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Port-Hamiltonian discretization of gas pipeline networks

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Abstract

In this thesis we describe how to simulate the 1-d isothermal Euler equations that can be used to model gas dynamics in pipeline networks, see [Grundel et al., 2014]. These equations form a set of partial differential equations (PDEs). When modelling physical quantities using partial differential equations, the equations can often be rewritten in a special form called a port-Hamiltonian system. The port-Hamiltonian formulation incorporates the energy of the system.

For simulation purposes the infinite dimensional port-Hamiltonian system has to be discretized to a finite dimensional system. We used a discretization method described in [Golo et al., 2004] which guarantees that the resulting finite dimensional system is actually a finite dimensional port-Hamiltonian system. This means that the structure of the equations is preserved during the discretization method and energy conservation still holds.

We succeeded in forming the equations for pipelines as a port-Hamiltonian system. The port-Hamiltonian formulation makes it possible to systematically connect the gas pipes together to form a pipeline network. This thesis describes the whole procedure from the initial PDEs to the resulting simulation.

The successful application of the port-Hamiltonian formulation and discretization to the equations of gas dynamics in pipelines can be seen as the result of this thesis.

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1 Introduction

In the past decades, the topic about the responsible use of natural resources got worldwide interest. The European Union set some targets to become partly independent of natural resources in the nearby future. Therefore many countries like the Netherlands, Germany, Spain and Denmark invested a lot of money to build photovoltaic (PV) cells and/or wind turbines. Power generation from wind or sun is however hard to predict and coupling these new energy sources to the already existing power sources generated by gas, coal or uranium soon became a new challenge [Klimstra and Hotakainen, 2011]. The main challenge is to distribute the energy in the energy grid in such a way, that production matches demand. The energy grid is changing from a centralized generating system to a system with distributed generation of energy [Larsen et al., 2013]. This substantially increases the complexity of controlling the energy grid, while the control of the grid, just as production, becomes more and more decentralized.

The port-Hamiltonian framework was especially developed for modelling the dynamics and interaction of physical quantities which are connected to each other (see [van der Schaft and Jeltsema, 2014]). The focus of this framework is to model components of a network of connected objects separately, but in such a way that they can be connected to each other afterwards while preserving conservation properties like energy conservation.

As explained in [Klimstra and Hotakainen, 2011], power generation in gas turbines will remain important in the future. The gas network should therefore be incorporated in the energy grid and precise control of the gas network becomes more important.

The port-Hamiltonian framework was used before to model power networks [Fiaz et al., 2013]. In this thesis we are going to describe how we can model gas dynamics in pipeline networks. We will use a port-Hamiltonian description of gas dynamics in a single pipe emerging from the isothermal Euler equations. The port-Hamiltonian framework then allows us to connect many pipes to each other as a full pipeline network and simulate global dynamics.

In chapter 2 we are going to describe the model of a pipeline emerging from the isothermal Euler equations. We will also find an energy density function that describes the dynamics of the model. This is the first step towards the port-Hamiltonian formulation. In chapter 3 we explain what a port-Hamiltonian system is and it turns out that our system is a port-Hamiltonian system. We also explain how to connect two pipes together. In chapter 4 we work towards a simulation by applying a port-Hamiltonian discretization to our system. In chapter 5 we can use everything we derived to build up the equations of a full pipeline network. In chapter 6 we look back at the results of this thesis and explain how this thesis can be extended with further research.

2 Model definition

Gas pipeline networks are interesting to simulate in order to find efficient ways of controlling the distribution of gas in the network. The pipeline networks under consideration are used for the distribution of gas over long distances from the drilling site to the end user. In this chapter we describe the model for gas dynamics in pipes of such a pipeline network.

2.1 Isothermal Euler equations

We will fully focus on a description of the dynamics of gas in a single straight pipe. We assume that the pipe has constant diameter and is placed horizontally without any height differences. Of particular interest are the density distribution and the flow speed along the pipe. We therefore choose to use a 1-dimensional model of a pipe, where we consider all the physical quantities like flow speed and density distribution to be depending on time and only one spatial coordinate.

Just as in the paper [Grundel et al., 2014] and based on our assumptions, we model the dynamics of gas in a pipe by a 1-dimensional version of the isothermal Euler equations, given by the following set of equations.

$$\begin{cases} \rho_t + q_z &= 0 \\ q_t + p_z + (\rho v^2)_z + g\rho h_z &= -\frac{\lambda}{2D}\rho v|v| \end{cases} \quad (1)$$

A subscript in these equations denotes a partial derivative with respect to the character in the subscript. All physical quantities depend on only one spatial coordinate which we will denote by $z \in \mathbb{R}$ and on time, denoted by $t \in \mathbb{R}_{\geq 0}$. To make the equations more readable, we normally omit the notation with dependencies. The meaning of all the characters in this set of equations can be found in the list below.

$\rho(z, t)$	Density
$v(z, t)$	Stream velocity
$q(z, t) := \rho(z, t)v(z, t)$	Mass flow
$p(z, t) := a^2\rho(z, t)$	Pressure
$h(z, t) := \text{constant}$	Geodesic height
$\frac{\lambda}{2D}$	Friction constant
g	Gravitational constant

The list also includes three equalities, which make it possible to simplify the equations in (1). The first equality from the list relates the mass flow to the density and stream velocity and is used to eliminate mass flow from the equations. The second equality from the list relates pressure to density using the

constant speed of sound, denoted by $a = 300$ m/s. This allows us to eliminate pressure from the equations. The third equality from the list is an assumption about the differences in the height of the pipe made in the beginning of this chapter. We neglect height differences and therefore the term including h_z vanishes.

With the simplifications just mentioned, the equations given in (1) can be rewritten to:

$$\begin{cases} \rho_t + \rho v_z + v \rho_z = 0 \\ \underbrace{\rho v_t + v \rho_t}_{q_t} + a^2 \rho_z + v^2 \rho_z + 2\rho v v_z = -\frac{\lambda}{2D} \rho v |v|. \end{cases}$$

Observe that from the first equation we find $v \rho_t = -v \rho v_z - v^2 \rho_z$, which can be substituted in the second equation as follows:

$$\begin{cases} \rho_t + \rho v_z + v \rho_z = 0 \\ \rho v_t + \underbrace{-v \rho v_z - v^2 \rho_z}_{v \rho_t} + a^2 \rho_z + v^2 \rho_z + 2\rho v v_z = -\frac{\lambda}{2D} \rho v |v|. \end{cases}$$

This greatly simplifies the second equation. We also immediately divide by ρ in the second equation and get:

$$\begin{cases} \rho_t + \rho v_z + v \rho_z = 0 \\ v_t + v v_z + \frac{a^2}{\rho} \rho_z = \underbrace{-\frac{\lambda}{2D} v |v|}_{\text{friction}}. \end{cases} \quad (2)$$

Here we recognize a friction term on the right hand side that is derived in section 11.5 of [Alexandrou, 2001]. We will not recall this derivation, while the friction is not of particular interest in the rest of this thesis. With this friction term in there, we call the last equations the lossy Euler equations, while without friction, we will call it the lossless Euler equations.

The equations in (2) can be written in system form, which in the lossless case is given by:

$$\begin{bmatrix} \dot{\rho} \\ \dot{v} \end{bmatrix} = - \begin{bmatrix} v & \rho \\ a^2/\rho & v \end{bmatrix} \begin{bmatrix} \rho_z \\ v_z \end{bmatrix}. \quad (3)$$

The dot on top of a function denotes its time derivative.

The equations of motion in (3) have only two unknowns, ρ and v . As mentioned before, these are physical quantities of the gas depending only on a 1-dimensional space coordinate z and time t . Therefore the unknowns in the equations of motion are functions of z and t . This means, that the solution at each fixed time t is still a function of z and in this sense still an infinite dimensional object. This fact together with the presence of the partial derivative of the unknowns with respect to z in the equations, make the system in (3) an infinite dimensional system.

