university of groningen

faculty of mathematics and natural sciences

Bachelor research

Higher Order Lagrangians
for classical mechanics and scalar fields

Floris Harmanni, s2592274

Theoretical Physics

Supervisor / First examiner:
prof. dr. Diederik Roest

Second examiner:
prof. dr. Daniel Boer

1st July 2016

Abstract

In this bachelor research report I look at the properties of Lagrangians containing terms with higher order derivatives. Normally only first order Lagrangians produce workable equations of motion, because higher order Lagrangians contain ghost degrees of freedom (i.e. the Hamiltonian contains a term linearly in momentum, meaning it is unbounded.), as shown by Ostrogradski.

For classical mechanics there are (under certain conditions) possibilities to write higher order Lagrangians, but the higher order terms can be taken into total derivatives, meaning all these possibilities are in fact equivalent to first order Lagrangians and are therefore, in the context of this research, trivial. For scalar fields there are possibilities to write second order Lagrangians, which carry the name (generalised) Galileons. Specific antisymmetric properties make non-trivial second order Lagrangians possible.

For higher order Lagrangians, I tried to construct third order (or higher) Lagrangians that produce workable equations of motion. It is shown that Lagrangians containing only higher order derivative terms are fatal, possibly with the exception of the case there a third derivative enters linear. For this case and for the case where a second order Lagrangian is added to a linear third order Lagrangian, conditions are found, but not yet solved.

Contents

1 Introduction **3**
1.1 Other formulations of classical mechanics 4
1.2 Scalar fields 6
1.3 Problems with Higher Order Lagrangians 6
1.4 Why higher order Lagrangians? 9
1.5 Research goal and approach 11
1.6 Restrictions, terminology and notation 11

2 Second Order Lagrangians in Classical Mechanics **13**
2.1 Constraints 13
2.1.1 Constraints in the Hamiltonian Formalism 14
2.1.2 Constraints in the Lagrangian Formalism 16
2.2 Equations of motion 19
2.3 Conclusion on second order Lagrangians in classical mechanics . 21

3 Second Order Lagrangians for Scalar Fields **22**
3.1 How do Galileons work? 23
3.1.1 Symmetric and antisymmetric tensors 23
3.1.2 Getting rid of the problematic terms 23
3.2 Constructing Galileon Lagrangians 25
3.2.1 Constructing the first Lagrangian 25
3.2.2 Other Lagrangians 27
3.3 Generalised Galileons 28
3.4 Do these Lagrangians contain new information? 28
3.5 Conclusion on second order Lagrangians for scalar fields 29

4 Higher Order Lagrangians in Classical Mechanics **30**
4.1 Third order Lagrangians 30
4.2 Higher order Lagrangians 32
4.3 Conclusion on higher order Lagrangian in classical mechanics . . 32

5 Higher Order Lagrangians for Scalar Fields **34**
5.1 A first attempt 34
5.2 Finding conditions for possibility (II) 36

5.2.1 Finding conditions to solve the first problem 37
5.2.2 Finding conditions to solve the second problem 38
5.3 Conditions for possibility (I) 40
5.4 Finding the tensors 40
5.5 Conclusions on higher order Lagrangians for scalar fields 40

6 Conclusions **42**

A Derivations **46**
A.1 Equations of motion for classical mechanics 46
A.2 Equations of motion for scalar field 50
A.3 Second order Lagrangians for scalar fields (Galileons) 51
A.3.1 Working out the Euler-Lagrange equation 51
A.3.2 Lagrangians $\mathcal{L}_N^{\mathrm{Gal},1}$ 53
A.3.3 Equations of motion from $\mathcal{L}_N^{\mathrm{Gal},1}$ 54
A.3.4 Relations between $\mathcal{L}_N^{\mathrm{Gal},1}, \mathcal{L}_N^{\mathrm{Gal},2}$ and $\mathcal{L}_N^{\mathrm{Gal},3}$ 56
A.4 Higher order Lagrangian for classical mechanics 57
A.4.1 Example of third order Lagrangian 57

Chapter 1

Introduction

In physics, we are often interested in the behaviour of a system in terms of its motion as a function of time. This system can be all sorts of things. In secondary school we all encountered the simplest systems of all; a single object with mass m falling towards the ground under the influence of a gravitational force $\mathbf{F}$. We learned that Sir Isaac Newton $(1643 - 1727)$ wrote his three laws of motion, describing how forces on systems act and how they produce motion. The second of these laws is[1]

$$\mathbf{F} = m\ddot{\mathbf{x}}, \tag{1.1}$$

where a dot on a variable indicates that it is the time-derivative of this variable. In this case, $\ddot{\mathbf{x}}$ is the second time-derivative of position vector, also known as the acceleration $\mathbf{a}$. Newton's second law tells us that the sum of all forces acting on an object gives to the mass an acceleration. Being a second order differential equation, this means that if we know $\mathbf{x}_0$ (the starting position) and $\dot{\mathbf{x}}_0$ (the starting velocity), the motion of the object can fully be determined. An equation like this one, that links the theory to the motion, is called an *equation of motion* (the term *field equations* is also used when dealing with fields instead of particles). It is this deterministic character that is central to classical physics. Newton himself wrote his three laws as 'axioms', meaning he regarded them as the starting point of the analyses of movement [15]. Until the development of quantum mechanics in the early 20th century, it was believed that if we were able to write the correct theory and know the initial conditions, all motion in nature can be fully determined. This idea can schematically be seen as follows:

$$\boxed{\begin{array}{c}\text{Discription}\\ \text{of system}\end{array}} \longrightarrow \boxed{\text{Equations of motion}} \xrightarrow[\text{conditions}]{\text{initial}} \boxed{\text{Motion}}$$

In *Newtonian mechanics*, forces play the central role in the theory. If we know exactly what forces act on an object (or even better, when we have an equation

[1] Newton himself did not use notation in symbols. His formulation is: *The change (mutatio) of the modus* (i.e. of the 'quantity of movement' or momentum) *is proportional to the working vis motrix* (force) *and is located along a straigh line, by which this force acts.*[15]

for the potential), equation (1.1) gives the motion. Although in principle this can be done for any classical system, it already becomes very difficult to produce equations of motions for a system that has more than two particles that have interactive forces between them.

1.1 Other formulations of classical mechanics

Fortunately, Newtonian mechanics is not the only way to come up with equations of motion for a system. There are other formulations of classical mechanics that produce that same kind of equations of motion and are therefore physically equivalent formulations. There are the so-called the *Lagrangian formulation* and *Hamiltonian formulation*. Let's first look at the Lagrangian formalism. The key concept in this formulation is the *principle of least action*. Instead of working with vectors, we work in configuration space. In this space, each vector coordinate $\mathbf{r}$ in Cartesian space has three coordinates q_i. This means that for N particles, there are the coordinates q_i with $i = 1, 2, \ldots, 3N$. We then say that there are $3N$ *degrees of freedom*, such that

$$\#\,\text{degrees of freedom} = \text{dimension of configuration space}. \tag{1.2}$$

The *Lagrangian* is then defined as

$$L(q_i, \dot{q}_i) = T(q, \dot{q}_i) - V(q_i), \tag{1.3}$$

where $T(q, \dot{q}_i)$ is the kinetic term and $V(q_i)$ is the potential term. This Lagrangian is a function of the coordinate q and only its first time derivative. I will refer to such a Lagrangian as a first order Lagrangian, where 'order' refers to the order of derivatives that appear in the Lagrangian. It has long be thought that only first order Lagrangian are acceptable. Why this was and that this is not true will become apparent later on. The motion of a system brings a system from an initial state $q_i(t_i)$ tot a final state $q_i(t_f)$. There are many paths through configurations space that the system can follow to get from the initial state to the final state. The question is, which path does the system take in reality? To find the out, there is a number assigned tot every path, called the action S:

$$S[q_i(t)] = \int_{t_i}^{t_f} L(q_i, \dot{q}_i)\, dt. \tag{1.4}$$

The principle of least action then states the the actual path taken is the one where S in minimised, while keeping the endpoints fixed. This results in equations of motion know as the *Euler-Lagrange equation*, developed by Joseph-Louis Lagrange $(1736 - 1813)$ and Leonhard Euler $(1707 - 1783)$ in the 1750s:

$$E_i = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{1.5}$$

A proof of this equation can be found in appendix A.1. It can easily be shown (see for example [9] and appendix A.1) that this Euler-Lagrange equation give

back Newton's second law, meaning the formulations are equivalent. Both the Euler-Lagrange equation and Newton's second law are second order differential equations. The great advantage of the Lagrangian formulation over the Newtonian one is that it does not involve working with vectors, which is rather cumbersome when the system becomes even slightly more complicated that an object falling to the ground. The reason Newtonian mechanics is still the preferred choice in secondary school is simple the fact the Lagrangian formulation is more abstract an the fact that students are not yet comfortable with (partial) differentiation. However, for the theoretical physicist it is much more useful and often the preferred choice.

The Hamiltonian formulation is very closely linked to the Lagrangian formalism. We again work in a different space. We define

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.6}$$

as the generalised momenta. The $\dot{q}_i$'s can then be eliminated in favour of p_i's and q_i and p_i can be placed on equal footing. The pair $\{q_i, p_i\}$ then defines a point in *phase space*, where the dimension of phase space for N particles is $2N$, meaning

$$\#\,\text{degrees of freedom} = \frac{1}{2}\,\text{dimension of phase space.} \tag{1.7}$$

Using this transformation from configuration space to phase space, we can define the *Hamiltonian* as

$$H(q_i, p_i) \equiv \sum_{i=1}^{N} p_i \dot{q}_i - L(q_i, \dot{q}_i), \tag{1.8}$$

which is a function of q_i and p_i only. Via the variation in H (see [9] for full proof), equations of motion called the *Hamilton equations* can be derived:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{1.9}$$

The motion is now described by a set of two first order differential equations, requiring both one initial condition. This means that the pair contains the same information as the one second order differential equation in the former two formulations. The equivalence between Newtonian and Hamiltonian (and thus also to Lagrangian) mechanics is shown in appendix A.1. Although the Hamiltonian formulation is not very helpful when it comes to solving concrete problems, it does provide some nice insights, as will become apparent further on.

There are thus three ways to get to the motion in classical mechanics. The schematic representation provided earlier can thus be further filled in:

$$\boxed{\text{Discription of system}} \xrightarrow[\substack{\textit{or} \\ \text{Hamilton equations}}]{\substack{\text{Newton's second law,} \\ \text{Euler-Lagrange eq's}}} \boxed{\text{Equations of motion}} \xrightarrow[\text{conditions}]{\text{initial}} \boxed{\text{Motion}}$$

Where the description of the system is either in terms of forces, a Lagrangian or a Hamiltonian.

1.2 Scalar fields

Another advantage of the Lagrangian and Hamiltonian formulations is that they can easily be extended to scalar fields. A scalar field $\phi(\mathbf{x}, t)$ is a physical quantity that has a specific (scalar) value at every point in space and time. The simplest example of a scalar field is temperature $T(\mathbf{x}, t)$. Every point in space and in time has a specific value for the temperature.

The scalar fields I will be working with, will be in Minkowski space (flat space-time). A position in Minkowski space is denoted as x^μ where $\mu = 0, 1, 2, 3$, or, equivalently, $\mu = t, x, y, z$.

When working with scalar field, the *Lagrangian density* $\mathcal{L}$ is used. This is defined as

$$L = \int \mathcal{L}\, d^3x, \tag{1.10}$$

although $\mathcal{L}$ is for convenience also often (as in this report) referred to as the Lagrangian. The Euler-Lagrange equations for scalar field are then given by

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0, \tag{1.11}$$

where the four-derivative ∂_μ is defined as

$$\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \tag{1.12}$$

For the Hamiltonian, there is also a *Hamiltonian density* $\mathcal{H}$, which is defined in the same way as the Lagrangian density. The Hamilton equations for field are then

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \phi} + \nabla \frac{\partial \mathcal{H}}{\partial(\nabla \phi)}, \qquad \dot{\phi} = \frac{\partial \mathcal{H}}{\partial \pi}, \tag{1.13}$$

where π is the conjugate momentum to ϕ, defined as[2]

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{1.14}$$

1.3 Problems with Higher Order Lagrangians

In the above, the Lagrangian was given as $L(q, \dot{q})$ (or $\mathcal{L}(\phi, \partial_\mu \phi)$ in the case of scalar fields), meaning that the Lagrangian only contains up to first order (time-)derivatives of some variable. The mean reason why it has been long thought that Lagrangians containing second (or higher) derivatives in time are not acceptable, is the theory by Mikhail Ostrogradski published 1850 [16]. This theory predicts that instabilities arise when Lagrangians contain these derivatives. In this chapter this theory will be addressed, as worked out in [2].

[2]Note that later on the letter π is used for the field itself, because of convention in the literature. The conjugate momentum to the field will not be used in what follows.

Ostrogradski instabilities

In classical mechanics a usual Lagrangian $L = L(q, \dot{q})$ gives rise to equations of motion through the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{1.15}$$

When $\frac{\partial L}{\partial \dot{q}}$ depends upon $\dot{q}$, this is known as non-degeneracy. Assuming this, the Euler-Lagrange equations result in equations of motion

$$\ddot{q} = \mathcal{F}(q, \dot{q}) \qquad \Rightarrow \qquad q(t) = \mathcal{Q}(t, q_0, \dot{q}_0). \tag{1.16}$$

This is the usual mechanics that Newton assumed back in the 17th century. Since there are two initial values required, there must be two canonical coordinates in phase space, P and Q, where

$$Q \equiv q \qquad ; \qquad P \equiv \frac{\partial L}{\partial \dot{q}}. \tag{1.17}$$

Assuming that L is non-degenerate implies that we can invert the expressions above and solve for $\dot{q}$. In other words, there exists a function $v(Q, P)$ such that

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{\substack{q=Q \\ \dot{q}=v}} = P, \tag{1.18}$$

leading to the canonical Hamiltonian

$$H(Q, P) \equiv P\dot{q} - L = Pv(Q, P) - L(Q, v(Q, P)), \tag{1.19}$$

which indeed generates time evolution. With this we mean that the Hamilton equations reproduce the inverse phase space transformation and the Euler-Lagrange equation:

$$\dot{Q} \equiv \frac{\partial H}{\partial p} = v + P\frac{\partial v}{\partial P} - \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial P} = v, \tag{1.20}$$

$$\dot{P} \equiv -\frac{\partial H}{\partial Q} = -P\frac{\partial v}{\partial Q} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial P} = \frac{\partial L}{\partial q}. \tag{1.21}$$

If one now considers a system with a Lagrangian $L = L(q, \dot{q}, \ddot{q})$ that has up to second order time derivatives, this gives the Euler-Lagrange equation[3]

$$E = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0. \tag{1.22}$$

Non-degeneracy implies that $\frac{\partial L}{\partial \ddot{q}}$ depends on $\ddot{q}$, which results in the fact that the equations of motions are of the form

$$q^{(4)} = \mathcal{F}(q, \dot{q}, \ddot{q}, q^{(3)}) \qquad \Rightarrow \qquad q(t) = \mathcal{Q}(t, q_0, \dot{q}_0, \ddot{q}_0, q_0^{(3)}). \tag{1.23}$$

[3] Again, see appendix A.1 for a derivation

Because the solutions depend now on four initial values, four canonical coordinates are needed. Ostrogradski's choices are [2]:

$$Q_1 \equiv q \quad ; \quad Q_2 \equiv \dot{q} \quad ; \quad P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \quad ; \quad P_2 \equiv \frac{\partial L}{\partial \ddot{q}}. \tag{1.24}$$

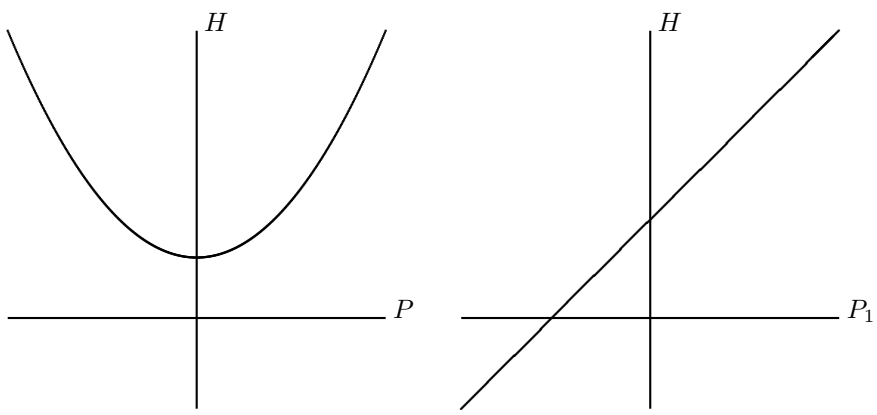
Again non-degeneracy is assumed, such that we can invert and solve for $\ddot{q}$. There is thus a function $a(Q_1, Q_2, P_2)$, such that

$$\left.\frac{\partial L}{\partial \ddot{q}}\right|_{\substack{q=Q_1 \\ \dot{q}=Q_2 \\ \ddot{q}=a}} = P_2. \tag{1.25}$$

This leads to[4]

$$\begin{aligned} H(Q_1, Q_2, P_1, P_2) &\equiv \sum_{i=1}^{2} P_i q^{(i)} - L \\ &= P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L(Q_1, Q_2, a(Q_1, Q_2, P_2)). \end{aligned} \tag{1.26}$$

Using the Hamilton equations, it can again be checked that this indeed generates time evolution [2]. In this Hamiltonian, the first term is what causes a so called *Ostrogradski instability* or an Ostrogradski *ghost*.



Non-linear; Bounded Linear; Unbounded

The fact that this term is linear in P_1 means that this degree of freedom does not feel any barrier that prevents it of going to arbitrary negative energies. This means that the Hamiltonian is not bounded from below[2, 8], see the figures above. Theories that include such instabilities are often referred to as "ghost-like".

