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Three-valued modal logic and the problem of future Contingents

Bachelorthesis Mathematics

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1 Introduction

Consider the statement 'It will rain tomorrow'. Is it true now? Or false? And if it is already true now, is it then possible that is does not rain tomorrow, or has the fact that it will rain tomorrow become a necessity? And if it is necessary, why do we perceive it as if it is not, as if it just as easily could not have rained tomorrow? And if we decide it is not true nor false now, is 'it will rain tomorrow or not' then still true, or is that also neither true nor false?

These questions were raised by Aristotle in the famous chapter 9 of his work $\Pi\varepsilon\rho\grave{\iota}$ $`E\rho\mu\eta\nu\varepsilon\acute{\iota}\alpha\varsigma$, De Interpretatione in Latin, and have come to be known as the problem of future contingents. In this thesis we will try to find a modern logic that can deal with this problem.

There are many possible solutions to the problem of future contingents. For example the idea of branching-time semantics has been employed to formalize the contingency of the future. In this thesis, however, we will consider three-valued modal logics as possible candidates to offer a solution. Three-valued logics are promising because they create the possibility of a truth-value gap, the possibility that a statement about a future contingent event is now neither true nor false. The modal features are desirable because they can give a precise meaning to the terms possible and necessary, and both these terms are often used when speaking about the future.

In this thesis some philosophical considerations play an important role. For example, the thesis of determinism is considered undesirable. While there is no logical objection against creating a deterministic system for time, in this thesis we hold that any logic of time should be indeterministic. The reason for this is that we believe that free will and determinism are mutually exclusive, and that we believe we have free will. For more information on the debate considering free will and determinism, which is closely connected to the problem of future contingents, see Kane (2005). Furthermore the law of excluded middle plays an important role. Again there is no logical motivation as to why this law should hold, it is just our basic intuition about logic and time.

In the first section Aristotle's text will be discussed and we derive criteria any solution should meet. In the next section we will consider the system Ł, named after its inventor Łukasiewicz. Ł is specifically designed to deal with the problem of future contingents and stays very close to the solution proposed by Aristotle. In section 4 the system Q, designed by A. N. Prior, is discussed. Q was initially created to deal with the logic of contingent beings, and so it is also interesting to apply it to the problem of future contingents. Finally, Q_t will be discussed. Q_t is a temporal extension of Q and thus seems an ideal candidate to solve the problem of future contingents.

Finally it should be noted that in the literature $1/2$, -1, 2, u and i have all been used to denote the third truth-value. In this thesis we will always use $1/2$, to create some consistency in notation.

2 Formulating the problem of future contingents

2.1 Aristotle's problem

The problem of future contingents was first raised by Aristotle. In his work *De Interpretatione* Aristotle deals with the relation between language and logic. In chapter 9 statements about future contingent events are discussed. All of the quotes below are from this chapter; the translation used is that of J. L. Ackrill (1963).

Aristotle starts his discussion with noting that every statement about what is or what has been must be either true or false. However, with statements about the future it is different.

> 'With regard to what is and what has been it is necessary for the affirmation or the negation to be true or false. And with universals taken universally it is always necessary for one to be true and the other false, and with particulars too, as we have said; but with universals not spoken of universally it is not necessary. But with particulars that are going to be it is different.' **18a28-33**

Statements about the future need not be either true or false. The reason for this is that if statements about the future would be already true or false, determinism would follow. Aristotle argues:

> 'Again, if it is white now it was true to say earlier that it would be white; so that it was always true to say of anything that has happened that it would be so. But if it was always true to say it was so, or would be so, it could not not be so, or not be going to be so. But if something cannot not happen it is impossible for it not to happen; and if it is impossible for something not to happen, it is necessary for it to happen. Everything that will be, therefore, happens necessarily.' **18b9-15**

His argument is thus that if it were true beforehand that something would happen it is necessary that it happens, for if it would not happen, it would not have been true to say that it would happen. If it were true yesterday that it would rain right now, it is a clear contradiction if it does not rain at this moment.

Furthermore, Aristotle asserts that not everything happens out of necessity. Some things happen as chance has it, he says, and we seem to be able to shape the future by our actions. Most of us believe that we have free will, and that choices we make can change the future. Aristotle shares this belief and he says about the conclusion that everything happens out of necessity:

> 'But what if this is impossible? For we see that what will be has an origin both in deliberation and in action, and that, in general, in things that are not always actual there is the possibility of being and not being.' **19a7-10**
> 'Clearly, therefore, not everything is or happens of necessity; some things happen as chance has it, and of the affirmation and the negation neither is true rather than the other.' **19a18-21**

Aristotle does not accept his own conclusion that everything happens out of necessity, and therefore he must be looking for a fault in the argument. This fault is the premise that statements about the future are already true or false. Aristotle denies this premise, by saying that statements about the future are neither true nor false.

Now Aristotle has asserted that statements about the future are neither true nor false, another problem arises. Sentences like 'tomorrow it will rain or not rain' seem definitely true. Therefore, necessarily tomorrow it will rain or not. This, however, does not mean that it will necessarily rain or necessarily not rain. When talking about the future, the disjunction of contradictory statements should be true, while each disjunct is neither true nor false.

> 'And the same account holds for contradictories: everything necessarily is or is not, and will be or will not be; but one cannot divide and say one or the other is necessary.' **19a29-31**

This last problem is the core of the problem of future contingents. Intuitively Aristotle is right when he says that it will necessary rain or not tomorrow, but that it is not necessary that it will rain and neither it is necessary that it will not rain. When formalizing this, it is not difficult to construct a logic in which statements about the future are neither true nor false. However, it is very difficult to ensure that $p \vee \neg p$ is always true, also when p is about the future.

2.2 Criteria any solutions should meet

In the above discussion of Aristotles text we came upon several principles well known in Logic.

The first is **the principle of bivalence**, saying that everything must be true or false. As we have seen, Aristotle does not think the principle of bivalence to be true, since statements about the future are neither true nor false. Logics in which the principle of bivalence does not hold are multivalued logics; there are more possibilities in truth-values then merely true or false.

The next principle we encountered is **the law of excluded middle**. This is the idea that every statement of the form $p \vee \neg p$ must be true. As we have seen, Aristotle does believe the law of excluded middle to hold, even in cases were p is a proposition about a future contingent event.