2.2 Energy equations

The first step towards the formulation of (3) as a port-Hamiltonian system, is the derivation of an energy density function $\mathcal{H}(\rho, v)$ such that the system (3) is equivalent to

$$\begin{cases} \dot{\rho} &= -\frac{\partial}{\partial z}(\mathcal{H}_v) \\ \dot{v} &= -\frac{\partial}{\partial z}(\mathcal{H}_\rho), \end{cases} \quad (4)$$

where as before the subscripts on \mathcal{H} denote partial derivatives. It is natural to include kinetic energy in the energy density function \mathcal{H} . The potential energy that we should consider in our system is the energy causing a compressed gas to expand and vice versa. This energy is therefore a function of the density of the gas only. To find the exact expression for the potential energy, we start with an initial guess of the form of the potential energy given as $\rho U(\rho)$. This, together with the kinetic energy, then gives the energy density function $\mathcal{H}(\rho, v)$:

$$\mathcal{H}(\rho, v) = \underbrace{\frac{1}{2}\rho v^2}_{\text{kinetic energy}} + \underbrace{\rho U(\rho)}_{\text{potential energy}}. \quad (5)$$

The next step is to substitute this energy density function in (4) and simplify:

$$\begin{aligned} \begin{cases} \dot{\rho} &= -\frac{\partial}{\partial z}(\rho v) \\ \dot{v} &= -\frac{\partial}{\partial z}\left(\frac{1}{2}v^2 + \rho U'(\rho) + U(\rho)\right) \end{cases} \\ \Downarrow \\ \begin{cases} \dot{\rho} &= -v\rho_z - \rho v_z \\ \dot{v} &= -\rho_z U'(\rho) - \rho U''(\rho)\rho_z - U'(\rho)\rho_z - vv_z \end{cases} \\ \Downarrow \\ \begin{cases} \dot{\rho} &= -v\rho_z - \rho v_z \\ \dot{v} &= -(2U'(\rho) + \rho U''(\rho))\rho_z - vv_z \end{cases} \\ \Downarrow \\ \begin{bmatrix} \dot{\rho} \\ \dot{v} \end{bmatrix} = - \begin{bmatrix} v & \rho \\ 2U'(\rho) + \rho U''(\rho) & v \end{bmatrix} \begin{bmatrix} \rho_z \\ v_z \end{bmatrix}. \end{aligned} \quad (6)$$

Now a compelling similarity is seen by comparing (6) to the lossless isothermal Euler equations as given in (3). It should be clear that (6) is equivalent to (3) if $U(\rho)$ satisfies

$$2U'(\rho) + \rho U''(\rho) = \frac{a^2}{\rho}.$$

This equation has the general solution

$$U(\rho) = a^2 \log \rho + \frac{c_1}{\rho} + c_2,$$

where c_1 and c_2 are integration constants. Substituting the solution of $U(\rho)$ into (5) leads to the following energy density function:

$$\begin{aligned}\mathcal{H}(\rho, v) &= \frac{1}{2}\rho v^2 + \rho U(\rho) \\ &= \frac{1}{2}\rho v^2 + a^2\rho \log \rho + c_1 + c_2\rho.\end{aligned}$$

The integration constant c_1 can be set to zero, since a constant can always be added to the energy density without changing any dynamics. The second integration constant c_2 is related to the equilibrium density of the gas and is renamed to c from here on.

We conclude by summarizing what we found in this subsection so far.

Result of subsection 2.2

By defining the energy density function $\mathcal{H}(\rho, v)$ to be

$$\mathcal{H}(\rho, v) = \frac{1}{2}\rho v^2 + a^2\rho \log \rho + c\rho, \quad (7)$$

we saw that the system

$$\begin{cases} \dot{\rho} &= -\frac{d}{dx}(\mathcal{H}_v) \\ \dot{v} &= -\frac{d}{dx}(\mathcal{H}_\rho), \end{cases}$$

is equivalent to

$$\begin{bmatrix} \dot{\rho} \\ \dot{v} \end{bmatrix} = - \begin{bmatrix} v & \rho \\ a^2/\rho & v \end{bmatrix} \begin{bmatrix} \rho_z \\ v_z \end{bmatrix}.$$

2.2.1 More about the energy density function

We called the function $\mathcal{H}(\rho, v)$ the energy density function. That's because it has the physical dimension of energy per distance. The total energy in the pipe segment ab from the point $z = a$ to $z = b$ is therefore defined as the following integral of \mathcal{H} :

$$H(\rho, v) = \int_a^b \mathcal{H}(\rho, v) dz.$$

It has the dimension of energy. From this integral, the change of energy with respect to time (power) inside a pipe segment without friction is easily calculated as follows:

$$\dot{H}(\rho, v) = \int_a^b \frac{d}{dt} \mathcal{H}(\rho, v) dz$$

$$\begin{aligned}
&= \int_a^b \{ \mathcal{H}_\rho \dot{\rho} + \mathcal{H}_v \dot{v} \} dz \\
&= \int_a^b \left\{ \mathcal{H}_\rho \left(-\frac{\partial}{\partial z} (\mathcal{H}_v) \right) + \mathcal{H}_v \left(-\frac{\partial}{\partial z} (\mathcal{H}_\rho) \right) \right\} dz \\
&= - \int_a^b \left\{ \mathcal{H}_\rho \frac{\partial}{\partial z} (\mathcal{H}_v) + \mathcal{H}_v \frac{\partial}{\partial z} (\mathcal{H}_\rho) \right\} dz \\
&= - \int_a^b \left\{ \frac{\partial}{\partial z} (\mathcal{H}_\rho \mathcal{H}_v) \right\} dz \\
&= \mathcal{H}_\rho \mathcal{H}_v \Big|_{z=a} - \mathcal{H}_\rho \mathcal{H}_v \Big|_{z=b}.
\end{aligned}$$

The term $\mathcal{H}_\rho = \frac{1}{2}v^2 + a^2 \log \rho + a^2 + c$ is called the hydrostatic pressure (or sometimes Bernoulli function) and has the physical dimension of energy per unit mass m^2/s^2 . The term $\mathcal{H}_v = \rho v$ is the mass flow and has the physical dimension of kg/s . Therefore the product $\mathcal{H}_\rho \mathcal{H}_v$ has dimension $kg m^2/s^3$ and hence has the meaning of (mechanical) power. One could say in words that the change of total energy inside the pipe segment is equal to the power supplied at the point $z = a$ minus the power subtracted at the point $z = b$. This is an important physical property of the lossless system.

Let us note here that the dimensional analysis above also reveals that the constant c in the energy function (7) should have the physical dimension m^2/s^2 .

3 Port-Hamiltonian formulation

For the port-Hamiltonian formulation of the equations of gas dynamics, the energy equations formulated in the previous section are really important. In this section we briefly discuss how we use the terminology of a port-Hamiltonian system and how our system of equations fits in this framework. In this explanation we mainly follow the way a port-Hamiltonian system was defined in Chapter 2 and 14 of [van der Schaft and Jeltsema, 2014] .

As mentioned before, the system of equations we are working with form an infinite dimensional system. This means that the unknowns of the system are functions of time and of some spatial coordinate. The equations are partial differential equations. For a finite dimensional system on the other hand, the unknowns only depend on time and the equations are ordinary differential equations. When (spatially) discretizing an infinite dimensional system, one tries to find a finite dimensional system that approximates the infinite dimensional system. Since we want to discretize our system in the next chapter, it is important to make the difference between finite and infinite dimensional systems clear. Furthermore, a port-Hamiltonian system can also be finite or infinite dimensional.

In section 2.3 and section 7.1 of the book [Jacob and Zwart, 2012] finite and infinite dimensional port-Hamiltonian systems are defined separately. We prefer however to simultaneously define both cases and explain the similarities and differences. In chapter 2 of [van der Schaft and Jeltsema, 2014] a port-Hamiltonian is defined in a geometric way, which gives in our opinion better possibilities to define a port-Hamiltonian system in a more general way.