[4] Where $q^{(i)}$ denotes the ith derivative of q

Looking now at the general case with the Lagrangian $L = L(q, \dot{q}, \ldots, q^{(n)})$, which depends non-degenerately upon $q^{(n)}$, the equation of motion is

$$\sum_{i=0}^{n} (-1)^i \frac{d^i}{dt^i} \left(\frac{\partial L}{\partial x^{(i)}} \right) = 0, \tag{1.27}$$

which contains up to $q^{(2n)}$ derivative terms. For the $2n$ canonical coordinates, Ostrogradski's choices are:

$$Q_i \equiv q^{(i-1)} \qquad P_i \equiv \sum_{j=i}^{n} \left(-\frac{d}{dt} \right)^{j-i} \frac{\partial L}{\partial q^{(j)}}. \tag{1.28}$$

These lead to the Hamiltonian

$$H \equiv \sum_{i=1}^{n} P_i q^{(i)} - L$$

$$= P_1 Q_2 + P_2 Q_3 + \cdots + P_{n-1} Q_n + P_n \mathcal{A} - L(Q_1, \cdots, Q_n, \mathcal{A}), \tag{1.29}$$

where the function $\mathcal{A}(Q_1, \ldots, Q_n, P_n)$ is again the function that implies

$$\left. \frac{\partial L}{\partial q^n} \right|_{\substack{q^{(i-1)} = Q_i \\ q^{(n)} = \mathcal{A}}} = P_n, \tag{1.30}$$

as the result of the assumption of non-degeneracy. This Hamiltonian is linear in P-terms, in this case $n-1$ of them, meaning that such a theory is unstable over half the classical phase space [2].

To summarise, the Ostrogradski Theorem can be states as follows [8]: *If the higher order time derivative Lagrangian is non-degenerate, there is at least one linear instability in the Hamiltonian of this system.*

1.4 Why higher order Lagrangians?

Ostrogradski's Theorem thus shows that Lagrangians that involve second derivatives or higher are of little use. To be more precise, they can be put in two classes: either they are trivial and do not include new physics or they are fatal and include additional ghost-like degrees of freedom. This does not seem to be a hopeful starting point to explore higher order Lagrangians. Why is it that people are interested in them?

In this section I will very briefly give some examples of application of theories involving such Lagrangians. It is by no means a full overview, but solely meant as a motivation for the analysis that follows. Of course, besides possible applications there is also the curiosity of the theoretical physicist. Some 19th century Ukrainian-Russian physicist telling us we cannot do this makes us wonder...

As will be worked out later on, in the case of fields there are theories that include second order derivatives in the Lagrangian that do not lead to Ostrogradski ghosts. They produce normal second order equations of motion. This very specific set of second order Lagrangians is referred to as *Lovelock gravity* for the metric tensor and *(generalised) Galileons* in the case of scalar field [1]. I will only focus on the latter.

Theoretical physicists have been looking at a number of ways to construct gravity in order to avoid having to deal with a cosmological constant or dark matter and still be true to the accelerating universe that we observe. An example of such a model is the Dvali-Gabadadze-Porrati (DGP) model. In this model, gravity is modified at large distances. This has a number of branches, of which one is ghost-free. In a particular 'decoupling' limit, the Lagrangian of this branch reduces to a theory of a single scalar field ϕ , with a cubic self-interacting term that is proportional to $(\partial\phi)^2\Box\phi$, which contains a second order derivative[5] [17]. There is a modification to this model that involves gravitational interaction via the so-called Vainshtein mechanism. This has a screening effect such that forces can compete with gravity on cosmological scales, but has little physical effect on local scales [17]. One thing that is interesting in this mechanism is that it includes again a scalar field whose action depends on second derivatives of the field [6, 7, 17]. However, the resulting equations stay second order.

Another typical application is in the primordial Universe. Theories including higher order derivatives in the Lagrangian can be used to form radiatively stable inflation models. It can also be used to construct stable alternatives to inflation.[6]

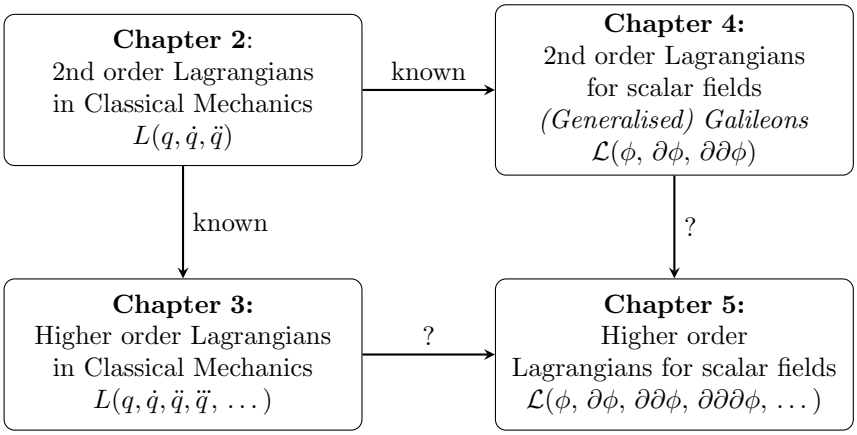
Another interesting application of Galileons in an alternative to inflation [6]. The Null Energy Condition (NEC) is one of the most robust energy conditions. In cosmology, this forbids a non-singular change from a contracting universe to an expanding universe, meaning the universe either expands or contracts [19]. Any alternative to inflation requires violation of the NEC. Often, the breaking of this condition comes with ghost instabilities, but Galileons provide a way to avoid this. This allows possibilities to use Galileons for alternatives to inflation [19].

There are thus a wide range of applications that motivates to look further in the properties of higher derivative theories. Although the above examples only include up to second order derivatives, theories with higher order derivatives are first of all interesting from a theoretical point of view. So far, the properties of Lagrangians containing higher than second order derivatives are, as far as I could find, not (yet) worked out. Scalar field are used very often in many areas of physics. It is therefore interesting to know what properties they have and in what possible ways they can be implemented. If the higher order derivative Lagrangians appear to lead to stable equations of motion, they may well find their application.

[5] Here the notation $\Box\phi \equiv \partial^\mu\partial_\mu\phi$ is used.

1.5 Research goal and approach

The main goal of this research is to explore the properties of Galileons and to see whether I can say something about the possibilities of extending the Galileon theory to higher orders. This means I am interested in Lagrangians of scalar field containing higher than first order derivatives, but still give workable (second order) equations of motion. These Lagrangians should be Lorentz invariant. However, looking also at higher order Lagrangians in classical mechanics may give some feeling of how the scalar field versions work. The approach I will use can be viewed schematically:



I will start by looking at second order Lagrangians in classical mechanics, which will be treated in chapter 2. After this, in chapter 3, I will look at the second order variant for scalar field, the (generalised) Galileons. Up to this point, everything is in principle known and this part of the research will be literature based. In chapter 4, I will go back to classical mechanics to see which ways there are to construct higher order Lagrangians in classical mechanics. In chapter 5, I will try to construct Lagrangians containing higher order derivatives of scalar field and, if possible, whether they do indeed contain new information. There are basically two ways to get to these new Lagrangians. As can be seen in the diagram above, I can generalise the higher order Lagrangians in classical mechanics to field. An other way is to promote the Galileons to higher order(s). Chapter 6 will contain an summary of the results from which I will draw my conclusions. Recommendations to further research will also be included in this last chapter.

1.6 Restrictions, terminology and notation

In the process of analysing Lagrangians, I will restrict the research field to keep it within the scope of a bachelor research. First of all I, will only look at flat spacetime (i.e. 3+1 dimensional Minkowski space). When I speak of higher order

Lagrangian, I will mainly focus on third and possibly fourth order. Sometimes it will prove to be easy to generalise things up to n^{th} order, but this is not a goal. The same holds for the number of particles or fields. In principle, I will restrict to single particles and field, but sometimes generalisations to multiple particles and field can easily be made.

An important note on the use of the word 'order' for Lagrangians should be made. Unless otherwise states, I will use this to refer to the highest derivative in the Lagrangian. For Lagrangians in classical mechanics an 'n^{th} order Lagrangian' means a Lagrangian that contains up to n^{th} *time*-derivatives, i.e. $L = L(q, \dot{q}, \ddot{q}, \ldots, q^{(n)})$. In the case of scalar field an 'n^{th} order Lagrangian' refers to four-derivatives ∂_μ, in stead of only time-derivatives, i.e. $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, \ldots, \partial^{(n)}\phi)$. When referring to linear, quadratic or cubic appearance of a variable, I will use the term 'polynomial order'. The terms 'workable' or 'healthy' equations of motion mean that these equations of motion are up to second order differential equations and do not contain any ghosts.

Different notations for derivatives are used, which fully equivalent:

$$\frac{dF}{dq} \equiv F_q\,; \qquad \frac{d^2F}{dq_i dq_j} \equiv F_{q_i q_j}\,; \qquad \partial_\mu\phi \equiv \phi_\mu\,; \qquad \partial_\mu\partial_\nu\phi \equiv \phi_{\nu\mu}. \tag{1.31}$$

The index notation will primarily used when other notation would make equations unnecessarily messy.

Chapter 2

Second Order Lagrangians in Classical Mechanics

As was shown in the previous chapter, Lagrangians of order two or higher in time derivatives generally cause instabilities due to the fact that they have an linear instability in the Hamiltonian of the system. The equations of motion are then higher than second order and require $2n$ initial conditions. In this chapter, possibilities to construct Lagrangians of second order that still lead to workable equations of motion will be discussed.

The equations of motion when working with a Lagrangian of second order is

$$E_i = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0. \tag{2.1}$$

In general, this contains up to fourth order in derivatives of q. For the theory to be healthy, the third and fourth order terms must vanish. The fourth order derivatives of q can only enter through the last term of equation (2.1). More specifically, they can only enter when $\frac{\partial L}{\partial \ddot{q}}$ contains $\ddot{q}$, in other words, when the Lagrangian is non-degenerate. The third order terms can enter the equation both through the second and last term of equation (2.1).

2.1 Constraints

The way to assure that all fourth and third derivatives vanish in the equations of motion, is that there should be certain restrictions in the system. This means that the canonical variables are not independent, but are in some way related to each other. These relations are called the *constraints* [12]. They will in the end cause the terms that are problematic to drop out. There are two approaches for working with these kind of constraints that will be discussed in this chapter. One in the Hamiltonian formalism and the other in the Lagrangian formalism. The former is quite a complex one and I will not elaborate on it very extensively.

It is merely shown for completeness and to give some insights in the way to get rid of the linear terms in the Hamiltonian and thereby making the connection to Ostrogradski's theorem.

I must make a brief comment on the notation I use in the text below. Different sources use different notations and symbols when working with constraints. I have tried to be as consistent as possible with the use of notation in this chapter and tried to avoid the use of single symbol for different things. This means that notation may differ from what is found in the sources used for this chapter.

2.1.1 Constraints in the Hamiltonian Formalism

A Lagrangian that is second order in time derivatives will generally need four initial conditions and therefore it will have four degrees of freedom in phase space. The Hamiltonian was given by

$$H(Q_1, Q_2, P_1, P_2) \equiv \sum_{i=1}^{2} P_i q^{(i)} - L$$

$$= P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L\left(Q_1, Q_2, a(Q_1, Q_2, P_2)\right). \tag{2.2}$$

The first linear term in this expression was problematic, as it means that the Hamiltonian is unbounded from below. This means we somehow want this term to vanish. If we can find a constraints where Q_2 is some function of P_1, this will do the job. The way to achieve this is fully worked out in [8]. I will give a brief summery of this in order to illustrate the approach.

For this the starting point is a second order time derivative Lagrangian with one auxiliary field λ, given by

$$L = f(q, \dot{q}, \ddot{q}, \lambda). \tag{2.3}$$

Since six initial conditions are needed to solve the equations of motion (q_0, $\dot{q}_0$, $\ddot{q}_0$, $q_0^{(3)}$, λ_0 and $\dot{\lambda}_0$), phase space will be six-dimensional. Following Ostrogradski's choices for the canonical variable [8]:

$$Q_1 \equiv q \qquad \longleftrightarrow \qquad P_1 \equiv \frac{\delta L}{\delta \dot{q}} = -\frac{d}{dt}\frac{\partial f}{\partial \ddot{q}} + \frac{\partial f}{\partial \dot{q}} \quad ; \tag{2.4}$$

$$Q_2 \equiv \dot{q} \qquad \longleftrightarrow \qquad P_2 \equiv \frac{\delta L}{\delta \ddot{q}} = \frac{\partial f}{\partial \ddot{q}} \quad ; \tag{2.5}$$

$$Q_3 \equiv \lambda \qquad \longleftrightarrow \qquad P_3 \equiv \frac{\delta L}{\delta \dot{\lambda}} = 0. \tag{2.6}$$

Here the last statement is called the *primary constraint*, which will be denoted as $\Phi_1 : P_3 = 0$. The notation ":" is used to denote "functional form given by". Using the assumption of non-degeneracy, (2.5) can be inverted to $\ddot{q} = h(Q_1, Q_2, Q_3, P_2)$. The total Hamiltonian can then be written as

$$H_T = P_1 Q_2 + P_2 h(Q_1, Q_2, Q_3, P_2) - f(Q_1, Q_2, Q_3, h) + u_1 \Phi_1. \tag{2.7}$$

The function u_1 is a function of the canonical variables that can in principle be found later. The fact that $P_3 = 0$ implies that its time derivative $\dot{P}_3 = \{P_3, H\}_{PB}$ must also vanish. This leads to a series of *constancy relations* from which further *secondary constraints*[1] can be found. In this case this gives [8]

$$\begin{aligned}\Phi_2 : \dot{\Phi}_1 = \{\Phi_1, H\}_{PB} &= -P_2 \frac{\partial h}{\partial Q_3} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial Q_3} + \frac{\partial f}{\partial Q_3} \\ &= -P_2 \frac{\partial h}{\partial Q_3} + \left[\frac{\partial f}{\partial \ddot{q}}\right]_{\ddot{q}=h} \frac{\partial h}{\partial Q_3} + \left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=Q_3} \\ &= \left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=Q_3} (Q_1, Q_2, Q_3, h) \approx 0.\end{aligned} \tag{2.8}$$

The *weak equality symbol* "$\approx$" means that this is numerically restricted to zero on the surface where the constraints hold. Since P_1 does not enter equation (2.8), it is necessary to generate further constraints via $\Phi_3 : \dot{\Phi}_2 = \{\Phi_2, H\}_{PB}$ etc.

Although it can be shown for a general Lagrangian with second order derivatives in time (which is done in [8]), I think it is more insightful to show the principle by means of an relatively simple example (again, from [8])

An example

Consider the Lagrangian

$$L = \frac{\dot{q}^2}{2} + \frac{(\ddot{q} - \lambda)^2}{2}. \tag{2.9}$$

This is a non-degenerate Lagrangian that contains a second time derivative of the variable q. The canonical coordinates are then

$$Q_1 \equiv q \qquad \longleftrightarrow \qquad P_1 \equiv \dot{q} - q^{(3)} + \dot{\lambda} \quad ; \tag{2.10}$$

$$Q_2 \equiv \dot{q} \qquad \longleftrightarrow \qquad P_2 \equiv \ddot{q} - \lambda \quad ; \tag{2.11}$$

$$Q_3 \equiv \lambda \qquad \longleftrightarrow \qquad P_3 \equiv 0 \tag{2.12}$$

and the total Hamiltonian is

$$H_T = P_1 Q_2 + P_2 Q_3 + \frac{P_2^2}{2} - \frac{Q_2^2}{2} + u_1 \Phi_1, \tag{2.13}$$

where $\Phi_1 : P_3 = 0$ is the primary constraint. The other (secondary) constraints then follow:

$$\Phi_2 \equiv \dot{\Phi}_1 = \{\Phi_1, H\}_{PB} : \quad -P_2 \qquad \approx 0 \quad ; \tag{2.14}$$

$$\Phi_3 \equiv \dot{\Phi}_2 = \{\Phi_2, H\}_{PB} : \quad P_1 - Q_2 \qquad \approx 0 \quad ; \tag{2.15}$$

$$\Phi_4 \equiv \dot{\Phi}_3 = \{\Phi_3, H\}_{PB} : \quad -Q_3 P_2 \qquad \approx 0 \quad . \tag{2.16}$$

[1] The difference between primary and secondary constraints is not in the physics of the constraints. It only refers to the way the constraints are found. Primary conditions are directly found from the canonical momenta. Secondary conditions follow from the consistency relations [12].

These constraints can now be inserted into the Hamiltonian. The combination (Φ_1, Φ_4) will reduce (Q_3, P_3) and (Φ_2, Φ_3) will reduce (Q_2, P_2). The reduced Hamiltonian then becomes

$$H_R = P_1^2 - \frac{P_1^2}{2} = \frac{P_1^2}{2}. \tag{2.17}$$

We have now lost the problematic linear term in P_1 and this Hamiltonian is stable. The effect of doing this was that the dimension of phase space has been reduced from (Q_1, Q_2, P_1, P_2) to (Q_1, P_1). This is also what is happens in [8] for the general case. The only way to make the Hamiltonian of a system whose Lagrangian depends on second order time derivatives stable, is to reduce the dimension of phase space by means of constraints. By doing so, some Lagrangians with second order time derivatives are in fact stable. It should however be noted that this is not possible for all Lagrangians. Only in the case that constraints can be found that make the terms linear in momentum drop out of the Hamiltonian, this can be achieved.

2.1.2 Constraints in the Lagrangian Formalism

In the Lagrangian formalism a different approach is used. This again is an algorithm that find the constraints in a system. The algorithm for a Lagrangian $L = L(q_i, \dot{q}_i)$ is as follows [1, 13]:

First one finds the equations of motion by use of the Euler-Lagrange equation and defines them as

$$E_i^0 = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \equiv W_{ij}^0(q, \dot{q})\, \ddot{q}^j + K_i^0(q, \dot{q}). \tag{2.18}$$

This system is constrained when some linear combination of these equation is not of second order, meaning there exist left null vectors $\lambda^i(q, \dot{q})$ to W_{ij}^0 [1].

$$\begin{aligned} \lambda^i E_i^0 &= \lambda^i W_{ij}^0 + \lambda^i K_i^0 \\ &= \lambda^i K_i^0 = 0. \end{aligned} \tag{2.19}$$

These are called the *Lagrangian constraints* $\psi_{q,1}(q, \dot{q})$, where we only look at the functionally independent ones. The procedure so far is known as step 0.

In step 1 the constraints are demanded to be preserved under time evolution

$$\frac{d}{dt}\psi_{q,1} = \frac{\partial \psi_{q,1}}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial \psi_{q,1}}{\partial q^i}\dot{q}^i = 0. \tag{2.20}$$

These are added to the equations of motion 2.18

$$E_i^1 = \begin{pmatrix} E_i^0 \\ \frac{d}{dt}\psi_{q,1} \end{pmatrix} = W_{ij}^1(q, \dot{q})\ddot{q}^j + K_i^1(q, \dot{q}) = 0. \tag{2.21}$$

Again, there are constraints when there is a linear combination of equations that is not of second order, meaning we are looking for left null vectors of W_{ij}^1, which

give us new constraints $\psi_{q,2}$. These are then again demanded to be preserved under time evolution, etc.