The difference between the law of excluded middle and the principle of bivalence is a subtle one and at first they may seem equivalent. If every proposition must be either true or false, then the conjunction $p \vee \neg p$, which says that either p must be true or not-p must be true, seems an obvious consequence. Furthermore, if we know that $p \vee \neg p$ is true, than we may conclude that p is true or not-p is true, which seems the same as saying that p is either true or false.

The reason that the principle of bivalence and the law of excluded middle are treated differently by Aristotle lies in the way Aristotle interprets truth. In modern logic, a proposition must be a singular statement, like 'it is raining on the first of July 2014'. For Aristotle, however, the sentence 'it is raining now' also represented a proper proposition. The truth-value of the latter proposition depends on the time at which it is uttered. It might be true today that it is raining now, but yesterday afternoon it was false to say it was raining now.

Since propositions are not fixed in time, it is thus possible that the truth-value of such a proposition varies. To Aristotle this meant that the proposition was not really true, for a true proposition should be always true. Therefore we can interpret Aristotle's truth as omni-temporal truth.

This explains the difference between the principle of bivalence and the law of excluded middle. To start with the latter, the law of excluded middle represents a omni-temporal truth. Independent of when it is said, it is always true that is raining now or it is not raining now. This, however, does not mean that it is true that it is raining now, or that it is not true that it is raining now. If truth means omni-temporal truth, this would mean that it is always raining or never raining, and it is clear that this is not the case. Therefore, the principle of bivalence should be rejected. Even though $p \vee \neg p$ is always true, this does not mean that p is always true, and neither does it mean that $\neg p$ is always true.

In their paper *Three-valued temporal logic Q_t and future contingtents* Akama, Nagata and Yamada say about this difference:

> 'By the law of excluded middle we mean the syntactic thesis of the form $A \vee \neg A$. The principle of bivalence is the semantic thesis that every proposition is either true or false. If the principle of bivalence holds, then these theses are equivalent.' (Akama et al., 2008, p. 216)

Finally, we have encountered **the principle of antecedent truth**. Unlike the last two this is not a well known principle but a term introduced by G. Ryle (1953, p. 16) to summarize the core of Aristotle's argument for determinism. The principle of antecedent truth is the idea that if something is true now, it was always true beforehand that it would be true. As Aristotle said it: 'If it is white now it was true to say earlier that it would be white'. Aristotle continues this argument to show that everything happens out of necessity. Since this is an unwanted conclusion, the principle of antecedent truth should not hold.

To summarize, we have thus encountered three criteria any logic attempting to solve the problem of future contingents should meet. The principle of bivalence and the principle of antecedent truth should not hold, since they entail determinism, but the law of excluded middle should hold. The logical systems that will be discussed in the following sections will be measured by these criteria.

3 Łukasiewicz' three-valued system

3.1 The system

The first three-valued system we will consider is from Łukasiewicz (1970). Łukasiewicz developed his three-valued logic in 1920 and he says that he constructed it in such a way that it deviates least from two-valued logic. His motivation for developing a three-valued system was the problem of future contingents. Therefore, the third truth-value is a truth-value gap, making it possible that statements are neither true nor false.

The functions for $\neg$ and $\rightarrow$ are defined by the following tables:

p	$\neg p$
1	0
$1/2$	$1/2$
0	1

$p \rightarrow q$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	1	1	$1/2$
0	1	1	1

$p \vee q$ and $p \wedge q$ are defined from the above functions as:

Definition 3.1
$p \vee q := (p \rightarrow q) \rightarrow q)$
$p \wedge q := \neg(\neg p \vee \neg q)$

Which results in the following tables:

$p \vee q$	1	$1/2$	0
1	1	1	1
$1/2$	1	$1/2$	$1/2$
0	1	$1/2$	0

$p \wedge q$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	0
0	0	0	0

Łukasiewicz mentions we should interpret the third truth-value as 'possible'. Furthermore he notes that some laws of Aristotelian Logic are only possible in this three-valued system. Especially the law of excluded middle is not a law in this system, but merely a possibility.

Łukasiewicz only formulated the three-valued system as given above. In his paper *Three-valued logic and future contingents* (1953), Prior added modal functions to this system. He defines Mp, it is possible that p, and Sp, it is necessary that p as follows:

Definition 3.2
$Mp := \neg p \rightarrow p$
$Sp := \neg M \neg p$

Prior motivates his definition for Mp by noting that $\neg p \rightarrow p$ is true when $\neg p$ is no closer to the truth than p is. This means that at least we know that p is not false, and therefore it seems likely to say that p is possible. The definition for necessary is the usual saying that something is necessary when it is impossible that it is false.

These definitions give rise to the following tables:

p	Mp
1	1
1/2	1
0	0

p	Sp
1	1
1/2	0
0	0

As we can see this means that whenever p is true, p is necessary, and whenever p is false, p is impossible. This seems strange. Is it really so that if it is false today to say that it is raining, that it is impossible that it might ever be raining?

Again the solution to this oddity lies in the way we interpret truth. If we, like Aristotle, take truth to mean omni-temporal truth, these meanings for Mp and Sp become much clearer. If it is false to say that it is raining, this means that it is always false to say that it is raining, i.e. it never rains. If this is the case, it seems quite right to say that it is impossible that it might be raining. Furthermore, if it is true to say that it is raining, this means that it is always raining. In this case it seems correct to say it is necessarily true that it is raining.

As we can see Łukasiewicz stays very close to Aristotle's idea's. The third truth-value means that something is neither true nor false, exactly as Aristotle's proposed. Furthermore Aristotle's idea that truth is omni-temporal is also preserved in Łukasiewicz' system.

3.2 Łukasiewicz' system and the problem of future contingents

In Ł, **The principle of bivalence** does not hold, as desired. Since this is a three-valued system, there is another possible truth-value besides true or false. It is therefore not the case that every proposition must be either true or false.

Unfortunately, **The principle of antecedent truth** does hold. The third truth-value can be a way to avoid the principle of antecedent truth. If statements about the future are neither true nor false, we cannot say that something was true forever beforehand. However, in this case the third truth-value is not ascribed to all statements about the future, it is ascribed to statements that have a varying truth-value, and are therefore neither always true, nor always false.

