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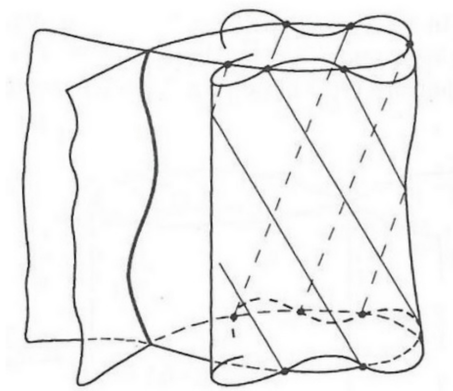
The Homoclinic Melnikov Method



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Abstract

A planar dynamical system with a hyperbolic equilibrium point, connected to itself by a homoclinic orbit, can behave chaotically when perturbed by a time-periodic function. Melnikov has developed a method, which can be used to check whether the system possesses chaotic dynamics. The method is based on the so called Melnikov function whose zeros correspond to homoclinic points which by Moser's theorem imply chaotic behavior. In this thesis the Melnikov method is described in some detail and applied to a *driven Morse oscillator*.

Contents

1 Introduction

In this thesis, the homoclinic Melnikov method is described and applied to a driven Morse oscillator. When considering a two-dimensional dynamical system, it is shown that a time periodic perturbation of this system can cause chaotic behaviour of the solutions. Transversal intersections of stable and unstable manifolds are the cause of this behaviour. When an unperturbed two-dimensional system contains a hyperbolic fixed point that is connected to itself by a homoclinic orbit, this homoclinc orbit can be interpreted as the union of the unstable and the stable manifold, and the fixed point itself. The perturbation can cause the stable and the unstable manifold to split up and intersect transversely.

When we consider a perturbed two-dimensional dynamical system that is periodic in time, the homoclinic Melnikov method can be used to prove that there exist transverse homoclinic orbits to hyperbolic periodic orbits. Using the Melnikov method, three steps must be followed. But before starting with the first step, we want to rewrite the system as an autonomous system. After this, we can move on with the Melnikov method. First, the unperturbed system must be considered. The homoclinic manifold, which could be expressed in terms of the union of the unstable manifold, the stable manifold and the hyperbolic fixed point, is parametrized in the first step. After having parametrized the homoclinic manifold, we consider the splitting of the manifolds. A measure is defined to express the distance between the stable and the unstable manifold on a certain cross section of the phase space. The third step involves deriving the Melnikov function. The zeros of this function prove the existence of transversal intersections. Then Moser's theorem can be used to prove that the system has chaotic dynamics.

After the homoclinic Melnikov method is described, it is also applied to the so-called *driven Morse oscillator*. This system has the property that it has a *nonhyperbolic* fixed point at $(\infty,0)$, which means we cannot directly apply Melnikov's method: we need a *hyperbolic* fixed point. But we use a coordinate transformation, which results in a system with a fixed point at the origin, which is also hyperbolic. Now we can derive the Melnikov function. After having derived this function, the chaotic dynamics will be described in the original coordinates. The chaotic dynamics will be confirmed by means of numerical testing.

In this thesis, the description of the homoclinic Melnikov function closely follows the chapters written about this subject in the book *Introduction to Applied Nonlinear Systems and Chaos* [Wiggins, 1990].

2 The Homoclinic Melnikov Method

In this chapter the homoclinic Melnikov method is explored. In the first subsection, we will describe the system the Melnikov method is applied to thoroughly. Then the three steps of the Melnikov method will be described. They consist of respectively the parametrization of the homoclinic manifold, the measurement of the distance between the stable and unstable manifold and finally the Melnikov function is derived.

2.1 Description of the System

We consider a two-dimensional system of the form

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x,y) + \varepsilon g_1(x,y,t,\varepsilon), \\ \dot{y} &= -\frac{\partial H}{\partial x}(x,y) + \varepsilon g_2(x,y,t,\varepsilon), \end{aligned} \tag{1}$$

where $(x,y) \in \mathbb{R}^2$, and the perturbation function g is periodic in time with period $T = 2\pi/\omega$. We can rewrite these equations in vector form, defining $q = (x,y)$, $DH = (\frac{\partial H}{\partial x},\frac{\partial H}{\partial y})$, $g = (g_1,g_2)$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the system will become

$$\dot{q} = JDH(q) + \varepsilon g(q,t,\varepsilon). \tag{2}$$

Taking $\varepsilon = 0$, the unperturbed system is given by

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x,y), \\ \dot{y} &= -\frac{\partial H}{\partial x}(x,y). \end{aligned} \tag{3}$$

This system is Hamiltonian[1]. A Hamiltonian system has the property that there exists a function $H = H(x,y)$ called the Hamiltonian, which generates the vector field according to equation 3. The Hamiltonian is an integral of motion, representing the total energy. So conservation of H means that the total energy is conserved.

[1] We assume that the system is Hamiltonian, but the Melnikov method also works if the system isn't Hamiltonian.

The solutions of the above system lie on level sets of H. Now rewriting the unperturbed system in vector form gives us

$$\dot{q} = JDH(q). \tag{4}$$

To show that the system will behave chaotically under perturbation, it has to satisfy some conditions. We want to prove the existence of transverse homoclinic orbits to hyperbolic periodic orbits. In order to prove this, we need the system to have a hyperbolic fixed point, connected to itself by a homoclinic orbit, so we assume our unperturbed system has one. We call this fixed point p_0, and the associated homoclinic orbit $q_0(t) = (x_0(t),y_0(t))$. We define the union of the homoclinic orbit $q_0(t)$ with the fixed point to be

$$\Gamma_{p_0} = \{q \in \mathbb{R}^2 | q = q_0(t), t \in \mathbb{R}\} \cup \{p_0\}, \tag{5}$$

which is equal to $W^s(p_0) \cap W^u(p_0) \cup \{p_0\}$. We assume that the inside of Γ_{p_0} contains periodic orbits, oriented in a way that is shown in figure 1. We parametrize the periodic orbits as q^α, where $\alpha \in$ (-1,0). Each $q^\alpha(t)$ has period T^α. When we take the limit of these periodic orbits as $\alpha \to 0$, the homoclinic orbit $q_0(t)$ is approached which has an infinite period, so $\lim\limits_{\alpha\to 0} q^\alpha(t) = q_0(t)$ and $\lim\limits_{\alpha\to 0} T^\alpha = \infty$. The last assumption we make, is that our perturbed system is sufficiently differentiable, in t as well as in ε.

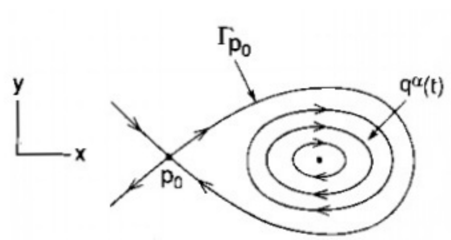

Figure 1: Γ_{p_0} and its internal periodic orbits

It is apparent that (1) is non-autonomous. In our case it is convenient to deal with autonomous system, so we will rewrite the system by introducing a new variable ϕ. We take this ϕ, such that $\dot{\phi} = \omega$. This way, $\phi(t) = \omega t + \phi_0$ (where ϕ_0 is just the integration constant). The system was periodic in time with period $2\pi/\omega$, and now it is periodic in ϕ with period $\omega{\cdot}\frac{2\pi}{\omega} = 2\pi$. Rewriting (1) and introducing the new variable ϕ we get

$$
\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial y}(x,y) + \varepsilon g_1(x,y,\phi,\varepsilon), \\
\dot{y} &= -\frac{\partial H}{\partial x}(x,y) + \varepsilon g_2(x,y,\phi,\varepsilon), \\
\dot{\phi} &= \omega,
\end{aligned} \tag{6}
$$

where $(x,y,\phi) \in \mathbb{R}^2 \times S^1$. Rewritten in vector form,

$$
\begin{aligned}
\dot{q} &= JDH(q) + \varepsilon g(q,\phi,\varepsilon), \\
\dot{\phi} &= \omega.
\end{aligned} \tag{7}
$$

The unperturbed system is of course again given by $\varepsilon = 0$.

Now we can start with the first step of the Melnikov method: parametrizing the homoclinic manifold.

2.2 Step 1: Parametrizing the Homoclinic Manifold

Recall that $\Gamma_{p_0} = W^s(p_0) \cap W^u(p_0) \cup \{p_0\}$. After having introduced the variable ϕ, we look at the hyperbolic fixed point p_0 in the three-dimensional phase space $\mathbb{R}^2 \times S^1$. Since ϕ is periodic in 2π, the hyperbolic fixed point will become a periodic orbit. We saw that by integrating ϕ with respect to t, we obtained $\phi(t) = \omega t + \phi_0$. Here ϕ_0 was just the integration constant. We name the periodic orbit $\gamma(t)$, and it is of the form

$$
\gamma(t) = (p_0, \phi(t) = \omega t + \phi_0). \tag{8}
$$

This is shown in figure 2, where the lines on the homoclinic manifold represent a trajectory.

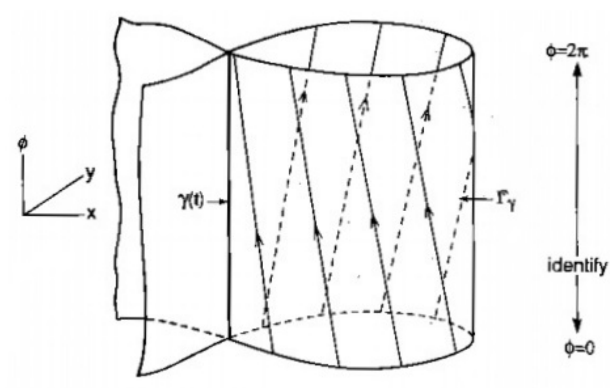

Figure 2: The homoclinic manifold Γ_γ

When we viewed the hyperbolic fixed point in the two-dimensional phase space, the homoclinic orbit that connected the fixed point to itself consisted of two one-dimensional manifolds: $W^s(p_0)$, and $W^u(p_0)$. Now, viewed in three dimensions, we denote those manifolds $W^s(\gamma(t))$ and $W^u(\gamma(t))$ respectively. When we now look at the perturbed system, the stable and the unstable manifold, which coincided in the unperturbed case, will break up (see Figure 3). We want to measure the distance between these two manifolds, but then we first have to parametrize Γ_γ.

