university of groningen

faculty of mathematics
and natural sciences

The predictability of extreme temperature differences in a model for El Niño

Bachelor Project Mathematics

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Student: J.N. van Dulken

First supervisor: Dr. A.E. Sterk

Second supervisor: Dr.ir. F.W. Wubs

Abstract

El Niño is a phenomenon that takes place when the Walker circulation weakens or reverse. Trade winds weakens, which lend to warmer water at the Peruvian coast. The intriguing and difficult part of an El Niño, is that is is irregular. It occurs mostly once every 3-7 years. Because of the effects of an El Niño event, there is a desire to be able to predict El Niño. Scientists try to do so with help of mathematical models. This paper will look at the predictability of extreme events (large temperature differences in the ocean which indicates an El Niño) for the Vallis model. With help of Lyapunov exponents λ, something can be said about the trajectory of small perturbations of initial conditions of the Vallis model. To say something about the predictability of a system, it is common to take a look at only one variable of interest, this is called *the observable*. In this paper the variable y is taken as observable. When the maximum FTLE λ is plotted against the observable ϕ, there can be a correlation between λ and ϕ. A positive correlation means that a negative observable is better predictable and the more the observable becomes positive, the more the observable is less predictable. A negative correlation is the same as a positive correlations, only then the other way round. With that knowledge, a conclusion can be drawn about the predictability of extreme events of the Vallis model. The conclusion is that the correlation between ϕ and λ is too small to really say something about the predictability of an El Niño event for the Vallis model.

1 Introduction

El Niño is a most remarkable inter-annual climate variability on a global scale. El Niño is known by most people as a phenomenon that brings warm water to the coastal waters of Peru. However, El Niño is much more than that. In an El Niño period, not only the water temperature by Peru changes, but the climate in half the world changes then. For example, El Niño leads to areas of low pressure and increased rainfall along the west coasts of North and South America. Another example of changed climate as a result of El Niño is in the western Pacific. There, the water temperature is cooler than normal, and causes higher air pressure and decreased rainfall. All those short climate changes provide a lot of inconvenience. There are examples of flooding and landslides in the western Pacific and forest fires and their ensuing air pollution and droughts in the eastern Pacific [2].

The effects of El Niño has consequence that policymakers, and all the other people that are affected by El Niño, would like to know when an El Niño will arrive. Here lies a problem. El Niño is not a regular phenomenon. It appears in random periods, the only thing known is that El Niño appears once every 3-7 years. However, policymakers, and all the other people that are affected by El Niño would still like to know when the next El Niño period will arrive. And that is where the mathematics comes in.

To predict El Niño, a lot of mathematical models are made. Those models try to simulate the normal and El Niño conditions. There are different kinds of models. The more simplistic models are easier to work with, but the simulations of the model will not always be as realistic as the simulations of that of a more complex model. The aim of the models is to be so accurate in the simulations, that they can predict when the next El Niño period will arrive.
The aim of my paper is that I will look for a specific model for El Niño, the Vallis model, at the predictability of extreme values (El Niño's) of the model.

In chapter 2 I will describe what El Niño is exactly, how it occurs and how to measure it. In chapter 3 the Vallis model is being explained. Chapter four will cover the predictability of the extreme values of the model. This will be done working with finite-time Lyapunov exponents, which will be explained as well in chapter 4. Chapter 5 will conclude this paper with a conclusion. All Matlab code used can be found in the appendix.

2 El Niño

El Niño is part of the El Niño-Southern Oscillation phenomenon (ENSO). The ENSO is the phenomenon that sea surface temperatures changes for long periods of time, once in several years, at the western coast of South America and causes climatic changes across the tropics and subtropics. In the past, Peruvian Fisherman noticed that in some years, there was less fishing due to warmer water and called this phenomenon El Niño. Later, scientists discovered there was more to this phenomenon than only the oceanic part. Therefore, the name El Niño refers to the oceanic part of the ENSO, and the Southern Oscillation stands for the atmospheric component of the ENSO. During an El Niño period, unusually warm water temperatures occur in the central and Eastern equatorial Pacific waters. Except for the rise of the water temperatures, there are more changes in the normal state of the climate which I will describe later. There is no regularity in the time that El Niño occurs, but mostly it occurs once in periods of 3-7 years.

2.1 The Walker circulation

Before I explain how El Niño arises, I will first talk about the neutral conditions in the equatorial Pacific. The two most important indicators responsible for the conditions in the equatorial Pacific are *air pressure* and *ocean temperature*.

During the normal state of the equatorial Pacific, due to the warm temperatures in the West (Australia), warm air ascends and therefore creates an area of low air pressure. The temperature in the central and eastern (Tahiti) Pacific is cooler than in the West, and has therefore a relatively higher air pressure. Because air moves from high pressure areas to low pressure areas, wind arises, which near the equator are called trade winds. This is called *The Walker circulation*. The trade wind we are looking at moves from the coast of South America toward the western Pacific Ocean.

