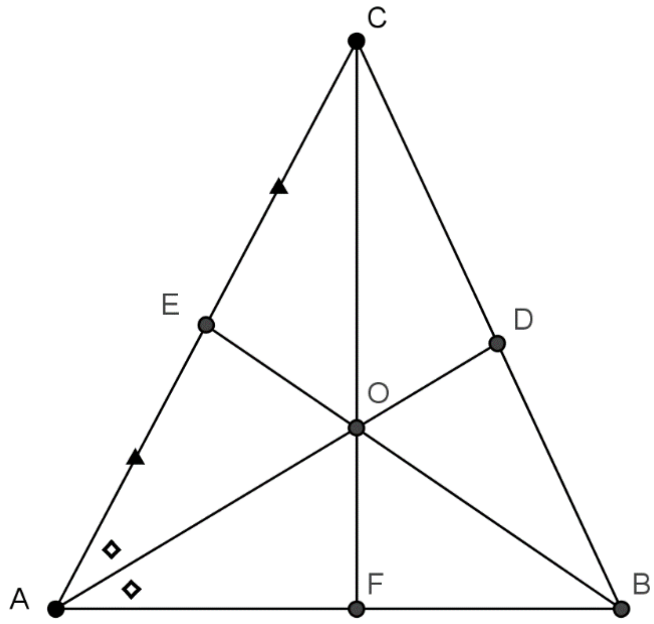
university of groningen

faculty of mathematics and natural sciences

Albime Triangles: Acute or Obtuse?



Bachelor Project Mathematics

June 2015

Student: J.F. Kooij

First supervisor: Prof. dr. J. Top

Second supervisor: Prof. dr. M. K. Camlibel

Abstract

A triangle ABC is called albime if a bisector in A, the median in B and the altitude in C are concurrent. We scale these triangles such that the points of the elliptic curve $y^2 = x^3 - 4x + 4$ that have an x-coordinate between zero and two correspond exactly to an albime triangle with internal bisector, and the point that have an x-coordinate between minus two and zero to an albime triangle with external bisector.

The rational points on this curve form a group that can be generated by one element, namely $P = (2, 2)$. This group of rational points can be mapped onto the unit circle using some map Ψ. Then there exists the element $\Psi(P)$ that generates the unit circle.

We can find how many of the rational points on this circle correspond to a rational albime triangle. We find that 36.12% of the points coincide with an albime triangle with inner bisector and 19.40% of them generate one with an external bisector. For the albime triangles with internal bisector we prove that 22.48% of rational points on the elliptic curve coincide with an acute albime triangle and 13.67% with an obtuse albime triangle.

Contents

Chapter 1

Introduction

This thesis is written to answer a question posed by Th.J. Kletter regarding an albime triangle with external bisector. Recall that, in a triangle, a bisector is a line which divides one of the angles into equal parts. A median is a line going through any of the vertices, which divides the opposite side into two sections of equal length. Lastly, the altitude of a triangle is the line which goes through one of the vertices and is perpendicular to the opposite side. This brings us to our first definition:

Definition 1.0.1. A triangle is called albime concurrent, or simply albime, if after possibly permuting the vertices $\{A, B, C\}$, we have a bisector in A, the median in B and the altitude in C to be concurrent.

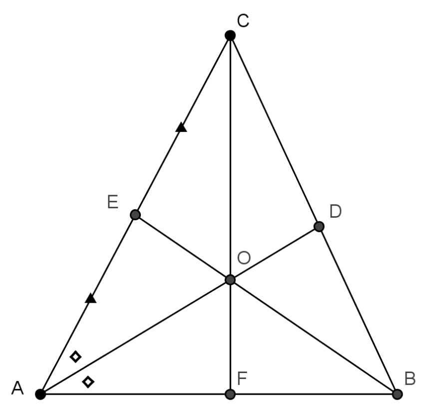


Figure 1.1: An albime triangle with internal bisector

The bisector can be internal or external. This gives totally different albime triangles, which will be shown in chapter 2 and 3, respectively. Elliptic curves are also needed to get to the objective, these will be discussed in chapter 4. In chapter 5 we will map this elliptic curve to a circle. The ratio will be defined in chapter 6. The conclusion can be found in chapter 7.

1.1 History

As far as we know, the earliest research on albime triangles happened when David L. Mackay posed a question in the American Mathematical Monthly about them in 1937 [11]. He posed another question about rational albime triangles two years later [12]. In 1940 the chemist Charles W. Trigg (1898-1989) published a "proof" that the only rational albime triangle is the equilateral triangle. In 1972 another false proof was published by the dentist Leon Bankoff (1908-1997). A picture of both of them can be found in figure 1.2 [2].

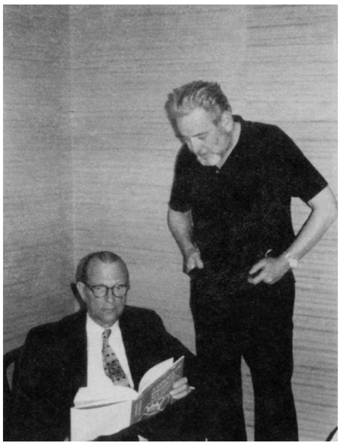


Figure 1.2: Charles W. Trigg and Leon Bankoff

John P. Hoyt asked in the Monthly of April 1991 whether it could be proven or disproven that there are infinitely many pairwise dissimilar rational albime triangles. With this he attached three examples, the triangles with sides $(12, 13, 15)$, $(35, 277, 308)$ and $(26598, 26447, 3193)$.

Richard Guy partially answered Hoyt's question in 1995. He gave almost all ingredients to prove that there are infinitely many rational albime triangles in an article [9] in the Monthly. Guy's article is so complete that almost everything that will be discussed in this thesis can be found in it.

Much earlier, in 1957, Th. J. Kletter was thinking about the problem of having a bisector, the median and the altitude be concurrent in a triangle. He was a teacher at het Mendelcollege in Haarlem, where he asked his students to draw such a triangle. He tried to find an example with integer sides, but never found one besides the equilateral triangle.

After his retirement Kletter moved to Gorssel, where he explained the problem to his neighbor, F. van der Blij, who was determined to solve it. Van der Blij mentions the triangles in December 2003 on an annual teacher day in Groningen for the first time. After this J. Top became intrigued by the subject and wrote an unpublished text about it. This text contained a lot of information already published by Guy.

The distinctive history on the subject was discovered years later. Together with J.S. Chahal, J. Top published three articles about albime triangles [6, 7, 8]. They managed to explicitly answer Hoyt's question; they proved that there are infinitely many pairwise dissimilar rational albime triangles. For more information on the history behind albime triangles, check [6, 7, 8].

In Dutch texts [3, 6] the triangles are called "Kletterdriehoeken", named after Th.J. Kletter. Here we call them albime triangles, named after the altitude, the bisector and the median, a name proposed by J.S. Chahal in May 2012.

1.2 Objective

The goal of this thesis is to extend the work on albime triangles to the case where an external bisector instead of an internal one is used. This idea comes from Kletter's manuscript [10]. Moreover, for the case of the internal bisector we make the distinction between acute albime triangles and obtuse ones.

First it will be shown that any albime triangle can be made from the points P on a certain elliptic curve, such that the x-coordinate of P is in some interval. Since any rational point on this curve can be written as a multiple of the point $P = (2, 2)$, we can use an equidistribution theorem to determine the fraction of those rational points that give rise to an albime triangle with the additional property that all sides have rational length.

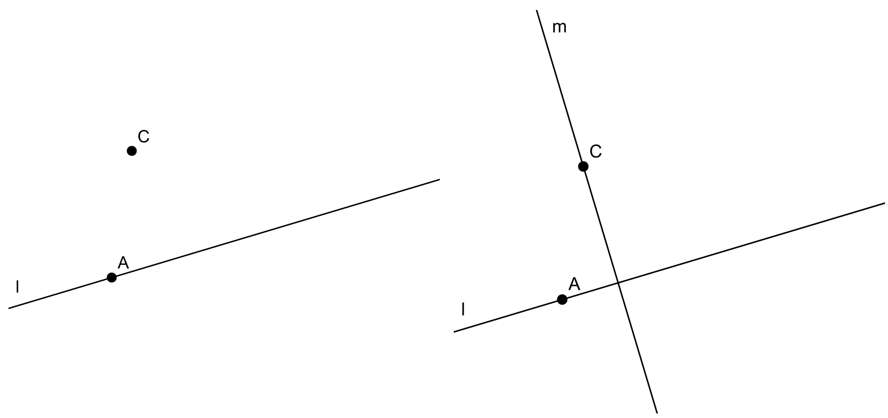
Chapter 2

Internal bisector

2.1 Construction

An albime triangle with an internal bisector can be constructed as follows.

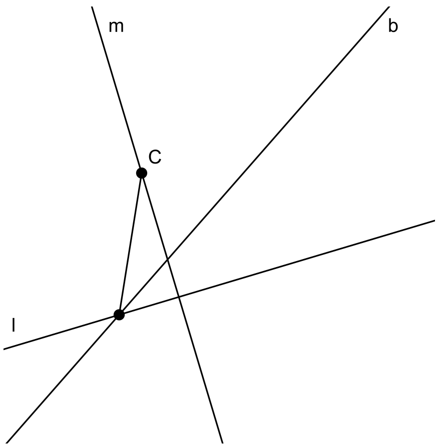
1. Fix a point A, a line ℓ containing A, and a point C not on ℓ.
2. Draw the line m containing C perpendicular to ℓ.



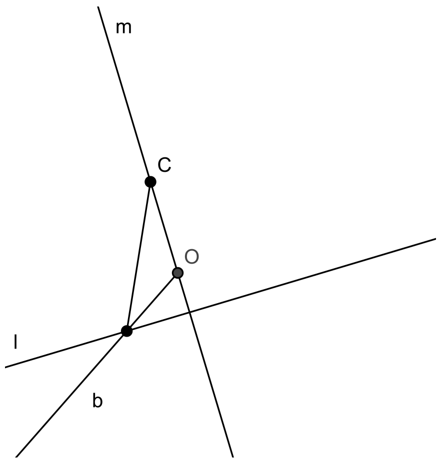
(a) Step 1 (b) Step 2

Figure 2.1: First two steps of making an albime triangle

3. Split the internal angle at A between the lines ℓ and AC in two equal parts using the line b.
4. Intersect the lines m and b and call this point O.



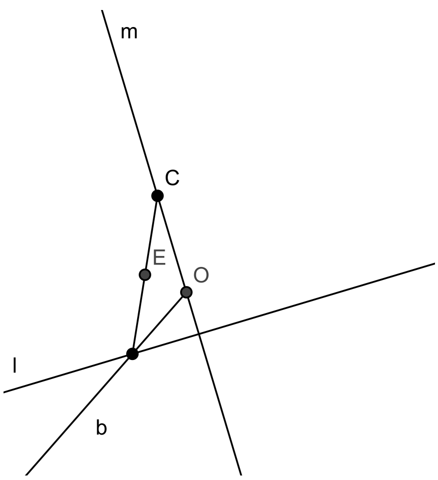
(a) Step 3



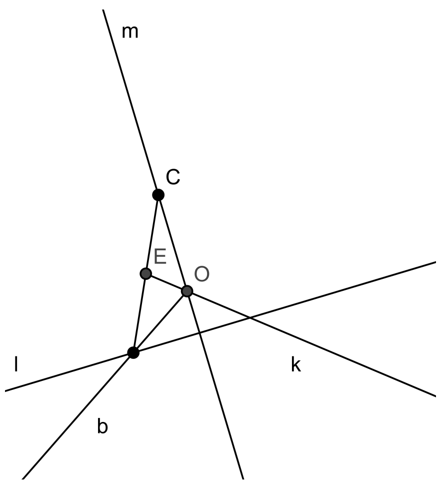
(b) Step 4

Figure 2.2: The next two steps of making an albime triangle

5. Fix the point E in the middle of segment AC.
6. Draw the line k containing E and O.



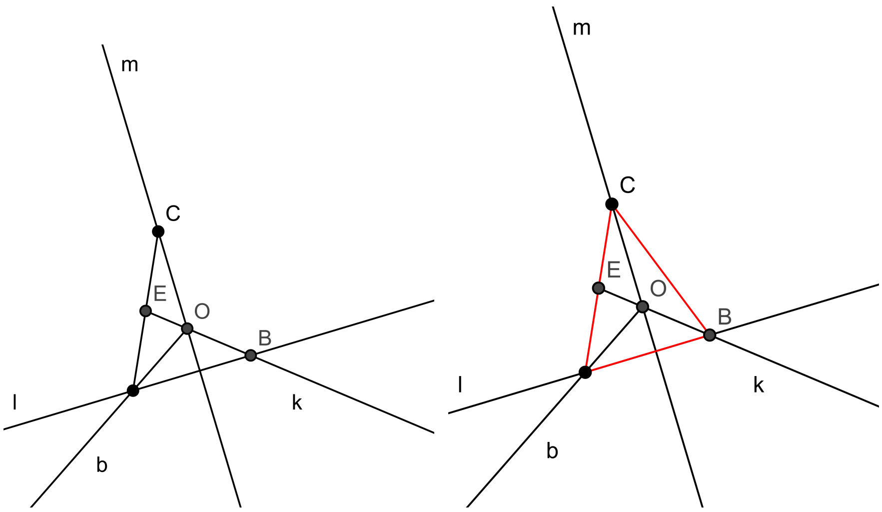
(a) Step 5



(b) Step 6

Figure 2.3: The next two steps of making an albime triangle

7. Use B to name the intersection of k and l.
8. Finish by drawing the albime triangle ABC.



(a) Step 7 (b) Step 8

Figure 2.4: The last two steps of making an albime triangle

We can scale these triangles by any positive real number. This can happen in such a manner that the length of segment AB is in the open interval $(0, 2)$, which we will call I. This will make it easier to say something about the angles in these triangles.