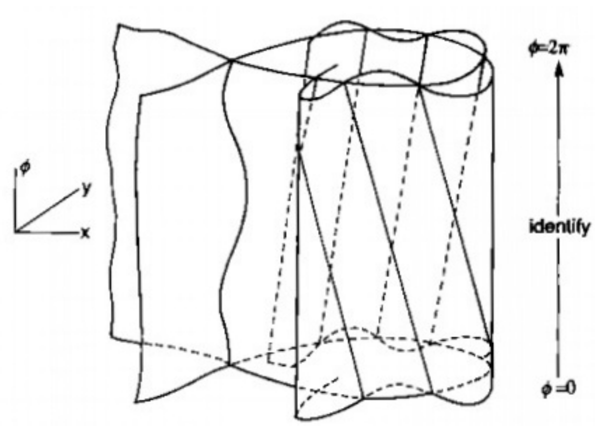

Figure 3: The stable and unstable manifold when a perturbation is inserted

The Parametrization

Looking at the three-dimensional homoclinic orbit Γ_γ, we can look at $q_0(t)$ in a different way. For any $t_0 \in \mathbb{R}$, following the unperturbed homoclinic trajectory, it takes exactly t_0 time to get from q_0(-t_0) to $q_0(0)$. So Γ_γ is the union of all points

$$(q_0(-t_0), \phi_0) \in \Gamma_\gamma, \tag{9}$$

with $\phi \in (0,2\pi]$ and $t_0 \in \mathbb{R}$. Given (t_0,ϕ_0) $\in \mathbb{R} \times S^1$, for ($q_0$(-$t_0$),$\phi_0$) to correspond to a unique point on Γ_γ, we need the map

$$h : (t_0, \phi_0) \to (q_0(-t_0), \phi_0) \tag{10}$$

to be bijective. Each point on Γ_γ can be represented by (q_0(-t_0), ϕ_0) for a certain t_0. So if we pick a point (q_0(-$\bar{t}_0$),$\overline{\phi}_0$) on Γ_γ, then h(-$\bar{t}_0$,$\overline{\phi}_0$) = (q_0(-$\bar{t}_0$),$\overline{\phi}_0$). For each image (q_0(-t_0), ϕ_0) there is an original (t_0, ϕ_0), so h is onto. There is only one trajectory that can be followed to get from $q_0(-t_0)$ to $q_0(0)$. This means that when (q_0(-t_1),ϕ_1) = (q_0(-t_2),ϕ_2), it follows that (t_1,ϕ_1) = (t_2,ϕ_2). So g is also one-to-one, and we have proven that h is bijective.

Now we can write

$$\Gamma_\gamma = \{(q, \phi) \in \mathbb{R}^2 \times S^1 | q = q_0(-t_0), t_0 \in \mathbb{R}; \phi = \phi_0 \in (0, 2\pi]\}. \tag{11}$$

We have predicted, that in the perturbed system, the stable and the unstable manifold will break up. To measure the distance between these two manifolds, which we wanted to do in step 2 of the Melnikov method, we define a vector π_p *transversal* to Γ_γ. The definition of a transversal intersection is as follows.

Definition 1. *Let M and N be differentiable (at least C^1) manifolds in $\mathbb{R}^n$. Let p be a point in $\mathbb{R}^n$; then M and N are said to be transversal at p if $p \notin M \cap N$; or, if $p \in M \cap N$, then $T_pM + T_pN = \mathbb{R}^n$, where T_pM and T_pN denote the tangent spaces of M and N, respectively, at the point p. M and N are said to be transversal if they are transversal at every point $p \in \mathbb{R}^n$.*

We defined π_p to be transversal to Γ_γ, so it is obvious that the intersection between π_p and the stable/unstable manifold is transversal at each point $p \in \Gamma_\gamma$. Under small perturbations, like in our case, the intersection between π_p and the stable/unstable manifold is still transversal.

Construct the distance vector

To construct a distance vector π_p normal to Γ_γ, we first note that each point on Γ_γ can be written as (q_0(-t_0),ϕ_0). A vector tangent to Γ_γ at each point is then

$$(\frac{\partial H}{\partial y}(x_0(-t_0), y_0(-t_0)), -\frac{\partial H}{\partial x}(x_0(-t_0), y_0(-t_0)), \phi_0). \tag{12}$$

A normal vector to this is[2]

$$(\frac{\partial H}{\partial x}(x_0(-t_0), y_0(-t_0), \frac{\partial H}{\partial y}(x_0(-t_0), y_0(-t_0)), 0) \tag{13}$$

which is the same as

$$\pi_p = (DH(q_0(-t_0)), 0). \tag{14}$$

Now that we have parametrized Γ_γ and defined a vector normal to Γ_γ (see Figure 4) at each point, we can move on with step 2 of the Melnikov method.

2.3 Step 2: Measure the Distance Between the Unstable and the Stable Manifolds

Before we can construct the distance vector, we will explore how Γ_γ behaves under perturbation. By understanding this, we need the following proposition.

[2]Since we look at the unperturbed trajectory, which is time-independent, $\phi = 0$.

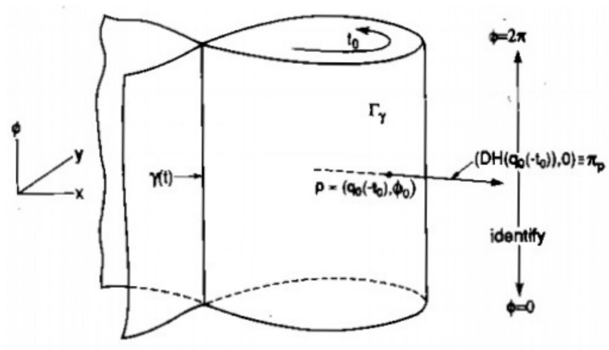

Figure 4: The vector π_p transversal to the homoclinic manifold

Proposition 1. *For ε sufficiently small, the periodic orbit $\gamma(t)$ of the unperturbed vector field (equation 7 with $\varepsilon = 0$) persists of a periodic orbit, $\gamma_\varepsilon(t) = \gamma(t) + \mathcal{O}(\varepsilon)$, of the perturbed vector field (7) having the same stability type as $\gamma(t)$ with $\gamma_\varepsilon(t)$ depending on ε in a $\mathbf{C}^r$ manner. Moreover, $W^s_{loc}(\gamma_\varepsilon(t))$ and $W^u_{loc}(\gamma_\varepsilon(t))$ are $\mathbf{C}^r$ ε-close to $W^s_{loc}(\gamma(t))$ and $W^s_{loc}(\gamma(t))$, respectively.*

Proving this is easy using a Poincaré map and the stable and unstable manifold theorem. The proof is not included in this thesis, but more information on the persistence of hyperbolic periodic orbits and their local stable and unstable manifolds can be found in [Fenichel, N. 1974].

Near the perturbed orbit $\gamma_\varepsilon(t)$, we want to describe the local stable and unstable manifolds. We do this in terms of the flow $\phi_t(\cdot)$ associated with (7). Now the global stable and unstable are defined as

$$\begin{aligned} W^s(\gamma_\varepsilon(t)) &= \bigcup_{t\leq 0} \phi_t(W^s_{loc}(\gamma_\varepsilon(t)), \\ W^u(\gamma_\varepsilon(t)) &= \bigcup_{t\geq 0} \phi_t(W^u_{loc}(\gamma_\varepsilon(t)). \end{aligned} \tag{15}$$

The flow is a $\mathbf{C}^r$ diffeomorphism that is also $\mathbf{C}^r$. Assuming that we only look at compact sets in $\mathbb{R}^2 \times S^1$ which contain $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$, we can now conclude from the following theorem that $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ are $\mathbf{C}^r$ functions of ε on these sets.We only consider the splitting of the two manifolds in an $\mathcal{O}(\epsilon)$ neighborhood of Γ_γ.

Theorem 1. *For $(t_0,\, x_0,\, \mu) \in U$ the solution $x(t,\, t_0,\, x_0,\, \mu)$ is a $\boldsymbol{C}^r$ function of t, t_0, x_0 and μ.*

We omit the proof of this theorem.

Proposition 1 requiers some explanation. It says that if we take a particular small ε called ε_0, we can define a neighborhood $\mathcal{N}(\varepsilon_0)$ in $\mathbb{R}^2 \times S^1$ of $\gamma(t)$ such that the distance between $\gamma(t)$ and the boundary of $\mathcal{N}(\varepsilon_0)$ is $\mathcal{O}(\varepsilon_0)$. Also, since $\gamma_\varepsilon(t) = \gamma(t) + \mathcal{O}(\varepsilon)$, if we pick an ε such that $0 < \varepsilon < \varepsilon_0$, it must be that $\gamma_\varepsilon(t)$ is in $\mathcal{N}(\varepsilon_0)$. Also $W^{u,s}_{loc}(\gamma(t))$ ($\equiv W^{u,s}(\gamma(t)) \cap \mathcal{N}(\varepsilon_0)$) are respectively $\mathbf{C}^r$ ε-close to $W^{u,s}_{loc}(\gamma_\varepsilon(t))$ ($\equiv W^{u,s}(\gamma_\varepsilon(t)) \cap \mathcal{N}(\varepsilon_0)$). A neighborhood that has the above mentioned properties is the solid torus

$$\mathcal{N}(\varepsilon_0) = \{(q,\phi) \in \mathbb{R}^2 \mid\mid q - p_0 \mid\leq C\varepsilon_0, \phi \in (0, 2\pi]\}, \tag{16}$$

where C is a constant (see figure 5).

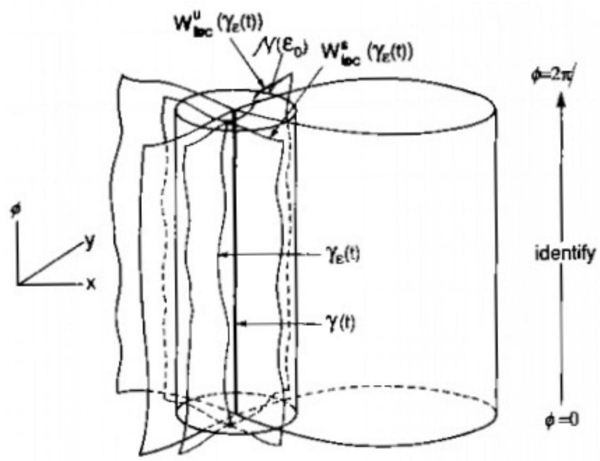

Figure 5: The solid torus $\mathcal{N}(\varepsilon_0)$ and $W^{u,s}(\gamma_\varepsilon(t))$

Since the unperturbed vector field is autonomous, we can just take a cross section of the phase space (by fixing ϕ_0). When we compare the unperturbed vector field to the perturbed one, it is sometimes easier to do this on a fixed ϕ_0 level. So we define the following cross-section of the phase space (see Figure 6)

$$\Sigma^{\phi_0} = \{(q,\phi) \in \mathbb{R}^2 | \phi = \phi_0\}. \tag{17}$$

Because the unperturbed system is autonomous, at a fixed ϕ_0 value, the hyperbolic fixed point p_0 is on this cross-section. Σ^{ϕ_0} also intersects Γ_γ in this value of ϕ_0. This intersection is exactly equal to Γ_{p_0}, again because the unperturbed system is autonomous. We have defined the trajectory of the unperturbed vector field to be ($q(t)$,$\phi(t)$). Now we can define the trajectory of the perturbed system in the same way, namely ($q_\varepsilon(t)$,ϕ_0).

The unperturbed system is nonautonomous, so $q_\varepsilon(t)$ is dependent on ϕ_0. Since we don't know what the unstable and the stable manifold will look like under perturbation, it is possible that they both intersect π_p more than once. That is

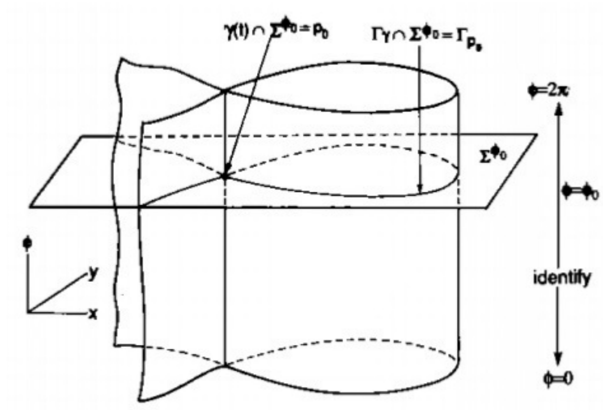

Figure 6: Cross-section of the phase space Σ^{ϕ_0}

why we define the distance between the manifolds to be the distance between certain points on π_p. These points are defined in the following way.

