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Kontsevich graphs and their weights *in* deformation quantization of Poisson structures

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Abstract.

To show the existence of deformation quantizations for arbitrary Poisson structures, M. Kontsevich gave in 1997 an explicit universal formula: a formal power series in the deformation parameter $\hbar$ with a sum of weighted graphs (wherein a Poisson structure can be implanted) at each order in $\hbar$. We outline a systematic and graphical approach, implemented in software, to the problem of expanding the power series for this $\star$-product, particularly the problem of finding the universal coefficients (weights of graphs) in terms of as few parameters as possible, and the problem of pictorially proving the associativity of $\star$ up to a given order in $\hbar$. We obtain the expansion of the star-product up to the order 4 in $\hbar$ in terms of 10 parameters (6 parameters modulo gauge-equivalence) and we verify that the star-product expansion is associative modulo $\bar{o}(\hbar^4)$ for every value of the 10 parameters. Jointly with A. Bouisaghouane and A.V. Kiselev, we confirm at the infinitesimal level the existence of a universal flow on the space of Poisson structures.

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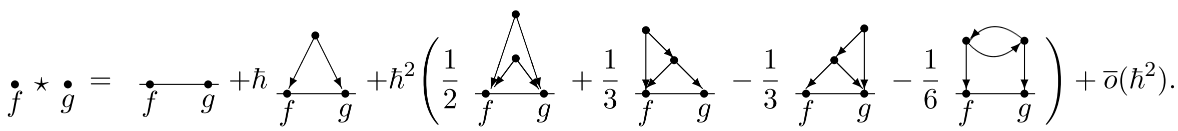
Summary

Ist das Kunst oder kann das weg?
German proverb

In this master's thesis we study the graphs and weights which appear in the deformation quantization formula for Poisson structures on $\mathbb{R}^n$ given by M. Kontsevich in [9]. The formula, a formal power series in the deformation parameter $\hbar$, looks like this:



The meaning of the graphs and the general definition of the weights are given in [9], of course, but also in [5] below. The main features of the Kontsevich $\star$-product are its associativity – $(f \star g) \star h = f \star (g \star h)$ – for *any* Poisson structure which is implanted in the graphs, and the universality – independence of Poisson structure – of the weights. The results of the thesis are summarized as follows:

- Calculated all the weights of Kontsevich graphs at the order 3.
  See [5, Example 21] for the method and [7] for the result.
- Found an explicit mechanism for the associativity of $\star$ at the third order [7].
- Expressed all the weights at order 4 in terms of 10 parameters [5, Theorem 9].
  Modulo gauge-equivalence, there remain 6 parameters [5, Theorem 14].
- Verified the associativity of $\star$ up to order 4 explicitly (as it was done in [7] for the third order) for all values of the parameters [5, Theorem 11].
- Developed software to handle series of Kontsevich graphs [5].
  This software [4] was used to obtain and/or verify the results listed here.
- Jointly with A. Bouisaghouane and A.V. Kiselev, confirmed at the infinitesimal level the existence of a universal flow on the space of Poisson structures [1].

The articles [1], [5], and [7] cited in this summary are included in the thesis. The reader is advised to check the `arXiv` links in the bibliography for the most recent versions of these articles and/or journal references.

Preface

Any event, once it has occurred, can be made to appear inevitable by a competent historian.
— Lee Simonson

As can be read on the title page, this master's thesis was written in the course of approximately two and a half years. The following is a brief recollection of the events.

Beginnings

The goal of the project, written in the original proposal, was to extend Kontsevich's deformation quantization formula to a variational (jet space) set-up, to the third order in $\hbar$, for some particular examples of variational Poisson structures. To prepare me for this – particularly the jet spaces – my supervisor Arthemy Kiselev encouraged me to attend

The 3rd *Summer School on the Geometry of Differential Equations.*
September 8 – 12, 2014. Malenovice, Czech Republic.
Organized by the Mathematical Institute of Silesian University in Opava.

Participating in this summer school was a wonderful experience. Besides learning a thing or two about jet spaces, I met some very nice people. When I returned from Malenovice, work on the master's thesis began in earnest. Through regular appointments with Arthemy in his office, I learned about jet spaces, the Jets package for Maple, (variational) Poisson structures, the Kontsevich graph technique, and the extension of the graph technique to jet spaces proposed by Arthemy. A particular problem for such a jet space extension was that Kontsevich's original proof of the associativity for $\star$ would not work in the new infinite-dimensional setting. However, the $\star$-product formula is explicit, so one should be able to check the associativity directly (up to some finite order) by expanding both sides of the equation. The main research question became

How does the associativity for $\star$ follow from the Jacobi identity for $\mathcal{P}$?

Following the direct approach, expanding the associativity equation for $\star$ by hand up to the order 2 in $\hbar$ and collecting similar terms, we found that associativity for $\star$ at order 2 is exactly $\frac{2}{3}$ times the Jacobi identity for $\mathcal{P}$ [2] (it was well-known that it should be equivalent to the Jacobi identity). In the variational set-up, the same logic applied. I also wrote some code to assign variational differential operators to graphs, by using a combination of Sage and Jets.

The result of associativity up to order 2, the existence of the software, and our plans to continue with this extension were presented at

> *Symmetries of Discrete Systems and Processes III.*
> August 3 – 7, 2015. Děčín, Czech Republic.
> Conference organized by the Czech Technical University in Prague.

By this time Anass Bouisaghouane was also working on his master's project under Arthemy's supervision. We had been working together on things related to jet spaces and variational Poisson structures, and attended this conference together.

While I did not pursue the jet space extension of the quantization formula further, the variational setting did inspire one of the subsequent results: the pictorial associativity mechanism for the $\star$-product at order 3 (and beyond), which is discussed below.

The star-product up to order 3

Meanwhile, there was the problem of passing to the third order in $\hbar$. Here already drawing all the graphs becomes difficult to do without the aid of a computer. After struggling for a while, I wrote some Sage code to do it. Arthemy asked me to find the weights of these graphs in the literature, or calculate them myself. To our surprise, the full expansion of the $\star$-product up to $\hbar^3$ could not be found in the literature. My code to generate graphs became the basis for the Sage package `kontsevich_graph_series` [3]. By using (elementary) relations found in the literature, the problem of calculating the weights of graphs at order 3 was reduced to the calculation of just 15 graph weights.

In the absence of other relations between these 15 weights (we would later use the associativity equation at order 4 and cyclic weight relations to determine them exactly), I turned to numerical methods. By expressing the weight integral in Cartesian coordinates, the integrand became a rational function of several variables. My attempts to numerically integrate this rational function (with singularities, over an unbounded domain) by using several different programs were unsuccessful. The numerical approach was saved by Cauchy's residue theorem, which let us integrate out 3 of the 6 (real) variables symbolically, so the rest of the integral (now over a 3-dimensional domain) could be done numerically. (This is described in more detail in [5, Appendix A].) The numerical approximations of the weights were very close to certain rational numbers, namely $\pm 1/48$ and $\pm 1/24$ (obtained by looking at the convergents of the continued fraction expansions of the approximations), which led us to conjecture that these were their true values. This method and the conjecture (which later turned out to be correct) were reported in the brief communication [6]. Having all these weights, we could build the $\star$-product expansion up to the third order.

Associativity at order 3

With this star product expansion in hand, the next task was to show its associativity up to order 3. The Sage program [3] could eventually build star-product expansions (given the weights), expand the associator $(f \star g) \star h - f \star (g \star h)$, and collect terms by using the skew-symmetry of $\mathcal{P}$. Expanding the associator up to the order 3 in $\hbar$ and collecting terms by using the skew-symmetry of $\mathcal{P}$, we were left with 39 mysterious terms at $\hbar^3$. They should vanish as a consequence of the Jacobi identity. But how? For 9 terms it was obvious, but the other 30 remained a mystery for several weeks. Finally I realized how differential consequences of the Jacobi identity could be produced in a pictorial way. The result was reported in [7] and presented at the workshop

> *Group Analysis of Differential Equations and Integrable Systems VIII.*
> June 12 – 17, 2016. Larnaca, Cyprus.
> Workshop organized by the Department of Mathematics and Statistics of the University of Cyprus and the Department of Applied Research of the Institute of Mathematics of the National Academy of Sciences of Ukraine.

The proof in [7] is based on the lemma that states a (tri-)linear differential operator with smooth coefficients vanishes identically if and only if its homogeneous components vanish. My idea of a proof of this fact was to use Borel's lemma (recalled at the beginning of a course on jet spaces) which says that every formal power series with real coefficients is the Taylor series of some smooth function. Arthemy quickly pointed out that this is major overkill; just consider the value of the differential operator on polynomial functions instead. Around this time I had translated my code to `C++` (see [4]) for efficiency reasons, and was working on proceeding to the next expansion order.

Universal flows and the ratio $1:6$

A parallel story is about universal flows $\dot{P} = Q_{a:b}(P)$ on the space of Poisson structures (also proposed by M. Kontsevich) which Anass was studying. Here the approach suggested by Arthemy was also to look for the explicit mechanism, first at the infinitesimal level: why does the cocycle condition $[\![P, Q_{a:b}(P)]\!] = 0$ hold for all Poisson bi-vectors P? Anass discovered by using explicit examples of Poisson structures that the condition $a : b = 1 : 6$ is necessary for the universal flow to exist, and furthermore at this ratio the cocycle condition is satisfied for some class of Poisson structures. The question remained whether the cocycle condition for $a : b = 1 : 6$ holds in general, as a consequence of the Jacobi identity $[\![P, P]\!] = 0$. By writing the Schouten bracket $[\![P, Q_{1:6}(P)]\!]$ in terms of graphs, the idea of pictorial differential consequences of the Jacobi identity from [7] could be applied. Using a perturbative approach, Anass was able to find some of the necessary differential consequences [1, Appendix E]. In the hope of finding the full solution, I worked together with Anass on a program `reduce_mod_jacobi` which would try to reduce a sum of graphs by subtracting with undetermined coefficients all possible pictorial differential consequences of the Jacobi identity, and solving the resulting linear system. Eventually the program worked and found a solution consisting of 27 differential consequences. By hand we collected the 27 terms into 8 terms modulo skew-symmetry; this solution is reported in [1].

Software and the fourth order

After using the same strategy as in [6] for the third order, there were 149 weights of graphs at order 4 left to be determined. Two ways to obtain weight relations which were not covered in [6] are to use the associativity equation for particular Poisson structures (e.g., at particular points), and the cyclic weight relations found in [8, Appendix E]. By these methods we verified the conjecture about the values of all the weights at order 3 from [6], and we expressed all the weights at order 4 in terms of only 10 weights. This result was presented at the

> *Symposium on advances in semi-classical methods in mathematics and physics.*
> October 19 – 21, 2016. Groningen, The Netherlands.
> Jointly organized by the Johann Bernoulli Institute for Mathematics and Computer Science (JBI) and the Van Swinderen Institute for Particle Physics and Gravity (VSI).

This meeting was held in remembrance of Hip Groenewold, the theoretical physicist at the University of Groningen who was the first to write down a star-product. It was an honor to be a part of this symposium, and I enjoyed speaking with people there.

In an effort to bring the number of parameters down from 10, I tried to get more relations by evaluating the associativity equation at higher orders (this was effective at the previous order) and by using different Poisson structures. No new relations were obtained. Arthemy suggested to inspect whether the star-product was already associative up to the fourth order. Indeed, it was! Furthermore, we checked which parameters could be gauged out, if any. It turned out that 4 of the 10 parameters could be removed by a gauge-transformation. These results, together with explanations of the software created to obtain these results, is contained in [5]. The process of writing about the programs has suggested many improvements. Some manipulations which would have to be done by hand were tedious to describe, so it was easier to modify or extend the programs to simplify the writing (which also benefits the reader/user).

What's next

In theory, one could go on expanding Kontsevich's formula for ever and ever. By itself, that is not the most interesting thing. However, the question whether all the weights of Kontsevich graphs are rational or not is an open problem. In [8] it is proved that a certain graph at order 7 has a weight which is, up to rationals, $\zeta(3)^2/\pi^6$. Hence it is interesting to keep on expanding and finding weights at least until the order 7. This can be done in part by using the software [5], but to really determine all the weights some new idea will be needed. I hope that my thesis inspires further work in this direction.

Acknowledgements

First of all I express my gratitude to my supervisor Arthemy Kiselev. Working with him was a wonderful experience. Besides him teaching me the mathematics that I needed to know for this project, I have learned from him a great deal about the art and craft of writing, rewriting and typesetting articles; practicing and giving talks; persistence in research; and the value of keeping notes. His connectedness allowed me to attend schools and conferences not only in the Netherlands, but also in the Czech Republic and in Cyprus. The great experiences I had there were made possible by Arthemy. When the project went in a different direction than originally intended, he was flexible. His dedication and hard work set an example that I have done my best to follow.

Secondly I thank Anass Bouisaghouane for all our fruitful collaboration as students of our common supervisor. We were often able to help one another, and we always had a good time at the faculty and at the conferences we visited abroad.

I also thank Jaap Top for his support and continued interest in the project. Thanks are due to Mark Jeeninga for an interesting discussion about the number-theoretical properties of the weights. Not least, I thank my family and friends for putting up with me and encouraging me to keep going. Thanks to the Center for Information Technology of the University of Groningen for providing access to the Peregrine high performance computing cluster. Thanks to the Graduate School of Science of the University of Groningen for financial support to participate in the conference in Děčín and the workshop in Larnaca.

Groningen, February 2017

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SOFTWARE MODULES AND COMPUTER-ASSISTED PROOF SCHEMES IN THE KONTSEVICH DEFORMATION QUANTIZATION

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Abstract. The Kontsevich deformation quantization combines Poisson dynamics, noncommutative geometry, number theory, and calculus of oriented graphs. To manage the algebra and differential calculus of series of weighted graphs, we present software modules: these allow generating the Kontsevich graphs, expanding the noncommutative $\star$-product by using *a priori* undetermined coefficients, and deriving linear relations between the weights of graphs. Throughout this text we illustrate the assembly of the Kontsevich $\star$-product up to order 4 in the deformation parameter $\hbar$. Already at this stage, the $\star$-product involves hundreds of graphs; expressing all their coefficients via 149 weights of basic graphs (of which 67 weights are now known exactly), we express the remaining 82 weights in terms of only 10 parameters (more specifically, in terms of only 6 parameters modulo gauge-equivalence). Finally, we outline a scheme for computer-assisted proof of the associativity, modulo $\bar{o}(\hbar^4)$, for the newly built $\star$-product expansion.

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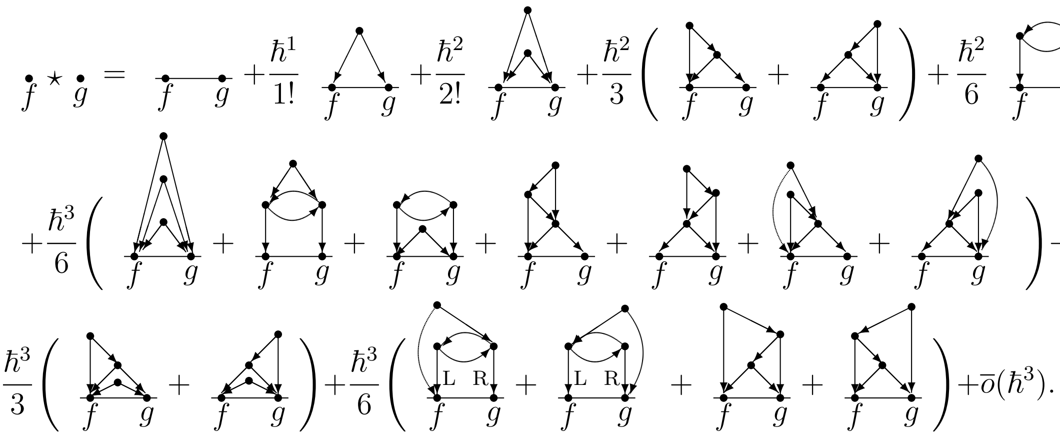
Introduction. On every finite-dimensional affine (i.e. piecewise-linear) manifold N^n, the Kontsevich star-product $\star$ is an associative but not necessarily commutative deformation of the usual product $\times$ in the algebra of functions $C^\infty(N^n)$ towards a given Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ on N^n. Specifically, whenever $\star = \times + \hbar\,\{\cdot,\cdot\}_{\mathcal{P}} + \bar{o}(\hbar)$ is an infinitesimal deformation, it can always be completed to an associative star-product $\star = \times + \hbar\,\{\cdot,\cdot\}_{\mathcal{P}} + \sum_{k\geqslant 2} \hbar^k B_k(\cdot,\cdot)$ in the space of formal power series $C^\infty(N^n)[[\hbar]]$; this was proven in [25]. An explicit calculation of the bi-linear bi-differential terms $B_k(\cdot,\cdot)$ at high orders $\hbar^k$ is a computationally hard problem. In this paper we reach the order $k = 4$ in expansion of $\star$ by using software modules for the Kontsevich graph calculus, which we presently discuss.

Convenient in practice, the idea from [25] (see also [21, 22, 24]) is to draw every derivation $\partial_i \equiv \partial/\partial x^i$ (with respect to a local coordinate x^i on a chart in the Poisson manifold N^n at hand) as decorated edge $\xrightarrow{\ i\ }$, so that large differential expressions become oriented graphs. For example, the Poisson bracket $\{f,g\}_{\mathcal{P}}(\boldsymbol{x}) = \sum_{i,j=1}^{n} (f)\overleftarrow{\partial_i}\big|_{\boldsymbol{x}} \cdot \mathcal{P}^{ij}(\boldsymbol{x}) \cdot \overrightarrow{\partial_j}\big|_{\boldsymbol{x}}(g)$ of two functions $f, g \in C^\infty(N^n)$ is depicted by the graph $(f) \xleftarrow{i} \mathcal{P}^{ij} \xrightarrow{j} (g)$; here $\mathcal{P}^{ij}$ is the skew-symmetric matrix of Poisson bracket coefficients and the summation over i, j running from 1 to the dimension n of N^n is implicit. In these terms, the known – from [6] – expansion of Kontsevich star-product looks as follows:[1]



(1)

By construction, every oriented edge carries its own index and every *internal* vertex (not containing the arguments f or g) is inhabited by a copy of the coefficient matrix $\mathcal{P} = (\mathcal{P}^{ij})$ of the Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$. This means that expansion (1) encodes the analytic formula

$$f \star g = f \times g + \hbar\mathcal{P}^{ij}\partial_i f \partial_j g + \hbar^2\big(\tfrac{1}{2}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_k\partial_i f \partial_\ell\partial_j g + \tfrac{1}{3}\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_k\partial_i f\partial_j g$$
$$- \tfrac{1}{3}\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_i f\partial_k\partial_j g - \tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_i f\partial_k g\big) + \hbar^3\big(\tfrac{1}{6}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_n\partial_\ell\partial_j g$$

[1]The indication L and R for Left $\prec$ Right, respectively, matches the indices – which the pairs of edges carry – with the ordering of indices in the coefficients of the Poisson structure contained in the arrowtail vertex. Note that exactly *two* edges are issued from every internal vertex in every graph in formula (1); not everywhere displayed in (1), the ordering $L \prec R$ in each term is determined from same object's expansion (2).

$$+ \tfrac{1}{3}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_\ell\partial_j g - \tfrac{1}{3}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_\ell\partial_j g$$
$$- \tfrac{1}{6}\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g + \tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_j g$$
$$+ \tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_m\partial_k\partial_j g - \tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_j g$$
$$- \tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_k\partial_j g - \tfrac{1}{6}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g$$
$$- \tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g + \tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_m\partial_k g$$
$$- \tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_i f\partial_k g - \tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_k g\big) + \bar{o}(\hbar^3). \quad (2)$$

We now see that the language of Kontsevich graphs is more intuitive and easier to percept than writing formulae. The calculation of the associator $\text{Assoc}_\star(f,g,h) = (f\star g)\star h - f\star(g\star h)$ can also be done in a pictorial way (see section 2.4 on p. 18). The coefficients of graphs at $\hbar^k$ in a star-product expansion are given by the Kontsevich integrals over the configuration spaces of k distinct points in the Lobachevsky plane $\mathbb{H}$, see [25] and [8]. Although proven to exist, such weights of graphs are very hard to obtain in practice.[2] Much research has been done on deriving helpful relations between the weights in order to facilitate their calculation [10, 27, 14, 11, 2]. In Example 21 on p. 25 we explain how expansion (1) modulo $\bar{o}(\hbar^3)$ was obtained in [6]. The techniques which were then sufficient are no longer enough to build the Kontsevich $\star$-product beyond the order $\hbar^3$; clearly, extra mathematical concepts and computational tools must be developed. In this paper we present the software in which several known relations between the Kontsevich graph weights are taken into account; we express the weights of all graphs at $\hbar^4$ in terms of 10 master-parameters. (To be more precise, the ten master-parameters are reduced to just 6 by taking the quotient over certain four degrees of gauge freedom in the associative star-products.)