These steps are repeated until no new constraints (that are not gauge identities) are found. Let me illustrate the procedure with an example from [13].

An example

Consider the Lagrangian

$$L(q_i, \dot{q}_i) = \dot{q}_1 q_2 - \dot{q}_2 q_1 - (q_1 - q_2) q_3. \tag{2.22}$$

The Euler-Lagrange equations give

$$E^0 = \begin{pmatrix} 2\dot{q}_1 + q_3 \\ -2\dot{q}_1 - q_3 \\ q_1 - q_2 \end{pmatrix} = W^0 \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} + K^0 = K^0. \tag{2.23}$$

Since there are no second derivatives, the lef null vectors of W^0 are the trivial

$$\lambda_1 = (1, 0, 0), \qquad \lambda_2 = (0, 1, 0), \qquad \lambda_3 = (0, 0, 1). \tag{2.24}$$

The contraction of E^0 with the nulvectors give

$$\lambda_i E^i = \lambda_i K^i = 0 \qquad \Rightarrow \qquad K^0 = 0 \tag{2.25}$$

and thus the first three constraints are

$$\psi_i : K^0_i = 0 \qquad i = 1, 2, 3. \tag{2.26}$$

In step 1 these constraints must be preserved under time evolution, so their time-derivatives are added to the equations of motion.

$$E^1 = \begin{pmatrix} E^0 \\ \frac{d}{dt}\psi_i \end{pmatrix} = \begin{pmatrix} 2\dot{q}_1 + q_3 \\ -2\dot{q}_1 - q_3 \\ q_1 - q_2 \\ 2\ddot{q}_2 + \dot{q}_3 \\ 2\ddot{q}_1 - \dot{q}_3 \\ \dot{q}_1 - \dot{q}_2 \end{pmatrix} \tag{2.27}$$

$$\begin{aligned} E^1 &= W^1 \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} + K^1 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} + \begin{pmatrix} 2\dot{q}_1 + q_3 \\ -2\dot{q}_1 - q_3 \\ q_1 - q_2 \\ \dot{q}_3 \\ -\dot{q}_3 \\ \dot{q}_1 - \dot{q}_2 \end{pmatrix}. \end{aligned} \tag{2.28}$$

The left null vectors of W^1 are

$$\begin{aligned} \lambda_1 &= (1,0,0,0,0,0) & \lambda_2 &= (0,1,0,0,0,0) \\ \lambda_3 &= (0,0,1,0,0,0) & \lambda_4 &= (0,1,0,0,0,1). \end{aligned} \tag{2.29}$$

Again, the contraction $\lambda_i E^i = 0$ is made to get the constraints. The first three null vectors give the same constraints as in step 0. The null vector λ_4 gives a new constraint

$$\psi_4 : \frac{d}{dt}\psi_3 = \dot{q}_1 - \dot{q}_2 = 0. \tag{2.30}$$

It can be checked that this last constraint is a gauge identity, meaning the algorithm stops.

Second order Lagrangians

When working with second order Lagrangians $L = L(q_i, \dot{q}_i, \ddot{q}_i)$, we in general have N equations of 4th order, meaning $4N$ degrees of freedom. In [1], the Lagrangian constraint analyses is used in the case of a second order Lagrangian. I shall use this analyses to show the connection to the equations of motion. In this analyses all the variables are treated on the same footing, meaning constraints always come in packages of N. Since this algorithm is for first order Lagrangians, to analyse the constraints our Lagrangian must first be put into first order by introducing auxiliary fields $A_i = \dot{q}_i$

$$L' = L(\dot{A}_i, A_i, q_i) + \mu^i(\dot{q}_i - A_i), \tag{2.31}$$

which had $3N$ degrees of freedom, where we want it to have N degrees of freedom in order to be healthy. The number of degrees of freedom (d.o.f) of the contraint system is given by:

$$\#\,\text{d.o.f}_{\text{ constraint}} = \#\,\text{d.o.f} - \frac{1}{2}\,\#\,\text{contraints} \tag{2.32}$$

The first step (step 0) of the Lagrange analysis then gives[2]:

$$E^0 = W^0 \begin{pmatrix} \ddot{A}_j \\ \ddot{q}_j \\ \ddot{\mu}_j \end{pmatrix} + K^0 \tag{2.33}$$

$$= \begin{pmatrix} L_{\dot{A}_i \dot{A}_j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{A}_j \\ \ddot{q}_j \\ \ddot{\mu}_j \end{pmatrix} + \begin{pmatrix} L_{\dot{A}_i A_j}\dot{A}_j + L_{\dot{A}_i q_j}\dot{q}_j - L_{A_i} + \mu^i \\ \dot{\mu}^i - L_{q_i} \\ -(\dot{q}_i - A_i) \end{pmatrix}. \tag{2.34}$$

The contraints are then

$$\begin{aligned} \psi^0_{q_i} &= K^0_{q_i} = \dot{\mu}^i - L_{q_i} = 0 \\ \psi^0_{\mu^i} &= K^0_{\mu^i} = -(\dot{q}_i - A_i) = 0. \end{aligned} \tag{2.35}$$

[2]Here I use index notation for derivatives, i.e. $\frac{\partial L}{\partial q} \equiv L_q$.

In general the analysis stops here, because no new constraints are generated. Finding $2N$ constraints here means that we still have $3N - \frac{1}{2}(2N) = 2N$ degrees of freedom. It is found that the only new ones are generated when the *primary condition*

$$L_{\dot{A}_i \dot{A}_j} = 0 \tag{2.36}$$

holds. In that case there is a third constraint in the this step

$$\psi^0_{C_i} = K_{A_i} = L_{\dot{A}_i A_j} \dot{A}_j + L_{\dot{A}_i q_j} \dot{q}_j - L_{A_i} + \mu^i, \tag{2.37}$$

bringing the total to $3N$ constraints. Since there are still N short to a healthy number of degrees of freedom, a further step is needed. This then gives

$$W^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ L_{\dot{A}_i A_j} - L_{A_i \dot{A}_j} & L_{\dot{A}_i \phi_j} & 0 \\ -L_{q_i \dot{A}_j} & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad K^1 = \begin{pmatrix} K^0_{A_i} \\ K^0_{q_i} \\ K^0_{\mu^i} \\ K^1_{C_i} \\ K^1_{q_i} \\ K^1_{\lambda^i} \end{pmatrix}. \tag{2.38}$$

There are an additional N constraints if and only if the *secondary condition*

$$L_{\dot{A}_i A_j} - L_{A_i \dot{A}_j} = 0 \tag{2.39}$$

holds. One could in principle continue, but we now have sufficient constraints to have no more Ostrogradski ghost degrees of freedom.

Going back to the original variables, there are thus two conditions[3] on the system:

$$\begin{aligned} \text{Primary condition} &: L_{\ddot{q}_i \ddot{q}_j} = 0 \\ \text{Secondary condition} &: L_{\ddot{q}_i \dot{q}_j} - L_{\dot{q}_i \ddot{q}_j} = 0. \end{aligned} \tag{2.40}$$

2.2 Equations of motion

Looking now at the equations of motion, one can see that they in general look like

$$\begin{aligned} E_{q_i} = L_{\ddot{q}_i \ddot{q}_j} \, q^{(4)}_j &+ (L_{\ddot{q}_i \dot{q}_j} - L_{\dot{q}_i \ddot{q}_j}) \, q^{(3)}_j \\ &+ \text{other third order terms} \\ &+ \text{lower order terms} = 0. \end{aligned} \tag{2.41}$$

If the primary condition $L_{\ddot{q}_i \ddot{q}_j} = 0$ is satisfied, the fourth order term disappears and only the third order terms specifically stated in the equation above remain.[4]

[3] Notice a difference between conditions and constraints!

[4] The fact that the others drop out can be seen from the fully worked out equation of motion in the appendix, equation A.20. All these terms contain the primary condition. These ones that remain are marked in blue, the ones that drop out in green.

The third order terms we are left with will drop out if the secondary condition $L_{\ddot{q}_i\dot{q}_j} - L_{\dot{q}_i\ddot{q}_j} = 0$ is satisfied. If and only if both these conditions hold, the equations of motions will be strictly of second order. Note that in the case of a single variable, the secondary condition is automatically met. [1]

For the equations of motion to remain up to second order, the primary condition implies that the $\ddot{q}$ term must enter the Lagrangian linearly, in order to avoid $q^{(4)}$ terms in the equations of motion. This means the Lagrangian is of the form[1]:

$$L(q_i, \dot{q}_i, \ddot{q}_i) = \ddot{q}_i f^i(q_i, \dot{q}_i) + g(q_i, \dot{q}_i). \tag{2.42}$$

For a single variable, this can be rewritten[5] in the form

$$L(q, \dot{q}, \ddot{q}) = \frac{d}{dt}F(q, \dot{q}) - \frac{\partial F(q, \dot{q})}{\partial q}\dot{q} + g(q, \dot{q}). \tag{2.43}$$

with $\frac{\partial F}{\partial \dot{q}} = f$. The only term that includes a second derivative of q is the first term. But since this is a total derivative, this will not be of any consequence in the equations of motion. This also implies that the third order terms are not existing for the equations of motion when $\ddot{q}$ enters linearly, which can also be seen as the fact that third order terms entering through the second term in equation 2.1 are exactly cancelled by those entering through the last term.

For the multi-variable case there is the additional secondary condition, which implies $\frac{\partial f^i}{\partial \dot{q}_j} = \frac{\partial f^j}{\partial \dot{q}_i}$. From this there is again an $F(q_i, \dot{q}_j)$ such that $\frac{\partial F}{\partial \dot{q}_i} = f^i$ and giving the Lagrangian in the form.

$$L(q_i, \dot{q}_i, \ddot{q}_i) = \frac{d}{dt}F(q_i, \dot{q}_i) - \frac{\partial F(q_i, \dot{q}_i)}{\partial q_j}\dot{q}_j + g(q_i, \dot{q}_i). \tag{2.44}$$

So we found a Lagrangian of second order in time derivatives that does lead to workable equations of motion. However, on further inspection, we see also that this Lagrangian only includes functions of q and $\dot{q}$ and therefore it is of trivial form.

The link between the Hamiltonian formalism and the equations of motion is far less straightforward than in the Lagrangian formalism. It is therefore not included in this project. However, I can say something about the equations of motion themselves. In the case where there are constraints, the Hamilton equations are given by [12]

$$\dot{P}_j = -\frac{\partial H}{\partial Q_j} - \sum_k u_k \frac{\partial \Phi_k}{\partial Q_j}, \qquad \dot{Q}_j = \frac{\partial H}{\partial P_j} + \sum_k u_k \frac{\partial \Phi_k}{\partial P_j}, \qquad \Phi_k(Q, P) = 0 \tag{2.45}$$

which are pairs of first order differential equations, which can be combined to second order differential equations. So, although not fully derived, we can get a feeling as to why a constraint theory can be healthy. The constraints themselves can get rid of the ghost degrees of freedom and the pairs of first order Hamilton equations make sure there are the usual amount of initial conditions required.

[5] Using a simple chain rule : $\frac{d}{dt}F(q, \dot{q}) = \frac{\partial F(q,\dot{q})}{\partial q}\dot{q} + \frac{\partial F(q,\dot{q})}{\partial \dot{q}}\ddot{q}$.

2.3 Conclusion on second order Lagrangians in classical mechanics

In this chapter we saw that in the case of classical mechanics, a system which is described by a Lagrangian containing second order time derivatives must have constraints in order to produce second order equation of motion. These constraints make sure that there are no unwanted ghost degrees of freedom.

There are two approaches for these constraints. In the Lagrangian formalism there is an algorithm to find the necessary constraints to reduce the number of degrees of freedom. The two conditions found can be linked one to one to the Euler-Lagrange equations of motion, making sure that third and fourth order terms drop out of it. In the Hamiltonian formalism the link between the constraints and the equations of motion is far more difficult to make and lies beyond the scope of this project. However, in the Hamiltonian formalism, the constraints reduce the dimension of phase space (by relating canonical coordinates to each other) and in this way one gets rid of the unwanted degrees of freedom. For each canonical coordinate that is left, the Hamilton equations produce a pair of first order differential equations that describe the motion.

Finally, we also saw that for a second order Lagrangian to produce second order equations of motion, it must be linear in $\ddot{q}$. This immediately implies that the Lagrangian can be rewritten into a normal first order Lagrangian and a total derivative, meaning no extra information is added. Ostrogradski told us that we can not write higher order Lagrangians. But, although trivial, we can write second order Lagrangians as long as they are linear in $\ddot{q}$. Does this mean Ostrogradski was (partially) wrong? No. When writing the Hamiltonians (1.26) and (1.29), it was assumed the theory is non-degenerate (i.e. linear in $\ddot{q}$, resp. $q^{(n)}$). The findings in this section are thus in line with Ostrogradski's theorem.

This leads to the general conclusion for the case of Lagrangian containing up to second order time derivatives in the classical mechanics case: the second derivative terms are either trivial or fatal [8].

Chapter 3

Second Order Lagrangians for Scalar Fields

For scalar field, as for particles, it is again not al all obvious that there are second order Lagrangians that lead to equations of motion that include up to second order derivatives of the field. Analogue to the literature [4, 7], π is the scalar field, and the derivatives of the field are denoted by:

$$\pi_\mu \equiv \partial_\mu \pi, \qquad \pi_{\mu\nu} \equiv \partial_\nu \partial_\mu \pi, \qquad \pi_{\mu\nu\rho} \equiv \partial_\rho \partial_\nu \partial_\mu \pi, \qquad \text{etc.} \tag{3.1}$$

The Euler-Lagrange equation

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \pi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \pi_\mu} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \pi_{\mu\nu}} \right) = 0 \tag{3.2}$$

would in general give rise to third and fourth order derivative terms, due to the last term. So far this is completely analogue to the Lagrangians in classical mechanics that we saw in the last chapter. However, there are known Lagrangians that do result in workable equations of motion that are not trivial. A set of these are called *Galileons*. Galileons have the properties:

- The Lagrangian contains up to second derivatives of the scalar field π;
- The equations of motion are polynomial in second order derivatives of π;
- The equations of motion contain only second derivatives of π.

The last property is required for the Galileons to be invariant under the Galilean transformation $\pi \to \pi + c + v_\mu x^\mu$ (hence the naming "Galileons") as well as under the usual Lorentz transformations. There are also theories that have the same properties as Galileons, with the exception that their equations of motion also contain the field π and its first derivative π_μ. Because these are more general theories, they are referred to as *generalised Galileons*. Because their equations of motion are no longer restricted to second order derivatives

only, they are not any more invariant under the Galilean transformation (but are still Lorentz invariant). In this chapter, these (generalised) Galileons will be analysed.

3.1 How do Galileons work?

Let's start with Galileons. The way to get rid of the problematic third and fourth order terms is to work with Lagrangians that have specific properties. When working with Lagrangians for scalar fields with up to second order time-derivatives, we can construct the general Lagrangian [4, 7]:

$$\mathcal{L} = \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n}. \tag{3.3}$$

The tensor $\mathcal{T}$ is a functional tensor that only depends on π and it's first derivative

$$\mathcal{T}_{(2n)} = \mathcal{T}_{(2n)}(\pi, \pi_\mu) \tag{3.4}$$

and is totally anti-symmetric in it's first n indices $\{\mu_1 \cdots \mu_n\}$ and separately totally anti-symmetric in it's last n indices $\{\nu_1 \cdots \nu_n\}$. The index n gives the number of second derivatives of π. It is this specific form of $\mathcal{T}_{(2n)}$ that makes sure that all terms that we do not want will not end up in the equations of motion. Before this can be shown in detail, it is important to first take a closer look at what an anti-symmetric tensor does when it is contracted with other tensors.

3.1.1 Symmetric and antisymmetric tensors

To be more specific, let's look at what an antisymmetric tensor does when working on an symmetric tensor.

Let $A^{\mu\nu}$ be an antisymmetric tensor and $S_{\mu\nu}$ a symmetric tensor. Then

$$A^{\mu\nu} S_{\mu\nu} = A^{\mu\nu} S_{\nu\mu} = -A^{\nu\mu} S_{\nu\mu} = -A^{\mu\nu} S_{\mu\nu}. \tag{3.5}$$

In the first two steps, the indices of respectively A and S were interchanged. For S this does nothing, but for A a minus sign is picked up. In the last step the renaming $\mu \to \nu, \nu \to \mu$ was used, which of course is of no consequence at all for the physics. The fact that the object is then minus itself means it must be zero.

3.1.2 Getting rid of the problematic terms

This property is the key of getting rid of most of the unwanted terms in the equations of motion. To illustrate how this comes to be, let's look at the structure of the terms generated by the Euler-Lagrange equation. Because of the lengthiness of the derivation, only the result will be shown in the equations

below. In appendix A.3.1 the full derivation is shown. Also the indices of $\mathcal{T}_{(2n)}$ will be left out all through[1].

$$\begin{aligned}\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial\pi_\rho}\right) = \partial_\rho\left(\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\rho}\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n} \\ +\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\rho}\pi_{\mu_1\nu_1\rho}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n} \\ +\cdots+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\rho}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\pi_{\mu_n\nu_n\rho}\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\partial_\rho\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial\pi_{\rho\sigma}}\right) = \\ n\cdot\Bigg(&\partial_\rho\left(\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\right)\pi_{\alpha\sigma}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma\rho}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}} \\ &+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma}\pi_{\mu_1\nu_1\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}\rho} \\ &+\partial_\rho\left(\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\right)\pi_\sigma\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_{\sigma\rho}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}} \\ &+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_\sigma\pi_{\mu_1\nu_1\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_\sigma\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}\rho} \\ &+\partial_\rho\left(\mathcal{T}_{(2n)}\right)\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma\rho}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}} \\ &+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\rho} \\ &+\quad\cdots \\ &+\partial_\rho\left(\mathcal{T}_{(2n)}\right)\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma}+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\rho}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma} \\ &+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma}+\cdots+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-2}\nu_{n-2}\rho}\pi_{\mu_{n-1}\nu_{n-1}\sigma} \\ &+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma\rho}\Bigg)\end{aligned} \tag{3.7}$$

In the above, the problematic terms are coloured. There are basically two ways for the third order terms to disappear. The first is quite similar to what we saw in the classical mechanics case. Some third order terms that appear in the second term (3.7) of the Euler-Lagrange equations cancel exactly to those appearing in the first term (3.6) of the Euler-Lagrange equation. These are the terms coloured in blue (note that this concerns n terms). The other way for the third order as well as the fourth order terms to drop out is through the contraction of third and fourth order derivatives with the anti-symmetric $\mathcal{T}_{(2n)}$ tensor. Since derivatives commute, the π's are symmetric under interchange of two indices. For example, the term

$$\frac{\partial\mathcal{T}_{(2n)}^{\mu_1\ldots\mu_{n-1}\rho\,\nu_1\ldots\nu_{n-1}\sigma}}{\partial\pi_\alpha}\pi_{\alpha\sigma}\pi_{\mu_1\nu_1\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}} \tag{3.8}$$

[1]The proper indices on the $\mathcal{T}_{(2n)}$'s in equation 3.6 are $\{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\}$ and the proper indices on the $\mathcal{T}_{(2n)}$'s in equation 3.7 are $\{\mu_1\ldots\mu_{n-1}\rho\,\nu_1\ldots\nu_{n-1}\sigma\}$, where the indices ρ and σ are included in the antisymmetric set of respectively the μ_i's and ν_i's.

contains the indices μ_1 and ρ on the second π, under which $\mathcal{T}_{(2n)}$ is antisymmetric. Thus the entire term must equal zero. This hold for all the term coloured red in the above.

