a boost $\exp(\beta(J_y - K_x))$. The orbit of O as measured in S'' is thus given by the following equation:

$$\begin{pmatrix} x_0'' \\ x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}_O^{''} = \begin{pmatrix} 1+\beta^2/2 & \beta & 0 & -\beta^2/2 \\ \beta & 1 & 0 & -\beta \\ 0 & 0 & 1 & 0 \\ \beta^2/2 & \beta & 0 & 1-\beta^2/2 \end{pmatrix} \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_O$$

$$= x_0 \cdot \begin{pmatrix} 1+1/2(\alpha^2+\beta^2) \\ \beta \\ \alpha \\ 1/2(\alpha^2+\beta^2) \end{pmatrix} = \frac{x_0''}{1+1/2(\alpha^2+\beta^2)} \begin{pmatrix} 1+1/2(\alpha^2+\beta^2) \\ \beta \\ \alpha \\ 1/2(\alpha^2+\beta^2) \end{pmatrix}$$

This gives a velocity of $\boldsymbol{w}_O'' = \frac{1}{1+1/2(\alpha^2+\beta^2)}(\beta, \alpha, \alpha^2/2+\beta^2/2)$ for the orbit of the lab frame, as seen from the electrons frame. We now need to transform this into the quasi-parallel system $\bar{S}''$. To do this, we note that each rotation it self results in a sign change in the z-element of the velocity vector, since the other rotated element is zero. As such, the combined rotations result in the z-component changing sign twice, which has a net result of nothing. Thus, it turns out that $S'' = \bar{S}''$ is the proper coordinate system for our particle. We can also do the reverse, i.e. measure the velocity of O'', such as seen in S:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}_{O''} = \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix}^{-1} \begin{pmatrix} 1+\beta^2/2 & \beta & 0 & -\beta^2/2 \\ \beta & 1 & 0 & -\beta \\ 0 & 0 & 1 & 0 \\ \beta^2/2 & 0 & \beta & 1-\beta^2/2 \end{pmatrix}^{-1} \begin{pmatrix} x_0'' \\ 0 \\ 0 \\ 0 \end{pmatrix}_{O''}^{''}$$

$$= x_0'' * \begin{pmatrix} 1+1/2(\alpha^2+\beta^2) \\ -\beta \\ -\alpha \\ -1/2(\alpha^2+\beta^2) \end{pmatrix} = \frac{x_0}{1+1/2(\alpha^2+\beta^2)} \begin{pmatrix} 1+1/2(\alpha^2+\beta^2) \\ -\beta \\ -\alpha \\ -1/2(\alpha^2+\beta^2) \end{pmatrix}$$

We find as velocity vector $\boldsymbol{w}_{O''} = -\frac{1}{1+1/2(\alpha^2+\beta^2)}(\beta, \alpha, \alpha^2/2+\beta^2/2) = -\boldsymbol{w}_O''$! Thus, the systems S and S'' are already quasi-parallel. As such, these transformations do not lead to some kind of Thomas-factor. We could have expected this, since $[J_x + K_y, J_y - K_x] = 0$. This motivates us to look at the other Cohen and Glashow subgroups, since those do have non-vanishing commutators.

5.3 The subgroup HOM(2)

The subgroup $HOM(2)$ is the subgroup generated by $J_x + K_y$, $J_y - K_x$ and K_z. Thus, the only transformations are boosts in each of the cardinal directions. However, the boosts in the x and y-directions require an additional rotation, if the transformation is to be a symmetry. I will look in particular at transformations formed by combining $J_x + K_y$ with K_z. The commutator of these two is $[J_x + K_y, K_z] = -(K_y + J_x)$, which is one of the transformations we are using. There are two different methods to look at these transformations. First, the transformation between the lab frame I and the particle rest frame at $t = 0$ (I') is a pure boost in the z-direction and we then apply a transformation generated by $J_x + K_y$. The other transformation is the opposite order, i.e. the transformation between I and I' is of the form $J_x + K_y$, and we then apply a z-boost.