This paper contains three chapters. In chapter 1 we introduce the software to encode and generate the Kontsevich graphs and operate with series of such graphs. In particular, the coefficients of graphs in series can be undetermined variables. The series are then reduced modulo the skew-symmetry of graphs (under the swapping of Left $\rightleftarrows$ Right in their construction). Thirdly, a series can be evaluated at a given Poisson structure: that is, a copy of the bracket is placed at every internal vertex.

Chapter 2 is devoted to the construction of Kontsevich $\star$-product: containing a given Poisson structure in its leading deformation term, this bi-linear operation is not necessarily commutative but it is required to be associative; hence the coefficients of a power series for $\star$ must be specified. For example, at order $k = 4$ of the deformation parameter $\hbar$ there are 149 parameters to be found. (The actual number of graphs at $\hbar^4$ is much greater; we here count the "basic" graphs only.) We review a number of methods to obtain the weights of Kontsevich graphs; the spectrum of techniques employed ranges from complex analysis and direct numeric integration [7] to finding linear relations between such weights by using abstract geometric reasonings. The associativity of Kontsevich $\star$-product is the main source of relations between the graph weights; at $\hbar^4$ such relations are *linear* because everything is known about the weights up to order three. We obtain these relations at order four in chapter 3 and we solve that system

[2]In fact, there are many other admissible graphs, not shown in (1), in which every internal vertex is a tail for two oriented edges but the weights of those graphs are found to be zero.

of linear algebraic equations for 149 unknowns. The solution is expressed in terms of only 10 master-parameters, see formula (11) on pp. 33–38.

The algebraic system constructed in section 3.1 was obtained by restricting the associativity for $\star$ to (a class of) specific Poisson structures. We want however to prove that for the newly found collection of graph weights, the $\star$-product is associative for *every* Poisson structure on *all* finite-dimensional affine manifolds. For that, in section 3.2 we design a computer-assisted proof scheme that is independent of the bracket (and of a manifold at hand). Specifically, in Theorem 11 on p. 29 we reveal how the associator for Kontsevich $\star$-product, taken modulo $\bar{o}(\hbar^4)$, is factorised via the Jacobiator $\mathrm{Jac}(\mathcal{P})$ or via its differential consequences that all vanish identically for Poisson structures $\mathcal{P}$ on the manifolds N^n. We discover in particular that such factorisation,

$$\mathrm{Assoc}_\star(f,g,h) = \Diamond\big(\mathcal{P}, \mathrm{Jac}(\mathcal{P}), \mathrm{Jac}(\mathcal{P})\big) \mod \bar{o}(\hbar^4),$$

is quadratic and has differential order two with respect to the Jacobiator. For all Poisson brackets $\{\cdot,\cdot\}_\mathcal{P}$ on finite-dimensional affine manifolds N^n our ten-parameter expression of the $\star$-product does agree up to $\bar{o}(\hbar^4)$ with previously known results about the values of Kontsevich graph weights at some fixed values of the ten master-parameters and about the linear relations between those weights at all values of the master-parameters.[3]

The software implementation [5] consists of a `C++` library and a set of command line calls. The latter are specified in what follows; a full list of new `C++` subroutines and their call syntaxis is contained in Appendix B. Whenever a command line call refers to just one particular function in `C++`, we indicate that in the text. The current text refers to version `0.16` of the software.

[3]From Theorem 11 we also assert that the associativity of Kontsevich $\star$-product does not carry on but it can leak at orders $\hbar^{\geqslant 4}$ of the deformation parameter, should one enlarge the construction of $\star$ to an affine bundle set-up of N^n-valued fields over a given affine manifold M^m and of variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ for local functionals $F, G, H \colon C^\infty(M^m \to N^n) \to \Bbbk$, see [16, 17, 18, 19] and [20].

1. Weighted graphs

In this section we introduce the software to operate with series of graphs.

1.1. **Normal forms of graphs and their machine-readable format.** As it was explained in the introduction, we consider graphs whose vertices contain Poisson structures and whose edges represent derivatives. To be precise, the class of graphs to deal with is as follows.

Definition 1. Let us consider a class of oriented graphs on $m+n$ vertices labelled 0, …, $m+n-1$ such that the consecutively ordered vertices 0, …, $m-1$ are sinks, and each of the internal vertices m, …, $m+n-1$ is a source for two edges. For every internal vertex, the two outgoing edges are ordered using $L \prec R$: the preceding edge is labeled L (Left) and the other is R (Right). An oriented graph on m sinks and n internal vertices is a Kontsevich graph of type (m, n). We denote by $G_{m,n}$ the set of all Kontsevich graphs of type (m, n), and by $\tilde{G}_{m,n}$ the subset of $G_{m,n}$ consisting of all those graphs having neither double edges nor tadpoles.

Remark 1. The class of graphs which we consider is not the most general type considered by Kontsevich in [25]. In the construction of the Formality morphism there also appear graphs with sources for more or fewer (than two) arrows. However, in our approach to the problem at hand, which is the construction of a $\star$-product expansion that would be associative modulo $\hbar^k$ for some $k \gg 0$, we shall only meet graphs from the class of Definition 1.

Remark 2. There can be tadpoles or cycles in a graph $\Gamma \in G_{m,n}$, see Fig. 1.

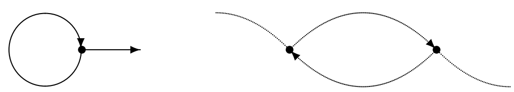


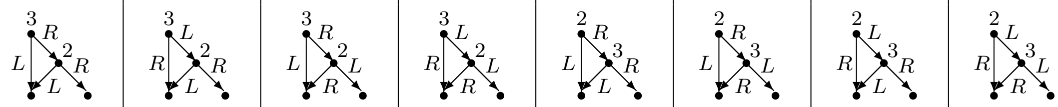
Figure 1. A tadpole and an “eye”.

A Kontsevich graph $\Gamma \in G_{m,n}$ is uniquely determined by the numbers n and m together with the list of ordered pairs of targets for the internal vertices. For reasons which will become clear immediately below, we now consider a Kontsevich graph Γ together with a *sign* $s \in \{0, \pm 1\}$, denoted by concatenation of the symbols: $s\Gamma$.

Implementation 1 (encoding)**.** The format to store a signed graph $s\Gamma$ with $\Gamma \in G_{m,n}$ is the integer number $m > 0$, the integer $n \geqslant 0$, the sign s, followed by the (possibly empty, when $n = 0$) list of n ordered pairs of targets for edges issued from the internal vertices m, …, $m+n-1$, respectively. The full format is then (m, n, s; list of ordered pairs).

We recall that to every Kontsevich graph one associates a polydifferential operator by placing a copy of the Poisson bracket at each vertex. To a signed graph one associates the polydifferential operator of the graph multiplied by the sign. The skew-symmetry of the Poisson bracket implies that the same polydifferential operator may be represented by several different signed graphs, all having different encodings.

Example 1. Taken with the signs in the first row, the graphs in the second row all represent the same polydifferential operator:



+1	−1	−1	+1	+1	−1	−1	+1
0 1 0 2	0 1 2 0	1 0 0 2	1 0 2 0	0 3 0 1	0 3 1 0	3 0 0 1	3 0 1 0

In the third row the target list (for internal vertices 2 and 3, respectively) is written.

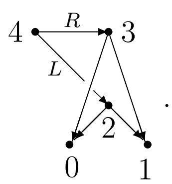
We would like to know whether two (encodings of) signed graphs specify the same topological portrait — up to a permutation of internal vertices and/or a possible swap $L \rightleftarrows R$ for some pair(s) of outgoing edges. To compare two given encodings of a signed graph, let us define its normal form. Such normal form is a way to pick the representative modulo the action of group $S_n \times (\mathbb{Z}_2)^n$ on the space $G_{m,n}$.

Definition 2 (normal form)**.** The list of targets of a graph $\Gamma \in G_{m,n}$ can be considered as a $2n$-digit integer written in base-$(n+m)$ notation. By running over the entire group $S_n \times (\mathbb{Z}_2)^n$, and by this over all the different re-labelings of Γ, we obtain many different integers written in base-$(n + m)$. The *absolute value* $|\Gamma|$ of Γ is the re-labeling of Γ such that its list of targets is *minimal* as a nonnegative base-$(n + m)$ integer. For a signed graph $s\Gamma$, the *normal form* is the signed graph $t|\Gamma|$ which represents the same polydifferential operator as $s\Gamma$. Here we let $t = 0$ if the graph is zero (see Remark 3 below).

Example 2. The minimal base-4 number in the third column of Example 1 is `0 1 0 2`. Hence the absolute value of each of the graphs in Example 1 is the first graph. The normal form of each of the signed graphs in Example 1 is the first graph taken with the appropriate sign ±1; the encodings of the normal forms are then `2 2` $\pm$`1  0 1 0 2`.

Remark 3. The graphs $\Gamma \in G_{m,n}$ for which the associated polydifferential operator vanishes, by being equal to minus itself, are called *zero*. This property can be detected during the calculation of the normal form of a signed graph. One starts with the encoding of a signed graph. Obtain a "sorted" encoding (representing the same polydifferential operator) by sorting the outgoing edges in every pair in nondecreasing order: each swap $L \rightleftarrows R$ entails a reversion of the sign. Now run over the group S_n of permutations of the internal vertices in the graph at hand, relabeling those vertices. Should the list of targets in the sorted encoding of a relabeling be equal to the list of targets in the original sorted encoding, but the sign be opposite, then the graph is zero. We will see in Chapter 2 (specifically, in Lemma 2 on p. 14) that the weights of these graphs also vanish, this time by the anticommutativity of certain differentials under the wedge product.

Example 3. Consider the graph



.

with the encoding `2 3 1  0 1 0 1 2 3`. For the identity permutation we obtain the initial sorted encoding `2 3 1  0 1 0 1 2 3` (it was already sorted). For the permutation $2 \leftrightarrows 3$ we obtain the encoding `2 3 1  0 1 0 1 3 2`; upon sorting the pairs it becomes `2 3 -1  0 1 0 1 2 3`. The list of pairs coincides with the initial sorted encoding but the sign is opposite; hence the graph is zero.

The notion of normal form of graphs allows one to generate lists of graphs with different topological portraits (e.g., Kontsevich graph series, see section 1.2 below) by using the following algorithm. Initially, the set of generated graphs is empty. For every encoding (according to Implementation 1) in a run-through, its normal form with sign $+1$ or 0 is added to the list if it is not contained there (otherwise, the offered encoding is skipped).

Implementation 2. To generate all the Kontsevich graphs with `m` sinks and `n` internal vertices in $\tilde{G}_{m,n}$ (without tadpoles or double edges), the command is

```
> generate_graphs n m
```

The procedure lists all such graphs (one per line) in the standard output. The second argument `m` may be omitted: the default value is `m = 2`.
Similarly, to generate only normal forms (with sign $+1$ or 0), the call is

```
> generate_graphs n m --normal-forms=yes
```

The optional argument `--with-coefficients` indicates that (numbered) coefficients should be listed along with the graphs.
(Accordingly, see `KontsevichGraph::graphs` in Appendix B.)

Example 4. The Kontsevich graphs in $\tilde{G}_{m,n}$ with *one* internal vertex

```
> generate_graphs 1
2 1 1   0 1
2 1 1   1 0
```

consist of the wedge with its two different labellings. We can check that the number of Kontsevich graphs on n internal vertices and two sinks is $(n(n+1))^n$:

```
> generate_graphs 2 | wc -l
36
> generate_graphs 3 | wc -l
1728
> generate_graphs 4 | wc -l
160000
> generate_graphs 5 | wc -l
24300000
```

Here, "`| wc -l`" counts the number of lines in the output.

Let us remember that while a graph series is generated, more options can be chosen to restrict the graphs: e.g., only *prime* graphs can be taken into account, or graphs but not their mirror copies can be allowed. On the same grounds, only those graphs in which the sink(s) is or are the target(s) for at least one arrow per sink can be taken. The implementation of these conventions will be explained in the next chapter (see p. 16 below).

1.2. **Series of graphs: file format.** We now specify how formal power series expansions of graphs are implemented in software. Denote by $\hbar$ the formal parameter; in machine-readable format, a power series in $\hbar$ is a list of coefficients of $\hbar^k$, $k \geqslant 0$. The coefficients are formal sums of graphs (see `KontsevichGraphSum` in Appendix B) in which the coefficients can be of any type, e.g.,

- integer or floating point numbers (e.g., 0.333),
- rational numbers (e.g., 1/3),
- undetermined variables (resp., `OneThird`).

To be precise, the library contains the class `KontsevichGraphSeries` which depends on a template parameter `T`; it specifies the type of all the coefficients of graphs in the series. In the command line programs, the external type `GiNaC::ex`, which is the expression type of the `GiNaC` library [1], allows all of the above values (and combinations of them). Hence a series under study can contain coefficients of all types at once; the coefficient of a graph itself can be a sum of different type objects (e.g., `p16` + 0.25).

In the file format for formal power series expansions, two kinds of lines are possible: either

```
h^k:
```

or (separated by whitespace)

```
<encoding of a graph> <coefficient>
```

The precision of the formal power series expansion is indicated by the highest `k` occurring in lines of the form "`h^k:`". Hence one can control this bound by adding such a line with a high `k` at the end of the file.

Implementation 3. The substitution of undetermined coefficients by their actual values, as well as re-expression of indeterminates via other such objects, is done by using the program

```
> substitute_relations <graph-series-file> <subsitutions-file>
```

Its command line arguments are two file names: the first file contains the series and the second file consists of a list of substitutions (one per line), each substitution written in the form

```
<variable>==<what it is set equal to>
```

The command line program sends the series with all those substitutions to the standard output.

Example 5. The Kontsevich $\star$-product (see §2) is a graph series given up to the second order in the deformation parameter $\hbar$ in the file `star_product2_w.txt` which reads

```
h^0:
2 0 1             1
h^1:
2 1 1   0 1       1
h^2:
2 2 1   0 1 0 1   1/2
2 2 1   0 1 0 2   w_2_1
2 2 1   0 1 1 2   w_2_2
2 2 1   0 3 1 2   w_2_3
```

In fact, the values of the three unknowns are written in weights2.txt:

```
w_2_1==1/3
w_2_2==-1/3
w_2_3==-1/6
```

Whence the star-product is given modulo $\bar{o}(\hbar^2)$ as follows:

```
> substitute_relations star_product2_w.txt weights2.txt
h^0:
2 0 1             1
h^1:
2 1 1   0 1       1
h^2:
2 2 1   0 1 0 1   1/2
2 2 1   0 1 0 2   1/3
2 2 1   0 1 1 2   -1/3
2 2 1   0 3 1 2   -1/6
```

In practice one may encounter graph series containing many graphs and undetermined coefficients. To split a graph series into parts, the following command is helpful.

Implementation 4. To extract the part of a graph series proportional to a given expression, use the call

```
> extract_coefficient <graph-series-file> <expression>
```

In the standard output one obtains a modification of the original graph series: each graph coefficient c is now replaced by the coefficient of <expression> in c. If the coefficient of <expression> in c is identically zero, then the graph is skipped. The special value <expression> = 1 yields the constant part of the graph series (all the undetermined variables in the input are set to zero).

Example 6. From the file in Example 5, we extract the part proportional to w_2_1:

```
> extract_coefficient star_product2_w.txt w_2_1
h^0:
h^1:
h^2:
2 2 1   0 1 0 2   1
```

It is just one graph.

1.3. **Reduction modulo skew-symmetry.** Let us recall that for every internal vertex in a Kontsevich graph, the pair of out-going edges is ordered by the relation Left $\prec$ Right and by a mark-up of those two edges using L and R. Starting from the vector space of formal sums of graphs with real coefficients, we pass to its quotient. Namely, we postulate that graphs which differ only by their internal vertex labeling are equal. Further, we proclaim that every reversal of the edge order in any pair entails the reversion of the graph sign. By construction, the sign is a part of the encoding for a graph, see Implementation 1 on p. 5 above; this part of a graph's description is used whenever formal sums of graphs with coefficients,

+(sign)<coeff> $\cdot\,\Gamma$raph + (sign)<coeff> $\cdot\,\Gamma$raph, are reduced. In particular, we let

$$\texttt{+(+1)<coeff>}\cdot\Gamma\ \texttt{+}\ \texttt{(-1)<coeff>}\cdot\Gamma \equiv 0$$

for a graph Γ with any coefficient `<coeff>`.

The ordering mechanism Left $\prec$ Right creates graphs that equal zero because they are equal to minus themselves (see Remark 3 and Example 3).

Remark 4. To avoid such comparison of graphs with zero all the time and so, to increase efficiency, every graph is brought to its normal form as soon as it is constructed. It is this moment when zero graphs acquire zero signs.

The algorithm to reduce a sum of graphs modulo skew-symmetry runs as follows. For the starting graph or every next graph in the list, its sign (if nonzero) is set equal to +1 and its coefficient is modified, if necessary, by using the rule

$$\texttt{<sign>}\cdot\texttt{<coeff> = (+1)}\cdot\texttt{<sign}\cdot\texttt{coeff>}. \tag{3}$$

Every graph with sign 0 is removed. Then the graph at hand (in its normal form, times a coefficient) is compared, disregarding signs, with all the graphs which follow in the list. A match found, its coefficient is added – using relation (3) – to the coefficient of the graph we started with; the match itself is removed. By this reduction procedure for graph sums, all vanishing graphs with zero signs are excluded from the list.

Implementation 5. To reduce a graph series expansion modulo skew-symmetry, call

```
> reduce <graph-series-file> [--print-differential-orders]
```

The resulting graph series is sent to the standard output. The optional argument `--print-differential-orders` controls whether the differential orders of the graphs (as operators acting on the sinks) are included in the output, with lines such as

```
# 2 1
```

indicating subsequent graphs have differential order $(2, 1)$. (The corresponding methods are `KontsevichGraphSeries<T>::reduce()` and `KontsevichGraphSum<T>::reduce()` in Appendix B.)

Example 7. We put the zero graph from Example 3 with the coefficient +1 into a file `zerograph.txt`:

```
h^3:
2 3 1  0 1 0 1 2 3    1
```

We confirm that `reduce` kills it:

```
> reduce zerograph.txt
h^3:
```

The output is an empty list of graphs.

Sometimes it is desirable to skew-symmetrize a graph series over the content of its sinks. For example, one may want to do this when dealing with first-order differential operators which represent (skew-symmetric) polyvectors (e.g., as the authors did jointly with A. Bouisaghouane in [3]).

Implementation 6. To skew-symmetrize a graph series over the content of its sinks, the command

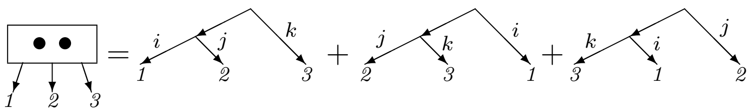
```
> skew_symmetrize <graph-series-file>
```

is available. The convention is that the sum over all permutations of the sinks is taken, with the signs of those permutations, without any pre-factor (such as $1/n!$). (Accordingly, see `KontsevichGraphSum<T>::skew_symmetrize()` in Appendix B, as well as `KontsevichGraphSeries<T>::skew_symmetrize()`, which calls the former.)

Remark 5. Sums of graphs may also be reduced modulo the (graphical) Jacobi identity and its (pictorial) differential consequences; this is the subject of section 3.2.

1.4. **Evaluate a given graph series at a given Poisson structure.** Let us recall that every Kontsevich graph contains at least one sink. Every edge (decorated with an index, say i, over which the summation runs from 1 to $n = \dim N^n$) denotes the derivation with respect to a local coordinate x^i at a given point $\boldsymbol{x}$ of the affine manifold N^n (hence the edge denotes $\partial/\partial x^i|_{\boldsymbol{x}}$). Every internal vertex (if any) encodes a copy of a given Poisson structure $\mathcal{P}$. Should the labellings of two outgoing edges be $\xrightarrow{i}$ and $\xrightarrow{j}$ so that the edge with i precedes that with j, the Poisson structure in that vertex is $\mathcal{P}^{ij}(\boldsymbol{x})$ (that is, the ordering $i \prec j$ is preserved; moreover, the reference to a point $\boldsymbol{x}$ is common to all vertices). Now, every Kontsevich graph (with a coefficient after it) represents a (poly)differential operator with respect to the content of sink(s); to build that operator, we apply the derivations (at $\boldsymbol{x} \in N^n$) to objects in the arrowhead vertices, multiply the content of all vertices at a fixed set of index values, and then sum over all the indices.