Like trade winds, water in the equatorial Pacific moves in a circulation as well. Surface water of the ocean is heated by the sun, and due to the trade winds this surface water is pushed from the East Pacific to the West pacific. To replace this water in the East along the equator and the coast of South America, water from deeper in the ocean moves upwards which is called *upwelling*. The upwelling water is cold and rich with nutrients.

The climate, as a result of the trade winds and the ocean temperature, in the equatorial Pacific is a so to say *coupled system*. The state of the ocean influences the atmosphere and the atmosphere influences the ocean.
The same as by the Walker circulation, the sea-surface temperature (SST) is warmer in the warmer West Pacific than the sea-surface temperature in the East Pacific. Low-level winds move from colder to warmer surface waters in the tropics, so the difference in sea-surface temperature in the equatorial Pacific

reinforces the trade winds from the East Pacific to the West pacific. Due to the reinforced trade winds, the effect of water pushed westwards by the trade winds is also reinforced, which result in a greater upwelling of cold ocean water in the East Pacific. In turn, because of the cold upwelling water the difference in sea-surface temperature between the East and the West increases, the increased difference in sea-surface temperature reinforces the trade winds once more. This process of the two indicators reinforcing each other is called *positive feedback*. This completes the coupled system [1, 3].

2.2 Start of an El Niño

El Niño is the result when the Walker circulation weakens or reverses. At the beginning of El Niño, trade winds of the Walker circulation weakens or reversed. Why those trade winds changes in strength is not always clear. The changed trade winds, depending on their strength and how long they last, can set off *Kelvin waves*. Kelvin waves are warm water that flows back from West to East, due to weakens or reversed trade winds. Those waves are 100-200 meters below the sea surface, 5-10 cm high and hundreds of kilometers wide. The temperature of such a wave is a few degrees warmer than the surrounding water. It can happen that the warm water of the Kelvin wave makes its way to the surface. Due to this process, the air pressure in the East Pacific lowers, which ensures a less difference in air pressure between East and West. Therefore, the trade wind is less strong and has as result that less water pushed from East to West, and so there is less upwelling. Less upwelling reinforces the difference in pressure gradient only more. This sets off an El Niño [1, 3, 8].

2.3 Measuring El Niño

To measure El Niño, an indicator is the Southern Oscillation Index (SOI). This indicator is obtained by the pressure differential between Tahiti (East Pacific) and Darwin (West Pacific) over a period of 5 months. The SOI is negative during El Niño. The SOI is not the only indicator used. The U.S. National Oceanographic and Atmospheric Administration (NOAA) measures also sea surface and sub-surface temperatures, atmospheric conditions, water currents and wind data in a specific region. This region is divided in sections (see figure 2.1). Each section has a different implication for the likelihood of ENSO. The NINO1+2 area may be the first to warm during El Niño, the NINO3 region experiences the most temperature variability, and NINO4 can gives a good indication for precipitation conditions over Indonesia.
Before measuring and forecasting El Niño, a definition is needed to describe what exactly an El Niño is. There is no one overall definition of El Niño, this differs per forecast center and country. The NOAA uses the following definition:

NOAA defines an El Niño event when the NINO3.4 area has sea-surface temperatures at least 0.5^{o} Celsius warmer than normal for five consecutive three-month-averaged periods. [1]

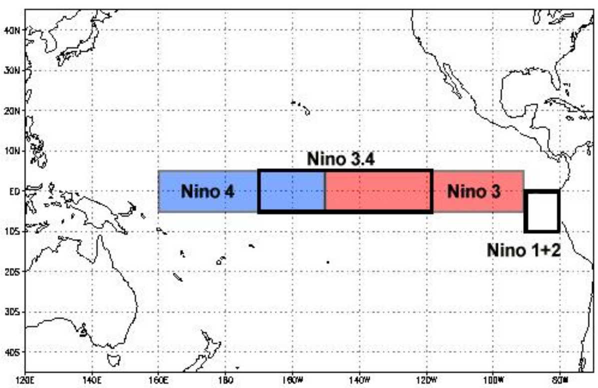

Figure 2.1: Location of NINO1+2, NINO3 and NINO4 region [1]

2.4 Models of El Niño

As we have seen, El Niño is an event that does not occur regularly. The main reason that we would like to know when El Niño will occur, is that in years of El Niño, disasters as floods, droughts and other climate-related disasters may be better predictable. With disasters better predictable, people can better prepare themselves for what is coming [2]. Therefore models of El Niño are quite useful.