$\mathbf{K_z}$ followed by $\mathbf{J_x + K_y}$

We first set up our coordinate systems in our frames. Since we do not have pure boosts in the x and y direction any more, we relabel our axes so that initially the transformation between the particle's frame and the lab frame is still a pure, invariant, boost. The system S in the lab frame I is such that at $t = 0$ the particle is moving in the z-direction with velocity v and rapidity $\beta = \tanh^{-1}\left(\frac{v}{c}\right)$, with the x- and y-axes in arbitrary directions, though all orthogonal to each other. The S' system in the particle rest frame also has the particle at its origin at $t' = 0$, and its axes are parallel to those of S at $t = t' = 0$. The boost between S and S' is thus given by [12]

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{45}$$

We then apply the transformation $\exp(\beta(J_x + K_y))$ from (43). Therefore, the origin O of S has an orbit given by

$$\begin{aligned}
\begin{pmatrix} x''_0 \\ x''_1 \\ x''_2 \\ x''_3 \end{pmatrix}''_O &= \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_O \\
&= x_0 \begin{pmatrix} 1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta \\ 0 \\ \alpha(\cosh\beta - \sinh\beta) \\ 1/2\alpha^2(\cosh\beta - \sinh\beta) + \sinh\beta \end{pmatrix} \\
&= \frac{x''_0}{1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta} \begin{pmatrix} 1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta \\ 0 \\ \alpha(\cosh\beta - \sinh\beta) \\ 1/2\alpha^2(\cosh\beta - \sinh\beta) + \sinh\beta \end{pmatrix}.
\end{aligned}$$

Thus the velocity of O, as seen by an observer in S'' is

$$\boldsymbol{w}''_O = \frac{1}{1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta}(0, \alpha(\cosh\beta - \sinh\beta), 1/2\alpha^2(\cosh\beta - \sinh\beta) + \sinh\beta).$$

Applying the correction to reach a quasi parallel system $\bar{S}''$ gives

$$\bar{\boldsymbol{w}}''_O = \frac{1}{1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta}(0, \alpha(\cosh\beta - \sinh\beta), -1/2\alpha^2(\cosh\beta - \sinh\beta) - \sinh\beta).$$

Proceeding as before, we next calculate the orbit of O'' as seen from S:

$$\begin{aligned}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}_{O''} &= \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix}^{-1} \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix}^{-1} \begin{pmatrix} x''_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}''_{O''} \\
&= x''_0 \begin{pmatrix} 1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta \\ 0 \\ -\alpha \\ 1/2\alpha^2(\cosh\beta - \sinh\beta) - \sinh\beta \end{pmatrix} \\
&= \frac{x_0}{1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta} \begin{pmatrix} 1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta \\ 0 \\ -\alpha \\ 1/2\alpha^2(\cosh\beta - \sinh\beta) - \sinh\beta \end{pmatrix}.
\end{aligned}$$

Again, we calculate the velocity of O'', as seen by S, as

$$\boldsymbol{w}_{O''} = \frac{1}{1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta}(0, -\alpha, 1/2\alpha^2(\cosh\beta - \sinh\beta) - \sinh\beta).$$

Observe that $\boldsymbol{w}_{O''} \neq -\bar{\boldsymbol{w}}''_O$. We next introduce the coordinate system $\bar{S}$ in I, which is quasi-parallel to $\bar{S}''$. If we are to compare the angle between $\boldsymbol{w}_{O''}$ and $\bar{\boldsymbol{w}}_{O''} = -\bar{\boldsymbol{w}}''_O$, then we do not need the scaling factor of $(1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta)^{-1}$, and I will thus neglect it in the future. Some further calculations also show that $|\boldsymbol{w}_{O''}| = |\bar{\boldsymbol{w}}''_O|$. The angle between the vectors is given by [11]

$$\cos\theta = \frac{w_y * \bar{w}_y + w_z * \bar{w}_z}{|\boldsymbol{w}_{O''}|^2} \tag{46}$$

$$\sin\theta = \frac{\bar{w}_y * w_z - \bar{w}_z * w_y}{|\boldsymbol{w}_{O''}|^2} \tag{47}$$

For reference sake, I repeat the two vectors we are comparing, without the scaling factors:

$$\begin{aligned}\boldsymbol{w}_{O''} &= (0, -\alpha, \alpha^2/2(\cosh\beta - \sinh\beta) - \sinh\beta)\\ \bar{\boldsymbol{w}}_{O''} = -\bar{\boldsymbol{w}}''_O &= (0, -\alpha(\cosh\beta - \sinh\beta), \alpha^2/2(\cosh\beta - \sinh\beta) + \sinh\beta)\end{aligned}$$

