Investigating the absence of Dirac bound states in a linear potential well

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31 january 2017

Abstract

There are no normalizable bound states for the Dirac equation for a linear confining potential in 1+1 dimensions, while its nonrelativistic approximation - the Schrödinger equation - has bound solutions. We investigate how well these Schrödinger solutions solve the Dirac equation by time-evolving them and determining their energy distribution. Their evolution under the Dirac equation can be explained in terms of unstable resonance solutions for the Dirac equation. We show that these resonances also appear as transmission resonances for positrons scattering off the linear potential.

1 Introduction

In 1928 Dirac introduced his famous Dirac equation, the relativistic counterpart to the quantummechanical Schödinger equation[1]. While very successful, the Dirac equation had some problematic features such as the existence of negative energy states, which were only fully resolved by quantum field theory. The solutions of the Dirac equation for a Coulomb potential in 3 dimensions are well known[2]. In 1 dimension a point charge produces a linear potential. For this potential no normalizable energy eigenstates exist[3], and the energy spectrum is purely continuous. However, in the nonrelativistic limit of the same problem normalizable solutions with a discrete energy spectrum do exist, and can be expressed in terms of Airy functions. We investigate the transition between these cases and qualitatively try to understand what causes the differences between them.
To this end we investigate the time-evolution under the Dirac equation for an initial state that solves the Schrödinger equation. In a parameter range where we expect relativistic effects to be small, this state will be approximately stationary. As we approach the region where the nonrelativistic approximation breaks down, the bound states become resonances with a finite lifetime and width.[4]

2 Theory

2.1 The Dirac equation

The Dirac equation is the relativistic equation of motion of a spin-half particle. In the relativistically covariant form it reads:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = (i\hbar\gamma^0\frac{\partial}{\partial ct} + i\hbar\gamma^1\frac{\partial}{\partial x} - mc)\psi(x) = 0 \tag{1}$$

The index μ is 0 for the time dimension and 1 for the spatial dimension. The gamma matrices γ^μ must satisfy the Clifford algebra anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1} = 2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\mu\nu}\mathbb{1}. \tag{2}$$

In 3+1 dimensions this requires 4×4 matrices, which is why Dirac spinors in that space have four components. However, in 1+1 dimensions we can find two 2×2 matrices γ^0, γ^1 that satisfy the Clifford algebra. We will use

$$\gamma^0 = \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \text{and} \qquad \gamma^1 = i\sigma_2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{3}$$

Since these matrices are 2×2 the Dirac equation acts on a spinorial wavefunction with only two components

$$\psi(x) = \begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix}. \tag{4}$$

Multiplying equation (1) by $c\gamma^0$ we find that

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} = \left[-i\hbar c\gamma^0\gamma^1\frac{\partial}{\partial x} + \gamma^0 mc^2\right]\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix}. \tag{5}$$

The Dirac equation conserves the current

$$\overline{\psi}\gamma^\mu\psi(x) := \psi^\dagger(x)\gamma^0\gamma^\mu\psi(x) \tag{6}$$

with its 0^{th} component $\overline{\psi}\gamma^0\psi(x) := \psi^\dagger(x)\psi(x) = \phi^*\phi + \chi^*\chi$ being positive definite and acting as a probability density.

2.2 Coupling to electromagnetic field

The Dirac equation can be modified to include interactions with an external electromagnetic field, by coupling it to the electromagnetic 2-vector potential A^μ. The usual way to introduce the field is the minimal coupling, given by $\gamma^\mu\partial_\mu \to \gamma^\mu(\partial_\mu + i\frac{e}{\hbar c}A_\mu)$. The Dirac equation (1) becomes

$$\left[\gamma^\mu(i\hbar\partial_\mu - \frac{e}{c}A_\mu) - mc\right]\psi(x) = 0. \tag{7}$$

Coupling the field to an electrostatic potential corresponds to setting $eA^0(x) = V(x)$ and $A^1 = 0$. If we add this term in equation (5) we find the equations of motion for the components $\phi(x,t)$ and $\chi(x,t)$ to be

$$\begin{aligned}\left[i\hbar\frac{\partial}{\partial t} - eA^0(x)\right]\phi &= -i\hbar c\tfrac{\partial}{\partial x}\chi + mc^2\phi \\ \left[i\hbar\frac{\partial}{\partial t} - eA^0(x)\right]\chi &= -i\hbar c\tfrac{\partial}{\partial x}\phi - mc^2\chi.\end{aligned} \tag{8}$$

These equations also describe the evolution of a Dirac spinor in 3+1 dimensions, as long as the wave function and potentials do not depend on the y and z (2 and 3) dimensions, and the y and z component of the vector potential are zero. In that case the two different spin states do not interact, which explains

the reduction from a four-component to a two-component spinor.
From now on we will use natural units where

$$\hbar = c = 1. \tag{9}$$

2.3 Nonrelativistic limit

We will demonstrate how and under what circumstances the Dirac equation in an external potential reduces to the Schrödinger equation, in the presence of an external potential $V(x)$ but in the absence of a magnetic field.
In the nonrelativistic limit the rest mass energy m is the largest term in the time evolution of the spinor components. We introduce more slowly varying functions Φ and X by

$$\begin{aligned} \Phi &= e^{imt}\varphi \\ X &= e^{imt}\chi. \end{aligned} \tag{10}$$

Making this replacement in the component equations (8) gives their time evolution:

$$\left[i\frac{\partial}{\partial t} - V(x)\right]\Phi = -i\tfrac{\partial}{\partial x}X \tag{11}$$

$$\left[i\frac{\partial}{\partial t} - V(x)\right]X = -i\tfrac{\partial}{\partial x}\Phi - 2mX \tag{12}$$

The operator for the nonrelativistic energy is $E^{NR} \equiv i\frac{\partial}{\partial t}$, and does not include the rest-mass energy. In the nonrelativistic limit this energy and the potential energy is small compared to the rest-mass energy, so

$$|E^{NR} - V(x)| = |i\frac{\partial}{\partial t} - V(x)| \ll 2m. \tag{13}$$

Then equation (12) gives

$$X \approx \frac{-i}{2m}\frac{\partial}{\partial x}\Phi \tag{14}$$

and we get the second order equation

$$i\frac{\partial}{\partial t}\Phi = \left[\frac{-1}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Phi \tag{15}$$

which is the Schrödinger equation for a potential $V(x)$.
We can rewrite the condition (13) in another way. Using the Schrödinger equation we get

$$|(E^{NR} - V(x))\Phi| = |\frac{p^2}{2m}\Phi| \ll 2m\Phi \tag{16}$$

$$|p^2\Phi| \ll (2m)^2\Phi \tag{17}$$

$$|p\Phi| = |\frac{\partial}{\partial x}\Phi| \ll 2m\Phi. \tag{18}$$

The nonrelativistic approximation breaks down if the wavefunction changes substantially over a distance of the order of the Compton wavelength $\frac{1}{m}$. Also, since E^{NR} is constant for an eigenfunction, if the potential changes by an amount comparable to the rest-mass energy condition 13 cannot hold everywhere. In those cases the evolution cannot be described by the Schrödinger equation and we expect relativistic effects to play a significant role.

2.3.1 Positron solutions

The Schrödinger equation describes the evolution of positive energy electrons. The Dirac equation also has negative energy solutions.
A negative energy electron state will in the nonrelativistic limit have an energy dominated by the negative rest mass energy $E = -mc^2$, so slow-moving component functions should be defined by inverting the sign in the exponent in equation (10). Then, assuming a large lower component $X \gg \Phi$ we can find its evolution is described by the Schrödinger equation for a positive energy particle but for an inverted potential $V(x) = -eA^0(x)$. We can interpret this as the equation of motion of a positron, which has an inverted charge.

2.3.2 Plane wave solutions

The Dirac equation in the absence of a potential admits a solution into plane waves. These plane are written as

$$\begin{pmatrix} \varphi(x,t) \\ \chi(x,t) \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} e^{\pm i(kx - Et)} \tag{19}$$

where all spacetime dependence is in the exponential, which is multiplied by a fixed spinor. The solutions are labeled by the momentum k. For any value of the momentum we have the following two solutions:

$$\begin{aligned} \psi(x,t) = u(k)e^{+(ik-Et)} &= \begin{pmatrix} \left(\frac{E+m}{2m}\right)^{1/2} \\ \frac{\mathbf{k}}{[2m(m+E)]^{1/2}} \end{pmatrix} e^{+(ik-Et)} \\ \psi(x,t) = v(k)e^{-(ik-Et)} &= \begin{pmatrix} \frac{\mathbf{k}}{[2m(m+E)]^{1/2}} \\ \left(\frac{E+m}{2m}\right)^{1/2} \end{pmatrix} e^{-(ik-Et)} \end{aligned} \tag{20}$$

In these solutions we have $E = \sqrt{k^2 + m^2}$ and the phase velocity is in the direction of k. However, the second solution actually has a negative energy $-E$, as can be seen from the minus sign in the exponent. Operating on it with the energy operator $i\frac{d}{dt}$ we get a negative value, and operating with the momentum operator $-i\frac{d}{dx}$ the momentum points in the opposite direction from k.
We can add a constant potential $V_0 = eA$. All this will do is shift the energy of all solutions up by V_0, as can be seen from the equation (8). Then the allowed eigenvalues of the energy operator $M \equiv i\frac{d}{dt}$ are in the range

$$M \leq -m + V_0 \text{ or } M \geq m + V_0. \tag{21}$$

2.4 Relativistic effects

In this section we will describe various relativistic effects that occur in problems involving the Dirac equation

2.4.1 Dirac sea

The Dirac equation has solutions of arbitrarily low energy. This is a problem when the electron is allowed to interact with an electromagnetic field, since it can then fall down into these negative energy states, emitting the infinite amount of energy it gains in the form of photons. This process is not observed.
Dirac proposed that all negative energy states are already filled by electrons in the vacuum state. Therefore a positive energy electron can not fall down into those already occupied states. If one electron is raised up from the negative energy Dirac sea to a positive energy the hole that is left behind will act as a positive energy positively charged particle. This hole can then be interpreted as a positron, the antiparticle of the electron. In the process of raising up an electron from the Dirac sea a positron-electron pair has been created.