To predict El Niño events, scientist use different kind of models. The models can be divided into three groups: *Conceptual and simple models*, *intermediate models* and *coupled general circulation models* [6, 7].
Conceptual models are simplified models that are reduced to a system with only a few variables and coupled equations that are only time depending. *Simple models* consist of simplified known partial differential equations. Both models show possible fundamental processes of the interactions in the coupled system. However, because these models are simplified, it is very difficult to compare the solutions with the outcomes in the real world.
Intermediate models are more complicated that the conceptual and simple models, therefore, the solutions of the model are more realistic than the solutions of the conceptual and simple models. The model consists of a set of coupled systems with realistic physical phenomena.
Coupled general circulation models are models that come closest to the real life observations. However, because these models are quite complex, it is not that easy to determine the major processes responsible for the model.
The model I will use this paper is explained next chapter.

3 The Vallis model

In this paper I will look at the model for El Niño introduced by Vallis in 1986. This is a simple, realistic model that differs from other models in that no stochastic or seasonally varying forcing is required to produce the aperiodicity, no explicit wave dynamics are required to explain the time scales, and no mid-latitude influences are needed to explain variations in intensity.
The model treats the ocean as one box with water with T_w the near surface temperature in the West of the ocean and T_e the near surface temperature in the East. The surface current u (wind driven horizontal advection) is a result of the Walker circulation (explained in Chapter 2). See figure 3.1.

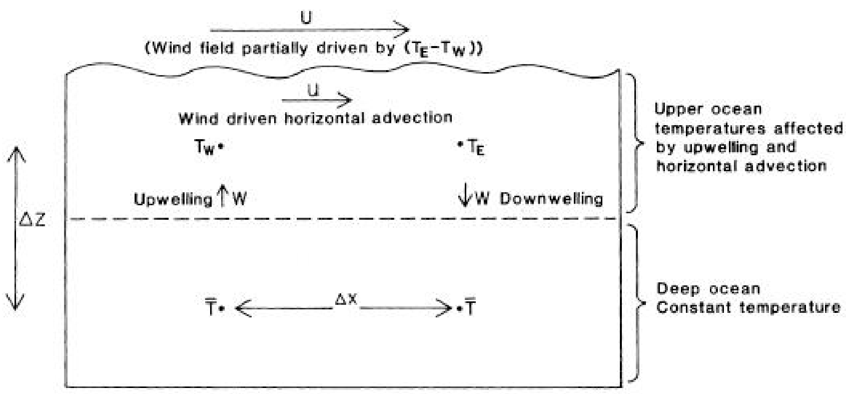

Figure 3.1: Scheme of the Vallis model [4]

The surface current u satisfies the following differential equation:

$$\frac{du}{dt} = \frac{B(T_e - T_w)}{2\Delta x} - C(u - u^*). \tag{3.1}$$

Here, B is a constant for the influence of ocean temperatures on the wind, C is a constant for the frictional time scale, and u^* represents the effect of the mean tropical surface in the east or in the west. Therefore, the term $\frac{B(T_e - T_w)}{2\Delta x} + Cu^*$ represents wind produced stress and $-Cu$ is the mechanical damping. Take $\overline{T}$ as the constant temperature of the deep ocean. With this, the simplest finite difference approximation to the temperature equation of fluid flow is

$$\frac{dT_w}{dt} = \frac{u}{2\Delta x}(\overline{T} - T_e) - A(T_w - T^*) \tag{3.2}$$

$$\frac{dT_e}{dt} = \frac{u}{2\Delta x}(T_w - \overline{T}) - A(T_e - T^*). \tag{3.3}$$

The first term of the right-hand side of (3.2) and (3.3) represents horizontal advection and upwelling and the second term represents forcing and thermal damping. A is a constant for the temperature decay time scale. T^* is the temperature to which the ocean would relax in the absence of motion and therefore represents radiative processes and heat exchange with the atmosphere [4].

To make the equations (3.1), (3.2) and (3.3) easier to work with, we simplify the equations in the following manner: First, we set $\overline{T} = 0$. This defines the zero level of the temperature. We also set $u^* = 0$, which means that there is a complete symmetry between T_e and T_w. Further, we make the following substitutions:

$$\hat{t} = At \qquad \hat{u} = \frac{u}{2A\Delta x} \qquad \hat{T_w} = \frac{T_w}{T^*} \qquad \hat{T_e} = \frac{T_e}{T^*}.$$