Define $\mathcal{S}$ to be the set of equivalence classes of similar triangles. Since we can scale our albime triangles, we have $\mathcal{A} \subset \mathcal{S}$. Here $\mathcal{A}$ is the set of equivalence classes of similar albime triangles with internal bisector. There is a theorem about the scaling of these triangles. In the proof we will see how we define our sides and why this works.

Theorem 2.1.1. *The map*

$$\Delta : I \to \mathcal{A} \tag{2.1}$$

given by $\Delta(c)$ = *the triangle with side lengths* $c, 2-c$ *and* $\sqrt{c^3 - 4c + 4}$ *is a bijection.*

Before we formally prove this theorem, two complementary theorems are needed. The main theorem needed is known as Ceva's theorem.

Theorem 2.1.2 (Ceva's Theorem)**.** *Given a triangle* ABC *with a point* D *on side* BC*, a point* E *on side* AC *and a point* F *on side* AB*, then the lines* AD*,* BE *and* CF *are concurrent, if and only if*

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1. \tag{2.2}$$

The first part of the proof of this theorem is based on the proof stated in the master thesis written by E. Bakker [3].

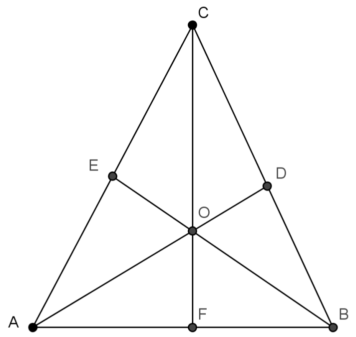


Figure 2.5: The triangle ABC

Proof. There are two cases, namely when the point where they coincide, lets call it O, is inside the triangle and when O is outside the triangle. We will start with the case when O is inside the triangle. We start by constructing the line k parallel to AB through the point C. The point G is where the extension of EB and k intersect. The point H is where the extension of AD and k intersect.

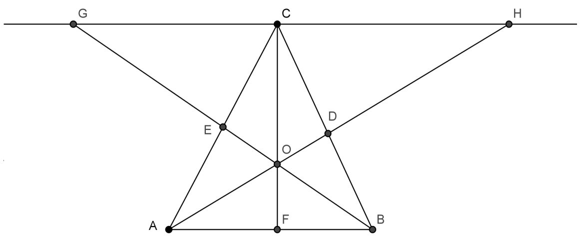


Figure 2.6: O is inside the triangle

This yields the next similar triangles

$$\triangle CDH \sim \triangle BDA \Rightarrow \frac{|BD|}{|DC|} = \frac{|AB|}{|HC|} \qquad *$$

$$\triangle CEG \sim \triangle AEB \Rightarrow \frac{|CE|}{|EA|} = \frac{|CG|}{|AB|} \qquad \bullet$$

$$\triangle COH \sim \triangle FOA \Rightarrow \frac{|HC|}{|AF|} = \frac{|CO|}{|FO|} \qquad \circ$$

$$\triangle COG \sim \triangle FOB \Rightarrow \frac{|CG|}{|FB|} = \frac{|CO|}{|FO|} \qquad \star$$

Combining $\circ$ and $\star$ gives

$$\frac{|HC|}{|AF|} = \frac{|CG|}{|FB|}, \tag{2.3}$$

which implies

$$\frac{|AF|}{|FB|} = \frac{|HC|}{|CG|}. \tag{2.4}$$

Equation (2.2) from Ceva's theorem can be filled in using $*$, $\bullet$ and equation (2.4).

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = \frac{|HC|}{|CG|} \cdot \frac{|AB|}{|HC|} \cdot \frac{|CG|}{|AB|} = 1 \tag{2.5}$$

This proves the first part of the theorem.

To prove the case when O is outside the triangle, we construct a line through C parallel to AB and label the points the same as above. As can be seen in figure 2.7, we have the same similar triangles as in the first part of our proof. Therefore, we may assume that the theorem also holds when O is outside the triangle.

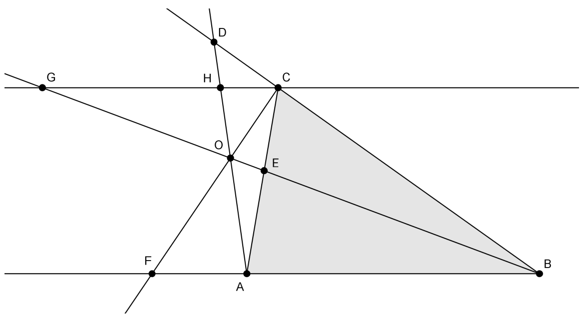


Figure 2.7: O is outside the triangle

The converse is also true. Let D, E and F be points on the lines BC, AC and AB, respectively, such that the equation holds. Let AD and BE intersect in O and let F' be the point where CO crosses AB. Then the theory should also hold for the point D, E and F'. This gives

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1 = \frac{|AF'|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|}, \tag{2.6}$$

which leads to

$$\frac{|AF|}{|FB|} = \frac{|AF'|}{|FB|}. \tag{2.7}$$

Since the ratio can't be the same for two different points, we have that $F = F'$. This concludes our proof. $\square$

Next to this theorem we need a theorem for the bisector, also known as the Angle Bisector Theorem.

Theorem 2.1.3 (Angle Bisector Theorem)**.** *Suppose that in triangle ABC the point D is on side BC. Then AD is an internal bisector, if and only if*

$$\frac{|AB|}{|AC|} = \frac{|BD|}{|DC|}. \tag{2.8}$$

Before using this theorem we state the proof. We will use the law of sines.

Theorem 2.1.4. *The law of sines in trigonometry states*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \tag{2.9}$$

Using the law of sines we can easily prove the Angle Bisector theorem.

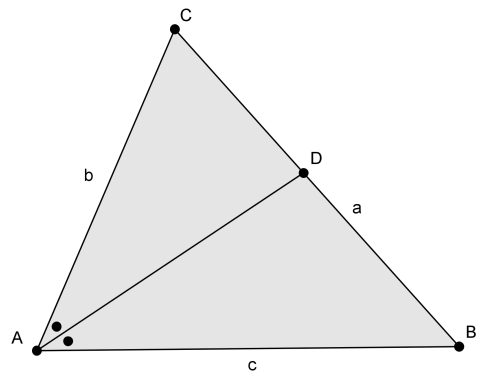


Figure 2.8: Point D is on side BC

Proof of Theorem 2.1.3. Proving the Angle Bisector theorem consists of using the law of sines in the triangles ΔABD and ΔACD. For the first triangle we get

$$\frac{\sin \angle BAD}{|BD|} = \frac{\sin \angle ADB}{|AB|}. \tag{2.10}$$

Rewriting this gives us

$$\frac{|AB|}{|BD|} = \frac{\sin \angle ADB}{\sin \angle BAD}. \tag{2.11}$$

We do the same for ΔACD, resulting in

$$\frac{|AC|}{|DC|} = \frac{\sin \angle ADC}{\sin \angle DAC}. \tag{2.12}$$

We can easily see that $\angle ADB$ and $\angle ADC$ are supplementary angles. This implies that $\sin\angle ADB = \sin\angle ADC$. Because AD is a bisector, we also have that $\angle DAC = \angle BAD$. Therefore the equations we found are equal.

$$\frac{|AB|}{|BD|} = \frac{|AC|}{|DC|} \Leftrightarrow \frac{|AB|}{|AC|} = \frac{|BD|}{|DC|} \tag{2.13}$$

This concludes our proof. □

Having this information, we will start with the proof that is most important in this section. This is a more detailed version of the proof found in [8].

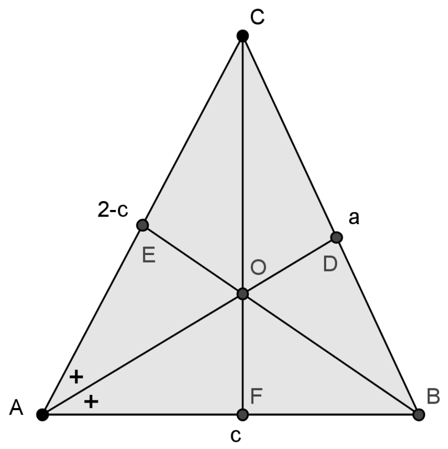


Figure 2.9: Albime triangle with sides a, $2-c$ and c

Proof of Theorem 2.1.1. We will start with describing all possible albime triangles. Let ΔABC be albime and set $a = |BC|$, $b = |AC|$, and $c = |AB|$. We may assume that $b + c = 2$ without loss of generality. This albime triangle can be found in figure 2.9. What we will try to do is find an expression for a in terms of c. Using the Angle Bisector theorem we get the following equality.

$$\frac{|AB|}{|AC|} = \frac{c}{2-c} = \frac{|BD|}{|DC|} \tag{2.14}$$

Since BE is a median, we have that $|CE| = |EA|$. This gives us

$$\frac{|CE|}{|EA|} = 1. \tag{2.15}$$

We also have the next equality.

$$\frac{|AF|}{|FB|} = \frac{|AF|}{c - |AF|} \tag{2.16}$$

Using this information and Ceva's theorem we can express $|AF|$ and $|FB|$ in terms of c.

$$\frac{|AF|}{c-|AF|} \cdot \frac{c}{2-c} \cdot 1 = 1 \tag{2.17}$$

Rewriting this gives

$$\frac{|AF|}{c-|AF|} = \frac{2-c}{c} \tag{2.18}$$

Now multiply both sides of the equation with $c(c-|AF|)$.

$$c \cdot |AF| = (2-c)(c-|AF|) \tag{2.19}$$

We eliminate the brackets and make sure that all the terms that contain $|AF|$ are on the left side of the equation.

$$2|AF| = c(2-c) \tag{2.20}$$

This leads us to an expression for $|AF|$, namely

$$|AF| = \frac{c(2-c)}{2}. \tag{2.21}$$

We know $|BF| = c - |AF|$, thus we can also determine $|BF|$.

$$|BF| = c - \frac{2c-c^2}{2} \tag{2.22}$$

After some simple calculations we get

$$|BF| = \frac{c^2}{2}. \tag{2.23}$$

To find the length of BC we use the two right angled triangles that are in ΔABC, namely ΔAFC and ΔFBC. See figure 2.10. We will apply Pythagoras' theorem in both triangles and use the according side FC to give a value for a in terms of c.

In ΔAFC the Pythagorean theorem states $|FC|^2 = |AC|^2 - |AF|^2$. We know $|AC| = 2-c$ and have already seen an expression for $|AF|$ in equation (2.21).

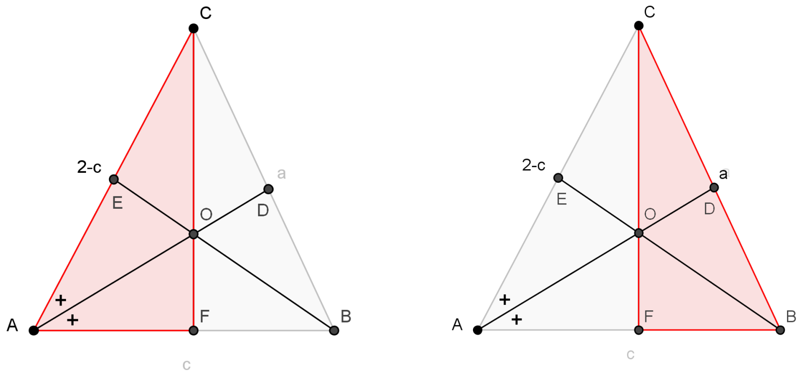
$$|FC|^2 = (2-c)^2 - \frac{c^2(2-c)^2}{4} \tag{2.24}$$

Eliminating the brackets gives

$$|FC|^2 = -\frac{c^4}{4} + c^3 - 4c + 4. \tag{2.25}$$

Doing the same for ΔFBC yields

$$|FC|^2 = a^2 - \frac{c^4}{4} \tag{2.26}$$



(a) The triangle AFC (b) The triangle FBC

Figure 2.10: Two triangles used in this proof

Combining the expressions of $|FC|^2$ found in equation (2.25) and (2.26) gives

$$a^2 - \frac{c^4}{4} = -\frac{c^4}{4} + c^3 - 4c + 4, \tag{2.27}$$

which can be written as

$$a^2 = c^3 - 4c + 4. \tag{2.28}$$

Hence, any albime triangle can be scaled to have sides c, $2 - c$ and $\sqrt{c^3 - 4c + 4}$.