In modern quantum-field theoretical treatments the picture of the Dirac sea is no longer needed, and antiparticles are not considered holes but are on equal footing with particles. It correctly describes

multiparticle processes. However, in solid state systems holes in filled valence bands actually occur. These are not real elementary particles, but it is useful to treat them as positively charged quasiparticles.

2.4.2 Supercritical potential well

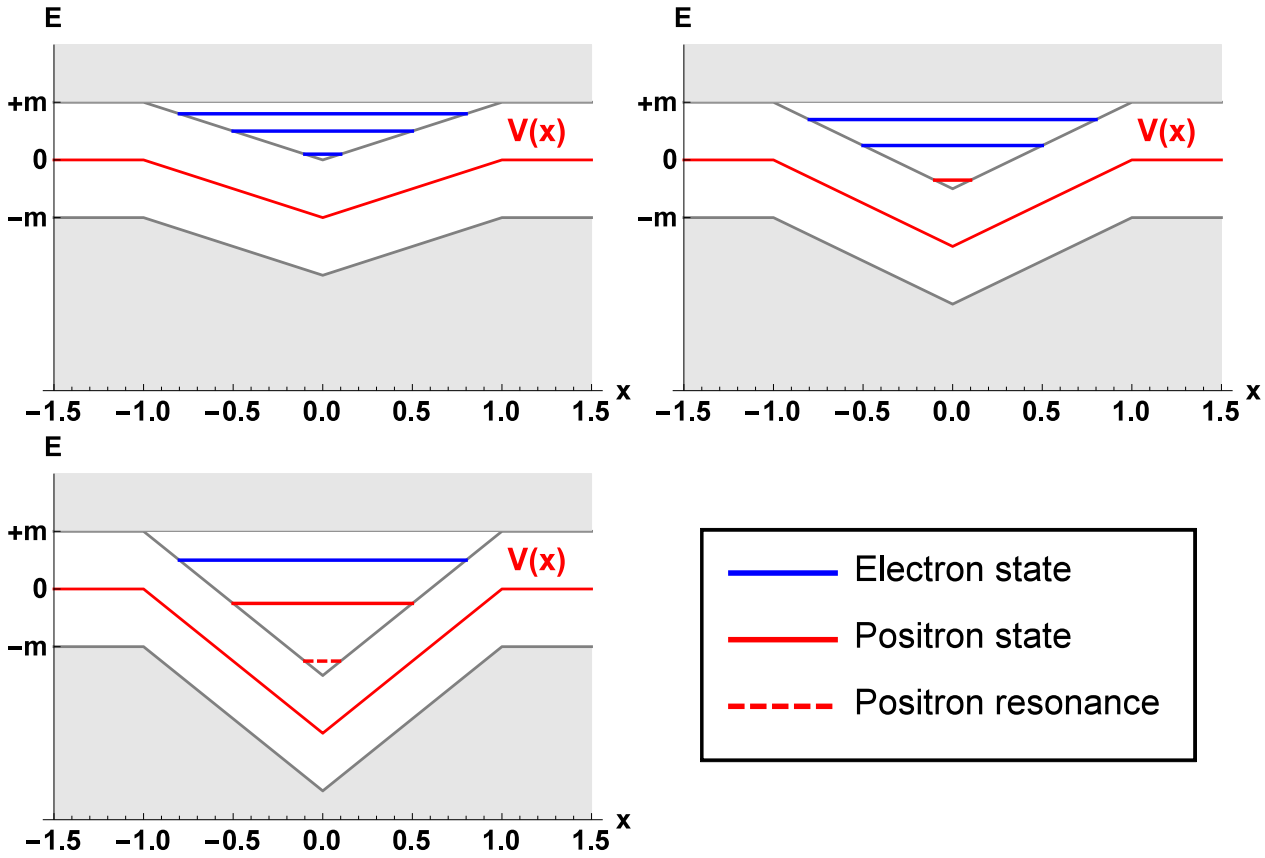


Figure 1: In a supercritical potential some bound states become resonances

For an electrostatic potential the Schrödinger equation we can distinguish three kinds of solutions. If the potential is sufficiently localized (meaning the potential is zero outside some region or falls off quickly like the Coulomb potential), there will still be a continuum of unbound states with energy $M > +m$. A state in the potential with $M < +m$ is a stable bound state, because an electron in such a state has insufficient energy to transition into the continuum. Sometimes there are also metastable resonances with energy $E > m$. An electron in such a state can remain bound for some time because it is separated from the continuum states by a potential barrier which the electron has to tunnel through before escaping. Because of their finite lifetime these resonances do not have a definite energy value.

For the Dirac equation there is also a continuum of unbound negative energy states, with energy $M < -m$. Therefore there can only be bound states in the energy range $-m < M < +m$. If a potential is turned on over time it can happen that an empty positive energy bound state is lowered into the negative energy continuum, becoming a resonance. This process can be thought of in terms of the Dirac sea. An empty bound state of negative energy is a hole in the Dirac sea, so as soon as the empty state has crossed from $M > 0$ to $M < 0$ we should interpret it as a filled positron bound state. If the potential is lowered further this positron bound state becomes a resonance. It has a finite lifetime because the state is still separated from the negative energy continuum by a forbidden region where $-m < M + V(x) < m$. We then say that the potential has become supercritical. The positron can escape (or alternatively the empty state can be filled by a negative energy electron from the Dirac sea). If the state is raised

to positive energy again the end result is the creation of an electron-positron pair.[5] If quantum field theory is taken into account the vacuum is interpreted somewhat differently, but the same process still leads to pair creation.[6] This is similar to the Schwinger pair creation due to a strong constant field, but in the case of a supercritical potential only a limited number of positron-electron pairs can be created before all available states are filled.

The potential of a nucleus is predicted to become supercritical at a nuclear charge of about 172. Nuclei of this charge have not been produced, but the same charge distribution can be created briefly in the collision of two heavy ions. There have been experimental searches for the production of positrons in these heavy ion collisions, but the results remain inconclusive[7]

2.4.3 Klein paradox

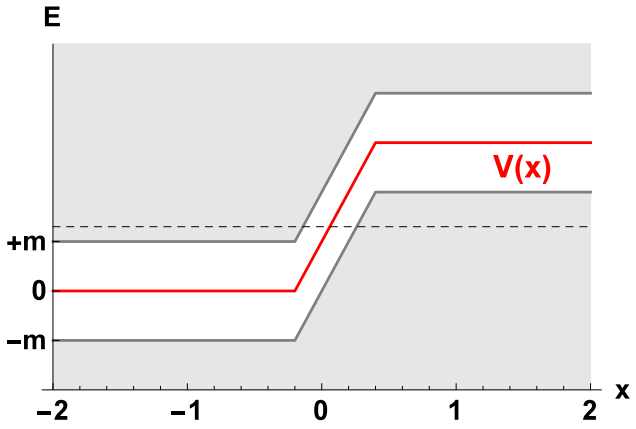


Figure 2: Klein paradox: an incoming positive energy plane wave can scatter to a negative energy state

The Klein paradox arises from the scattering of a plane wave off a potential step. When a particle hits an extremely high and abrupt potential step, it may propagate into the classically forbidden region.[8]. On the left of the potential step, we have a propagating wave if $M > m$. On the right side, a positive energy wave can be transmitted if $M > m + V$. If the potential step is higher than $2m$ and the energy of the incoming wave lies in the range

$$m < M < V - m \tag{22}$$

a positive energy wave on the left side of the potential step can have the same energy as a negative energy wave on the right side of the potential

In that case an incoming positive energy can be partially transmitted through the barrier. This situation is shown in figure 2. For a heaviside step potential

$$V(x) = 0 \text{ if } x < 0 \text{ and } V(x) = V \text{ if } x > 0 \tag{23}$$

the transmission and reflection coefficients can be found by taking a superposition of an incoming wave with momentum k and amplitude 1 and a reflected wave with momentum $-k$ on the left side of the potential step, and a right-moving transmitted wave on the right side. Here care must be taken, since the potential has shifted the energy of the transmitted wave from a negative to a positive value. This has inverted the direction of the phase velocity with respect to the momentum. To get a wave propagating to the right, we should take the negative value of the momentum in equation (21) consistent with the energy conservation: $q = -\sqrt{(M+V)^2 - m^2}$. [9]
Solving the waves to be continuous at x=0 we indeed find a nonzero transmission coefficient for the energy M in the range (22).
If the height of the step is taken to infinity and the energy kept fixed the probability for transmission tends to a constant value. This is unexpected, nonrelativistically the probability for tunneling through a potential barrier goes down exponentially with the height of the barrier.
This effect has been shown to be due to the abrupt potential step, which localizes the particle within a