If we also substitute the constants B and C for the constants $\hat{B} = T^* B/(\Delta x A^2)$ and $\hat{C} = C/A$, we gain the equations:

$$\frac{d\hat{u}}{d\hat{t}} = \hat{B}(\hat{T_e} - \hat{T_w}) - \hat{C}\hat{u} \tag{3.4}$$

$$\frac{d\hat{T_w}}{d\hat{t}} = -\hat{u}\hat{T_e} - (\hat{T_w} - 1) \tag{3.5}$$

$$\frac{d\hat{T_e}}{d\hat{t}} = \hat{u}\hat{T_w} - (\hat{T_e} - 1). \tag{3.6}$$

Then, as final step of our simplification, we make the substitution:

$$y = \frac{T_e - T_w}{2} \qquad z = \frac{T_e + T_w}{2}.$$

This gives us the following system:

$$\frac{d\hat{u}}{d\hat{t}} = \hat{B}y - \hat{C}u \tag{3.7}$$

$$\frac{dy}{d\hat{t}} = uz - y \tag{3.8}$$

$$\frac{dz}{d\hat{t}} = -uy - (z - 1). \tag{3.9}$$

As I have have said in the beginning of this chapter, the Vallis model is a simple model. This means that not all physical processes have been taken into account. So are small-scale structures, certain types of wave propagation and the role of absolute temperature (in latent heat release) not covered in the model. The role of Kelvin waves and a equatorial wave guide are not specific taken into the model, but are implicit included in the simplistic way the model works [4].

The rest of the paper I will work with the model consisting of equations (3.7), (3.8) and (3.9) [5].

4 The predictability of the Vallis model

As said in the introduction, the aim of this paper is to take a look at the predictability of the extreme values of the Vallis model. I will do this with help of the Lyapunov exponents.

4.1 Finite-time Lyapunov exponents

The Vallis system as given in equations (3.7), (3.8) and (3.9) can be integrated, which gives the attractor of the system (see figure 4.1).

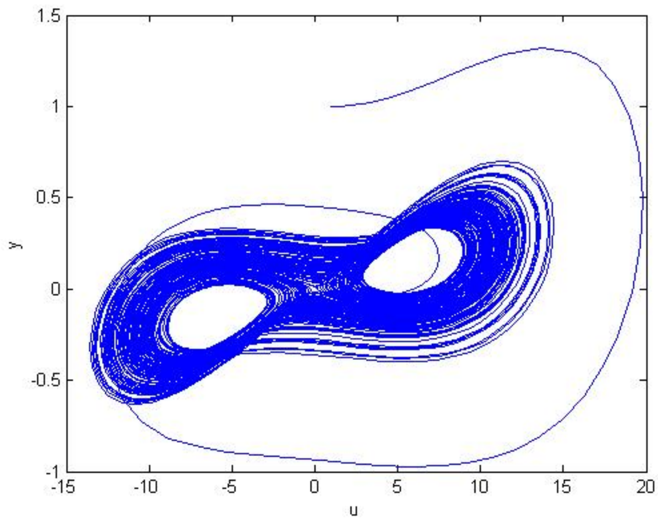

Figure 4.1: Attractor of the Vallis model till $t = 250$

Lyapunov exponents measure the growth or decay of infinitesimal perturbations of the initial conditions of an orbit of dynamical systems. Mathematically, this is written as [14]:

$$\|\Phi_\tau(\mathbf{x}) - \Phi_\tau(\mathbf{y})\|_f = e^{\lambda\tau}\|\mathbf{x} - \mathbf{y}\|_i. \tag{4.1}$$

Here, $\mathbf{x}$ is the initial point and $\mathbf{y}$ is the perturbed initial condition of $\mathbf{x}$ ($\mathbf{y} = \mathbf{x} + \epsilon\mathbf{v}$). ϵ is the infinitesimal perturbation in the direction of $\mathbf{v}$. $\|\mathbf{x} - \mathbf{y}\|_i$ is the distance between the initial point and the perturbation at time $t = 0$ ($\|\,\|_i$ is the norm at $t = 0$). $\|\Phi_\tau(\mathbf{x}) - \Phi_\tau(\mathbf{y})\|_f$ is the distance between the initial point and the perturbed point at time $t = \tau$ ($\|\,\|_f$ is the norm at $t = \tau$). λ is the Lyapunov exponent. Equation 4.1 is visualised in figure 4.2. The Lyapunov exponent depends on the initial condition $\mathbf{x}$ and τ. Equation 4.1 can also be written as

$$\frac{\|\Phi_\tau(\mathbf{x}) - \Phi_\tau(\mathbf{y})\|_f^2}{\|\mathbf{x} - \mathbf{y}\|_i^2} = e^{\lambda\tau}$$