Conversely, any pair of real numbers (a, c) satisfying $0 < c < 2$ and $a^2 = c^3 - 4c + 4$ should give an albime triangle with sides of length $|a|$, $b = 2 - c$ and c, respectively. This can be checked using the triangle inequalities and the fact that $c \in (0, 2)$. The inequality $a \leq b + c$ yields

$$\sqrt{c^3 - 4c + 4} \leq 2 - c + c = 2. \tag{2.29}$$

This can be squared.

$$c^3 - 4c + 4 \leq 4 \tag{2.30}$$

Rewriting equation (2.30) gives us the next inequality.

$$c(c^2 - 4) \leq 0 \tag{2.31}$$

Since c cannot be smaller than zero, we have that $c \geq 0$ and $c^2 - 4 \leq 0$. Rewriting this and taking the square root gives $-2 \leq c \leq 2$, which satisfies $c \in (0, 2)$.

Now we check the inequality $c \leq a + b$. Substituting the values for a and b, we get

$$c \leq \sqrt{c^3 - 4c + 4} + 2 - c. \tag{2.32}$$

We make sure that the square root is on one side of the equation.

$$2c - 2 \leq \sqrt{c^3 - 4c + 4} \tag{2.33}$$

Now we square both sides of the inequality and get

$$4c^2 - 8c + 4 \leq c^3 - 4c + 4. \tag{2.34}$$

Bringing all the terms to one side and factoring out c we get

$$0 \leq c(c-2)^2. \tag{2.35}$$

We know $(c-2)^2$ is always positive and we must have $c \geq 0$. Thus the inequality in (2.35) holds for all c in the interval $(0, 2)$.

The inequality $b \leq c + a$ follows in the same way as we have seen for $c \leq a + b$ above.

Therefore, we know that the map Δ maps to every class of albime triangles in $\mathcal{A}$, since for every albime triangle these theorems and calculations hold. We conclude from this that the map is surjective. Note that for every $c \in (0, 2)$ we get a different albime triangle. Hence, the map is also injective. Thus the map $\Delta : I \to \mathcal{A}$ is bijective and this concludes our proof. $\square$

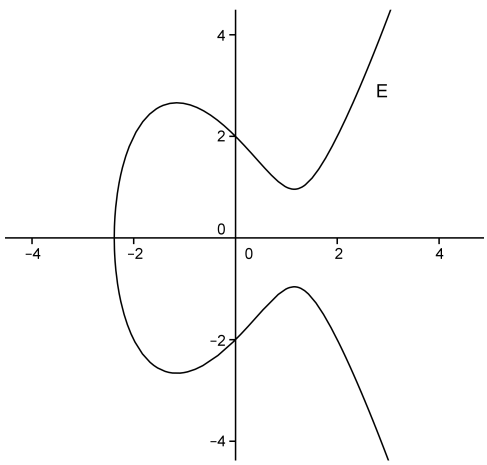


Figure 2.11: The elliptic curve defined by $a^2 = c^3 - 4c + 4$.

The equation $a^2 = c^3 - 4c + 4$ together with a point O at infinity defines an elliptic curve, as seen in figure 2.11. It is also known as Guy's favorite elliptic curve [9]. This curve will be discussed in chapter 4. Next, we will show for which values of c we will have an obtuse albime triangle.

2.2 Obtuse and acute

Using the given lengths of the sides of these triangles we can say something about their angles. To do this we use the generalisation of the Pythagorean Theorem, namely the law of cosines.

Theorem 2.2.1 (The law of cosines)**.** *The law of cosines states*

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C. \tag{2.36}$$

Theorem 2.2.2. *For any albime triangle $\{A, B, C\}$ in $\mathcal{A}$ we have that A and B are acute.*

Proof. First, we will look at the angle of A. We will try to find for which values of $c \in (0, 2)$ we can get that A is obtuse. This angle is obtuse if and only if $\cos A < 0$. Using the law of cosines, we get

$$\frac{b^2 + c^2 - a^2}{2bc} < 0, \tag{2.37}$$

which is equivalent to

$$b^2 + c^2 - a^2 < 0. \tag{2.38}$$

We found the different values for the sides of our albime triangle in the previous section. Substituting $a = \sqrt{c^3 - 4c + 4}$ and $b = 2 - c$ yields

$$(2 - c)^2 + c^2 - (\sqrt{c^3 - 4c + 4})^2 < 0 \tag{2.39}$$

With some simple calculations, this can be written as

$$c^2(2 - c) < 0 \tag{2.40}$$

Because $c \in (0, 2)$, we have that $c^2 > 0$. To satisfy equation (2.40), it is needed to have $2 - c < 0$. This results in c being greater than 2, which is not possible. The other possibility is when $2 - c > 0$. Then $c < 2$, but we would get $c^2 < 0$. This is impossible for any real value of c. Hence, A can never be obtuse.

For the second part of the proof we assume B is obtuse. The angle of B is obtuse if and only if $\cos B < 0$. Using the law of cosines, we get

$$\frac{a^2 + c^2 - b^2}{2ac} < 0, \tag{2.41}$$

which is equivalent to

$$a^2 + c^2 - b^2 < 0. \tag{2.42}$$

Substituting the values for a and b yields

$$(\sqrt{c^3 - 4c + 4})^2 + c^2 - (2 - c)^2 < 0, \tag{2.43}$$

which is equivalent to

$$c^2(c + 2) < 0. \tag{2.44}$$

We know that $c^2 < 0$ is not possible for any real value of c. Hence, c^2 must be bigger than zero which is equivalent to having c greater than zero. Then at the same time $c + 2$ must be smaller than zero, which means $c < -2$. But $c \in (0, 2)$, thus this is a contradiction. Hence, the angle of B can never be obtuse. This concludes our proof. □

This means that an albime triangle can only be obtuse if and only if the angle C is obtuse.

Theorem 2.2.3. *Any albime triangle $\{A, B, C\}$ in $\mathcal{A}$ is obtuse for $c \in (\sqrt{5} - 1, 2)$.*

Proof. Assume C is obtuse, then $\cos C < 0$. The cosine law gives us the following result.

$$\frac{a^2 + b^2 - c^2}{2ab} < 0, \tag{2.45}$$

which is equivalent to

$$a^2 + b^2 - c^2 < 0. \tag{2.46}$$

Substituting the values of a and b in equation (2.46) gives

$$(\sqrt{c^3 - 4c + 4})^2 + (2 - c)^2 - c^2 < 0, \tag{2.47}$$

which can be written as

$$(c - 2)(c^2 + 2c - 4) < 0. \tag{2.48}$$

The case when $c - 2 > 0$, i.e. when $c > 2$ does not occur since c is in the interval $(0, 2)$. Thus $c - 2$ must be smaller than zero. Then equation (2.48) holds if and only if $c^2 + 2c - 4 > 0$.

To make it easier, we will first solve $c^2 + 2c - 4 = 0$ using the quadratic formula and then use the graph to see where the inequalities hold. The quadratic formula gives

$$c = -\sqrt{5} - 1 \ \vee \ c = \sqrt{5} - 1. \tag{2.49}$$

Thus, as seen in figure 2.12, when $c < -\sqrt{5} - 1$ and $c > \sqrt{5} - 1$ inequality (2.48) holds. Hence, we can say that an albime triangle is obtuse if and only if $\sqrt{5} - 1 < c < 2$. This concludes our proof. □

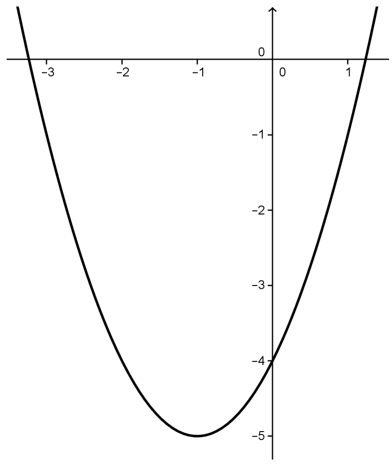


Figure 2.12: Graph of $c^2 + 2c - 4$

From the proof of theorem 2.2.3 we can also deduce that C is right angled for $c = \sqrt{5} - 1$ and that C is acute when $0 < c < \sqrt{5} - 1$. This result has also been discussed in [9].

With this information we can give a percentage of how much albime triangles are obtuse. From a real way of looking at this problem we can already give these percentages.

$$\frac{2 - (\sqrt{5} - 1)}{2} \cdot 100 = 38.1966\ldots\% \tag{2.50}$$

We have 38.2 % of obtuse triangles.

$$\frac{(\sqrt{5} - 1) - 0}{2} \cdot 100 = 61.8034\ldots\% \tag{2.51}$$

This means that from all the triangles that are albime, we have 61.8 % triangles that are acute. Our definition of the percentage will be different and will be more clear after chapter 6.

Chapter 3

External bisector

3.1 Construction

In the previous chapter we saw a method for scaling all albime triangles with an internal bisector so we can say something about their angles. In this chapter we will do something that is similar for the albime triangles with an external bisector. Drawing such a triangle is almost the same as seen in chapter 2. Instead of splitting the internal angle at A, we need to split the external angle at A.

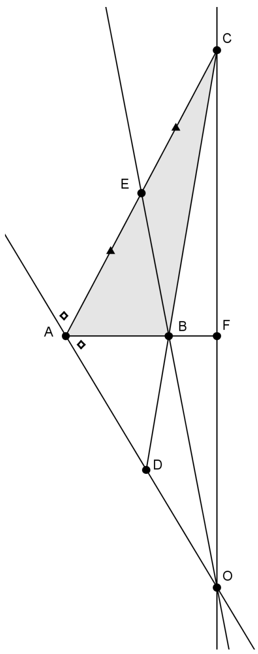


Figure 3.1: An albime triangle with external bisector

We again have that $\mathcal{S}$ is the set of equivalence classes of similar triangles. Then for this case take $\mathcal{B}$ as the set of equivalence classes of albime triangles with external bisector. Then take an interval $J = (-2, 0)$. After scaling these triangles we found the next theorem to be true.

Theorem 3.1.1. *The map*

$$\Theta : J \to \mathcal{B} \tag{3.1}$$

given by $\Theta(c) =$ *the triangle with side lengths* $-c, 2-c$ *and* $\sqrt{c^3 - 4c + 4}$ *is a bijection.*

Instead of using the Angle Bisector Theorem, we use a theorem called the Exterior Angle Bisector Theorem.

Theorem 3.1.2 (Exterior Angle Bisector Theorem)**.** *Suppose we have a triangle* ABC *and the point* D *is on the extention of side* BC*. Then* AD *is an external bisector, if and only if*

$$\frac{|AB|}{|AC|} = \frac{|BD|}{|DC|}. \tag{3.2}$$

Together with Ceva's theorem, we will use the Angle Bisector theorem to prove that theorem 3.1.1 holds.

Proof. We will start with describing all possible albime triangles with external bisector. Let ΔABC be albime and set $a = |BC|$, $b = |AC|$, and $c = |AB|$. Note that we assume that c is positive.

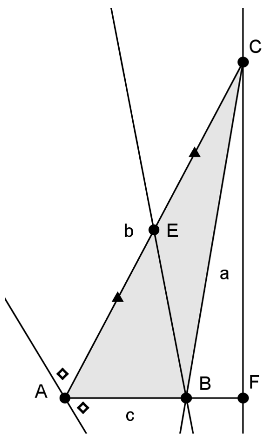


Figure 3.2: The triangle has sides a, b and c

We again want a scaling where b is dependent on c. To do this we want to know something about $|AE|$ and $|GC|$. Because BE is a median, we have

$$\frac{|CE|}{|EA|} = 1. \tag{3.3}$$

Using the exterior angle bisector theorem, we also get a ratio for $|BD|$ and $|DC|$.

$$\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|} = \frac{c}{b} \tag{3.4}$$

With this information we can use equation (2.2) in Ceva's theorem, which gives

$$\frac{|AF|}{|FB|} = \frac{b}{c}. \tag{3.5}$$

In figure 3.3, we have drawn G such that BG is perpendicular to AF. Hence, $CF \parallel BG$. Because $\angle FAC = \angle BAG$ and $\angle AFC = 90^\circ = \angle ABG$, we have that the triangles ABG and AFC are similar.