We will now interpret coordinate system S'' as the particle rest system at time $t = \Delta t$, with Δt infinitesimally small. Then the angle θ must also be small and $\sin\theta \approx \theta$. Thus performing the calculation in (47) gives us

$$\theta = \frac{2\alpha(e^\beta - 1)}{\alpha^2 + (e^\beta - 1)^2}. \tag{48}$$

Since Δt is small, α is also small and thus we can look at the Taylor expansion of (48) and truncate it after terms of order α^2 to get

$$\theta = \frac{2\alpha}{e^\beta - 1} - \frac{2\alpha^3}{(e^\beta - 1)^3} + O(\alpha^4) \approx \frac{2\alpha}{e^\beta - 1}. \tag{49}$$

Now the original velocity, before the second transformation, was $\boldsymbol{w}_{O'} = (0, 0, -\tanh\beta)$. The new velocity is

$$\begin{aligned}\boldsymbol{w}_{O''} &= \frac{1}{1/2\alpha^2(\cosh\beta - \sinh\beta) + \cosh\beta}(0, -\alpha, 1/2\alpha^2(\cosh\beta - \sinh\beta) - \sinh\beta)\\ &\approx (0, -\frac{\alpha}{\cosh\beta}, -\tanh\beta).\end{aligned}$$

In this last equation, I have again neglected terms of the order α^2. Thus the change in velocity is $(0, \frac{-\alpha}{\cosh\beta}, 0)$. We now look at the variable α. This parameter arises in $\exp(\alpha(J_x + K_y))$ and thus we have $\alpha = \tanh^{-1}(\frac{\Delta u}{c})$ for some change in velocity Δu in the y-direction. For small α, as we are dealing with, we can reduce this to $\alpha \approx \frac{\Delta u}{c}$ using the Taylor expansion of tanh. Thus, we have an acceleration $\frac{a}{c} = \frac{-1}{\cosh\beta}\frac{\alpha}{\Delta t}$, or in vector form $\boldsymbol{a} = (0, a, 0)$. We can therefore rewrite (49) in terms of this acceleration as

$$\theta = -\frac{a\cosh\beta\Delta t}{c(e^\beta - 1)},$$

and calculate the angular velocity as

$$\Omega = \frac{\theta}{\Delta t} = -\frac{a\cosh\beta}{c(e^\beta - 1)}. \tag{50}$$

Since $\beta = \tanh(v/c)^{-1}$, this equation can be further expressed as

$$\Omega = -\frac{a}{c}\frac{1}{a + v/c - \sqrt{1-(v/c)^2}}. \tag{51}$$

Since we are working at non-relativistic speeds ($v \approx c/137$), this last part can be simplified, using a Taylor expansion at $v/c \ll 1$:

$$\Omega = -\frac{a}{c}\left(\frac{c}{v} - \frac{1}{2} + \frac{v}{4c} - \frac{v^2}{4c^2} + \frac{3v^3}{16c^3} + O(v^4)\right) \approx -\frac{a}{v} + \frac{a}{2c} - \frac{av}{4c^2} + \frac{av^2}{4c^3}. \tag{52}$$

If we were to apply our transformation at each instant $t = \Delta t, 2\Delta t, 3\Delta t, \ldots,$ ensuring that at each instant the acceleration is orthogonal to the velocity $\boldsymbol{v}$, we achieve an angular velocity of

$$\boldsymbol{\Omega} \approx -\boldsymbol{v} \times \boldsymbol{a}\left(\frac{1}{v^2} - \frac{1}{2cv} + \frac{1}{4c^2} - \frac{v}{4c^3}\right) \tag{53}$$

However, this is not the only rotation happening. Our transformation itself is rotational, and thus induces an additional rotation about the x-axis. The amount of rotation in radians is the angle α [12]. This angle α is related to the acceleration by $\frac{\alpha}{\Delta t} = -\frac{a}{c}\cosh\beta$. We can turn this into an angular velocity by

$$\Omega_\alpha = -\frac{a\cosh\beta}{c} = \frac{-a}{c\sqrt{1-(v/c)^2}}. \tag{54}$$

Again, we take the Taylor expansion and ignore terms after v^2, since v is non-relativistic:

$$\Omega_\alpha = -\frac{a}{c}\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right). \tag{55}$$