Example 8 (Jacobi identity)**.** For all Poisson structures $\mathcal{P}$ and all triples of arguments from the algebra $C^\infty(N^n)$ of functions on the Poisson manifold at hand, we have that

$\mathrm{Jac}(\mathcal{P})(\mathit{1},\mathit{2},\mathit{3}) =$  $= 0.$ (4)

In formulae, by ascribing the index ℓ to the unlabeled edge, the identity reads

$$(\partial_\ell \mathcal{P}^{ij}\mathcal{P}^{\ell k} + \partial_\ell \mathcal{P}^{jk}\mathcal{P}^{\ell i} + \partial_\ell \mathcal{P}^{ki}\mathcal{P}^{\ell j})\partial_i(\mathit{1})\partial_j(\mathit{2})\partial_k(\mathit{3}) = 0.$$

Indeed, the coefficient of $\partial_i \otimes \partial_j \otimes \partial_k$ is the familiar form of the Jacobi identity.

In fact, the graph itself is the most convenient way to transcribe the formulae which one constructs from it, see [18, §2.1] for more details.[4] The computer implementation is straightforward. We acknowledge however that it is one of the most needed instruments.

[4]In the variational set-up of Poisson field models, the affine manifold N^n is realised as fibre in an affine bundle π over another affine manifold M^m equipped with a volume element. The variational Poisson brackets $\{\cdot,\cdot\}_{\mathcal{P}}$ are then defined for integral functionals that take sections of such bundle π to numbers. The encoding of variational polydifferential operators by the Kontsevich graphs now reads as follows. Decorated by an index i, every edge denotes the variation with respect to the ith coordinate along the fibre. By construction, the variations act by first differentiating their argument with respect to the fibre variables (or their derivatives along the base M^m); secondly, the integrations by parts over the underlying space M^m are performed. Whenever two or more arrows arrive at a graph vertex, its content is first differentiated the corresponding number of times with respect to the jet fibre variables in $J^\infty(\pi)$ and only then it can be differentiated with respect to local coordinates on the base manifold M^m. The assumption that both the manifolds M^m and N^n be affine makes the construction coordinate-free, see [16, 20] and [17, 19].

Implementation 7. The call is

```
> poisson_evaluate <graph-series-filename> <poisson-structure>
```

and options for `<poisson-structure>` are[5]

- `2d-polar`,
- `3d-generic`,
- `3d-polynomial`,
- `4d-determinant`,
- `4d-rank2`,
- `9d-rank6`.

The output is a list of coefficients of the differential operator that the graph series represents, filtered by (a) powers of $\hbar$, (b) the differential order as an operator acting on the sinks, and (c) the actual derivatives falling on the sinks.

Example 9. Put the graph sum for the Jacobiator $\mathrm{Jac}(\mathcal{P})$ in `jacobiator.txt`:

```
3 2 1   0 1 2 3    -1
3 2 1   0 2 1 3    1
3 2 1   0 4 1 2    -1
```

We evaluate it at a Poisson structure:

```
> poisson_evaluate jacobiator.txt 2d-polar
Coordinates: r t
Poisson structure matrix:
[[0, r^(-1)]
[-r^(-1), 0]]

h^0:
# 1 1 1
# [ r ] [ r ] [ r ]
0
# [ r ] [ r ] [ t ]
0
# [ r ] [ t ] [ r ]
0
# [ r ] [ t ] [ t ]
0
# [ t ] [ r ] [ r ]
0
# [ t ] [ r ] [ t ]
0
# [ t ] [ t ] [ r ]
0
# [ t ] [ t ] [ t ]
0
```

[5]The current version of the software does not allow specification of an arbitrary Poisson structure at runtime (e.g. input as a matrix of functions); however, in the source file `util/poison_structure.hpp` the list of Poisson structures (as matrices) can be extended to one's heart's desire.

For example, the pair of lines

```
# [ r ] [ t ] [ r ]
0
```

indicates that the coefficient of $\partial_r \otimes \partial_t \otimes \partial_r$ is zero in the polydifferential operator.

Restriction of graph series to Poisson structures will be essential in section 3.1 below where systems of linear algebraic equations between the Kontsevich graph weights in $\star$ will be obtained by restricting the associativity equation $\mathrm{Assoc}_\star(f,g,h)=0$ to a given Poisson bracket.

2. The Kontsevich $\star$-product

The star-product $\star = \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + \bar{o}(\hbar)$ in $C^\infty(N^n)[[\hbar]]$ is an associative unital noncommutative deformation of the associative unital commutative product $\times$ in the algebra of functions $C^\infty(N^n)$ on a given affine manifold N^n of dimension $n<\infty$. The bi-linear bi-differential $\star$-product is realized as a formal power series in $\hbar$ by using the weighted Kontsevich graphs. In fact, the bi-differential operator at $\hbar^k$ is a sum of all Kontsevich graphs $\Gamma \in G_{2,k}$ without tadpoles, with k internal vertices (and two sinks) taken with some weights $w(\Gamma)$. Let us recall their original definition [25].

Definition 3. Every Kontsevich graph $\Gamma \in \tilde{G}_{2,k}$ can be embedded in the closed upper half-plane $\mathbb{H} \cup \mathbb{R} \subset \mathbb{C}$ by placing the internal vertices at pairwise distinct points in $\mathbb{H}$ and the external vertices at 0 and 1; the edges are drawn as geodesics with respect to the hyperbolic metric, i.e. as vertical lines and circular segments. The angle $\varphi(p,q)$ between two distinct points $p,q\in\mathbb{H}$ is the angle between the geodesic from p to q and the geodesic from p to ∞ (measured counterclockwise from the latter):

$$\varphi(p,q) = \mathrm{Arg}\left(\frac{q-p}{q-\bar{p}}\right),$$

and it can be extended to $\mathbb{H}\cup\mathbb{R}$ by continuity. The *weight* of a Kontsevich graph $\Gamma \in \tilde{G}_{2,k}$ is given by the integral[6]

$$w(\Gamma) = \frac{1}{(2\pi)^{2k}} \int_{C_k(\mathbb{H})} \bigwedge_{j=1}^{k} \mathrm{d}\varphi(p_j, p_{\mathrm{Left}(j)}) \wedge \mathrm{d}\varphi(p_j, p_{\mathrm{Right}(j)}), \tag{5}$$

over the *configuration space* of k points in the upper half-plane $\mathbb{H}\subset\mathbb{C}$,

$$C_k(\mathbb{H}) = \{(p_1,\ldots,p_k)\in\mathbb{H}^k : p_i \text{ pairwise distinct}\};$$

the integrand is defined pointwise at $(p_1,\ldots,p_k)$ by considering the embedding of Γ in $\mathbb{H}$ that sends the jth internal vertex to p_j; the numbers $\mathrm{Left}(j)$ and $\mathrm{Right}(j)$ are the left and right targets of jth vertex, respectively.

For every Poisson bi-vector $\mathcal{P}$ on N^n and an infinitesimal deformation $\times \mapsto \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + \bar{o}(\hbar)$ towards the respective Poisson bracket, the $\hbar$-linear star-product

$$\star = \times + \sum_{k\geqslant 1} \frac{\hbar^k}{k!} \sum_{\Gamma\in\tilde{G}_{2,k}} w(\Gamma)\,\Gamma(\mathcal{P})(\cdot,\cdot) \colon C^\infty(N^n)[[\hbar]] \times C^\infty(N^n)[[\hbar]] \to C^\infty(N^n)[[\hbar]] \tag{6}$$

[6] We omit the factor $1/k!$ that was written in [25], to make the weight multiplicative (see Lemma 5).

is associative.

Lemma 1. Permuting the internal vertex labels of a Kontsevich graph leaves the weight unchanged.

Proof. Such a permutation re-orders the factors in a wedge product of two-forms. □

Lemma 2. Swapping $L \rightleftarrows R$ at an internal vertex of a Kontsevich graph $\Gamma \in \tilde{G}_{2,k}$ implies the reversal of the sign of its weight.

Proof. Anticommutativity of wedge product of two differentials in formula (5) for $w(\Gamma)$. □

Lemma 3. The weight of a graph $\Gamma \in \tilde{G}_{2,k}$ and its reflection $\bar{\Gamma}$ are related by $w(\bar{\Gamma}) = (-)^k w(\Gamma)$.

Proof. Taking the reflection of a graph (with respect to the vertical line $\Re(z) = 1/2$) corresponds to an orientation-reversing coordinate change on each "factor" $\mathbb{H}$ in $C_n(\mathbb{H})$. □

Lemma 4 ([9])**.** For a Kontsevich graph such that at least one sink receives no edge(s), its weight is zero.[7]

Lemma 5. The map $w\colon \sqcup_k \tilde{G}_{2,k} \to \mathbb{R}$ that assigns weights to graphs is multiplicative,

$$w(\Gamma_i \bar{\times} \Gamma_j) = w(\Gamma_i) \times w(\Gamma_j), \tag{7}$$

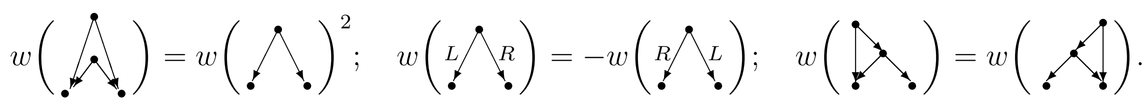
with respect to the product $\bar{\times}$ of graphs,

$$\Gamma_i \bar{\times} \Gamma_j = (\Gamma_i \sqcup \Gamma_j) \big/ \{a^{\text{th}} \text{ sink in } \Gamma_i = a^{\text{th}} \text{ sink in } \Gamma_j, \quad 0 \leqslant a \leqslant 1\},$$

which identifies the respective sinks.

Proof. The integrals converge absolutely [25]; apply Fubini's theorem and linearity. □

Example 10. Some weight relations obtained from the lemmas above:



Lemma 5 motivates the following definition.

Definition 4. A Kontsevich graph $\Gamma \in \tilde{G}_{2,k}$ is called *prime* if Γ is not equal to the $\bar{\times}$ product of any Kontsevich graphs on two sinks and positive number of internal vertices in either of the co-factors. Otherwise (if such a realization is possible), the graph is called *composite*.

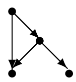
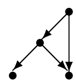
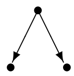
Using Lemma 5 and induction, we obtain that the weight of a composite graph $\Gamma = \Gamma_1 \bar{\times} \cdots \bar{\times} \Gamma_t$ is the product of the weights of its factors: $w(\Gamma) = w(\Gamma_1) \times \cdots \times w(\Gamma_t)$.

[7]The fact that the differential order of $\star$ is positive with respect to either of its arguments should be expected, in view of the required property of $\star$-product to be unital: $f \star 1 = f = 1 \star f$.

2.1. **Basic set of graphs.** We identify a set of graphs such that the weights of those graphs would suffice to determine all the other weights.

Definition 5. A *basic* set of graphs on k internal vertices is a set of pairwise distinct normal forms (the signs of which are discarded) of only those Kontsevich graphs $\Gamma \in \tilde{G}_{2,k}$ which are prime, and in which every sink receives at least one edge. By definition, the basic set contains the normal form of a graph but not its mirror reflection if it differs from the graph at hand. To decide whether a graph or its mirror-reflection $\bar{\Gamma} \neq \Gamma$ is included into a basic set, we take the graph whose absolute value is *minimal* as a base-$(k+2)$ number. Note that a basic set on $k \geqslant 3$ vertices *does* contain zero graphs.

Corollary 6. To build $\star$-product (6) up to $\bar{o}(\hbar^k)$ for some power $k \geqslant 1$, knowing the Kontsevich weights $w(\Gamma_i)$ only for a *basic* set of graphs $\Gamma_i \in \tilde{G}_{2,\ell}$ at all $\ell \leqslant k$ is enough. Indeed, the weights of all other graphs with ℓ internal vertices are calculated from Lemmas 1, 2, 3, 4, and 5.

Example 11. Consider the prime graph  and its mirror-reflection . The encodings of their normal forms are `2 2 1  0 1 0 2` and `2 2 1  0 1 1 2` respectively. Since `0 1 0 2` < `0 1 1 2` as base-4 numbers, only the first graph is included in the basic set. The fork graph  is mirror-symmetric hence it is included anyway.

The basic set at order 3 is displayed in Figure 2.

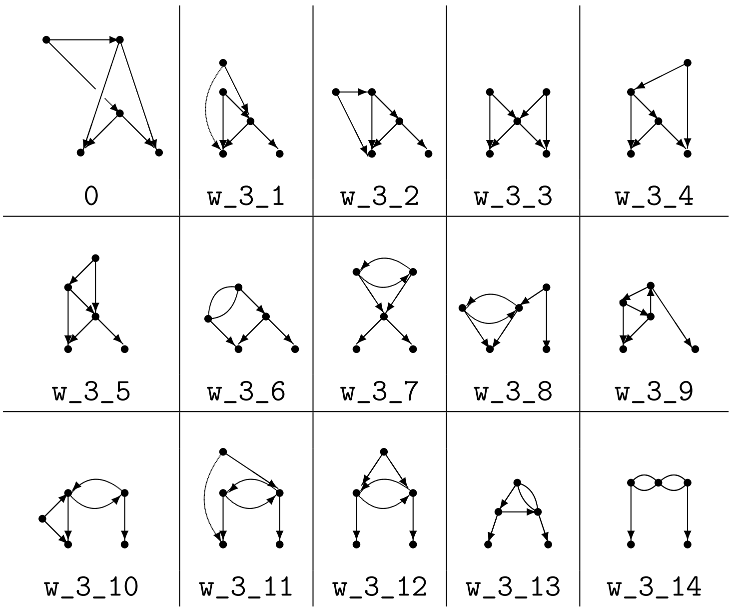


FIGURE 2. Basic set at order 3, with undetermined coefficients for nonzero graphs.

TABLE 1. How many basic graphs there are at low orders k.

Order $= k$	0	1	2	3	4	5	6
#(Basic set)	0	1	2	15	156	2307	–
#(Nonzero in basic set)	0	1	2	14	149	2218	42050

2.2. **"All" graphs in $\star$ mod $\bar{o}(\hbar^4)$.** In Table 1 we list the number of basic graphs at every order $k \leqslant 6$ in the Kontsevich $\star$-product. The actual number of graphs with respect to which the sums in formula (6) expand is of course much greater.

Implementation 8. To obtain the list of normal forms for graphs from a basic set, the following command is available:

```
> generate_graphs k --basic=yes
```

The list of normal forms is then sent to the standard output. This command is equivalent to

```
> generate_graphs k --prime=yes --normal-forms=yes \
     --postive-differential-order=yes --modulo-mirror-images=yes
```

Starting from a basic set, the $\star$-product is built up to a certain order $k \geqslant 0$ in $\hbar$.

Implementation 9. The program

```
> star_product <basic-set-filename>
```

takes as its input a graph series with a basic set of graphs at each order; the graphs go with coefficients of any nature (i.e. number or indeterminate). The program's output is an expansion of the $\star$-product up to the order that was specified by the input. In other words, all the graphs which are produced from the ones contained in a given basic set are generated and their coefficients are (re)calculated from the ones in the input (using Lemmas 2, 3, and 5).

Example 12. To generate the star-product up to order 3 with all weights of nonzero basic graphs undetermined, one proceeds as follows:

```
$ cat > basic_graphs_undetermined3.txt
h^0:
2 0 1     1
h^1:
^D (press Ctrl+D)
$ generate_graphs 1 --basic=yes --with-coefficients=yes \
      >> basic_graphs_undetermined3.txt
$ echo 'h^2:' >> basic_graphs_undetermined3.txt
$ generate_graphs 2 --basic=yes --with-coefficients=yes \
      >> basic_graphs_undetermined3.txt
$ echo 'h^3:' >> basic_graphs_undetermined3.txt
$ generate_graphs 3 --basic=yes --with-coefficients=yes \
      >> basic_graphs_undetermined3.txt
$ star_product basic_graphs_undetermined3.txt \
      > star_product_undetermined3.txt
```

The file `star_product_undetermined3.txt` now contains the desired star-product.

2.3. **Methods to obtain the weights of basic graphs.** We deduce that to build the $\star$-product modulo $\bar{o}(\hbar^4)$ as many as 149 weights of nonzero basic graphs $\Gamma_i \in \tilde{G}_{2,4}$ at $k = 4$ must be found (or at least expressed in terms of as few master-parameters as possible). In fact, direct calculation of all of the 149 Kontsevich integrals is not needed to solve the problem in full because there exist more algebraic relations between the weights of basic graphs. In the following proposition we recall a class of such relations.[8]

Proposition 7 (cyclic weight relations [10])**.** Let Γ be a Kontsevich graph on $m = 2$ ground vertices. Let $E \subset \text{Edge}(\Gamma)$ be a subset of edges in Γ such that for every $e \in E$, $\text{target}(e) \neq 0$. (That is, every edge from the subset E lands on the sink 1 or an internal vertex.) For a given subset E, define the graph Γ_E as follows: let its vertices be the same as in Γ and for every edge $e \in \text{Edge}(\Gamma)$, preserve it in Γ_E if $e \neq E$, but if $e \in E$ replace that edge by a new edge in Γ_E going from source(e) to the sink 0. By definition, the ordering $L \prec R$ of outgoing edges is inherited in Γ_E from E even if the targets of any of those edges are new. Thirdly, denote by $N_0(\Gamma_E)$ the number of edges in Γ_E such that their target is the sink 0. Then the Kontsevich weight of a graph Γ is related to the weights of all such graphs Γ_E obtained from Γ by the formula

$$w(\Gamma) = (-)^n \sum_{E \subseteq \text{Edge}(\Gamma)} (-)^{N_0(\Gamma_E)} w(\Gamma_E). \tag{8}$$

Note that this relation is linear in the weights of all graphs.

If the graph Γ or, in practice, some of the new graphs Γ_E in (8) is composite, Lemma 5 provides a further, nonlinear reduction of $w(\Gamma)$ by using graphs with fewer internal vertices.

Example 13. Consider the graph $\Gamma_{3,8}$ in Figure 2 with weight $w(\Gamma_{3,8}) =$ `w_3_8`. For every non-empty subset E (with $\text{target}(e) \neq 0$ for every $e \in E$) the graph $(\Gamma_{3,8})_E$ is a zero-weight graph by virtue of one of the Lemmas at the beginning of this chapter. Hence the only term in the sum on the right-hand side in (8) is the weight of the graph corresponding to the empty set: $w((\Gamma_{3,8})_\varnothing) = w(\Gamma_{3,8})$. Since $n = 3$ and $N_0(\Gamma_{3,8}) = 2$ we get the cyclic relation $w(\Gamma_{3,8}) = -w(\Gamma_{3,8})$; whence $w(\Gamma_{3,8}) = 0$.

Remark 6. It is readily seen that only *prime*, that is, non-composite graphs Γ need be used to generate *all* relations (8). Indeed, every subset E of edges for a composite graph $\Gamma = \Gamma^1 \bar{\times} \Gamma^2$ splits to a disjoint union $E^1 \sqcup E^2$ of such subsets for the graphs Γ^1 and Γ^2 separately. Therefore the re-direction of edges in a composite graph would inevitably yield the composite graph $\Gamma^1_{E^1} \bar{\times} \Gamma^2_{E^2}$. Now, the multiplicativity of Kontsevich weights and the additivity of the count $N_0(\Gamma_E) = N_0(\Gamma^1_{E^1}) + N_0(\Gamma^2_{E^2})$ can be used to conclude that the relations obtained from composite graphs are redundant.

Implementation 10. The command

```
> cyclic_weight_relations <star-product-file>
```

[8]A convenient approach to calculation of Kontsevich weights (5) at order 3 by using direct integration (and for that, using methods of complex analysis such as the Cauchy residue theorem) was developed in [7], see Appendix A on p. 41 below. However, we note that most successful at $k = 3$, this method is no longer effective for all graphs at $k \geqslant 4$. More progress is badly needed to allow $k \geqslant 5$.

treats the input $\star$-product as a clothesline for graphs and their weights. For each graph Γ in the $\star$-product, it outputs the relation (8) between the weights of the respective graphs in the form `LHS - RHS == 0`.