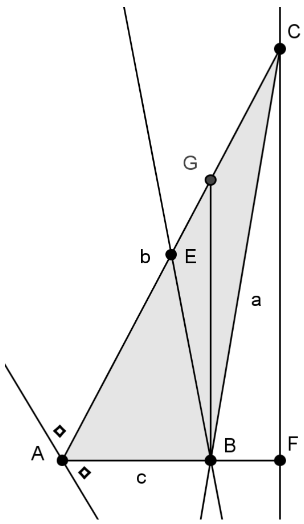


Figure 3.3: The line BG parallel to CF

This means we have the following ratio.

$$\frac{|GC|}{|AC|} = \frac{|FB|}{|AF|} \tag{3.6}$$

Using equation (3.5) and that $|AC| = b$ we get

$$\frac{|GC|}{b} = \frac{c}{b}, \tag{3.7}$$

which leads us to $|GC| = c$.

This means that when we set $|AE| = y$, we have that $b = y + c$. The value of y can be scaled without loss of generality. This result has also been found in [10]. We choose $y = 2$ and get that $b = 2 + c$. Next, an expression for a has to be found.

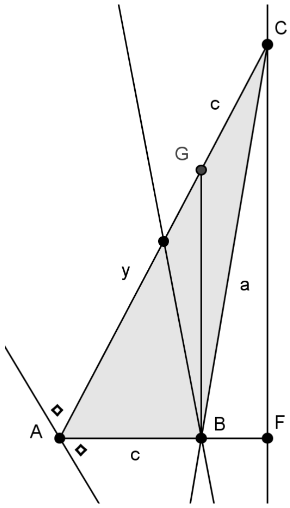


Figure 3.4: Side AC had length $c + y$

To find a value for $|BC|$, use the two right angled triangles that both have FC and where one of them contains BC. These are ΔBFC and ΔAFC, as seen in figure 3.4. We will use Pythagoras' theorem in both triangles and use the according side FC to give a value for a in terms of c. Before we start calculating anything we make the notation easier; we define $|FB| = d$.

In ΔBFC the Pythagorean theorem states $|FC|^2 = |BC|^2 - |FB|^2$, which gives

$$|FC|^2 = a^2 - d^2. \tag{3.8}$$

The Pythagorean theorem in ΔAFC states $|FC|^2 = |AC|^2 - |AF|^2$. We fixed $|AC|$ to be $c + 2$ and we know $|AF| = c + d$. This gives

$$|FC|^2 = (c+2)^2 - (c+d)^2, \tag{3.9}$$

which can be written as

$$|FC|^2 = 4c + 4 - 2cd - d^2. \tag{3.10}$$

Combining equation (3.8) and (3.10) gives

$$a^2 - d^2 = 4c + 4 - 2cd - d^2, \tag{3.11}$$

which can be rewritten as

$$a^2 = 4c + 4 - 2cd. \tag{3.12}$$

We combined the two lengths and got an expression for a^2, but it contains d. As we don't know exactly what $|FB|$ is, this is not an option. Instead we will write it as d expressed in c and a and use Ceva's theorem. Rewriting equation (3.12) gives

$$d = \frac{4c + 4 - a^2}{2c}. \tag{3.13}$$

As seen in equation (3.5), for Ceva's theorem we already have that

$$\frac{|AF|}{|FB|} = \frac{b}{c}. \tag{3.14}$$

Substituting $|FB|$ and $|AF|$ in terms of c and d yields

$$\frac{c+d}{d} = \frac{2+c}{c}. \tag{3.15}$$

We can rewrite the last equality such that it gives d in terms of c. First multiply both sides with cd.

$$(c+d)c = (2+c)d \tag{3.16}$$

We eliminate the brackets and after some simple calculations we get

$$d = \frac{c^2}{2}. \tag{3.17}$$

Next, we combine the expressions for d found in equation (3.13) and (3.17).

$$\frac{4c+4-a^2}{2c} = \frac{c^2}{2} \tag{3.18}$$

Multiplying both sides by $2c$ gives

$$4c+4-a^2 = c^3. \tag{3.19}$$

Hence, we can write

$$a^2 = -c^3 + 4c + 4. \tag{3.20}$$

This is our expression for a, since $-c^3 + 4c + 4 = (-c)^3 - 4(-c) + 4$. So far we have found that for some positive c, Θ maps to a triangle with sides c, $2+c$ and $\sqrt{(-c)^3 - 4(-c) + 4}$.

In the theorem, we talk about the interval $J = (-2, 0)$. Having c in this interval, makes that it is always negative. The length of a side of a triangle can't be negative. Therefore we define our triangle to have sides with lengths $-c$, $2-c$ and $\sqrt{c^3 - 4c + 4}$, which are all positive. This shows that for any albime triangle we have an expression for its sides.

Conversely, any pair of real numbers (a, c) satisfying $-2 < c < 0$ and $a^2 = c^3 - 4c + 4$ gives a triangle as desired, with sides of length $|a|$, $b = 2-c$ and $-c$, respectively. This can be shown in more detail by using the triangle inequality. The inequality $a \leq b + c$ yields

$$\sqrt{c^3 - 4c + 4} \leq 2 - c - c = 2 - 2c. \tag{3.21}$$

This can be squared.

$$c^3 - 4c + 4 \leq 4c^2 - 8c + 4 \tag{3.22}$$

Rewriting equation (3.22) gives us the next inequality.

$$c(c-2)^2 \leq 0 \tag{3.23}$$

Since $(c-2)^2$ is always larger than zero and c cannot be larger than zero, we have that $c \in (-2, 0)$ implies that equation (3.23) holds.

Now we check the inequality $c \leq a + b$. Substituting the values for a, b and c, we get

$$-c \leq \sqrt{c^3 - 4c + 4} + 2 - c. \tag{3.24}$$

We make sure that the square root is on one side of the equation.

$$-2 \leq \sqrt{c^3 - 4c + 4} \tag{3.25}$$

We square both sides of the inequality and get

$$4 \leq c^3 - 4c + 4. \tag{3.26}$$

Bringing all the terms to one side and factoring out c we get

$$c(c^2 - 4) \geq 0. \tag{3.27}$$

For $c(c^2 - 4)$ to be positive, c and $c^2 - 4$ most either be both positive of both negative. Since we know c must be smaller than zero, we get that $c^2 - 4$ must be smaller than zero as well. This happens when $-2 \leq c \leq 2$. Hence, we have that $c \in (-2, 0)$ satisfies this equation.

The inequality $b \leq c + a$ follows in the same way as we have seen for $c \leq a + b$ above.

Therefore, we know that the function maps to every albime triangle in $\mathcal{B}$, since for every albime triangle these theorems and calculations hold. We can conclude from this that the map is surjective. Note that for every $c \in (-2, 0)$ we get a different albime triangle. Hence, the map is also injective. Thus the map $\Theta : I \to \mathcal{B}$ is bijective and this concludes our proof. $\square$

This proof shows that when we take $c \in (-2, 0)$, we define the same elliptic curve as for the albime triangles with an internal bisector. This will be discussed in more detail in chapter 4.

3.2 Obtuse

In this section it will be shown that any albime triangle with an external bisector is obtuse by proving the next theorem.

Theorem 3.2.1. *Any albime triangle $\{A, B, C\}$ with external bisector in $\mathcal{B}$ is obtuse.*

This theorem will be proven by showing that B is always obtuse.

Proof. In this proof a result from the proof of theorem 3.1.1 will be used. We take $c \in (0, 2)$ such that we get the albime triangle with sides c, $c + 2$ and $\sqrt{(-c)^3 - 4(-c) + 4}$. Assume B is acute, then we have that the cosine of B is larger than zero. Using the cosine law seen in theorem 2.2.1, we have that

$$\frac{a^2 + c^2 - b^2}{2ac} > 0 \tag{3.28}$$

which is equivalent to

$$a^2 + c^2 - b^2 > 0. \tag{3.29}$$

Filling in the values of a, b and c we get

$$(\sqrt{(-c)^3 - 4(-c) + 4})^2 + c^2 - (c + 2)^2 > 0. \tag{3.30}$$

Rewriting this gives

$$-c^3 > 0. \tag{3.31}$$

Looking at the graph of this function in figure 3.5 gives that c must be smaller than zero. This cannot happen since we assumed $c \in (0, 2)$, hence B is always obtuse. This concludes our proof. $\square$

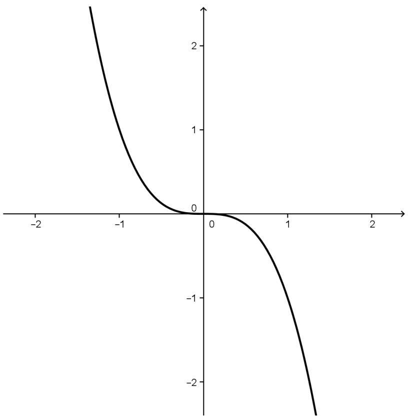


Figure 3.5: Graph of $-c^3$

This means 100% of our albime triangles with an external bisector are obtuse. In the next chapter we will start with looking at the elliptic curves that we found.

Chapter 4

Elliptic curves

Some basic knowledge of the subject elliptic curves will be discussed here. The points of any elliptic curve form a group, together with a point $\mathcal{O}$ at infinity. Moreover, this group is an Abelian group with zero element $\mathcal{O}$, which we will denote by $E(\mathbb{R})$. The addition in these groups is defined by the so called chord and tangent method.

To add two points, P and Q, in $E(\mathbb{R})$ we have to construct a line ℓ going through those two points. We obtain another point in $E(\mathbb{R}) \cap \ell$ which is neither P nor Q. Let us call it R. Now, take the reflection of R in the x-axis and call it R'. Then $P+Q = R'$ is our addition for this group. For more basic knowledge about elliptic curves, one can check [14].

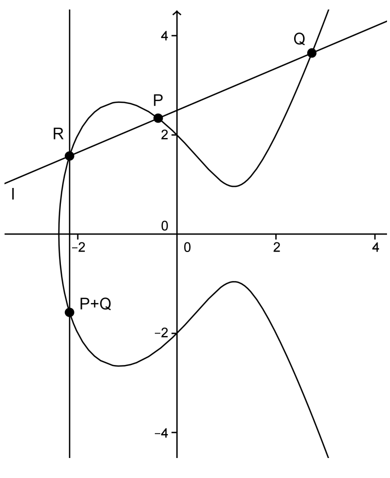


Figure 4.1: Addition on $E(\mathbb{R})$

4.1 Albime triangles

First we will show that $E(\mathbb{R})$ is a group when $E : y^2 = x^3 - 4x + 4$. Then it will be shown that $E(\mathbb{Q})$ is a subgroup of $E(\mathbb{R})$. After that some finale statements about this elliptic curve will be made.

Theorem 4.1.1. *When E is defined by $y^2 = x^3 - 4x + 4$,*

$$E(\mathbb{R}) = \{(x, y) \in \mathbb{R} \mid y^2 = x^3 - 4x + 4\} \tag{4.1}$$

is a group.

Proof. To show $E(\mathbb{R})$ is a group, we have to show the closure, its associativity, and the inverse element according to the addition defined in section 4.1. To do this, we first make a formula which describes the addition in terms of coordinates in the xy-plane. This has also been done by E. Bakker in [3] and R. Guy in [14] for an arbitrary elliptic curve. Start with the points $P = (a, b)$ and $Q = (c, d)$ on $E(\mathbb{R})$. There are three cases:

1. $a \neq c$ and $b \neq d$
2. $a = c$ and $b \neq d$
3. $P = Q$.

The first case is the one with the most calculations. We start with defining a line ℓ through P and Q,

$$\ell : y = \alpha x + \beta \tag{4.2}$$

with

$$\alpha = \frac{b - d}{a - c} \tag{4.3}$$

and

$$\beta = b - a\alpha. \tag{4.4}$$

To find an expression for the third intersection point $R = (e, f)$, we have to equate this with the elliptic curve.

$$y^2 = (\alpha x + \beta)^2 = x^3 - 4x + 4 \tag{4.5}$$

Rewriting this gives

$$x^3 - \alpha^2 x^2 - (4 + 2\alpha\beta)x + 4 - \beta^2 = 0. \tag{4.6}$$

Since we know the x-coordinates of all three intersection points, we can factorize the expression in equation (4.6) to get

$$x^3 - \alpha^2 x^2 - (4 + 2\alpha\beta)x + 4 - \beta^2 = (x - a)(x - c)(x - e). \tag{4.7}$$

Reworking the left side of equation (4.7) gives us

$$x^3 - (a + c + e)x^2 + (e(a + c) + ac)x - ace. \tag{4.8}$$

Looking at the constants belonging to the x^2 term gives us an expression for e, which is the x-coordinate of R.

$$e = \alpha^2 - a - c \tag{4.9}$$

The y-coordinate is then given by $\alpha^3 - a\alpha - c\alpha + \beta$. The only thing left to do is taking the reflection of R in the x-axis. This gives the equation to find $P + Q$ for two points P and Q in $E(\mathbb{R})$

$$P + Q = (\alpha^2 - a - c, -\alpha^3 + a\alpha + c\alpha - \beta), \tag{4.10}$$

where α and β are still given by the values in equations (4.3) and (4.4).