3.1 Definition of a port-Hamiltonian system

The definition as how we present it here may or may not be fully exhaustive, but we tried to capture the defining features that are necessary to understand the rest of this thesis. For the readers who already worked with port-Hamiltonian systems before: we omit the existence of resistive ports and control ports here.

A **port-Hamiltonian system** is a collection of ports which are internally connected to each other. This is called the (internal) interconnection structure, and it forms a Dirac structure on the ports, denoted by \mathcal{D} , see figure 1.

A **port** is a pair of two port variables, one is called the flow f and the other is called the effort e . The product $f \cdot e$ has the meaning of power.

The essential feature of a **Dirac structure** is that the total power is zero, which means that energy is preserved.

How **total power** is defined depends slightly on whether the system is finite or infinite dimensional. This will be explained in the next subsection.

A port-Hamiltonian system can have **different types of ports**. Here we will only consider external ports and storage ports. A port-Hamiltonian system has at least one storage port. A storage port is drawn with a small box in the figure below. An external port is recognizable by the capital B in the superscript of the corresponding variable.

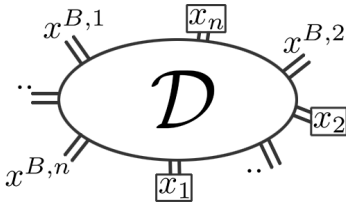


Figure 1: Abstract port-Hamiltonian system

There is always an **energy function** $H(x) : x \mapsto \mathbb{R}$ associated to the storage ports. This function is also called the Hamiltonian (-function). The Hamiltonian is a function of the state variables of the system, denoted by x here. The state variables may also be called energy variables. In general x can be a collection of several different variables, let's say $\{x_i\}$. For every state variable x_i , there is a storage port i . Each storage port i has two port variables (just as every other port). The flow of the i -th storage port will be denoted f^{x_i} and is equal to the derivative of the state variable x_i with respect to time. The corresponding effort will be denoted e^{x_i} and is equal to the partial derivative of the Hamiltonian with respect to the state variable x_i . That is:

$$\begin{cases} f^{x_i} &= \dot{x}_i \\ e^{x_i} &= \frac{\partial H}{\partial x_i}(x). \end{cases} \quad (8)$$

The equations in (8) are sometimes also called the constitutive relations of the port-Hamiltonian system.

The external ports are necessary ports if the port-Hamiltonian system should be able to be connected to other port-Hamiltonian systems. Every external port k has a flow denoted by $f^{B,k}$ and an effort denoted by $e^{B,k}$. The B in the superscript stands for 'Boundary'. The external port of the first port-Hamiltonian system is then connected to one of the external ports of the second port-Hamiltonian system. A connection of two external ports should be made in such a way, that the energy of the two port-Hamiltonian systems together is still preserved.

A system with the properties described above is called a port-Hamiltonian system. Now we should elaborate a bit more on the differences between finite and infinite dimensional port-Hamiltonian systems.

3.1.1 Finite vs infinite dimensional port-Hamiltonian systems

In this section we will describe the differences between a finite and an infinite dimensional port-Hamiltonian system.

In a finite dimensional port-Hamiltonian system the state variables are only depending on time, i.e. $x_i \in L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ and the Hamiltonian is a rather simple algebraic equation of the state variables. As such, the flows and efforts of the storage ports as defined in (8), are also in $L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$. The total power is then simply defined as:

$$\sum_i f^{x_i} \cdot e^{x_i} + \sum_k f^{B,k} \cdot e^{B,k} \in L^2(\mathbb{R}_{\geq 0}, \mathbb{R}),$$

which is a function of time.

In an infinite dimensional port-Hamiltonian system on the other hand, the state variables are not only depending on time, but also on some spatial coordinate. For the spatial domain we will restrict ourself to a subset $U \subseteq \mathbb{R}^n$. This means that the state variables x_i are functions in $L^2(U \times \mathbb{R}_{\geq 0}, \mathbb{R})$. The energy function is a mapping from the state variables x to \mathbb{R} and therefore is an integral over the spatial domain of some energy density function $\mathcal{H}(x)$ of the state variables. That is, a function $\mathcal{H} : x \mapsto \mathcal{H}(x) \in L^2(U \times \mathbb{R}_{\geq 0}, \mathbb{R})$, such that:

$$H(x) = \int_U \mathcal{H}(x) \, dz.$$

H is implicitly depending on time, but for every time t , we have $H(x) \in \mathbb{R}$. In the infinite dimensional case, the H in the constitutive relations of the port-Hamiltonian system, as defined in (8), should be replaced by the \mathcal{H} of the energy density function. That is:

$$\begin{cases} f_i &= \dot{x}_i \\ e_i &= \frac{\partial \mathcal{H}}{\partial x_i}(x). \end{cases} \quad (9)$$

As a consequence, the flows and efforts of the storage ports are in $L^2(U \times \mathbb{R}_{\geq 0}, \mathbb{R})$. The total power is therefore defined as:

$$\sum_i \left(\int_U f^{x_i} \cdot e^{x_i} \, dz \right) + \sum_k f^{B,k} \cdot e^{B,k} \in L^2(\mathbb{R}_{\geq 0}, \mathbb{R}), \quad (10)$$

which is, because of the added integrals, a function of time.

The differences between a finite and infinite dimensional port-Hamiltonian system are subtle, but important if you want to talk about a port-Hamiltonian discretization. Before turning to the port-Hamiltonian discretization, we first formulate the gas equations as a port-Hamiltonian system.

3.2 Gas equations as a port-Hamiltonian system

In this section we show that the gas equations are a infinite dimensional port-Hamiltonian system. Recall the equations of motion of a gas written in the form we derived in section 2.2, using an energy density function \mathcal{H} :

$$\begin{cases} \dot{\rho} &= -\frac{d}{dx}(\mathcal{H}_v) \\ \dot{v} &= -\frac{d}{dx}(\mathcal{H}_\rho), \end{cases} \quad \text{where} \quad \mathcal{H}(\rho, v) = \frac{1}{2}\rho v^2 + a^2 \rho \log \rho + c\rho. \quad (11)$$

There are two state variables ρ and v and both elements in $L^2(\mathcal{Z} \times \mathbb{R}_{\geq 0}, \mathbb{R})$, where $\mathcal{Z} = [a, b] \subseteq \mathbb{R}$. And so they depend on a spatial variable z , meaning the system is an infinite dimensional system. There are two storage ports. The first is a storage port for ρ , where the flow and effort are defined as (see (9)):

$$\begin{cases} f^\rho &= \dot{\rho} \\ e^\rho &= \frac{\partial \mathcal{H}}{\partial \rho}(\rho, v) = \frac{1}{2}v^2 + a^2 \log \rho + a^2 + c \end{cases} \quad (12)$$

and the second is a storage port for v where the flows and efforts are:

$$\begin{cases} f^v &= \dot{v} \\ e^v &= \frac{\partial \mathcal{H}}{\partial v}(\rho, v) = \rho v. \end{cases} \quad (13)$$

The equations above are the constitutive relations of our system. Now formulate the internal interconnection structure. A part of the internal interconnection structure consists of the equations of motion in (11). In these equations we can substitute the constitutive relations and find a part of the interconnection structure:

$$\begin{cases} f^\rho &= -\frac{\partial}{\partial z} e^v \\ f^v &= -\frac{\partial}{\partial z} e^\rho. \end{cases} \quad (14)$$

For a correct port-Hamiltonian system, the total power should be zero. Right now we did not yet define any external ports and the total power as defined in (10) of the system would be:

$$\begin{aligned} \int_a^b f^\rho \cdot e^\rho dz + \int_a^b f^v \cdot e^v dz &\stackrel{(14)}{=} \int_a^b \underbrace{-\frac{\partial}{\partial z} e^v}_{f^\rho} \cdot e^\rho dz + \int_a^b \underbrace{-\frac{\partial}{\partial z} e^\rho}_{f^v} \cdot e^v dz \\ &= -\int_a^b \frac{\partial}{\partial z} (e^\rho \cdot e^v) dz \\ &= (e^\rho \cdot e^v)|_{z=a} - (e^\rho \cdot e^v)|_{z=b} \\ &\neq 0. \end{aligned} \quad (15)$$