At first it seems promising that omni-temporal truths are treated as necessary and that propositions with a varying truth-value are contingent. For example, we do not mind that the statement '$2+2=4$' is true forever beforehand and therefore necessary, as long as the statement 'It will be raining tomorrow' is treated as contingent, which it is.

However, this system does not treat all contingents statements about the future in the same way. As an example take the sentence 'It is raining at de Grote Markt in Groningen at 12:15 of July 3th 2014'. Unlike the sentence 'It is raining now', this very specific statement does not have a varying truth-value. This means it cannot be ascribed the third truth-value, so it should be either true or false; eternally true or eternally false. This means that if it is true, it was true forever beforehand. And this is precisely the principle of antecedent truth, which entails determinism.

Finally, **The law of excluded middle** does not hold. In this case the problem does not lie with the specific, singular statements, but with the statements that have a varying truth value. As an example consider the statement 'it will be raining tomorrow'. This is not always true, and neither is it always false. Therefore it is not true nor false and should be appointed the third truth-value. This means that its negation will also be neither true nor false, and a conjunction of which both conjuncts are neither true nor false, is in this system neither true nor false itself. This means that $p \vee \neg p$ is not true, and therefore not a logical law.

It should be noted, however, that Prior in his discussion of the problem of future contingents did not think it possible to create an indeterministic three-valued system in which the law of excluded middle does hold. As we have seen $p \vee \neg p$ should be always true, while neither p nor $\neg p$ is required to be always true. This means we want the 'always' to operate on the entire conjunction, and not on both the conjuncts separately. If we use parentheses to signify 'always' this means that we want $(p \vee \neg p)$ to be true, while $(p) \vee (\neg p)$ does not have to be true in the case of contingent events. The system is not capable of making this distinction, both these conjunctions will be evaluated in the same way. For this reason Prior believes that Aristotle was, excusably, mistaken, when he asserted that the former should be true while the latter should be false.

To conclude, this means that two out of three criteria are not met. Even though the principle of bivalence does not hold, the principle of antecedent truth is still true, and therefore this system is deterministic. Furthermore, the law of excluded middle is not valid. This means that despite the fact that this system is very true to Aristotle's ideas about the subject, it is not capable of solving the problem of future contingents in a satisfactory way.

4 The system Q

4.1 Prior's system Q

In this section the three-valued modal logic Q will be discussed. This logic was developed by A. N. Prior (1957). Prior himself does not use Q to solve the problem of future contingents. He designed Q to deal with contingent and necessary beings and with the possibility that some things can come in and out of existence. However, Akama, Nagata and Yamada (2008) have extended Q with an explicit temporal component to create the system Q_t. The system Q_t is specifically designed to solve the problem of future contingents and will be discussed in the next section.

The system Q under the assumption that there are only two times
Unlike Łukasiewicz and Aristotle, Prior does not interpret truth as omni-temporal truth. However, he accepts that some propositions can be true at one time and false at another, and he tries to formalize this. He starts developing his system with the assumption that there are only two times, today and yesterday, and therefore every proposition needs to be evaluated at both these times to give it a complete truth-value. Furthermore, he introduces a third value, next to true and false, namely unstatable. This means a statement can be true, false or unstatable.

This third truth-value, unstatable, is not to be interpreted as indeterminate. As an example we consider the sentence 'tomorrow it will rain'. We consider this indeterminate, since it is a contingent statement about the future, and therefore we might be tempted to give it the third truth-value. In Prior's system, however, this is not the route to take, since the third truth-value means unstatable, and the sentence 'tomorrow it will rain' is perfectly statable today. If I did not exist yesterday, a sentence that is unstatable could be something like 'I am hungry'. This might be true today, but due to my non-existence, this was unstatable yesterday.

Prior himself says:

> 'A 2 (unstatable) at a point in a sequence meant that there is no such proposition as the one in question in the world represented by that point.' (Prior, 1967, p. 154)

and

> 'I have myself sometimes put it by saying that at the times in question no such proposition is statable. This latter locution has the disadvantage of suggesting that the difficulty here is simply with our mechanisms of reference; I want to say rather that there are no facts about x to be stated except when x exists.' (Prior, 1968, p. 147)

To summarize, a statement is unstatable if it truly cannot be stated. This also means that every statement containing a part that is unstatable, is unstatable as a whole. If I did not exist yesterday, the statement 'I am hungry' was unstatable, and so the statement 'I am hungry and the sun is shining' would also be unstatable. Since part of this sentence cannot be stated, the entire sentence cannot be stated. Furthermore, any statement we are considering today, must be statable today, since we clearly need to be able to state it before considering it.

Now we know what it means for a statement to be unstatable, we can return to the beginning of the system Q in which there are only two times. As we have seen, every statement we consider today is statable today, so there are 6 possible truth-values a statement can take.

1.	True today and true yesterday	Tt
2.	True today and unstatable yesterday	Tu
3.	True today and false yesterday	Tf
4.	False today and true yesterday	Ft
5.	False today and unstatable yesterday	Fu
6.	False today and false yesterday	Ff

The designated values in this system, i.e. the values that a tautology may take, are Tt and Tu, since, as Prior puts it, 'a formula expresses a logical law if its concrete substitutions are true whenever they are statable'. (Prior, 1957, p. 41)

If we keep in mind that any proposition containing a part that is unstatable, is unstable itself, it is easy to construct a truth-table for the negation.

Example 4.1 *If the value of p is Tu, i.e. p is true today but unstatable yesterday, the negation of p will be false today, but still unstatable yesterday, so the value of $\neg p$ is Fu.*

The modal operators in this two-timed system are given very intuitive meanings. It is possible that p, Mp, is taken to mean that p is true at some time. It is necessary that p, Lp, is taken to mean that p is true at both times, i.e. all times.