In vector form:

$$\boldsymbol{\Omega}_\alpha = -\boldsymbol{v} \times \boldsymbol{a}\left(\frac{1}{cv} + \frac{v}{2c^3}\right). \tag{56}$$

Our last task is then to simply sum these two angular velocities to get the total (approximate) velocity.

$$\boldsymbol{\Omega}_{combined} \approx -\boldsymbol{v} \times \boldsymbol{a}\left(\frac{1}{v^2} + \frac{1}{2cv} + \frac{1}{4c^2} + \frac{v}{2c^3}\right) \tag{57}$$

When we compare this to (31), we see that we have a quite different expression. This difference could be explained by looking at the cause of the second transformation. In normal Thomas precession, the second transformation is another pure boost caused by the Coulomb force, while in this case, the second transformation is itself partially rotational. As such, the observed rotation is different.

$\mathbf{J_x + K_y}$ followed by $\mathbf{K_z}$

The recipe is clear by now, we calculate the orbits of the origins as viewed from the different coordinate systems, and try to calculate the angle between the proper coordinate systems. Thus, I shall proceed with speed. The orbit of O in S'' is given by:

$$\begin{aligned}
\begin{pmatrix} x_0'' \\ x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}''_O &= \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_O \\
&= x_0 \begin{pmatrix} \alpha^2/2(\cosh\beta+\sinh\beta)+\cosh\beta \\ 0 \\ \alpha \\ \alpha^2/2(\cosh\beta+\sinh\beta)+\sinh\beta \end{pmatrix}''_O \\
&= \frac{x_0''}{\alpha^2/2(\cosh\beta+\sinh\beta)+\cosh\beta} \begin{pmatrix} \alpha^2/2(\cosh\beta+\sinh\beta)+\cosh\beta \\ 0 \\ \alpha \\ \alpha^2/2(\cosh\beta+\sinh\beta)+\sinh\beta \end{pmatrix}''_O,
\end{aligned}$$

while the orbit of O'' in S is given by:

$$\begin{aligned}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}_{O''} &= \begin{pmatrix} 1+\alpha^2/2 & 0 & \alpha & -\alpha^2/2 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2/2 & 0 & \alpha & 1-\alpha^2/2 \end{pmatrix}^{-1} \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix}^{-1} \begin{pmatrix} x_0'' \\ 0 \\ 0 \\ 0 \end{pmatrix}''_{O''} \\
&= x_0'' \begin{pmatrix} \alpha^2/2(\cosh\beta+\sinh\beta)+\cosh\beta \\ 0 \\ -\frac{\alpha}{\cosh\beta-\sinh\beta} \\ \alpha^2/2(\cosh\beta+\sinh\beta)-\sinh\beta \end{pmatrix}''_{O''} \\
&= \frac{x_0}{\alpha^2/2(\cosh\beta+\sinh\beta)+\cosh\beta} \begin{pmatrix} \alpha^2/2(\cosh\beta+\sinh\beta)+\cosh\beta \\ 0 \\ -\frac{\alpha}{\cosh\beta-\sinh\beta} \\ \alpha^2/2(\cosh\beta+\sinh\beta)-\sinh\beta \end{pmatrix}''_{O''}.
\end{aligned}$$

Moving on, if we set $d = (\alpha^2/2(\cosh\beta+\sinh\beta)+\cosh\beta)^{-1}$, we can give the velocities of interest as $\boldsymbol{w}''_O = d(0, \alpha, \alpha^2/2(\cosh\beta+\sinh\beta)+\sinh\beta)$ and $\boldsymbol{w}_{O''} = d(0, -\frac{\alpha}{\cosh\beta-\sinh\beta}, \alpha^2/2(\cosh\beta+\sinh\beta)-\sinh\beta)$. Since S' is the proper coordinate system at $t = t' = 0$ (we defined it as such), and the transformation between S' and S'' is a pure boost, these two systems are quasi-parallel and thus at $t = \Delta t$ S'' is the proper coordinate system. However, both S' and S'' are rotated w.r.t. the lab frame, i.e. there is an initial rotation. Taking this into account, we find that we are to compare the following velocities:

$$\bar{w}_{O''} = d(0, -\frac{\alpha}{\cosh\beta - \sinh\beta}, -\alpha^2/2(\cosh\beta + \sinh\beta) + \sinh\beta)$$
$$\hat{\bar{w}}_{O''} = -\boldsymbol{w}_O'' = d(0, -\alpha, -\alpha^2/2(\cosh\beta + \sinh\beta) - \sinh\beta)$$