The condition that $\mathcal{T}_{(2n)}$ is totally antisymmetric in it's first and last n indices, is thus sufficient to ensure that the equations of motion are of second order in derivatives. In [7], it reverse is also proven. The most general Galileon has this condition on it's indices.

3.2 Constructing Galileon Lagrangians

There are a number of ways to construct the Galileon action. They depend on the choice of $\mathcal{T}_{(2n)}$. In this section I will show three of these Lagrangians that are often found in the literature [4, 7] and I will show that although these Lagrangians are different, they only differ by a total derivative through an integration by parts and therefore give the same equations of motion. Some definitions will hold for a general D dimensions, but when working things out further, I will restrict the analyses to $D = 4$. I will also restrict to flat space-time only.

3.2.1 Constructing the first Lagrangian

First of all, a new tensor must be defined:

$$\mathcal{A}_{(2m)}^{\mu_1\mu_2\ldots\mu_m\nu_1\nu_2\ldots\nu_m} \equiv \frac{1}{(D-m)!}\,\varepsilon^{\mu_1\mu_2\ldots\mu_m\sigma_1\sigma_2\ldots\sigma_{D-m}}\,\varepsilon^{\nu_1\nu_2\ldots\nu_m}{}_{\sigma_1\sigma_2\ldots\sigma_{D-m}}, \tag{3.9}$$

where ε is the totally anti-symmetric Levi-Civita tensor. This means that $\mathcal{A}_{(2m)}$ is antisymmetric in its first m as wel as, separately, in its last m indices. Just as the $\mathcal{T}_{(2n)}$ tensor is.

A first way to construct the Lagrangian is then [4, 7]

$$\begin{aligned}\mathcal{L}_N^{\mathrm{Gal},1} &= \left(\mathcal{A}_{(2n+2)}^{\mu_1\ldots\mu_{n+1}\nu_1\ldots\nu_{n+1}}\pi_{\mu_{n+1}}\pi_{\nu_{n+1}}\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\\ &\equiv \mathcal{T}_{(2n),\mathrm{Gal},1}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}.\end{aligned} \tag{3.10}$$

The index N denotes the number of π's appearing in the Lagrangian. Since there are n second derivatives of π and an additional 2 first derivatives of π, this is given by

$$N = n + 2. \tag{3.11}$$

The index "Gal, 1" simply indicates that this is the first possibility for an Galilean Lagrangian. The largest number of products of fields (and thus the value of N) is $D + 1$. This means that for this Lagrangian, we can construct five Galileon Lagrangians in 4-dimensional Minkowski space, four of which are non-trivial[2]:

[2] An example of a derivation from equation A.41 can be found in appendix A.3.2

$$
\begin{aligned}
\mathcal{L}_2^{\mathrm{Gal},1} &= \mathcal{A}_{(2)}^{\mu_1\nu_1}\pi_{\mu_1}\pi_{\nu_1} \\
&= -\pi^\mu\pi_\mu \qquad (3.12)\\
\mathcal{L}_3^{\mathrm{Gal},1} &= \mathcal{A}_{(4)}^{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_2}\pi_{\nu_2}\pi_{\mu_1\nu_1} \\
&= \pi^\mu\pi^\nu\pi_{\mu\nu} - \pi^\mu\pi_\mu\Box\pi \qquad (3.13)\\
\mathcal{L}_4^{\mathrm{Gal},1} &= \mathcal{A}_{(6)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\pi_{\mu_3}\pi_{\nu_3}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2} \\
&= -(\Box\pi)^2(\pi_\mu\pi^\mu) + 2(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_\nu) \\
&\quad + (\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) - 2(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho) \qquad (3.14)\\
\mathcal{L}_5^{\mathrm{Gal},1} &= \mathcal{A}_{(8)}^{\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}\pi_{\mu_4}\pi_{\nu_4}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3} \\
&= -(\Box\pi)^3(\pi_\mu\pi^\mu) + 3(\Box\pi)^2(\pi_\mu\pi^{\mu\nu}\pi_\nu) + 3(\Box\pi)(\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) \\
&\quad - 6(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho) - 2(\pi_\mu{}^\nu\pi_\nu{}^\rho\pi_\rho{}^\mu)(\pi_\lambda\pi^\lambda) \\
&\quad - 3(\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^{\rho\lambda}\pi_\lambda) + 6(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^{\rho\lambda}\pi_\lambda). \qquad (3.15)
\end{aligned}
$$

Equations of motion

These Lagrangians lead to the following equations of motion:

$$
\begin{aligned}
\mathcal{E}_2 &= \Box\pi \qquad (3.16)\\
\mathcal{E}_3 &= (\Box\pi)^2 - \pi_{\mu\nu}\pi^{\mu\nu} \qquad (3.17)\\
\mathcal{E}_4 &= (\Box\pi)^3 - 3\Box\pi\pi_{\mu\nu}\pi^{\mu\nu} + 2\pi^\mu{}_\nu\pi^\nu{}_\rho\pi^\rho{}_\mu \qquad (3.18)\\
\mathcal{E}_5 &= (\Box\pi)^4 - 6(\Box\pi)^2\pi_{\mu\nu}\pi^{\mu\nu} + 3(\pi_{\mu\nu}\pi^{\mu\nu})^2 \qquad (3.19)\\
&\quad + 8(\Box\pi)\pi^\mu{}_\nu\pi^\nu{}_\rho\pi^\rho{}_\mu - 6\pi^\mu{}_\nu\pi^\nu{}_\rho\pi^\rho{}_\sigma\pi^\sigma{}_\mu.
\end{aligned}
$$

The equations above can be derived by use of the Euler-Lagrange equation, but for this specific Lagrangian $\mathcal{L}_N^{\mathrm{Gal},1}$ a general formula can be found:

$$
\mathcal{E}_N = -\mathcal{A}_{2n+2}^{\mu_1\ldots\mu_{n+1}\nu_1\ldots\nu_{n+1}}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n+1}\nu_{n+1}}, \qquad (3.20)
$$

where

$$
\mathcal{E} = -N \times \mathcal{E}_N = 0. \qquad (3.21)
$$

The derivation of the first two equations of motion is fully worked out using both methods in appendix A.3.3.

3.2.2 Other Lagrangians

As said before, there is more than one possibility to construct a Galileon Lagrangian. Two other Lagrangian found in the literature [4, 7] are:

$$\begin{aligned}\mathcal{L}_N^{\text{Gal},2} &= \left(\mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_{\mu_1}\pi_\lambda\pi^\lambda{}_{\nu_1}\right)\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}, \qquad (3.22)\\ &\equiv \mathcal{T}_{(2n),\text{Gal},2}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\\ \mathcal{L}_N^{\text{Gal},3} &= \left(\mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_\lambda\pi^\lambda\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n} \qquad (3.23)\\ &\equiv \mathcal{T}_{(2n),\text{Gal},3}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}.\end{aligned}$$

These Lagrangians can be related to each each other. To do so, one must first define

$$J_N^\mu = \pi_\lambda\pi^\lambda\mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n}\pi_{\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}. \qquad (3.24)$$

From this we get

$$\mathcal{L}_N^{\text{Gal},2} = -\frac{1}{2}\mathcal{L}_N^{\text{Gal},3} + \frac{1}{2}\partial_\mu J_N^\mu. \qquad (3.25)$$

Using the properties of $\mathcal{A}_{(2n)}$ it can be proven[3] that there is a relation between the three Galileon Lagrangians:

$$(N-2)\mathcal{L}_N^{\text{Gal},2} = \mathcal{L}_N^{\text{Gal},3} - \mathcal{L}_N^{\text{Gal},1}, \qquad (3.26)$$

from which it follows that

$$\begin{aligned}\mathcal{L}_N^{\text{Gal},1} &= \frac{N}{2}\mathcal{L}_N^{\text{Gal},3} - \frac{N-2}{2}\partial_\mu J_N^\mu, \qquad (3.27)\\ \mathcal{L}_N^{\text{Gal},1} &= -N\mathcal{L}_N^{\text{Gal},2} + \partial_\mu J_N^\mu. \qquad (3.28)\end{aligned}$$

Equations (3.25), (3.27) and (3.28) show that the three Lagrangians are all equal up to a total derivative and a constant. This means that all three Lagrangians generate the same equations of motion and that the ones we saw in equations (3.16) − (3.19) also hold for $\mathcal{L}_N^{\text{Gal},2}$ and $\mathcal{L}_N^{\text{Gal},3}$.

Finally there is also the formulation where the Lagrangian is only a function of π and π_{μ_ν}:

$$\mathcal{L}_N^{\text{Gal},4} = -\pi\,\mathcal{A}_{(2n)}^{\mu_1,\ldots\mu_{n+1}\nu_1\ldots\mu_{n+1}}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n+1}\nu_{n+1}}, \qquad (3.29)$$

where the connection to the other Lagrangians is

$$\mathcal{L}_N^{\text{Gal},3} = \frac{1}{N}\mathcal{L}_N^{\text{Gal},4} + \text{total derivative} \qquad (3.30)$$

The most general Galileon Lagrangian is a linear combination of the Galileon Lagrangians shown above:

$$\mathcal{L}^{\text{Gal,general}} = \sum_j C_j\left(\mathcal{L}_N^{\text{Gal},i}\right)_j \qquad (3.31)$$

[3] See Appendix A.3.4 for the derivations of the equations relating the three Lagrangians.

3.3 Generalised Galileons

The Lagrangians analysed for far are all Galileons. Their equations of motion only contain second order derivatives $\pi_{\mu\nu}$. For completeness, let's take a quick look at generalised Galileons. They produce equations of motion that contain *up to* second order derivatives.

The most general theory that satisfies the conditions for generalised Galileons in D dimensions are proved to be given by [4, 7]

$$\mathcal{L} = \sum_{n=0}^{D-1} \tilde{\mathcal{L}}_n\{f_n\}, \tag{3.32}$$

where f_n are arbitrary functions of π and $\pi_\mu \pi^\mu \equiv X$. The curly brackets indicate that $\tilde{\mathcal{L}}_n\{f\}$ is an functional of f, which is given by

$$\begin{aligned} \tilde{\mathcal{L}}_n\{f\} &\equiv f(\pi, X)\mathcal{L}^{\mathrm{Gal},3}_{N=n+2} && (3.33)\\ &= f(\pi, X)\left(X\mathcal{A}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}_{(2n)}\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}. && (3.34) \end{aligned}$$

The equations of motion corresponding to each $\tilde{\mathcal{L}}_n\{f\}$ are

$$\begin{aligned} 0 = &2(f + Xf_X)\mathcal{E}_N + 4(2f_X + Xf_{XX})\mathcal{L}^{\mathrm{Gal},2}_{N+1}\\ &+ X[2Xf_{X\pi} - (n-1f_\pi)]\mathcal{E}_{N-1}\\ &- n(4Xf_{X\pi} + 4f_\pi)\mathcal{L}^{\mathrm{Gal},2}_{N+1} - nXf_{\pi\pi}\mathcal{L}^{\mathrm{Gal},1}_{N-1}. \end{aligned} \tag{3.35}$$

When the function f is constant, these equations reduce to $\mathcal{E}_N = 0$ as we saw in (3.20). For non-constant f they depend on $\pi_{\mu\nu}$, π_μ as well as π itself. The full prove of the above is quite extensive and can be found in [7].

3.4 Do these Lagrangians contain new information?

There is still one question for these Lagrangians that needs to be addressed. In the case of second order Lagrangians in classical mechanics, the Lagrangian could be rewritten as a normal first order Lagrangian plus a total derivative containing the second derivative terms:

$$L(q_i, \dot{q}_i, \ddot{q}_i) = \frac{d}{dt}F(q_i, \dot{q}_i) - \frac{\partial F(q_i, \dot{q}_i)}{\partial q_j}\dot{q}_j + g(q_i, \dot{q}_i). \tag{3.36}$$

This meant that these Lagrangians contained no new information, since the total derivative does not influence the equations of motions. But what about the second order Lagrangians for scalar fields? Can they be written in this way and do they really give something new?

The answer is yes, they do give new information. This has to do with the fact that in the classical mechanics case, there are only time derivatives $\frac{d}{dt}$. In the

case of scalar field however, we are working with derivatives ∂_μ. These contain a time derivative, but also three[4] space derivatives. There is in principle nothing that holds us from rewriting the Lagrangian with a total time-derivative:

$$\mathcal{L}(\pi, \pi_\mu, \pi_{\mu\nu}) \sim \frac{\partial}{\partial t} F(\pi, \pi_\mu) + \bar{\mathcal{L}}(\pi, \dot{\pi}, \pi_x, \pi_{xx}, \dot{\pi}_x, \dots). \tag{3.37}$$

However, it can clearly be seen that $\bar{\mathcal{L}}$ does not treat space and time on equal footing and is therefore not Lorentz invariant. This is a property that we want in these theories to have in order to be useful. The Galileons are in fact the special cases in which the Lagrangians can written with second derivatives $\partial_\mu \partial_\nu$ and are therefore Lorentz invariant.

It is because of this derivatives ∂_μ in stead of only time derivatives $\frac{d}{dt}$ that the Galileons, in contrast to their classical mechanical counterparts, do contain something new.

3.5 Conclusion on second order Lagrangians for scalar fields

We have so far seen that for scalar fields in flat space-time, we can construct Galileon Lagrangians that have the properties:

- The Lagrangian can contain the field π as well as its first and second derivatives π_μ and $\pi_{\mu\nu}$;
- The equations of motion are polynomials in second order derivatives of the fields and contain no first order derivatives or the undifferentiated field.

These Lagrangians can be written in general as

$$\mathcal{L} = \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n}, \qquad \text{with} \qquad \mathcal{T}_{(2n)} = \mathcal{T}_{(2n)}(\pi, \pi_\mu), \tag{3.38}$$

where the key feature lies in the fact that $\mathcal{T}_{(2n)}$ is antisymmetric in both its first n and its last n indices. This property makes sure that no unwanted (higher derivative) terms end up in the equations of motion.

The Lagrangians that have these properties are known as *Galileons*. In the literature [4, 7] there are multiple ways to construct the Lagrangian explicitly that are used often. These Lagrangians are in fact equal up to a total derivative and therefore give the same equations of motion.

A generalisation can be made such that the equations of motion contain *up to* (in stead of only) second order derivatives of the field. These Lagrangians are called *generalised Galileons* and are linear combination of a Galileon multiplied by an arbitrary function of π and $\pi_\mu \pi^\mu$.

Finally we saw that, in contradiction to the classical mechanics case, these Galileon do have new properties compared to the normal first order Lagrangians.

[4] Remember that we are looking at $D = 4$ only.

Chapter 4

Higher Order Lagrangians in Classical Mechanics

In section 2.2 we saw that second order Lagrangian in classical mechanics need to be linear in $\ddot{q}$, in order to give equations of motion of second order. We then saw that this linear term could be written as a total derivative, making the Lagrangian first order up to a total derivative. In this chapter, I will analyse this principle for higher order Lagrangian in classical mechanics. I will do this by first starting at third order and then generalise this towards Lagrangian of nth order. This will hopefully give an idea of how the structure of these higher order Lagrangians work, which may be useful for the scalar field version.

4.1 Third order Lagrangians

In third order we have the Lagrangian $L = L(q, \dot{q}, \ddot{q}, \dddot{q})$. The equations of motion corresponding to this Lagrangian are given by the Euler-Lagrange equation

$$E = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial \dddot{q}}\right) = 0, \tag{4.1}$$

which in general will give rise to equations of motion containing $q, \dot{q}, \ddot{q}, \dddot{q}, q^{(4)}, q^{(5)}$ and $q^{(6)}$. The first concern is to get rid of $q^{(6)}$ term. As can be seen from the Euler-Lagrange equation, the only way for this term to appear is when $\frac{\partial L}{\partial \dddot{q}}$ is a function of $\dddot{q}$. In other words, when the Lagrangian in degenerate in $\dddot{q}$. This is avoided when $\dddot{q}$ appears linearly in the Lagrangian and is completely analogue to the second order case.