Example 4.2 *If p has the value Ft, i.e. is false today but true yesterday, Mp will be true both today and yesterday, since p is true at some time, so the value of Mp is Tt. Lp will be false both today and yesterday, since p is false at some time, so the value of Lp is Ff.*

Together these rules give us the following truth-tables for $\neg p$, Mp, Lp and $\neg M \neg p$.

p	$\neg p$
Tt	Ff
Tu	Fu
Tf	Ft
Ft	Tf
Fu	Tu
Ff	Tt

p	Mp
Tt	Tt
Tu	Tu
Tf	Tt
Ft	Tt
Fu	Fu
Ff	Ff

p	Lp
Tt	Tt
Tu	Fu
Tf	Ff
Ft	Ff
Fu	Fu
Ff	Ff

p	$\neg M \neg p$
Tt	Tt
Tu	Tu
Tf	Ff
Ft	Ff
Fu	Fu
Ff	Ff

As we can see, if the value of p is Tu or Fu, the values of $\neg p$, Mp, Lp and $\neg M \neg p$ are also either Tu or Fu, since if something is unstatable, every statement that contains it must also be unstatable.

Furthermore, $\neg M \neg p$ is included in this table since in most modal logics we have $Lp \leftrightarrow \neg M \neg p$; something is necessary when it is not possible that it is not the case. As we can see this does not hold in Q. In Q, $p \to \neg M \neg p$, as we can see by the fact that whenever p has designated values Tt or Tu, $\neg M \neg p$ also has designated values Tt or Tu. In natural language this says that if p is the case it is impossible that p is not the case, and this seems indeed right.

Unlike in Łukasiewicz system we do not have $p \to Lp$, as is seen by the fact that when p has designated value Tu, Lp takes value Fu, which is not designated. In natural language this says that it is not so that if p is the case it is necessary that p is the case. Again this seems right; we do not regard something as necessary simply because it is so now. The fact that Lp and $\neg M \neg p$ are not equivalent in Q thus fits our natural understanding of the terms necessary and impossible.

A table for $\wedge$ can also be constructed by keeping in mind that a statement is always unstatable when some of its parts are unstatable, and further following the standard for $\wedge$.

Example 4.3 *Suppose p has the value Ft, i.e. is false today and true yesterday, and q has the value Tu, i.e. is true today but unstatable yesterday. Then today p is false and q is true, so $p \wedge q$ is false today. Yesterday p was true but q unstatable, so $p \wedge q$ was unstatable as well yesterday. This means $p \wedge q$ is false today, and unstatable yesterday, and therefore takes value Fu.*

This gives the following table for $\wedge$:

$p \wedge q$	Tt	Tu	Tf	Ft	Fu	Ff
Tt	Tt	Tu	Tf	Ft	Fu	Ff
Tu	Tu	Tu	Tu	Fu	Fu	Fu
Tf	Tf	Tu	Tf	Ff	Fu	Ff
Ft	Ft	Fu	Ff	Ft	Fu	Ff
Fu	Fu	Fu	Fu	Fu	Fu	Fu
Ff	Ff	Fu	Ff	Ff	Fu	Ff

Prior defines $\vee$ and $\to$ in the usual way:

Definition 4.1
$p \vee q := \neg(\neg p \wedge \neg q)$
$p \to q := \neg p \vee q$

This gives the following truth-tables for $\vee$ and $\to$:

$p \vee q$	Tt	Tu	Tf	Ft	Fu	Ff
Tt	Tt	Tu	Tt	Tt	Tu	Tt
Tu	Tu	Tu	Tu	Tu	Tu	Tu
Tf	Tt	Tu	Tf	Tt	Tu	Tf
Ft	Tt	Tu	Tt	Ft	Fu	Ft
Fu	Tu	Tu	Tu	Fu	Fu	Fu
Ff	Tt	Tu	Tf	Ft	Fu	Ff

$p \to q$	Tt	Tu	Tf	Ft	Fu	Ff
Tt	Tt	Tu	Tf	Ft	Fu	Ff
Tu	Tu	Tu	Tu	Fu	Fu	Fu
Tf	Tt	Tu	Tt	Ft	Fu	Ft
Ft	Tt	Tu	Tf	Tt	Tu	Tf
Fu	Tu	Tu	Tu	Tu	Tu	Tu
Ff	Tt	Tu	Tt	Tt	Tu	Tt

The system Q for infinitely many times

We have described the system Q under the assumption that there are only two times, today and yesterday. Since this is obviously false, we now omit this assumption and develop the system Q for infinitely many times. This means every statement needs to be evaluated at infinitely many times, and the truth-value of a statement therefore becomes an infinite sequence where each term is the truth-value at a specific time. Prior lets this sequence start in the present. The first number is thus the truth-value of the proposition today, and all the other numbers in the sequence are in the future. At each time, there are three possible truth-values: True (1), unstatable ($1/2$) and false (0).

Definition 4.2 *The rules for determining the truth-value-sequence of $\neg p$, $p \wedge q$, Mp and Lp are defined as follows:*

- *The sequence for $\neg p$ is determined by the sequence for p by considering each element separately. If the truth-value-sequence for p is $a\ b\ c\ \ldots$, the truth-value-sequence for $\neg p$ is thus $\neg a\ \neg b\ \neg c\ \ldots$. $\neg p$ is defined by the following rule:*

p	$\neg p$
1	0
$1/2$	$1/2$
0	1

- *The sequence for $p \wedge q$ is again determined by the sequences of p and q where each pair is considered separately. The rule for evaluating each pair is given by the following table:*

$p \wedge q$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
0	0	$1/2$	0

- *The sequences for $p \vee q$ and $p \rightarrow q$ are again determined by considering each pair separately. The rules for $\vee$ and $\rightarrow$ are defined as in definition 4.1.*
- *The sequence for Lp is determined by the sequence for p as follows:*
  - *If the sequence for p contains only 1's, the sequence for Lp is the same*
  - *If the sequence for p contains any $1/2$'s, then in the sequence for Lp these $1/2$'s keep their place unaltered, and all other places are occupied by 0's.*
  - *If the sequence contains no $1/2$'s, but it does contain 0's, the sequence for Lp is that consisting of 0's only*
- *The sequence for Mp is determined by the sequence for p as follows:*
  - *If the sequence for p consists either of 0's only, or of $1/2$'s and 0's only, the sequence for Mp is the same as the sequence for p.*
  - *If the sequence for p contains 1's or $1/2$'s, whether it contains 0's or not, in the sequence for Mp the $1/2$'s keep their place unaltered and the other places are occupied by 1's.*
  - *Where the sequence for p contains no $1/2$'s, but does contain 1's, the sequence for Mp is that consisting of 1's only.*
- *The designated sequences are all those which contain no 0's, i.e. are true whenever they are statable.*