We calculate the angle (47), again using that the angle is small and thus $\theta \approx \sin\theta$.

$$\theta = -\frac{2\alpha(e^\beta - 1)}{\alpha^2 \sinh\beta + (\alpha^2 + 2)\cosh\beta - 2}. \tag{58}$$

We then relate β to a small change in velocity, $\tanh\beta = \frac{\Delta u}{c}$, to expand (58) to

$$\theta = \frac{2\alpha\left(\sqrt{1 - (\frac{\Delta u}{c})^2} - \frac{\Delta u}{c} - 1\right)}{\alpha^2(\frac{\Delta u}{c} + 1) - 2\left(\sqrt{1 - (\frac{\Delta u}{c})^2}\right) + 2}. \tag{59}$$

We can reduce this equation to something simpler by taking the Taylor expansion and considering that $\frac{\Delta u}{c} \ll 1$, and thus ignore terms of order $(\Delta u)^2$ or of larger order. We then get (60).

$$\theta = -\frac{2\Delta u}{\alpha c} + \frac{(\Delta u)^2}{\alpha c^2} + O((\Delta u)^3) \approx -\frac{2\Delta u}{\alpha c} \tag{60}$$

From this we can calculate an angular velocity as

$$\Omega = \frac{\theta}{\Delta t} = -\frac{2}{\alpha c}\frac{\Delta u}{\Delta t}. \tag{61}$$

We look at the change in velocity. The original velocity of O', as viewed from S, was $\boldsymbol{w}_{O'} = \frac{1}{1+\alpha^2/2}(0, -\alpha, \alpha^2/2)$. After the boost, this changed to $\boldsymbol{w}_{O''} = d(0, -\frac{\alpha}{\cosh\beta - \sinh\beta}, \alpha^2/2(\cosh\beta + \sinh\beta) - \sinh\beta)$. Since the boost was small, $\beta \ll 1$, we can take a Taylor expansion and ignore terms of order greater than 2, to get $\boldsymbol{w}_{O''} = (0, \frac{-\alpha}{1+\alpha^2/2} + \frac{\alpha\beta}{(1+\alpha^2/2)^2}, \frac{\alpha^2}{2+\alpha^2} + \frac{2\beta(\alpha^4-2)}{(\alpha^2+2)^2})$. This constitutes a change of velocity of $\Delta\boldsymbol{w} = (0, \frac{\alpha\beta}{(1+\alpha^2/2)^2}, \frac{2\beta(\alpha^4-2)}{(\alpha^2+2)^2})$. If we divide the size of this change by the difference in time, and then do $\Delta t \to 0$, we get $a = c\sqrt{|\Delta\boldsymbol{w}|^2} = \frac{\beta\sqrt{\alpha^8 - 4\alpha^4 + 4\alpha^2 + 4}}{(1+\alpha^2/2)\Delta t} \approx \frac{\Delta u\sqrt{\alpha^8 - 4\alpha^4 + 4\alpha^2 + 4}}{(1+\alpha^2/2)\Delta t}$. We can use this in (61):

$$\frac{\Delta u}{\Delta t} = a\frac{(1+\alpha^2/2)}{\sqrt{\alpha^8 - 4\alpha^4 + 4\alpha^2 + 4}}.$$

$$\Omega = -\frac{2}{\alpha c}\frac{\Delta u}{\Delta t} = -\frac{2a(1+\alpha^2/2)}{\alpha c\sqrt{\alpha^8 - 4\alpha^4 + 4\alpha^2 + 4}} \tag{62}$$

We can then take take the Taylor expansion of this term for α, to achieve

$$\Omega = -\frac{2a}{c}\left(\frac{1}{2\alpha} + \frac{5\alpha^3}{16} - \frac{5\alpha^5}{16} + O(\alpha^6)\right) \approx -\frac{a}{v} - \frac{5av^3}{8c^4}, \tag{63}$$

in which we have used that $\alpha = \tanh^{-1}(v/c) \approx v/c$ for small (non-relativistic) velocities v. In this case, the difference between this precession and normal, Thomas precession is due to the initial orientation of our lab frame with respect to the frame of the particle. As such, this is the effect that would be observed, if the atom is in a certain alignment with respect to the lab.