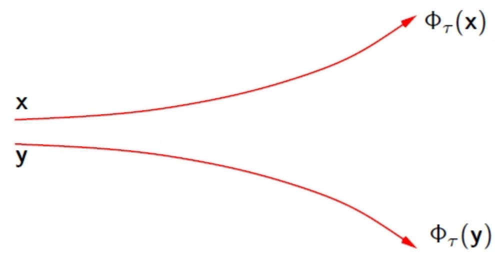

Figure 4.2: Trajectory difference for different initial conditions [14]

$$\frac{\|\Phi_\tau(\mathbf{x}) - \Phi_\tau(\mathbf{x} + \epsilon\mathbf{v})\|_f^2}{\epsilon\|\mathbf{v}\|_i^2} = e^{\lambda\tau}$$

$$\frac{\|\frac{1}{\epsilon}(\Phi_\tau(\mathbf{x}) - \Phi_\tau(\mathbf{x} + \epsilon\mathbf{v}))\|_f^2}{\|\mathbf{v}\|_i^2} = e^{\lambda\tau}. \tag{4.2}$$

Now, the limit can be taken to approach the infinitesimally small perturbation [10, 14].

$$\lim_{\epsilon \to 0} \frac{\Phi_\tau(\mathbf{x} + \epsilon\mathbf{v}) - \Phi_\tau(\mathbf{x})}{\epsilon} = D_x\Phi_\tau(x)\mathbf{v} = L_\tau(x)\mathbf{v}. \tag{4.3}$$

The meaning of $L_\tau(x)$ will be explained next paragraph. So, equation 4.2 now becomes

$$\frac{\|L_\tau(x)\mathbf{v}\|_f^2}{\|\mathbf{v}\|_i^2} = e^{\lambda\tau}. \tag{4.4}$$

For finding the Lyapunov exponents, at first, the Vallis system needs to be linearised. Say, $\mathbf{x_r}(t)$ is a particular solution of the Vallis system ($\mathbf{x}$ is here the vector $(u,\, y,\, z)$). This solution is called the reference trajectory. The dynamics of infinitesimally small perturbations $\boldsymbol{\epsilon}\mathbf{v}$ about the reference trajectory $\mathbf{x_r}(t)$ is described by the tangent-linear system

$$\dot{X} = Df(x)X. \tag{4.5}$$

Here, Df is the jacobi matrix of f and $X(\tau) = L_\tau(x)$ is the solution matrix. The proof that equation 4.5 indeed describes the dynamics of infinitesimally small perturbation is given below [14]:

$$\begin{aligned}
L_0(x) &= I \\
\frac{d}{dt}L_\tau(x) &= \frac{d}{dt}\{D_x\Phi_\tau(x)\} \\
&= D_x\{\frac{d}{dt}\Phi_\tau(x)\} \\
&= D_x\{\mathbf{F}(\Phi_\tau(x))\} \\
&= D_x\mathbf{F}(\Phi_\tau(x))L_\tau(x).
\end{aligned}$$

So, due to the linearization, we need to integrate $\dot{X} = Df(x)X$ with $X(0) = I$. The matrix X is written as:

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}.$$

In the Vallis system the matrices $f(x)$ and $Df(x)$ are the following:

$$f(u,y,z) = \begin{pmatrix} \hat{B}y - \hat{C}u \\ uz - y \\ -uy - (z-1) \end{pmatrix} \quad \text{and} \quad Df(u,y,z) = \begin{pmatrix} -\hat{C} & \hat{B} & 0 \\ z & -1 & u \\ -y & -u & -1 \end{pmatrix}.$$

As seen in equation 4.2, the distance between two points is given by the norm. For the norm at $t = 0$, the initial metric carries the information about the distribution of initial errors. Therefore, take C as an estimate of the initial error covariance. Then, the initial norm can be written as

$$\|\mathbf{v}\|_i^2 = \mathbf{v}^\top C^{-1}\mathbf{v} = \|C^{-1/2}\mathbf{v}\|^2. \tag{4.6}$$

The right hand side is here the Euclidean norm [9].

To say something about the predictability of a system, it is common to take a look at only one variable of interest, this is called *the observable*. The concept of the Lyapunov exponent that I use in this paper is based on this, by means of a projection matrix P. The matrix P is chosen in such a way that the singular vectors will point in the direction of maximal growth of the observable [9]. Here is P is the orthogonal projection matrix ($P = P^2 = P^\top$). For the Vallis model I will take the observable y. As stated in chapter 3, the variable $y = \frac{T_e - T_w}{2}$. For simplicity we take for C the identity matrix. C and P will then become of the form:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, the matrix P finds its way to $\|L_\tau(x)\mathbf{v}\|_f^2$ in the following manner [14]:

$$\begin{aligned}
\|L_\tau(x)\mathbf{v}\|_f^2 &= \mathbf{v}^\top L_\tau(x)^\top P L_\tau(x)\mathbf{v} \\
&= \mathbf{v}^\top L_\tau(x)^\top P^\top P L_\tau(x)\mathbf{v} \\
&= \|P L_\tau(x)\mathbf{v}\|^2.
\end{aligned}$$