Compton wavelength. Since the particle is localized, it will be coupled to negative energy states, and particle-antiparticle pairs can be created. When the potential increases linearly over some interval, as first worked out by Sauter, the transmission probability does decrease exponentially unless the potential step occurs in a distance of the order of or smaller than the Compton wavelength of the particle. [10]
In a quantum field theoretical treatment a similar situation exists. However, in the fully quantized theory the negative energy states aren't present, and are replaced by positive energy positrons. Charge is conserved, so the incoming electron cannot tunnel to the oppositely charged positron states on the other side of the potential, but is completely reflected. However, in that case, the vacuum is unstable, and the supercritical electric field in the potential step will continuously create electron-positron pairs out of the vacuum[11]. The electron will temporarily decrease the pair creation rate because it partially screens the potential. This results in less positrons being emitted to the right, the net effect looking like the partial transmission of the electron.[9]

2.4.4 Zitterbewegung

A wave packet of Dirac plane waves with a width of the order of its Compton wavelength is predicted to undergo an extremely fast trembling motion known as Zitterbewegung[12][2]. This Zitterbewegung is caused by interference between the positive and negative energy plane waves that make up the wave packet. To derive this oscillation we must expand the initial state into plane wave solutions of the Dirac equation.
As an initial state we take a Gaussian wave packet with the lower component zero:

$$\psi(0,\mathbf{x}) = \frac{1}{(\pi d^2)^{3/4}} e^{-x^2/2d^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

This can be expanded into an integral over momentum eigenstates:

$$\psi(t,\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{E} \left[b(p)u(p)e^{-ip\cdot x} + d^*(p)v(p)e^{ip\cdot x} \right] \tag{25}$$

We can find the coefficient functions b and d by Fourier transforming this state. This involves the Fourier transform of a Gaussian

$$\int d^3x e^{-\mathbf{x}^2/2d^2} e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi d^2)^{3/2} e^{-\mathbf{p}^2 d^2/2}. \tag{26}$$

We can construct a wave-packet that starts with its small component identically zero. However, this requires some negative frequency solutions in the expansion to cancel the lower component of the positive frequency solutions. Since they oscillate with opposite frequency, at later times they will interfere constructively, giving rise to an oscillating negative-frequency component in the wave packet.
The oscillation frequency is larger than $2m$, an extremely high frequency, and has never been observed in experiments.
A wide wave-packet does not contain plane waves with a large momentum. Therefore the oscillating small component will have a small amplitude. But if the wave packet is constrained to a size of the order of or smaller than its Compton wavelength the Zitterbewegung will have a considerable amplitude.

2.4.5 Foldy-Wouthuysen transformation

All these relativistic effects have their origin in the interaction between positive-energy and negative-energy solutions. The cause of this coupling is the fact that the upper and lower component of the spinor cannot be identified with positive or negative energy states, and are coupled by the Hamiltonian. A transformation known as the Foldy-Wouthuysen transformation exists that diagonalizes the Hamiltonian and therefore separates the positive and negative energy states. For a free Dirac spinor the transformation is relatively easy, since there are no interactions. After the transformation the upper and lower components correspond to the plane-wave energy eigenstates we have already found.
For a weak field the first few terms of the transformation can be calculated. They lead to some corrections of the hamiltonian of an electron, giving it a magnetic moment (the spin-orbit coupling term) and the Darwin term.

For a strong field the Foldy-Wouthuysen transformation is unfortunately not useful, since both components are coupled intricately and not easily separated[13].

2.5 The linear potential

The potential we will investigate is the linear potential. This is the potential generated by a point source. A point source with charge $+Ze$ at the origin has the charge distribution $Ze\delta(x)$. By Gauss' law in one dimension

$$\nabla \cdot \overrightarrow{E} = \frac{\partial \overrightarrow{E}}{\partial x} = \rho/\epsilon_0 = Ze\delta(x)/\epsilon_0 \tag{27}$$

the resulting electric field is $\overrightarrow{E}(x) = Ze\,\mathrm{sgn}(x)/2\epsilon_0$ and the electrostatic potential is $A^0(x) = Ze|x|/2\epsilon_0$. The potential energy is

$$V(x) = Ze^2|x|/2\epsilon_0 = g|x|/2 \tag{28}$$

where the field strength is given by $g = Ze^2/\epsilon_0$. However, the vacuum permittivity in 1+1 dimensions is different from that in three dimensions, since it has a different dimension.
Alternatively, the same potential arises from a 3+1 dimensional problem which is symmetric in two directions. Then the field is generated by a planar charge density at $x = 0$, and g denotes the electrostatic potential in Joule per meter. The electric field strength is g/e.

2.5.1 Constants and parameters

The Dirac equation for a linear potential says that

$$\left[i\frac{\partial}{\partial t} - \frac{g|x|}{2}\right]\psi(x,t) = -i\frac{\partial}{\partial x}\gamma^0\gamma^1\psi(x,t) + \gamma^0 m\psi(x,t). \tag{29}$$

Here we see that $g|x|$ has the same dimension as m, so it has the dimension of energy. So indeed g has the dimension of energy per meter.
There are two parameters in the problem: the mass of the electron m and the strength of the potential $g = Ze^2/\epsilon_0$. We can choose our units so that the fields strength $g \equiv 1$, in which case all lengths and energies become dimensionless. This means the units will depend on the field strength, but it has the advantage that only a single free parameter remains in the problem. We can return to dimensionful quantities to physical units by multiplying by the appropriate powers of g, $\hbar$ and c. Our units are:

$$\begin{aligned} [l] &= \sqrt{\hbar c/g} \\ [t] &= \sqrt{\hbar/gc} \\ [E] &= \sqrt{\hbar g c} \\ [m] &= \sqrt{\hbar g/c^3} \end{aligned} \tag{30}$$

If the mass of our particle equals the electron mass, the electric field strength for which the dimensionless parameter m equals one is

$$g = 0.21 J/m; |E| = 1.3 \cdot 10^{18} V/m$$

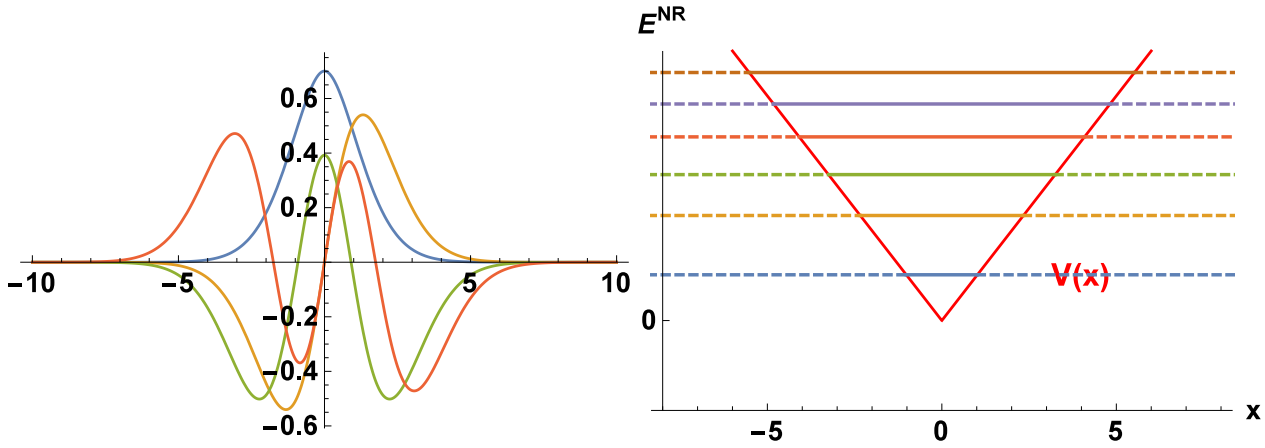
which is equal to the Schwinger limit.

Since we work with natural units $\hbar = c = 1$, the nonrelativistic limit $c \to \infty$ can no longer be taken. Instead we should consider the limit $m \to \infty$, since this is the limit in which the rest-mass energy is the main contribution to the total energy. Because the unit of mass depends on the field strength as above, this is also the limit of the field strength going to zero, as should be expected.

2.5.2 Nonrelativistic limit

In the nonrelativistic limit $m \to \infty$ the Dirac equation reduces to the Schrödinger equation, which has a discrete spectrum of normalizable bound states, which can be expressed in terms of the Airy functions.