Definition 2. *Let $p^s_{\varepsilon,i} \in W^s(\gamma_\varepsilon(t)) \cap \pi_p$ and $p^u_{\varepsilon,i} \in W^u(\gamma_\varepsilon(t)) \cap \pi_p$, $i \in \mathcal{I}$, where $\mathcal{I}$ is some index set. Let $(q_{\varepsilon,i}(t),\phi(t)) \in W^s(\gamma_\varepsilon(t))$ and $(q^u_{\varepsilon,i}(t),\phi(t)) \in W^u(\gamma_\varepsilon(t))$ denote orbits of the perturbed vector field (6) satisfying $(q^s_{\varepsilon,i}(0),\phi(0)) = p^s_{\varepsilon,i}$ and $(q^u_{\varepsilon,i}(0),\phi(0)) = p^u_{\varepsilon,i}$ respectively. Then we have the following.*

1. *For some $i = \bar{i} \in \mathcal{I}$ we say that $p^s_{\varepsilon,\bar{i}}$ is the point in $W^s(\gamma_\varepsilon(t)) \cap \pi_p$ that is closest to $\gamma_\varepsilon(t)$ in terms of positive time of flight along $W^s(\gamma_\varepsilon(t))$ if, for all $t > 0$, $(q^s_{\varepsilon,\bar{i}}(t),\phi_0) \cap \pi_p = \emptyset$.*

2. *For some $i = \bar{i} \in \mathcal{I}$ we say that $p^u_{\varepsilon,i}$ is the point in $W^u(\gamma_\varepsilon(t)) \cap \pi_p$ that is closest to $\gamma_\varepsilon(t)$ in terms of negative time of flight along $W^s(\gamma_\varepsilon(t))$ if, for all $t < 0$, $(q^u_{\varepsilon,\bar{i}}(t),\phi_0) \cap \pi_p = \emptyset$.*

Now that we have defined the points between which we want to measure the distance, we can start defining the distance between these points. Suppose we want to measure the distance between $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$ in a point $p \in \Gamma_\gamma$. We already know that $W^s(\gamma_\varepsilon(t))$ en $W^u(\gamma_\varepsilon(t))$ intersect π_p transversely at p. First we define the distance between these two intersections p^s_ε and p^u_ε respectively, assuming both manifolds only intersect π_p once. It makes sense to define the distance between these points to be

$$d(p,\varepsilon) = |p^u_\varepsilon - p^s_\varepsilon| \tag{18}$$

(see Figure 7).

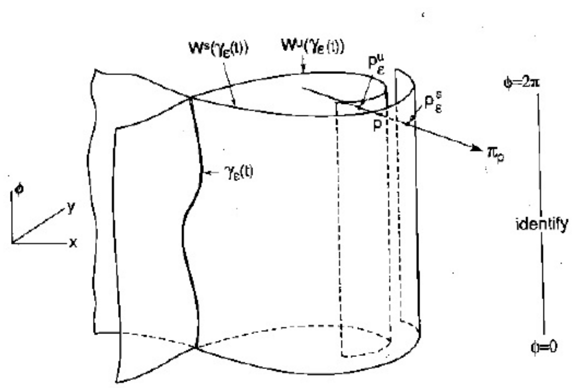

Figure 7: The intersections $p_\varepsilon^{u,s}$ between π_p and respectively the unstable and stable manifold

But it might be useful to know which way the manifolds are oriented. This can be done by defining the distance to be signed. We will therefore redefine the distance between the two points to be

$$d(p,\varepsilon) = \frac{(p_\varepsilon^u - p_\varepsilon^s) \cdot (DH(q_0(-t_0)), 0)}{\|DH(q_0(-t_0))\|}. \tag{19}$$

Because we normed the vector by the $\|DH(q_0(\text{-}t_0))\|$ term, it is obvious that (18) and (19) have the same magnitude. Later on, it will become clear why we inserted the fraction $\dfrac{DH(q_0(-t_0)), 0)}{\|DH(q_0(-t_0))\|}$. Since $p_\varepsilon^{u,s}$ are the intersections of $q_\varepsilon^{u,s}$ and π_p in a certain ϕ_0 value, we can also write

$$p_\varepsilon^u = (q_\varepsilon^u, \phi_0) \tag{20}$$

and

$$p_\varepsilon^s = (q_\varepsilon^s, \phi_0). \tag{21}$$

So now the distance can be redefined as

$$d(t,\phi_0,\varepsilon) = \frac{DH(q_0(-t_0)) \cdot (q_\varepsilon^u - q_\varepsilon^s)}{\|DH(q_0(-t_0))\|}. \tag{22}$$

In the above mentioned definitions of the distance between the intersection points of the (un)stable manifold and π_p, we assumed that there were only two intersection points between which the distance could be measured. But we did not know the structure of ($q_\varepsilon(t)$,ϕ_0); it is possible that this curve intersects π_p several times. We also defined which intersection points to use, in definition 1. We denote these points $p^s_{\varepsilon,\bar{i}} \in W^s(\gamma_\varepsilon(t)) \cap \pi_p$ and $p^u_{\varepsilon,\bar{i}} \in W^u(\gamma_\varepsilon(t)) \cap \pi_p$. So when we want to measure the distance between the intersections, we use these points.

But why did we choose just those points, that are defined in definition 1? This will be explained by the following Lemma plus its proof.

Lemma 1 (lemma 28.1.3)**.** *Let* $p^s_{\varepsilon,\bar{i}}$ *(resp.* $p^u_{\varepsilon,\bar{i}}$*) be a point on* $W^s(\gamma_\varepsilon(t)) \cap \pi_p$ *(resp.* $W^u(\gamma_\varepsilon(t) \cap \pi_p)$*) that is not closest to* $\gamma_\varepsilon(t)$ *in the sense of Definition 1, and let* $\mathcal{N}(\varepsilon_0)$ *denote the neighborhood of* $\gamma(t)$ *and* $\gamma_\varepsilon(t)$ *described following Proposition 1. Let* $(q^s_{\varepsilon,\bar{i}}(t),\phi(t))$ *(resp.* $(q^u_{\varepsilon,\bar{i}}(t),\phi(t))$*) be a trajectory in* $W^s(\gamma_\varepsilon(t))$ *(resp.* $W^u(\gamma_\varepsilon(t))$*) satisfying* $(q^s_{\varepsilon,\bar{i}}(0),\phi(0)) = p^s_{\varepsilon,\bar{i}}$ *(resp.* $(q^u_{\varepsilon,\bar{i}}(0),\phi(0)) = p^u_{\varepsilon,\bar{i}}$*). Then, for* ε *sufficiently small, before* $(q^s_{\varepsilon,\bar{i}}(t),\phi_0)$*,* $t > 0$*, (resp.* $(q^u_{\varepsilon,\bar{i}}(t),\phi_0)$*,* $t < 0$*) can intersect* π_p *(as it must by Definition 1, it must pass through* $\mathcal{N}(\varepsilon_0)$*.*

Proof: In this proof, we only consider trajectories in $W^s(\gamma_\varepsilon(t))$. The proof for trajectories in $W^u(\gamma_\varepsilon)$ will follow from the proof of the unstable case.

Take any point $(q^s_0,\phi_0) \in W^s(\gamma(t)) \cap \mathcal{N}(\varepsilon_0)$. We now call the trajectory in $W^s(\gamma(t))$ associated with this point $(q^s_0(t),\phi(t))$, where $(q^s_0(0),\phi(0)) = (q^s_0,\phi_0)$. When we follow this unperturbed trajectory until we reenter $\mathcal{N}(\varepsilon_0)$, it will take a finite time[3] $-\infty < T^s < 0$.

Now we look at the unperturbed case, where the trajectory will be less smooth. We choose a point $(q^s_\varepsilon,\phi_0) \in W^s_{loc}(\gamma_\varepsilon(t)) \cap \mathcal{N}(\varepsilon_0)$, with corresponding trajectory $(q^s_\varepsilon(t),\ \phi(t)) \in W^s(\gamma_\varepsilon(t))$. Just like in the unperturbed case, $(q^s_\varepsilon(0),\phi(0)) = (q_\varepsilon,\phi_0)$.
Now

$$\mid (q^s_\varepsilon(t),\phi(t)) - (q^s_0(t),\phi(t)) \mid = \mathcal{O}(\varepsilon_0) \tag{23}$$

for $0 \leq t \leq \infty$. Using Gronwall's inequality (which can be found in Appendix A), it can be shown that also

$$\mid (q^s_\varepsilon(t),\phi(t)) - (q^s_0(t),\phi(t)) \mid = \mathcal{O}(\varepsilon) \tag{24}$$

for $T^s \leq t \leq 0$. We can conclude from the above equality that every trajectory in $W^s(\gamma_\varepsilon(t))$ from (q^s_ε,ϕ_0) to $(q_\varepsilon(T^s),\phi_0(T^s))$ in negative time, must be $\mathcal{O}(\varepsilon)$ close to a trajectory in $W^s(\gamma(t))$ from (q^s_0,ϕ) to $(q^s_0(T),\phi(T))$ in negative time, where T is the time it takes to reenter $\mathcal{N}(\varepsilon_0)$ from (q^s_0,ϕ) following the trajectory in $W^s(\gamma(t))$. In other words, $(q^s_0(t),\phi(t))$ and $(q^s_\varepsilon(t),\phi(t))$ are ε-close when $T^s \leq t \leq \infty$ between (q_ε,ϕ_0) and $(q_\varepsilon(T^s),\phi(T^s))$ (in negative time).

We want to know if it is still possible that π_p get intersected more than once by the stable and the unstable manifold. Since we know that $(q^s_0(t),\phi(t))$ and

[3] This finite time depends on the choice of ε_0. This means a fixed ε_0 should be considered to define $\mathcal{N}(\varepsilon_0)$

$(q^s_\varepsilon(t),\phi(t))$ are ε-close when $T^s \le t \le \infty$, we can rewrite $(q^s_\varepsilon(t),\phi(t))$ to

$$(q^s_\varepsilon(t),\phi(t)) = (q^s_0(t) + \mathcal{O}(\varepsilon),\phi(t)). \tag{25}$$

A vector tangent to the above vector is

$$\begin{aligned} \dot{q^s_\varepsilon} &= JDH(q^s_\varepsilon)_+\varepsilon g(q^s_\varepsilon,\phi(t)) \\ \dot{\phi} &= \omega \end{aligned} \tag{26}$$

, and taking into account equation (25), we get

$$\begin{aligned} \dot{q^s_\varepsilon} &= JDH(q^s_0 + \mathcal{O}(\varepsilon)) + \varepsilon g(q^s_0 + \mathcal{O}(\varepsilon),\phi(t)) \\ \dot{\phi} &= \omega \end{aligned} \tag{27}$$

We now do a Taylor expansion on the above equation around $\varepsilon = 0$, so we get

$$\begin{aligned} \dot{q^s_\varepsilon} &= JDH(q^s_0)_+\mathcal{O}(\varepsilon). \\ \dot{\phi} &= \omega \end{aligned} \tag{28}$$

This means that a tangent vector to $(q_0(t),\phi(t))$ is

$$\begin{aligned} \dot{q^s_0} &= JDH(q^s_0) \\ \dot{\phi} &= \omega. \end{aligned} \tag{29}$$

For $T^s \le t < \infty$, it is clear that equation (28) and (29) are $\mathcal{O}(\varepsilon)$ close. This means that it is not possible for $(q_\varepsilon(t),\phi(t))$ to intersect π_p more than once. This situation is sketched in figure 8.