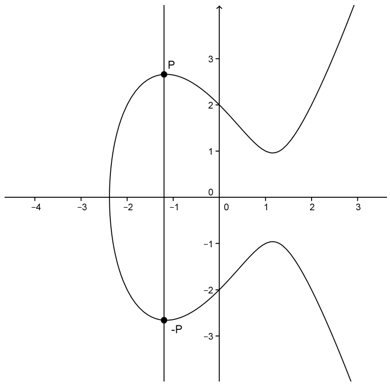


Figure 4.2: Adding P to $-P$

The second case is somewhat easier. When a is equal to c we have that $b = -d$, since Guy's favorite elliptic curve is mirrored in the x-axis. Then $P = (a, b)$ gives $Q = (a, -b)$, which we define as $-P$. It is clear that this line is vertical, which means the third intersection point with the elliptic curve is $\mathcal{O}$ at infinity. We get that $P + (-P) = \mathcal{O}$, which defines the inverse element in $E(\mathbb{R})$.

When P is equal to Q, we have to calculate $P + P$. Then ℓ will be the tangent to P on $E(\mathbb{R})$.

We find the formula for the line $\ell : y = \lambda x + \mu$ by determining the implicit derivative of E at P. This gives

$$2y\frac{dy}{dx} = 3x^2 - 4, \tag{4.11}$$

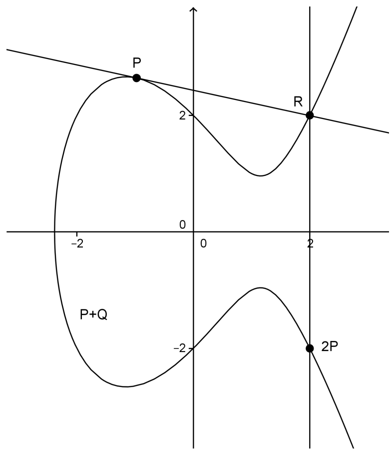


Figure 4.3: Addition of P to itself

which results in

$$\lambda = \frac{3a^2 - 4}{2b}. \tag{4.12}$$

After finding an expression for μ we get a formula for the tangent ℓ.

$$\ell : y = \frac{3a^2 - 4}{2b}x + \frac{2b^2 - 3a^3 - 4a}{2b}. \tag{4.13}$$

Equating this with our elliptic curve gives

$$x^3 - \lambda^2 x^2 - (4 + 2\lambda\mu)x + 4 - \mu^2 = 0, \tag{4.14}$$

which can be factorized into

$$(x - a)(x - a)(x - e). \tag{4.15}$$

Rewriting this results in

$$x^3 - (e + 2a)x^2 + (a^2 + 2ae)x + a^2 e. \tag{4.16}$$

Comparing the x^2 term in equations (4.14) and (4.16), gives

$$e = \lambda^2 - 2a = \frac{9a^2 - 24a^2 - 8ab^2 + 16}{4b^2}. \tag{4.17}$$

Using $b^2 = a^3 - 4a + 4$, we get

$$e = \frac{1}{4}\frac{(a - 2)(a^3 + 2a^2 + 12a - 8)}{a^3 - 4a + 4} \tag{4.18}$$

Using $y = \lambda x + \mu$ and that we have to take the mirror imaged of the y-coordinate in the x axis, we get

$$y_{2P} = -\frac{1}{4}\lambda\left(\frac{(a - 2)(a^3 + 2a^2 + 12a - 8)}{a^3 - 4a + 4}\right) - \mu. \tag{4.19}$$

Then we have an expression for $2P$, namely

$$\left(\frac{1}{4}\frac{(a-2)(a^3+2a^2+12a-8)}{a^3-4a+4}, \frac{1}{8}\frac{b(a^6-20a^4+80a^3-80a^2+64a-64)}{(a^3-4a+4)^2}\right).$$

We defined the addition in such a way that for any P and Q in $E(\mathbb{R})$, $P+Q$ is also in $E(\mathbb{R})$. This satisfies the closure condition of a group. We have already seen that for any P, there exist an element $-P \in E(\mathbb{R})$ such that

$$P + (-P) = \mathcal{O}. \tag{4.20}$$

This means we have fulfilled the necessity of having an inverse. The identity element is set to be $\mathcal{O}$ at infinity. Rewriting equation (4.20) gives us $P + \mathcal{O} = P$, which verifies that $\mathcal{O}$ is indeed the identity.

Finally, we need to show the associativity of this addition. That is, for any points $P = (a, b)$, $Q = (c, d)$ and $R = (g, h)$ in $E(\mathbb{R})$ we have

$$(P + Q) + R = P + (Q + R). \tag{4.21}$$

Using the addition defined above we can express $P+Q$, $Q+R$, $(P+Q)+R$ and $P + (Q + R)$ in terms of a, b, c, d, g and h. Let us define $P + Q = (p, q)$. Then

$$p = \left(\frac{b-d}{a-c}\right)^2 - a - c \tag{4.22}$$

and

$$q = \left(\frac{b-d}{a-c}\right) \cdot \left(-\left(\frac{b-d}{a-c}\right)^2 + 2a + c\right) - b. \tag{4.23}$$

Doing the same for $Q + R = (r, s)$ yields

$$r = \left(\frac{d-f}{c-e}\right)^2 - c - e \tag{4.24}$$

and

$$s = \left(\frac{d-f}{c-e}\right) \cdot \left(-\left(\frac{d-f}{c-e}\right)^2 + 2c + e\right) - d. \tag{4.25}$$

Let $(P + Q) + R$ be denoted by (u, v) and $P + (Q + R)$ by (w, z). For these to be equal, it is sufficient to show that $u - w = 0$. This can be done by giving an expression for both u and w. Using the addition as seen before, we get

$$u = \left(\frac{q-f}{p-e}\right)^2 - p - e \tag{4.26}$$

and

$$w = \left(\frac{b-s}{a-r}\right)^2 - a - r, \tag{4.27}$$

where p, q, r and s are defined as above.

Calculating the whole expression for u and w can be done in Maple 17. Using the next script we find that $u - w$ is indeed zero.

$alias(b =$RootOf$(-a^3 + _Z^2 + 4 \cdot a - 4))$;
$alias(d =$RootOf$(-c^3 + _Z^2 + 4 \cdot c - 4))$;
$alias(h =$RootOf$(-g^3 + _Z^2 + 4 \cdot g - 4))$;
$p := \left(\frac{(b-d)}{(a-c)}\right)^2 - a - c;$
$q := \left(\frac{(b-d)}{(a-c)}\right) \cdot \left(-\left(\frac{(b-d)}{(a-c)}\right)^2 + 2 \cdot a + c\right) - b;$
$r := \left(\frac{(d-f)}{(c-e)}\right)^2 - c - e;$
$s := \left(\frac{(d-f)}{(c-e)}\right) \cdot \left(-\left(\frac{(d-f)}{(c-e)}\right)^2 + 2 \cdot c + e\right) - d;$
$u := \left(\frac{(q-f)}{(p-e)}\right)^2 - p - e;$
$w := \left(\frac{(b-s)}{(a-r)}\right)^2 - a - r;$
$simplify(u - w);$

This proves that the addition is associative and concludes our proof. □

We can now show that $E(\mathbb{Q})$ is a subgroup of $E(\mathbb{R})$. The group $E(\mathbb{Q})$ contains the albime triangles with sides of rational length. Since there are infinitely many elements in $\mathbb{Q}$, we have that the same curve denotes all these points.

Theorem 4.1.2. *$E(\mathbb{Q})$ is a subgroup of $E(\mathbb{R})$.*

Proof. $E(\mathbb{Q})$ is a subgroup of $E(\mathbb{R})$ if and only if it is closed under products and inverses. The addition will be defined the same as for $E(\mathbb{R})$. The identity element is $\mathcal{O}$ at infinity.

Take $P = \left(\frac{a}{a_1}, \frac{b}{b_1}\right)$ and $Q = \left(\frac{c}{c_1}, \frac{d}{d_1}\right)$ in $E(\mathbb{Q})$. Then the addition gives

$$P + Q = \left(\lambda^2 - \frac{a}{a_1} - \frac{c}{c_1}, -\lambda^3 + \frac{a}{a_1}\lambda + \frac{c}{c_1}\lambda - \mu\right), \tag{4.28}$$

with

$$\lambda = \frac{(bd_1 - db_1)a_1c_1}{(ac_1 - ca_1)b_1d_1} \tag{4.29}$$

and

$$\mu = \frac{b}{b_1} - \frac{a}{a_1}\lambda. \tag{4.30}$$

It can easily be seen that these coordinates are in $\mathbb{Q}$. The coordinates also satisfy the equation $x^2 = y^3 - 4y + 4$ by definition of the addition. Thus, $E(\mathbb{Q})$ is closed under products.

Since the indentity element is the same, we know for any $P \in E(\mathbb{Q})$ that $P + \mathcal{O} = P$. This can also be written as $P + (-P) = \mathcal{O}$. Then $-P$ is our inverse element and is defined as

$$-P = \left(\frac{a}{a_1}, -\frac{b}{b_1} \right). \tag{4.31}$$

This means that for every element in $E(\mathbb{Q})$ we have constructed an inverse element that is also in $E(\mathbb{Q})$, see figure 4.2. This means $E(\mathbb{Q})$ is a subgroup of $E(\mathbb{R})$. This concludes our proof. □

Finally, we will discuss where the albime triangles are on this curve and how we can say something meaningfull about these points. To do this we need to show that $E(\mathbb{Q})$ is generated by one element, namely $P = (2, 2)$.

Theorem 4.1.3. *All elements in $E(\mathbb{Q})$ are multiples of the point $P = (2, 2)$.*

The first time it was shown that this curve can be generated by one element was in 1963 in the Journal für die reine und angewandte Mathematik [5]. These notes by B.J. Birch and H.P.F. Swinnerton-Dyer tell us about calculations that where done for elliptic curves of the form $\Gamma : y^2 = x^3 - Ax - B$. Checking the table on page 21 gives that when $A = 4$ and $B = -4$, we get a rank of 1. This means that the group is generated by one element.

The element that is the generator for this group can be found in the book Modular Functions of One Variable IV [4]. In table 1 it can be seen that our curve, given by 88A, has only one generator which has infinite order. The point $(2, 2)$ can be found in table 2. The book states that this is either the generator or the multiple of the generator that is not given by integers. This has been disproven by Joseph H. Silverman in his book; Rational Points on Elliptic Curves [15].

We can verify that this generator is correct by using the magma calculator [1]. The following code

```
E:=EllipticCurve([-4,4]);
MordellWeilGroup(E);
Generators(E);
```

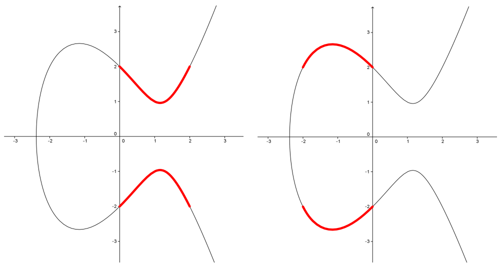
defines the elliptic curve $y^2 = x^3 - 4x + 4$, then gives the number of generators and all these generators. The output is

```
Abelian Group isomorphic to Z
Defined on 1 generator (free)
[ (2 :  2 :  1) ],
```

which means that the group $E(\mathbb{Q})$ is infinite cyclic, with $P = (2, 2)$ as generator. Then it is also isomorphic to $\mathbb{Z}$.

$$E(\mathbb{Q}) = \{nP \,|\, n \in \mathbb{Z}\} \cong \mathbb{Z} \tag{4.32}$$

Eventually we want to say something about which points generated by $P = (2, 2)$ give an albime triangle. In the chapters above we have seen that all these albime triangles are created using points in the red parts of figure 4.4. We will calculate some multiples of $P = (2, 2)$ in chapter 6. These can also be found in table 1 of Guy's article [9].



(a) Internal bisector (b) External bisector

Figure 4.4: The points on the curve that result in an albime triangle

Chapter 5

Mapping to a circle

There are infinitely many rational points on $y^2 = x^3 - 4x + 4$. We are interested in the points on this curve that give an albime triangle. In chapter 2 and 3 it has been shown that this is when $0 < c < 2$ for the inner bisector and $-2 < c < 0$ for the outer bisector.

To say something about how many points of this curve are in this interval, we will have to map the elliptic curve to a circle. This will be done by mapping to the interval $(0, 1)$ first. The idea for this comes from [16], which has been executed in [3, 8].