2.5.3 Nonrelativistic Schrödinger solutions



(a) First four Airy solutions (for $m = 1$)

(b) Solutions oscillate in the allowed region (solid lines) and decay exponentially in the forbidden region (dashed lines)

Figure 3: Nonrelativistic solutions

In the nonrelativistic limit we have to solve the Schrödinger equation for a linear potential $V(x) = |x|/2$, for energy E^{NR}. The equation is

$$E^{NR}\Phi(x) = \left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + |x|/2\right]\Phi(x). \tag{31}$$

For $x > 0$ we can write this as

$$\frac{\partial^2}{\partial x^2}\Phi(x) = m(x - 2E^{NR})\Phi(x). \tag{32}$$

We can change variables to

$$z \equiv m^{1/3}(x - 2E^{NR}) \tag{33}$$

and find that

$$\frac{\partial^2}{\partial z^2}\Phi(z) = m^{1/3}(x - 2E^{NR})\Phi(x) = z\Phi(z). \tag{34}$$

This is Airy's differential equation[14]. The solutions are the two linearly independent Airy functions $Ai(z)$ and $Bi(z)$. However, since we want a normalizable wavefunction and $Bi(z)$ grows exponentially as $x \to \infty$, only the $Ai(z)$ solution should be used.
The potential is symmetric, so the solutions are either even or odd, and $\Phi(-x) = \pm\Phi(x)$. The continuity condition at $x = 0$ - for odd solutions Φ has to vanish at $x = 0$, for even solutions its first derivative has to vanish - then determines the allowed discrete energy eigenvalues.

$$\Phi(x)|_{x=0} = Ai(m^{1/3}(x - 2E^{NR}))|_{x=0} = Ai(-2m^{1/3}E^{NR}) \tag{35}$$

so for every zero $Ai(z_n) = 0$ there is an odd solution, and for every zero of the derivative $Ai'(z_n) = 0$ there is an even solution. The energy of the solutions is $E^{NR} = -z_n m^{-1/3}/2$. $Ai(z)$ and its derivative have an infinite number of zeros, so there are an infinite number of even and odd solutions, all with positive energy. The ground state is even and has energy $E^{NR} \approx 1.01879m^{-1/3}/2$
We can use this solution to predict at what value of m we expect relativistic effects to become important. From section 2.3 we know that the nonrelativistic approximation only works as long as the wavefunction

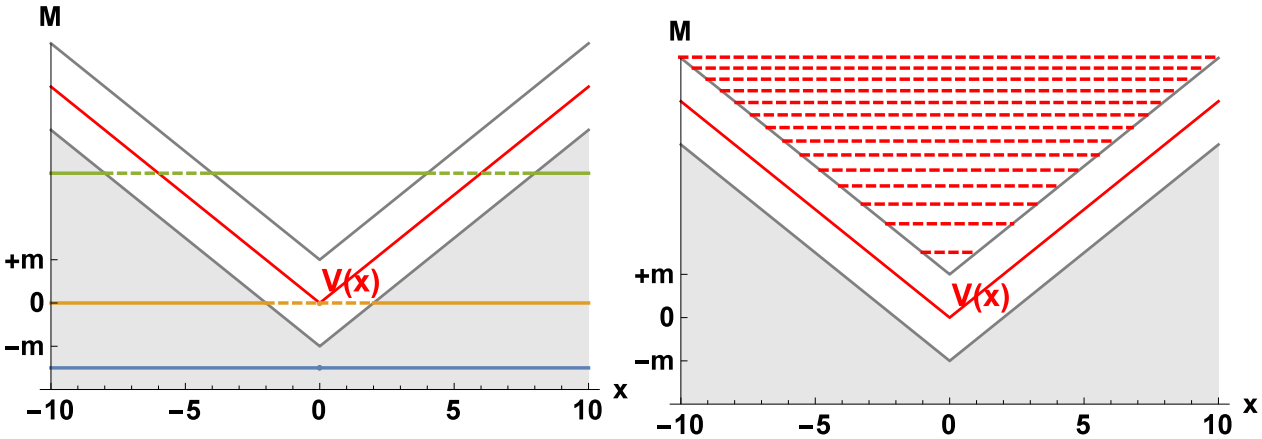
does not vary significantly over one Compton wavelength. For $m = 1$ the Compton wavelength is one, and the FWHM is 2.7. Since the FWHM scales as $m^{-1/3}$ we solve

$$2.7/m^{1/3} = 1/m \Rightarrow m = (1/2.7)^{2/3} = 0.51.$$

So we expect relativistic effects to be important when m is of the order one, which occurs when the field strength is close to the Schwinger limit.
Comparing the Schrödinger and Dirac solutions we should remember the rest-mass energy,

$$M \approx E^{NR} + m. \tag{36}$$

2.5.4 Form of solutions of the linear potential



(a) Form of solutions. Solid line: oscillating. Dashed line: Combination of exponential growth and decay

(b) The linear potential; Dashed: resonances

Figure 4: Solutions of the relativistic Dirac equation

The Dirac equation, like the Schrödinger equation, has two linearly independent solutions for the linear potential. In the Schrödinger problem one of the two solutions (the Airy Bi function) could be discarded because it grows exponentially in the forbidden region, which extends all the way to $x \to \pm\infty$. Continuity at $x = 0$ is therefore only possible at those energies where the Airy Ai function is (anti) symmetric at $x = 0$. Therefore there are normalizable solutions at certain discrete energy values.

For the Dirac equation the situation is different. There are two classically allowed regions, where $M + V(X) < -m$ or $M + V(x) > m$, where the solutions will oscillate. The solutions will grow or decay exponentially in the classically forbidden region where $-m < M + V(x) < m$. Crucially, the tails of the solutions do not decay or grow exponentially in the region $x \to \pm\infty$ but will oscillate with an increasing wavenumber and approach a constant probability density. This situation is shown in figure 4a. Therefore both solutions are equally valid, and neither can be discarded. Because there are two linearly independent solutions we can find both a symmetric and an antisymmetric linear combination at $x = 0$ for every energy value. For this reason there is a continuous spectrum of nonnormalizable stationary states for the linear potential.

The electron states under the potential can also be viewed as resonances. This can be seen clearly by comparing the linear potential in figure 4b to the supercritical potential in figure 1. when the potential levels off after some distance x there are some resonances and some bound states. The bound states are similar to the Airy solutions of figure 3a, but changed somewhat if their energy is much larger than the rest-mass energy near $x = 0$. But for the linear potential every bound state has become a resonance because for every energy value there is a classically allowed region at $x \to \pm\infty$. These resonances will be visible in the solutions when the energy is close to the corresponding bound state energy.

2.5.5 Stationary states of the linear potential

For completeness we will solve the problem following Dietrich[15] and Hoyer[16].
The solutions are labeled by the energy eigenvalue M. The stationary states solve the Dirac equation with $i\frac{\partial}{\partial t} = M$ in the component equations

$$\begin{aligned}[M - |x|/2]\,\varphi &= -i\frac{\partial}{\partial x}\chi + m\phi \\ [M - |x|/2]\,\chi &= -i\frac{\partial}{\partial x}\varphi - m\chi\end{aligned} \tag{37}$$

which has two linearly independent solutions for every value of M.
We first solve this differential equation in the region $x \geq 0$. In this region we can change variables from x to $\sigma = (M - V(x))^2 = (M - x/2)^2$. Then $M - V(x) = \pm\sqrt{\sigma}$. We first solve for the region where $M - V(x) = \sqrt{\sigma} \geq 0$. The partial differentials can be replaced according to

$$\frac{\partial}{\partial x} = \frac{d\sigma}{dx}\frac{\partial}{\partial \sigma} = -(M - x/2)\frac{\partial}{\partial \sigma} = \mp\sqrt{\sigma}\frac{\partial}{\partial \sigma} \tag{38}$$

and the differential equation becomes

$$\begin{aligned}i\frac{\partial}{\partial \sigma}\chi(\sigma) &= \left(1 \mp \frac{m}{\sqrt{\sigma}}\right)\varphi(\sigma) \\ i\frac{\partial}{\partial \sigma}\varphi(\sigma) &= \left(1 \pm \frac{m}{\sqrt{\sigma}}\right)\chi(\sigma)\end{aligned} \tag{39}$$

These are two coupled first-order differential equations. From these equations it can be seen that χ and φ both obey a real second-order differential equation. Since the overall phase factor is unimportant φ can be taken real everywhere. Then the above equation shows that χ is purely imaginary. The two components can then be combined into a single complex function to simplify the solution process. We define the function

$$\phi(\sigma) \quad \equiv \quad [\varphi(\sigma) + \chi(\sigma)]e^{i\sigma} \tag{40}$$

Adding both equations (39) we find

$$\begin{aligned}i\frac{\partial}{\partial \sigma}\left[\chi(\sigma) + \phi(\sigma)\right] &= i\frac{\partial}{\partial \sigma}\left[\phi(\sigma)e^{-i\sigma}\right] = \left[i\frac{\partial\phi(\sigma)}{\partial\sigma} + \phi(\sigma)\right]e^{-i\sigma} \\ &= \phi(\sigma)e^{-i\sigma} \mp \frac{m}{\sqrt{\sigma}}\left[\phi(\sigma)e^{-i\sigma}\right]^* \\ \frac{\partial\phi(\sigma)}{\partial\sigma} &= \mp i\frac{m}{\sqrt{\sigma}}\phi(\sigma)^* e^{2i\sigma}\end{aligned} \tag{41}$$

This finally leads to the second order equation

$$2\sigma\frac{\partial^2\phi(\sigma)}{\partial\sigma^2} + (1 - 4i\sigma)\frac{\partial\phi(\sigma)}{\partial\sigma} - 2m^2\phi(\sigma) = 0 \tag{42}$$