To illustrate these rules consider the following examples:

Example 4.4 *a) V(p) = 1 1 1 1 1 .... In this case p is always statable and true, so Lp is true. Therefore V(Lp) = 1 1 1 1 1 ....*

b) V(p) = 1 1 ½ ½ 1 .... In this case p is true whenever it is statable. However, Lp is only true if p is true at all possible times. This is not the case, since at some times p is not statable. Therefore, Lp is false whenever it is statable, thus V(Lp) = 0 0 ½ ½ 0 ....

c) V(p) = 0 0 1 1 0 .... In this case p is always statable, but false at some times. This means Lp is false whenever it is statable, but since p is always statable, Lp is always statable. Therefore V(Lp) = 0 0 0 0 0 ....

d) V(p) = 0 ½ ½ 0 0 .... In this case p is never true, so Mp is false whenever it is statable. Therefore V(Mp) = 0 ½ ½ 0 0 ....

e) V(p) = 1 ½ 1 1 0 .... In this case p is true at some times, so Mp is true whenever it is statable. Therefore V(Mp) = 1 ½ 1 1 1 ....

f) V(p) = 1 0 0 1 1 .... In this case p is always statable and true at some times. This means Mp is true at all times, i.e. true whenever statable. Therefore V(Mp) = 1 1 1 1 1 ....

Note that the tables used to describe the two-times case are a special case of definition 4.2, where we assume that everything we can consider today is statable today. Therefore we do not need to consider the truth-values Ut, Uu and Uf.

4.2 Possible world semantics for Q

To account for multiple times, Prior let the truth-value of a proposition be a sequence of the truth-values of that proposition at all possible times. This creates quite a complicated system. In their paper *On Prior's Three-Valued Modal Logic Q* (2005) Akama and Nagata give a possible-world semantics for Q. They view each time in the infinite truth-value-sequence as a possible world, and with this they greatly simplify Prior's system.

As we have seen in the previous section, the key features of the system Q are that once some part of a proposition is unstatable, the entire proposition is unstatable, that something is possible when it is true at some time, and that something is necessary when it is true at all times. A possible world semantics for Q should also have these properties.

Definition 4.3 *The Kripke model for Q is a triple $\langle W, stat, V\rangle$ where:*

- *W is a non-empty set, the set of all possible worlds*
- *stat is the statability relation, where stat(A,w) means that A is statable at world w*

- *V is the three-valued valuation-function* $V : FOR \times W \rightarrow \{1, 1/2, 0\}$*, where FOR is the set of all possible formulas*

Definition 4.4 *stat(A,w) iff* $V(p, w) \neq 1/2$ *for all atoms p in A*

Like in the last section there are three truth-values; True (1), unstatable (1/2) and false (0). A proposition is thus statable whenever it has truth-value 1 or 0. The valuation of other formulas can be defined as follows:

Definition 4.5

$V(\neg A, w) = 1$	*iff*	$V(A,w) = 0$
$V(\neg A, w) = 0$	*iff*	$V(A,w) = 1$
$V(\neg A, w) = 1/2$	*iff*	$V(A,w) = 1/2$
$V(A \wedge B, w) = 1$	*iff*	$V(A,w) = V(B,w) = 1$
$V(A \wedge B, w) = 0$	*iff*	*stat(A,w) and stat(B,w) and* ($V(A,w) = 0$ *or* $V(B,w) = 0$)
$V(A \wedge B, w) = 1/2$	*iff*	$V(A, w) = 1/2$ *or* $V(B, w) = 1/2$
$V(MA,w) = 1$	*iff*	*stat(A,w) and* $\exists v \in W$ *such that* $V(A,v) = 1$
$V(MA,w) = 0$	*iff*	*stat(A,w) and* $\forall v \in W\ V(A, v) \neq 1$
$V(MA,w) = 1/2$	*iff*	$V(A, w) = 1/2$
$V(SA,w) = 1$	*iff*	$\forall v \in W\ V(A, v) \neq 1/2$
$V(SA,w) = 0$	*iff*	$V(A, w) \neq 1/2$ *and* $\exists v \in W$ *such that* $V(A, v) = 1/2$
$V(SA,w) = 1/2$	*iff*	$V(A, w) = 1/2$

S can be interpreted as necessary statability. SA reads as A is statable in all possible worlds. As we have seen LA and $\neg M \neg A$ are not equivalent in Q. Prior defines necessity as being always true, not just true whenever statable. For this reason necessity of A is not just that it is impossible that not-A, but also that A is always statable. We define:

Definition 4.6 $LA := \neg M \neg A \wedge SA$

With this definition the valuation function for LA becomes:

Definition 4.7

$V(LA,w) = 1$	*iff*	$\forall v \in W\ V(A, v) = 1)$
$V(LA,w) = 0$	*iff*	$V(A, w) \neq 1/2$ *and* $\exists v \in W$ *such that* $V(A, v) \neq 1$
$V(LA,w) = 1/2$	*iff*	$V(A, w) = 1/2$

$\vee$ and $\rightarrow$ are again defined as in definition 4.1.

Deviations from Akama and Nagata
Definition 4.5 differs in several points from the valuation function given by Akama and Nagata (2005). Every deviation from the definition of Akama and Nagata is motivated by a desire to stay closer to the system Q as proposed by Prior.

Firstly Akama and Nagata define $V(A \wedge B, w) = 0$ iff $V(A, w) = 0$ or $V(B, w) = 0$. By this definition $V(A \wedge B, w) = 0$ when for example $V(A, w) = 0$ and $V(B, w) = 1/2$. However, Prior clearly emphasises that when some part of a proposition is unstatable, the entire proposition is unstatable, and Akama and Nagata agree with this as is shown by definition 4.4. For this reasons we have added the conditions stat(A,v) and stat(B,v).

Furthermore, Akama and Nagata define $V(MA, w) = 0$ iff stat(A,w) and $\forall v \in WV(A, v) = 0$. However, as we can see from definition 4.2 and example 4.4, Prior thinks something is possible when it is true at at least one time point. This means that MA is false when A is never true. This does not mean, however, that A must be always false; it is also allowed that A is unstatable. Because of this the condition $\forall v \in W\ V(A, v) = 0$ is changed into the condition $\forall v \in W\ V(A, v) \neq 1$.