Since it is not yet zero, we need to add external ports to the system. The power balance above and the physical shape of the gas pipe, suggest that we should

add two external ports: one at the point $z = a$ and the other at the point $z = b$. Let us connect these two external ports to the storage ports as follows:

$$\begin{cases} f^{B,a} &= -e^\rho|_{z=a} \\ e^{B,a} &= e^v|_{z=a} \end{cases} \quad \text{and} \quad \begin{cases} f^{B,b} &= e^\rho|_{z=b} \\ e^{B,b} &= e^v|_{z=b} \end{cases} \quad (16)$$

The total power as defined in (10) now reads:

$$\begin{aligned} & \int_a^b f^\rho \cdot e^\rho dz + \int_a^b f^v \cdot e^v dz + f^{B,a} \cdot e^{B,a} + f^{B,b} \cdot e^{B,b} \\ & \stackrel{(15)}{=} (e^\rho \cdot e^v)|_{z=a} - (e^\rho \cdot e^v)|_{z=b} + f^{B,a} \cdot e^{B,a} + f^{B,b} \cdot e^{B,b} \\ & \stackrel{(16)}{=} 0 \end{aligned}$$

and is indeed zero.

We can now conclude that the equations describing gas dynamics form a port-Hamiltonian system. Following the definition as formulated in section 3.1 this port-Hamiltonian system has four ports. Two storage ports and two external ports. The storage ports are defined in (12) and (13). All the ports are internally connected by the equations given in (14) and (16). The total power of the system is zero.

3.3 Connecting two gas pipes

Included in the description of a port-Hamiltonian system, an external port makes it possible to connect the port-Hamiltonian system to another port-Hamiltonian system. In this section we are going to demonstrate how to connect two port-Hamiltonian systems using the external ports. We will use the example of connecting two pipes.

Imagine that we have two gas pipes and we want to connect them. Each individual pipe can be described by the port-Hamiltonian system above on its own spatial domain. We connect both port-Hamiltonian systems by connecting the external port on the right hand side of the first pipe to the external port on the left hand side of the second pipe, see figure 2.

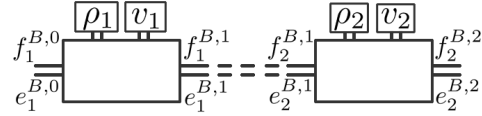


Figure 2: Connection of two pipes

Connecting two external ports means that the port variables of one of the ports are expressed in terms of the port variables of the other external port. Since

physical properties like mass conservation and energy conservation should hold, there is normally only one canonical choice to connect ports. In the case of our port-Hamiltonian system the boundary flows and efforts are defined in (16). The boundary flows represent the hydrostatic pressures at the boundary, since e^ρ is the hydrostatic pressure of the gas. The boundary efforts represent the mass flows at the boundary, while e^v is equal to mass flow. From a physical point of view we should therefore connect the boundary flows to boundary flows and the boundary efforts to boundary efforts.

For the gas equations we should have that the hydrostatic pressure on the right hand side of the first pipe is equal to the hydrostatic pressure at the left hand side of the second pipe. Because the boundary flow at the left hand side of each pipe was chosen to be minus the hydrostatic pressure, while it was defined with a plus on the right hand side (see (16)), we need to set $f_2^{B,1} = -f_1^{B,1}$. We use the notational convention that the subscript indicates the number of the pipe and the superscript denotes the number of the interconnection node. The spatial point z_i is referred to as the i -th node. Furthermore we want conservation of mass and therefore the mass flow at the right hand side of the first pipe should be equal to the mass flow at the left hand side of the second pipe. Since the boundary efforts represent mass flow, we will set $e_2^{B,1} = e_1^{B,1}$.

Observe that the two pipes together form again a port-Hamiltonian system. Now it has 4 storage ports, two for each pipe. Since we connected two external ports to each other, these ports are no longer externally connectable and should therefore no longer be seen as external ports. The two connected pipes have therefore only two external ports, one at the left hand side of the first pipe and one at the right hand side of the second pipe. Each pipe has its own Hamiltonian density depending on its own state variables, denoted by $\mathcal{H}_1(\rho_1, v_1)$ and $\mathcal{H}_2(\rho_2, v_2)$. The overall new energy density function is the sum of the old ones: $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. The interconnection structure of the new system is the composition of the interconnection structures of both separated systems, together with the interconnection of the two previously connected external ports. Now the total power of the connected pipes as defined in (10) reads:

$$\int_{z_0}^{z_1} f_1^\rho \cdot e_1^\rho + f_1^v \cdot e_1^v dz + \int_{z_1}^{z_2} f_2^\rho \cdot e_2^\rho + f_2^v \cdot e_2^v dz + f_1^{B,0} \cdot e_1^{B,0} + f_2^{B,2} \cdot e_2^{B,2}$$

The first integral is equal to (see (15)):

$$(e_1^\rho \cdot e_1^v)|_{z=z_0} - (e_1^\rho \cdot e_1^v)|_{z=z_1} = -f_1^{B,0} \cdot e_1^{B,0} - f_1^{B,1} \cdot e_1^{B,1}$$

and similarly the second integral is equal to:

$$(e_2^\rho \cdot e_2^v)|_{z=z_1} - (e_2^\rho \cdot e_2^v)|_{z=z_2} = -f_2^{B,1} \cdot e_2^{B,1} - f_2^{B,2} \cdot e_2^{B,2}.$$

Substituting this into the total power yields:

$$\begin{aligned}
& \int_{z_0}^{z_1} f_1^\rho \cdot e_1^\rho + f_1^v \cdot e_1^v \, dz + \int_{z_1}^{z_2} f_2^\rho \cdot e_2^\rho + f_2^v \cdot e_2^v \, dz + f_1^{B,0} \cdot e_1^{B,0} + f_2^{B,2} \cdot e_2^{B,2} \\
&= -f_1^{B,1} \cdot e_1^{B,1} - f_2^{B,1} \cdot e_2^{B,1} \\
&= 0,
\end{aligned}$$

since we have chosen to connect the pipes using $f_2^{B,1} = -f_1^{B,1}$ and $e_2^{B,1} = e_1^{B,1}$. The power in the new system is zero and the properties of a port-Hamiltonian system are satisfied.

In this chapter we defined a port-Hamiltonian system and saw that the equations of motion of gas in a pipe form a port-Hamiltonian system. We also saw that the framework of a port-Hamiltonian systems makes it possible to connect pipes to each other. For simulation purposes we need to discretize the port-Hamiltonian system. By doing this we want to preserve the structure of flows and efforts and this means we want to turn the infinite dimensional port-Hamiltonian system into a finite dimensional port-Hamiltonian system. The next chapter is dedicated to explain this procedure.

4 Port-Hamiltonian discretization

In this chapter we describe how the port-Hamiltonian discretization of the port-Hamiltonian system of the gas equations described in section 3.2 works.

For the discretization we need to cut the gas pipe into small gas pipe segments. We use the convention that the geometry of the pipe reaches from the spatial point z_0 to z_n and the pipe is cut at the nodes z_1, z_2, \dots, z_{n-1} . The result is a pipe divided into n equally long pipe segments. The first pipe segment reaches from z_0 to z_1 , the second pipe segment reaches from z_1 to z_2 and so on. A pipe segment may also be called a lump. In every pipe segment the gas equations

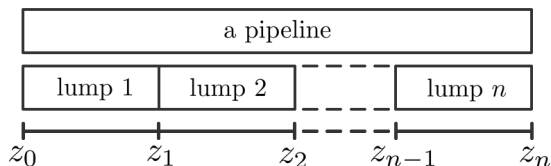


Figure 3: Division of the pipeline into lumps

hold and can be described by the port-Hamiltonian system from section 3.2. All these identical port-Hamiltonian systems will be discretized and connected afterwards as in section 3.3 to get the full discretization of the pipe.