5.1 Mapping to (0,1)

To construct this map we will use the only real zero Guy's favorite elliptic curve has. To calculate this point, we will use a method described in a lecture by J. Top [17]. This method was invented by N. Tartaglia (1500–1557). It exists of substituting $a + b$ for some of the x terms. To find our real zero we have to solve $x^3 - 4x + 4 = 0$. Using the substitution, we get

$$(a + b)^3 = 4x - 4 \tag{5.1}$$

which can also be written as

$$a^3 + b^3 + 3ab(a + b) = 4x - 4. \tag{5.2}$$

Here $(a + b)$ can be replaced by x again. This gives us

$$3abx + a^3 + b^3 = 4x - 4. \tag{5.3}$$

Solving $x^3 - 4x + 4 = 0$ is then the same as solving the system

$$\begin{cases} 3ab = 4 \\ a^3 + b^3 = -4 \end{cases} \tag{5.4}$$

Rewriting the first of those two equations by cubing both sides yields the system

$$\begin{cases} a^3b^3 = \frac{64}{27} \\ a^3 + b^3 = -4 \end{cases} \tag{5.5}$$

Setting up an equation that is quadratic containing both these expressions is then easier to solve.

$$(t - a^3)(t - b^3) = t^2 - (a^3 + b^3)t + a^3b^3 = 0 \tag{5.6}$$

Combining this with the system in (5.5) gives

$$t^2 + 4t + \frac{64}{27} = 0, \tag{5.7}$$

which can be solved using the quadratic method. Then we get

$$t = -2 \pm \frac{2}{9}\sqrt{33}. \tag{5.8}$$

This means that we can take $a^3 = -2 + \frac{2}{9}\sqrt{33}$. Taking the cubic root of this expression gives $a = (-2 + \frac{2}{9}\sqrt{33})^{\frac{1}{3}}$. Combining this with the first case in the system found in (5.5) yields an expression for b.

$$b = \frac{4}{3 \cdot (-2 + \frac{2}{9}\sqrt{33})^{\frac{1}{3}}} \tag{5.9}$$

Our zero is then given by $a + b$. We call this zero α.

$$\alpha = a + b = (-2 + \frac{2}{9}\sqrt{33})^{\frac{1}{3}} + \frac{4}{3 \cdot (-2 + \frac{2}{9}\sqrt{33})^{\frac{1}{3}}} \tag{5.10}$$

Using a program like Matlab R2012b, Maple 17 or an ordinary calculator gives us an idea of the value of α.

$$\alpha \approx -2.382975767\ldots \tag{5.11}$$

Using this real zero, we define

$$\Omega := 2\int_{\alpha}^{\infty} \frac{\mathrm{d}t}{\sqrt{f(t)}}, \tag{5.12}$$

with $f(t) = t^3 - 4t + 4$. For any point $P = (x, y)$ which is not infinity, we can construct $\varphi : E(\mathbb{R}) \to (0, 1)$.

$$\varphi(P) := \begin{cases} \frac{1}{\Omega}\int_x^{\infty} \frac{\mathrm{d}t}{\sqrt{f(t)}} & \text{for } y \geq 0 \\ 1 - \frac{1}{\Omega}\int_x^{\infty} \frac{\mathrm{d}t}{\sqrt{f(t)}} & \text{for } y \leq 0 \end{cases} \tag{5.13}$$

We define $\varphi(\mathcal{O}) = 0$. For our only real zero we find

$$\varphi(\alpha, 0) = \frac{1}{2}. \tag{5.14}$$

As seen in figure 2.11, our x-coordinate can only get bigger than α. This means the interval on which we integrate will get smaller. For $y \geq 0$ this leads to $\varphi(x, y)$ getting smaller and for $y \leq 0$ it will get larger.

Hence, if we take $\alpha \leq x_1 < x_2$ and $y_i = \sqrt{x_i^3 - 4x_i + 4}$ it holds that

$$0 < \varphi(x_2, y_2) < \varphi(x_1, y_1) \leq \frac{1}{2} \tag{5.15}$$

and

$$\frac{1}{2} \leq \varphi(x_1, -y_1) < \varphi(x_2, -y_2) < 1. \tag{5.16}$$

Then we have that φ indeed maps to $(0, 1)$. This also shows that the map is bijective.

5.2 Mapping to a circle

The unit circle group under multiplication will be denoted by

$$\mathbb{T} := \{z \in \mathbb{C}^* \mid |z| = 1\}. \tag{5.17}$$

The map $\Psi : E(\mathbb{R}) \to \mathbb{T}$ is given by

$$\Psi(P) := e^{2\pi i \varphi(P)}. \tag{5.18}$$

We have already seen in chapter 4 that $E(\mathbb{R})$ is a group. It is well-known that $\mathbb{T}$ is also a group. To use this map in the way we want, the next theorem has to be proven.

Theorem 5.2.1. *The map*

$$\Psi : E(\mathbb{R}) \to \mathbb{T} \tag{5.19}$$

defines an isomorphism of groups.

A map is called an isomorphism of groups, if it is a group homomorphism that is also bijective. To show that a map is a group homomorphism we use the next lemma.

Lemma 5.2.2. Ψ *is a group homomorphism if and only if*

(i) for all points P on E we have $\varphi(-P) + \varphi(P) \in \mathbb{Z}$,

(ii) for all points P_1, P_2 and P_3 on E we have if $P_1 + P_2 + P_3 = \mathcal{O}$, then $\varphi(P_1) + \varphi(P_2) + \varphi(P_3) \in \mathbb{Z}$.

The proof of this lemma can be found in [3]. It is also stated here.

Proof. ($\Rightarrow$) Assume Ψ is a group homomorphism, then

$$\Psi(P+Q) = \Psi(P) \cdot \Psi(Q). \tag{5.20}$$

Rewriting this gives

$$e^{2\pi i\varphi(P+Q)} = e^{2\pi i\varphi(P)} \cdot e^{2\pi i\varphi(Q)}, \tag{5.21}$$

which implies

$$\varphi(P+Q) - \varphi(P) - \varphi(Q) \in \mathbb{Z}. \tag{5.22}$$

This holds for any P and Q on E. Hence, we can choose P and Q to be any point. This means we have

$$\varphi(\mathcal{O}) - \varphi(P) - \varphi(-P) \in \mathbb{Z} \tag{5.23}$$

and

$$\varphi(\mathcal{O}) - \varphi(\mathcal{O}) - \varphi(\mathcal{O}) \in \mathbb{Z}. \tag{5.24}$$

Which implies $-\varphi(\mathcal{O}) \in \mathbb{Z}$ and therefore $\varphi(P) + \varphi(-P) \in \mathbb{Z}$. This proves the first part of the theorem.

Using equation (5.22), $P = P_1$ and $Q = P_2 + P_3$ we get

$$\varphi(\mathcal{O}) - \varphi(P_1) - \varphi(P_2 + P_3) \in \mathbb{Z}. \tag{5.25}$$

Hence, $\varphi(P_1) + \varphi(P_2 + P_3)$ is in $\mathbb{Z}$. Choosing $P = P_2$ and $Q = P_3$ gives

$$\varphi(P_2 + P_3) - \varphi(P_2) - \varphi(P_3) \in \mathbb{Z}. \tag{5.26}$$

Subtracting the second from the first equation gives

$$\varphi(P_1) + \varphi(P_2) + \varphi(P_3) \in \mathbb{Z}, \tag{5.27}$$

which proves the second part of the theorem.
($\Leftarrow$) We need to show that $\Psi(P+Q) = \Psi(P) \cdot \Psi(Q)$. By definition of Ψ we have

$$\Psi(P+Q) = e^{2\pi i\varphi(P+Q)}. \tag{5.28}$$

Also, we have $(P+Q)+(-P)+(-Q) = 0$. By (ii), we get $\varphi(P+Q)+\varphi(-P)+\varphi(-Q) \in \mathbb{Z}$. Rewriting this gives

$$\varphi(P+Q) = -\varphi(-P) - \varphi(-Q) + z, \tag{5.29}$$

with $z \in \mathbb{Z}$. Combining this with equation (5.28) gives us

$$\Psi(P+Q) = e^{2\pi i(-\varphi(-P)-\varphi(-Q)+z)}. \tag{5.30}$$

In the same fashion, using (i) gives

$$\varphi(-P) = -\varphi(P) + z_p \tag{5.31}$$

for any P on E. Then combining this again with equation (5.30) yields

$$\Psi(P+Q) = e^{2\pi i(\varphi(P) - z_P + \varphi(Q) - z_Q + z)}. \tag{5.32}$$

This can be written as

$$e^{2\pi i\varphi(P)} \cdot e^{-2\pi i z_P} \cdot e^{2\pi i\varphi(Q)} \cdot e^{2\pi i(-z_Q + z)}. \tag{5.33}$$

It is known that any e to the power $2\pi i$ multiplied by an integer reduces to 1. Then our equation will become

$$\Psi(P+Q) = e^{2\pi i\varphi(P)} \cdot e^{2\pi i\varphi(Q)}. \tag{5.34}$$

Which is exactly what we wanted to prove. $\square$

To show that a group homomorphism is bijective, it has to be shown that this map takes inverses to inverses. With this knowledge a proof of theorem 5.2.1 can be stated.

Proof of theorem 5.2.1. The proof of this theorem consist of three parts. We will start with showing that part (i) of lemma 5.2.2 holds. We take an arbitrary point $P = (a, b)$ in $E(\mathbb{R})$. As seen in chapter 4, we have that $-P$ is defined to be $(a, -b)$. Using the definition of the map φ we get

$$\varphi(-P) + \varphi(P) = \left(1 - \frac{1}{\Omega}\int_x^\infty \frac{\mathrm{d}t}{\sqrt{f(t)}}\right) + \frac{1}{\Omega}\int_x^\infty \frac{\mathrm{d}t}{\sqrt{f(t)}} = 1 \in \mathbb{Z}, \tag{5.35}$$

when b is greater than or equal to zero. If we take b less than or equal to zero, we have

$$\varphi(-P) + \varphi(P) = \frac{1}{\Omega}\int_x^\infty \frac{\mathrm{d}t}{\sqrt{f(t)}} + \left(1 - \frac{1}{\Omega}\int_x^\infty \frac{\mathrm{d}t}{\sqrt{f(t)}}\right) = 1 \in \mathbb{Z}. \tag{5.36}$$

Hence, $\varphi(-P) + \varphi(P) \in \mathbb{Z}$ holds for any P.

Next, we will show that Ψ takes inverses to inverses. Define $P = (a, b)$, then the inverse is defined to be $-P = (a, -b)$, where $b \geq 0$. By definition of Ψ, we have

$$\Psi(-P) = e^{2\pi i\varphi(-P)}. \tag{5.37}$$

Using how the map φ is defined together with equation (5.37) , we get

$$\Psi(-P) = e^{2\pi i\left(1 - \frac{1}{\Omega}\int_x^\infty \frac{\mathrm{d}t}{\sqrt{f(t)}}\right)}. \tag{5.38}$$

Rewriting this gives

$$\Psi(-P) = 1 \cdot e^{-\left(2\pi i \frac{1}{\Omega} \int_x^\infty \frac{\mathrm{d}t}{\sqrt{f(t)}}\right)}, \tag{5.39}$$

where $\varphi(P)$ can be recognized. Substituting this yields

$$\Psi(-P) = e^{-(2\pi i\varphi(P))} = \left(e^{2\pi i\varphi(P)}\right)^{-1}. \tag{5.40}$$

Hence, we have $\Psi(-P) = (\Psi(P))^{-1}$. This is exactly what we wanted to show.

Finally, it needs to be shown that part (ii) of lemma 5.2.2 holds. A readable version can be found in the master thesis of E. Bakker [3]. For completeness one can check the article [8]. $\square$

This result will be used in the next chapter to show how many elements generated by the point $P = (2, 2)$ coincide with an albime triangle.

Chapter 6

The percentage

In this chapter the percentage will be clarified. First, it will be shown which points on the unit circle coincide with albime triangles. This will be done using the maps that were defined in chapter 5. After that we will define our percentage using the fact that the points on $E(\mathbb{Q})$ are generated by the point $P = (2, 2)$.

6.1 Albime triangles on the unit circle

As seen in figure 4.4, the points on $E(\mathbb{R})$ that generate an albime triangle, with an internal bisector, have x- coordinates between 0 and 2. Using the maps φ and Ψ on the two points $(0, 2)$ and $(2, 2)$ gives the image of them on the unit circle.