According to Akama and Nagata $V(SA, w) = 0$ when there is some possible world at which A is not statable. However, again we need to keep in mind that according to Prior the entire proposition is unstatable if some part of it is unstatable. Therefore it is needed to ensure that $V(A, w) \neq 1/2$. In this same light, we have specified $V(SA, w) = 1/2$ iff $V(A, w) = 1/2$ to clarify the definition given by Akama and Nagata.

Since we have changed some of the valuations for MA and SA, our valuation function for LA also differs from the function given by Akama and Nagata.

4.3 Q and the problem of future contingents

How we need to apply Q to the problem of future contingents is not immediately clear. We cannot ascribe the third truth-value to propositions about the contingent future, since this value means unstatable, and most propositions about the future are perfectly statable.

It is therefore fruitful to consider Prior's infinite truth-value-sequence. The first term of this sequence is today, so every next term is in the future. Considering a statement about the future is thus the same as considering the entire truth-value sequence of the present-tense of that statement.

For example take the statement 'It will be raining'. This means that at some point in the future the statement 'It is raining' will be true, i.e. at some point in the sequence the truth-value of 'it is raining' should be a 1. And as we can see by definition 4.4 this is the same as possibility. 'It will be that p' can therefore be equated by Mp.

Now we know how to interpret Q we can take a look at our criteria for a solution to the problem of future contingents.

As desired, **the principle of bivalence** does not hold. Q is a three-valued logic, and therefore there is another possible truth-value besides true and false. It is therefore not necessary for every proposition to be either true or false.

Next consider the **The law of excluded middle**. Let us consider the infinite truth-value sequence for 'it is raining'. As long as the world exists, it seems correct to assume this sentence is statable. At every time, it is therefore either true or false. The fact that 'it is raining' is a contingent statement is shown by the fact that its valuation varies. Sometime it is true, sometime it is false. By definition 4.5 this shows that 'it is raining' is possible, but not necessary.

If it is raining the first three days from now, and it is not raining the first two days after that, the beginning of the truth-value sequence for 'it is raining' will be 1 1 1 0 0 . . .. The sequence for it is not raining will be 0 0 0 1 1 . . .. The rule for $\vee$ shows us that the sequence for 'it is raining or not' is 1 1 1 1 1 . . .. Since 'it is raining' is always statable, the sequence for 'it is raining or not' will consists of only 1s. 'It is raining or not' is therefore a necessary statements, as desired.

This means that as long as we are considering something that is always statable in the future, $p \vee \neg p$ is necessarily true.

Furthermore, consider the truth-table for $p \vee \neg p$:

p	p	$\vee$	$\neg p$
1	1	**1**	0
$1/2$	$1/2$	$1/2$	$1/2$
0	0	**1**	1

As we can see, the value for $p \vee \neg p$ is either 1 or $1/2$, meaning that $p \vee \neg p$ is true whenever it is statable. This is exactly how Prior defined a logical law, so $p \vee \neg p$ is a law. This means that the law of excluded middle holds in Q, as desired, even if p is not always statable in the future.

Unfortunately, **The principle of antecedent truth** holds in this system as well. Statements about the future are ascribed definite truth-values in the present. The truth-value-sequence for 'It is raining' is determined, so the truths of 'it will be raining tomorrow', 'it will be raining the day after tomorrow' and so on are already set. This is also the case when the truth-value-sequence of 'It is raining' was evaluated yesterday, so if it is raining today it was true yesterday that it would be raining. Since this argument also holds for any other day in the past, the fact that it is raining today implies that it was always true that it would be raining. This means that, even though the principle of bivalence does not hold, we still have the principle of antecedent truth because, like in Łukasiewicz' system, the third truth-value is not assigned to statements about contingent future events.

A different interpretation of $1/2$

We could interpret the third truth-value in a different way. This is done by Akama et al. (2008) (see next section) and therefore it is worth considering for Q also. On this interpretation a proposition is only truly statable if we know if it is true or false. If we cannot determine this, we are not truly stating a proposition, since propositions must be either true or false. This means that with this interpretation indeterminate statements are assigned truth-value $1/2$.

Using this interpretation, statements about contingent future events are assigned truth-value $1/2$, since we consider the truth of future contingents indeterminate. In this case, the principle of antecedent truth no longer holds. If a statement about a future contingent event is indeterminate, it was not true yesterday to say it would rain today, for yesterday this was a proposition about a contingent future event which evaluates as $1/2$ and not 1. Therefore, the fact that something is true now does not imply that is has always been the case that it would be true, and so determinism does not follow.

On the interpretation of unstatable as indeterminate, the law of excluded middle still holds. The truth-table for $p \vee \neg p$ does not change and neither does the definition of a law; something is still a law if its truth-value is always 1 or $1/2$. However, in this case this is not entirely satisfactory. It seems reasonable that something is a law if it is true whenever it is statable, i.e. true whenever the objects spoken of exist. That something is a law if it is true whenever it is determinate is less intuitive. Something that is as of yet indeterminate could turn out to be false in the future. The statement 'it will rain tomorrow' is indeterminate right now, but if is does not rain tomorrow, tomorrow the statement will be determined and false. In this light it seems unreasonable to claim that something is a law if its truth-values are always $1/2$ or 1, since the truth-value $1/2$ means it might turn out false.

The law of excluded middle still holds, its values are always $1/2$ or 1. However, when p is the statement 'it will rain tomorrow', p is a statement about a future contingent event, so the truth-value of p is $1/2$. Therefore, the truth-value of $p \vee \neg p$ is also $1/2$.

Why should something that can be indeterminate be a law? Should not a law be always true? Certainly our intuition tells us $p \vee \neg p$ should be true. It will rain tomorrow, or not. The fact that it will rain or not does not seem indeterminate. Changing the interpretation of the third truth-value to make it apply to a wider range of propositions has diminished the justification for the notion of a law being something that is true whenever it is statable.

To conclude, on Prior's interpretation Q thus satisfies two of our three criteria. The principle of bivalence does not hold, while the law of excluded middle does. However, the principle of antecedent truth also holds, and therefore the system is still deterministic.