For the discretization we should focus on one lump. We will use the paper of Golo [Golo et al., 2004], where they discretized the same port-Hamiltonian system as we have, but for another physical example with another Hamiltonian function. The discretization method was especially developed for this type of port-Hamiltonian system and assures that the resulting finite dimensional system is also a port-Hamiltonian system. We will therefore use this discretization method. How the discretized system results in a port-Hamiltonian system for the whole pipeline with n segments is not described in the paper of Golo and we will do it in the last section of this chapter.

4.1 Discretizing one lump

We will first explain how to discretize the k -th lump, located between $z = z_{k-1}$ and $z = z_k$. We first discretize the constitutive relations and then the internal interconnection structure.

We start with the constitutive relations. Instead of having infinite dimensional states ρ and v in $L^2([z_{k-1}, z_k] \times \mathbb{R}_{\geq 0}, \mathbb{R})$, we want to have finite dimensional states ρ_k and v_k in $L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$. Golo applies an approximation of the infinite dimensional objects ρ and v as is usually done in finite volume approximations:

$$\rho(z, t) \approx \rho_k(t) \cdot w_k^\rho(z) \quad \text{and} \quad v(z, t) \approx v_k(t) \cdot w_k^v(z), \quad (17)$$

where $w_k^\rho(z), w_k^v(z) \in L^2([z_{k-1}, z_k], \mathbb{R})$ are the basis functions and $\rho_k(t), v_k(t) \in L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ will be the new state variables. The basis functions can be chosen freely, but should satisfy the following properties:

$$\int_{z_{k-1}}^{z_k} w_k^\rho(z) dz = 1 \quad \text{and} \quad \int_{z_{k-1}}^{z_k} w_k^v(z) dz = 1.$$

Because of this requirement, the integrals of the infinite dimensional states $\rho(z, t)$ and $v(z, t)$ over the spatial domain $[z_{k-1}, z_k]$ are approximated by $\rho_k(t)$ and $v_k(t)$, respectively.

Just as in the infinite dimensional system, the finite dimensional system will have two energy ports. Since the flows will be equal to the time derivatives of the states, we approximate the flows as:

$$f^\rho(z, t) \approx f_k^\rho(t) \cdot w_k^\rho(z) \quad \text{and} \quad f^v(z, t) \approx f_k^v(t) \cdot w_k^v(z),$$

i.e. with the same basis functions as in the approximation of $\rho(z, t)$ and $v(z, t)$. Therefore the finite dimensional equivalent of the constitutive relations $f^\rho = \dot{\rho}$ and $f^v = \dot{v}$ for the flows of the energy ports are:

$$f_k^\rho(t) := \dot{\rho}_k(t) \quad \text{and} \quad f_k^v(t) := \dot{v}_k(t).$$

For the efforts of the energy ports, we need to discretize the Hamiltonian function. We substitute the approximations for $\rho(z, t)$ and $v(z, t)$ from (17) into the infinite dimensional Hamiltonian $H(\rho, v) = \int_{z_{k-1}}^{z_k} \mathcal{H}(\rho, v) dz$ (with \mathcal{H} as in (11)) and find the following discrete energy function:

$$\begin{aligned} H_k(\rho_k(t), v_k(t)) &:= H(\rho_k(t)w_k^\rho(z), v_k(t)w_k^v(z)) \\ &= \frac{1}{2}\rho_k(t)v_k(t)^2 \cdot c_{1,k} + a^2\rho_k(t) \cdot (\log \rho_k(t) + c_{2,k}) + c \cdot \rho_k(t), \end{aligned}$$

where

$$\begin{aligned} c_{1,k} &:= \int_{z_{k-1}}^{z_k} w_k^\rho(z) \cdot w_k^v(z)^2 dz \\ c_{2,k} &:= \int_{z_{k-1}}^{z_k} w_k^\rho(z) \cdot \log w_k^\rho(z) dz. \end{aligned}$$

Therefore the equation for the efforts of the energy ports become:

$$\begin{aligned} e_k^\rho &= \frac{\partial H_k}{\partial \rho_k} = \frac{1}{2}v_k(t)^2 \cdot c_{1,k} + a^2 \cdot (\log \rho_k(t) + c_{2,k} + 1) + c \\ e_k^v &= \frac{\partial H_k}{\partial v_k} = \rho_k(t)v_k(t) \cdot c_{1,k} \end{aligned}$$

This leads to the discrete equivalent of (12) and (13):

$$\begin{cases} f_k^\rho &= \dot{\rho}_k \\ e_k^\rho &= \frac{\partial H_k}{\partial \rho_k} = \frac{1}{2}v_k(t)^2 \cdot c_{1,k} + a^2 \cdot (\log \rho_k(t) + c_{2,k} + 1) + c \end{cases} \quad (18)$$

$$\begin{cases} f_k^v &= \dot{v}_k \\ e_k^v &= \frac{\partial H_k}{\partial v_k} = \rho_k(t)v_k(t) \cdot c_{1,k}. \end{cases} \quad (19)$$

These equations form the constitutive relations of the finite dimensional port-Hamiltonian system and we now should try to find the internal interconnection structure. Just as in the infinite dimensional case, the finite dimensional system will have two external ports, one at the point $z = z_{k-1}$, called node $k-1$ and one at $z = z_k$, called node k . The superscript on the external variables will denote the number of the node and the subscript will denote the number of the lump. Therefore we have on lump k the boundary port variables $f_k^{B,k-1}$ and $e_k^{B,k-1}$ at the left hand side and the boundary port variables $f_k^{B,k}$ and $e_k^{B,k}$ at the right hand side.

Since the interconnection structure of the infinite dimensional system we have is exactly equal to the infinite dimensional system that Golo discretized, we use his resulting discretized interconnection, without detailed derivation. The interested reader can look-up the paper [Golo et al., 2004]. The interconnection structure obtained of our discretized port-Hamiltonian system obtained from the paper of Golo is:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & \alpha_{ab} & \alpha_{ba} \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_k^\rho \\ e_k^v \\ e_k^B \\ e_{k+1}^B \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\alpha_{ba} & \alpha_{ab} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f_k^\rho \\ f_k^v \\ f_k^B \\ f_{k+1}^B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (20)$$

where $\alpha_{ab} = \int_{z_{k-1}}^{z_k} \{w_k^\rho(z) \cdot (1 - \int_0^z w_k^v(\bar{z}) d\bar{z})\} dz$ and $\alpha_{ba} = 1 - \alpha_{ab}$. When choosing the most simple choice of basis functions, given by $w_k^\rho(z) = w_k^v(z) = 1/(z_k - z_{k-1})$, the constants will be $\alpha_{ab} = \alpha_{ba} = 1/2$.

The interconnection structure is slightly different from the paper of Golo since we use another sign convention, the idea and derivation are however exactly the same. The interconnection structure is a simple linear equation and it defines how the boundary port variables are related to the storage port variables. We ultimately want to simulate the state variables ρ_k and v_k . Therefore we are going to express the storage flows in terms of the storage efforts, while that will give a system in the form $\dot{\rho}_k = f_1(\rho_k, v_k)$ and $\dot{v}_k = f_2(\rho_k, v_k)$. First rewrite the equations in (20) to:

$$\underbrace{\begin{bmatrix} e_k^\rho \\ e_k^v \end{bmatrix}}_{e_k} = \underbrace{\begin{bmatrix} -\alpha_{ba} & 0 & \alpha_{ab} & 0 \\ 0 & \alpha_{ab} & 0 & \alpha_{ba} \end{bmatrix}}_{A = \begin{bmatrix} A_\ell & A_r \end{bmatrix}} \underbrace{\begin{bmatrix} f_k^{B,k-1} \\ e_k^{B,k-1} \\ f_k^{B,k} \\ e_k^{B,k} \end{bmatrix}}_{x_k^B}, \quad (21)$$

$$\underbrace{\begin{bmatrix} f_k^\rho \\ f_k^v \\ f_k \end{bmatrix}}_{f_k} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix}}_{B=[B_\ell \mid B_r]} \underbrace{\begin{bmatrix} f_k^{B,k-1} \\ e_k^{B,k-1} \\ f_k^{B,k} \\ e_k^{B,k} \end{bmatrix}}_{x_k^B}. \quad (22)$$