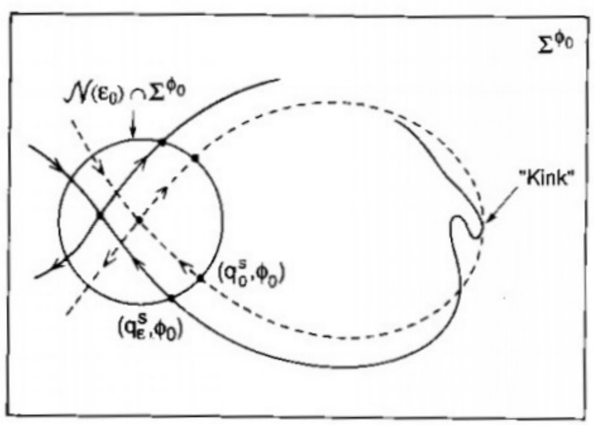

Figure 8: The case where $(q^s_\varepsilon(t),\,\phi(t))$ re-intersects π_p

Now, let's consider a point in $W^s(\gamma_\varepsilon(t)) \cap \pi_p$ that in positive time of flight is not closest to $\gamma_\varepsilon(t)$ explained in definition 1. We call this point $p^s_{\varepsilon,a}$, and its corresponding orbit in $W^s(\gamma_\varepsilon(t))$ we call $(q^s_{\varepsilon,a}(t),\phi(t))$, with $(q^s_{\varepsilon,a}(0),\phi(0))$ $= p^s_{\varepsilon,a}$. This means that in some finite (positive) time t_a, $(q^s_{\varepsilon,a}(t),\phi(t))$ must intersect π_p. And we argued above that $(q^s_{\varepsilon,a}(t),\phi(t))$ must have entered $\mathcal{N}(\epsilon_0)$ for some $0 < t < t_a$.

□

From the above proof it follows that the points closest to $\gamma_\varepsilon(t)$, as described in definition 1 are unique. To show this, consider a point $p^s_{\varepsilon,a}$ that is closest to $\gamma_\varepsilon(t)$, as described in definition 1. Let $(q^s_{\varepsilon,a}(t),\phi(t))$ satisfy $(q^s_{\varepsilon,a}(0),\phi(0)) = p^s_{\varepsilon,a}$. Now, what if there was another point $p^s_{\varepsilon,b}$ on $(q^s_{\varepsilon,a}(t),\phi(t))$ closest to $\gamma_\varepsilon(t)$, as described in definition 1. Then, since for all $t > 0$ it holds that $(q^s_{\varepsilon,a}(t),\phi_0) \cap \pi_p = \emptyset$, it must be that $p^s_{\varepsilon,b}$ only can be reached following $(q^s_{\varepsilon,a}(t),\phi(t))$ in a negative time of flight. But this means, that following $(q^s_{\varepsilon,a}(t),\phi(t))$ in a positive time flight from $p^s_{\varepsilon,b}$, $(q^s_{\varepsilon,a}(t),\phi(t)) \cap \pi_p = p^s_{\varepsilon,a} \neq \emptyset$. So $p^s_{\varepsilon,b}(t)$ cannot be closest to $\gamma_\varepsilon(t)$, as described in definition 1. A similar argument can be used for the unstable case. So the points closest to to $\gamma_\varepsilon(t)$, as described in definition 1 are unique.

In step 3, we are going to derive the Melnikov function. We saw in the proof of Lemma 1 that if $p^s_\varepsilon = (q^s_\varepsilon,\phi_0) \in W^s(\gamma_\varepsilon(t)) \cap \pi_p$ and with the corresponding trajectory $(q^s_\varepsilon(t),\, \phi(t)) \in W^s(\gamma_\varepsilon(t))$ that satisfies $(q_\varepsilon(0),\, \phi(0)) = (q^s_\varepsilon,\phi_0)$, is closest to to $\gamma_\varepsilon(t)$, as described in definition 1, we saw that

$$| \, q^s_\varepsilon(t) - q_0(t - t_0) \, |= \mathcal{O}(\varepsilon), \qquad t \in [0, \infty), \tag{30}$$
$$| \, \dot{q}^s_\varepsilon(t) - \dot{q}_0(t - t_0) \, |= \mathcal{O}(\varepsilon), \qquad t \in [0, \infty). \tag{31}$$

Of course something similar also holds for the unstable case. When we are deriving the Melnikov function, from the unperturbed solutions in $W^s(\gamma(t))$ and $W^u(\gamma(t))$ we approximate the unperturbed solutions in $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$ for finite time intervals, where the distance between these solutions will be $\mathcal{O}(\varepsilon)$. This explains why the Melnikov function only detects these points on $W^s(\gamma(t))$ and $W^u(\gamma(t))$ that intersect π_p, which are closest to γ_ε as described in definition 1.

2.4 Step 3: Deriving the Melnikov function

The first step in the derivation of the Melnikov function is the Taylor expansion of our in equation 22 defined distance, about $\varepsilon = 0$. This will give us

$$d(t_0, \phi_0, \varepsilon) = d(t_0, \phi_0, 0) + \frac{\partial d}{\partial \varepsilon}(t_0, \phi_0, 0) \cdot \varepsilon + h.o.t. \tag{32}$$

When $\varepsilon = 0$, the distance between the stable and the unstable manifold will be zero, since this corresponds to the unperturbed case. This means that $d(t_0,\phi_0,0) = 0$. Also, since $DH(q_0(\text{-}t_0))$ isn't dependent on ε, and

$\|DH(q_0(\text{-}t_0))\|$ is a scalar,

$$\frac{\partial d}{\partial \varepsilon}(t_0,\phi_0,0) = \frac{DH(q_0(-t_0)) \cdot (\frac{\partial q^u_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} - \frac{\partial q^s_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0})}{\|DH(q_0(-t_0))\|}. \tag{33}$$

Now we have come to the point where we can define the Melnikov function. The Melnikov function is dependent on t_0 and ϕ_0, and is defined to be

$$M(t_0,\phi_0) \equiv DH(q_0(-t_0)) \cdot (\frac{\partial q^u_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} - \frac{\partial q^s_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}). \tag{34}$$

The time-dependent Melnikov function, which looks a lot like the original Melnikov function, is to be derived. The only difference is the replacement of $q^{u,s}_\varepsilon$ with $q^{u,s}_\varepsilon(t)$. This gives

$$M(t,t_0,\phi_0) \equiv DH(q_0(t-t_0)) \cdot (\frac{\partial q^u_\varepsilon(t)}{\partial \varepsilon}|_{\varepsilon=0} - \frac{\partial q^s_\varepsilon(t)}{\partial \varepsilon}|_{\varepsilon=0}). \tag{35}$$

In the above equation, $q^{u,s}_\varepsilon(t)$ satisfies

$$q^{u,s}_\varepsilon(0) = q^{u,s}_\varepsilon, \tag{36}$$

since those are just the points on orbits in $W^s(\gamma_\varepsilon(t))$, that intersect π_p. Also, the expression $DH(q_0(t \text{ - } t_0))$ is different from the one described in equation 22. There $q_0(t \text{ - } t_0)$ just denotes the homoclinic orbit from the unperturbed system. We can now conlcude that when the time $t = 0$, the time dependent Melnikov function has the same value as the time-independent Melnikov function.

Next, we use a Lemma to show how we can rewrite the Melnikov function to the integral we need. But first we rewrite the time-dependent Melnikov function a little bit. We define

$$\frac{\partial q^{u,s}_\varepsilon(t)}{\partial \varepsilon}|_{\varepsilon=0} \equiv q^{u,s}_1(t). \tag{37}$$

Then, equation 35 will become

$$M(t,t_0,\phi_0) = DH(q_0(t-t_0)) \cdot (q^u_1(t) - q^s_1(t)). \tag{38}$$

We can also simplify the above equation by substituting

$$\Delta^{u,s}(t) \equiv DH(q_0(t-t_0)) \cdot q^{u,s}_1(t). \tag{39}$$

Now, the time-dependent Melnikov function can also be written as

$$M(t,t_0,\phi_0) \equiv \Delta^u(t) - \Delta^s(t). \tag{40}$$

Differentiating this equation with respect to t gives

$$\frac{d}{dt}(\Delta^{u,s}(t)) = (\frac{d}{dt}(DH(q_0(t-t_0)))) \cdot q^{u,s}_1(t) + DH(q_0(t-t_0)) \cdot \frac{d}{dt} q^{u,s}_1(t). \tag{41}$$

Since $q_\varepsilon^{u,s}(t)$ is the solution of equation 2, we have

$$\frac{d}{dt}(q_\varepsilon^{u,s}(t)) = JDH(q_\varepsilon^{u,s}(t)) + \varepsilon g(q_\varepsilon^{u,s}(t), \phi(t), \varepsilon), \tag{42}$$

where $\phi(t) = \omega t + \phi_0$. We know that $q_\varepsilon^{u,s}(t)$ is $\mathbf{C}^r$ in both ε and t so we can also differentiate 42 with respect to ε, to get

$$\frac{d}{dt}(\frac{\partial q_\varepsilon^{u,s}(t)}{\partial \varepsilon}|_{\varepsilon=0}) = JD^2H(q_0(t-t_0))\frac{\partial q_\varepsilon^{u,s}(t)}{\partial \varepsilon}|_{\varepsilon=0} + q(q_0(t-t_0), \phi(t), 0) \tag{43}$$

which is the same as

$$\frac{d}{dt}q_1^{u,s}(t) = JD^2H(q_0(t-t_0))q_1^{u,s}(t) + q(q_0(t-t_0), \phi(t), 0). \tag{44}$$

Since $q_1^s(t)$ corresponds to the derivative of $q_\varepsilon^s(t)$ with respect to ε, $q_1^s(t)$ is a solution of equation 44 for $t \in (0,\infty]$, and since $q_1^u(t)$ corresponds to the derivative of $q_\varepsilon^u(t)$ with respect to ε, $q_1^u(t)$ is a solution of equation 44 for $t \in (-\infty,0]$. Now that we have equation 44, equation 41 becomes

$$\begin{aligned}\frac{d}{dt}(\Delta^{u,s}(t)) = (&\frac{d}{dt}(DH(q_0(t-t_0)))) \cdot q_1^{u,s}(t) \\ &+ DH(q_0(t-t_0)) \cdot JD^2H(q_0(t-t_0))q_1^{u,s}(t) \\ &+ DH(q_0(t-t_0)) \cdot q(q_0(t-t_0), \phi(t), 0).\end{aligned} \tag{45}$$

Now we use the following Lemma, to simplify the above expression.