On the second interpretation all three criteria are met. However, Prior's definition of validity is in this case rather dubious, and therefore the law of excluded middle is not met in a satisfactory way.

5 The system Qt

5.1 Expansion of Q to a temporal system Qt

As we have seen in the previous section, applying Prior's system Q to a temporal problem like the problem of future contingents is not trivial. To simplify this, Akama, Nagata and Yamada (2008) have provided Q with an explicit temporal component. In this section the system Q_t will be explained. We look at the Kripke model for Q_t and its validation function.

The primary functions of Q_t are $\neg$ (negation), $\wedge$ (conjunction), F (future possibility), P (past possibility), S_P (past statability) and S_F (future statability).

As we can see, the modal operator possibility of the system Q is replaced with the temporal operators past possibility and future possibility. The statability operators are based on Q, but should be interpreted slightly different.

> 'The formula $S_F A$ reads "A is statable at all future time points" or "A has a truth-value at all future time points." ' (Akama et al., 2008, p.219)

Furthermore they say that 'contingent sentences are indentified with unstatable ones' (p. 222)

Definition 5.1 *The Kripke model for Q_t is denoted by $\langle T, <, stat, V\rangle$ where:*

- *T is the set of all possible time points*
- *$<$ is an ordering relation on $T \times T$*
- *stat is the statabillity relation*
- *V is the three-valued valuation-function $V : FOR \times T \rightarrow \{1, 1/2, 0\}$, where FOR is the set of all possible formulas*

Akama et al. do not specify anything about the ordering relation ¡, but the chosen notation and the nature of time suggest that this ordering is irreflexive and transitive.

Like Q, Q_t is a three-valued system with truth-values true (1), unstatable (1/2) and false (0). The valuation of the primary functions is defined as follows:

Definition 5.2

$V(\neg A, t) = 1$	*iff*	$V(A, t) = 0$
$V(\neg A, t) = 0$	*iff*	$V(A, t) = 1$
$V(\neg A, t) = 1/2$	*iff*	$V(A, t) = 1/2$
$V(A \wedge B, t) = 1$	*iff*	$V(A, t) = V(B, t) = 1$
$V(A \wedge B, t) = 0$	*iff*	*stat(A, t) and stat(B, t) and ($V(A, t) = 0$ or $V(B, t) = 0$)*

$V(A \wedge B,\ t) = 1/2$ iff $V(A,t) = 1/2$ or $V(B,t) = 1/2$

$V(FA,\ t) = 1$ iff $stat(A,\ t)$ and $\exists s \in T$ such that $s > t$ and $V(A,\ s) = 1$
$V(FA,\ t) = 0$ iff $stat(A,\ t)$ an $\forall s \in T\ (s > t \rightarrow V(A,s) \neq 1)$
$V(FA,\ t) = 1/2$ iff $V(A,t) = 1/2$

$V(PA,\ t) = 1$ iff $stat(A,\ t)$ and $\exists s \in T$ such that $s < t$ and $V(A,\ s) = 1$
$V(PA,\ t) = 0$ iff $stat(A,\ t)$ an $\forall s \in T\ (s < t \rightarrow V(A,s) \neq 1)$
$V(PA,\ t) = 1/2$ iff $V(A,t) = 1/2$

$V(S_F A,\ t) = 1$ iff $stat(A,\ t)$ and $\forall s \in T\ (s > t \rightarrow stat(A,s))$
$V(S_F A,\ t) = 0$ iff $stat(A,\ t)$ and $\exists s \in T$ such that $s > t$ and $\neg stat(A,s)$
$V(S_F A,\ t) = 1/2$ iff $V(A,t) = 1/2$

$V(S_P A,\ t) = 1$ iff $stat(A,\ t)$ and $\forall s \in T\ (s < t \rightarrow stat(A,s))$
$V(S_P A,\ t) = 0$ iff $stat(A,\ t)$ and $\exists s \in T$ such that $s < t$ and $\neg stat(A,s)$
$V(S_P A,\ t) = 1/2$ iff $V(A,t) = 1/2$

We can add the temporal necessity operators Gp (it will always be the case that p) and Hp (it has always been the case that p) to the system Q_t. Like in Prior's system Q, necessity of p means that it is always true, not just true whenever p is statable. Because of this the temporal necessity operators Gp and Hp are not dual to the temporal possibility operators Pp and Fp. We also need to ensure that p is statable. This leads to the following definition:

Definition 5.3
$Gp := S_F p \wedge \neg F \neg p$
$Hp := S_P p \wedge \neg P \neg p$

$\vee$ (disjunction) and $\rightarrow$ (implication) are again defined in the usual way:

Definition 5.4
$p \vee q := \neg(\neg p \wedge \neg q)$
$p \rightarrow q := \neg p \vee q$

The designated values in this system are again 1 and 1/2. Something is thus a law if it is true whenever it is statable.

5.2 Qt and the problem of future contingents

As mentioned above Akama et al. identify unstatable with undefined and indeterminate. This makes it plausible to assign propositions about future contingent events the third truth-value, since we consider these to be neither true nor false and therefore indeterminate.

The principle of bivalence does not hold. Again we are dealing with a three-valued logic, so there is another possibility besides true or false. Therefore it is not the case that every proposition must be either true or false.

Since Q_t has an explicit temporal component, we can now formalize **The principle of antecedent truth**. We have seen that the principle of antecedent truth is the theses that if something is true now it has always been the case that it would be true. We do not want this principle to be true, since we believe this would entail determinism.

The principle of antecedent truth can be translated as $p \rightarrow HFp$. One simple counterexample suffices to show this is not a law in Q_t. If p is a statement about a contingent event, like 'it is raining', Fp, 'it will be raining' will evaluate indeterminate; Fp will get the value $1/2$.

As we have seen Hp, it has always been the case that p, is only true if p is true at all points before now. It is not enough that p is true whenever it is statable, it needs to be always true. This means that HFp is false in the case of a contingent event, since Fp is indeterminate. Therefore, it is possible that p is true, while HFp is not true. Therefore $p \rightarrow HFp$ is not a law.