Another way to see that this must be the case (also for multiple particles) is through the Lagrangian constraint analyses. When doing this, the Lagrangian must first be put into first order by introducing auxiliary field (two in this case):

$$L(q_i, A_i, B_i, \dot{B}_i) + \mu^i(\dot{q}_i - A_i) + \nu^i(\dot{A}_i - B_i). \tag{4.2}$$

The equations of motion in step 0 then give

$$E^0 = W^0 + \begin{pmatrix} \ddot{B}_j \\ \ddot{A}_j \\ \ddot{q}_j \\ \ddot{\mu}_j \\ \ddot{\nu}_j \end{pmatrix} + K^0 \tag{4.3}$$

$$= \begin{pmatrix} L_{\dot{B}_i \dot{B}_j} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{B}_j \\ \ddot{A}_j \\ \ddot{q}_j \\ \ddot{\mu}_j \\ \ddot{\nu}_j \end{pmatrix} + \begin{pmatrix} L_{\dot{B}_i B_j} \dot{B} + L_{\dot{B}_i A_j} \dot{A} + L_{\dot{B}_i q_j} \dot{q} - L_{B_i} + \nu^i \\ -L_{A_i} + \dot{\nu}^i + \mu^i \\ \dot{\mu}^i - L_{q_i} \\ -(\dot{q}_i - A_i) \\ -(\dot{A}_i - B_i) \end{pmatrix},$$

where the first four constraints are the last four rows of K^0. Further constraints are needed in order to reach a healthy amount of degrees of freedom. This is only possible when $L_{\dot{B}_i \dot{B}_j} = 0$. If we change back to the original variables, we get the *primary condition*

$$L_{\dddot{q}_i \dddot{q}_j} = 0, \tag{4.4}$$

which immediately requires the Lagrangian to be linear in $\dddot{q}$. This gives (for the single particle case)

$$L(q, \dot{q}, \ddot{q}, \dddot{q}) = f(q, \dot{q}, \ddot{q})\, \dddot{q} + g(q, \dot{q}, \ddot{q}). \tag{4.5}$$

This linear term can be rewritten in a total derivative and

$$L(q, \dot{q}, \ddot{q}, \dddot{q}) = \frac{d}{dt} F(q, \dot{q}, \ddot{q}) - F_{\dot{q}}(q, \dot{q}, \ddot{q})\, \dddot{q} - F_q(q, \dot{q}, \ddot{q}) + g(q, \dot{q}, \ddot{q}) \tag{4.6}$$

$$= \frac{d}{dt} F(q, \dot{q}, \ddot{q}) + \tilde{g}(q, \dot{q}, \ddot{q}), \tag{4.7}$$

where $F_{\ddot{q}} = f$. When looking at the equations of motion of this Lagrangian, the total derivative can be ignored. The $\tilde{g}$ term is exactly the kind of second order Lagrangian analysed in chapter 2, where we saw that this must be linear in $\ddot{q}$, which again can be written as a total derivative:

$$L(q, \dot{q}, \ddot{q}, \dddot{q}) = \frac{d}{dt} F(q, \dot{q}, \ddot{q}) + h(q, \dot{q})\, \ddot{q} + k(q, \dot{q}) \tag{4.8}$$

$$= \frac{d}{dt} F(q, \dot{q}, \ddot{q}) + \frac{d}{dt} H(q, \dot{q}) - H_q(q, \dot{q})\, \dot{q} + k(q, \dot{q}) \tag{4.9}$$

$$= \frac{d}{dt} F(q, \dot{q}, \ddot{q}) + \frac{d}{dt} H(q, \dot{q}) + \tilde{k}(q, \dot{q}). \tag{4.10}$$

It should be stresses that although the function $\tilde{g}(q, \dot{q}, \ddot{q})$, must be linear in $\ddot{q}$, this does not mean the Lagrangian $L(q, \dot{q}, \ddot{q}, \dddot{q})$ as a whole must be linear in $\ddot{q}$. For example, the Lagrangian

$$L(q, \dot{q}, \ddot{q}, \dddot{q}) = \left(2q^2 \dot{q} \ddot{q} - 2q\dot{q}^2 \ddot{q} + 3q\ddot{q}^2\right) \dddot{q} + q^2 \ddot{q} - \dot{q}\ddot{q}^2 + \ddot{q}^3 \tag{4.11}$$

is perfectly fine, although not being linear in $\ddot{q}$. It only means that there is a specific condition on our choices of $f(q, \dot{q}, \ddot{q})$ and $g(q, \dot{q}, \ddot{q})$. The combination

$$- F_{\dot{q}}(q, \dot{q}, \ddot{q}) - F_q(q, \dot{q}, \ddot{q}) + g(q, \dot{q}, \ddot{q}) \tag{4.12}$$

must be linear in $\ddot{q}$, which it is[1].

4.2 Higher order Lagrangians

The procedure for third order Lagrangians can very easily be extended to higher order Lagrangian. Taking the Lagrangian $L = L(q, \dot{q}, \ldots, q^{(n)})$, it must be linear in $q^{(n)}$, in order to make sure no $q^{(2n)}$ terms end up in the equation of motion. This can again be written as a total derivative

$$\begin{aligned} L(q, \dot{q}, \ldots, q^{(n)}) &= f(q, \dot{q}, \ldots, q^{(n-1)})\, q^{(n)} + g(q, \dot{q}, \ldots, q^{(n-1)}) \\ &= \frac{d}{dt} F(q, \dot{q}, \ldots, q^{(n-1)}) - F_{q^{(n-2)}} q, \dot{q}, \ldots, q^{(n-1)} \\ &\qquad - \cdots - F_q(q, \dot{q}, \ldots, q^{(n-1)}) + g(q, \dot{q}, \ldots, q^{(n-1)}) \\ &= \frac{d}{dt} F(q, \dot{q}, \ldots, q^{(n-1)}) + \tilde{g}(q, \dot{q}, \ldots, q^{(n-1)}), \end{aligned} \tag{4.13}$$

where $F_{q^{(n-1)}} = f$ and where $\tilde{g}$ (but not the Lagrangian as a whole) must again be linear in $q^{(n-1)}$. This is a process that can be iterated until we get

$$\begin{aligned} L(q, \dot{q}, \ldots, q^{(n)}) = \frac{d}{dt} F(q, \dot{q}, \ldots, q^{(n-1)}) + \frac{d}{dt} G(q, \dot{q}, \ldots, q^{(n-2)}) \\ + \cdots + \frac{d}{dt} H(q, \dot{q}) + \tilde{k}(q, \dot{q}). \end{aligned} \tag{4.14}$$

This Lagrangian is then linear in $q^{(n)}$ and the appearance of derivative terms of order $n - 1$ and lower need not be linear, but series of extended versions of (4.12) must be linear in the respective derivative of q.

4.3 Conclusion on higher order Lagrangian in classical mechanics

From the above it can be concluded that the classical mechanics case (for a single variable) of higher order Lagrangian is a generalised version of what we saw for second order Lagrangian. The first condition for a Lagrangian to produce workable equations of motion is that it is linear in the highest order. This makes it possible to take this higher order into a total derivative. The same can iteratively be done for the lower order, until one remains with a normal first order Lagrangian modulo total derivative terms. Conditions need to be

[1] See appendix A.4.1 for explicit proof.

met to do so, but in theory this is possible for n^{th} order Lagrangians. Second order Lagrangians are then just the special case $n = 2$. This extends the conclusion made for the second order case in chapter 2 to: higher order terms in Lagrangians in classical mechanics are either trivial of fatal.

Chapter 5

Higher Order Lagrangians for Scalar Fields

So far, I have only looked at things that are already known. The main question for this research is whether there are possibilities to write higher than second order Lagrangians for scalar field, i.e. higher order versions of Galileons. This is a very broad question and it is not easy to find a closed set of these kinds of Lagrangians. In the short time that I had for this research, I have looked at some specific cases. These attempts are inspired on what is described in the earlier chapters about second order Lagrangians (both mechanics and fields) and higher order Lagrangians in mechanics. In this chapter I will discuss the attempts and how they were inspired. I will explain whether they are successful or fail and how this comes to be.

5.1 A first attempt

An obvious first attempt to construct a Lagrangian for scalar fields π up to third order is to promote the general Lagrangian (3.3) discussed in section 3.1 to third order. This would mean that we have

$$\mathcal{L} = \mathcal{R}_{(3n)}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n \rho_1 \dots \rho_n} \pi_{\mu_1 \nu_1 \rho_1} \cdots \pi_{\mu_n \nu_n \rho_n}, \tag{5.1}$$

where the tensor $\mathcal{R}_{(3n)}$ is fully antisymmetric in the indices $\mu_1 \dots \mu_n$ and separately in $\nu_1 \dots \nu_n$ as well as separately in $\rho_1 \dots \rho_n$. The tensor would be a functional of the field itself as well as its first and second derivatives:

$$\mathcal{R}_{(3n)} = \mathcal{R}_{(3n)}(\pi, \pi_\mu, \pi_{\mu\nu}). \tag{5.2}$$

Unfortunately, this general Lagrangian immediately fails disastrously. The Euler-Lagrange equation for third order field Lagrangians is given by

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \pi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \pi_\mu} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \pi_{\mu\nu}} \right) - \partial_\mu \partial_\nu \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial \pi_{\mu\nu\rho}} \right) = 0, \tag{5.3}$$

where the first two terms give

$$\frac{\partial \mathcal{L}}{\partial \pi} = \frac{\partial \mathcal{R}_{(3n)}}{\partial \pi} \pi_{\mu_1\nu_1\rho_1} \cdots \pi_{\mu_n\nu_n\rho_n} \quad ; \tag{5.4}$$

$$\begin{aligned} \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \pi_\alpha} \right) &= \partial_\alpha \left(\frac{\partial \mathcal{R}_{(3n)}}{\partial \pi_\alpha} \right) \pi_{\mu_1\nu_1\rho_1} \cdots \pi_{\mu_n\nu_n\rho_n} + \frac{\partial \mathcal{R}_{(3n)}}{\partial \pi_\alpha} \pi_{\mu_1\nu_1\rho_1\alpha} \cdots \pi_{\mu_n\nu_n\rho_n} \\ &\quad + \cdots + \frac{\partial \mathcal{R}_{(3n)}}{\partial \pi_\alpha} \pi_{\mu_1\nu_1\rho_1} \cdots \pi_{\mu_n\nu_n\rho_n\alpha}. \end{aligned} \tag{5.5}$$

Here the problem can immediately be spotted. In the case of second order field Lagrangians, all terms of the Euler-Lagrange equation contained second derivatives and some also a third or fourth derivative. The latter were destroyed by the antisymmetry of the tensor (the red ones in equation (3.7)) or cancelled between the terms in the Euler-Lagrange equations (the blue term in equations (3.6) and (3.7)). The fact that the remaining terms (the black ones) had only (up to) second order derivative terms was that the ∂_μ's and ∂_ν's from the Euler-Lagrange equation only acted on the tensor and left the series of $\pi_{\mu_i\nu_i}$'s untouched. This can impossibly be used in the general third order case. As can be seen for the term above, the terms where the π's remain untouched contain (obviously) third order terms. In other cases an extra derivative is added, brining us further from where we want to be. In this approach there are, for the general case, thus no terms in the equation of motion that contain up to second order derivatives only. This means that even if a tensor $\mathcal{R}_{(3n)}$ could be found that has the appropriate properties, it won't do the job, because then all terms would vanish and the equation of motion would be $0 = 0$. The exact same argument holds for a generalisation to fourth or higher order Lagrangians of this type.

There is one way possible way out of this problem of non-existing healthy terms. In the case that the Lagrangian is linear in third order, i.e.

$$\mathcal{L} = \mathcal{R}^{\alpha\beta\gamma}(\pi, \partial\pi, \partial\partial\pi)\, \pi_{\alpha\beta\gamma}, \tag{5.6}$$

there are a few terms that are healthy. The first three terms of the Euler-Lagrange equations still contain no healthy terms, but the last term does. They result from the fact that

$$\begin{aligned} \partial_\mu\partial_\nu\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial \pi_{\mu\nu\rho}} \right) &= \partial_\mu\partial_\nu\partial_\rho \left(\mathcal{R}^{\alpha\beta\gamma}(\pi, \partial\pi, \partial\partial\pi) \frac{\partial \pi_{\alpha\beta\gamma}}{\partial \pi_{\mu\nu\rho}} \right) \\ &= \partial_\mu\partial_\nu\partial_\rho \left(\mathcal{R}^{\alpha\beta\gamma}(\pi, \partial\pi, \partial\partial\pi) \delta^\mu_\alpha \delta^\nu_\beta \delta^\rho_\gamma \right) \\ &= \partial_\alpha\partial_\beta\partial_\gamma \left(\mathcal{R}^{\alpha\beta\gamma}(\pi, \partial\pi, \partial\partial\pi) \right). \end{aligned} \tag{5.7}$$

When m is the number of third derivatives in the Lagrangian, the $\frac{\partial \mathcal{L}}{\partial \pi_{\mu\nu\rho}}$ term generally has $m - 1$ third derivatives[1]. This is why only Lagrangian linear in third order can produce healthy terms. In the linear case of (5.7), the function within the derivatives is a one comparable to the general second order

[1] This can be seen as being analogue to the differentiation rule $\frac{d}{dx} x^n = n x^{n-1}$.

Lagrangian in (3.3) and can thus can be written as

$$\mathcal{R}(\pi, \partial\pi, \partial\partial\pi) = \tilde{\mathcal{R}}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}(\pi, \partial\pi)\, \pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}. \tag{5.8}$$

Healthy terms are then possible if all three derivatives in (5.7) act on the tensor $\tilde{\mathcal{R}}_{(2n)}$, although this will require restrictions on $\tilde{\mathcal{R}}_{(2n)}$.

Before looking further in this possibility, there is an issue that must be addressed. Looking back to the third order Lagrangian in classical mechanics, linear in third order meant that this third order could be taken into a total derivative. If this is also the case for fields, then linear in third order is trivial. Analogue to classical mechanics (see section 4.1), this means there must be some F such that

$$\mathcal{R}^{\mu\nu\rho}(\pi, \partial\pi, \partial\partial\pi) = F^{\mu}_{\partial_\nu\partial_\rho\pi}. \tag{5.9}$$

However, this differential equation can not be solved, making it impossible to take the third order term into a total derivative. This linear case thus has potential.

The question is now whether it is possible that in some way only the healthy terms remain. Two possible ways to achieve this are:

I. Finding a tensor $\mathcal{R}$ that makes sure that in the linear case of (5.6) only the healthy terms remain.

II. An extra general second order Lagrangian could be added whose terms in the equations of motion cancel (part of) the unwanted terms resulting from the third order Lagrangian. In general this would look like

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{3rd order}} + \mathcal{L}_{\text{2nd order}} \\ &= \mathcal{R}_{(3n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\rho_1\ldots\rho_n}\pi_{\mu_1\nu_1\rho_1}\cdots\pi_{\mu_n\nu_n\rho_n} + \mathcal{S}_{(2m)}^{\mu_1\ldots\mu_m\nu_1\ldots\nu_m}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_m\nu_m},\end{aligned} \tag{5.10}$$

where in this case, the $\mathcal{L}_{\text{3rd order}}$ part need not necessarily be linear in third order. The $\mathcal{L}_{\text{2nd order}}$ part is *not* a Galileon, because it is required that is produces unhealthy terms, such that cancellation with others can take place.

5.2 Finding conditions for possibility (II)

The question remains whether these two possibilities can actually succeed. Regarding possibility (II), I will restrict the third order part to be linear, i.e.

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{linear in 3rd order}} + \mathcal{L}_{\text{2nd order}} \\ &= \mathcal{R}_{(2n+3)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho} + \mathcal{S}_{(2m)}^{\mu_1\ldots\mu_m\nu_1\ldots\nu_m}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_m\nu_m}\end{aligned} \tag{5.11}$$

For both possibilities, conditions on the tensor(s) must be found. The way to find them appear to be quite similar. I will first go through the procedure of possibility (II) and after this I will make the connection to possibility (I). Let's

first take a closer look at what kind of terms appear in the equations of motion for the general tensors $\mathcal{R}$ and $\mathcal{S}$. Because the third order Lagrangian part is linear in third order, no sixth order terms appear in the equations of motions. The terms that do appear are found in the table below.

Orders of derivatives	**Terms in $\mathcal{E}$ of $\mathcal{L}_{\text{3rd order, linear}}$**	**Terms in $\mathcal{E}$ of $\mathcal{L}_{\text{2nd order}}$**
Linear in 3rd	X	X
Quadratic in 3rd	X	X
Cubic in 3rd	X	
Linear in 4th	X	X
Linear in 3rd and 4th	X	
Linear in 5th	X	

Table 5.1: Unwanted terms in Lagrangian parts, by structure

There are a number of problems to be solved that will lead to conditions on the tensors, which can be divided in two:

1. All term structures that do appear equations of motion from $\mathcal{L}_{\text{3rd order}}$, but do not appear those from $\mathcal{L}_{\text{2nd order}}$, must be cancelled or set to zero internally in $\mathcal{L}_{\text{3rd order}}$ part's equations of motion. These are the terms cubic in 3rd order, terms linear in both 3rd and 4th and those linear in 5th.

2. The other unwanted terms for the equations of motion of $\mathcal{L}_{\text{3rd order}}$ and $\mathcal{L}_{\text{2nd order}}$ should cancel between them or be zero.

For this procedure to be successful, conditions on the tensors should be found such that the above problems are solved *and* tensors must be found that satisfy all these conditions.

5.2.1 Finding conditions to solve the first problem

The first concern is to remove all terms that appear in the equations of motion of $\mathcal{L}_{\text{3rd order}}$ only. This can be done in two ways. One is to require them to be zero. A way to do this, is by imposing the same kind of antisymmetries as the Galileons. A logical step analogue to this is imposing $\mathcal{R}_{(2n+3)}$ to be fully antisymmetric in all the μ_i's, including the μ of the third order and in all the ν_i's, including ν. However, to succeed, ρ must also be antisymmetric to either the μ's or the ν's. This is where it fails, because this would mean that $\mathcal{R}_{(2n+3)}$ on contraction with $\pi_{\mu\nu\rho}$, the entire $\mathcal{L}_{\text{3rd order}}$ is destroyed.

The other possibility is to let the terms cancel between the terms of the Euler-Lagrange equations. This proves to be more successful.

Cancelling 5th terms in $\mathcal{L}_{\text{3rd order}}$

There are 5th order terms appearing in both the third and fourth term of the Euler-Lagrange equation. Because in both terms there are n of them, they can

be cancelled. However, this required a symmetry in some of the indices. The third term of the Euler-Lagrange equation gives as 5th order terms

$$\mathcal{R}_{(2n+3)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\pi_{\mu\nu\rho\mu_n\nu_n}, \tag{5.12}$$

and the fourth term of the Euler-Lagrange equation gives

$$\begin{aligned}\mathcal{R}_{(2n+3)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}\pi_{\mu_1\nu_1\rho\mu\nu}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n} \quad ; \\ \mathcal{R}_{(2n+3)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2\rho\mu\nu}\pi_{\mu_3\nu_3}\cdots\pi_{\mu_n\nu_n} \quad ; \quad \ldots\end{aligned} \tag{5.13}$$

So for these terms to cancel between each other, the tensor $\mathcal{R}_{(2n+3)}$ must be symmetric in the indices $\{\mu_1\mu_2\ldots\mu_n\mu\}$ and in $\{\nu_1\nu_2\ldots\nu_n\nu\}$. Note that this inclused all μ_i's *and* μ and the same for the ν's.