Using a script in Maple the integrals that define φ can be calculated for each point. This script can be found in the appendix. For the left bound of the albime triangles with internal bisector we get

$$\varphi(0,2) = \frac{\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}}{2\int_\alpha^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}} \approx 0.3612080156\ldots \tag{6.1}$$

Mapping this point to the unit circle can be done using the map Ψ.

$$\Psi(0,2) = e^{2\pi i \cdot 0.3612080156} \approx -0.6432539111 + 0.7656529278i \tag{6.2}$$

This point can be plotted on the unit circle in the complex plane. We do the same for the right bound of this interval.

$$\varphi(2,2) = \frac{\int_2^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}}{2\int_\alpha^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}} \approx 0.1806040078\ldots \tag{6.3}$$

Using the map Ψ we get

$$\Psi(2,2) = e^{2\pi i \cdot 0.3612080156} \approx 0.4223423309 + 0.9064364046i. \tag{6.4}$$

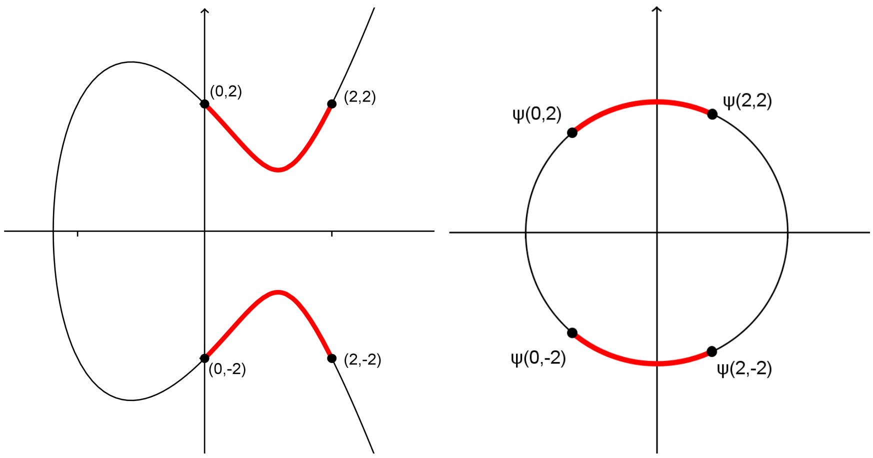
For the negative interval seen in figure 4.4b we have

$$\varphi(0,-2) = 1 - \frac{\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}}{2\int_\alpha^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}} \approx 0.6387919844\ldots \tag{6.5}$$

and

$$\varphi(2,-2) = 1 - \frac{\int_2^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}}{2\int_\alpha^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}} \approx 0.8193959922\ldots \tag{6.6}$$

Using the map Ψ, we get the complex conjugates of the points $\Psi(0,2)$ and $\Psi(2,2)$. The result of applying these maps to those points can be seen in figure 6.1.



(a) Elliptic Curve (b) Unit Circle

Figure 6.1: The points that result in an albime triangle with internal bisector

The albime triangles with external bisector can be mapped to the unit circle in the same way. Computing the map φ and Ψ for the points $(-2,2)$ and $(-2,-2)$ gives us enough information to show a similar mapping as in figure 6.1. Computing the map φ gives

$$\varphi(-2,2) = \frac{\int_{-2}^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}}{\int_\alpha^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}} \approx 0.4581879766\ldots \tag{6.7}$$

and

$$\varphi(-2,-2) = 1 - \frac{\int_{-2}^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}}{\int_\alpha^\infty \frac{\mathrm{d}t}{\sqrt{t^3-4t+4}}} \approx 0.5418120234\ldots \tag{6.8}$$

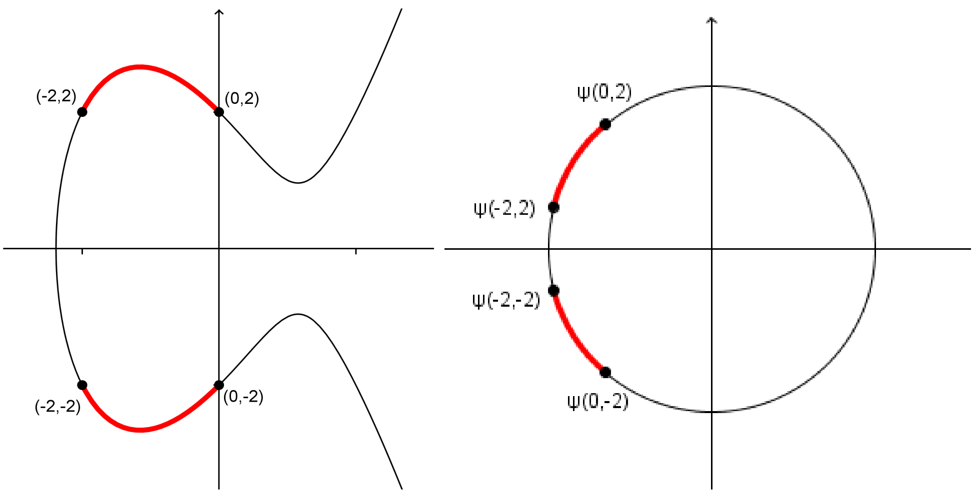
Then applying the map Ψ yields

$$\Psi(-2,2) = e^{2\pi i \cdot 0.4581879766} \approx -0.9656890432 + 0.2597011203i \tag{6.9}$$

and

$$\Psi(-2,-2) = e^{2\pi i \cdot 0.5418120234} \approx -0.9656890432 - 0.2597011203i. \tag{6.10}$$

This results in the mapping to the unit circle seen in figure 6.2. In the next section, it will be shown how many points of the elliptic curve coincide with an albime triangle.



(a) Elliptic Curve

(b) Unit Circle

Figure 6.2: The points that result in an albime triangle with external bisector

6.2 The percentage

The percentage can be defined as how many of all multiples of $P = (2, 2)$ are such that $0 < x(nP) < 2$ for albime triangles with internal bisector. This percentage has been given in [8]. We will first verify that this percentage is 0.361208 and after that we will give a percentage for the albime triangles with external bisector.

The set of multiples of P limited by some value N that satisfy the condition can be written as

$$A(N) = \{n \cdot P \ : \ |n| \leq N \ \& \ 0 < x(nP) < 2\} . \tag{6.11}$$

It is clear that the set

$$\{n \cdot P \ : \ |n| \leq N\} \tag{6.12}$$

contains $2N + 1$ elements. So the percentage equals

$$P_N = \frac{\#A(N)}{2N + 1} \cdot 100 \tag{6.13}$$

which depends on N. To get all the multiples of P it is needed to take the limit of N to infinity. Then our total percentage is

$$\lim_{N\to\infty} P_N = \lim_{N\to\infty} \frac{\# \{n \cdot P \ : \ |n| \leq N \ \& \ 0 < x(nP) < 2\}}{2N + 1} \tag{6.14}$$

if this limit exists.

The elliptic curve that contains all these points has points at infinity. This makes calculating the number of points that have some property difficult. This is where we use the isomorphism of groups of Ψ. All real points on our curve are being mapped to the circle, where $\Psi(\mathcal{O}) = 1$. Note that only the rational points can be generated by our point $P = (2, 2)$.

Using Ψ , we get that $\Psi(P)$ generates an infinite cyclic subgroup of $\mathbb{T}$, which is a multiplicative group. L. Kronecker proved in 1884 that this subgroup is dense in $\mathbb{T}$ [8, 18]. Then by using a result from the proof of theorem 5.2.1 we have

$$\Psi(nP) = \Psi(P)^n \tag{6.15}$$

so that

$$E(\mathbb{Q}) = \{nP \ : \ n \in \mathbb{Z}\} \cong \{\Psi(P)^n \ : \ n \in \mathbb{Z}\} \tag{6.16}$$

To show that the limit in (6.14) exists and to calculate its value, we use a special equidistribution theorem. The general case of the theorem is sometimes called Kronecker's theorem [13], improved around 1909-1910 by Piers Bohl from Latvia, Waclaw Sierpiński from Poland, and the German Hermann Weyl individually.

Figure 6.3: Leopold Kronecker (1865)

Theorem 6.2.1 (Kronecker's Theorem)**.** *If α is an irrational real number then the numbers*

$$e^{i\pi k\alpha}$$

with k = 0, 1, 2,... are uniformly distributed on the circle $\mathbb{T}$ in the sense that for any continuous function g on the circle,

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(e^{i\theta}\right) d\theta = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} g\left(e^{i\pi k\alpha}\right). \tag{6.17}$$

To get to a special case of this theorem we use the function

$$f(e^{i\theta}) = \begin{cases} 1 & \text{if } \theta \in (a,b) \\ 0 & \text{elsewhere} \end{cases} \tag{6.18}$$

where $(a, b) \in (0, 2\pi)$ such that $f(a, b)$ is an arc on the unit circle of length $b - a$. Then the equation in Kronecker's theorem can be written as

$$\lim_{N\to\infty} \frac{\#\{n \cdot P \ : \ |n| \leq N \,\&\, a < \theta < b\}}{2N+1} = \frac{b-a}{2\pi}. \tag{6.19}$$

To apply this to our set, we have to define the multiples of P on an arc. This can be done by applying only φ to the end points and multiplying them with 2π. Then we get a value for the endpoints according to their place on the circumference of the unit circle.

Using that the length of these arcs together is approximately 0.722416π as seen in figure 6.4 and the equality from equation (6.19) we get

$$\lim_{N\to\infty} \frac{\#\{n \cdot P \ : \ |n| \leq N \,\&\, 0 < x(nP) < 2\}}{2N+1} = \frac{0.722416\pi}{2\pi} \approx 0.361208\ldots \tag{6.20}$$

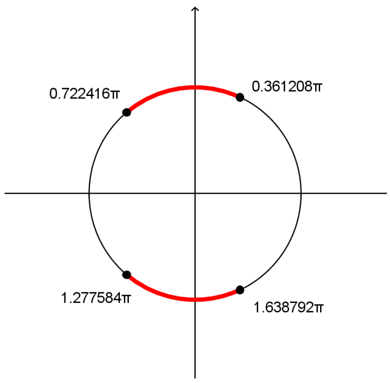


Figure 6.4: The arc that represents the albime triangles with internal bisector

We do the same for the albime triangles with external bisector. The arc length is approximately 0.387920π as seen in figure 6.5. Then our percentage becomes

$$\lim_{N\to\infty} \frac{\#\{n \cdot P \ : \ |n| \leq N \,\& \, -2 < x(nP) < 0\}}{2N+1} = \frac{0.387920\pi}{2\pi} \approx 0.193960\ldots \tag{6.21}$$

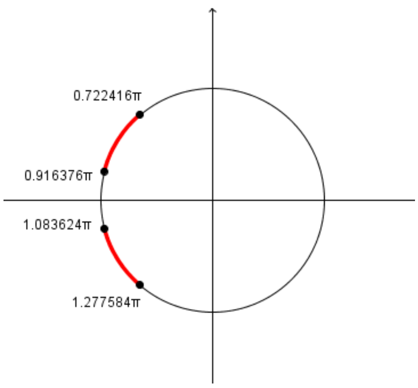


Figure 6.5: The arc that represents the albime triangles with external bisector

This means that if we take the first ten multiples of $P = (2, 2)$, we should find three or four albime triangles with internal bisector and two with external bisector.

$n \cdot P$	Point	Albime triangle
P	$(2, 2)$	None
$2 \cdot P$	$(0, 2)$	None
$3 \cdot P$	$(-2, -2)$	None
$4 \cdot P$	$(1, -1)$	Internal
$5 \cdot P$	$(6, -14)$	None
$6 \cdot P$	$(8, 22)$	None
$7 \cdot P$	$(\frac{10}{9}, \frac{26}{27})$	Internal
$8 \cdot P$	$(-\frac{7}{4}, \frac{19}{8})$	External
$9 \cdot P$	$(-\frac{6}{25}, -\frac{278}{125})$	External
$10 \cdot P$	$(\frac{88}{49}, -\frac{554}{343})$	Internal

Table 6.1: The first ten multiples of P

Looking at table 6.1, we indeed find three albime triangles with internal bisector and two with external bisector. We can take more multiples of P and see that it comes closer to the percentages found in equation (6.20) and (6.23).

Using the script in A.1 and A.2, we can check how many albime triangles will be found when we take more multiples of P. This can be done by changing the value of `upb` in the script. We want the output to only give the number of triangles that are found. To do this we change `Albs;` into `#Albs;`. The result can be found in table 6.2. The script that was used can be found below.