Lemma 2.

$$\frac{d}{dt}(DH(q_0(t\text{-}t_0)))\cdot\ q_1^{u,s}(t)\ +\ DH(q_0(t\text{-}t_0))\cdot\ JD^2H(q_0(t\text{-}t_0))q_1^{u,s}(t)\ =0.$$

Proof: When we differentiate $DH(q_0(t\ \text{-}\ t_0))$ with respect to t, we get

$$\frac{d}{dt}(DH(q_0(t-t_0))) = D^2H(q_0(t-t_0))\dot{q_0}(t-t_0) \tag{46}$$

which is equal to

$$(D^2H(q_0(t-t_0)))(JDH(q_0(t-t_0))). \tag{47}$$

Rewriting the above equation in vector/matrix form

$$\begin{aligned}(D^2H)(JDH)\cdot q_1^{u,s} &= \begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} x_1^{u,s} \\ y_1^{u,s} \end{pmatrix} \\ &= x_1^{u,s}\left(\frac{\partial^2 H}{\partial x^2}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial x \partial y}\frac{\partial H}{\partial x}\right) \\ &+ y_1^{u,s}\left(\frac{\partial^2 H}{\partial x \partial y}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial y^2}\frac{\partial H}{\partial x}\right).\end{aligned} \tag{48}$$

and

$$
\begin{aligned}
DH \cdot (JD^2H)q_1^{u,s} &= \begin{pmatrix} \dfrac{\partial H}{\partial x} \\ \\ \dfrac{\partial H}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial^2 H}{\partial x \partial y} & \dfrac{\partial^2 H}{\partial y^2} \\ \\ -\dfrac{\partial^2 H}{\partial x^2} & -\dfrac{\partial^2 H}{\partial x \partial y} \end{pmatrix} \begin{pmatrix} x_1^{u,s} \\ \\ y_1^{u,s} \end{pmatrix} \\
&= x_1^{u,s} \left(\frac{\partial^2 H}{\partial x^2} \frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial x \partial y} \frac{\partial H}{\partial x} \right) \\
&+ y_1^{u,s} \left(\frac{\partial^2 H}{\partial x \partial y} \frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial y^2} \frac{\partial H}{\partial x} \right).
\end{aligned} \tag{49}
$$

Equations 48 and 49 add up to zero. So $\frac{d}{dt}(DH(q_0(t\text{-}t_0)))\cdot\, q_1^{u,s}(t) + DH(q_0(t\text{-}t_0))\cdot\, JD^2H(q_0(t\text{-}t_0))q_1^{u,s}(t) = 0$.

□

Now equation 45 becomes

$$
\frac{d}{dt}(\Delta^{u,s}(t)) = DH(q_0(t - t_0)) \cdot g(q_0(t - t_0), \phi(t), 0). \tag{50}
$$

Since $\Delta^u(t)$ corresponded to the unstable manifold and $\Delta^s(t)$ to the stable one, we integrate $\Delta^u(t)$ from -τ to 0 (where $\tau > 0$), and $\Delta^s(t)$ from 0 to τ. This gives

$$
\Delta^u(0) - \Delta^u(-\tau) = \int_{-\tau}^{0} DH(q_0(t - t_0)) \cdot g(q_0(t - t_0), \omega t + \phi_0, 0)dt \tag{51}
$$

and

$$
\Delta^s(\tau) - \Delta^s(0) = \int_{0}^{\tau} DH(q_0(t - t_0)) \cdot g(q_0(t - t_0), \omega t + \phi_0, 0)dt. \tag{52}
$$

We saw that $M(t,\, t_0,\, \phi_0)$ was equal to $\Delta^u(t)$ - $\Delta^s(t)$, and that $M(0,\, t_0,\, \phi_0)$ was equal to the time-independent Melnikov function $M(t_0,\, \phi_0)$. So

$$
M(t_0, \phi_0) = \Delta^u(0) - \Delta^s(0) = \int_{-\tau}^{\tau} DH(q_0(t - t_0)) \cdot g(q_0(t - t_0), \omega t + \phi_0, 0)dt + \Delta^s(\tau) - \Delta^u(-\tau). \tag{53}
$$

We want to take the limit of the above equation as $\tau \to \infty$. To compute this limit, we need the following two Lemmas.

Lemma 3.

$$
\lim_{\tau \to \infty} \Delta^s(\tau) = \lim_{\tau \to \infty} \Delta^u(-\tau) = 0. \tag{54}
$$

Proof: We knew that

$$\Delta^{u,s}(t) = DH(q_0(t - t_0)) \cdot q_1^{u,s}(t).$$

Now we want to take the limit of the above equation as $t \to \infty$. First consider the unstable manifold. As $t \to \infty$, $q_0(t - t_0)$ approaches the hyperbolic fixed point, which means that $DH(q_0(t - t_0)) \to 0$. So now the limit can only go to zero if $q_1^u(t)$ is bounded.
When t approaches infinity, $q_\varepsilon^u(t)$ approaches γ_ε. So

$$\lim_{t\to\infty} q_1^u(t) = \frac{\partial q_\varepsilon^u(t)}{\partial \varepsilon}|_{\varepsilon=0} \to \frac{\partial \gamma_\varepsilon^u(t)}{\partial \varepsilon}|_{\varepsilon=0}. \tag{55}$$

And since $\gamma_\varepsilon(t) = \gamma(t) + \mathcal{O}(\varepsilon)$, it follows that $\frac{\partial \gamma_\varepsilon^u(t)}{\partial \varepsilon}|_{\varepsilon=0}$ is bounded.
The same argument can be used for the stable manifold, since for $t \to -\infty$, $DH(q_0(t - t_0)) \to 0$, and $q_\varepsilon^s(t)$ again approaches γ_ε.

□

Lemma 4. *The integral*

$$\int_{-\infty}^{\infty} DH(q_0(t - t_0)) \cdot g(q_0(t - t_0), \omega t + \phi_0, 0)\ dt$$

converges absolutely.

Proof: We already stated that $DH(q_0(t - t_0))$ goes to zero when $t \to \pm\infty$. Also g is bounded for all t. So the integral converges absolutely.

□

Now we can conclude

$$M(t_0, \phi_0) = \int_{-\infty}^{\infty} DH(q_0(t - t_0)) \cdot g(q_0(t - t_0), \omega t + \phi_0, 0) dt. \tag{56}$$

We can make the transformation

$$t \to t + t_0$$

to change (56) into

$$M(t_0, \phi_0) = \int_{-\infty}^{\infty} DH(q_0(t)) \cdot g(q_0(t), \omega t + \omega t_0 + \phi_0, 0) dt. \tag{57}$$

We knew that g was periodic in t_0 with period $2\pi/\omega$, so $M(t_0,\, \phi_0)$ must also be periodic in t_0 with the same period. $M(t_0,\, \phi_0)$ is also dependent on ϕ_0 and is therefore also periodic in ϕ_0 with period 2π. In the end, we want to show that the zeros of the Melnikov function correspond to homoclinic points of a two-dimensional map. Then, we can apply Moser's theorem, to show that the system behaves chaotically. In order to do that, we need to prove the following theorem.

Theorem 2. *Suppose we have a point* $(t_0,\phi_0) = (\bar{t}_0,\overline{\phi}_0)$ *such that*

1. $M(\bar{t}_0,\overline{\phi}_0) = 0$ *and*

2. $\frac{\partial M}{\partial t_0}|_{(\bar{t}_0,\overline{\phi_0})} \neq 0$.[4]

Then $W^s(\gamma_\varepsilon(t))$ *and* $W^u(\gamma_\varepsilon(t))$ *intersect transversely at* $(q_0(\text{-}\bar{t}_0) + \mathcal{O}(\varepsilon),\overline{\phi}_0)$. *Moreover, if* $M(t_0,\phi_0) \neq 0$ *for all* $(t_0,\phi_0) \in \mathbb{R} \times S^1$, *then* $W^s(\gamma_\varepsilon(t)) \cap W^u(\gamma_\varepsilon(t)) = \emptyset$.

Proof: Combining the information we get from equations 32, 33 and 34 we can write

$$d(t_0,\phi_0,\varepsilon) = \varepsilon \frac{M(t_0,\phi_0)}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\varepsilon^2). \tag{58}$$

For the sake of simpler notation, we define

$$d_2(t_0,\phi_0,\varepsilon) = \frac{M(t_0,\phi_0)}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\varepsilon^2). \tag{59}$$

This way

$$d(t_0,\phi_0,\varepsilon) = \varepsilon d_2(t_0,\phi_0,\varepsilon). \tag{60}$$

It should be clear that when we look at the zeros of $d_2(t_0,\phi_0,\varepsilon)$, these are also the zeros of $d(t_0,\phi_0,\varepsilon)$.

From part 1 of the above stated theorem, we know that $M(\bar{t}_0,\, \bar{\phi}_0) = 0$. So

$$d_2(\bar{t}_0,\bar{\phi}_0,0) = \frac{M(\bar{t}_0,\bar{\phi}_0)}{\|DH(q_0(-\bar{t}_0))\|} = 0. \tag{61}$$

From part 2 of the above stated theorem, we can conclude that

$$\frac{\partial d_2}{\partial t_0}\,|_{(\bar{t}_0,\bar{\phi}_0,0)} = \frac{1}{\|DH(q_0(-\bar{t}_0))\|}\frac{\partial M}{\partial t_0}\,|_{(\bar{t}_0,\bar{\phi}_0)} \neq 0. \tag{62}$$

[4]Note that $\frac{\partial M}{\partial \phi_0}(t_0,\phi_0) = \omega \frac{\partial M}{\partial t_0}(t_0,\phi_0)$. So we could also have chosen $\frac{\partial M}{\partial t_0}|_{(\bar{t}_0,\bar{\phi}_0)} \neq 0$.

The implicit function theorem now says that for $|\phi$ - $\phi_0|$ and ε sufficiently small there exists a function

$$t_0 = t_0(\phi_0, \varepsilon) \tag{63}$$

such that

$$d_2(t_0(\phi_0, \varepsilon), \phi_0, \varepsilon) = 0. \tag{64}$$

Now we know that $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$ intersect $\mathcal{O}(\varepsilon)$ close to $(q_0(\text{-}t_0))$. But is this intersection also transversal? We know that points on $W^{u,s}(\gamma_\varepsilon(t))$ that are closest to $\gamma_\varepsilon(t)$ (ε sufficiently small) can be parametrized by t_0 and ϕ_0. To have transversal intersections, from definition 1 we must have

$$T_pW^s(\gamma_\varepsilon(t)) + T_pW^u(\gamma_\varepsilon(t)) = \mathbb{R}^3. \tag{65}$$

Now, since q^s_ε respectively q^u_ε are points on the trajectories $q^s_\varepsilon(t) \in W^s(\gamma_\varepsilon(t))$ and $q^u_\varepsilon(t) \in W^s(\gamma_\varepsilon(t))$ evaluated at $t = 0$, and these trajectories are parametrized by t_0 and ϕ_0, a basis for $T_pW^u(\gamma_\varepsilon(t)$ is equal to

$$(\frac{\partial q^u_\varepsilon}{\partial t_0}, \frac{\partial q^u_\varepsilon}{\partial \phi_0}) \tag{66}$$

and a basis for $T_pW^s(\gamma_\varepsilon(t)$ is equal to

$$(\frac{\partial q^s_\varepsilon}{\partial t_0}, \frac{\partial q^s_\varepsilon}{\partial \phi_0}). \tag{67}$$

It is clear that the intersection between $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$at point p won't be transversal (which means they are tangent at p) if

$$\frac{\partial q^u_\varepsilon}{\partial t_0} = \frac{\partial q^s_\varepsilon}{\partial t_0} \tag{68}$$

or

$$\frac{\partial q^u_\varepsilon}{\partial \phi_0} = \frac{\partial q^s_\varepsilon}{\partial \phi_0}. \tag{69}$$

So if we now differentiate $d(t_0$,ϕ_0,$\varepsilon)$ with respect to t_0 and ϕ_0 and evaluate this equation at the zero of the Melnikov function (wich is the intersection point ($\overline{t_0}$ + $\mathcal{O}(\varepsilon)$, $\overline{\phi_0}$) gives us

$$\begin{aligned}\frac{\partial d}{\partial t_0}(\overline{t_0}, \overline{\phi_0}, \varepsilon) &= \frac{DH(q_0(-\overline{t_0})) \cdot ((\partial q^u_\varepsilon)/(\partial t_0) - (\partial q^s_\varepsilon)/(\partial t_0))}{\|DH(q_0(-\overline{t_0}))\|} \\ &= \frac{\partial M/\partial t_0(\overline{t_0}, \overline{\phi_0})}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\varepsilon),\end{aligned} \tag{70}$$

$$\begin{aligned}\frac{\partial d}{\partial \phi_0}(\overline{t_0}, \overline{\phi_0}, \varepsilon) &= \frac{DH(q_0(-\overline{t_0})) \cdot ((\partial q^u_\varepsilon)/(\partial t_0) - (\partial q^s_\varepsilon)/(\partial t_0))}{\|DH(q_0(-\overline{t_0}))\|} \\ &= \frac{\partial M/\partial t_0(\overline{t_0}, \overline{\phi_0})}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\varepsilon),\end{aligned} \tag{71}$$

So for $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$ not to be tangent at p, a condition is

$$\frac{\partial M}{\partial \phi_0}(\bar{t}_0, \overline{\phi}_0) = \omega \frac{\partial M}{\partial t_0}(\bar{t}_0, \overline{\phi}_0) \neq 0. \tag{72}$$

Now that we have proven that the intersection between $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$ is not tangent but transversal, Moser's theorem [Moser, 1973] can be applied to show that the system possesses chaotic behaviour. To be able to use Moser's theorem, the system has to satisfy the following.