However, there is a problem with this counterexample. If we look at the valuation function of definition 5.2, we can see that Fp only gets the value $1/2$ if p is unstatable itself, i.e. if p also has the value $1/2$. However, when we are talking about a future contingent event, the truth of the present tense of that statement is usually perfectly determined. It might not be determined yet if it will rain tomorrow or not, but it is easy enough to see whether it is raining right now or not. This means that if p represents the statement 'it is raining now', the value of p is either 1 or 0, and therefore by definition 5.2 the value of Fp cannot be $1/2$.

In their solution for the problem of future contingents, Akama et al. thus follow the intuition that statements about future contingent events should be indeterminate, but unfortunately they have not succeeded in embedding this in their semantics.

The law of excluded middle holds in Q_t, like desired. As in Prior's systems Q, the designated values in Q_t are 1 and $1/2$. Something is a law if it is true whenever it is statable. The truth-table for $p \vee \neg p$ is the same as in Q:

p	p	$\vee$	$\neg p$
1	1	**1**	0
$1/2$	$1/2$	$1/2$	$1/2$
0	0	**1**	1

As we can see, $p \vee \neg p$ is never false, and therefore the law of excluded middle holds.

However, since Akama et al. do not interpret statable is the same way as Prior, their definition of validity is rather ad hoc. In Prior's case, a proposition is unstatable if it truly cannot be stated. Unstatable propositions are typically about things that do not exist. In this case, it is very plausible to say something is a law if it is true whenever it is statable. However, if we change the interpretation of the third truth-value to indeterminate, this definition is less plausible. Why would something that is true whenever it is determined be a law?

As is shown by the above truth-table, $p \vee \neg p$ is indeterminate if p is indeterminate. So if p is a proposition about something contingent, Fp is indeterminate, and therefore $Fp \lor \neg Fp$, it will rain tomorrow or it will not rain tomorrow, is also indeterminate. However, this latter proposition is precisely what Aristotle wanted to be true when he asserted the law of excluded middle should hold. That it is not false is not enough, it should really be true.

Akama et al. offer no real justification for their definition of validity, while this is needed since they have changed the interpretation of the third truth-value. They say:

> 'The notion of validity in Q_t is different from that of standard temporal logic. It is needed to validate classical tautologies when the valuation of every propositional variable in them is undefined. As a result, all classical tautologies are shown to be valid in Q_t'(Akama et al., 2008, p. 222)

This means that the main reason for defining validity in the way they do is to make sure the law of excluded middle, together with other classical tautologies, holds. To me this seems like begging the question.

To conclude, the system Q_t seems very promising in solving the problem of future contingents. All three criteria are met. However, the way Akama et al. evaluate Fp is not justified by their valuation function. Furthermore, their definition of validity lacks justification and under their interpretation of the third truth-value this definition seems arbitrary. Therefore, Q_t does not offer a satisfactory solution to the problem of future contingents.

6 Conclusion

We have considered three systems that could possibly be used to solve the problem of future contingents, Łukasiewicz' system Ł, Prior's system Q and the system Q_t by Akama, Nagata and Yamada.

In the first section we derived three criteria any solution should meet. Firstly the principle of bivalence should not hold, secondly the principle of antecedent truth should not hold and thirdly the law of excluded middle should hold.

Łukasiewicz' system meets only the first of these criteria and is therefore not suitable as a solution to the problem of future contingents.

Q meets two of the criteria, but is still a deterministic system. Therefore Q is also not suited to solve the problem of future contingents.

Q_t seems very promising and according to Akama et al. it meets all three of our criteria. However, the valuation of statements about contingent future events as $1/2$ is not justified by the semantics of Q_t and the definition of validity that ensures that the law of excluded middle holds is rather ad hoc. Therefore, Q_t does not satisfactory solve the problem of future contingents.

This means that none of the proposed three-valued modal logics is able to solve the problem of future contingents. However, Q_t came very close. Perhaps it is possible to modify the valuation function of Q_t in such a way that it justifies ascribing statements about future contingent events the third truth-value. This would result in a system in which the principle of antecedent truth does not hold, and therefore a likely candidate for an indeterministic logic. Unfortunately, even if this is remedied, the problem with the law of excluded middle remains.

Therefore, regrettably, we are bound to conclude that three-valued modal logics are not likely candidates to solve the problem of future contingents in a satisfactory way.

7 Literature

Articles:

1. Akama, S. & Nagata, Y. (2005). On Prior's Three-Valued Modal Logic Q. *Proceedings of the 35th International Symposium on Multiple-Valued Logic*, pp. 14-19.

2. Akama. S., Nagata, Y. & Yamada, C. (2008). Three-valued temporal logic Qt and future contingents. *Studia Logica, vol. 88* (no. 2), pp. 215-231.

3. Prior, A. N. (1953). Three-valued logic and Future Contingents. *The Philosophical Quarterly, vol. 3* (no. 13), pp. 317-326.

4. Strang, C. (1960). Aristotle and the Sea-battle. *Mind, New Series, vol. 69* (no. 276), pp. 447-465.

5. Hintikka, J. (1964). The once and future Sea Fight: Aristotle's Discussion of Future Contingents in De Interpretatione IX. *The Philosophical Review, vol. 73* (no. 4), pp. 461-492

6. Bourne, C. (2004). Future Contingents, Non-Contradiction, and the Law of Excluded Middle Muddle. *Analysis, vol. 64* (no. 2), pp. 122-128.

Articles 4, 5 and 6 are not referred to in the text but they are mentioned here since they provided valuable background knowledge about the debate concerning Aristotle's text.

Books:

1. Kane, R. (2005). *A contemporary introduction to free will.* New York: Oxford University Press

2. Aristotle (1963). *Categories and De Interpretatione, Translated with notes and glossary by J. L. Ackrill.* Oxford: Clarendon Press

3. Ryle, G. (1953). *Dilemmas.* Cambridge: Cambridge University Press

4. Prior, A. N. (1968). *Papers on Time and Tense.* Oxford: Clarendon Press

5. Prior, A. N. (1957). *Time and Modality.* Oxford: Clarendon Press

6. Prior, A. N. (1967). *Past, Present and Future.* Oxford: Clarendon Press

7. Łukasiewicz, J. (1970). *Selected works.* Amsterdam, London: North-Holland Publishing Company.