Example 14. At the order four with the $\star$-product from Table 5 in Appendix C:

```
> cyclic_weight_relations star_product4.txt
-1/1728 - 1/2*w_4_1==0
...
-1/3456-1/12*w_4_2==0
...
-1/384+1/8*w_4_6-1/4*w_4_8==0
```

Remark 7. For some (basic) graphs it happens that the weight integrand in (5), as a differential $2k$-form, vanishes identically, even if the graph is not zero due to skew-symmetry. This is the case for 21 out of 149 nonzero basic graphs at $k = 4$; see also Appendix A.

For calculations of particular weight integrals we refer to the literature in section 3.3.

Remark 8 (rationality). Willwacher and Felder [10] relate the questions of (ir)rationality of the Kontsevich graph weights to the questions of (ir)rationality of the value $\zeta(3)/\pi^3$ (similarly, $\zeta(5)$ and $\zeta(7)$) of the Riemann ζ-function. We note that if, by running all the algorithms from the present paper, one attains the order $\bar{o}(\hbar^7)$ in the deformation parameter (such that the associativity of the Kontsevich $\star$-product is ascertained at least mod $\bar{o}(\hbar^7)$) and if all the values $w(\Gamma_i)$ of all the graphs Γ_i involved appear to be rational numbers, then this result about the graph weight values would resolve the (ir)rationality (more precisely, algebraicity) problem for $\zeta(3)/\pi^3$.

All the above being said about methods to obtain the values $w(\Gamma)$ for Kontsevich graph weights and about the schemes to generate linear relations between these numbers, we observe that the requirement of associativity for the $\star$-product modulo $\bar{o}(\hbar^k)$, whenever that structure is completely known at all orders up to $\hbar^k$, is an ample source of relations of that kind. This will be used intensively in chapter 3 from p. 21 onwards. In particular, we mention here that the values weights of graphs at order k may be restricted by the associativity requirement at orders $> k$, by restriction to fixed differential orders (i, j, k) (see Lemma 10 on p. 28).

2.4. **How graphs act on graphs.** Let us have a closer look at the equation of associativity for the sought-for $\star$-product:

$$\mathrm{Assoc}_\star(f, g, h) = (f \star g) \star h - f \star (g \star h) = 0.$$

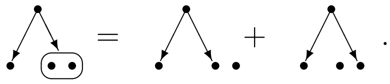
We see that the graph series $f \star g$ and $g \star h$ serve as the left- and right co-multiples of h and f, respectively, in yet another copy of the star-product. To realize the associator by using the Kontsevich graphs, we now explain how graphs act on graphs (here, in every composition $\star \circ \star$ the graph series acts on a graph series by linearity).

We postulate that the action of graph series on graph series is $\Bbbk[[\hbar]]$-linear and $\Bbbk[G_{*,*}]$-linear with respect to both the graphs that act and that become the arguments.

Recall that every Poisson bracket is a derivation in each of its arguments. In consequence, every derivation falling on a sink – in a graph Γ_1 that acts on a given graph Γ_2

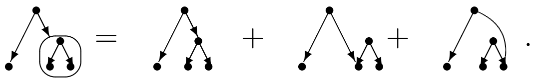
taken as the new content of that sink – acts on the sink's content via the Leibniz rule; all the Leibniz rules for the derivations in-coming to that sink work independently from each other.

Example 15. Consider the action of a wedge graph Λ on the two-sinks graph $(\bullet\bullet) \in G_{2,0}$, taken as its second argument. We have that

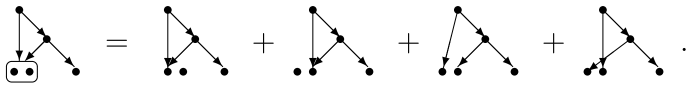


The result is a sum of Kontsevich graphs of type $(3, 1)$. Let us remember that the sinks are distinguished by their ordering; in particular the two Kontsevich graphs on the right-hand side are not equal.

Example 16. Now let the wedge graph act on a wedge graph (again, as the former's second argument):



Example 17. Finally, consider a graph in which two arrows fall on the first sink and let its content be $(\bullet\bullet) \in G_{2,0}$:



These three examples basically cover all the situations; we shall refer to them again, namely, from the next chapter where the restrictions by using the total differential orders are discussed.

So far, we have focused on graphs; under the action of a graph on a graph, their coefficients are multiplied. (This is why the associativity of the $\star$-product is an infinite system of quadratic equations for the coefficients of all the graphs).

Implementation 11. In the respective class, the method is

```
KontsevichGraphSeries<T>::operator()
```

and its argument is `std::vector` (that is, a list) of the Kontsevich graph series in $\hbar$; these are the m respective arguments for a Kontsevich graph series. The method is called for the object of the class, that is, for the graph series which is evaluated at the m specified arguments.

2.5. **Gauge transformations.** At first glance, the concept of gauge transformations for (graphs in the) $\star$-products is an extreme opposite of plugging a list of graph series as arguments of a given graph series. Namely, the idea of a gauge transformation is that a graph series (possibly of finite length) is towered over a single vertex $\bullet \in G_{1,0}$. By definition, a gauge transformation of a vertex $\bullet$ is a map of the form $\bullet \mapsto [\bullet] = \bullet + \hbar \cdot (\ldots)$ taking $G_{1,0} \to \Bbbk[G_{1,*}][[\hbar]]$.

Example 18. The map $\bullet \mapsto \bullet + \frac{\hbar^2}{12}$  is a gauge transformation of $\bullet \in G_{1,0}$. This graph series is encoded in the following file:

```
h^0:
1 0 1     +1
h^2:
1 2 1  0 2 1 0  +1/12
```

The construction of gauge transformations is extended from $G_{1,0}$ by $\Bbbk[[\hbar]]$- and $\Bbbk[G_{*,*}]$-linearity. This effectively means that in the course of action by a gauge transformation $\mathtt{t}$ on a graph series $f \in \Bbbk[G_{*,*}][[\hbar]]$, all the arrows work over the vertices in every graph in f via the Leibniz rule (as it has been explained in the previous section). This is how one expands $[f] \star [g]$, that is, the Kontsevich $\star$-product (6) of two gauged arguments $[f]$ and $[g]$. Let us recall further that the shape $[\bullet] = \bullet + \hbar \cdot (\ldots)$, where the gauge tail of $\bullet$ is given by some graphs from $\Bbbk[G_{1,*}][[\hbar]]$, guarantees the existence of a formal left inverse $\mathtt{t}^{-1}$ to the original transformation $\mathtt{t}$, so that $(\mathtt{t}^{-1} \circ \mathtt{t})(\bullet) = \bullet$.

Lemma 8. If $\bullet \mapsto \blacksquare = \mathtt{t}(\bullet) = \bullet + \hbar\Gamma_1(\bullet) + \ldots + \hbar^\ell\Gamma_\ell(\cdot) + \bar{o}(\hbar^\ell)$ is a gauge transformation, let

$$\mathtt{t}^{-1}(\blacksquare) = \blacksquare + \hbar\gamma_1(\blacksquare) + \ldots + \hbar^\ell\gamma_\ell(\blacksquare) + \bar{o}(\hbar^\ell)$$

by setting

$$\gamma_m(\blacksquare) := -\sum_{k=0}^{m-1} \gamma_k(\Gamma_{m-k}(\blacksquare)).$$

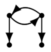
Then $\mathtt{t}^{-1}(\mathtt{t}(\bullet)) = \bullet$, that is, the transformation $\mathtt{t}^{-1} \colon \Bbbk[G_{1,*}][[\hbar]] \to \Bbbk[G_{1,*}][[\hbar]]$ is the left inverse of $\mathtt{t}$ up to $\bar{o}(\hbar)$.

It is readily seen that the assembly of the entire $\mathtt{t}^{-1}$ can require infinitely many operatioins even if the direct transformation $\mathtt{t}$ took only finitely many of them, e.g., as in Example 18.

In these terms, for the Kontsevich $\star$-product (6) we obtain, by operating with gauge transformations and their formal inverses, a class of star products $\star'$ which are defined by the relation

$$\mathtt{t}(f \star' g) = \mathtt{t}(f) \star \mathtt{t}(g), \qquad f, g \in C^\infty(N^n)[[\hbar]]. \tag{9}$$

Clearly, all these gauged star-products $\star'$ remain associative (because $\star$ was) but the coefficients of graphs at an order $k \geqslant 2$ in $\hbar$ are no longer necessarily equal to the respective values in (6). The use of gauge transformations for products allows to gauge out *some* graphs, often at a certain order $\hbar^k$ in the star-product expansion.

Example 19. The graph  with a loop is gauged out from the Kontsevich $\star$-product (6) by using the gauge transformation $\mathtt{t} \colon \bullet \mapsto \bullet + \frac{\hbar^2}{12}$  , see Example 18. Note that taking the formal inverse $\mathtt{t}^{-1}$ does create loop-containing graphs at higher orders $\hbar^{\geqslant 3}$ in the gauged star-product $\star'$ which is specified by (9).

Remark 9. Not every graph taken in the Kontsevich star-product $\star$ at a particular order $\hbar^k$ can be gauged out. For example, such are the graphs $\Gamma \in \tilde{G}_{2,*}$ containing an internal vertex v with edges running from it to both the ground vertices.

Implementation 12. The command for gauge transformation is

```
> gauge <star-product-filename> <gauge-transformation-filename>
```

where

- the file `<star-product-filename>` contains a machine-format graph encoding of star-product $\star$ truncated modulo $\bar{o}(\hbar^k)$ for some $k \geqslant 0$;
- the content of `<gauge-transformation-filename>` is a gauge transformation $\mathtt{t}(\bullet)$, that is, a truncated modulo $\bar{o}(\hbar^{\ell \geqslant 0})$ series in $\hbar$ consisting of the Kontsevich graphs built over one sink vertex $\bullet$.

In the standard output one obtains the truncation, modulo $\bar{o}(\hbar^{\min(k,\ell)})$, of the graph series for the gauged star-product $\star'$ defined by $f \star' g = \mathtt{t}^{-1}(\mathtt{t}(f) \star \mathtt{t}(g))$.

(The corresponding method is `KontsevichGraphSeries<T>::gauge_transform()` in Appendix B.)

Example 20. Let the gauge transformation from Example 19 be stored in the file `gaugeloop.txt`, and recall the $\star$-product up to order two from Example 5 in the file `star_product2.txt`. The gauge transformation kills the loop graph:

```
> gauge star_product2.txt gaugeloop.txt > star_product2_gauged.txt
> reduce star_product2_gauged.txt
h^0:
2 0 1          1
h^1:
2 1 1  0 1     1
h^2:
2 2 1  0 1 0 1   1/2
2 2 1  0 1 0 2   1/3
2 2 1  0 1 1 2   -1/3
```

Indeed, we see that the line

```
2 2 1  0 3 1 2   -1/6
```

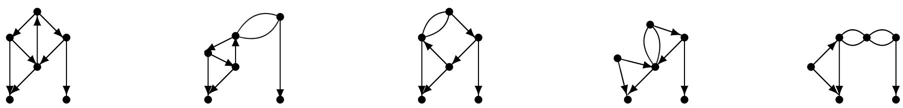
containing the loop graph has disappeared.

Let us note at once that every gauge transformation $\mathtt{t}$ given by a Kontsevich graph polynomial in $\hbar$ of degree ℓ can clearly be viewed formally as a polynomial transformation of any degree greater or equal than ℓ. This is why by using the same software we can actually obtain the gauged star-product $\star'$ modulo $\bar{o}(\hbar^4)$ starting with the Kontsevich star-product $\star$ modulo $\bar{o}(\hbar^4)$ and applying the gauge transformation of nominal degree $\ell = 2$ from Example 18. In other words, the precision in $\star'$ with respect to $\hbar$ is the same as in $\star$ even though the degree of the polynomial gauge transformation $\mathtt{t}$ is smaller. In practice, this is achieved by adding an empty list of graphs at powers $\hbar^k$ to a given gauge transformation of degree $\ell < k$.

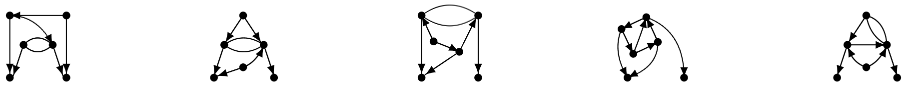
3. Associativity of the Kontsevich $\star$-product

In the final section of this paper we explore two complementary matters. One the one hand, we analyse how the associativity postulate for the Kontsevich $\star$-product contributes to finding the values of weights $w(\Gamma)$ for graphs Γ in $\star$. On the other hand, a point is soon reached when no new information can be obtained about the values of $w(\Gamma)$: specifically, neither from the fact of associativity of the $\star$-product nor from any proven properties of the Kontsevich weights. We outline a computer-assisted scheme of reasoning that, working uniformly over the set of all Poisson structures under study, reveals the associativity of $\star$-product on the basis of our actual knowledge about the weights $w(\Gamma)$ of graphs Γ in it.

In [6] we reported an exhaustive description of the Kontsevich $\star$-product up to $\bar{o}(\hbar^3)$. At the next expansion order $\bar{o}(\hbar^4)$ in $\star$, we now express the weights of all the 160.000 graphs $\Gamma \in \tilde{G}_{2,4}$ in terms of only 10 parameters; those ten master-parameters themselves are the (still unknown) Kontsevich weights of the four internal vertex graphs portrayed in Fig. 3. By following the second strategy we prove that for any values of those ten parameters the $\star$-product expansion modulo $\bar{o}(\hbar^4)$ is associative, also up to $\bar{o}(\hbar^4)$.



$p_1 =$ `w_4_100` $p_2 =$ `w_4_101` $p_3 =$ `w_4_102` $p_4 =$ `w_4_103` $p_5 =$ `w_4_104`



$p_6 =$ `w_4_107` $p_7 =$ `w_4_108` $p_8 =$ `w_4_109` $p_9 =$ `w_4_119` $p_{10} =$ `w_4_125`

FIGURE 3. The ten graphs whose unknown weights[9] are taken as the master-parameters p_i; in fact, the four graphs whose weights are underlined can be gauged out from $\star$ so that there remain only 6 parameters that determine it modulo $\bar{o}(\hbar^4)$.

3.1. **Restriction of the $\star$-product associativity equation** $\mathrm{Assoc}_\star(f,g,h) = 0$ **to a Poisson structure** $\mathcal{P}$**.** We now view the postulate of associativity for the Kontsevich $\star$-product as an equation for coefficients in the graph expansion of $\star$. Whenever an expansion modulo $\bar{o}(\hbar^\ell)$ is known for the $\star$-product, one passes to the next order $\bar{o}(\hbar^{\ell+1})$ by taking all the graphs $\Gamma \in \tilde{G}_{2,\ell+1}$ with undetermined coefficients, and then expands (with respect to graphs) the associator $\mathrm{Assoc}_\star(f,g,h)$ up to the order $\bar{o}(\hbar^{\ell+1})$. This expansion now runs over all the graphs with at most $\ell+1$ internal vertices. It is readily seen that by construction this associativity equation $\mathrm{Assoc}_\star(f,g,h) = \bar{o}(\hbar^{\ell+1})$ is always *linear*[10] with respect to the coefficients of graphs from $\tilde{G}_{2,\ell+1}$.

Remark 10. One can still get *linear* relations between the weights $w(\Gamma)$ of graphs $\Gamma \in \tilde{G}_{2,\ell+1}$ at order $\hbar^{\ell+1}$ in $\star$ by inspecting the associativity of $\star$ at *higher* orders – ranging from $\ell+2$ till $2\ell+1$ – in $\hbar$. Indeed, a linear relation containing the unknown weights (and the already known lower-order part of $\star$ as coefficients) but not the weights of graphs with $\geqslant \ell+2$ internal vertices can appear whenever a properly chosen homogeneous component of the tri-differential operator $\mathrm{Assoc}_\star(f,g,h)$ does not contain any weights from higher orders. For instance, this is the component at homogeneity orders (i,j,k)

[9]Numerical approximations of two of these weights are listed in Table 3 in Appendix A.

[10]Should a graph $\Gamma \in \tilde{G}_{2,\ell+1}$ be composite so that its Kontsevich weight is factorized using formula (7), the resulting nonlinearity with respect to the weights would actually involve only the graphs with at most ℓ internal vertices.

such that prime graphs $\Gamma \in \tilde{G}_{2,\geqslant \ell+2}$ of homogeneity orders $(i+j,k)$ and $(i,j+k)$ (when viewed as bi-differential operators) do not exist or if the weights of all such graphs are known in advance.

3.1.1. Let us also note that in the graph equation $\mathrm{Assoc}_\star(f,g,h) = 0$ that holds by virtue of the Jacobi identity $\mathrm{Jac}(\mathcal{P}) = 0$, not every coefficient of every graph in the expansion should be expected to vanish. Indeed, the Jacobiator is a vanishing sum of three graphs that evaluates to zero at every Poisson structure $\mathcal{P}$ which we put into every internal vertex. This is why the restriction of associativity equation to a given Poisson structure (or to a class of Poisson structures) is a practical way to proceed in solution of the problem of finding the coefficients of graphs in $\star$. More specifically, after the restriction of associator $\mathrm{Assoc}_\star(f,g,h)$ to a structure $\mathcal{P}$ which is known to be Poisson so that all the instances and all derivatives of the Jacobiator $\mathrm{Jac}(\mathcal{P})$ are automatically trivialized, the left-hand side of the associativity equation $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}} = 0 \mod \bar{o}(\hbar^{\ell+1})$ becomes an analytic expression (*linear* with respect to the unknowns $w(\Gamma)$ for $\Gamma \in \tilde{G}_{2,\ell+1}$). At this point one can proceed in several ways.

We now outline three methods to obtain systems of linear equations upon the unknown weights $w(\Gamma)$ of basic graphs $\Gamma \in \tilde{G}_{2,\ell+1}$. Working in local coordinates, we ensure that the unknowns' coefficients in the equations which we derive are *real* numbers.[11]

The three methods which we presently describe can be compared as follows. On the one hand, as far as maximizing the rank of the algebraic system which are obtained by using the respective methods is concerned, Method 1 is the least effective whereas Method 3 is the most productive. One the other hand, Method 1 is the least computationally expensive, so it can be used effectively at the initial stage, e.g., to detect the zero values of certain graph weights: once found, such trivial values allow to decrease the number of unknowns in the further reasoning. We finally note that the linear algebraic systems which are produced by each method should be merged. Indeed, the goal is to maximize the rank and by this, reduce the number of free parameters in the solution.[12]

Method 1. Let the associator's arguments be given functions $f,g,h \in C^\infty(N^n)$. Restrict the analytic expression $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}}$ to a point $\boldsymbol{x}$ of the manifold N^n equipped with a Poisson structure $\mathcal{P}$. For every choice of $f,g,h \in C^\infty(N^n)$ and of a point $\boldsymbol{x} \in N^n$, the restriction $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}}(\boldsymbol{x}) = 0 \mod \bar{o}(\hbar^{\ell+1})$ yields *one* linear relation between the weights of graphs at order $\hbar^{\ell+1}$. Taking the restriction at several points $\boldsymbol{x}_1$, $\ldots$, $\boldsymbol{x}_k \in N^n$, one obtains a system of such equations, the rank of which does not exceed the number k of such points in N^n. Bounded by the number of unknowns $w(\Gamma)$, the rank would always stabilize as $k \to \infty$.

[11]From the factorization of associator for $\star$ via differential consequences of the Jacobi identity for a Poisson structure $\mathcal{P}$, which will be revealed in section 3.2 below, it will be seen in hindsight that the construction of linear relations between the graph weights is overall insensitive to a choice of local coordinates in a chart within a given Poisson manifold. Indeed, the factorization will have been achieved simultaneously for all Poisson structures on all the manifolds at once, irrespective of any local coordinates.

[12]If the rank of the resulting linear algebraic system is equal to the number of unknowns – and if all the coefficients coming from lower orders $\leqslant \ell$ within the $\star$-product expansion with respect to $\hbar$ are also rational – then all the solution components are rational numbers as well, cf. [10].