Let

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

be a C^r $(r \geq 1)$ diffeomorphism satisfying the following hypotheses.

1. f has a hyperbolic periodic point p.
2. $W^s(p)$ and $W^u(p)$ intersect transversely.

One of the assumptions we made about the system was that the system has a hyperbolic fixed point. We have just proven that $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$ will intersect transversely. The function f is our case the Poincaré map. Now it is safe to apply Moser's theorem [Moser, 1973], which goes beyond the scope of this thesis.

3 Application to the Driven Morse Oscillator

In theoretical chemistry, the driven Morse oscillator is used to describe the photolysis of molecules [Goggin and Milonni, 1988]. This is a chemical reaction where molecular bonds are broken by the interaction with photons. The molecule in question is subject to a sinusoidal driving force. We consider the case where besides a driving force, also a friction force is involved. In practice, friction might arise from a liquid surrounding the molecule. This problem was originally solves by B. Bruhn [Bruhn, 1989].

3.1 Description of the system

The equations of motion of the driven Morse oscillator with friction are

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\mu(e^{-x} - e^{-2x}) - \varepsilon c y + \varepsilon\gamma\cos\omega t, \end{aligned} \tag{73}$$

with μ, γ, $\omega > 0$.

Here the cy-term corresponds to the friction force, proportional to the velocity. The dissociation energy is given by $\mu/2$. The γ and the ω are constants associated with the driving force. Here γ is the amplitude, and ω is the frequency of the driving force.

The unperturbed system is Hamiltonian where the Hamiltonian function is given by

$$H(x, y) = \frac{y^2}{2} + \mu(-e^{-x} + \frac{1}{2}e^{-2x}). \tag{74}$$

We can see that in the unperturbed system, where $\varepsilon = 0$, $\dot{x}$ and $\dot{y}$ are both zero at (∞,0). So (∞,0) is an equilibrium point of the above mentioned system. When we set det(Df - λI) to zero, we obtain the eigenvalues $\lambda_{1,2} = \pm\sqrt{2\mu e^{-2x} - \mu e^{-x}}$. Both of these eigenvalues are zero in (∞,0), so (∞,0) is a nonhyperbolic equilibrium point of (73). Now we want to show that (∞,0) is connected to itself by a homoclinic orbit. From the Hamiltonian, taking $H(x,y)$ to be a fixed value h, we can see that $y = \pm\sqrt{2h - \mu(-2e^{-x} + e^{-2x})}$. The homoclinic orbit is the solution for wich $h = 0$, see figure 9.

3.2 Coordinate Transformation

To desingularize the equilibrium point, we do the following McGehee-type transformation [McGehee, 1973]:

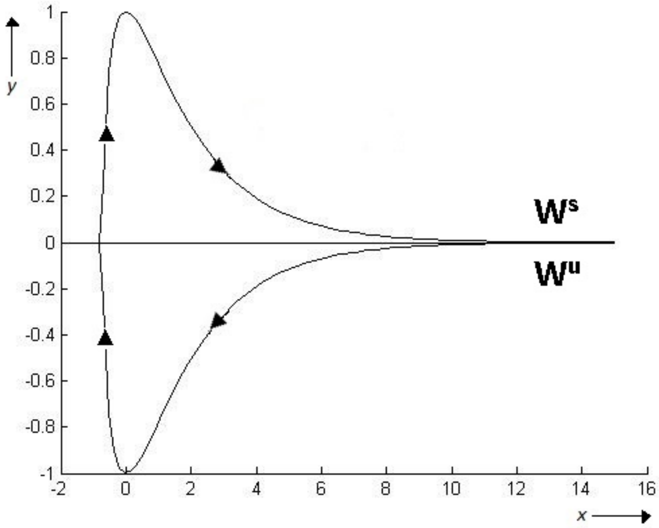

Figure 9: The homoclinic orbit

$$
\begin{aligned}
x &= -2\log(u) \\
y &= v \\
\frac{ds}{dt} &= -\frac{u}{2}
\end{aligned}
\tag{75}
$$

Rewriting the equations of motion in the new coordinates gives

$$
\begin{aligned}
\frac{du}{ds} &= v \\
\frac{dv}{ds} &= \mu(2u - 2u^3) + \varepsilon\left(\frac{2cv}{u} - \frac{2\gamma}{u}\cos\omega t(s)\right)
\end{aligned}
\tag{76}
$$

where

$$
t(s) = -2\int \frac{ds}{u(s)}. \tag{77}
$$

In these new coordinates, when $\varepsilon = 0$, we can see that there is an equilibrium point at $(u,v) = (0,0)$. This equilibrium point is also hyperbolic, since the eigenvalues of $Df(0,0)$ are $\pm\sqrt{2\mu}$, which both have non-zero real part.

Rewriting the Hamiltonian in these new coordinates gives the following first integral

$$
I = \frac{v^2}{2} + \mu u^2\left(\frac{1}{2}u^2 - 1\right). \tag{78}
$$

We can rewrite this first integral, to get $v = \pm\sqrt{2I - \mu u^2(u^2-2)}$. The homoclinic solution again corresponds to $I = 0$. A plot of $v = \pm\sqrt{2I - \mu u^2(u^2-2)}$ with $I = 0$ (and $\mu = 1$, which is just a random choice) against u can be found in figure 10.

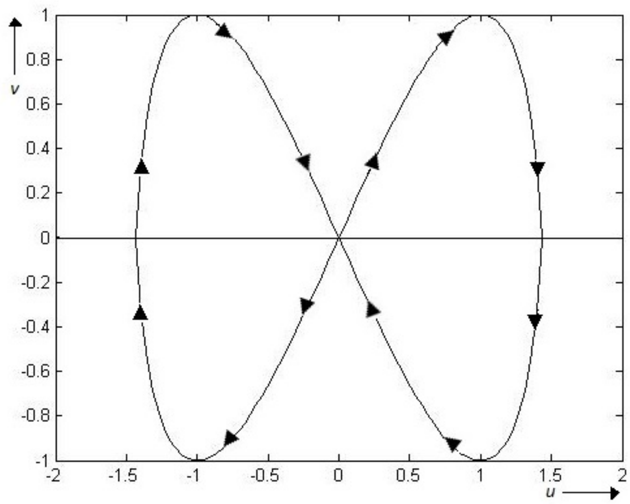

Figure 10: The homoclinic solution, with $\mu = 1$

The homoclinic solution corresponds to

$$\begin{aligned} u_0(s,s_0) &= \sqrt{2}\,\mathrm{sech}(\sqrt{2\mu}(s-s_0) \\ v_0(s,s_0) &= -2\sqrt{\mu}\,\mathrm{sech}(\sqrt{2\mu}(s-s_0))\tanh(\sqrt{2\mu}(s-s_0)) \end{aligned} \tag{79}$$

This is indeed a solution, since

$$\begin{aligned} \frac{d(u_0(s,s_0))}{ds} &= -2\sqrt{\mu}\,\mathrm{sech}(\sqrt{2\mu}(s-s_0))\tanh(\sqrt{2\mu}(s-s_0)) = v_0(s,s_0) \\ \frac{d(v_0(s,s_0))}{ds} &= -2\sqrt{\mu}(-\sqrt{2\mu}\,\mathrm{sech}(\sqrt{2\mu}(s-s_0))\tanh^2(\sqrt{2\mu}(s-s_0)) \\ &+ \sqrt{2\mu}\,\mathrm{sech}(\sqrt{2\mu}(s-s_0))(1-\tanh^2(\sqrt{2\mu}(s-s_0)))) \\ &= -2\sqrt{\mu}(\sqrt{2\mu}\,\mathrm{sech}(\sqrt{2\mu}(s-s_0))(2\,\mathrm{sech}^2(\sqrt{2\mu}(s-s_0))-1) \\ &= -2\mu u_0(u_0^2-1). \end{aligned} \tag{80}$$

Now after some calculations, we see that $t(s,s_0)$ is equal to

$$t(s,s_0) = t_0 - \frac{1}{2\sqrt{\mu}}\sinh(\sqrt{2\mu}(s-s_0)). \tag{81}$$

Here t_0 is the integration constant, which corresponds to the initial time.

3.3 Derivation of the Melnikov Function

Now we want to apply Melnikov's method. Recall from equation 56 that the Melnikov function was given by

$$M(t_0, \phi_0) = \int_{-\infty}^{\infty} DH(q_0(t)) \cdot g(q_0(t), \omega(t) + \omega t_0 + \phi_0, 0)dt. \tag{82}$$

We already have the perturbation function g, which has a zero u-component, and is equal to

$$g(q_0(t), \omega, \gamma, s, 0) = (0, \frac{2cv_0}{u_0} - \frac{2}{u_0}\gamma \cos \omega t(s, s_0)). \tag{83}$$

Since we named the Hamiltonian in the new coordinates I, $DI(q_0(t))$ is equal to

$$DI(q_0(t)) = \begin{pmatrix} 2\mu u_0^3 - 2\mu u_0 \\ v_0 \end{pmatrix}. \tag{84}$$

Now, the Melnikov function will be

$$M(t_0, \phi_0) = \int_{-\infty}^{\infty} \left(\frac{2cv_0^2}{u_0} - \frac{2v_0}{u_0}\gamma \cos \omega t(s, s_0) \right) ds. \tag{85}$$

Before we compute this integral, we make the substitution

$$s' = \sinh(\sqrt{2\mu}(s - s_0)). \tag{86}$$

After rewriting the integral in terms of s', it becomes

$$M(s') = \int_{-\infty}^{\infty} \left(4c\sqrt{\mu} \cdot \frac{s'^2}{(1 + s'^2)^2} + \frac{2s'}{1 + s'^2} \cdot \gamma \cos \omega t(s') \right) ds' \tag{87}$$

where

$$t(s') = t_0 - \frac{s'}{2\sqrt{\mu}}. \tag{88}$$

We solve this integral in two parts. The first part is obtained from observing that

$$\int_{-\infty}^{\infty} \frac{x^2}{(1 + x^2)^2} dx = \frac{\pi}{2}. \tag{89}$$

So

$$\int_{-\infty}^{\infty} 4c\sqrt{\mu} \cdot \frac{s'^2}{(1+s'^2)^2} ds' = 2\pi c\sqrt{\mu}. \tag{90}$$