This is the confluent hypergeometric differential equation, and it has two linearly independent solutions. A general solution will be a linear combination of these solutions. They are

$$\phi(\sigma) = (a_1 + ib_1)\,{}_1F_1\left(-\frac{1}{2}im^2, \frac{1}{2}, 2i\sigma\right) + (a_2 + ib_2)\sqrt{\sigma}\,{}_1F_1\left(\frac{1}{2} - \frac{1}{2}im^2, \frac{3}{2}, 2i\sigma\right) \tag{43}$$

where we separated the constants into their real and imaginary parts. The second solution depends on $\sqrt{\sigma}$. If we want continuity around $\sigma = 0$ we have to set $M - V = -\sqrt{\sigma}$ for $M - V < 0$.
Continuity and differentiability at $x = 0$ imposes a restriction on the parameters. By expanding the

solutions around this point in terms of σ and matching the largest terms one can show that $a_2 = 2mb_1$ and $b_2 = 2ma_1$. Using this restriction the general solution in the region $x \geq 0$ has the form

$$\begin{aligned}\varphi(x) + i\chi(x) = \Bigg[(a+ib)_1F_1\left(-\frac{im^2}{2};\frac{1}{2};2i(M-\frac{x}{2})^2\right) + \\ +2m(M-\frac{x}{2})(b+ia)_1F_1\left(\frac{1}{2}-\frac{im^2}{2};\frac{3}{2};2i(M-\frac{x}{2})^2\right)\Bigg]e^{-i(M-\frac{x}{2})^2}\end{aligned} \tag{44}$$

This shows that - up to multiplication by an overall constant - there are two linearly independent solutions to the Dirac equation for the linear potential, parameterized by the ratio of a to b in equation (44). These solutions have to be extended to $x < 0$ respecting continuity and differentiability at $x = 0$. This can be arranged straightforwardly by finding the two solutions of definite parity. One solution, which we label with $\eta = +1$, can be found by adjusting a and b such that $\phi'(0) = \chi(0) = 0$ and is extended to $x < 0$ according to

$$\phi(-x) = \phi(x); \chi(-x) = -\chi(x) \tag{45}$$

The solution of opposite parity labeled $\eta = -1$ found by adjusting a and b to satisfy $\phi(0) = \chi'(0) = 0$ and is extended to $x < 0$ by

$$\phi(-x) = -\phi(x); \chi(-x) = \chi(x) \tag{46}$$

Since we can find these two linearly independent solutions for any value of M the spectrum is entirely continuous.

Some of the solutions are shown in figure 5. They show the expected behavior, oscillating only in the regions where $M + V(x) > m$ or $M + V(x) < -m$. In figures 5a and 5b the amplitude of the oscillation near $x = 0$ is much smaller than the amplitude of the tail for the green solution with $M > m$. This is the case for most values of the energy, but there exists some energy values around which the oscillation near the origin becomes much larger. The wavefunction at the maximum of two such resonances are shown in figures 5c and 5d. As m is increased the shape of these resonances approaches the shape of the Airy solutions and the amplitude of the oscillating tails goes to zero with respect to the value of the wavefunction around $x = 0$.

All solutions approach a constant probability density as $x \to \pm\infty$. In fact, in the limit they approach

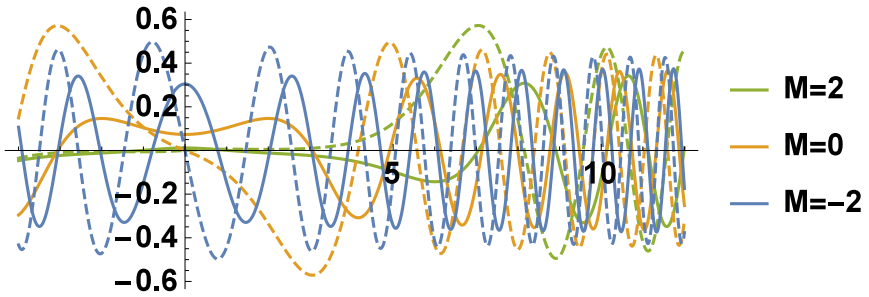
$$\lim_{x\to\infty}[\phi(x) + i\chi(x)] = C_{M,\eta}(M - x/2)^{im^2}e^{-i(m-x/2)^2} \tag{47}$$

with a proportionality constant depending on M and η. The solutions are orthogonal, as shown in the paper by Dietrich[15]. The inner product between two solutions satisfies

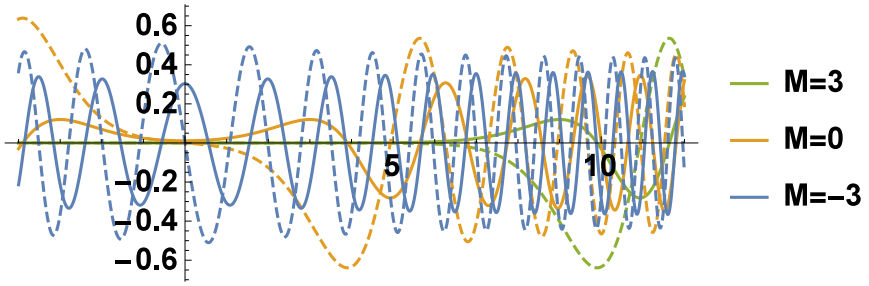
$$\int_{-\infty}^{\infty} dx[\phi^*_{M',\eta'}\phi_{M,\eta} + \chi^*_{M',\eta'}\chi_{M,\eta}] = \delta_\eta^{\eta'}\delta(M - M')2\pi Re\left[C^*_{M',\eta'}C_{M,\eta}exp[i(C^2_{M',\eta'} - C^2_{M,\eta})]\right] \tag{48}$$

2.5.6 Nonrelativistic limit of spectrum

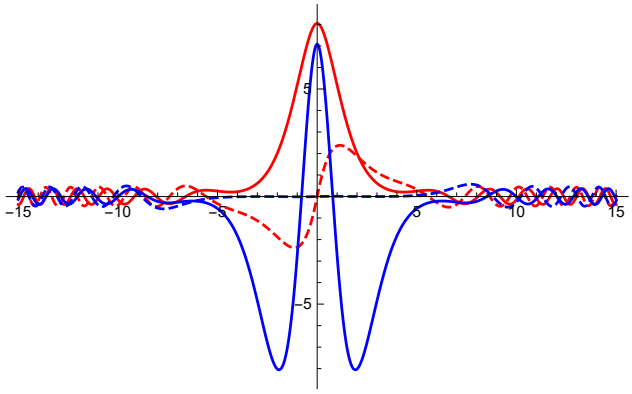
One might wonder how the purely continuous relativistic spectrum of solutions can limit to the discrete nonrelativistic spectrum. The continuous spectrum consists approximately of a sum of various Breit-Wigner resonance peaks. In the nonrelativistic limit the width of these peaks decreases and they turn into a sum of Dirac delta functions at the allowed discrete energy values. For energy values in between the allowed discrete energies the density of states limits to zero[17][4]



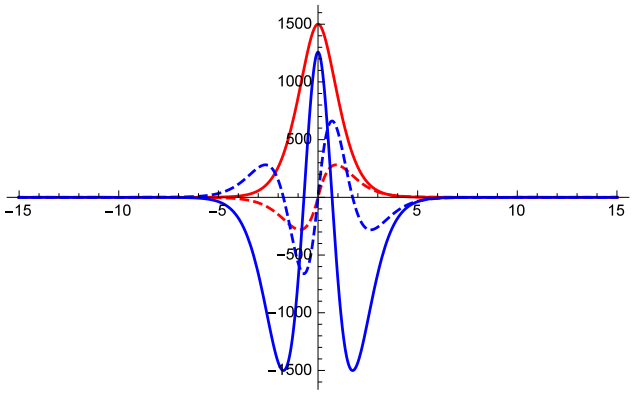
(a) Solutions for $m = 1$



(b) Solutions for $m = 1.5$



(c) Two solutions at the resonance energy for $m = 0.8$



(d) Two solutions at the resonance energy for $m = 1.5$

Figure 5: Some solutions with $\eta = +1$, (φ even, χ odd)

3 Results

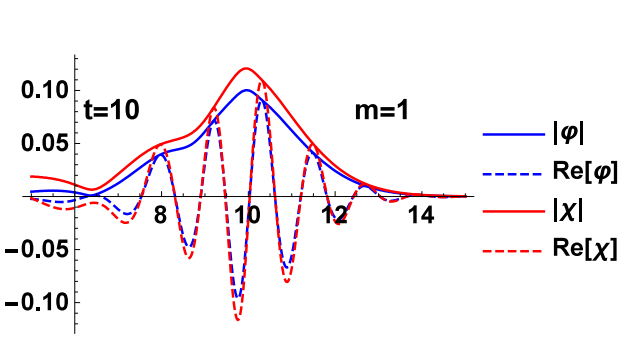
The results are split into four sections. In section 3.1 we numerically investigate the time-evolution of the Schödinger solution in the Dirac equation. In section 3.2 we analyze the traveling wave packet that appears in the time evolution. We show that part of the initial state behaves like relativistic positrons or negative energy electron accelerated by a constant force. In section 3.3 we investigate the properties of our initial state by expanding it in terms of energy eigenstates. Finally, in section 3.4 we find the transmission and reflection coefficients for scattering off the linear potential.