The flows and efforts of the storage port are related to each other by the flows and efforts of the external ports. We therefore rewrite the equation in (21) and express the boundary variables in terms of the efforts. We can only do this by prescribing two boundary variables. We choose to prescribe the boundary flow at the left hand side and the boundary effort at the right hand side as $f_k^{B,k-1} = u_k^1$ and $e_k^{B,k} = u_k^2$ for two inputs u_k^1 and u_k^2 . Then partition the equations of e_k in (21) and find:

$$\begin{aligned} \begin{bmatrix} e_k^\rho \\ e_k^v \end{bmatrix} &= \begin{bmatrix} -\alpha_{ba} \\ 0 \end{bmatrix} u_k^1 + \underbrace{\begin{bmatrix} 0 & \alpha_{ab} \\ \alpha_{ab} & 0 \end{bmatrix}}_A \begin{bmatrix} e_k^{B,k-1} \\ f_k^{B,k} \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_{ba} \end{bmatrix} u_k^2 \\ \Leftrightarrow \begin{bmatrix} e_k^{B,k-1} \\ f_k^{B,k} \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & \alpha_{ab}^{-1} \\ \alpha_{ab}^{-1} & 0 \end{bmatrix}}_{A^{-1}} \left(\begin{bmatrix} e_k^\rho \\ e_k^v \end{bmatrix} + \begin{bmatrix} \alpha_{ba} & 0 \\ 0 & -\alpha_{ba} \end{bmatrix} \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \right). \end{aligned} \quad (23)$$

In a similar fashion we can also partition and rewrite the equation for f_k in (22):

$$\begin{bmatrix} f_k^\rho \\ f_k^v \\ f_k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} u_k^1 + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\bar{B}} \begin{bmatrix} e_k^{B,k-1} \\ f_k^{B,k} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u_k^2.$$

Fill in the expression for $e_k^{B,k-1}$ and $f_k^{B,k}$ from (23):

$$\begin{aligned} \begin{bmatrix} f_k^\rho \\ f_k^v \\ f_k \end{bmatrix} &= \bar{B} \cdot \bar{A}^{-1} \cdot \left(\begin{bmatrix} e_k^\rho \\ e_k^v \end{bmatrix} + \begin{bmatrix} \alpha_{ba} & 0 \\ 0 & -\alpha_{ba} \end{bmatrix} \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \right) + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} f_k^\rho \\ f_k^v \\ f_k \end{bmatrix} &= \begin{bmatrix} 0 & \alpha_{ab}^{-1} \\ -\alpha_{ab}^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} e_k^\rho \\ e_k^v \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_{ba}\alpha_{ab}^{-1} - 1 \\ -\alpha_{ba}\alpha_{ab}^{-1} - 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \alpha_{ab}^{-1} \\ -\alpha_{ab}^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} e_k^\rho \\ e_k^v \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_{ab}^{-1} \\ -\alpha_{ab}^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix}, \end{aligned}$$

where in the last equality we used the relation $\alpha_{ba} = 1 - \alpha_{ab}$. We successfully expressed the flows in terms of efforts, however there is also a term including u_k^1 and u_k^2 . If we now see u_k^1 and u_k^2 as the input of the lump, we found the expression for the flow f_k in terms of the effort e_k and the input. If we

furthermore see the other two external port variables $e^{B,k-1}$ and $f^{B,k}$ in (23) as output variables, then we also found an expression for the output in terms of the effort e_k and the input. If we denote the output by y_k^1 and y_k^2 , respectively, then the equations can be seen to be in input-output form:

$$\begin{bmatrix} f_k^\rho \\ f_k^v \\ y_k^1 \\ y_k^2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{ab}^{-1} & 0 & -\alpha_{ab}^{-1} \\ -\alpha_{ab}^{-1} & 0 & -\alpha_{ab}^{-1} & 0 \\ 0 & \alpha_{ab}^{-1} & 0 & \alpha_{ba}\alpha_{ab}^{-1} \\ \alpha_{ab}^{-1} & 0 & -\alpha_{ba}\alpha_{ab}^{-1} & 0 \end{bmatrix} \begin{bmatrix} e_k^\rho \\ e_k^v \\ u_k^1 \\ u_k^2 \end{bmatrix}. \quad (24)$$

By the anti-symmetry of the matrix, we easily find that the net power of lump k is zero:

$$\begin{aligned} f_k^\rho \cdot e_k^\rho + f_k^v \cdot e_k^v + f_k^{B,k-1} \cdot e_k^{B,k-1} + f_k^{B,k} \cdot e_k^{B,k} \\ &= f_k^\rho \cdot e_k^\rho + f_k^v \cdot e_k^v + u_k^1 \cdot y_k^1 + y_k^2 \cdot u_k^2 \\ &= [e_k^\rho \quad e_k^v \quad u_k^1 \quad u_k^2] \cdot [f_k^\rho \quad f_k^v \quad y_k^1 \quad y_k^2]^T \\ &= 0, \end{aligned}$$

implying that the discretization is indeed a finite dimensional port-Hamiltonian system.

In the next section we will show how to connect multiple lumps to form the equations of gas in a pipeline.

4.2 Coupling lumps to form a pipeline

We discretized one lump, for which we have as state variables $\rho_k(t)$ and $v_k(t)$, that represents the original distributed density $\rho(z, t)$ and $v(z, t)$. In order to simulate a pipeline, we split the pipeline into n lumps. We assume here that all the lumps are equally long and that the basis functions on each lump are the same. This implies that the equations on the each lump are exactly the same. Every lump has a left and a right boundary port. We are going to connect them in a long line as in figure 3, where each connection is similar to the connection as illustrated in figure 2.

We will connect the boundary ports at all the nodes $1, 2, \dots, n - 1$. This works exactly the same as in the infinite dimensional case described in section 3.3. There we found that the physically correct connection is given by the following equations:

$$\begin{bmatrix} f_k^{B,k-1} \\ e_k^{B,k-1} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_P \begin{bmatrix} f_{k-1}^{B,k-1} \\ e_{k-1}^{B,k-1} \end{bmatrix}, \quad (25)$$

where $k = 2, 3, \dots, n$. In the input-output terminology this means that the input of a lump is equal to the output of the neighboring lumps. However the output of the neighboring lumps depends on their own input because of the feed-through term (the non-zero lower-right block in (24)). This input in turn depends on the output of the next neighbour and so on. After all, the value of each lump in the pipe influences the dynamics of each other lump. When modelling compressible fluids, this property of the discretization often arises. Because of this, it is not that easy to couple all the lumps using the input-output representation. The procedure of finding the input-output representation is however instructive for finding the equations of the whole pipeline, using another method. We will now explain how to do this.