Examples of Poisson structures $\mathcal{P}$ – for instance, on the manifolds $\mathbb{R}^n$ – are available from [13] (here $n \geqslant 3$) and [28]; from Proposition 2.1 on p. 74 in the latter one obtains a class of Poisson (in fact, symplectic) structures with polynomial coefficients on even-dimensional affine spaces $\mathbb{R}^{2k}$. Besides, there is a regular construction (by using the R-matrix formalism, see [26, p. 287]) of Poisson brackets on the vector space of square matrices $\mathrm{Mat}(\mathbb{R}, k \times k) \cong \mathbb{R}^{k^2}$ (e.g., in this way one has a rank-six Poisson structure on $\mathbb{R}^9$).

Method 2. Now let $f, g, h \in \Bbbk[x^1, \ldots, x^n]$ be polynomials referred to local coordinates x^1, …, x^n on N^n. On that coordinate chart $U_\alpha \subset N^n$, take a Poisson structure the coefficients $\mathcal{P}^{ij}(\boldsymbol{x})$ of which would also be polynomial. In consequence, the left-hand side of the equation $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}} = 0 \mod \bar{o}(\hbar^{\ell+1})$ then becomes polynomial as well. Linear in the unknowns $w(\Gamma)$, all the coefficients of this polynomial equation vanish (independently from each other). Again, this yields a system of linear algebraic equations for the unknown weights $w(\Gamma)$ of the Kontsevich graphs $\Gamma \in \tilde{G}_{2,\ell+1}$ in the $\star$-product.

We observe that the linear equations obtained by using Method 2 better constrain the set of unknowns $w(\Gamma)$, that is, the rank of this system is typically higher than in Method 1. Intuitively, this is because the polynomials at hand are not collapsed to their values at points $\boldsymbol{x} \in N$.

Method 3. Keep the associator's arguments f, g, h unspecified and consider a class of Poisson structures $\mathcal{P}[\psi_1, \ldots, \psi_m]$ depending in a differential polynomial way on functional parameters ψ_α, that is, on arbitrary functions, whenever $\mathcal{P}$ is referred to local coordinates. (For example, let $n = 3$ and on $\mathbb{R}^3$ with Cartesian coordinates x, y, z introduce the class of Poisson brackets using the Jacobian determinants,

$$\{u, v\}_{\mathcal{P}} = p \cdot \det\bigl(\partial(q,u,v)/\partial(x,y,z)\bigr), \qquad q \in C^\infty(\mathbb{R}^3), \tag{10}$$

supposing that the density $p(x,y,z)$ is also smooth on $\mathbb{R}^3$.) Now view the associator $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}[\psi_1,\ldots,\psi_m]}$ as a polydifferential operator in the parameters f, g, h (with respect to which it is linear) and in ψ_1, …, ψ_m from $\mathcal{P}$. By splitting the associator, which is postulated to vanish modulo $\bar{o}(\hbar^{\ell+1})$, into homogeneous differential-polynomial components, we obtain a system of linear algebraic equations upon the graph weights.

It is readily seen that, whenever the parameters ψ_1, …, ψ_m are chosen to be polynomials (here let us suppose for definition that the resulting Poisson structure $\mathcal{P}(\boldsymbol{x})$ itself is polynomial), the rank of the algebraic system obtained by Method 3 can be greater than the rank of an analogous system from Method 2. This is because the analytic expression $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}[\psi_1,\ldots,\psi_m]}$ keeps track of all the parameters, whereas in Method 2 they are merged to a single polynomial.

Implementation 13. To calculate the associator $\mathrm{Assoc}_\star(f,g,h)$ for a given $\star$-product and ordered objects f, g, h, the call is

```
> star_product_associator <star-product-filename>
```

where the input file `<star-product-filename>` contains the (truncated) power series for the $\star$-product. In the standard output one obtains a (truncated at the same order in $\hbar$ as in the input) power series containing, at each power $\hbar^k$, the sums of graphs from $G_{3,k}$ with coefficients (their admissible types were introduced in §1.2 above).

The next step – namely, restriction of the associator to a given Poisson structure – can be performed by using a call `poisson_evaluate` as it has been explained in §1.4. However, the further restriction as described in the Methods has been implemented in a separate program (similar to `poisson_evaluate`) which directly outputs the desired relations, as follows.

Implementation 14. The command

```
> poisson_make_vanish <graph-series-file> <poisson-structure>
```

sends to the standard output relations such as

```
-1/24+w_3_1+4*w_3_2==0
```

between the undetermined coefficients in the input, which must hold if the input graph series is to vanish as a consequence of the Jacobi identity for the specified Poisson structure. The implementation is described in the Methods above. The choice of Poisson structure is made in the same way as in Implementation 7. If the optional extra argument `--linear-solve` is specified, the program will assume that the relations which will be obtained are linear, and attempt to solve the linear system.

Example 21. To obtain all the weights of basic graphs $\Gamma \in \tilde{G}_{2,3}$ at $\hbar^3$ in the Kontsevich star-product $\star$, it was enough to build the linear system of algebraic equations that combined (*i*) cyclic relations (8), (*ii*) the relations which Method 3 produces for generic Poisson structure (10), and (*iii*) those linear relations between the weights of $\Gamma \in \tilde{G}_{2,3}$ which – in view of Remark 10 on p. 22 – still do appear at the next power $\hbar^4$ in $\text{Assoc}_\star(f, g, h) = 0$, by using the same generic Poisson structure (10). The resulting expansion of $\star$-product modulo $\bar{o}(\hbar^3)$ is shown in formula (1) on p. 2. This result is achieved by using the software as follows. Starting from the sets of basic graphs up to the order 2 (with known weights) in the file `basic_graphs_2.txt`, generate lists of basic graphs (with undetermined weights) up to the order four:

```
$ cp basic_graphs_2.txt basic_graphs_34w.txt
$ echo 'h^3:' >> basic_graphs_34w.txt
$ generate_graphs 3 --basic=yes --with-coefficients=yes \
       >> basic_graphs_34w.txt
$ echo 'h^4:' >> basic_graphs_34w.txt
$ generate_graphs 4 --basic=yes --with-coefficients=yes \
       >> basic_graphs_34w.txt
```

Build the $\star$-product expansion up to the order 4 from these basic sets:

```
$ star_product basic_graphs_34w.txt > star_product_34w.txt
```

Generate cyclic weight relations:

```
$ cyclic_weight_relations star_product_34w.txt \
       > weight_relations_34w-cyclic.txt
```

Build the associator expansion up to the order 4 from the $\star$-product expansion:

```
$ star_product_associator star_product_34w.txt > associator_34w.txt
```

Obtain relations from the requirement of associativity for the Poisson structure (10):

```
$ poisson_make_vanish associator_34w.txt 3d-generic \
      > weight_relations_34w-3d.txt
```

Merge the systems of linear relations:

```
$ cat weight_relations_34w-* > weight_relations_34w_all.txt
```

Solving the linear system in `weight_relations_34w_all.txt` yields the solution

```
w_3_1=1/24,   w_3_2=0,      w_3_3=0, w_3_4=-1/48, w_3_5=-1/48
w_3_6=0,      w_3_7=0,      w_3_8=0, w_3_9=0,     w_3_10=0
w_3_11=-1/48, w_3_12=-1/48, w_3_13=0, w_3_14=0.
```

Instead of evaluating the associator in full, we could also have selected (e.g. by reading the file `associator_34w.txt`, which also contains lines of the form "`# i j k`") those differential orders (i, j, k) at which only weights from order 3 appear, in view of Remark 10: such orders are $(1, 3, 2)$, $(2, 3, 1)$, $(2, 1, 3)$, $(3, 2, 1)$, $(3, 1, 2)$, $(1, 2, 3)$ and $(2, 2, 2)$.

Remark 11. A substitution of the values of certain graph weights expressed via other weights is tempting but not always effective. Namely, we do not advise repeated running of any of the three methods with such expressions taken into account in the input. Usually, the gain is disproportional to the time consumed; for instead of a coefficient to-express the program now has to handle what typically is a linear combination of several coefficients. This shows that the only types of substitutions which are effective are either setting the coefficients to fixed numeric values (e.g., to zero) or the shortest possible assignments of a weight value via a single other weight value (like $w(\Gamma_1) = -w(\Gamma_2)$ for some graphs Γ_1 and Γ_2).

3.1.2. *The $\star$-product expansion at order four.* At order four in the expansion of the Kontsevich $\star$-product with respect to $\hbar$, there are 149 basic graphs $\Gamma \in \tilde{G}_{2,4}$. The knowledge of their coefficients would completely determine the $\star$-product modulo $\bar{o}(\hbar^4)$. By using Methods 1–3 from §3.1, we found the exact values of 67 basic graphs and we expressed the remaining 82 weights in terms of the 10 master-parameters (themselves the weights of certain graphs from $\tilde{G}_{2,4}$; the other 72 weights are linear functions of these ten).

Theorem 9. *The weights of* 27 *basic Kontsevich graphs are equal to zero. Of these* 27, *the integrands of* 21 *weights are identically zero*, *and the other* 6 *weight values were found to be equal to zero. The remaining* 122 *weights of basic graphs* $\Gamma \in \tilde{G}_{2,4}$ *are arranged as follows*:

- 40 *nonzero weights are known explicitly*;
- *the values of the remaining* 82 *weights are expressed linearly in terms of the weights of those ten graphs which are shown in Fig.* 3.

• *The encoding of entire $\star$-product modulo $\bar{o}(\hbar^4)$, that is, its part up to $\bar{o}(\hbar^3)$ known from formula* (1) *plus $\hbar^4$ times the sum of all the prime and composite weighted graphs with four internal vertices, is given in Appendix* C. (*In that table the weights of composite graphs are numbers*; *for they are expressed via the known coefficients of graphs from $\tilde{G}_{2,\leqslant 3}$.*) *The weights of basic graphs at $\hbar^4$ are expressed in Table* 6 *in terms of the ten master-parameters, see p.* viii *in Appendix* C.

Moreover (as stated in Theorem 11 on p. 29 below), the associativity $\mathrm{Assoc}_\star(f, g, h) = 0 \mod \bar{o}(\hbar^4)$ is established (up to order four) for the star product $\star \mod \bar{o}(\hbar^4)$ at all values of the ten master-parameters.

Proof scheme (*for Theorem* 9). We run the software as follows. First one generates the sets of basic graphs up to order 4, with undetermined weights at order 4 (the weights at order 2 and 3 are known from e.g. Example 5 and Example 21):

```
$ cp basic_graphs_3.txt basic_graphs_4w.txt
$ echo 'h^4:' >> basic_graphs_4w.txt
$ generate_graphs 4 --basic=yes --with-coefficients=yes \
      >> basic_graphs_4w.txt
```

Build the $\star$-product expansion up to order 4:

```
$ star_product basic_graphs_4w.txt > star_product_4w.txt
```

Generate the linear cyclic weight relations at order 4:

```
$ cyclic_weight_relations star_product_4w.txt > \
      > weight_relations_4w-cyclic.txt
```

Find relations of the form `w_4_xxx==0` which hold by virtue of the weight integrand vanishing in formula (5), by using Implementation 16 in Appendix A, and place these relations in the file `weight_relations_4w-integrandvanishes.txt`.
Build the expansion of the associator for the $\star$-product up to the order 4:

```
$ star_product_associator star_product_4w.txt > associator_4w.txt
```

Obtain relations from the requirement of associativity for the Poisson structure (10):

```
$ poisson_make_vanish associator_4w.txt 3d-generic \
      > weight_relations_4w-3d.txt
```

Merge the systems of linear equations:

```
$ cat weight_relations_4w-* > weight_relations_4w_total.txt
```

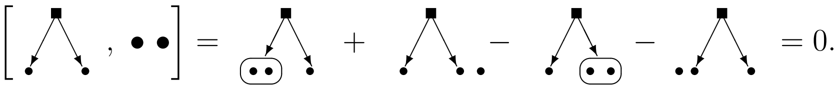
Solve the resulting system (contained in `weight_relations_4w_total.txt`) by using any relevant software. One obtains the relations listed in Table 6 in Appendix C, e.g. in the file `weight_relations_4w_intermsof10.txt`. To express the star-product (respectively, the associator for the $\star$-product) in terms of the 10 parameters, run

```
$ substitute_relations star_product_4w.txt \
      weight_relations_4w_intermsof10.txt
```

(respectively, `associator_4w.txt`); see Implementation 3. □

Remark 12. Numerical approximations of weights are listed in Tables 2 and 3 in Appendix A. In particular, we have the approximate values of the master-parameters $p_4 =$ `w_4_103` $\approx -1/11520$ and $p_5 =$ `w_4_104` $\approx 1/2880$.

Remark 13. Out of the 149 weights of basic graphs in the Kontsevich $\star$-product, as many as 28 weights do not appear in the equation $\mathrm{Assoc}_\star(f,g,h) = 0$ at $\hbar^4$. A mechanism which works towards such disappearance is that some graphs $\Gamma \in \tilde{G}_{2,4}$ which do not show up are bi-derivations with respect to the sinks. Combined at order four in the associator with only the original undeformed product $\times$, every such graph is cancelled out from $(f \star g) \star h - f \star (g \star h)$ according to the mechanism which we illustrate here:



In this way the ten master-parameters are split into the six which do show up in the associativity equation and the four weights which do not show up in $\mathrm{Assoc}_\star(f, g, h) = 0$ at $\hbar^4$ but which do appear through the cyclic weight relations (see formula (8) on p. 17).

3.2. **Computer-assisted proof scheme for associativity of $\star$ for all $\{\cdot, \cdot\}_\mathcal{P}$.** In practice, the methods from §3.1 stop producing linear relations that would be new with respect to the already known constraints for the graph weights. As soon as such "saturation" is achieved, the number of master-parameters in $\star$-product expansion can in effect be minimal. That is, the $\star$-product, known so far up to a certain order $\bar{o}(\hbar^k)$, can in fact always be associative – modulo $\bar{o}(\hbar^k)$ – irrespective of a choice of the Poisson structure(s) $\mathcal{P}$.

In this section we outline a scheme of computer-assisted reasoning that allows to reveal the factorization $\mathrm{Assoc}_\star(f, g, h) = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))(f, g, h)$ of associator for $\star$ via the Jacobiator $\mathrm{Jac}(\mathcal{P})$ that vanishes by definition for every Poisson structure $\mathcal{P}$. At order $k = 2$ the factorization $\Diamond(\mathrm{Jac}(\mathcal{P}))$ is readily seen; the factorizing operator $\Diamond(\mathrm{Jac}(\mathcal{P})) = \frac{2}{3}\hbar^2\,\mathrm{Jac}(\mathcal{P}) + \bar{o}(\hbar^2)$ is a differential operator of order zero, acting on its argument $\mathrm{Jac}(\mathcal{P})$ by multiplication. Involving the Jacobi identity and only seven differential consequences from it at the next expansion order $k = 3$, the factorization $\mathrm{Assoc}_\star(f, g, h) = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))(f, g, h)$ was established by hand in [6]. For higher orders $k \geqslant 4$ the use of software allows to extend this line of reasoning; the scheme which we now provide[13] works uniformly at all orders $\geqslant 2$.

Let us first inspect how sums of graphs can vanish by virtue of differential consequences of the Jacobi identity $\mathrm{Jac}(\mathcal{P}) = 0$ for Poisson structures $\mathcal{P}$ on finite-dimensional affine real manifolds N^n.

Lemma 10 ([6]). A tri-differential operator $C = \sum_{|I|,|J|,|K|\geqslant 0} c^{IJK}\ \partial_I \otimes \partial_J \otimes \partial_K$ with coefficients $c^{IJK} \in C^\infty(N^n)$ vanishes identically if and only if all its homogeneous components $C_{ijk} = \sum_{|I|=i,|J|=j,|K|=k} c^{IJK}\ \partial_I \otimes \partial_J \otimes \partial_K$ vanish for all differential orders (i, j, k) of the respective multi-indices (I, J, K); here $\partial_L = \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}$ for a multi-index $L = (\alpha_1, \ldots, \alpha_n)$.

Lemma 10 states in practice that for every arrow falling on the Jacobiator (for which, in turn, a triple of arguments is specified), the expansion of the Leibniz rule yields four fragments which vanish separately. Namely, there is the fragment such that the derivation acts on the content $\mathcal{P}$ of the Jacobiator's two internal vertices, and there are three fragments such that the arrow falls on the first, second, or third argument of the Jacobiator. It is readily seen that the action of a derivative on an argument of the Jacobiator effectively amounts to an appropriate redefinition of its respective argument. Therefore, a restriction to the order $(1, 1, 1)$ is enough in the run-through over all the

[13]In [3] this computer-assisted scheme of reasoning and the corresponding software were applied to solution of a similar factorization problem in the Kontsevich graph calculus.

graphs which contain Jacobiator (4) and which stand on the three arguments f, g, h of the tri-vector $\Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$ at hand.

Example 22. Consider the associator $\mathrm{Assoc}_\star(f, g, h) \mod \bar{o}(\hbar^3)$ for the $\star$-product which is fully known up to order 3. The assembly of factorizing operator $\Diamond(\mathcal{P}, \cdot)$ acting on $\mathrm{Jac}(\mathcal{P})$ is explained in [6]; linear in its argument, the operator $\Diamond$ has differential order one with respect to the Jacobiator.

Implementation 15. Let the input file `<graph-series-filename>` contain a graph series S with constant (e. g., rational, real or complex) coefficients; here S is supposed to vanish by virtue of the Jacobi identity and its differential consequences. Now run the command

```
> reduce_mod_jacobi <graph-series-filename>
```

The program finds a particular solution $\Diamond$ of the factorization problem

$$S(f, g, h) = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}), \ldots, \mathrm{Jac}(\mathcal{P}))(f, g, h).$$

In the standard output one obtains the list of encodings of Leibniz graphs in $\Diamond$ that specify differential consequences of the Jacobi identity; every such graph encoding is followed in the output by its sought-for nonzero coefficient.[14] Specifically, for a Leibniz-rule graph with ℓ Jacobiators appearing as arguments for the Leibniz rules, and with $n-2\ell$ internal vertices from the factorizing operator $\Diamond$, the target vertices $n-2\ell, \ldots, n-\ell-1$ are the ℓ placeholders for the Jacobiators with internal vertices $(n-2\ell, n-2\ell+1), \ldots, (n-1, n)$, respectively. (We emphasize that the (poly)differential operator $\Diamond$ can be nonlinear in $\mathrm{Jac}(\mathcal{P})$, so that $\ell \geqslant 1$.)

Two extra options can be set equal to nonnegative integer values, by passing these two numbers as extra command-line arguments. Namely,

- the parameter `max-jacobiators` restricts the number of Jacobiators in each Leibniz-rule graph, so that by the assignment `max-jacobiators = 1` one has that the right-hand side $\Diamond\bigl(\mathcal{P}, \mathrm{Jac}(\mathcal{P})\bigr)$ is linear in the Jacobiator, whereas if `max-jacobiators = 2`, the right-hand side $\Diamond\bigl(\mathcal{P}, \mathrm{Jac}(\mathcal{P}), \mathrm{Jac}(\mathcal{P})\bigr)$ can be quadratic in $\mathrm{Jac}(\mathcal{P})$, and so on;
- independently, the parameter `max-jac-indegree` restricts (from above) the number of arrows falling on the Jacobiator(s) in each of the Leibniz-rule graphs that constitute the factorizing operator $\Diamond$.

Furthermore, if `--solve` is specified as the third extra argument, the input graph series is allowed to contain undetermined coefficients; these are then added as variables to-solve-for in the linear system.

Theorem 11. *For every component $S^{(i)}$ of the associator*

$$\mathrm{Assoc}_\star(f, g, h) \mod \bar{o}(\hbar^4) =: S^{(0)} + p_1 S^{(1)} + \ldots + p_{10} S^{(10)},$$

there exists a factorizing operator $\Diamond^{(i)}$ such that

$$S^{(i)}(f, g, h) = \Diamond^{(i)}\bigl(\mathcal{P}, \mathrm{Jac}(\mathcal{P})\bigr)(f, g, h), \qquad 0 \leqslant i \leqslant 10.$$

- *At no values of the master-parameters p_i would the solution $\Diamond = \sum_i \Diamond^{(i)}$ of factorization problem be a* first*-order differential operator acting on the Jacobiator.*

[14]Sample outputs of specified type are contained in Table 8 in Appendix D.