3.1 Time-dependent behavior

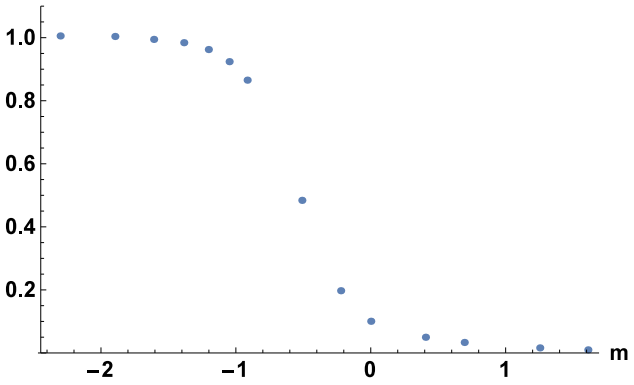
Figue 6 shows the time evolution of a localized initial Schrödinger solution when it is evolved by the Dirac equation. The figure only shows the probability density in the region $x \geq 0$. The probability density is symmetric about $x = 0$. To obtain the time-evolution the Dirac equation was discretized on a grid, and the time-evolution numerically estimated. As an initial state we took the large component φ equal to the ground state Airy solution, and set the lower component χ to zero.

As can be seen, for $m = 5$ the time evolution is almost stationary. This is to be expected because we are far in the non-relativistic regime. As we lower the mass towards $m = 1$ part of the probability density can be seen to move away from the origin. There is also an oscillation visible for the remaining bound state.
For even lower mass $m = 0.2$ practically all of the initial state quickly decays and moves away from the origin.

3.2 The escaping wave packet



(a) Zoom of wave packet



(b) Fraction of probability density inside packet

Figure 7: The escaping wave packet

From the numerical time-evolution of the Dirac equation in section 3.1 it is apparent that part or all of the initial state does not stay near the origin, but almost immediately starts to move away in the form of two traveling wave packets. Figure 7a shows the wavefunction of such a wave packet. The lower component χ is larger than the upper component φ. In the plane wave solutions for a constant potential from equation (21) this is indicative of negative energy solutions. The fact that the wave packet is not

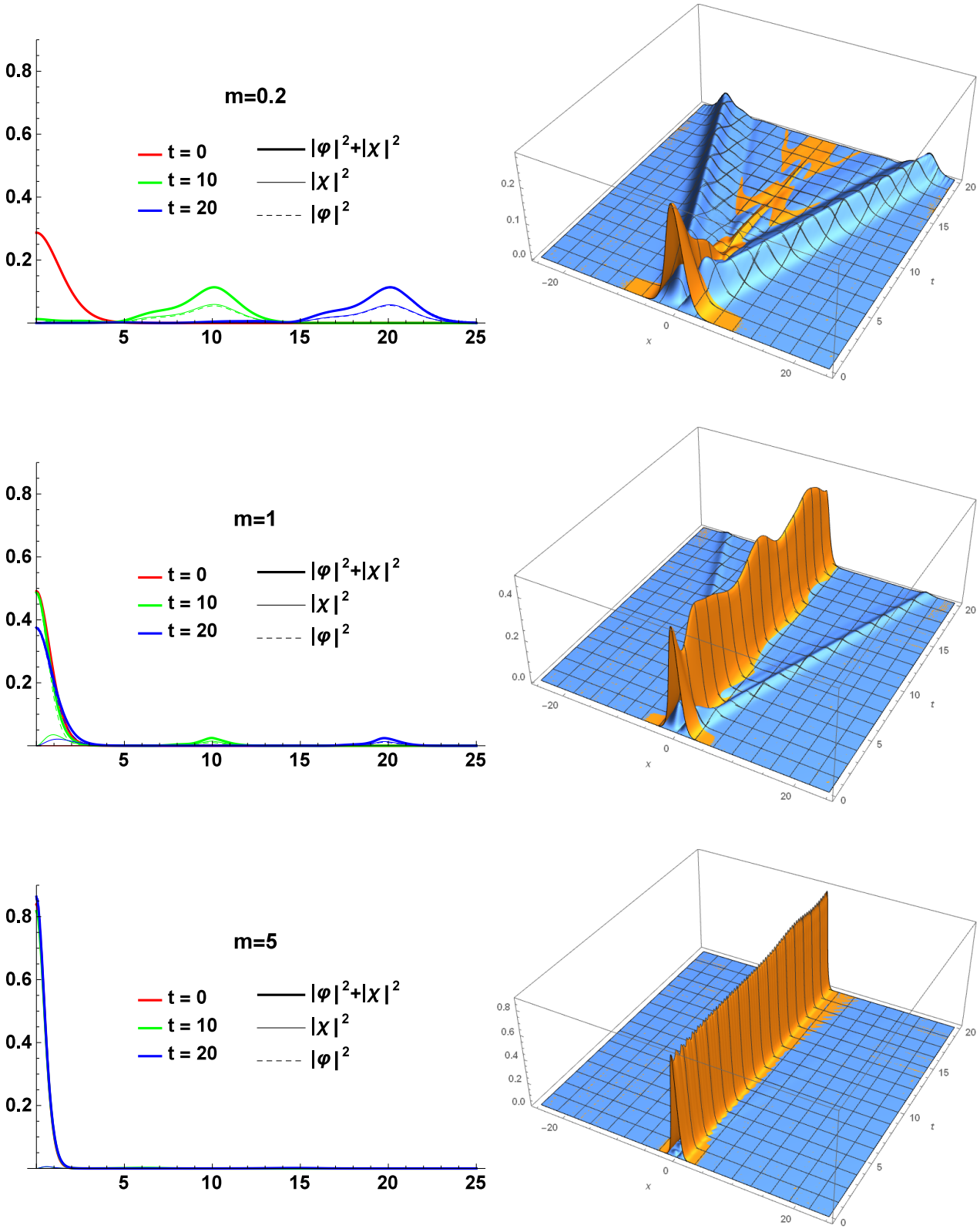
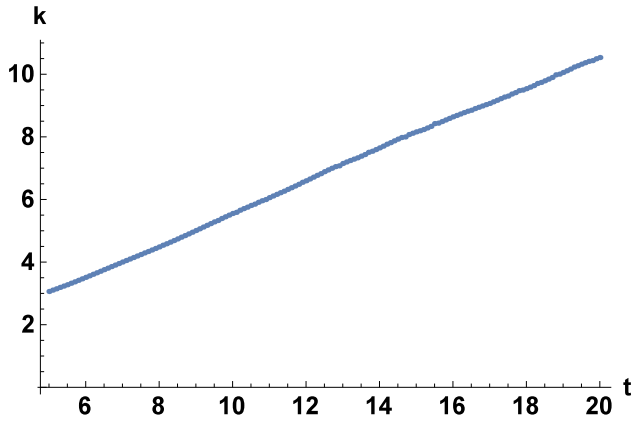
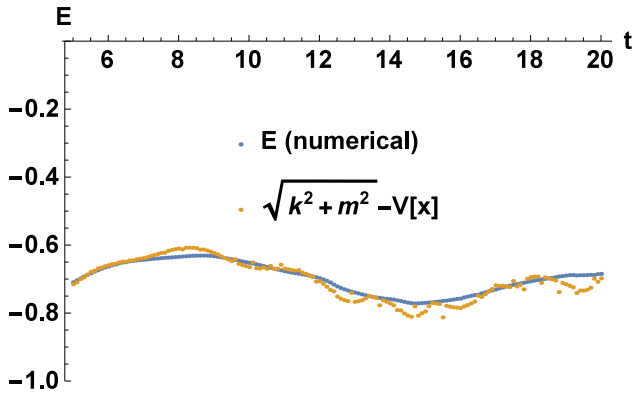


Figure 6: Evolution of the Schrödinger solution in the Dirac equation. For the surface plots $|\varphi|^2$ in orange, $|\chi|^2$ in blue

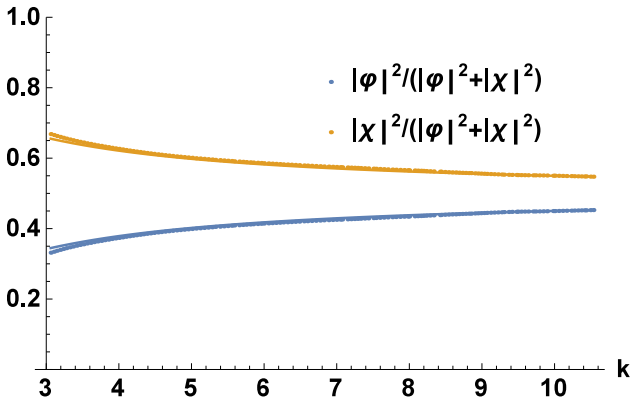
bound by the potential also indicates that it behaves like a positron.



(a) Wavenumber



(b) Energy of wave packet



(c) Ratio of $|\varphi|^2$ and $|\chi|^2$ to total density $|\varphi|^2 + |\chi|^2$

Figure 8: Analysis of the wavefunction at maximum of wave packet for $m = 1$

Figure 8 shows an analysis of the wavefunction at the maximum of the wave packet. The wavevector k increases linearly with time. This agrees with the momentum of a relativistic positron accelerated by a constant electric field, which increases according to

$$\overrightarrow{F} = e\overrightarrow{E} = \frac{d}{dt}\overrightarrow{p} = \hbar\frac{d}{dt}\overrightarrow{k} \tag{49}$$

The slope of this curve is 0.498, we expect it to be 0.5.
The negative energy plane wave in a constant electric field has energy eigenvalue

$$M = V - E(k) = V - \sqrt{m^2 + k^2} \tag{50}$$

In figure 8b two values for the energy are shown. One value is directly calculated from the frequency at the maximum of the wave packet, the other is calculated from the wavenumber according to the above equation. As can be seen these numbers agree reasonably well. The energy is negative in this case. This suggests that the initial state already contained some negative energy component. This could be a consequence of the initial state being constrained to within its Compton wavelength. In narrow Gaussian wave packet in the absence of a potential the negative energy component interferes and gives rise to the Zitterbewegung. In the presence of a linear potential there could be a similar negative energy component for narrow initial states, but instead of interfering with the positive energy component it is accelerated away from $x = 0$.
In a plane wave the probability density is the sum of the two components $|\varphi|^2$ and $|\chi|^2$. The relative fraction of the total density contained in each of the components is given by the spinor $v(k)$ In figure 8c this ratio is shown as a function of k, as well as the ratio numerically determined at the maximum of the wave packet at various times, plotted against the numerically determined value of k. From these three graphs we conclude that the escaping wave packet should be interpreted as a negative energy electron or positron.