Then we are left with the second part of the integral. Note that

$$\begin{aligned} \cos \omega t(s') &= \cos \omega(t_0 - \frac{s'}{2\sqrt{\mu}}) \\ &= \cos \omega t_0 \cos \frac{\omega s'}{2\sqrt{\mu}} + \sin \omega t_0 \sin \frac{\omega s'}{2\sqrt{\mu}}. \end{aligned} \tag{91}$$

Letting Mathematica compute

$$\int_{-\infty}^{\infty} \frac{s'}{1+s'^2} \cdot \cos \frac{\omega s'}{2\sqrt{\mu}} ds' = 0 \tag{92}$$

and

$$\int_{-\infty}^{\infty} \frac{s'}{1+s'^2} \cdot \sin \frac{\omega s'}{2\sqrt{\mu}} ds' = \pi e^{-\frac{\omega}{2\sqrt{\mu}}}. \tag{93}$$

We can combine equations 91, 92 and 93 to obtain

$$\int_{-\infty}^{\infty} \frac{2s'}{1+s'^2} \cdot \gamma \sin \omega t(s') ds' = 2\pi\gamma e^{-\frac{\omega}{2\sqrt{\mu}}} \sin \omega t_0. \tag{94}$$

Combining equations 87, 90 and 94, we obtain the following Melnikov function

$$M(t_0) = 2\pi(c\sqrt{\mu} + \gamma e^{-\frac{\omega}{2\sqrt{\mu}}} \sin \omega t_0). \tag{95}$$

3.4 Zeros of the Melnikov Function

We are interested in the zeros of the Melnikov function. In order for this function to have zeros, $c\sqrt{\mu}$ should be equal to or smaller than the amplitude $\gamma e^{-\frac{\omega}{2\sqrt{\mu}}}$. This is shown in figure 11 .

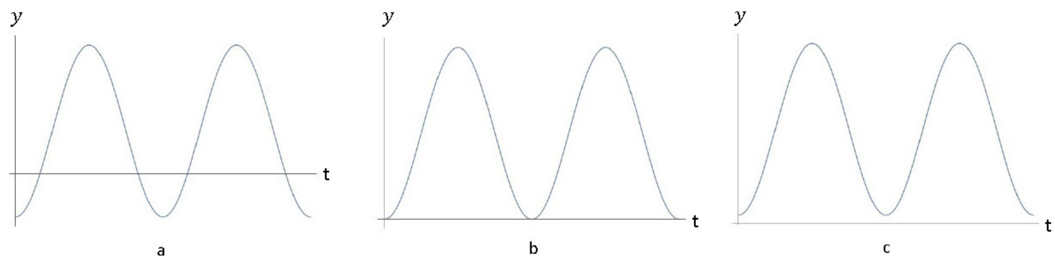

Figure 11: The amplitude of the Melnikov function for different values of $\sqrt{\mu}$

In figure 11a the Melnikov function is plotted for $c\sqrt{\mu} < \gamma e^{-\frac{\omega}{2\sqrt{\mu}}}$. There are infinitely many zeros, corresponding to infinitely many transversal intersections of the stable and the unstable manifold. In figure 11b, the situations is shown where $c\sqrt{\mu}$ is equal to $\gamma e^{-\frac{\omega}{2\sqrt{\mu}}}$. Here the Melnikov function touches the t_0 axis infinitely many times. The case shown in the last graph, figure 11c, is where $c\sqrt{\mu} > \gamma e^{-\frac{\omega}{2\sqrt{\mu}}}$. Here the function does not intersect with the t_0 axis anymore, so the stable and unstable manifold do not intersect transversally.

So the threshold condition for the Melnikov function to have zeros is $c\sqrt{\mu} = \gamma e^{-\frac{\omega}{2\sqrt{\mu}}}$. When we divide both sides by γ and define $\tilde{c} = \dfrac{c\sqrt{\mu}}{\gamma}$ and $\tilde{\omega} = \dfrac{\omega}{\sqrt{\mu}}$, this threshold takes on the following form

$$\tilde{c} = e^{-\frac{\tilde{\omega}}{2}}. \tag{96}$$

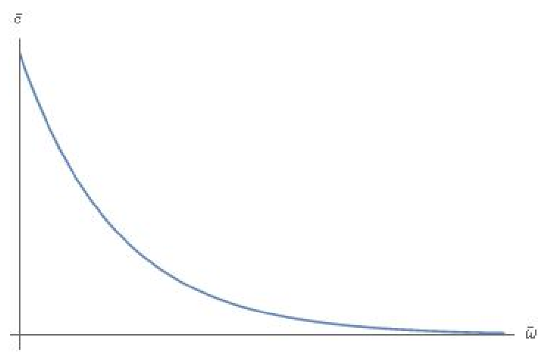

Figure 12: The threshold condition as a function of $\tilde{\omega}$

A graph of this is shown in figure 12.

3.5 Numerical validation of the predicted behaviour

We have predicted that when a planar dynamical system possessing a hyperbolic equilibrium point connected to itself by a homoclinic orbit is perturbed by a time-periodic function, chaotic behaviour can occur. Due to the perturbation the stable and unstable manifold will break up. The Melnikov method applied to a system of this form, can be used to show that the stable and unstable manifold intersect transversely. Now that we have derived the Melnikov function for the driven Morse oscillator, we can make some phase portraits that will reveal that the stable and unstable manifold will indeed intersect transversely.

The first plot in figure 13 represents the phase portrait of the unperturbed system (equation 73). Here the stable and the unstable manifold still coincide, and the equilibrium point is at $(\infty, 0)$. The phase portrait of the unperturbed system can be explained by observing that it is a plot of $y = \pm\sqrt{2h - \mu(-2e^{-x} + e^{-2x})}$ for different fixed values of h. Here y represents the velocity, and $V(x) = \mu(\text{-}2\mathrm{e}^{-x} + \mathrm{e}^{-2x})$ represents the potential energy. In the phase portrait, there are some grey lines, and there is a black line. The black line represents the separatrix, which corresponds to the homoclinic orbit (which was earlier shown in figure 9). As mentioned earlier, the solutions of the system lie on the level sets of the Hamiltonian. The separatrix corresponds to the value where h= 0. Taking h to be zero, and letting x approach infinity, y will approach the equilibrium point $(\infty,0)$. The periodic orbits inside the separatrix intersect the x-axis twice, so these orbits correspond to the values of h where v has exactly two zeros. This is the case when $-\frac{\mu}{2} \leq h < 0$. The solutions outside the separatrix have exactly one zero, which is the case for $h > 0$, and letting x approach infinity, the manifold will approach a constant equal to $\sqrt{2h}$.

Now let us see what happens if we insert a friction force, where we leave out the driving for now. A plot of this can be found in figure 14. The black lines correspond to the threshold case again. Since the Hamiltonian is the sum of the potential energy and the kinetic energy, it follows that if a friction force is inserted and the molecule's initial velocity is not high enough to begin with, the molecule will stop oscillating eventually. This explains that the unstable manifold will spiral towards the origin. In order for the unstable manifold to get to the equilibrium at $(\infty,0)$, it needs to start with a negative initial velocity. After some time it will lose energy due to the friction force, such that following the unstable manifold, it ends up with a lower velocity.

Now we only have the driving force left, where we expect the stable and unstable manifolds to break up, and intersect transversely. The plots of this can be found in figure 15 and 16. We have made the distinction between below and above the threshold condition. Recall that the threshold condition was

$$\tilde{c} = e^{-\frac{\tilde{\omega}}{2}}.$$

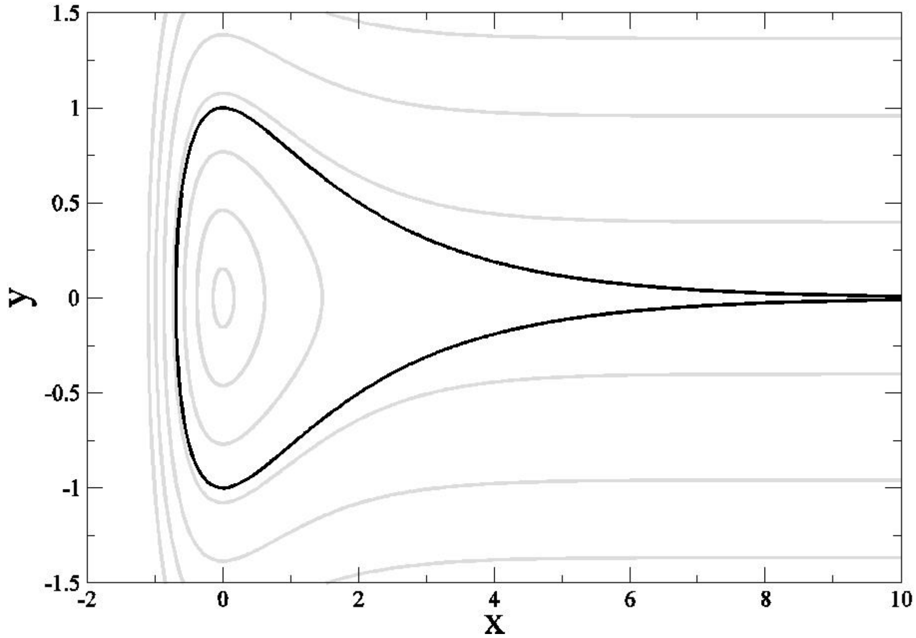

Figure 13: The stable and unstable manifolds of the unperturbed system. Parameters: $\mu = 1$, $\gamma = 1$, $\omega = 1$, $\varepsilon = 0$, $c = \mathrm{e}^{-\frac{1}{2}}/2$

Inserting besides the friction force also a driving force, we need to consider a Poincaré map of the point $(\infty,0)$ of the stable and unstable manifold, representing a cross-section of the phase space. In figure 15 the Poincaré map is shown where $\tilde{c}$ is equal $3\mathrm{e}^{-\frac{1}{2}}/2$. Referring to equation 96 and figure 12, this is above threshold case. Here $\tilde{c}$ will not have any zeros, so we do not expect there to be any transversal intersections. This is not really clear from figure 15a, but in 15b a piece of the stable and unstable manifolds is magnified to reveal that they indeed do not intersect. In figure 16 the Poincaré map is shown where $\tilde{c}$ is equal $\mathrm{e}^{-\frac{1}{2}}/2$, which is the below threshold case. Referring to equation 96 and figure 12 again, we do we expect transversal intersections. Again, in figure 16b a piece of the stable and unstable manifolds is magnified, to reveal that when $\tilde{c} = \mathrm{e}^{-\frac{1}{2}}/2$, the stable and unstable manifolds indeed intersect transversely.

Now that we have seen that the stable and unstable manifold indeed intersect transversely, we can apply Moser's theorem [Moser, 1973] to show that the system indeed possesses chaotic behaviour. To do this, recall that the system had to satisfy two conditions:

1. f has a hyperbolic periodic point, p.