Proof scheme. Take the associator $\mathrm{Assoc}_\star(f,g,h) \mod \bar{o}(\hbar^4)$ for the $\star$-product expansion modulo $\bar{o}(\hbar^4)$, in the file `associator4_intermsof10.txt` which was obtained in Theorem 9. The associator is linear in the ten master-parameters. Let us split it into the constant term (e.g., at the zero value of every parameter) plus the ten respective components $S^{(i)}$:

```
$ extract_coefficient associator4_intermsof10.txt 1 \
     > associator4_intermsof10_constantpart.txt
$ extract_coefficient associator4_intermsof10.txt w_4_100 \
     > associator4_intermsof10_part100.txt
$ extract_coefficient associator4_intermsof10.txt w_4_101 \
     > associator4_intermsof10_part101.txt
```

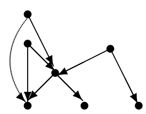
(and so on, for each parameter p_i). In fact, four of the parameters do not show up in the associator (see Remark 13): the corresponding files do not contain any graphs. Now run the command `reduce_mod_jacobi` for each input file with $S^{(i)}$, e.g., for $S^{(1)}$:

```
$ reduce_mod_jacobi associator4_intermsof10_part100.txt
```

For each $S^{(i)}$ a solution is found: the series vanishes modulo the Jacobi identity. The output for $S^{(1)}$ is written in Table 8 in Appendix D. For the second part of the theorem, we run `reduce_mod_jacobi` with the options `max-jac-indegree = 1` and `--solve`:

```
$ reduce_mod_jacobi associator4_intermsof10.txt 1 1 --solve
```

(Our setting of `max-jacobiators = 1` here makes no difference.) No solution is found. Inspecting the output, we find that the following term in the associator cannot be produced by a first-order differential consequence of the Jacobi identity:

$-\frac{2}{15}$ 

Indeed one can show this graph arises only in a differential consequence of order two. ☐

Corollary 12 ($\star$-product non-extendability from $\{\cdot,\cdot\}_{\mathcal{P}}$ to $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ at order $\hbar^4$)**.** Because there are at least two arrows falling on the object $\mathrm{Jac}(\mathcal{P})$ in $\Diamond$ at every value of the ten master-parameters p_i, the associativity can be broken at order $\hbar^4$ for extensions of the $\star$-product to infinite-dimensional set-up[4 on p. 11] of N^n-valued fields $\phi \in C^\infty(M^m \to N^n)$ over a given affine manifold M^m, of local functionals F, G, H taking such fields to numbers, and of variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ on the algebra of local functionals.

Indeed, the Jacobiator $\mathrm{Jac}(\boldsymbol{\mathcal{P}}) \cong 0$ for a variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ is a cohomologically trivial variational tri-vector on the jet space $J^\infty(M^m \to N^n)$, whence the first variation of $\mathrm{Jac}(\boldsymbol{\mathcal{P}})$ brought on it by a unique arrow would of course be vanishing identically. Nevertheless, that variational tri-vector's density is not necessarily equal to zero on $J^\infty(M^m \to N^n)$ over M^m for those variational Poisson structures whose coefficients $\mathcal{P}^{ij}$ explicitly depend on the fields ϕ or their derivatives along M^m. This is why the second and higher variations of the Jacobiator $\mathrm{Jac}(\boldsymbol{\mathcal{P}})$ would not always vanish. (Such higher-order variations of functionals are calculated by using the techniques from [16, 20].) We know from [6] that $\mathrm{Assoc}_\star(F,G,H) \cong 0 \mod \bar{o}(\hbar^3)$, i.e. the associator is trivial up to order $\hbar^3$ for all variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ but we now see that it can contain cohomologically nontrivial terms proportional to $\hbar^4$. Consequently, it is the order four at which the associativity of $\star$-products can start to leak in the course of deformation quantization of Poisson field models.

We now claim that four master-parameters can simultaneously be gauged out of the star-product. (That is, either some of the four or all of them at once can be set equal to zero, although this may not necessarily be their true value given by formula (5).)[15]

Theorem 13. *For each $j \in \{2,3,9,10\}$ there exists a gauge transformation $\mathrm{id} + \hbar^4 p_j Z_j$ (listed in Table 9 in Appendix E) such that the master-parameter p_j is reset to zero in the deformed star-product $\star'$. This is achieved in such a way that no graph coefficients which initially did not contain the parameter to gauge out would change at all.*
• *Moreover, the gauge transformation $\mathrm{id} + \hbar^4 \cdot \left(\sum_j p_j Z_j\right)$ removes at once all the four master-parameters, still preserving those coefficients of graphs in $\star$ which did not depend on any of them.*

Proof scheme. Let the $\star$-product expansion in terms of 10 parameters (obtained in Theorem 9) be contained in `star_product4_intermsof10.txt`. Construct a gauge transformation of the form $\mathrm{id} + \hbar^4 G$, where G is the sum over all possible graphs with four internal vertices over one sink which are nonzero, without double edges, without tadpoles, and with positive differential order, taken with undetermined coefficients g_i:

```
$ cat > gauge4.txt
1 0 1          1
h^4:
^D (press Ctrl+D)
$ generate_graphs 4 1 --normal-forms=yes --zero=no \
      --positive-differential-order=yes \
      --with-coefficients=yes >> gauge4.txt
$ sed -i 's/w/g/' gauge4.txt # replace coefficient prefix 'w' by 'g'
```

Obtain gauged star-product expansion $\star'$ by applying the gauge transformation to $\star$:

```
$ gauge star_product4_intermsof10.txt gauge4.txt \
      > star_product4_intermsof10_gauged.txt
```

Reduce the graph series for $\star'$ modulo skew-symmetry:

```
$ reduce star_product4_intermsof10_gauged.txt \
      > star_product4_intermsof10_gauged_reduced.txt
```

Inspect which of the 10 parameters p_j cannot be gauged out, by checking for the existence of graph coefficients containing p_j but not any g_i. For example, for $p_1 =$ `w_4_100`:

[15] Let us recall that the property of a parameter in a family of star-products to be removable by some gauge transformation is not the same as setting such parameter to zero (or any other value). Indeed, other graph coefficients, not depending on the parameter at hand, might get modified by that gauge transformation. However – and similarly to the removal of the loop graph at $\hbar^2$ in the Kontsevich $\star$-product (see Examples 19 and 20) – the trivialization of four parameters at no extra cost is the case which Theorem 13 states.

```
$ grep w_4_100 star_product4_intermsof10_gauged_reduced.txt \
      | grep -v g | wc -l
17
```

There are 17 graphs with such coefficients, so p_1 = `w_4_100` cannot be gauged out. Following this procedure for all the 10 parameters, we find that the only candidates to be gauged out are p_2 = `w_4_101`, p_3 = `w_4_102`, p_9 = `w_4_119`, and p_{10} = `w_4_125`. Now inspect the file `star_product4_intermsof10_gauged_reduced.txt` for the lines containing these p_j and (necessarily, some) g_i. For each p_j, find a choice of g_i so that p_j is completely removed from the file. (The g_i will be of the form $g_i = \alpha_{ij} p_j$ for $\alpha_{ij} \in \mathbb{R}$.) It turns out that this is always possible. Hence this choice of g_i defines the sought-for gauge transformation $\mathrm{id} + \hbar^4 p_j Z_j$ which gauges out the parameter p_j. The gauge-transformations which kill the (four) parameters separately may be combined into the gauge-transformation $\mathrm{id} + \hbar^4(\sum_j p_j Z_j)$ that kills all (four) of them simultaneously. □

Remark 14. The master-parameters which we can gauge out are exactly the ones which do not show up in the associativity equation (see Remark 13).

Let us finally address a possible origin of so ample a freedom in the ten-parameter family of star-products (now known up to $\bar{o}(\hbar^4)$). We claim that the mechanism of vanishing via differential consequences of the Jacobi identity, which was recalled in Lemma 10 and used in Theorem 11, starts working for the associator built over $\star$ but it does yet not apply to the coefficient of each master-parameter in the star-product itself.

Theorem 14. *Neither the coefficient $\star^{(i)}$ of ith master-parameter in the $\star$-product $\star = \cdots + \hbar^4 \cdot \big(\star^{(0)} + \sum_{i=1}^{10} p_i \star^{(i)}\big) + \bar{o}(\hbar^4)$ nor its free term $\star^{(0)}$ at order $\hbar^4$, corresponding to the zero value of all the ten parameters, admits a factorization $\star^{(i)} = \nabla^{(i)}\big(\mathcal{P}, \mathrm{Jac}(\mathcal{P})\big) \in G_{2,4}$ through some operators $\nabla^{(i)}$ that would store the three-sink Jacobiator $\mathrm{Jac}(\mathcal{P}) \in G_{3,2}$ in the Kontsevich graphs on two sinks in $\star$.*

In all the eleven cases at $\hbar^4$ we establish the absence of such factorization by using the same computer-assisted scheme of reasoning which worked in the proof of Theorem 11. Nevertheless, let us keep in mind that there could already be enough room to store five-vertex graphs (4) in the Kontsevich graphs on four internal vertices and two sinks. It is also clear that such null spaces of graphs, not contributing to the analytic realization of bi-differential structure $\star$ for any Poisson bi-vector $\mathcal{P}$, can be added to the $\star$-product at higher orders $\hbar^{\geqslant 5}$ of its further expansion.

3.3. **Discussion.** The values of weights for the Kontsevich graphs at orders $\hbar^3$ and $\hbar^4$ in the $\star$-product which we obtained in this paper agree with those from the literature and numerical experiment. The vanishing of three graph weights at order 3 is stated in [29]: they are `w_3_7`, `w_3_13`, `w_3_14` in Figure 2; this agrees with our calculation in Example 21. The graphs in the Bernoulli family have scaled Bernoulli numbers as weights (see [2, Corollary 6.3] or [14, Proposition 4.4.1]), e.g. `w_3_2` $= B_3/3! = 0$ and `w_4_12` $= B_4/4! = -1/720$. The weights of a family of graphs containing cycles are obtained in [2, Corollary 6.3], e.g. `w_3_9` $= \pm B_3/(2{\cdot}3!) = 0$ and `w_4_72` $= -B_4/(2{\cdot}4!) = 1/1440$. In Table 2 and 3 in Appendix A we list numerical approximations of weights which are consistent with the exact weights (and relations) obtained in this paper.

Conclusion

The expansion of Kontsevich star-product modulo $\bar{o}(\hbar^4)$ is (here $f, g \in C^\infty(N^n)$)

$$
\begin{aligned}
f \star g ={}& f \times g + \hbar \mathcal{P}^{ij} \partial_i f \partial_j g + \hbar^2 \big(\tfrac{1}{2} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_k \partial_i f \partial_\ell \partial_j g + \tfrac{1}{3} \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_k \partial_i f \partial_j g \\
& - \tfrac{1}{3} \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_i f \partial_k \partial_j g - \tfrac{1}{6} \partial_\ell \mathcal{P}^{ij} \partial_j \mathcal{P}^{k\ell} \partial_i f \partial_k g \big) + \hbar^3 \big(\tfrac{1}{6} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_k \partial_i f \partial_n \partial_\ell \partial_j g \\
& + \tfrac{1}{3} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{3} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g \\
& - \tfrac{1}{6} \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_j g + \tfrac{1}{6} \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_k \partial_i f \partial_j g \\
& + \tfrac{1}{6} \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_m \partial_k \partial_j g - \tfrac{1}{6} \partial_m \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_k \partial_i f \partial_j g \\
& - \tfrac{1}{6} \partial_m \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_k \partial_j g - \tfrac{1}{6} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_j g \\
& - \tfrac{1}{6} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_j g + \tfrac{1}{6} \partial_n \partial_\ell \mathcal{P}^{ij} \partial_j \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_m \partial_k g \\
& - \tfrac{1}{6} \partial_\ell \mathcal{P}^{ij} \partial_n \partial_j \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_i f \partial_k g - \tfrac{1}{6} \partial_m \partial_\ell \mathcal{P}^{ij} \partial_n \partial_j \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_k g \big) \\
& + \hbar^4 \big(\tfrac{1}{6} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_n \partial_\ell \partial_j g - \tfrac{1}{6} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_n \partial_\ell \partial_j g \\
& + \tfrac{1}{6} \partial_q \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_\ell \partial_j g + \tfrac{1}{6} \partial_q \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_\ell \partial_j g \\
& - \tfrac{1}{6} \partial_p \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{6} \partial_p \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g \\
& - \tfrac{1}{6} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g - \tfrac{1}{6} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g \\
& + \tfrac{1}{6} \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g - \tfrac{1}{6} \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_k \partial_i f \partial_m \partial_j g \\
& - \tfrac{1}{6} \mathcal{P}^{ij} \partial_p \partial_n \mathcal{P}^{k\ell} \partial_q \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_j g - \tfrac{1}{9} \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g \\
& - \tfrac{1}{9} \partial_p \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{9} \partial_p \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g \\
& + \tfrac{1}{24} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_q \partial_n \partial_\ell \partial_j g - \tfrac{1}{12} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g \\
& + \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{18} \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
& + \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_\ell \partial_j g + \tfrac{1}{18} \partial_q \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g \\
& + \tfrac{1}{72} \partial_\ell \mathcal{P}^{ij} \partial_j \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_i f \partial_p \partial_k g - \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_p \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g \\
& - \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_p \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g + \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
& - \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g + \tfrac{2}{45} \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
& - \tfrac{2}{45} \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g - \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
& + \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g - \tfrac{7}{90} \partial_p \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g \\
& + \tfrac{7}{90} \partial_p \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_m \partial_k \partial_j g + \tfrac{1}{30} \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
& - \tfrac{1}{30} \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g - \tfrac{1}{45} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
& + \tfrac{1}{45} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g + \tfrac{1}{45} \partial_q \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
& - \tfrac{1}{45} \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g + \tfrac{1}{30} \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
& - \tfrac{1}{30} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g - \tfrac{1}{90} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
& + \tfrac{1}{90} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g + \tfrac{1}{90} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g \\
& - \tfrac{1}{90} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_m \partial_k \partial_j g - \tfrac{1}{30} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g \\
& + \tfrac{1}{30} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_m \partial_k \partial_j g - \tfrac{2}{45} \partial_\ell \mathcal{P}^{ij} \partial_p \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{2}{45}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g-\tfrac{1}{30}\partial_q\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad+\tfrac{1}{30}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g-\tfrac{1}{15}\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad+\tfrac{1}{15}\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g-\tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad+\tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g-\tfrac{1}{45}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad+\tfrac{1}{45}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g+\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad-\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g+\tfrac{2}{45}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad-\tfrac{2}{45}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g+\tfrac{1}{15}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad-\tfrac{1}{15}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m\partial_j g+\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad-\tfrac{1}{90}\partial_q\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g+\tfrac{1}{90}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad-\tfrac{1}{90}\partial_q\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g+\tfrac{1}{90}\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad-\tfrac{1}{90}\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g+\tfrac{1}{90}\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad-\tfrac{1}{90}\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g-\tfrac{1}{30}\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_j g\\
&\quad+\tfrac{1}{30}\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k\partial_j g+\tfrac{1}{90}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g\\
&\quad+\tfrac{1}{90}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g+\tfrac{1}{90}\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g\\
&\quad+\tfrac{1}{90}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g+\tfrac{1}{90}\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g\\
&\quad+\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g-\tfrac{1}{30}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad-\tfrac{1}{30}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g-\tfrac{1}{45}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad-\tfrac{1}{45}\partial_q\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g-\tfrac{1}{90}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&\quad-\tfrac{1}{90}\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g-\tfrac{1}{60}\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad-\tfrac{1}{60}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g-\tfrac{1}{45}\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad-\tfrac{1}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g+\tfrac{1}{30}\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&\quad+\tfrac{1}{30}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g-\tfrac{1}{90}\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&\quad-\tfrac{1}{45}\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g-\tfrac{1}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&\quad-\tfrac{1}{20}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_i f\partial_k g-\tfrac{1}{20}\partial_q\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&\quad-\tfrac{13}{90}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g+\tfrac{13}{90}\partial_q\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad+\tfrac{13}{90}\partial_q\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g-16p_4\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+16p_4\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g-16p_5\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+16p_5\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g-16p_4\partial_p\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+16p_4\partial_p\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g+16p_4\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&-16p_4\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g+16p_5\partial_p\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&-16p_5\partial_p\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g+16p_1\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&-16p_1\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g+16p_2\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&-16p_2\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g+16p_3\partial_q\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g
\end{aligned}
$$

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\begin{aligned}
&- 16p_3\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + 16p_4\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&- 16p_4\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g + 16p_5\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&- 16p_5\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + 16p_6\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+ 16p_6\partial_q\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g + 16p_7\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&- 16p_7\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + 16p_8\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+ 16p_8\partial_n\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + 16p_9\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&- 16p_9\partial_q\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g - 32p_4\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+ 32p_4\partial_q\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + 16p_{10}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ 16p_{10}\partial_p\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g + 32p_5\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+ 32p_5\partial_q\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + (\tfrac{1}{12} + 8p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (-\tfrac{1}{12} - 8p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (-\tfrac{1}{12} - 8p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (\tfrac{1}{12} + 8p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (-\tfrac{1}{12} - 8p_7)\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ (\tfrac{1}{12} + 8p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ (\tfrac{1}{45} + 8p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ (-\tfrac{1}{60} - 24p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (\tfrac{1}{60} + 24p_7)\partial_p\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (\tfrac{4}{45} - 16p_6)\partial_q\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&+ (\tfrac{4}{45} - 16p_6)\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+ (\tfrac{17}{90} + 24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g\\
&+ (-\tfrac{1}{12} + 16p_6 + 48p_5)\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g\\
&+ (\tfrac{7}{90} - 16p_6 - 48p_5)\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (-\tfrac{7}{90} + 16p_6 + 48p_5)\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&+ (-\tfrac{1}{36} + 16p_4 - 16p_5 - 8p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (\tfrac{1}{36} - 16p_4 + 16p_5 + 8p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (-\tfrac{1}{9} + 16p_6 - 48p_4 + 48p_5)\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&+ (-\tfrac{1}{9} + 16p_6 - 48p_4 + 48p_5)\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+ (-\tfrac{1}{36} + 16p_4 - 16p_5 - 8p_7)\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ (\tfrac{1}{36} - 16p_4 + 16p_5 + 8p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+ (-\tfrac{13}{180} + 8p_6 + 24p_5 - 8p_1)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (\tfrac{13}{180} - 8p_6 - 24p_5 + 8p_1)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (\tfrac{4}{45} - 16p_6 + 48p_4 - 48p_5)\partial_q\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g
\end{aligned}
$$

$$+ (\tfrac{4}{45} - 16p_6 + 48p_4 - 48p_5)\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m\partial_k g$$
$$+ (-\tfrac{1}{18} + 32p_4 - 32p_5 - 16p_7)\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g$$
$$+ (-\tfrac{1}{18} + 32p_4 - 32p_5 - 16p_7)\partial_q\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g$$
$$+ (\tfrac{17}{90} - 32p_6 + 96p_4 - 96p_5)\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g$$
$$+ (\tfrac{23}{360} + 16p_4 - 8p_1 + 12p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g$$
$$+ (-\tfrac{23}{360} - 16p_4 + 8p_1 - 12p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g$$
$$+ (\tfrac{49}{180} - 16p_6 - 48p_5 + 24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g$$
$$+ (-\tfrac{19}{180} - 32p_4 + 16p_1 - 16p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g$$
$$+ (-\tfrac{31}{180} + 32p_6 - 96p_4 + 96p_5)\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g$$
$$+ (-\tfrac{1}{90} - 8p_6 - 24p_5 + 8p_1 - 8p_7)\partial_p\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g$$
$$+ (\tfrac{1}{90} + 8p_6 + 24p_5 - 8p_1 + 8p_7)\partial_p\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g$$
$$+ (\tfrac{2}{9} - 48p_8 + 96p_4 - 96p_5 + 48p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g$$
$$+ (\tfrac{2}{9} - 48p_8 + 96p_4 - 96p_5 + 48p_7)\partial_n\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g$$
$$+ (\tfrac{1}{6} - 32p_8 + 64p_4 - 64p_5 + 32p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g$$
$$+ (-\tfrac{1}{6} + 32p_8 - 64p_4 + 64p_5 - 32p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g$$
$$+ (-\tfrac{1}{9} + 16p_8 - 32p_4 + 32p_5 - 16p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g$$
$$+ (\tfrac{1}{9} - 16p_8 + 32p_4 - 32p_5 + 16p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g$$
$$+ (\tfrac{1}{120} - 8p_8 + 16p_4 - 24p_5 + 4p_7)\partial_p\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g$$
$$+ (\tfrac{1}{120} - 8p_8 + 16p_4 - 24p_5 + 4p_7)\partial_k\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g$$
$$+ (\tfrac{5}{18} - 48p_8 + 96p_4 - 96p_5 + 48p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g$$
$$+ (\tfrac{5}{18} - 48p_8 + 96p_4 - 96p_5 + 48p_7)\partial_q\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g$$
$$+ (\tfrac{4}{15} - 48p_8 + 96p_4 - 96p_5 + 48p_7)\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g$$
$$+ (-\tfrac{4}{15} + 48p_8 - 96p_4 + 96p_5 - 48p_7)\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g$$
$$+ (-\tfrac{1}{18} + 16p_8 - 32p_4 + 32p_5 - 16p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g$$
$$+ (-\tfrac{1}{18} + 16p_8 - 32p_4 + 32p_5 - 16p_7)\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g$$
$$+ (-\tfrac{7}{36} + 32p_8 - 64p_4 + 64p_5 - 32p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g$$
$$+ (-\tfrac{7}{36} + 32p_8 - 64p_4 + 64p_5 - 32p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g$$
$$+ (-\tfrac{1}{12} + 16p_8 - 32p_4 + 32p_5 - 16p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g$$
$$+ (-\tfrac{1}{12} + 16p_8 - 32p_4 + 32p_5 - 16p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g$$
$$+ (-\tfrac{1}{90} - 16p_8 + 32p_4 - 80p_5 + 16p_1)\partial_p\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g$$
$$+ (\tfrac{1}{360} - 16p_8 - 16p_3 + 32p_4 - 48p_5)\partial_p\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g$$
$$+ (\tfrac{11}{120} - 16p_8 + 32p_4 - 16p_5 + 16p_7)\partial_k\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g$$
$$+ (\tfrac{1}{90} + 8p_6 + 16p_4 + 24p_5 - 8p_1 + 8p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g$$