We will expand the matrices A and B in equations (21) and (22). By introducing the notation:

$$x_k^{B,m} = \begin{cases} f_k^{B,m} & \text{for } m = k - 1 \\ e_k^{B,m} & \text{or } m = k \end{cases}$$

for the external port variable pairs at node m belonging to lump k , we may rewrite (21) and (22) to:

$$e_k = [A_\ell \quad A_r] \begin{bmatrix} x_k^{B,k-1} \\ x_k^{B,k} \end{bmatrix} \quad f_k = [B_\ell \quad B_r] \begin{bmatrix} x_k^{B,k-1} \\ x_k^{B,k} \end{bmatrix}.$$

The equations for the first lump are therefore:

$$e_1 = [A_\ell \quad A_r] \begin{bmatrix} x_1^{B,0} \\ x_1^{B,1} \end{bmatrix} \quad f_1 = [B_\ell \quad B_r] \begin{bmatrix} x_1^{B,0} \\ x_1^{B,1} \end{bmatrix}.$$

For the second lump, we eliminate $x_2^{B,1}$ by using the equation $x_2^{B,1} = P \cdot x_1^{B,1}$, that is the connection from (25), and we write:

$$e_2 = [A_\ell \quad A_r] \begin{bmatrix} P \cdot x_1^{B,1} \\ x_2^{B,2} \end{bmatrix} \quad f_2 = [B_\ell \quad B_r] \begin{bmatrix} P \cdot x_1^{B,1} \\ x_2^{B,2} \end{bmatrix}.$$

We can proceed in this manner by also eliminating $x_k^{B,k-1}$ by using the equation $x_k^{B,k-1} = P \cdot x_{k-1}^{B,k-1}$ for all k upto n . This will ultimately makes it possible to write:

$$\underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}}_e = \underbrace{\begin{bmatrix} A_\ell & A_r & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_\ell P & A_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & A_\ell P & A_r \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1^{B,0} \\ x_1^{B,1} \\ x_2^{B,2} \\ \vdots \\ x_n^{B,n} \end{bmatrix}}_{x^B}, \quad (26)$$

$$\underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_f = \underbrace{\begin{bmatrix} B_\ell & B_r & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & B_\ell P & B_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & B_\ell P & B_r \end{bmatrix}}_B \underbrace{\begin{bmatrix} x_1^{B,0} \\ x_1^{B,1} \\ x_2^{B,2} \\ \vdots \\ x_n^{B,n} \end{bmatrix}}_{x^B}. \quad (27)$$

These two matrix equalities define the interconnection structure of n lumps connected in a row, just as the matrix equalities (21) and (22) did this for one lump. We now apply the same strategy as before to express f in terms of e . As before we prescribe two boundary values, one at the left hand side and one at the right hand side. That is, we choose $f_1^{B,0} = u_1$ and $e_n^{B,n} = u_2$. Then split the matrices A and B by taking the first and last column apart, these correspond to the two prescribed boundary values. For A denote the first column by A_{first} , the last column by A_{last} and the remaining columns by \bar{A} . We use similar notation for the matrix B . That is, we write:

$$A = [A_{\text{first}} \mid \bar{A} \mid A_{\text{last}}] \quad B = [B_{\text{first}} \mid \bar{B} \mid B_{\text{last}}].$$

In a similar fashion we write \bar{x}^B for the vector x^B without its first and last entry (the prescribed entries).

The equation for e now yields:

$$\begin{aligned} e &= A_{\text{first}} \cdot u_1 + \bar{A} \cdot \bar{x}^B + A_{\text{last}} \cdot u_2 \\ \implies \bar{x}^B &= \bar{A}^{-1} \cdot (e - A_{\text{first}} \cdot u_1 - A_{\text{last}} \cdot u_2) \end{aligned}$$

The inversion of \bar{A} is possible because it has a full upper and full lower diagonal and all other entries are zero. Rewrite the equation for f in the same way and substitute the expression for \bar{x}^B :

$$\begin{aligned} f &= B_{\text{first}} \cdot u_1 + \bar{B} \cdot \bar{x}^B + B_{\text{last}} \cdot u_2 \\ \implies f &= B_{\text{first}} \cdot u_1 + \bar{B} \cdot \bar{A}^{-1} \cdot (e - A_{\text{first}} \cdot u_1 - A_{\text{last}} \cdot u_2) + B_{\text{last}} \cdot u_2 \end{aligned}$$

This last equation expresses the flow of the pipeline in terms of the prescribed values and the efforts of the pipeline. Together with the definition of the efforts and the flows of each lump as given in equation (18) and (13) this ultimately gives the equations describing the dynamics of a gas pipe. This is the result of this chapter and we will state it explicitly here.

Result of section 4.2

The result of the port-Hamiltonian discretization of the equations of motion

of gas in a pipeline is given by the equation:

$$f = B_{\text{first}} \cdot u_1 + \bar{B} \cdot \bar{A}^{-1} \cdot (e - A_{\text{first}} \cdot u_1 - A_{\text{last}} \cdot u_2) + B_{\text{last}} \cdot u_2,$$

where

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f_1^\rho \\ f_1^v \\ f_2^\rho \\ f_2^v \\ \vdots \\ f_n^\rho \\ f_n^v \end{bmatrix} = \begin{bmatrix} \dot{\rho}_1 \\ \dot{v}_1 \\ \dot{\rho}_2 \\ \dot{v}_2 \\ \vdots \\ \dot{\rho}_n \\ \dot{v}_n \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} e_1^\rho \\ e_1^v \\ e_2^\rho \\ e_2^v \\ \vdots \\ e_n^\rho \\ e_n^v \end{bmatrix} = \begin{bmatrix} \frac{\partial H_1}{\partial \rho_1} \\ \frac{\partial H_1}{\partial v_1} \\ \frac{\partial H_2}{\partial \rho_2} \\ \frac{\partial H_2}{\partial v_2} \\ \vdots \\ \frac{\partial H_n}{\partial \rho_n} \\ \frac{\partial H_n}{\partial v_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial \rho_1} \\ \frac{\partial H}{\partial v_1} \\ \frac{\partial H}{\partial \rho_2} \\ \frac{\partial H}{\partial v_2} \\ \vdots \\ \frac{\partial H}{\partial \rho_n} \\ \frac{\partial H}{\partial v_n} \end{bmatrix},$$

where $H = \sum_k H_k$ and

$$H_k = \frac{1}{2} \rho_k v_k^2 \cdot c_{1,k} + a^2 \rho_k \cdot (\log \rho_k + c_{2,k}) + c \cdot \rho_k.$$

Furthermore, u_1 is the prescribed boundary flow at the left hand side of the pipe and u_2 is the prescribed boundary effort at the right hand side of the pipe. The matrices A and B are defined as before.

In this chapter we successfully discretized the port-Hamiltonian system describing the gas dynamics in a pipeline. The used methods and resulting equations form a basis for the next chapter, where we will form the port-Hamiltonian system describing the gas dynamics of a full pipeline network.

5 Simulating pipeline junctions and networks

To this end we succeeded in discretizing the port-Hamiltonian formulation of the gas dynamics in a pipe. This includes the connection of several pipe segments to form a pipeline. There is however more to explore, while pipes are normally part of a network of pipes. Connecting several pipes using pipeline junctions starts to make the simulations far more exciting. In this chapter we will explain how to model a junction of pipelines using the port-Hamiltonian systems that we previously derived for pipe segments. Finally this can be generalized to simulate pipeline networks.

5.1 Pipeline junctions

Already in the introduction we mentioned that the port-Hamiltonian formulation of physical objects is very useful for connecting them together as a network. Now since we formulated the pipeline as a port-Hamiltonian system, we are able to connect pipes to form a network. To understand how to do this, we need to find equations for modelling junctions. In this section we are going to formulate the equations of motion of a Y-junction, see figure 4.

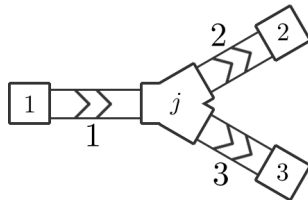


Figure 4: A sketch of a Y-junction

In section 4.2 we discussed how to eliminate port variables when connecting two or more lumps connected in a row. This was necessary to make the inversion of the matrix A in (26) possible. For junctions this is not that easy and we will explain why. Therefore we will instead add the equations describing the connection to the matrix A . This turns out to give the exact same result, but can easily be generalized to model pipeline junctions and networks.

For instructional purposes we assume that we have the Y-junction shown above, with each pipe consisting of only one lump. One can replace each lump by a long pipe easily. The interconnection structure of each lump is given by the two equations (21) and (22) and read in abbreviated form, using a subscript on A to indicate it corresponds to lump k :

$$e_k = A_k \cdot x_k^B \qquad f_k = B_k \cdot x_k^B$$

We take these interconnection together in the matrix A and B as in:

$$\underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_e = \underbrace{\begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1^B \\ x_2^B \\ x_3^B \end{bmatrix}}_{x^B} \quad \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_f = \underbrace{\begin{bmatrix} B_1 & & \\ & B_2 & \\ & & B_3 \end{bmatrix}}_B \underbrace{\begin{bmatrix} x_1^B \\ x_2^B \\ x_3^B \end{bmatrix}}_{x^B}, \quad (28)$$

where A and B are both 6×12 matrices.