3.3 Investigating the stationary solutions

3.3.1 Expansion of initial state in terms of energy eigenstates

We can expand an arbitrary wavefunction in terms of energy eigenstates. The energy eigenstates $\Psi_\eta(M, x)$ are given by the general solution (44), and also depend on m. The ratio between the free parameters a and b in the general solution is determined by the parity condition

$$\begin{aligned} \varphi'(0) = \chi(0) &= 0 && \text{if } \eta = +1 \\ \varphi(0) = \chi'(0) &= 0 && \text{if } \eta = -1 \end{aligned}$$

The states have to be normalized so that their inner product is a delta Dirac function:

$$\int_{-\infty}^{\infty} dx \Psi^*_{\eta'}(M', x)\Psi_\eta(M, x) = \delta_\eta^{\eta'}\delta(M - M') \tag{51}$$

This can be achieved by dividing a nonnormalized state by $\sqrt{|C_{M,\eta}|}$, where the proportionality constant is found as the limiting value of equation (47). These conditions specify the parameters a and b up to a sign. The expansion of an initial state can be written as

$$\psi(x) = \int_{-\infty}^{\infty} dM \tilde{\psi}(M)\Psi(M, x) \tag{52}$$

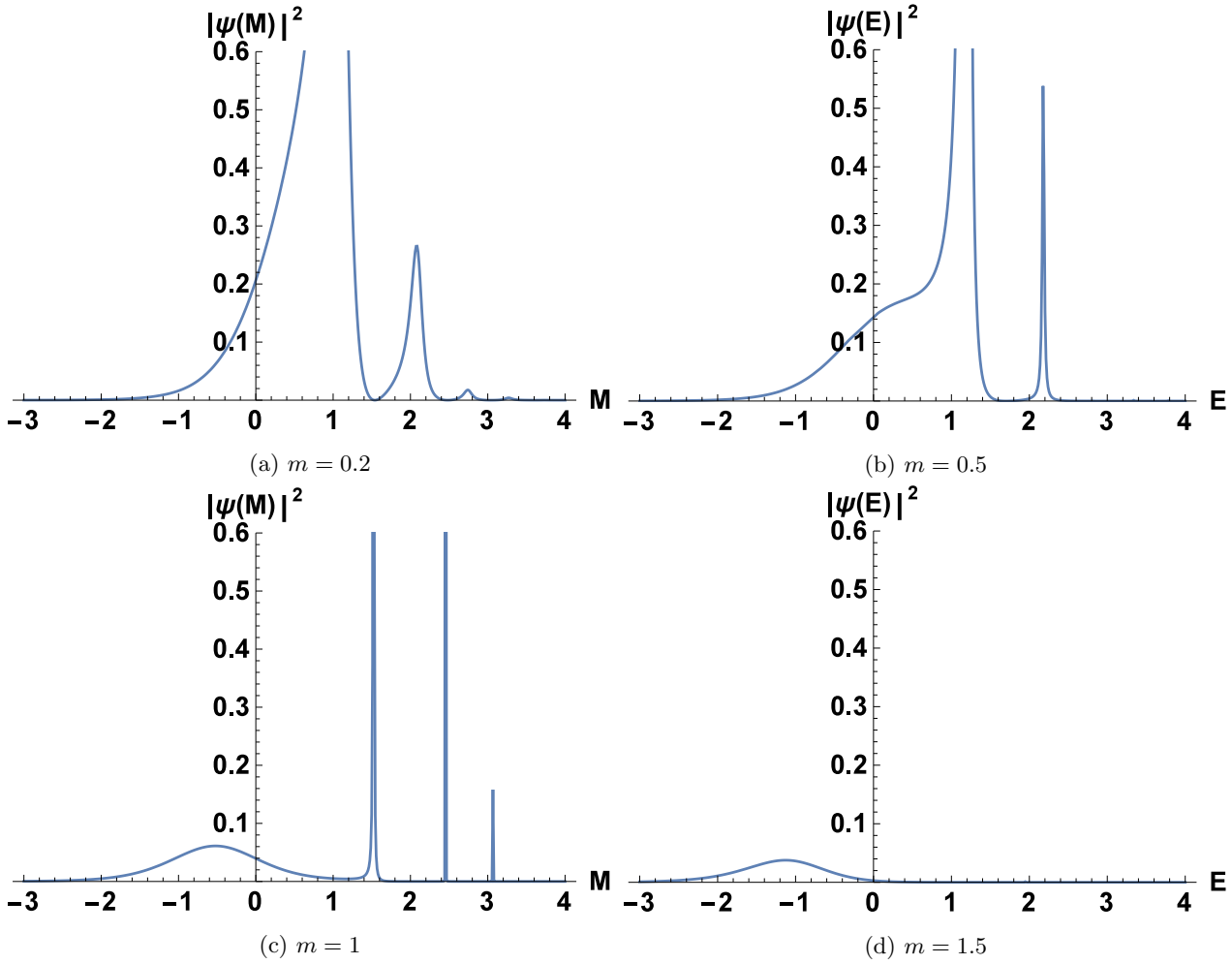
where all states are assumed to be of the same parity and η is suppressed. The orthogonality can be used to show that

$$\tilde{\psi}(M) = \int_{-\infty}^{\infty} dx \Psi^*(M, x)\psi(x) \tag{53}$$

3.3.2 Energy distribution of Airy states

Figure 9 shows the energy distribution of the Schrödinger initial state for different values of m. For $m = 1$ three sharp peaks with energy $M > m$ are visible, as well as a broad peak with energy $M < 0$. A fraction 0.899 of the probability density is in the peak at $M \approx 1.52$, and a fraction 0.093 is in the broad peak with it's maximum at $M \approx -0.65$. As m is increased the broad peak moves to lower values of M and it's area decreases. At $m = 1.5$ the other peaks are too narrow to be resolved on the plot. As m is decreased all peaks become wider and the broad peak merges with the first narrow peak.

We know there are some unstable resonance states. Probably there are three narrow peaks in the energy distribution for $m = 1$ because the initial state overlaps significantly with the first three symmetric resonance states. We only expect the shape of the first resonance to agree closely with the first Airy solution for a weak field (high m), so it isn't surprising that there are multiple peaks for $m = 1$.



(a) $m = 0.2$ (b) $m = 0.5$

(c) $m = 1$ (d) $m = 1.5$

Figure 9: Energy distribution of the Airy ground state for Dirac equation

Interference between the different resonances could also explain the oscillation that is seen in the time evolution in figure 6.

The probability density inside the two wave packets is approximately equal to the probability density inside the broad negative energy peak in the energy distribution for $1 < m < 2$. These two quantities are plotted in figure 10. It seems the resonance states do not decay significantly during the simulated time, and the wave packet is entirely due to an overlap of the initial wavepacket with negative energy states.

For $m = 1$ the energy difference between the two highest peaks is $\Delta M = 2.46 - 1.52 = 0.93$. we expect these two peaks to interfere with a period of $P = 2\pi/0.93 = 6.74$. In figure 11 the probability density at $x = 0$ is shown. There are oscillations with different frequencies visible. A sine has been fitted to find the period of the slowest oscillation, and has a period of $P = 6.72$, close to the expected value.

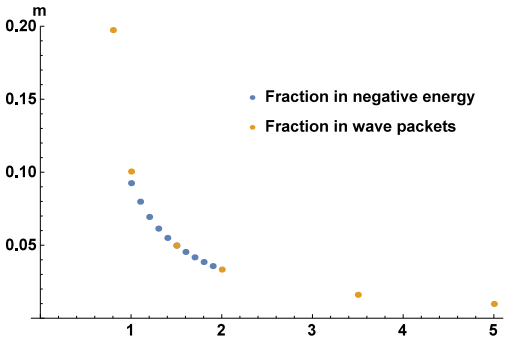


Figure 10: Comparison of the energy distribution and time evolution

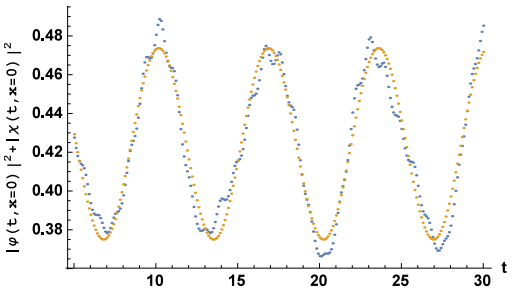


Figure 11: Blue: Numerical probability density at $x = 0$ for $m = 1$ Yellow: fitted sine wave