$$
\begin{aligned}
&+ (-\tfrac{1}{90} - 8p_6 - 16p_4 - 24p_5 + 8p_1 - 8p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (-\tfrac{1}{15} + 8p_8 + 8p_4 + 8p_2 + 16p_{10} - 16p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ (-\tfrac{1}{15} + 8p_8 + 8p_4 + 8p_2 + 16p_{10} - 16p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+ (\tfrac{1}{20} - 8p_8 + 24p_4 - 16p_5 - 8p_2 + 8p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+ (-\tfrac{1}{20} + 8p_8 - 24p_4 + 16p_5 + 8p_2 - 8p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+ (-\tfrac{1}{40} + 8p_8 + 16p_4 + 8p_5 + 16p_{10} - 12p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ (\tfrac{1}{40} - 8p_8 - 16p_4 - 8p_5 - 16p_{10} + 12p_7)\partial_p\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+ (\tfrac{11}{90} + 8p_6 - 16p_4 + 40p_5 - 8p_1 + 24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (-\tfrac{11}{90} - 8p_6 + 16p_4 - 40p_5 + 8p_1 - 24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (\tfrac{1}{5} - 32p_8 - 48p_5 - 32p_{10} + 16p_1 + 48p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ (\tfrac{1}{5} - 32p_8 - 48p_5 - 32p_{10} + 16p_1 + 48p_7)\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ (-\tfrac{1}{6} + 16p_8 - 16p_3 + 32p_4 - 16p_1 - 32p_7)\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+ (\tfrac{1}{6} - 16p_8 + 16p_3 - 32p_4 + 16p_1 + 32p_7)\partial_q\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+ (\tfrac{1}{9} - 16p_8 + 16p_4 - 32p_5 + 16p_1 + 16p_7)\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+ (-\tfrac{1}{9} + 16p_8 - 16p_4 + 32p_5 - 16p_1 - 16p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g\\
&+ (-\tfrac{1}{9} + 16p_8 - 32p_4 + 48p_5 + 16p_2 - 16p_7)\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+ (-\tfrac{1}{9} + 16p_8 - 32p_4 + 48p_5 + 16p_2 - 16p_7)\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+ (\tfrac{1}{9} - 16p_8 + 32p_4 - 48p_5 + 16p_2 + 16p_7)\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+ (-\tfrac{1}{9} + 16p_8 - 32p_4 + 48p_5 - 16p_2 - 16p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g\\
&+ (\tfrac{7}{90} - 16p_8 + 40p_4 - 40p_5 + 8p_2 + 12p_7)\partial_k\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+ (\tfrac{7}{90} - 16p_8 + 40p_4 - 40p_5 + 8p_2 + 12p_7)\partial_m\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+ (\tfrac{1}{180} - 16p_8 - 16p_4 - 16p_5 - 32p_{10} + 8p_7)\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ (-\tfrac{1}{180} + 16p_8 + 16p_4 + 16p_5 + 32p_{10} - 8p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ (\tfrac{37}{90} - 48p_8 - 16p_6 + 96p_4 - 96p_5 + 48p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (-\tfrac{37}{90} + 48p_8 + 16p_6 - 96p_4 + 96p_5 - 48p_7)\partial_p\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (\tfrac{29}{360} - 16p_8 - 16p_4 - 16p_{10} + 8p_1 + 20p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ (\tfrac{29}{360} - 16p_8 - 16p_4 - 16p_{10} + 8p_1 + 20p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+ (\tfrac{34}{45} - 96p_8 - 32p_6 + 240p_4 - 288p_5 + 96p_7)\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (-\tfrac{34}{45} + 96p_8 + 32p_6 - 240p_4 + 288p_5 - 96p_7)\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&+ (-\tfrac{2}{45} + 8p_9 + 4p_6 - 8p_3 + 4p_5 - 4p_1 - 4p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ (\tfrac{2}{45} - 8p_9 - 4p_6 + 8p_3 - 4p_5 + 4p_1 + 4p_7)\partial_q\partial_m\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+ (\tfrac{1}{30} + 8p_6 + 32p_4 + 8p_5 + 32p_{10} - 8p_1 + 8p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ (-\tfrac{1}{30} - 8p_6 - 32p_4 - 8p_5 - 32p_{10} + 8p_1 - 8p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g
\end{aligned}
$$

$$
\begin{aligned}
&+(-\tfrac{7}{90}+8p_8-16p_3+16p_4-8p_5-8p_1-16p_7)\partial_p\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(\tfrac{7}{90}-8p_8+16p_3-16p_4+8p_5+8p_1+16p_7)\partial_q\partial_k\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+(-\tfrac{7}{180}+16p_8-8p_6-16p_4+8p_5+8p_1-16p_7)\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+(\tfrac{7}{180}-16p_8+8p_6+16p_4-8p_5-8p_1+16p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+(\tfrac{13}{360}-8p_8+24p_4-32p_5-8p_2+8p_1+4p_7)\partial_m\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+(-\tfrac{13}{360}+8p_8-24p_4+32p_5+8p_2-8p_1-4p_7)\partial_p\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{1}{15}-32p_8+8p_6+48p_4-72p_5+24p_1+24p_7)\partial_p\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{1}{15}+32p_8-8p_6-48p_4+72p_5-24p_1-24p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(-\tfrac{11}{180}-16p_9+16p_8-8p_6+8p_5+8p_1-16p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_n\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(-\tfrac{17}{180}+16p_8-8p_6-32p_4+40p_5-8p_1-16p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+(\tfrac{17}{180}-16p_8+8p_6+32p_4-40p_5+8p_1+16p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(\tfrac{61}{180}-48p_8-8p_6+96p_4-120p_5+24p_1+48p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(-\tfrac{61}{180}+48p_8+8p_6-96p_4+120p_5-24p_1-48p_7)\partial_q\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&+(\tfrac{53}{90}-96p_8-16p_6+192p_4-240p_5+48p_1+96p_7)\partial_n\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(-\tfrac{49}{90}+48p_8+24p_6-144p_4+168p_5-24p_1-72p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(\tfrac{49}{90}-48p_8-24p_6+144p_4-168p_5+24p_1+72p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(\tfrac{1}{90}-16p_8+8p_6-16p_3+16p_4-24p_5-8p_1+8p_7)\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{1}{90}+16p_8-8p_6+16p_3-16p_4+24p_5+8p_1-8p_7)\partial_q\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+(\tfrac{3}{20}+16p_9-32p_8+8p_6+16p_4-40p_5-8p_1+32p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{3}{20}-16p_9+32p_8-8p_6-16p_4+40p_5+8p_1-32p_7)\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+(-\tfrac{7}{180}-16p_9+16p_8-8p_6+16p_4+8p_5-8p_1-16p_7)\partial_q\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&+(\tfrac{7}{180}+16p_9-16p_8+8p_6-16p_4-8p_5+8p_1+16p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(\tfrac{7}{120}+16p_9-8p_8+8p_6+16p_4+16p_{10}-8p_1+12p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(-\tfrac{7}{120}-16p_9+8p_8-8p_6-16p_4-16p_{10}+8p_1-12p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(8p_9-8p_8+4p_6-8p_3-8p_4+4p_5-8p_2-4p_1+4p_7)\partial_p\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(-8p_9+8p_8-4p_6+8p_3+8p_4-4p_5+8p_2+4p_1-4p_7)\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_n\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{23}{360}+8p_9-16p_8+4p_6-8p_3+8p_4-20p_5+8p_2-16p_{10}-4p_1+16p_7)\\
&\qquad\partial_p\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(-\tfrac{23}{360}-8p_9+16p_8-4p_6+8p_3-8p_4+20p_5-8p_2+16p_{10}+4p_1-16p_7)\\
&\qquad\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\big)+\bar{o}(\hbar^4).
\end{aligned}
\tag{11}
$$

The ten master-parameters in (11) are the still unknown weights of the prime graphs which are portrayed in Fig. 3 on p. 22. The four underlined parameters can be gauged out (*without* modifying the coefficients of any other Kontsevich graphs with four internal vertices), see Theorem 13 on p. 31. At all values of the ten master-parameters, that is, irrespective of their true values given by formula (5), the $\star$-product is proven in Theorem 11 to be associative modulo $\bar{o}(\hbar^4)$.

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Appendix A. Numerical approximation of weight integrals

The material presented here is an expanded version of section 3 of the note [7] by the authors.

A.1. **The weight integral in Cartesian coordinates.** Recall the integral formula for the weight of a graph $\Gamma \in \tilde{G}_{2,k}$ (see section 2):

$$w(\Gamma) = \frac{1}{(2\pi)^{2k}} \int_{C_k(\mathbb{H})} \bigwedge_{j=1}^{k} \mathrm{d}\varphi(p_j, p_{\mathrm{Left}(j)}) \wedge \mathrm{d}\varphi(p_j, p_{\mathrm{Right}(j)}), \tag{5}$$

such that the integral is taken over the configuration space of k points in the upper half-plane $\mathbb{H} \subset \mathbb{C}$,

$$C_k(\mathbb{H}) = \{(p_1, \ldots, p_k) \in \mathbb{H}^k : p_i \text{ pairwise distinct}\},$$

and where $\varphi\colon C_2(\mathbb{H}) \to [0, 2\pi)$ was defined by $\varphi(p,q) = \mathrm{Arg}\left(\frac{q-p}{q-\bar{p}}\right)$.

For nonzero $z = x + iy$ in $\mathbb{H}$ we have $\mathrm{Arg}(x+iy) \cong \arctan(y/x)$, where $\cong$ denotes equality of functions up to a constant. Since $\frac{d}{dt}\arctan(t) = 1/(1+t^2)$, the weight integrand is a rational function of the Cartesian coordinates: for $p = a+ib$ and $q = x+iy$,

$$\varphi(p,q) \cong \arctan\left(\frac{2b(a-x)}{(a-x)^2 + (y+b)(y-b)}\right). \tag{12}$$

In Cartesian coordinates $(x_1, y_1, \ldots, x_k, y_k)$, the weight integrand can now be written as the Jacobian determinant of the map $\Phi_\Gamma\colon C_k(\mathbb{H}) \to [0, 2\pi)^{2k}$ defined by[16]

$$\Phi_\Gamma(p_1, \ldots, p_k) = (\varphi(p_1, p_{\mathrm{Left}(1)}), \varphi(p_1, p_{\mathrm{Right}(1)}), \ldots, \varphi(p_k, p_{\mathrm{Left}(k)}), \varphi(p_k, p_{\mathrm{Right}(k)}))$$

considered as a function of the (x_j, y_j) through $p_j = x_j + iy_j$.

Implementation 16. The command

```
> weight_integrands <graph-series-file>
```

takes as input a list of graphs $\Gamma \in \tilde{G}_{2,k}$ with (possibly undetermined) coefficients, and sends to the standard output lines of the following form:

```
(* <graph encoding>    <coefficient> *)
<weight integrand of the graph above>
```

where the weight integrands are written in Mathematica format, as `Det[...]`.

We can take integration domain to be $\mathbb{H}^k$, since for any $i \neq j$ the set $\{(p_1, \ldots, p_k) \in \mathbb{C}^k : p_i = p_j\}$ is a strict linear subspace of $\mathbb{C}^k$, which has measure zero. The weight integral is absolutely convergent [25], so by the Fubini–Tonelli theorem we may evaluate it as an iterated integral in any order. We can use the residue theorem[17] to integrate out the Cartesian coordinates corresponding to the k real parts, halving the dimension. It then remains to integrate the result (a function of the k imaginary parts) over $\mathbb{R}^k$.

---

[16] Called a Gauss map by M. Polyak [27].

[17] G. Dito used the residue method for one graph [9] at $k = 2$, and remarked that that it becomes unpractical for $k \geq 3$.

Example 23. For the wedge graph Λ we have the Cartesian coordinates $x + iy$ in the upper half-plane and the integrand (obtained using Implementation 16)

$$f(x,y) = \frac{4y}{((x-1)^2 + y^2)(x^2 + y^2)}.$$

To apply the residue theorem we interpret $f(x, y)$ as a rational function in a single *complex* variable x. Its poles are then $\pm iy$ and $1 \pm iy$, so the poles in the upper half-plane are iy and $1 + iy$ (since $y > 0$). The residues at these poles are $r_1 = 2/(i + 2y)$ and $r_2 = -2/(2y - i)$ respectively. Hence the residue theorem yields that the integral of $f(x, y)$ with respect to x over the real line is $2\pi i(r_1 + r_2) = 8\pi/(1 + 4y^2)$. When we integrate this over $y > 0$ and divide by $(2\pi)^2$ we obtain $1/2$, as desired.

This is of course a toy example. For higher k the expressions become larger, but also one has to consider more carefully which poles are in the upper half-plane. From the expression (12) for φ one can see that this issue depends on the relative position of the coordinates on the imaginary axis (y and b in that formula).

For $k = 3$ with coordinates on $\mathbb{H}^3$ given by

$$a + bi, \quad c + di, \quad e + fi,$$

let us agree to call a, c, e the real coordinates and b, d, f the imaginary coordinates. We now split the integral into a sum of integrals over $3! = 6$ regions, one for each possible ordering of the imaginary coordinates:

$$b < d < f; \quad b < f < d; \quad d < b < f; \quad d < f < b; \quad f < b < d; \quad f < d < b.$$

In each such region it is known for every (complexified) real coordinate which poles are in the upper half-plane, so we can apply the residue theorem three times. The result can be numerically integrated more effectively than the original expression, for one because we have halved the dimension of the integration domain.

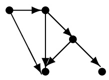
Remark 15. To integrate over the region of $\mathbb{H}^3$ defined by $b < d < f$, one can choose integration bounds as follows: $\int_0^\infty \mathrm{d}b \int_b^\infty \mathrm{d}f \int_b^f \mathrm{d}d$ (and similarly for the other permutations). For the region of $\mathbb{H}^4$ defined by $b < d < f < h$ one can choose the integration bounds $\int_0^\infty \mathrm{d}b \int_b^\infty \mathrm{d}h \int_b^h \mathrm{d}d \int_d^h \mathrm{d}f$, and so on.

Implementation 17. The strategy above is implemented by the following Mathematica code (for the order 4, but it can be adapted for others), where W is the weight integrand.

```
W = ...

integrationvariables = {a, b, c, d, e, f, g, h};
imaginaryvariables =
 integrationvariables[[2 #1]] & /@
  Range[1, Length[integrationvariables]/2];
realvariables =
 integrationvariables[[2 #1 - 1]] & /@
  Range[1, Length[integrationvariables]/2];

basicAssumptions =
```

```
  Element[a, Reals] && Element[c, Reals] && Element[e, Reals] &&
   Element[g, Reals] && b > 0 && d > 0 && f > 0 && h > 0;

ContourIntegrate[function_, variable_, assumptions_] :=
 2*Pi*I*Total[
   Map[
    Function[{p}, (Numerator[Together[function]]/
        D[Denominator[Together[function]], variable]) /. {variable ->
        p}],
    Select[
     ReplaceList[variable,
      Assuming[assumptions,
       Flatten[FullSimplify[
         Solve[Denominator[Together[function]] == 0, variable,
          Complexes]]]]],
     Function[{r}, Simplify[ComplexExpand[Im[r]] > 0, assumptions]]]]]

IteratedContourIntegrate[function_, variables_, assumptions_] :=
 Fold[ContourIntegrate[Together[#1], #2, assumptions] &, function,
  variables]

integrals = Map[
  NIntegrate[
     Simplify[
      IteratedContourIntegrate[W, realvariables,
       basicAssumptions && #1[[1]] < #1[[2]] < #1[[3]] < #1[[4]]]
     TimeConstraint -> Infinity],
     Evaluate[
      Sequence @@
       {{#1[[1]], 0, Infinity}, {#1[[3]], #1[[1]],
         Infinity}, {#1[[2]], #1[[1]], #1[[3]]}, {#1[[4]], #1[[3]],
         Infinity}}
      ],
     Method -> {GlobalAdaptive, MaxErrorIncreases -> 10^4}
     ] &, Permutations[imaginaryvariables]]

Print[integrals]
Print[Total[integrals]]
Print[Total[integrals]/N[(2 Pi)^8]]
```

Remark 16. This strategy allows effective numerical integration of all weights up to order 3. At the order 4, it works for some weights but not others: see Tables 2 and 3. The call(s) to `Map` may be replaced by `ParallelMap` to parallelize the computation.

Example 24. The second Bernoulli graph  [10] has the weight integrand

$$\frac{64bfd\left(c\left((a-c)^2+b^2\right)+d^2(c-2a)\right)\left(f^2(e-2c)+e\left((e-c)^2+d^2\right)\right)}{\left(a^2+b^2\right)\left(f^2+(e-1)^2\right)\left(f^2+e^2\right)\left(c^2+d^2\right)\left((a-c)^2+(b-d)^2\right)\left((a-c)^2+(b+d)^2\right)\left((f+d)^2+(e-c)^2\right)}$$

The residue calculation followed by the numerical integration leads to the estimate $5.71871 \times 10^{-9} - 5.92495 \times 10^{-21} i$ of the weight; this leads to the guess that it is zero and in fact it is true.

Table 2. Verified values

Weight	Approximation	True value
`w_4_1`	$-0.0069444401170 \pm 0.000000906189$	$-1/144 \approx -0.00694444$

Table 3. Conjectured values

Weight	Approximation	Conjectured true value
`w_4_103`	$-0.000086894703 \pm 0.000000681076$	$-1/11520 \approx 0.000086805$
`w_4_104`	$0.000347214860 \pm 0.000000371598$	$1/2880 \approx 0.000347222$
`w_4_112`	$-0.000347219933 \pm 0.000000042901$	$-1/2880 \approx -0.000347222$
`w_4_113`	$0.000694441623 \pm 0.000000093136$	$1/1440 \approx 0.000694444$
`w_4_133`	$0.000694443060 \pm 0.000000078774$	$1/1440 \approx 0.000694444$
`w_4_138`	$-0.001041664533 \pm 0.000000095465$	$-1/960 \approx -0.001041666$
`w_4_147`	$-0.000043376821 \pm 0.000000095465$	?
`w_4_148`	$0.000173611294 \pm 0.000000015063$	$1/5760 \approx 0.000173611$

In particular, this table lists the approximate value of the master-parameters $p_4 =$ `w_4_103` and $p_5 =$ `w_4_104`. The relation `w_4_133` $= 2 \cdot$ `w_4_104` which was found in Theorem 9 and listed in Table 6 of Appendix C is satisfied approximately. Furthermore, the relation `w_4_103` $= 2 \cdot$ `w_4_147` seems to hold approximately.

Appendix B. C++ classes and methods

Class KontsevichGraph

Summary: a (signed) Kontsevich graph.