So now equation 4.4 can be written as

$$e^{\lambda\tau} = \frac{\|P L_\tau(x)\mathbf{v}\|^2}{\|C^{-1/2}\mathbf{v}\|^2}. \tag{4.7}$$

In this paper, only the maximum Lyapunov exponent plays a role. Therefore, to get the maximum Lyapunov exponent, equation 4.7 becomes

$$\begin{aligned}
e^{\lambda\tau} &= \max_{v\neq 0} \frac{\|P L_\tau(x)\mathbf{v}\|^2}{\|C^{-1/2}\mathbf{v}\|^2} \\
&= \max_{v\neq 0} \frac{\mathbf{v}^\top L_\tau(x)^\top P^\top P L_\tau(x)\mathbf{v}}{\mathbf{v}^\top C^{-1}\mathbf{v}} \\
&= \max_{v\neq 0} \frac{\mathbf{v}^\top L_\tau(x)^\top P^\top P L_\tau(x)\mathbf{v}}{\mathbf{v}^\top \mathbf{v}}.
\end{aligned} \tag{4.8}$$

For a given matrix M, that is symmetric and positive definite, the definition of the maximum Rayleigh quotient is

$$R(M, v_{max}) = \frac{\mathbf{v}_{max}^\top M \mathbf{v}_{max}}{\mathbf{v}_{max}^\top \mathbf{v}_{max}} = \lambda_{max}. \tag{4.9}$$

λ_{max} (not the same λ as the Lyapunov exponent) is the maximum eigenvalue of M and $\mathbf{v}_{max}$ is the corresponding eigenvector of M [12, 13]. The right hand side of 4.8 looks quite similar to the definition of the Rayleigh quotient, if $L_\tau^\top P^\top P L_\tau$ is taken as the matrix M. Then, v_{max} is an eigenvector of $L_\tau^\top P^\top P L_\tau$ which is the right singular vector of $P L_\tau$. λ_{max} becomes the maximum eigenvalue of $L_\tau^\top P^\top P L_\tau$. Hence $\sqrt{\lambda_{max}} = \sigma_{max}$ where σ_{max} is the maximum singular value of $P L_\tau$. In other words

$$\max_{v\neq 0} \frac{\|P L_\tau(x)\mathbf{v}\|^2}{\|C^{-1/2}\mathbf{v}\|^2} = \max_{v\neq 0} \frac{\mathbf{v}^\top L_\tau(x)^\top P^\top P L_\tau(x)\mathbf{v}}{\mathbf{v}^\top \mathbf{v}} = \sigma_{max}^2. \tag{4.10}$$

Now, equation 4.4 can be written as

$$\max_{v\neq 0} \frac{\|L_\tau \mathbf{v}\|_f^2}{\|\mathbf{v}\|_i^2} = e^{\lambda_{max}\tau}$$

$$\max_{v\neq 0}\frac{\|PL_\tau \mathbf{v}\|^2}{\|C^{-1/2}\mathbf{v}\|^2} = e^{\lambda_{max}\tau}$$
$$\sigma_{max}^2 = e^{\lambda_{max}\tau}$$
$$\ln\,\sigma_{max}^2 = \ln(e^{\lambda_{max}\tau}) = \lambda_{max}\tau$$
$$\frac{2}{\tau}\ln\,\sigma_{max} = \lambda_{max}.$$

So, the maximum finite-time Lyapunov exponents (FTLE) are given as:

$$\lambda_{max}(x,\tau) = \frac{2}{\tau}\ln\,\sigma_{max}(x,\tau). \tag{4.11}$$

4.2 Predictability of extreme values

Now that is known what FTLE are and how to calculate them, I can use the FTLE to say something about the predictability of the extreme values of the Vallis model. As said in the previous section, the Lyapunov exponent will tell the growth or decay of infinitesimal perturbations of the initial conditions. This means that with a negative Lyapunov exponent, there is less distance between the trajectory of the initial condition and the trajectory of the perturbed initial point at time is τ. So this means that if the Lyapunov exponent is known and the trajectory of an initial condition there is something to say about the predictability. With a negative Lyapunov exponent, an infinitesimal perturbation does not make that of a difference, and hence the trajectory is predictable. However, with a positive Lyapunov exponent the trajectory is not predictable. An infinitesimal perturbation here lead till a greater distance between the the trajectory of the initial condition and the trajectory of the perturbed initial point, so you can not really tell where the trajectory of the perturbed initial point will end up at time τ. Because I focus on the extreme events in this paper, I am only interested in the maximum finite Lyapunov exponent. The maximum Lyapunov exponent measures the maximum distance between the initial point and the perturbed point at $t = \tau$.