Summary of results

Since these two subgroups $E(2)$ and $SIM(2)$ only add additional rotations to the other subgroups, I have not considered these in my calculations, since the classic Thomas effect arises from the combination of two non-colinear boosts [11].

Now, in summary, we found the following precession frequencies:

- For pure Thomas precession
$$\Omega_T = -\frac{av}{2c^2}.$$
- For the subgroup $T(2)$, it was proven that no Thomas-like precession exists.
- For the subgroup $HOM(2)$, two distinct precessions were found. If the infinitesimal, second boost was $J_x + K_y$, then
$$\Omega_{comb} = -\frac{a}{v} - \frac{a}{2c} - \frac{av}{4c^2} - \frac{av^2}{2c^3}.$$
- If instead the boost K_z is applied infinitesimally, the resulting precession frequency is given by
$$\Omega_K = -\frac{a}{v} - \frac{5av^3}{8c^4}.$$

These results are all quite different, and indicate that there should be a measurable difference between the different (sub)groups of the Lorentz group, provided that this subgroup is the real symmetry group of Nature. Especially in the case of $HOM(2)$, the order of operations matter.

At this point perhaps, it is worth to look at the physical processes that take place. In case of the electron orbiting an atom's nucleus, the infinitesimal transformation is due to the Coulomb interaction. This interaction causes a pure acceleration, and not a rotation. The fact that the Coulomb interaction acts continuously, is what causes the rotation of the electron around the core. The fact that the lab-frame itself does not rotate, is then what causes the perceived Thomas precession. As such, when confined to $HOM(2)$, it is the frequency Ω_K, that belongs to K_z being applied last, that is most likely to be observed.

6 Conclusion

I have looked at the Lorentz Group and derived its subgroup structure. A total of 15 subgroups were found, including the trivial subgroup, containing only the identity, and the complete group. I then looked at specific subgroups, in particular I looked at those subgroups that included the subgroup generated by $h_4^N = T(2)$. I found that the generating algebras were solvable. I also derived the frequency of Thomas precession, if restricted to these subgroups. I found that the angular frequency is quite different in these subgroups, due to the fact that at least one of the transformations must include a rotation. Comparison of the classical Thomas precession frequency and the one that is most likely to occur, Ω_K, shows that while both are linear in the acceleration, they both behave very differently when the velocity of the electron is changed, as long as the velocity is non-relativistic. As such, a further (experimental) study of the Thomas precession could lead to a difference between the observed and theoretical precession frequency and thus a hint to which subgroup is the symmetry group of nature.

7 Acknowledgements

I would like to thank dr. A.V. Kiselev and R.G.E. Timmermans from the University of Groningen for supervising me throughout this thesis. They were available for questions and were very helpful in spotting errors and shortcomings in my reasoning.

Special thanks goes to dr. A. Nikitin from the Institute of Mathematics, National Academy of Sciences of Ukraine for pointing me towards some useful references and for willing to discuss my thesis with him. His explanation as to the importance of solvable algebras was especially helpful.

References

[1] Andrew G. Cohen and Sheldon L. Glashow. Very special relativity. *Phys. Rev. Lett.*, 97:021601, Jul 2006.

[2] M. Toller. Homogeneous Spaces of the Lorentz Group. *ArXiv Mathematical Physics e-prints*, January 2003.

[3] J.J. de Swart. Lecture notes. Lecture Notes about the Lorentz Group, given to me by dr. Timmermans.

[4] Daniël Boer. Symmetry in physics. 2013-2014.

[5] Sigurdur Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. Academic Press.

[6] G.S. Hall. *Symmetries and geometry in general relativity*, volume 1. World Scientific Publishing Co. Pte. Ltd., 1991.

[7] Nobelprize.org. The Nobel prize in physics 1957, 2013. Retrieved 2014-06-17.

[8] J. Patera, P. Winternitz, and H. Zassenhaus. Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group. *Journal of Mathematical Physics*, 16(8), 1975.

[9] A.V. Bocharov, V.N. Chetverikov, and S.V. et al. Duzhin. *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*. AMS, 1999.

[10] A.G. Nikitin. Private conversation. Date 12th July 2014.

[11] Sin-itiro Tomonaga and Takeshi Oka. *The story of spin*. The University of Chicago Press, 1997.

[12] Ben Bakker and Daniël Boer. Group theory in phyics. Lecture notes from a master course at the University of Amsterdam, 2010.