3.4 Construction of scattering states

The energy eigenstates $\Psi_\eta(M, x)$ are (anti)symmetric around $x = 0$. For these solutions the 1-component of the conserved probability current (6)

$$\overline{\psi}\gamma^1\psi = (\varphi^*, \chi^*)\gamma^0\gamma^1(\varphi, \chi)^\dagger = \varphi^*\chi + \chi^*\varphi \tag{54}$$

vanishes everywhere. They are standing waves, in which $\phi(x)$ is real and $\chi(x)$ is imaginary everywhere. While the total probability density tends to a constant for these solutions, the components $\phi^*\phi$ and $\chi^*\chi$ separately do not. We would like to take a linear combination of the symmetric and anti-symmetric states which corresponds to a scattering situation, in which a current incoming from the left is partially transmitted and partially reflected. Far to the right right linear combinations of the even and odd solution satisfy

$$\lim_{x\to\infty}[\phi(x) + i\chi(x)] = Cf_l(x) \tag{55}$$

for some complex constant C and a complex function $f_l(x)$ that depend on m and M, as given in equation (47). If we find a linear combination ψ_c with this constant equal to one, and another ψ_s where it is equal to i, we can use these to construct our traveling wave $\psi_{o,R} = \psi_c - i\psi_s$. This is similar to the way a traveling plane wave can be built from a linear combination of a cosine and a sine standing wave. The subscript o, R labels this solution as the outgoing plane wave to the right.
Separating the two components the asymptotic form of this solution will be

$$\begin{aligned}\lim_{x\to\infty}\psi_{o,R}(x) &= \lim_{x\to\infty}\begin{pmatrix}\varphi_{o,R}(x)\\ \chi_{o,R}(x)\end{pmatrix} = \begin{pmatrix}\Re[f_l(x)] - iRe[if_l(x)]\\ i\left(\Im[f_l(x)] - i\Im[if_l(x)]\right)\end{pmatrix}\\ &= \begin{pmatrix}\Re[f_l(x)] + i\Im[f_l(x)]\\ i\left(\Im[f_l(x)] - i\Re[f_l(x)]\right)\end{pmatrix} = f_l\begin{pmatrix}1\\1\end{pmatrix}\end{aligned} \tag{56}$$

so that in the limit the probability density and probability current are constant and equal to

$$\overline{\psi}_{o,R}\gamma^0\psi_{o,R} = \overline{\psi}_{0,R}\gamma^1\psi_{o,R} = 2f_l^*(x)f_l(x) \tag{57}$$

Since this limit corresponds to the wavefunction moving at c, in the limit there is only a traveling wave and no standing wave.

The incoming traveling wave from the right is given by $\psi_{i,R}(x) = \psi_c(x) + i\psi_s(x)$, with a plus instead of a minus. This has the same probability density as $\psi_{o,R}$ but an inverted probability current.

3.4.1 Transmission coefficients and transmission resonances

We have shown that the wavefunction $\psi_{o,R}$ corresponds to an outgoing traveling wave in the limit $x \to \infty$. In the other direction - in the limit $x \to -\infty$ - it is a linear combination of an incoming traveling wave $\psi_{i,L}$ and an outgoing traveling wave $\psi_{o,L}$.
Since it is as a linear combination of the even and odd solutions

$$\psi_{o,R}(x) = C_e \Psi_{+1}(M, x) + C_o \Psi_{-1}(M, x) \tag{58}$$

we have

$$\psi_{o,R}(-x) = \psi_{o,L}(x) = C_e \Psi_{+1}(M, x) - C_o \Psi_{-1}(M, x) \tag{59}$$

for a wave traveling to the left. In the same way we can find the incoming solution from the left $\phi_{i,left}$. Then the coefficients are found by solving

$$t\psi_{o,R} = \psi_{i,L} + r\psi_{o,L} \tag{60}$$

subject to the condition $R + T = |r|^2 + |t|^2 = 1$.

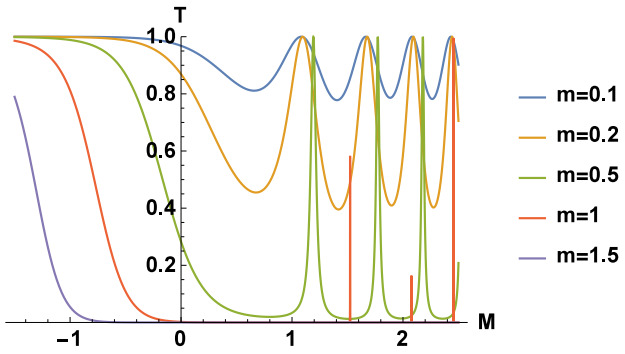


Figure 12: Transmission coefficient

The transmission coefficient for various values of the mass is shown in figure 12. There are transmission resonances for various energies. The energy of the transmission resonances increases with increasing m, while the width decreases. For $m = 1$ and $m = 1.5$ the resonances are so narrow they are not clearly resolved in the plot.

These transmission resonances seem to occur at the same value of M as the peaks in the energy distributions in figure 9.

4 Conclusion

We investigated the relativistic solutions for a linear confining potential. The nonrelativistic solutions can be found by solving the Schrödinger equation. The degree to which the nonrelativistic approximation fails depends on a parameter m, which is proportional to the electron mass over the square root of the electric field strength. High values of m correspond to a weak potential which gives rise to a wide bound state with low binding energy in the Schrödinger equation. In these cases the nonrelativistic approximation works well, as can be verified by taking the nonrelativistic solution as an initial wavefunction and evolving it with the Dirac equation. If the approximation works well the wavefunction will be approximately stationary.

When the value of m is decreased to around $m = 1$ - which is when the field strength equals the Schwinger limit - nonrelativistic effects are important. Part of the initial state is not bound but escapes as a wave packet emitted to both sides. The part of the initial state that remains bound initially displays some oscillations. As m is decreased further all probability density escapes the potential quickly.

The time evolution can be explained further from the energy distribution of the initial state. The Dirac equation does not have bound state solutions, but there are resonances with a long lifetime. For high m the initial state aligns with the first resonance, so almost all of the probability density in the energy distribution is concentrated in a single narrow peak. If m is close to one the initial state does not align well with the first resonance anymore. In its energy distribution multiple higher resonances are excited, which will interfere and cause the observed oscillations in the bound part of the wavefunction. There also is some density in unbound negative energy states, which explains the unbound wave packet.

When m is less than one the lifetime of the resonances becomes so short they do not remain bound during the simulation. This agrees with the spectrum, which will not have narrow peaks.

We have also found scattering solutions for the linear potential and used them to calculate the transmission and reflection coefficients. As expected an incoming positron will be transmitted when its energy allows it to penetrate to $x = 0$. The positron also has a high probability to be transmitted if it is close in energy to of one of the metastable resonance states, so there are transmission resonances. These transmission resonances are increasingly wide as the potential becomes more relativistic.

5 Discussion

In this investigation we have pushed the single-particle Dirac equation somewhat to its limits by attempting to use it to describe situations where both particles and antiparticles, as well as pair creation and interactions with the electromagnetic field, play an integral role. As a result, it is not immediately clear how the Dirac wavefunction should be interpreted and to what extent its evolution is correctly calculated.

An attempt could be made to solve the problem using a quantum field theoretical treatment, which naturally incorporates multiparticle processes. Such a treatment has yielded new insights into related situations such as the Klein paradox and supercritical atoms.

It could be that the predicted decay of an electron in the linear potential is unphysical, because it predicts the transition of the electron to a state in the negative energy continuum. Since we should normally assume all negative energy states to be filled this process could not take place. Then the evolution of the same wavefunction would describe the process of a positron escaping, in the same way an empty electron state in a potential is reinterpreted as being filled with a positron when it crosses $M = 0$, after which the positron can escape when the potential becomes supercritical. The ambiguity arises because there is no definite potential energy level at infinity, so there is no way to tell empty positron states apart from filled electron states. In any way it seems more plausible all electron states will be filled after some time, because they are surrounded by a continuum of negative energy states that we should assume to be filled with electrons.

In this picture the transmission resonance can also be explained. We assume that in the steady state all resonances are filled electron states (or equivalently empty positron states). An incoming positron of the right energy can then annihilate one of these electrons (fill one of the positron states). After this the electron state is filled again (the positron escapes), with the possibility of a transmission.

The initial Schrödinger state that is evolved has the benefit of being easy to calculated, but it has an unnecessarily complicated energy distribution. If possible it would probably be better to find a localized initial state that is only concentrated at one resonance in the energy distribution.

Perhaps it would be better to turn the potential on explicitly in the simulation. As an initial potential a linear potential that levels off at some distance such as displayed in figure 1 could be used. This has the benefit that for the right field strength its lowest energy state is a stable bound state. One could take just this state to be filled initially and then turn the potential further away on adiabatically, making the state unstable.

The field investigated in this report are probably too strong to be experimentally realized. It would be good to calculate the lifetime of the first resonance for more reasonable field strengths, in an attempt to connect to experiment. This lifetime could be calculated from the width of the resonance. It could be more difficult to obtain the lifetime from the simulated time evolution because of the limited size of the grid. When the grid is too small the wavefunction will reach the boundary of the grid during the simulation, but increasing the grid size is computationally expensive.

Finally, the study of the linear potential could perhaps be tested in graphene systems. Due to its linear dispersion relation electrons in graphene have to be described using the Dirac equation. This could provide new insights into the interesting and often paradoxical physics of the Dirac equation.

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