To find the singular values of PL_τ and hence the maximum FTLE of the Vallis model, I used the program Matlab. With help of the SVD, the singular values of PL_τ were found. The code used is find in the appendix. I calculated the maximum FTLE for different $\tau' s$ and for more than 2000 points on the attractor. In figure 4.3 a plot is seen for $\tau = 0.2$, where the maximum FTLE λ is plotted against the observable ϕ.

When the maximum FTLE λ is plotted against the observable ϕ, there can be a correlation between λ and ϕ. A positive correlation means that a negative observable is better predictable and the more the observable becomes positive, the more the observable is less predictable. So, if the water temperature in the east is higher than in the west, the extreme values are better predictable. A negative correlation is the same as a positive correlations, only then with T_e

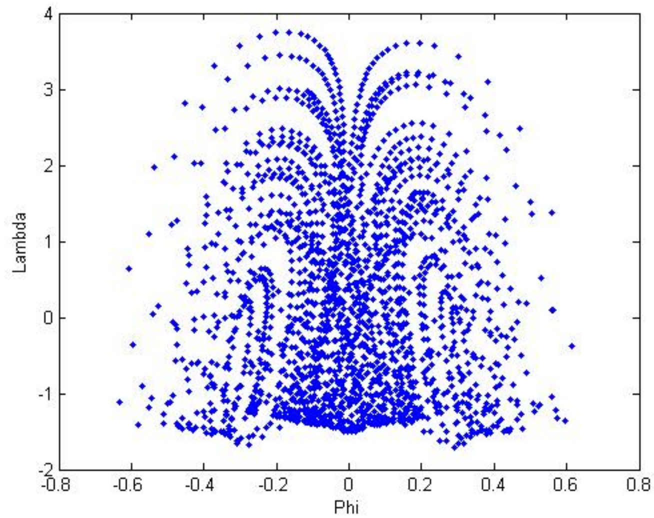

Figure 4.3: Plot of FTLE with observable ϕ for $\tau = 0.2$

and T_w switched. So, if the water temperature in the west is higher than in the east, the extreme values are better predictable.

Figure 4.3 shows that there is hardly any correlation between λ and ϕ for $\tau = 0.2$. This means that for $\tau = 2$, it does not matter which value the observable y has (which temperature difference there is between T_e and T_w), because there is no correlation, there is nothing to say about the predictability of the extreme values.

Per point of time $t = \tau$, the correlation between ϕ and λ can be calculated. I have done this for $\tau = 0.1 - \tau = 4$ with steps of 0.1. I plotted all those correlations for the different $\tau' s$ (see figure 4.4).

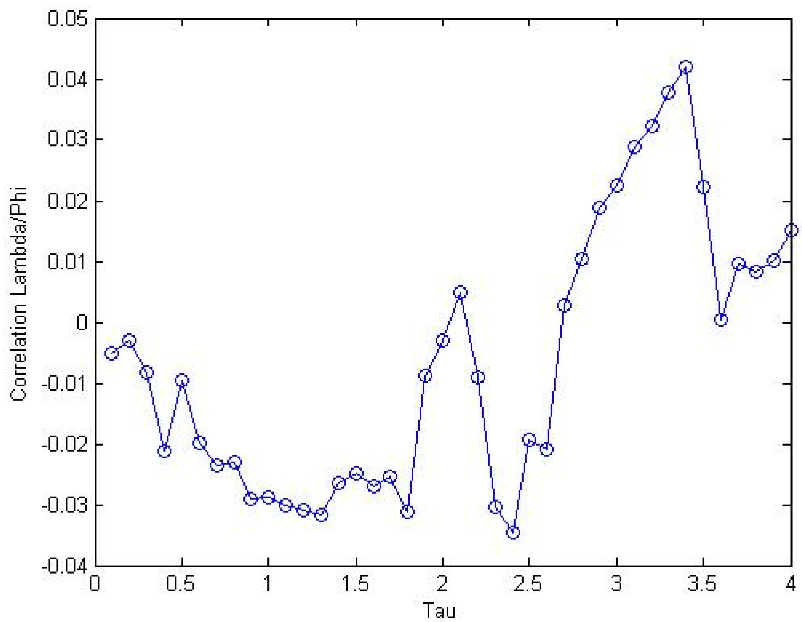

Figure 4.4: Correlation between ϕ and λ for τ

5 Conclusion and discussion

As said at the end of last chapter, I plotted the the correlation between ϕ and λ for $\tau \in [0, 4]$, as seen in figure 4.4. Until $\tau = 2.6$, there is a negative correlation, and from $\tau \in [2.7, 4]$ there is a positive correlation. In paragraph 4.2 I explained the meaning of the correlation between ϕ and λ. A positive correlation means that a negative observable is better predictable and the more the observable becomes positive, the more the observable is less predictable. For a negative correlation is it the other way round.
This means for $\tau \in [0, 2.6]$, a more negative observable is better predictable.
For $\tau \in [2.7, 4]$, this means that a a more positive observable is better predictable.
The conclusion of this paper is that for $\tau \in [0, 2.6]$, the correlation is negative and hence El Niño would be predictable. However, as seen in figure 4.4, the correlation is very small. This means that although there is a negative correlation, the correlation is too small to really say something about the predictability of an El Niño event for the Vallis model. Big differences in the ocean temperature between east and west are not better or worse predictable than small differences.