For internal bisector:

```
Z:=Integers(); E:=EllipticCurve([-4,4]);
P:=E![2,2]; Albs:={@ @}; upb:=??;
for n in [1..upb]
     do Q:=n*P; c:=Q[1];
     if c gt 0 and c lt 2
     then a:=Abs(Q[2]); d:=Denominator(a);
          a:=Z!(d*a); b:=Z!(d*(2-c)); c:=Z!(d*c);
          g:=Gcd(Gcd(a,b),c);
          Albs:=Albs join {[a/g,b/g,c/g]};
     end if;
end for; #Albs;
```

For external bisector:

```
Z:=Integers(); E:=EllipticCurve([-4,4]);
P:=E![2,2]; Albs:={@ @}; upb:=??;
for n in [1..upb]
     do Q:=n*P; mc:=Q[1];
     if mc gt -2 and mc lt 0
     then a:=Abs(Q[2]); d:=Denominator(a);
          a:=Z!(d*a); b:=Z!(d*(2-mc)); c:=Z!(-d*mc);
          g:=Gcd(Gcd(a,b),c);
          Albs:=Albs join {[a/g,b/g,c/g]};
     end if;
end for; #Albs;
```

upb	#Internal	Percentage	#External	Percentage
10	3	0.3000	2	0.2000
20	6	0.3000	4	0.2000
30	10	0.3333	5	0.1667
40	14	0.3500	6	0.1500
50	17	0.3400	8	0.1600
60	21	0.3500	10	0.1667
70	25	0.3571	12	0.1714
80	28	0.3500	15	0.1875
90	32	0.3556	16	0.1778
100	35	0.3500	18	0.1800
500	180	0.3600	96	0.1920
1000	360	0.3600	194	0.1940

Table 6.2: More multiples of P

We cannot find a percentage that is closer to our calculations, because the MAGMA calculator is limited to 120 seconds of calculating. But it does give us an idea of the limit of taking infinite multiples of P.

For the albime triangles with internal bisector we have made a distinction between acute and obtuse triangles. Using the image of $x = \sqrt{5} - 1$ on the elliptic curve we can define a percentage like above. Using the map φ and multiplying with 2π we get that $(\sqrt{5} - 1, \sqrt{4\sqrt{5} - 8})$ is mapped to 0.497930π on the circumference of the unit circle. This can also be seen in figure 6.6.

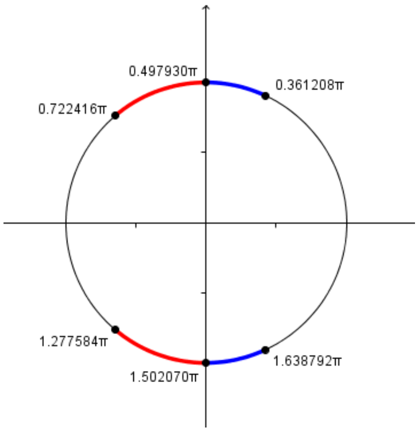


Figure 6.6: The acute and obtuse albime triangles

The ratio of rational points in $E(\mathbb{Q})$ corresponding to acute rational albime triangles then becomes

$$\lim_{N\to\infty} \frac{\#\left\{n \cdot P \,:\, |n| \leq N \,\&\, 0 < x(nP) < \sqrt{5} - 1\right\}}{2N + 1} \approx 0.224486\ldots \quad (6.22)$$

and for obtuse albime triangles

$$\lim_{N\to\infty} \frac{\#\left\{n \cdot P \,:\, |n| \leq N \,\&\, \sqrt{5} - 1 < x(nP) < 2\right\}}{2N + 1} \approx 0.136722\ldots \quad (6.23)$$

Note that there doesn't exist a right angled rational triangle, because c would have to be $\sqrt{5}-1$. Also, adding these percentages gives exactly the percentage we found for the total albime triangles with internal bisector.

Chapter 7

Conclusion

Recall that a triangle is called albime concurrent, or simply albime, if after possibly permuting the vertices $\{A, B, C\}$, we have a bisector in A, the median in B and the altitude in C to be concurrent. When we scale these triangles to have the length of AB in the interval $(0, 2)$ and $|AC| = 2 - c$, we get an expression for side BC. This happens to be an expression that generates an elliptic curve.

The rational points on this curve, Guy's favorite elliptic curve, form a group that can be generated by one element ($P = (2, 2)$). The group of real points on this curve can be mapped to the unit circle in the imaginary plane using the map Ψ seen in chapter 5. Since this is a bijection, there exists an element $\Psi(P)$ that generates a subgroup of the points on the unit circle.

The points of the elliptic curve that have an x-coordinate between zero and two generate an albime triangle with internal bisector. If the x-coordinate is between minus two and zero it gives an albime triangle with external bisector. These intervals can be mapped onto the unit circle.

To find the number of rational points on the curve that actually makes such a triangle we used a special case of Kronecker's theorem. This gives us a ratio of how many points on this curve coincide with an albime triangle.

We proved that 36.12% of the rational points on the elliptic curve coincide with an albime triangle with internal bisector and that 19.40% of them generate an albime triangle with external bisector. This answers Th.J. Kletter's question.

For the rational albime triangles with internal bisector some additional work has been done. It had been proven that 22.48% of rational points on the elliptic curve coincide with an acute albime triangle and 13.67% with an obtuse albime triangle. There is no right angled albime triangle with internal bisector.

Appendix A

Appendix

A.1 Albime triangles with internal bisector

In this section a script from MAGMA [1] is shown together with its output. This script takes the first 30 multiples of $P = (2, 2)$ and gives as output the albime triangles with internal bisector that can be made from those points.

```
Z:=Integers(); E:=EllipticCurve([-4,4]);
P:=E![2,2]; Albs:={@ @}; upb:=30;
for n in [1..upb]
     do Q:=n*P; c:=Q[1];
     if c gt 0 and c lt 2
     then a:=Abs(Q[2]); d:=Denominator(a);
          a:=Z!(d*a); b:=Z!(d*(2-c)); c:=Z!(d*c);
          g:=Gcd(Gcd(a,b),c);
          Albs:=Albs join {[a/g,b/g,c/g]};
     end if;
end for; Albs;
```

We take a multiple of P and define c to be its x-coordinate. Then when c is in the correct interval, it calculates a and b. To make sure the numbers are integers, we use the greatest common divisor and the denominator of a. This gives the next ten albime triangles.

```
{@
  [ 1, 1, 1 ],
  [ 13, 12, 15 ],
  [ 277, 35, 308 ],
  [ 26447, 26598, 3193 ],
  [ 587783, 610584, 482143 ],
```

```
    [ 162391823, 130866415, 204835960 ],
    [ 289288420681, 79912701162, 350627203989 ],
    [ 645323489165837, 658005416204774, 166519969756778 ],
    [ 327440745430922603, 343267427901969245,
    207812162799646548 ],
    [ 781764216317555155 7089, 5125420011844488724914,
    1023493025931067156673 5 ]
@}
```

A.2 Albime triangles with external bisector

In this section a script from MAGMA [1] is shown together with its output. This script takes the first 30 multiples of $P = (2,2)$ and gives as output the albime triangles with external bisector that can be made from those points.

```
Z:=Integers(); E:=EllipticCurve([-4,4]);
P:=E![2,2]; Albs:={@ @}; upb:=30;
for n in [1..upb]
    do Q:=n*P; mc:=Q[1];
    if mc gt -2 and mc lt 0
    then a:=Abs(Q[2]); d:=Denominator(a);
        a:=Z!(d*a); b:=Z!(d*(2-mc)); c:=Z!(-d*mc);
        g:=Gcd(Gcd(a,b),c);
        Albs:=Albs join {[a/g,b/g,c/g]};
    end if;
end for; Albs;
```

We take a multiple of P and define $-c$ to be its x-coordinate. Then when $-c$ is in the correct interval, it calculates a and b. To make sure the numbers are integers, we use the greatest common divisor and the denominator of a. This gives the next five albime triangles.

```
  {@
    [ 19, 30, 14 ],
    [ 139, 140, 15 ],
    [ 2511886033, 3360926870, 1421830647 ],
    [ 17691806567, 18257812083, 3677614045 ],
    [ 2951949073068081132558 71, 3513781498811512967144 33,
    1294753727151403271319 84 ]
@}
```

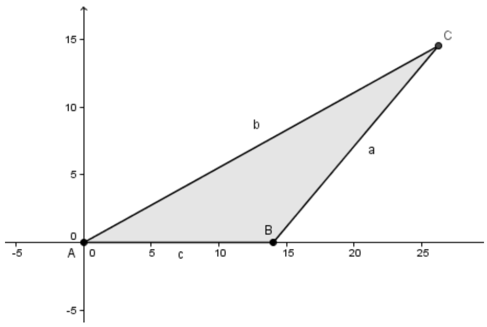


Figure A.1: The albime triangle [19,30,14]

A.3 Obtuse and acute

We will test the percentages found in chapter 2 and 3 of this thesis. First, we will calculate the angle C for all ten albime triangles found in section A.1. This can be done as we have seen in section 2.2. That is checking whether $a^2 + b^2 - c^2$ is smaller or bigger than zero. This can be done by using Matlab R2012b. The results are given in the next table.

Triangle (value of c)	$a^2 + b^2 - c^2$	Result
1	1	Acute
15	88	Acute
308	-16910	Obtuse
3193	1.3967×10^9	Acute
482143	4.8584×10^{11}	Acute
204835960	1.5394×10^{15}	Acute
350627203989	-3.2866×10^{22}	Obtuse
166519969756778	8.2168×10^{29}	Acute
207812162799646548	1.8186×10^{35}	Acute
10234930259310671566735	-1.7368×10^{43}	Obtuse

We see that three out of the ten albime triangles are obtuse. This percentage will get closer to 38.2% as we take more triangles. We will do the same for the five albime triangles found in section A.2. In that case we should look at the angle B, and it should always be obtuse. The results are given in the next table.

Triangle (value of c)	$a^2 - b^2 + c^2$	Result
14	-343	Obtuse
15	-54	Obtuse
1421830647	-2.9647×10^{18}	Obtuse
3677614045	-6.8228×10^{18}	Obtuse
129475372715140327131984	-1.9563×10^{46}	Obtuse

A.4 The map φ

To calculate the value of $\varphi(P)$ for some point P on the elliptic curve, we have to solve some difficult integrals. This can be done in Maple 17 using the next script. This script calculates the value for the point $(0, 2)$.

$alpha$:=fsolve($t^3 - 4 \cdot t + 4$);
$omega := 2 \cdot$evalf$\Big(Int\left(\frac{1}{\sqrt{t^3-4\cdot t+4}}, t = alpha..infinity\right)\Big)$;
$varphi$:=evalf$\Big(Int\left(\frac{1}{\sqrt{t^3-4\cdot t+4}}, t = 0..infinity\right)\Big)$;
$varphi02 := \frac{varphi}{omega}$;
$varphi02eval$:=evalf(%);

A.5 Multiples of $P = (2, 2)$

This is a script from MAGMA [1]. The script makes the polynomial ring generated by the point $(2, 2)$. It multiplies the point with integers, which gives us the possibility to define whether a point defines an albime triangle.

```
Q:=Rationals();
P<x>:=PolynomialRing(Q);
E:=EllipticCurve([-4,4]);
P:=E![2,2];
2*P;
3*P;
4*P;
5*P;
6*P;
7*P;
8*P;
9*P;
10*P;
```

The results can be found in table 6.1.

Bibliography

[1] Magma calculator. http://magma.maths.usyd.edu.au/calc/.

[2] G. L. Alexanderson. A conversation with leon bankoff. *The College Mathematics Journal*, 23(2):98–117, 1992.

[3] Erika Bakker. Congruente getallen en concurrente lijnen. Master's thesis, University of Groningen, 2012.

[4] B.J. Birch and W. Kuyk. *Modular Functions of One Variable IV*. Springer, 1972.

[5] B.J. Birch and H.P.F. Swinnerton-Dyer. Notes on elliptic curves. i. *Journal für die reine und angewandte Mathematik*, 212:7–25, 1963.

[6] Jasbir Chahal and Jaap Top. F. van der blij en de kletter-driehoeken. *Nieuw Archief voor Wiskunde*, 5/16(2):83–87, 2015.

[7] Jasbir S. Chahal and Jaap Top. Albime triangles over quadratic fields. *To be published in Rocky Mountain Journal of Mathematics*, 46, 2015.

[8] Erika Bakker et al. Albime triangles and guy's favourite elliptic curve. *Expo. Math.*, 2015.

[9] Richard K. Guy. My favorite elliptic curve: A tale of two types of triangles. *The American Mathematical Monthly*, 102(9):771–781, 1995.

[10] Th.J. Kletter. Een merkwaardige relatie tussen twee klassieke krommmen. 1957. Manuscript, 6 p.

[11] D.L. Mackay. Problem 263. *The American Mathematical Monthly*, 44:104, 1937.

[12] D.L. Mackay. Problem 374. *The American Mathematical Monthly*, 46:168, 1939.

[13] Jeffrey Rauch. Kronecker's theorem. Website, 2009. http://math.lsa.umich.edu/ rauch/558/Kronecker.pdf.

[14] Joseph H. Silverman. *Rational Points on Elliptic Curves*. Springer-Verlag, 1992.

[15] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. Springer Science and Business Media, 2009.

[16] Jaap Top. Albime triangles. Colloquium lecture, 2014. http://www.math.rug.nl/ top/lectures/Utrechtalbime.pdf.

[17] Jaap Top. Complexe getallen. Teachers seminar, 2014. http://www.math.rug.nl/ top/lectures/LerarenDag2014ComplexeGetallen.pdf.

[18] Jaap Top. (re)creative wiskunde. Alumniday lecture, 2014. http://www.math.rug.nl/ top/lectures/alumni2014.pdf.