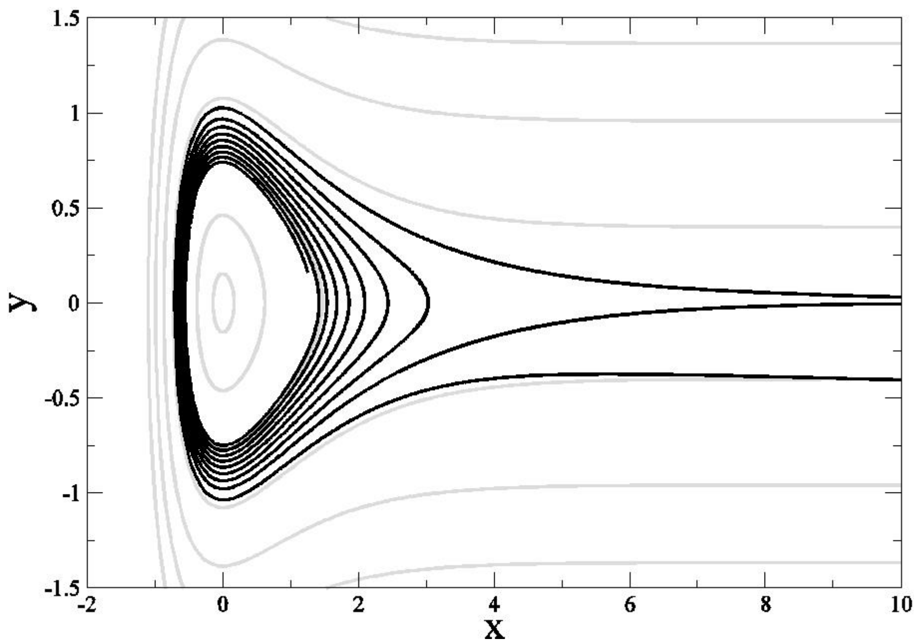

Figure 14: Phase portrait of the system with only friction.
Parameters: $\mu = 1$, $\gamma = 0$, $\omega = 1$, $\varepsilon = 0.01$, $c = 3\mathrm{e}^{\frac{1}{2}}/2$.

2. $W^s(p)$ and $W^u(p)$ intersect transversely.

Here the function f again represents the Poincaré map. Our system contains a hyperbolic periodic point, and we have seen that $W^s(p)$ and $W^u(p)$ intersect transversely. Now it is safe to apply Moser's theorem to prove that the system possesses chaotic dynamics, which is beyond the scope of this thesis.

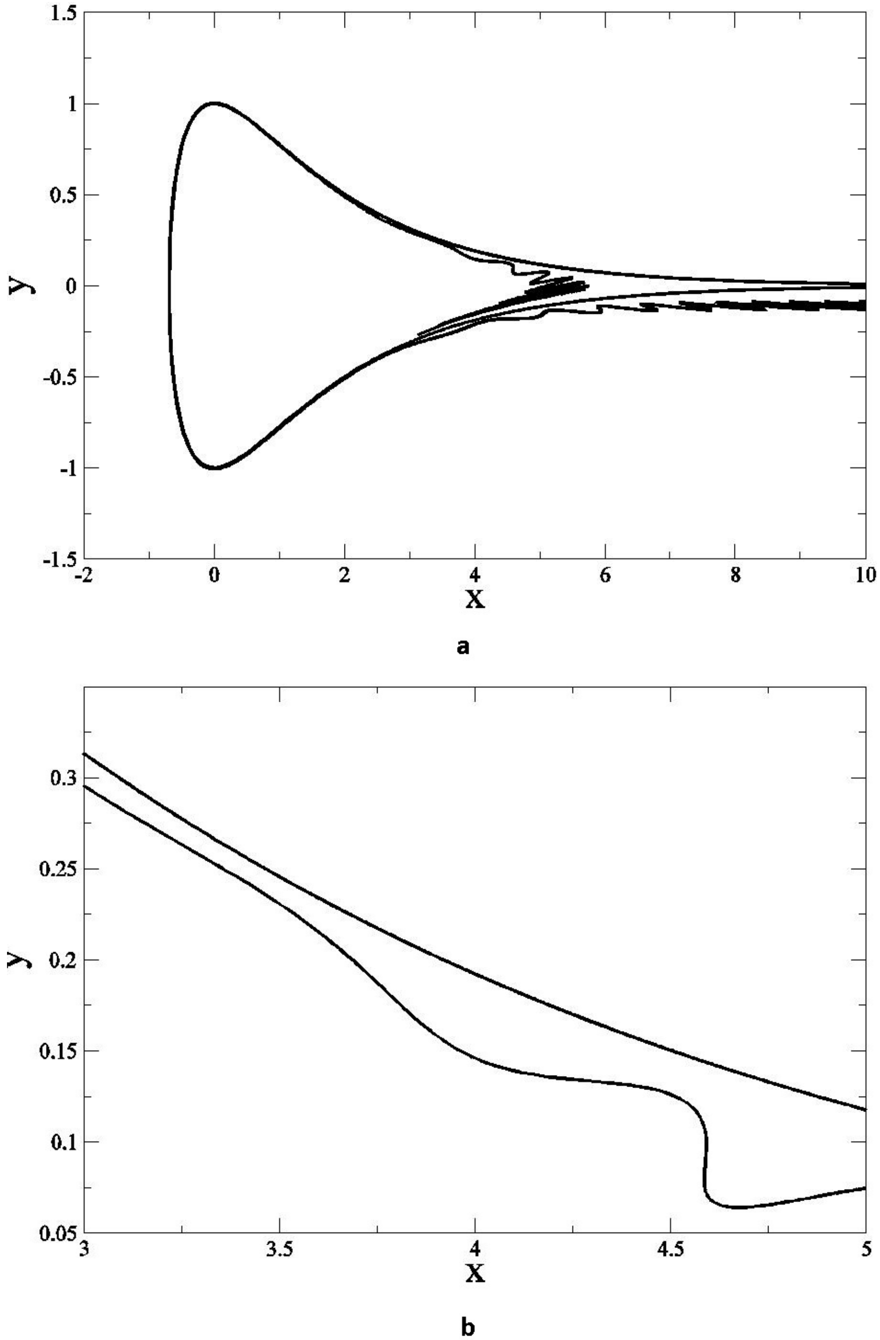

Figure 15: Poincaré map of the system with friction and driving, above the threshold condition.
Parameters: $\mu = 1$, $\gamma = 1$, $\omega = 1$, $\varepsilon = 0.001$, $c = 3\mathrm{e}^{\frac{1}{2}}/2$.

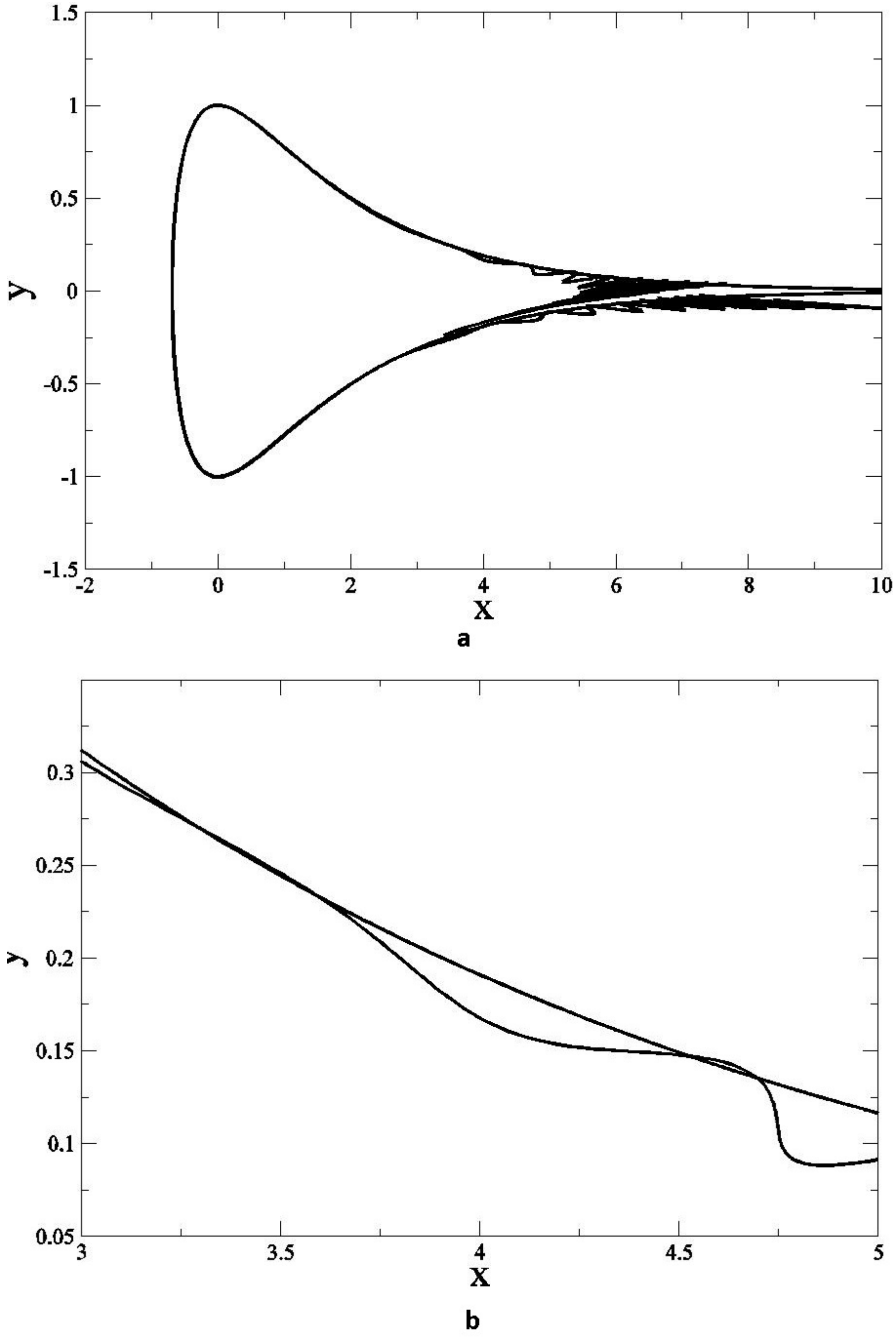

Figure 16: Poincaré map of the system with friction and driving, below the threshold condition.
Parameters: $\mu = 1$, $\gamma = 1$, $\omega = 1$, $\varepsilon = 0.001$, $c = \mathrm{e}^{\frac{1}{2}}/2$.

4 Conclusion

After describing the homoclinic Melnikov method thoroughly, we were able to apply it to the driven Morse oscillator. We derived the Melnikov function, and showed that the zeros of this function indicated that the stable and unstable manifold intersected transversely. After this, the predictions were numerically verified. Now Moser's theorem [Moser,1973] can be applied to prove that the system possesses chaotic dynamics, which is beyond the scope of this thesis.

5 Appendix A: Theorems and Definitons

Gronwall's Type Integral Inequality [Agarwal and O'Regan, 2008] Let $u(x)$, $p(x)$ and $q(x)$ be nonnegative continuous functions in the interval
$\mid x$ - $x_0 \mid \leq a$ and

$$u(x) \leq p(x) + \mid \int_{x0}^{x} q(t)u(t)dt \mid for \mid x - x_0 \mid \leq a. \tag{97}$$

Then the following inequality holds:

$$u(x) \leq p(x) + \mid \int_{x_0}^{x} p(t)q(t)\exp(\mid \int_{t}^{x} q(s)ds \mid)dt \mid for \mid x - x_0 \mid \leq a. \tag{98}$$

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