The model of Vallis is quite similar to the Lorenz-63 model. In the article of Sterk [9], the correlation for the Lorenz-63 model is calculated. The conclusion in the article was that there was hardly any correlation as well. So therefore, my conclusion for the Vallis model was not that surprising.

The predictability depends on the choice of the observable, and therefore the predictability may change when the observable is changed. This should be taken into consideration when studying the predictability of a given model.

That there is no predictability for the Vallis model, does not say that there is no predictability for El Niño at all. As said in chapter 3, the Vallis model is a simple model, not all physical processes have been taken into account. So, the simulations are not so realistic as the real El Niño. There could be other models, more realistic models, that shows another outcome about the predictability of an El Niño event.

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A Matlab code

A.1 Vallis system

```
function f = Vallis(t,x)

%parameters
f = zeros(3,1);
B=127;
C=3;

f(1)=B*x(2)-C*x(1);
f(2)=x(1)*x(3)-x(2);
f(3)=-x(1)*x(2)-x(3)+1;

[t,x] = ode45('Vallis', [0 250], [1 1 1]);

plot(x(:,1),x(:,2));
xlabel ('u');
ylabel ('y');
```

A.2 Linearized Vallis system

```
function f = VallisLin(t,x)

% Parameters
B = 127; C = 3;

% x is a 12-dimensional vector
% x contains 3 variables of the model
% x contains 9 variables of the linearized model
f = zeros(12,1);

% Right hand side not linearized system
f(1) = B*x(2)-C*x(1);
f(2) = x(1)*x(3)-x(2);
f(3) = -x(1)*x(2)-x(3)+1;

% Right hand side linearized system
Y = x(4:12);


f(4)  = -C*Y(1) + B*Y(4);
f(5)  = -C*Y(2) + B*Y(5);
f(6)  = -C*Y(3) + B*Y(6);

f(7)  = x(3)*Y(1) - Y(4) + x(1)*Y(7);
f(8)  = x(3)*Y(2) - Y(5) + x(1)*Y(8);
f(9)  = x(3)*Y(3) - Y(6) + x(1)*Y(9);

f(10) = -x(2)*Y(1) - x(1)*Y(4) - Y(7);
f(11) = -x(2)*Y(2) - x(1)*Y(5) - Y(8);
f(12) = -x(2)*Y(3) - x(1)*Y(6) - Y(9);
```

A.3 Maximum FTLE

```
function [lambda, phi] = MaxFTLE(x, tau)

% not linearized en linearized system integrating at the same time
[t,Y] = ode45('VallisLin3', [0 tau], [x(1); x(2); x(3); 1; 0; 0; 0; 1; 0; 0; 0; 1]);

% if t  n points of time contains, than Y  is a n x 12 matrix
% Y(n,4:12)contains the components of the matrix L

n = length(t);
L = reshape(Y(n,4:12), [3,3])';

P = [0 0 0; 0 1 0; 0 0 0];
L = P*L;

% calculate maximal FTLE from the singular value decomposition
lambda = log(max(svd(L))) / tau;

% value of the observable on time t = tau
phi = Y(n, 2);
```

A.4 Plot FTLE/ϕ for fixed τ

```
function [A] = PlotFTLE(tau)

[t,x] = ode45('Vallis', [0 10], [1 1 1]);
n = length(t);

x0 = x(n,1:3);

[t,x] = ode45('Vallis', [0 50], x0);

n=length(t)

A=zeros(n,2);

for k=1:n
    y=x(k,1:3);
    [lambda, phi] = MaxFTLE(y, tau);
    A(k,1)=phi;
    A(k,2)=lambda;
end

plot(A(:,1), A(:,2), '.')
xlabel ('Phi');
ylabel ('Lambda');
```

A.5 Correlation between ϕ and λ for fixed τ

```
function Correlation

tau = 0.1:0.1:4;
n = length(tau);
C = zeros(n,1);

for k=1:n
    [A] = PlotFTLE(tau(k));
    C(k,1) = corr(A(:,1), A(:,2));
end


plot(tau, C(:,1), '-o')
xlabel ('Tau');
ylabel ('Correlation Lambda/Phi');
```