Cancelling other terms that only appear in $\mathcal{L}_{\textbf{3rd order}}$

There are two types of terms that only appear in the equation of motion of $\mathcal{L}_{\text{3rd order}}$ and do not in any $\mathcal{L}_{\text{2nd order}}$. There are terms cubic in third order and terms linear in both third and fourth order. The issue of the first type is already solved by the condition above. In the same way as above they cancel between the third and fourth term of the Euler-Lagrange equation. The issue of terms of the second case is somewhat more difficult. There are $3n(n-1)$ of them in each term of the Euler-Lagrange equation, but only $n(n-1)$ of them cancel due to the condition above. The extra condition that the index ρ is symmetric to both $\{\mu_1\mu_2\ldots\mu_n\mu\}$ *and* to $\{\nu_1\nu_2\ldots\nu_n\nu\}$ is needed to ensure that the other $2n(n-1)$ terms also cancel between each other.

5.2.2 Finding conditions to solve the second problem

Now that the terms that only appear in the equations of motions of $\mathcal{L}_{\text{3rd order}}$ are gone, the second problem can be solved: making sure all other terms are removed. This can be done by cancellation between the $\mathcal{L}_{\text{3rd order}}$ and the $\mathcal{L}_{\text{2nd order}}$ part or by requiring terms to be zero.

The procedure to do so is as follows:

1. Identify all different kind of term and classify them according to the exponential order at which 4th, 3rd, 2nd and 1st order derivatives appear in the term. Do this for both $\mathcal{L}_{\text{3rd order}}$ (keeping in mind the symmetries already imposed on the tensor) and $\mathcal{L}_{\text{2nd order}}$.

2. Set the terms that appear in a certain class in $\mathcal{L}_{\text{3rd order}}$ and in $\mathcal{L}_{\text{2nd order}}$ equal to each other.

3. Set the terms that appear only in one of the Lagrangian parts equal to zero.

This leads eight conditions on the tensors $\mathcal{R}$ and $\mathcal{S}$. The tensor $\mathcal{S}$ is still free to choose, where $\mathcal{R}$ already has the symmetry conditions that followed from solving the first problem.

Conditions on the tensors $\mathcal{R}$ and $\mathcal{S}$

The conditions found are:

$$-2\left(\frac{\partial\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}_{(2n+3)}}{\partial\pi}\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho}=0 \tag{5.14}$$

$$\begin{aligned}&\Bigg(-\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}_{(2n+3)}}{\partial\pi_\alpha\partial\pi}-3\cdot\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\alpha\rho}_{(2n+3)}}{\partial\pi_\nu\partial\pi}\\&\quad-\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\alpha}_{(2n+3)}}{\partial\pi_\rho\partial\pi}-\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\alpha\nu\rho}_{(2n+3)}}{\partial\pi_\mu\partial\pi}\Bigg)\pi_\alpha\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho}\\&\quad=\left(2\cdot\frac{\partial\mathcal{S}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho\alpha}_{(2n+4)}}{\partial\pi}\right)\pi_\alpha\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho}\end{aligned} \tag{5.15}$$

$$\begin{aligned}&\Bigg(-\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}_{(2n+3)}}{\partial\pi_\alpha\partial\pi_\beta}-2\cdot\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\alpha\rho}_{(2n+3)}}{\partial\pi_\nu\partial\pi_\beta}\\&\quad-\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\alpha\nu\rho}_{(2n+3)}}{\partial\pi_\mu\partial\pi_\beta}\Bigg)\pi_{\alpha\beta}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho}\\&\quad=\left(2\cdot\frac{\partial\mathcal{S}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho\alpha}_{(2n+4)}}{\partial\pi_\beta}\right)\pi_{\alpha\beta}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho}\end{aligned} \tag{5.16}$$

$$\left(-2\cdot\frac{\partial^2\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}_{(2n+3)}}{\partial\pi\partial\pi}\right)\pi_\nu\pi_\rho\pi_{\mu_1\nu_1\mu}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}=0 \tag{5.17}$$

$$\begin{aligned}&\Bigg(-2\cdot\frac{\partial\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}_{(2n+3)}}{\partial\pi_\alpha}-\frac{\partial\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\alpha\nu\rho}_{(2n+3)}}{\partial\pi_\mu}\\&\quad-\frac{\partial\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\alpha\rho}_{(2n+3)}}{\partial\pi_\nu}\Bigg)\pi_{\alpha\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho}\\&\quad=\left(\mathcal{S}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho\alpha}_{(2n+4)}\right)\pi_{\alpha\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}\pi_{\mu\nu\rho}\end{aligned} \tag{5.18}$$

$$\left(-2\cdot\frac{\partial\mathcal{R}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n\mu\nu\rho}_{(2n+3)}}{\partial\pi}\right)\pi_\rho\pi_{\mu_1\nu_1\mu}\pi_{\mu_2\nu_2\nu}\pi_{\mu_3\nu_3}\cdots\pi_{\mu_n\nu_n}=0 \tag{5.19}$$

$$\left(-2 \cdot \frac{\partial \mathcal{R}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n \mu \nu \rho}_{(2n+3)}}{\partial \pi_\alpha} - \frac{\partial \mathcal{R}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n \mu \alpha \rho}_{(2n+3)}}{\partial \pi_\nu} \right) \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n} \pi_{\mu \nu \rho \alpha}$$

$$= \left(\mathcal{S}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n \mu \nu \rho \alpha}_{(2n+4)} \right) \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n} \pi_{\mu \nu \rho \alpha} \tag{5.20}$$

$$\left(- \frac{\partial \mathcal{R}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n \mu \nu \rho}_{(2n+3)}}{\partial \pi} \right) \pi_\mu \pi_{\mu_1 \nu_1 \nu \rho} \pi_{\mu_2 \nu_2} \cdots \pi_{\mu_n \nu_n} = 0 \tag{5.21}$$

5.3 Conditions for possibility (I)

For possibility (I), where there is only a $\mathcal{L}_{\text{3rd order}}$ part linear in $\pi_{\mu\nu\rho}$, conditions are quite the same. The difference is that no terms are cancelled by a second Lagrangian part and therefore should all be set to zero. This implies exactly the same conditions as (5.14)−(5.21), but all of them have 0 on the right hand side of the equation.

5.4 Finding the tensors

Unfortunately, due to limitation in time for this research, I have not been able to further solve the problem of third order Lagrangians for scalar fields. Let me finish by outlining the procedure that should be followed to do so.

- First of all, from the set of eight conditions for both possibilities (I) and (II), restrictions on tensors $\mathcal{R}_{(2n+3)}$ and $\mathcal{S}_{(4n)}$ should be found. This includes restriction in the indices (such as (anti-)symmetries) as well as restrictions of the tensors' dependency on π and $\partial\pi$.
- The restrictions found must be in compatible with the restrictions found earlier (i.e. the symmetry in $\{\mu_1 \dots \mu_n \mu \rho\}$ and $\{\nu_1 \dots \nu_n \nu \rho\}$). If this is not the case, the possibilities fail.
- If a set of compatible restrictions on the tensors can be found, the tensors should be explicitly build.

If all this can be achieved, a third order Lagrangians for scalar field is found.

5.5 Conclusions on higher order Lagrangians for scalar fields

I would have hoped to conclude by stating whether or not higher order Lagrangians for scalar field can lead to normal, workable equations of motion. However, due to time restrictions I cannot give such a conclusion. What can be said

based on the analyses above, is that Lagrangians containing only higher order derivative terms are generally not possible. The only possible exception to this is when a third order derivative enters the Lagrangian terms linearly. For this case and for the case where a second order Lagrangian is added to a linear third order Lagrangian, conditions were found. It is for later research to determine whether these conditions can be satisfied.

Chapter 6

Conclusions

Ostrogradski's Theorem stated that Lagrangians containing derivative terms higher then first order will either be trivial or fatal, but there are certain ways out of this. It was the goal of this research to analyse the known possibilities for second order Lagrangians and to see whether possibilities could be found to write higher order Lagrangians for scalar fields.

In chapter 2 constraints were used to find conditions on second order Lagrangians in classical mechanics. These constraints would make a theory ghost-free. These constraints implied linearity in $\ddot{q}$, which proved to be trivial. This is indeed in line with Ostrogradski's theorem. In chapter 4 higher order Lagrangians for classical mechanics were analysed. They appeared to be completely analogue to the second order case. There are possibilities (under a number of conditions) to write n^{th} order Lagrangians, but these also appeared to be trivial.

In chapter 3 Galileons were analysed. These second order Lagrangians for scalar fields arose from specific antisymmetric properties on the tensor included in the Lagrangian. There are multiple ways to choose the tensor, leading to multiple Galileon Lagrangians. However, they are all equal up to a total derivative, therefore giving the same equations of motions. For Galileons these contain only second order derivative terms. The generalised Galileons were briefly mentionted, which also contain first derivative terms and the field itself. (Generalised) Galileons are non-trivial subset of second order Lagrangians and do contain new information with respect to normal Lagrangians.

In chapter 5 an attempt was made to write third order Lagrangians for scalar fields, that produce workable equations of motion. Unfortunately, I cannot make a conclusion whether there is a possibility or not, mainly due to the short time frame of this research. However, it can be concluded that it is not possible to write Lagrangians that contain only third order derivative terms that are quadratic or higher in polynomial order. They will produce no healthy terms at all. The same holds for all Lagrangians containing only fourth or higher derivative order (including linear this order). For the case linear in third order there are two possibilities. Either all unwanted terms of the equations of motion are cancelled between the terms of the Euler-Lagrange or are set to zero in a

purely third order Lagrangian, or cancellation of (part of these) terms is achieved by adding a second order Lagrangian. Conditions for both these cases were found, but not yet solved.

It has thus become clear that generalisation of Galileons to higher derivative order is not a straightforward task. The characteristic properties of (anti-)symmetry in indices are required, but extra cancellations with lower order Lagrangian terms are most likely needed. However, there is still hope. Perhaps further research will prove that there are possibilities to write higher order Lagrangians which may then find their applications. After all, it was long thought that second order Lagrangians were impossible, but see where we are now!

Recommendations for further research

First of all, I have not had the time to finish the analyses on linear third order Lagrangians for scalar field. This will be the first thing that can be further investigated. The conditions found for the two cases should be properly analysed to see whether they can lead to compatible restrictions on tensors and if so, whether tensors can be found.

Then there is the possibility of quadratic third order Lagrangians, which I have not looked at. In this case terms in the equations of motion must be cancelled with other, lower order Lagrangian terms. Perhaps even a generalisation to higher polynomial order can be made if both the linear and quadratic case are well understood.

Finally, maybe third order is not the right choice at all. Looking at the antisymmetry in second order, there are reasons to assume that higher order Lagrangians only work for even orders of derivatives of the field. For Galileons, there is the so called Palatini formalism (see for example [6]), that for this research I have not thoroughly looked into. By defining a field strength $S_{\mu\nu} = \phi^{-1}\partial_\mu\partial_\nu\phi$, the Galilean Lagrangians can be written as $\mathcal{L}(\phi, \partial_\mu\partial_\nu\phi)$. This may be an indication that generalisation to a Lagrangian containing only the field and it's second and fourth derivative is a possibility. It is interesting for further research to take a look at this possibility.

Bibliography

[1] Remko Klein, Diederik Roest, *Exorcising the Ostrogradsky ghost in coupled systems*, arXiv: 1604.01719v1 (2016)

[2] R.P. Woodard, *Avoiding Dark Energy with 1/R Modifications of Gravity*, arXiv:astro-ph/0601672v2 (2006)

[3] Alberto Nicolis, Riccardo Rattazzi, Enrico Trincherini, *The Galileon as a Local Modification of Gravity* , arXiv:0811.2197v2 (2009)

[4] Cédric Deffayet, Danièle A. Steer, *A formal introduction to Horndeski and Galileon theories and their generalizations*, arXiv:1307.2450v1 (2013)

[5] D.B. Fairly, J. Govaerts, A. Morozov, *Universal Field Equations with Covariant Solutions* arXiv:hep-th/9110022v1 (1991)

[6] Remko Klein, Mehmet Ozkan, Diederik Roest, *Galileons as the Scalar Analogue of General Relativity* arXiv:1510.08864v1 (2015)

[7] C. Deffayet, Xian Gao, D.A. Steer, G. Zahariade, *From k-essence to generalised Galileons*, arXiv:1103.3260v1 (2011).

[8] Tai-jun Chen, Metteo Fasiello, Eugene A. Lim, Andrew J. Tolley, *Higher derivative theories with constraints: exorcising Ostrogradski's ghost* arXiv: 1209.0583v4 (2013)

[9] David Tong, *Classical Dynamics, University of Cambridge Part II Mathematical Tripos* http://www.damtp.cam.ac.uk/user/tong/dynamics.html (2004)

[10] David Tong, *Quantum Field Theory, University of Cambridge Part III Mathematical Tripos* http://www.damtp.cam.ac.uk/user/tong/qft.html (2006)

[11] C. Armendariz-Picon, V. Mukhanov, Paul J. Steinhardt *Essentials of k-essence* arXiv:astro-ph/0006373v1 (2008)

[12] Marc Henneaux, Claudio Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992)

[13] Bogar Días, Daniel Higuita, Merced Montesinos *Lagrangian approach to the physical degree of freedom count* arXiv: 1406.1156v2 (2014)

[14] Susskind, Leonard; Hrabovsky, George, *The Theoretical Minimum, what you need to know to start doing physics* (New York, 2013)

[15] Dijksterhuis, E.J., *De mechanisering van het wereldbeeld* (3rd print, Amsterdam, 1977)

[16] M. Ostrogradsky. Mem. Ac. St Petersbourg **VI 4** (1850) 385

[17] Mark Trodden, *Theoretical Aspects of Cosmic Acceleration* arXiv: 1604.08899v1 (2016)

[18] Kurt Hinterbichlen, Austin Joyce, *A Hidden Symmetry of the Galileon* arXiv: 1501.07600v2 (2015)

[19] Kurt Hinterbichler, Austin Joyce, Justin Khoury, Godfrey E.J. Miller, *DBI Genesis: An Improves Violation of the Null Energy Condition*, arXiv: 1212.3607v1 (2012)

Appendix A

Derivations

A.1 Equations of motion for classical mechanics

First order Lagrangian [9]

Consider the Lagrangian $L = L(q, \dot{q})$, with the action

$$S = \int_{t_i}^{t_f} L(q, \dot{q})\, dt. \tag{A.1}$$

Now the path is varied slightly, so

$$q(t) \to q(t) + \delta q(t), \tag{A.2}$$

where the end points of the path are fixed by demanding $\delta q(t_i) = \delta q(t_f) = 0$. The the change in the action is

$$\begin{aligned} \delta S &= \delta \left[\int_{t_i}^{t_f} L\, dt \right] \\ &= \int_{t_i}^{t_f} \delta L\, dt \\ &= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt. \end{aligned} \tag{A.3}$$

Integration by parts on the second term gives:

$$\delta S = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q\, dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_i}^{t_f}. \tag{A.4}$$

The last term vanishes because the endpoints are fixed. The principle of least actions requires $\delta S = 0$, which means that the term in parentheses must be zero. This gives the equation of motion:

$$E = \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0. \tag{A.5}$$

Equivalence between Lagrangian and Newtonian mechanics

Let the kinetic energy be given by

$$T = \frac{1}{2}m\dot{x}^2 \tag{A.6}$$

and the potential by the function $V(x)$. The Lagrangian of this system is then

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \tag{A.7}$$

The terms of the Euler-Lagrange equation become

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} \tag{A.8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \tag{A.9}$$

Combining and realising that $-\frac{\partial V}{\partial x} \equiv F$, gives back Newton's second law

$$F = m\ddot{x}. \tag{A.10}$$

The same argument holds in three dimensions.

Equivalence between Hamiltonian and Newtonian mechanics

We have the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \tag{A.11}$$

and the canonical momentum

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x}. \tag{A.12}$$

The Hamiltonian then becomes

$$H \equiv p\dot{x} - L = \frac{1}{2}m\dot{x}^2 + V(x) = \frac{p^2}{2m} + V(x). \tag{A.13}$$

The Hamilton equations give

$$\dot{p} \equiv -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} \tag{A.14}$$

$$\dot{x} \equiv \frac{\partial H}{\partial p} = \frac{p}{m}. \tag{A.15}$$

Combining these two equations and inserting $-\frac{\partial V}{\partial x} \equiv F$ gives back Newton's second law $F = m\ddot{x}$.