In the case of two lumps, the interconnection is given by equations (25), which reads:

$$\begin{bmatrix} f_k^{B,k-1} \\ e_k^{B,k-1} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_P \begin{bmatrix} f_{k-1}^{B,k-1} \\ e_{k-1}^{B,k-1} \end{bmatrix},$$

which made it easy to eliminate some of the boundary port variables. Remember that the boundary flows represent hydrostatic pressure and the boundary efforts represent mass flow. The interconnection was chosen such that the hydrostatic pressure is equal at the intersection and mass is conserved. For the junction of our example this should also hold. Remember that the lumps have an orientation in the sense that the boundary flow at the left hand side is minus the hydrostatic pressure and the boundary flow at the right hand side is plus the hydrostatic pressure. The hydrostatic pressures of the three pipes should be exactly the same at the junction. Keeping in mind the orientation of the pipes, this means we will need to set:

$$+f_1^{B,j} = -f_2^{B,j} = -f_3^{B,j}. \quad (29)$$

The constraint on the effort comes from the conservation of mass. The amount of mass entering the junction should be equal to the total amount of mass leaving the junction. In terms of boundary efforts and again keeping in mind the orientation of the pipes, this means we will need to set:

$$+e_1^{B,j} - e_2^{B,j} - e_3^{B,j} = 0. \quad (30)$$

These equations cannot easily be substituted in the equations of motion to eliminate some of the boundary port variables. Therefore we will instead simply add the equations to the system of equations in (31) as follows. The constraints on the flows and efforts form together 3 equations and can be written in the form:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = D \cdot \begin{bmatrix} x_1^B \\ x_2^B \\ x_3^B \end{bmatrix},$$

where D is a 3×12 matrix.

As before, we also need to prescribe some of the boundary flows and efforts to make the inversion of A possible. In the previous chapter we explicitly took

out the columns corresponding to the prescribed values of x^B . For a junction, we will instead add these input assignments as equations. For this particular junction we need to prescribe the flow at the left hand side of pipe 1, that is $f_1^{B,1} := u_1$ and the efforts at the right hand side of pipe 2 and 3, that is $e_2^{B,2} := u_2$ and $e_3^{B,3} := u_3$. These input assignment form three equations and can be written in the form:

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_u = C \cdot \begin{bmatrix} x_1^B \\ x_2^B \\ x_3^B \end{bmatrix},$$

where C is a 3×12 matrix.

Now it's time to form the equations of motion of this junction. We augment the equation of the effort in (31) as:

$$\begin{bmatrix} e \\ \bar{0} \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ D \\ C \end{bmatrix}}_G \begin{bmatrix} x_1^B \\ x_2^B \\ x_3^B \end{bmatrix},$$

where $\bar{0}$ at the left hand side represents an all zero vector with three entries. The matrix G is a 12×12 matrix which should be invertible. If for some junction it turns out that G is not invertible, one probably prescribes the wrong boundary variables. For the example given above, we chose everything correctly and it turns out that G is indeed invertible. Now invert the equation and substitute it in the equation of the flow to find:

$$f = B \cdot G^{-1} \cdot \begin{bmatrix} e \\ \bar{0} \\ u \end{bmatrix}$$

This equations expresses the internal flows f in terms of the internal efforts e and the inputs u and therefore describes the equations of motion of the junction. Note that this expression look more elegant than the expression of the flow at the end of the previous chapter, even though the underlying network structure is more complex.

In the next section we briefly describe how to generalize the pipeline junctions to pipeline networks.

5.2 Pipeline networks

For pipeline networks we can generalize the method from the previous section. A pipeline network can be described by a graph with edges representing pipelines

and vertices representing junctions. The pipes have a dishtinguisable left and right side, because the boundary flow at the left hand side is minus the hydrostatic pressure and at the right hand side it is plus the hydrostatic pressure. Therefore the pipeline network can be represented by an oriented graph, described by an incidence matrix. Assume that we have m junctions and n pipes in our network. The incidence matrix is then a $m \times n$ matrix. We assume that the definition of an incidence matrix of an oriented graph is known to the reader.

Each edge represents a pipeline and each pipeline can be modelled by the equations (26) and (27) derived in section 4.2. From a global perspective, these equations have the same structure as the equations of a lump and therefore we stack the equations of all the edges together in big matrices A and B just as we did in equations (31):

$$\underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}}_e = \underbrace{\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1^B \\ x_2^B \\ \vdots \\ x_n^B \end{bmatrix}}_{x^B} \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_3 \end{bmatrix}}_f = \underbrace{\begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_n \end{bmatrix}}_B \underbrace{\begin{bmatrix} x_1^B \\ x_2^B \\ \vdots \\ x_n^B \end{bmatrix}}_{x^B}. \quad (31)$$

Then add for each junction in the incidence matrix a set of equations representing the interconnection as we explained for the junction in the previous section to the matrix D . In a pipeline network one can also have junctions with more than three pipes. The equations in (29) and (30) then generalize easily to these cases and results in the case of a junction with k pipes to exactly k constraints ($k-1$ constraint on the flows and 1 constraint on the efforts). Furthermore prescribe as before the necessary boundary port variables and write it with the matrix C . We get:

$$\bar{0} = D \cdot x^B \quad u = C \cdot x^B.$$

If the controls u pcribe the correct boundary variables we can invert the matrix G in

$$\begin{bmatrix} e \\ \bar{0} \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ D \\ C \end{bmatrix}}_G \begin{bmatrix} x_1^B \\ x_2^B \\ x_3^B \end{bmatrix},$$

and write as before

$$f = B \cdot G^{-1} \cdot \begin{bmatrix} e \\ \bar{0} \\ u \end{bmatrix}$$

From the previous description it should be clear how one can find the equations of motion of a whole pipeline network. The details about how to explicitly compose the matrices C and D would make this explanation technical instead of instructional and therefore we omitted it. We wrote however a script that creates C and D and ultimately the full matrices B and G from the incidence matrix of any graph. It turns out the resulting equations are power conserving, since the resulting matrix is skew-symmetric. This indicates that we connected everything in such a way, that the resulting system is again a port-Hamiltonian system.

6 Conclusion and further research

In this thesis we found out that the 1-dimensional isothermal Euler equations can be rewritten as a port-Hamiltonian system. The energy function consists of the kinetic energy and potential energy that arises from compression or expansion of the gas. For the purpose of simulation, we discretized the port-Hamiltonian system using a port-Hamiltonian discretization method. We first explained how these finite dimensional port-Hamiltonian systems on lumps can be coupled to form a pipeline. Next, the port-Hamiltonian formulation made it possible to couple pipelines together to form a pipeline network, while preserving energy conservation and mass conservation.

The used methods for discretizing and coupling lumps together to form a network are illustrative for port-Hamiltonian systems in general. By replacing only the energy density function, one could for example simulate electrical energy networks. Therefore this thesis could serve as a lookup for anybody who wants to do simulations of port-Hamiltonian systems.

The work done in this thesis could also be extended. We neglected the friction term from section 2.1 that was originally in the isothermal Euler equations from [Grundel et al., 2014]. It would be interesting to add friction to the system, while friction causes a pressure drop inside real gas pipes. We worked on ways to incorporate the friction as a resistive port to the port-Hamiltonian system, but did not succeed. The resistive port is a concept often used in the port-Hamiltonian formulation. The port-Hamiltonian discretization methods that we found in the literature did however not incorporate a friction port. Therefore, a suggestion for more general research on port-Hamiltonian systems is to find port-Hamiltonian discretization methods for port-Hamiltonian systems including resistive ports.

Another point of interest would be the comparison of the port-Hamiltonian discretization method used in this thesis to other commonly applied discretization methods. For example finite volume, finite difference, finite element methods or spectral discretization methods.

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