Second order Lagrangian

For a second order Lagrangian $L = L(q, \dot{q}, \ddot{q})$ the procedure for getting the equation of motion is exactly the same as above, but requires an extra integration by parts. The variation of S gives

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} \right) dt \\
&= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\delta \dot{q} \right) dt + \left[\frac{\partial L}{\partial \ddot{q}}\delta \dot{q} \right]_{t_i}^{t_f} \\
&= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \right) \delta \dot{q} \right) dt + \left[\frac{\partial L}{\partial \ddot{q}}\delta \dot{q} \right]_{t_i}^{t_f} \\
&= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} \right) \delta q \right) dt \\
&\qquad + \left[\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \right) \delta q \right]_{t_i}^{t_f} + \left[\frac{\partial L}{\partial \ddot{q}}\delta \dot{q} \right]_{t_i}^{t_f} \\
&= \int_{t_i}^{t_f} \left(\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} \right) \delta q \right) dt \\
&\qquad + \left[\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \right) \delta q \right]_{t_i}^{t_f} + \left[\frac{\partial L}{\partial \ddot{q}}\delta \dot{q} \right]_{t_i}^{t_f} = 0.
\end{aligned} \tag{A.16}$$

As before, the last two terms are zero because the endpoints are fixed. Setting the variation of the action to zero means that the term in the inner parentheses under the integral must be zero

$$E = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}} \right) = 0, \tag{A.17}$$

which is the equation of motion for a second order Lagrangian. For multi-variables this simply becomes

$$E_i = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i} \right) = 0. \tag{A.18}$$

General equation of motion for second order Lagrangians

The equation of motion can be computed for the general case as follows:

$$
\begin{aligned}
E = \frac{\partial L}{\partial q_i} &- \left[\frac{\partial}{\partial \ddot{q}_j}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial \ddot{q}_j}{\partial t} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial \dot{q}_j}{\partial t} + \frac{\partial}{\partial q_j}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_j}{\partial t}\right] \\
&+ \frac{d}{dt}\left[\frac{\partial}{\partial \ddot{q}_j}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\frac{\partial \ddot{q}_j}{\partial t} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\frac{\partial \dot{q}_j}{\partial t} + \frac{\partial}{\partial q_j}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\frac{\partial q_j}{\partial t}\right] \\
= \frac{\partial L}{\partial q_i} &- \left[\frac{\partial^2 L}{\partial \ddot{q}_j \partial \dot{q}_i} q_j^{(3)} + \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\dot{q}_j\right] \\
&+ \frac{d}{dt}\left[\frac{\partial^2 L}{\partial \ddot{q}_j \partial \ddot{q}_i} q_j^{(3)} + \frac{\partial^2 L}{\partial \dot{q}_j \partial \ddot{q}_i}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \ddot{q}_i}\dot{q}_j\right] \\
= \frac{\partial L}{\partial q_i} &- \left[\frac{\partial^2 L}{\partial \ddot{q}_j \partial \dot{q}_i} q_j^{(3)} + \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\dot{q}_j\right] \\
&+ \frac{\partial^2 L}{\partial \ddot{q}_j \partial \ddot{q}_i} q_j^{(4)} + \left(\frac{\partial^3 L}{\partial \ddot{q}_k \partial \ddot{q}_j \partial \ddot{q}_i} q_k^{(3)} + \frac{\partial^3 L}{\partial \dot{q}_k \partial \ddot{q}_j \partial \ddot{q}_i}\ddot{q}_k + \frac{\partial^3 L}{\partial q_k \partial \ddot{q}_j \partial \ddot{q}_i}\dot{q}_k\right) q_j^{(3)} \\
&+ \frac{\partial^2 L}{\partial \dot{q}_j \partial \ddot{q}_i} q_j^{(3)} + \left(\frac{\partial^3 L}{\partial \ddot{q}_k \partial \dot{q}_j \partial \ddot{q}_i} q_k^{(3)} + \frac{\partial^3 L}{\partial \dot{q}_k \partial \dot{q}_j \partial \ddot{q}_i}\ddot{q}_k + \frac{\partial^3 L}{\partial q_k \partial \dot{q}_j \partial \ddot{q}_i}\dot{q}_k\right)\ddot{q}_j \\
&+ \frac{\partial^2 L}{\partial q_j \partial \ddot{q}_i}\ddot{q}_j + \left(\frac{\partial^3 L}{\partial \ddot{q}_k \partial q_j \partial \ddot{q}_i} q_k^{(3)} + \frac{\partial^3 L}{\partial \dot{q}_k \partial q_j \partial \ddot{q}_i}\ddot{q}_k + \frac{\partial^3 L}{\partial q_k \partial q_j \partial \ddot{q}_i}\dot{q}_k\right)\dot{q}_j = 0.
\end{aligned}
\tag{A.19}
$$

When using the shorthand notation $\frac{\partial L}{\partial q} \equiv L_q$ this looks like [1]

$$
\begin{aligned}
E = L_{q_i} &- \left[L_{\ddot{q}_j \dot{q}_i} q_j^{(3)} + L_{\dot{q}_j \dot{q}_i}\ddot{q}_j + L_{q_j \dot{q}_i}\dot{q}_j\right] \\
&+ L_{\ddot{q}_j \ddot{q}_i} q_j^{(4)} + \left(L_{\ddot{q}_k \ddot{q}_j \ddot{q}_i} q_k^{(3)} + L_{\dot{q}_k \ddot{q}_j \ddot{q}_i}\ddot{q}_k + L_{q_k \ddot{q}_j \ddot{q}_i}\dot{q}_k\right) q_j^{(3)} \\
&+ L_{\dot{q}_j \ddot{q}_i} q_j^{(3)} + \left(L_{\ddot{q}_k \dot{q}_j \ddot{q}_i} q_k^{(3)} + L_{\dot{q}_k \dot{q}_j \ddot{q}_i}\ddot{q}_k + L_{q_k \dot{q}_j \ddot{q}_i}\dot{q}_k\right)\ddot{q}_j \\
&+ L_{q_j \ddot{q}_i}\ddot{q}_j + \left(L_{\ddot{q}_k q_j \ddot{q}_i} q_k^{(3)} + L_{\dot{q}_k q_j \ddot{q}_i}\ddot{q}_k + L_{q_k q_j \ddot{q}_i}\dot{q}_k\right)\dot{q}_j = 0.
\end{aligned}
\tag{A.20}
$$

nth order Lagrangians

The principle of the the derivations above can be generalised to a Lagrangian $L = L(q, \dot{q}, \ldots, q^{(n)})$ of derivatives up to order n. This means that we need $n-1$ integration by parts, which result in an alternating sign in front of the terms of the equation of motion:

$$
E = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \cdots \pm \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial q^{(n)}}\right) = 0, \tag{A.21}
$$

[1] The fourth order term is indicated in red. The two third order terms that survive the primary condition are indicated in blue, while the ones that do net are indicated in green. The colours used are also referred to in section 2.2.

$$E = \sum_{i=0}^{n} (-1)^i \frac{d^i}{dt^i} \left(\frac{\partial L}{\partial q^{(i)}} \right) = 0. \tag{A.22}$$

A.2 Equations of motion for scalar field

The derivation of the Euler Lagrange equation for Lagrangians of scalar field is completely analogue to its classical mechanics counterpart. I shall therefore only fully derive the second order case and simply give the first order and nth order for completeness.

First order Lagrangian

For the first order case the Euler Lagrange equation is [10]

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{A.23}$$

Second order Lagrangian

Consider the Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi)$ and the action

$$S = \int_{t_i}^{t_f} dt \int d^3x \, \mathcal{L} = \int \mathcal{L} \, d^4x. \tag{A.24}$$

The variation in the action is then

$$\begin{aligned}
\delta S &= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \delta(\partial_\mu \partial_\nu \phi) \right] d^4x \\
&= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right) \delta(\partial_\mu \phi) \right] d^4x \\
&\qquad + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \delta(\partial_\mu \phi) \right]_{\mathbf{x}, t_i}^{\mathbf{x}, t_f} \\
&= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right) \right) \delta(\partial_\mu \phi) \right] d^4x \\
&\quad + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \delta(\partial_\mu \phi) \right]_{\mathbf{x}, t_i}^{\mathbf{x}, t_f} \\
&= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right) \right] \delta\phi \; d^4x \\
&\quad + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \delta(\partial_\mu \phi) \right]_{\mathbf{x}, t_i}^{\mathbf{x}, t_f} + \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right) \right) \delta\phi \right]_{\mathbf{x}, t_i}^{\mathbf{x}, t_f}.
\end{aligned} \tag{A.25}$$

As in the classical mechanics case, the last two terms are zero because the endpoints are fixed. Setting the variation of the action to zero means that the

term in the inner parentheses under the integral must be zero, leading to the equation of motion

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right) = 0. \tag{A.26}$$

nth order Lagrangian

Generalising the procedure in the subsection above for a Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, \ldots, \partial^{(n)} \phi)$ gives (appropriate indices are assumed):

$$\mathcal{E} = \sum_{i=0}^{n} (-1)^i \, \partial^{(i)} \left(\frac{\partial \mathcal{L}}{\partial(\partial^{(i)} \phi)} \right) = 0. \tag{A.27}$$

A.3 Second order Lagrangians for scalar fields (Galileons)

A.3.1 Working out the Euler-Lagrange equation

The Euler-Lagrange equation, with derivatives as indices is

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \pi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \pi_\mu} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \pi_{\mu\nu}} \right) = 0, \tag{A.28}$$

where the Lagrangian is

$$\mathcal{L} = \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n} \qquad ; \qquad \mathcal{T}_{(2n)} = \mathcal{T}_{(2n)}(\pi, \pi_\mu). \tag{A.29}$$

The terms in the Euler-Lagrange equation the give:

$$\frac{\partial \mathcal{L}}{\partial \pi} = \frac{\partial \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}}{\partial \pi} \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n}, \tag{A.30}$$

$$\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial \pi_\rho} \right) = \partial_\rho \left(\frac{\partial \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}}{\partial \pi_\rho} \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n} \right) \tag{A.31}$$

$$\begin{aligned} &= \partial_\rho \left(\frac{\partial \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}}{\partial \pi_\rho} \right) \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_n \nu_n} \\ &\quad + \frac{\partial \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}}{\partial \pi_\rho} \pi_{\mu_1 \nu_1 \rho} \pi_{\mu_2 \nu_2} \cdots \pi_{\mu_n \nu_n} \\ &\quad + \cdots + \frac{\partial \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}}{\partial \pi_\rho} \pi_{\mu_1 \nu_1} \cdots \pi_{\mu_{n-1} \nu_{n-1}} \pi_{\mu_n \nu_n \rho}, \end{aligned} \tag{A.32}$$

$$\partial_\rho\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial\pi_{\rho\sigma}}\right)$$

$$= \partial_\rho\partial_\sigma\left(\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(\frac{\partial\pi_{\mu_1\nu_1}}{\partial\pi_{\rho\sigma}}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}+\cdots+\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\frac{\partial\pi_{\mu_n\nu_n}}{\partial\pi_{\rho\sigma}}\right)\right) \tag{A.33}$$

$$= \partial_\rho\partial_\sigma\left(\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(\delta^\rho_{\mu_1}\delta^\sigma_{\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}+\cdots+\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\delta^\rho_{\mu_n}\delta^\sigma_{\nu_n}\right)\right) \tag{A.34}$$

$$= \partial_\rho\partial_\sigma\left(\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(\delta^\rho_{\mu_n}\delta^\sigma_{\nu_n}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\delta^\rho_{\mu_n}\delta^\sigma_{\nu_n}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\right)\right) \tag{A.35}$$

$$= \partial_\rho\partial_\sigma\left(n\cdot\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\delta^\rho_{\mu_n}\delta^\sigma_{\nu_n}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\right) \tag{A.36}$$

$$= n\cdot\partial_\rho\partial_\sigma\left(\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_{n-1}\rho\nu_1\cdots\nu_{n-1}\sigma}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\right) \tag{A.37}$$

(*For convenience, all* $\mathcal{T}_{(2n)}$ *terms below will be assumed to have the indices.* $\{\mu_1\cdots\mu_{n-1}\rho\,\nu_1\cdots\nu_{n-1}\sigma\}$)

$$= n\cdot\partial_\rho\left(\left(\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma}+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_\sigma\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\right. \tag{A.38}$$
$$\left.+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-2}\nu_{n-2}}\pi_{\mu_{n-1}\nu_{n-1}\sigma}\right)$$

$$= n\cdot\partial_\rho\left(\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_\sigma\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\right. \tag{A.39}$$
$$\left.+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-2}\nu_{n-2}}\pi_{\mu_{n-1}\nu_{n-1}\sigma}\right)$$

$$\begin{aligned}
= n\cdot\Bigg(&\partial_\rho\left(\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\right)\pi_{\alpha\sigma}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma\rho}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\\
&\quad+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma}\pi_{\mu_1\nu_1\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi_\alpha}\pi_{\alpha\sigma}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}\rho}\\
&+\partial_\rho\left(\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\right)\pi_\sigma\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_{\sigma\rho}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\\
&\quad+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_\sigma\pi_{\mu_1\nu_1\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\frac{\partial\mathcal{T}_{(2n)}}{\partial\pi}\pi_\sigma\pi_{\mu_1\nu_1}\cdots\pi_{\mu_{n-1}\nu_{n-1}\rho}\\
&+\partial_\rho\left(\mathcal{T}_{(2n)}\right)\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma\rho}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}}\\
&\quad+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}}+\cdots+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\sigma}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\rho}\\
&+\qquad\cdots\\
&+\partial_\rho\left(\mathcal{T}_{(2n)}\right)\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma}+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1\rho}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma}\\
&\quad+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2\rho}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma}+\cdots+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-2}\nu_{n-2}\rho}\pi_{\mu_{n-1}\nu_{n-1}\sigma}\\
&\quad+\mathcal{T}_{(2n)}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n-1}\nu_{n-1}\sigma\rho}\Bigg).
\end{aligned} \tag{A.40}$$

A.3.2 Lagrangians $\mathcal{L}_N^{\mathbf{Gal},1}$

The Galileons are defined as:

$$\mathcal{L}_N^{\mathrm{Gal},1} = \left(\mathcal{A}_{(2n+2)}^{\mu_1\ldots\mu_{n+1}\nu_1\ldots\nu_{n+1}}\pi_{\mu_{n+1}}\pi_{\nu_{n+1}}\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}, \tag{A.41}$$

where the tensor $\mathcal{A}_{(2m)}$ is defined as

$$\mathcal{A}_{(2m)}^{\mu_1\mu_2\ldots\mu_m\nu_1\nu_2\ldots\nu_m} \equiv \frac{1}{(D-m)!}\,\varepsilon^{\mu_1\mu_2\ldots\mu_m\sigma_1\sigma_2\ldots\sigma_{D-m}}\,\varepsilon^{\nu_1\nu_2\ldots\nu_m}{}_{\sigma_1\sigma_2\ldots\sigma_{D-m}} \tag{A.42}$$

and

$$N \equiv n+2. \tag{A.43}$$

The first non-trivial Lagrangian is when $N=2$, where $n=0$ and $D=4$:

$$\begin{aligned}
\mathcal{L}_2^{\mathrm{Gal},1} &= \mathcal{A}_{(2)}^{\mu_1\nu_1}\pi_{\mu_1}\pi_{\nu_1} \\
&= \frac{1}{(4-1)!}\varepsilon^{\mu_1\sigma_1\sigma_2\sigma_3}\varepsilon^{\nu_1}{}_{\sigma_1\sigma_2\sigma_3}\pi_{\mu_1}\pi_{\nu_1} \\
&= \frac{1}{6}\varepsilon^{\mu_1\sigma_1\sigma_2\sigma_3}\varepsilon_{\nu_1\sigma_1\sigma_2\sigma_3}\pi_{\mu_1}\pi^{\nu_1} \\
&= \frac{1}{6}\cdot -3!\,\delta^{\mu_1}_{\nu_1}\pi_{\mu_1}\pi^{\nu_1} \\
&= -\pi^{\mu}\pi_{\mu}.
\end{aligned} \tag{A.44}$$

In the fourth line there was the use of the identity[2]

$$\varepsilon_{i_1\ldots i_k i_{k+1}\ldots i_n}\varepsilon^{j_1\ldots j_k j_{k+1}\ldots j_n} = -k!\;\delta^{j_{k+1}\ldots j_n}_{i_{k+1}\ldots i_n}. \tag{A.45}$$

The second non-trivial Lagrangian is when $N=3$, where $n=1$ and $D=4$:

$$\begin{aligned}
\mathcal{L}_3^{\mathrm{Gal},1} &= \mathcal{A}_{(4)}^{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_2}\pi_{\nu_2}\pi_{\mu_1\nu_1} \\
&= \frac{1}{(4-2)!}\varepsilon^{\mu_1\mu_2\sigma_1\sigma_2}\varepsilon^{\nu_1\nu_2}{}_{\sigma_1\sigma_2}\pi_{\mu_2}\pi_{\nu_2}\pi_{\mu_1\nu_1} \\
&= \frac{1}{2}\varepsilon^{\sigma_1\sigma_2\mu_1\mu_2}\varepsilon_{\sigma_1\sigma_2\nu_1\nu_2}\pi_{\mu_2}\pi^{\nu_2}\pi_{\mu_1}{}^{\nu_1} \\
&= \frac{1}{2}\cdot -2!\;\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\;\pi_{\mu_2}\pi^{\nu_2}\pi_{\mu_1}{}^{\nu_1} \\
&= -\left(\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\pi_{\mu_2}\pi^{\nu_2}\pi_{\mu_1}{}^{\nu_1} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\pi_{\mu_2}\pi^{\nu_2}\pi_{\mu_1}{}^{\nu_1}\right) \\
&= -\left(\pi_{\mu_2}\pi^{\mu_2}\pi_{\mu_1}{}^{\mu_1} - \pi_{\mu_2}\pi^{\mu_1}\pi_{\mu_1}{}^{\mu_2}\right) \\
&= \pi^{\mu}\pi^{\nu}\pi_{\mu\nu} + \pi_{\mu}\pi^{\mu}\Box\pi.
\end{aligned} \tag{A.46}$$

Where the delta-function in the fourth line is the generalised Kronecker delta which is defined as:

$$\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p} = \sum_{\sigma\in\mathfrak{S}_p}\mathrm{sgn}(\sigma)\,\delta^{\mu_1}_{\nu_{\sigma(1)}}\cdots\delta^{\mu_p}_{\nu_{\sigma(p)}}. \tag{A.47}$$

[2]The minus sign here is included to match the convension used in the Lagrangian en Euquations of motions in [4].

and $\mathfrak{S}_p$ is the symmetric group of degree p. This expression means that there is summed over all permutations of the ν's where odd permutation are negative terms and even permutations are positive terms.

A.3.3 Equations of motion from $\mathcal{L}_N^{\mathbf{Gal},1}$

Here the derivations for $\mathcal{E}_2$ and $\mathcal{E}_3$ are fully worked out. The other equations follow the exact same procedure. They can be found by using the Euler-Lagrange equation or by working out the general equation for the equations of motion of second order Galileons:

$$\mathcal{E}_N = -\mathcal{A}_{(2n+2)}^{\mu_1\ldots\mu_{n+1}\nu_1\ldots\nu_{n+1}}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_{n+1}\nu_{n+1}}. \tag{A.48}$$

Filling in this equation gives

$$\begin{aligned}
\mathcal{E}_2 &= -\mathcal{A}_{(2)}^{\mu_1\nu_1}\pi_{\mu_1\nu_1} \\
&= -\frac{1}{(4-1)!}\varepsilon^{\mu_1\sigma_1\sigma_2\sigma_3}\varepsilon^{\nu_1}{}_{\sigma_1\sigma_2\sigma_3}\pi_{\mu_1\nu_1} \\
&= -\frac{1}{6}\varepsilon^{\mu_1\sigma_1\sigma_2\sigma_3}\varepsilon_{\nu_1\sigma_1\sigma_2\sigma_3}\pi_{\mu_1}{}^{\nu_1} \\
&= -\frac{1}{6}\cdot -3!\,\delta^{\mu_1}_{\nu_1}\pi_{\mu_1}{}^{\nu_1} \\
&= \pi_{\mu_1}{}^{\mu_1} = \Box\pi.
\end{aligned} \tag{A.49}$$

Using the Euler-Lagrange equation, we first rewrite the Lagrangian as

$$\mathcal{L}_2^{\mathrm{Gal},1} = -\pi^\mu\pi_\mu = -\eta^{\mu\nu}\pi_\nu\pi_\mu. \tag{A.50}$$

This gives

$$\begin{aligned}
\mathcal{E} &= \frac{\partial\mathcal{L}}{\partial\pi} - \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial\pi_\rho}\right) + \partial_\rho\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial\pi_{\rho\sigma}}\right) \\
&= -\partial_\rho\left(-\eta^{\mu\nu}\frac{\partial\pi_\nu}{\partial\pi_\rho}\pi_\mu - \eta^{\mu\nu}\pi_\nu\frac{\partial\pi_\mu}{\partial\pi_\rho}\right) \\
&= -\partial_\rho\left(-\eta^{\mu\nu}\delta^\rho_\nu\pi_\mu - \eta^{\mu\nu}\pi_\nu\delta^\rho_\mu\right) \\
&= -\partial_\rho\left(-\eta^{\mu\rho}\pi_\mu - \eta^{\rho\nu}\pi_\nu\right) \\
&= -\partial_\rho\left(-2\pi^\rho\right) = 2\Box\pi.
\end{aligned} \tag{A.51}$$

Which, when remembering that $\mathcal{E} = N\times\mathcal{E}_N = 0$ is the same as above.

Using equation (A.48) for $N = 3$ one gets

$$\begin{aligned}
\mathcal{E}_3 &= -\mathcal{A}_{(4)}^{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2} \\
&= -\frac{1}{(4-2)!}\,\varepsilon^{\mu_1\mu_2\sigma_1\sigma_2}\varepsilon^{\nu_1\nu_2}{}_{\sigma_1\sigma_2}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2} \\
&= -\frac{1}{2}\varepsilon^{\sigma_1\sigma_2\mu_1\mu_2}\varepsilon_{\sigma_1\sigma_2\nu_1\nu_2}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2} \\
&= -\frac{1}{2}\cdot -2!\ \delta^{\mu_1\mu_2}_{\nu_1\nu_2}\ \pi_{\mu_1}{}^{\nu_1}\pi_{\mu_2}{}^{\nu_2} \\
&= \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\pi_{\mu_1}{}^{\nu_1}\pi_{\mu_2}{}^{\nu_2} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\pi_{\mu_1}{}^{\nu_1}\pi_{\mu_2}{}^{\nu_2} \\
&= \pi_{\mu_1}{}^{\mu_1}\pi_{\mu_2}{}^{\mu_2} - \pi_{\mu_1}{}^{\mu_2}\pi_{\mu_2}{}^{\mu_1} \\
&= (\Box\pi)^2 - \pi_{\mu\nu}\pi^{\mu\nu}.
\end{aligned} \tag{A.52}$$

As in the previous case, this can also be obtained from the Euler-Lagrange equations. To do so, I first rewrite the Lagrangian as

$$\mathcal{L}_3^{\mathrm{Gal},1} = \pi^\mu\pi^\nu\pi_{\mu\nu} - \pi^\mu\pi_\mu\Box\pi = \pi^\mu\pi^\nu\pi_{\mu\nu} - \eta^{\alpha\beta}\pi^\mu\pi_\mu\pi_{\alpha\beta}. \tag{A.53}$$

Since this is slightly longer then the $N = 2$ case, let me compute the individual terms first

$$\begin{aligned}
\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial\pi_\rho}\right) &= \partial_\rho\left(\frac{\partial\pi^\mu}{\partial\pi_\rho}\pi^\nu\pi_{\mu\nu} + \pi^\mu\frac{\partial\pi^\nu}{\partial\pi_\rho}\pi_{\mu\nu} - \eta^{\alpha\beta}\frac{\partial\pi^\mu}{\partial\pi_\rho}\pi_\mu\pi_{\alpha\beta} - \eta^{\alpha\beta}\pi^\mu\frac{\partial\pi_\mu}{\partial\pi_\rho}\pi_{\alpha\beta}\right) \\
&= \partial_\rho\left(\eta^{\mu\rho}\pi^\nu\pi_{\mu\nu} + \pi^\mu\eta^{\nu\rho}\pi_{\mu\nu} - \eta^{\alpha\beta}\eta^{\mu\rho}\pi_\mu\pi_{\alpha\beta} - \eta^{\alpha\beta}\pi^\mu\delta^\rho_\mu\pi_{\alpha\beta}\right) \\
&= \partial_\rho\left(\pi^\nu\pi^\rho{}_\nu + \pi^\mu\pi_\mu{}^\rho - 2\pi^\rho\pi^\alpha{}_\alpha\right) \\
&= \pi^\nu{}_\rho\pi^\rho{}_\nu + \pi^\nu\pi^\rho{}_{\nu\rho} + \pi^\mu{}_\rho\pi_\mu{}^\rho + \pi^\mu\pi_\mu{}^\rho{}_\rho - 2\pi^\rho{}_\rho\pi^\alpha{}_\alpha - 2\pi^\rho\pi^\alpha{}_{\alpha\rho} \\
&= 2\pi^{\mu\nu}\pi_{\mu\nu} - 2(\Box\pi)^2,
\end{aligned} \tag{A.54}$$

$$\begin{aligned}
\partial_\rho\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial\pi_{\rho\sigma}}\right) &= \partial_\rho\partial_\sigma\left(\pi^\mu\pi^\nu\frac{\partial\pi_{\mu\nu}}{\partial\pi_{\rho\sigma}} - \eta^{\alpha\beta}\pi^\mu\pi_\mu\frac{\partial\pi_{\alpha\beta}}{\partial\pi_{\rho\sigma}}\right) \\
&= \partial_\rho\partial_\sigma\left(\pi^\mu\pi^\nu\delta^\rho_\mu\delta^\sigma_\nu - \eta^{\alpha\beta}\pi^\mu\pi_\mu\delta^\rho_\alpha\delta^\sigma_\beta\right) \\
&= \partial_\rho\partial_\sigma\left(\pi^\rho\pi^\sigma - \eta^{\rho\sigma}\pi^\mu\pi_\mu\right) \\
&= \partial_\rho\left(\pi^\rho{}_\sigma\pi^\sigma + \pi^\rho\pi^\sigma{}_\sigma - \eta^{\rho\sigma}\pi^\mu{}_\sigma\pi_\mu - \eta^{\rho\sigma}\pi^\mu\pi_{\mu\sigma}\right) \\
&= \pi^\rho{}_{\sigma\rho}\pi^\sigma + \pi^\rho{}_\sigma\pi^\sigma{}_\rho + \pi^\rho{}_\rho\pi^\sigma{}_\sigma + \pi^\rho\pi^\sigma{}_{\sigma\rho} - \eta^{\rho\sigma}\pi^\mu{}_{\sigma\rho}\pi_\mu \\
&\qquad - \eta^{\rho\sigma}\pi^\mu{}_\sigma\pi_{\mu\rho} - \eta^{\rho\sigma}\pi^\mu{}_\rho\pi_{\mu\sigma} - \eta^{\rho\sigma}\pi^\mu\pi_{\mu\sigma\rho} \\
&= -\pi^{\mu\nu}\pi_{\mu\nu} + (\Box\pi)^2.
\end{aligned} \tag{A.55}$$

The total equation of motion is then

$$\begin{aligned}
\mathcal{E} &= \frac{\partial\mathcal{L}}{\partial\pi} - \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial\pi_\rho}\right) + \partial_\rho\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial\pi_{\rho\sigma}}\right) \\
&= 0 - (2\pi^{\mu\nu}\pi_{\mu\nu} - 2(\Box\pi)^2) + (-\pi^{\mu\nu}\pi_{\mu\nu} + (\Box\pi)^2) \\
&= 3(\Box\pi)^2 - 3\pi^{\mu\nu}\pi_{\mu\nu}.
\end{aligned} \tag{A.56}$$

Which again by using $\mathcal{E} = N \times \mathcal{E}_N = 0$ is the same as found in (A.46).

A.3.4 Relations between $\mathcal{L}_N^{\mathbf{Gal},1}$, $\mathcal{L}_N^{\mathbf{Gal},2}$ and $\mathcal{L}_N^{\mathbf{Gal},3}$

The relation between $\mathcal{L}_N^{\mathrm{Gal},2}$ and $\mathcal{L}_N^{\mathrm{Gal},3}$ can be found by starting from the definition of J^μ

$$J_N^\mu = \pi_\lambda \pi^\lambda \mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \pi_{\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}. \tag{A.57}$$

Differentiating J^μ gives

$$\begin{aligned}
\partial_\mu J_N^\mu &= \pi_{\lambda\mu}\pi^\lambda \mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \pi_{\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n} \\
&\quad + \pi_\lambda \pi^\lambda{}_\mu \mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \pi_{\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n} \\
&\quad + \pi_\lambda \pi^\lambda \mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \pi_{\nu_1\mu}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n} \qquad \text{(A.58)} \\
&= 2\cdot \pi_\lambda \pi^\lambda{}_\mu \mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \pi_{\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n} + \mathcal{L}_N^{\mathrm{Gal},3} \\
&= 2\mathcal{L}_N^{\mathrm{Gal},2} + \mathcal{L}_N^{\mathrm{Gal},3} \qquad \text{(A.59)}
\end{aligned}$$

$$\Longrightarrow \qquad \mathcal{L}_N^{\mathrm{Gal},2} = -\frac{1}{2}\mathcal{L}_N^{\mathrm{Gal},3} + \frac{1}{2}\partial_\mu J_N^\mu. \tag{A.60}$$

Normally, by the product rule, there would be n terms generated when differentiating the π's behind the $\mathcal{A}$ tensor. However, due to the antisymmetry of the tensor, all but the first one drop out. For example, the second term would be

$$\pi_\lambda \pi^\lambda \mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \pi_{\nu_1}\pi_{\mu_2\nu_2\mu}\cdots\pi_{\mu_n\nu_n}. \tag{A.61}$$

But since interchanging the derivatives μ_2 and μ on π is symmetric, but interchanging these indices in $\mathcal{A}$ is antisymmetric, the term is zero.

To find the relation between the three Lagrangians, the properties of $\mathcal{A}$ that were already used in appendix A.3.2 are used. When the relations of equations (A.45) and (A.47) are combined, this becomes [7]

$$\mathcal{A}_{(2n)}^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_n} = -\delta^{\mu_1\mu_2\ldots\mu_n}_{\nu_1\nu_2\ldots\nu_n} \tag{A.62}$$

and

$$\begin{aligned}
\delta^{\mu_1\mu_2\ldots\mu_{n+1}}_{\nu_1\nu_2\ldots\nu_{n+1}} &= \sum_{i=1}^{n+1}(-1)^{i-1}\delta^{\mu_1}_{\nu_i}\delta^{\mu_2\mu_3\ldots\mu_{n+1}}_{\nu_1\nu_2\ldots\nu_{i-1}\nu_{i+1}\ldots\nu_{n+1}} \qquad \text{(A.63)} \\
&= \delta^{\mu_1}_{\nu_1}\delta^{\mu_2\ldots\mu_{n+1}}_{\nu_2\ldots\nu_{n+1}} + \sum_{i=2}^{n+1}(-1)^{i-1}\delta^{\mu_1}_{\nu_i}\delta^{\mu_2\mu_3\ldots\mu_{i-1}\mu_i\mu_{i+1}\ldots\mu_{n+1}}_{\nu_1\nu_2\ldots\nu_{i-1}\nu_{i+1}\ldots\nu_{n+1}}. \qquad \text{(A.64)}
\end{aligned}$$

I first rewrite $\mathcal{L}_N^{\text{Gal},1}$ using the above relations

$$\begin{aligned}
\mathcal{L}_N^{\text{Gal},1} &= \left(\mathcal{A}_{(2n+2)}^{\mu_1\dots\mu_{n+1}\nu_1\dots\nu_{n+1}}\pi_{\mu_{n+1}}\pi_{\nu_{n+1}}\right)\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n} && \text{(A.65)}\\
&= -\delta^{\mu_1\dots\mu_{n+1}}_{\nu_1\dots\nu_{n+1}}\pi_{\mu_{n+1}}\pi^{\nu_{n+1}}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n} && \text{(A.66)}\\
&= \overbrace{-\delta^{\mu_{n+1}}_{\nu_{n+1}}\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}\pi_{\mu_{n+1}}\pi^{\nu_{n+1}}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n}}^{A} && \text{(A.67)}\\
&\quad \underbrace{-\sum_{i=1}^{n}(-1)^{i-1}\delta^{\mu_{n+1}}_{\nu_i}\delta^{\mu_1\mu_2\dots\mu_{i-1}\mu_i\mu_{i+1}\dots\mu_n}_{\nu_1\nu_2\dots\nu_{i-1}\nu_{i+1}\dots\nu_{n+1}}\pi_{\mu_{n+1}}\pi^{\nu_{n+1}}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n}}_{B}.
\end{aligned}$$

The terms A and B in (A.67) will, for practical purposes, be worked out separately, which gives

$$\begin{aligned}
A &= -\delta^{\mu_{n+1}}_{\nu_{n+1}}\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}\pi_{\mu_{n+1}}\pi^{\nu_{n+1}}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n}\\
&= \delta^{\mu_{n+1}}_{\nu_{n+1}}\mathcal{A}_{(2n)}^{\mu_1\dots\mu_n\nu_1\dots\nu_n}\pi_{\mu_{n+1}}\pi^{\nu_{n+1}}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n}\\
&= \pi_{\mu_{n+1}}\pi^{\mu_{n+1}}\mathcal{A}_{(2n)}^{\mu_1\dots\mu_n\nu_1\dots\nu_n}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n}\\
&= \mathcal{L}_N^{\text{Gal},3}, && \text{(A.68)}
\end{aligned}$$

$$\begin{aligned}
B &= \delta^{\mu_{n+1}}_{\nu_1}\delta^{\mu_1\mu_2\dots\mu_n}_{\nu_2\nu_3\dots\nu_{n+1}}\pi_{\mu_{n+1}}\pi^{\nu_{n+1}}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n}\\
&\quad + \delta^{\mu_{n+1}}_{\nu_2}\delta^{\mu_1\mu_2\dots\mu_n}_{\nu_1\nu_3\nu_4\dots\nu_{n+1}}\pi_{\mu_{n+1}}\pi^{\nu_{n+1}}\pi_{\mu_1}{}^{\nu_1}\cdots\pi_{\mu_n}{}^{\nu_n}\\
&\quad + \cdots\\
&= n\cdot\delta^{\mu_{n+1}}_{\nu_{n+1}}\delta^{\mu_1\mu_2\dots\mu_n}_{\nu_1\nu_2\dots\nu_n}\pi_{\mu_1}\pi^{\nu_{n+1}}\pi_{\mu_{n+1}}{}^{\nu_1}\pi_{\mu_2}{}^{\nu_2}\cdots\pi_{\mu_n}{}^{\nu_n}\\
&= n\cdot\delta^{\mu_1\mu_2\dots\mu_n}_{\nu_1\nu_2\dots\nu_n}\pi_{\mu_1}\pi^{\mu_{n+1}}\pi_{\mu_{n+1}}{}^{\nu_1}\pi_{\mu_2}{}^{\nu_2}\cdots\pi_{\mu_n}{}^{\nu_n}\\
&= n\cdot\mathcal{A}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\pi_{\mu_1}\pi_{\lambda}\pi^{\lambda}{}_{\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}\\
&= n\cdot\mathcal{L}_N^{\text{Gal},2} = (N-2)\mathcal{L}_N^{\text{Gal},2}. && \text{(A.69)}
\end{aligned}$$

Putting the result for A and B back in (A.67) then gives the relation we were after:

$$\mathcal{L}_N^{\text{Gal},1} = \mathcal{L}_N^{\text{Gal},3} - (N-2)\mathcal{L}_N^{\text{Gal},2} \quad\Rightarrow\quad (N-2)\mathcal{L}_N^{\text{Gal},2} = \mathcal{L}_N^{\text{Gal},3} - \mathcal{L}_N^{\text{Gal},1}. \quad \text{(A.70)}$$

A.4 Higher order Lagrangian for classical mechanics

A.4.1 Example of third order Lagrangian

In the text the Lagrangian

$$L(q,\dot{q},\ddot{q},\dddot{q}) = \left(2q^2\dot{q}\ddot{q} - 2q\dot{q}^2\ddot{q} + 3q\ddot{q}^2\right)\dddot{q} + q^2\ddot{q} - \dot{q}\ddot{q}^2 + \ddot{q}^3 \qquad \text{(A.71)}$$

was taken as an example of a Lagrangian that is linear in $\dddot{q}$, but not in $\ddot{q}$. Still, it will give normal equations of motion, because of the specific choice of $f(q, \dot{q}, \ddot{q})$ and $g(q, \dot{q}, \ddot{q})$.

First we can take

$$F(q, \dot{q}, \ddot{q}) = q^2 \dot{q} \ddot{q}^2 - q \dot{q}^2 \ddot{q}^2 + q \ddot{q}^3 \tag{A.72}$$

and identify

$$F_q(q, \dot{q}, \ddot{q}) = 2q\dot{q}\ddot{q}^2 - \dot{q}^2\ddot{q}^2 + \ddot{q}^3 \tag{A.73}$$

$$F_{\dot{q}}(q, \dot{q}, \ddot{q}) = q^2\ddot{q}^2 - 2q\dot{q}\ddot{q}^2 \tag{A.74}$$

$$F_{\ddot{q}}(q, \dot{q}, \ddot{q}) = 2q^2\dot{q}\ddot{q} - 2q\dot{q}^2\ddot{q} + 3q\ddot{q}^2 \tag{A.75}$$

where it can be seen that indeed $F_{\ddot{q}} = f$.The Lagrangian can then be written as

$$L(q, \dot{q}, \ddot{q}, \dddot{q}) = \frac{d}{dt} F(q, \dot{q}, \ddot{q}) - F_{\dot{q}}(q, \dot{q}, \ddot{q})\, \ddot{q} - F_q(q, \dot{q}, \ddot{q}) + g(q, \dot{q}, \ddot{q}) \tag{A.76}$$

$$= \frac{d}{dt}\left(q^2\dot{q}\ddot{q}^2 - q\dot{q}^2\ddot{q}^2 + q\ddot{q}^3\right) - q^2\ddot{q}^2 + \dot{q}^2\ddot{q}^2 - \ddot{q}^3 + g(q, \dot{q}, \ddot{q}) \tag{A.77}$$

Taking

$$g(q, \dot{q}, \ddot{q}) = q^2\ddot{q}^2 - \dot{q}^2\ddot{q}^2 + \ddot{q}^3 + h(q, \dot{q}) \tag{A.78}$$

will make the $\tilde{g}(q, \dot{q}, \ddot{q}) \equiv -F_{\dot{q}}(q, \dot{q}, \ddot{q})\, \ddot{q} - F_q(q, \dot{q}, \ddot{q}) + g(q, \dot{q}, \ddot{q})$ up to linear in $\ddot{q}$ One can see that these are indeed the f and g in the Lagrangian of this example.