Data members (private):

```
size_t d_internal = 0;
size_t d_external = 0;
std::vector< std::pair<char, char> > d_targets;
int d_sign = 1;
```

Public typedefs:

```
typedef char Vertex;
typedef std::pair<Vertex, Vertex> VertexPair;
```

Constructors:

```
KontsevichGraph() = default;
KontsevichGraph(size_t internal, size_t external,
                std::vector<VertexPair> targets,
                int sign = 1, bool normalized = false);
```

Accessor methods:

```
std::vector<VertexPair> targets() const;
VertexPair targets(Vertex internal_vertex) const;
int sign() const;
int sign(int new_sign);
size_t internal() const;
size_t external() const;
```

Methods to obtain numerical information:

```
size_t vertices() const;
std::vector<Vertex> internal_vertices() const;
std::pair< size_t, std::vector<VertexPair> > abs() const;
size_t multiplicity() const;
size_t in_degree(KontsevichGraph::Vertex vertex) const;
std::vector<size_t> in_degrees() const;
std::vector<Vertex> neighbors_in(Vertex vertex) const;
KontsevichGraph mirror_image() const;
std::string as_sage_expression() const;
std::string encoding() const;
std::vector< std::tuple<KontsevichGraph, int, int> > permutations() const;
```

Methods that modify the graph:

```
void normalize();
KontsevichGraph& operator*=(const KontsevichGraph& rhs);
```

Methods that test for graph properties:

```
bool operator<(const KontsevichGraph& rhs) const;
bool is_zero() const;
```

```
bool is_prime() const;
bool positive_differential_order() const;
bool has_cycles() const;
bool has_tadpoles() const;
bool has_multiple_edges() const;
bool has_max_internal_indegree(size_t max_indegree) const;
```

Static methods:

```
static std::set<KontsevichGraph> graphs(size_t internal,
    size_t external = 2, bool modulo_signs = false,
    bool modulo_mirror_images = false,
    std::function<void(KontsevichGraph)> const& callback = nullptr,
    std::function<bool(KontsevichGraph)> const& filter = nullptr);
```

Private methods:

```
friend std::ostream& operator<<(std::ostream &os, const KontsevichGraph& g);
friend std::istream& operator>>(std::istream& is, KontsevichGraph& g);
friend bool operator==(const KontsevichGraph &lhs, const KontsevichGraph& rhs);
friend bool operator!=(const KontsevichGraph &lhs, const KontsevichGraph& rhs);
```

Functions defined outside the class:

```
KontsevichGraph operator*(KontsevichGraph lhs, const KontsevichGraph& rhs);
std::ostream& operator<<(std::ostream &os, const KontsevichGraph::Vertex v);
```

Class `KontsevichGraphSum<T>`

- Template parameter `T`: type of the coefficients (e.g. `KontsevichGraphSum<int>`).
- Publically extends: `std::vector< std::pair<T, KontsevichGraph> >`.

Summary: a sum of Kontsevich graphs, with method to reduce modulo skew-symmetry.

Data members: inherited.
Public typedefs:

```
typedef std::pair<T, KontsevichGraph> Term;
```

Constructors (inherited):

```
using std::vector< std::pair<T, KontsevichGraph> >::vector;
```

Accessor methods:

```
using std::vector< std::pair<T, KontsevichGraph> >::operator[];
KontsevichGraphSum<T> operator[](std::vector<size_t> indegrees) const;
T operator[](KontsevichGraph) const;
```

Arithmetic operators:

```
KontsevichGraphSum<T> operator()(std::vector< KontsevichGraphSum<T> > args) const;
KontsevichGraphSum<T>& operator+=(const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T>& operator-=(const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T>& operator=(const KontsevichGraphSum<T>&) = default;
```

Methods:

```
std::vector< std::vector<size_t> > in_degrees(bool ascending = false) const;
KontsevichGraphSum<T> skew_symmetrization() const;
```

Methods that modify the graph sum:

```
void reduce();
```

Comparison operators:

```
bool operator==(const KontsevichGraphSum<T>& other) const;
bool operator==(int other) const;
bool operator!=(const KontsevichGraphSum<T>& other) const;
bool operator!=(int other) const;
```

Friend operators:

```
friend std::ostream& operator<< <>(std::ostream& os,
                                   const KontsevichGraphSum<T>::Term& term);
friend std::ostream& operator<< <>(std::ostream& os,
                                   const KontsevichGraphSum<T>& gs);
friend std::istream& operator>> <>(std::istream& is,
                                   KontsevichGraphSum<T>& sum);
```

Functions defined outside the class:

```
KontsevichGraphSum<T> operator+(KontsevichGraphSum<T> lhs,
                                const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T> operator-(KontsevichGraphSum<T> lhs,
                                const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T> operator*(T lhs,
                                KontsevichGraphSum<T> rhs);
std::ostream& operator<<(std::ostream&, const std::pair<T, KontsevichGraph>&);
std::ostream& operator<<(std::ostream&, const KontsevichGraphSum<T>&);
std::istream& operator>>(std::istream&, KontsevichGraphSum<T>&);
```

Class `KontsevichGraphSeries<T>`

- Template parameter `T`: type of the coefficients (e.g. `KontsevichGraphSeries<int>`).
- Publically extends: `std::map< size_t, KontsevichGraphSum<T> >`

Summary: a formal power series expansion; sums of Kontsevich graphs as coefficients.

Data members: inherited, plus (private):

```
size_t d_precision = std::numeric_limits<std::size_t>::max();
```

Constructors (inherited):

```
using std::map< size_t, KontsevichGraphSum<T> >::map;
```

Accessor methods:

```
size_t precision() const;
size_t precision(size_t new_precision);
```

Arithmetic operators:

```
KontsevichGraphSeries<T> operator()(std::vector< KontsevichGraphSeries<T> >) const;
KontsevichGraphSeries<T>& operator+=(const KontsevichGraphSeries<T>& rhs);
KontsevichGraphSeries<T>& operator-=(const KontsevichGraphSeries<T>& rhs);
```

Methods:

```
KontsevichGraphSeries<T> skew_symmetrization() const;
KontsevichGraphSeries<T> inverse() const;
KontsevichGraphSeries<T> gauge_transform(const KontsevichGraphSeries<T>& gauge);
```

Comparison operators:

```
bool operator==(int other) const;
bool operator!=(int other) const;
```

Methods that modify the graph series:

```
void reduce();
```

Static methods:

```
static KontsevichGraphSeries<T> from_istream(std::istream& is,
    std::function<T(std::string)> const& parser,
    std::function<bool(KontsevichGraph, size_t)> const& filter = nullptr);
```

Friend methods:

```
friend std::ostream& operator<< <>(std::ostream& os,
                                   const KontsevichGraphSeries<T>& series);
```

Functions defined outside the class:

```
KontsevichGraphSeries<T> operator+(KontsevichGraphSeries<T> lhs,
                                   const KontsevichGraphSeries<T>& rhs);
KontsevichGraphSeries<T> operator-(KontsevichGraphSeries<T> lhs,
                                   const KontsevichGraphSeries<T>& rhs);
std::ostream& operator<<(std::ostream&, const KontsevichGraphSeries<T>&);
```

Appendix C. Encoding of the entire $\star$-product modulo $\bar{o}(\hbar^4)$

In the following two tables, containing the sets of basic graphs and the $\star$-product expansion respectively, encodings of graphs (see Implementation 1 on p. 5) are followed by their coefficients.

Table 4. Basic sets of Kontsevich graphs, up to order 4, including zero graphs.

h^0:		
2 0 1		1
h^1:		
2 1 1	0 1	1/2
h^2:		
2 2 1	0 1 0 2	1/12
2 2 1	0 3 2 1	1/24
h^3:		
2 3 1	0 1 0 2 3 2	1/48
2 3 1	0 1 0 2 0 2	1/24
2 3 1	0 1 2 1 0 3	1/48
2 3 1	0 3 2 1 2 1	1/48
2 3 1	0 3 2 1 2 3	1/48
2 3 0	0 1 0 1 2 3	0
h^4:		
2 4 1	0 1 0 1 0 2 2 3	w_4_1
2 4 1	0 1 0 1 0 2 3 4	w_4_2
2 4 0	0 1 0 1 0 5 2 3	0
2 4 1	0 1 0 1 2 3 2 3	w_4_3
2 4 1	0 1 0 1 2 3 2 4	w_4_4
2 4 1	0 1 0 1 2 5 3 4	w_4_5
2 4 1	0 1 0 2 0 2 0 2	w_4_6
2 4 1	0 1 0 2 0 2 0 3	w_4_7
2 4 1	0 1 0 2 0 2 1 2	w_4_8
2 4 1	0 1 0 2 0 2 1 3	w_4_9
2 4 1	0 1 0 2 0 2 2 3	w_4_10
2 4 0	0 1 0 2 0 2 3 4	0
2 4 1	0 1 0 2 0 3 0 3	w_4_11
2 4 1	0 1 0 2 0 3 0 4	w_4_12
2 4 1	0 1 0 2 0 3 1 2	w_4_13
2 4 1	0 1 0 2 0 3 1 3	w_4_14
2 4 1	0 1 0 2 0 3 1 4	w_4_15
2 4 1	0 1 0 2 0 3 2 3	w_4_16
2 4 1	0 1 0 2 0 3 2 4	w_4_17
2 4 1	0 1 0 2 0 3 3 4	w_4_18
2 4 1	0 1 0 2 0 5 1 2	w_4_19
2 4 1	0 1 0 2 0 5 1 3	w_4_20
2 4 1	0 1 0 2 0 5 2 3	w_4_21
2 4 1	0 1 0 2 0 5 2 4	w_4_22
2 4 1	0 1 0 2 0 5 3 4	w_4_23
2 4 1	0 1 0 2 1 2 2 3	w_4_24
2 4 1	0 1 0 2 1 2 3 4	w_4_25
2 4 1	0 1 0 2 1 3 1 3	w_4_26
2 4 1	0 1 0 2 1 3 1 4	w_4_27
2 4 1	0 1 0 2 1 3 2 3	w_4_28
2 4 1	0 1 0 2 1 3 2 4	w_4_29
2 4 1	0 1 0 2 1 3 3 4	w_4_30
2 4 1	0 1 0 2 1 5 2 3	w_4_31
2 4 1	0 1 0 2 1 5 2 4	w_4_32
2 4 1	0 1 0 2 1 5 3 4	w_4_33
2 4 1	0 1 0 2 2 3 2 3	w_4_34
2 4 1	0 1 0 2 2 3 2 4	w_4_35
2 4 1	0 1 0 2 2 3 3 4	w_4_36
2 4 1	0 1 0 2 2 5 2 4	w_4_37
2 4 1	0 1 0 2 2 5 3 4	w_4_38
2 4 1	0 1 0 2 3 5 3 4	w_4_39
2 4 1	0 1 0 4 0 3 2 3	w_4_40
2 4 1	0 1 0 4 0 5 2 3	w_4_41
2 4 1	0 1 0 4 0 5 2 4	w_4_42
2 4 1	0 1 0 4 1 3 2 3	w_4_43
2 4 1	0 1 0 4 1 5 2 3	w_4_44
2 4 1	0 1 0 4 1 5 2 4	w_4_45
2 4 1	0 1 0 4 2 3 0 4	w_4_46
2 4 1	0 1 0 4 2 3 1 4	w_4_47
2 4 1	0 1 0 4 2 3 2 3	w_4_48
2 4 1	0 1 0 4 2 3 2 4	w_4_49
2 4 1	0 1 0 4 2 3 3 4	w_4_50
2 4 1	0 1 0 4 2 5 2 3	w_4_51
2 4 1	0 1 0 4 2 5 2 4	w_4_52
2 4 1	0 1 0 4 2 5 3 4	w_4_53
2 4 1	0 1 0 4 3 5 2 3	w_4_54
2 4 1	0 1 0 4 3 5 2 4	w_4_55
2 4 1	0 1 2 4 2 3 2 3	w_4_56
2 4 0	0 1 2 4 2 3 3 4	0
2 4 1	0 1 2 4 2 5 2 3	w_4_57
2 4 1	0 1 2 4 2 5 3 4	w_4_58
2 4 0	0 1 2 4 3 5 2 4	0
2 4 1	0 1 2 4 3 5 3 4	w_4_59
2 4 1	0 3 0 2 0 2 1 2	w_4_60
2 4 1	0 3 0 2 0 2 1 3	w_4_61
2 4 1	0 3 0 2 0 2 1 4	w_4_62
2 4 1	0 3 0 2 0 5 1 2	w_4_63
2 4 1	0 3 0 2 1 2 1 2	w_4_64
2 4 1	0 3 0 2 1 2 1 3	w_4_65
2 4 1	0 3 0 2 1 2 1 4	w_4_66
2 4 1	0 3 0 2 1 2 2 3	w_4_67
2 4 1	0 3 0 2 1 2 2 4	w_4_68
2 4 1	0 3 0 2 1 2 3 4	w_4_69
2 4 0	0 3 0 2 1 5 2 3	0
2 4 1	0 3 0 2 1 5 2 4	w_4_70
2 4 1	0 3 0 4 0 2 1 2	w_4_71
2 4 1	0 3 0 4 0 5 1 2	w_4_72
2 4 1	0 3 0 4 0 5 1 3	w_4_73
2 4 1	0 3 0 4 0 5 1 4	w_4_74
2 4 1	0 3 0 4 1 2 0 3	w_4_75
2 4 1	0 3 0 4 1 2 0 4	w_4_76
2 4 1	0 3 0 4 1 2 1 2	w_4_77
2 4 1	0 3 0 4 1 2 1 3	w_4_78
2 4 1	0 3 0 4 1 2 1 4	w_4_79
2 4 1	0 3 0 4 1 2 2 3	w_4_80
2 4 1	0 3 0 4 1 2 2 4	w_4_81
2 4 1	0 3 0 4 1 2 3 4	w_4_82
2 4 1	0 3 0 4 1 3 0 3	w_4_83
2 4 1	0 3 0 4 1 3 0 4	w_4_84
2 4 1	0 3 0 4 1 3 1 2	w_4_85
2 4 1	0 3 0 4 1 3 1 3	w_4_86
2 4 1	0 3 0 4 1 3 1 4	w_4_87
2 4 1	0 3 0 4 1 3 2 3	w_4_88
2 4 1	0 3 0 4 1 3 2 4	w_4_89
2 4 1	0 3 0 4 1 3 3 4	w_4_90
2 4 1	0 3 0 4 1 5 0 4	w_4_91
2 4 1	0 3 0 4 1 5 1 2	w_4_92
2 4 1	0 3 0 4 1 5 2 3	w_4_93
2 4 1	0 3 0 4 1 5 2 4	w_4_94
2 4 1	0 3 0 4 1 5 3 4	w_4_95
2 4 1	0 3 0 4 2 3 1 2	w_4_96
2 4 1	0 3 0 4 2 3 1 3	w_4_97
2 4 1	0 3 0 4 2 3 1 4	w_4_98
2 4 1	0 3 0 4 2 5 1 2	w_4_99
2 4 1	0 3 0 4 2 5 1 3	w_4_100
2 4 1	0 3 0 4 2 5 1 4	w_4_101
2 4 1	0 3 0 4 3 5 1 2	w_4_102
2 4 1	0 3 0 4 3 5 1 3	w_4_103
2 4 1	0 3 0 4 3 5 1 4	w_4_104
2 4 1	0 3 1 2 0 3 0 3	w_4_105
2 4 1	0 3 1 2 0 3 1 2	w_4_106
2 4 1	0 3 1 2 0 3 1 4	w_4_107
2 4 1	0 3 1 2 0 3 2 3	w_4_108
2 4 1	0 3 1 2 0 3 2 4	w_4_109
2 4 1	0 3 1 2 0 3 3 4	w_4_110
2 4 1	0 3 1 2 0 5 2 3	w_4_111
2 4 1	0 3 1 2 0 5 2 4	w_4_112
2 4 1	0 3 1 2 0 5 3 4	w_4_113
2 4 1	0 3 1 2 2 3 2 3	w_4_114
2 4 1	0 3 1 2 2 3 2 4	w_4_115
2 4 1	0 3 1 2 2 5 2 4	w_4_116
2 4 1	0 3 1 2 2 5 3 4	w_4_117
2 4 1	0 3 1 4 0 5 1 2	w_4_118
2 4 1	0 3 1 4 0 5 2 3	w_4_119
2 4 1	0 3 1 4 0 5 2 4	w_4_120
2 4 1	0 3 1 4 0 5 3 4	w_4_121
2 4 1	0 3 1 4 2 3 0 3	w_4_122
2 4 1	0 3 1 4 2 3 0 4	w_4_123
2 4 1	0 3 1 4 2 3 1 4	w_4_124
2 4 1	0 3 1 4 2 3 2 3	w_4_125
2 4 1	0 3 1 4 2 3 2 4	w_4_126
2 4 1	0 3 1 4 2 3 3 4	w_4_127
2 4 0	0 3 1 4 2 5 0 3	0
2 4 1	0 3 1 4 2 5 0 4	w_4_128
2 4 1	0 3 1 4 2 5 1 4	w_4_129
2 4 1	0 3 1 4 2 5 2 3	w_4_130
2 4 1	0 3 1 4 2 5 2 4	w_4_131
2 4 1	0 3 1 4 2 5 3 4	w_4_132
2 4 1	0 3 1 4 3 5 0 4	w_4_133
2 4 1	0 3 1 4 3 5 1 4	w_4_134
2 4 1	0 3 1 4 3 5 2 3	w_4_135
2 4 1	0 3 1 4 3 5 2 4	w_4_136
2 4 1	0 3 1 4 3 5 3 4	w_4_137
2 4 0	0 3 2 4 0 3 1 3	0
2 4 1	0 3 2 4 1 3 0 3	w_4_138
2 4 1	0 3 2 4 1 3 2 3	w_4_139
2 4 1	0 3 2 4 1 3 2 4	w_4_140
2 4 1	0 3 2 4 1 5 2 3	w_4_141
2 4 1	0 3 2 4 1 5 2 4	w_4_142
2 4 1	0 3 2 4 1 5 3 4	w_4_143
2 4 1	0 3 2 4 2 3 1 3	w_4_144
2 4 1	0 3 2 4 2 3 1 4	w_4_145
2 4 1	0 3 2 4 2 5 1 3	w_4_146
2 4 1	0 3 2 4 3 5 1 3	w_4_147
2 4 1	0 3 2 4 3 5 1 4	w_4_148
2 4 1	0 3 4 5 1 5 2 3	w_4_149

Table 5. Kontsevich's star product up to order 4.

```
h^0:
2 0 1       1
h^1:
2 1 1   0 1    1
h^2:
2 2 1   0 1 0 1    1/2
2 2 1   0 1 0 2    1/3
2 2 1   0 1 1 2    -1/3
2 2 1   0 3 1 2    -1/6
h^3:
2 3 1   0 1 0 1 0 1    1/6
2 3 1   0 1 0 1 0 2    1/3
2 3 1   0 1 0 1 1 2    -1/3
2 3 1   0 1 0 4 1 3    -1/6
2 3 1   0 1 0 2 0 2    1/6
2 3 1   0 1 1 2 1 2    1/6
2 3 1   0 1 0 2 2 3    -1/6
2 3 1   0 1 1 2 2 3    -1/6
2 3 1   0 1 0 4 1 2    -1/6
2 3 1   0 1 0 2 1 3    -1/6
2 3 1   0 3 1 2 1 2    1/6
2 3 1   0 3 1 2 0 3    -1/6
2 3 1   0 3 1 2 2 3    -1/6
h^4:
2 4 1   0 1 0 1 0 1 0 1    1/24
2 4 1   0 1 0 1 0 1 0 2    1/6
2 4 1   0 1 0 1 0 1 1 2    -1/6
2 4 1   0 1 0 1 0 5 1 4    -1/12
2 4 1   0 1 0 1 0 2 0 2    1/6
2 4 1   0 1 0 1 1 2 1 2    1/6
2 4 1   0 1 0 1 0 2 2 4    -1/6
2 4 1   0 1 0 1 1 2 2 4    -1/6
2 4 1   0 1 0 1 0 5 1 2    -1/6
2 4 1   0 1 0 1 0 2 1 4    -1/6
2 4 1   0 1 0 4 1 3 1 3    1/6
2 4 1   0 1 0 4 1 3 0 4    -1/6
2 4 1   0 1 0 4 1 3 3 4    -1/6
2 4 1   0 1 0 1 0 2 0 3    1/18
2 4 1   0 1 0 1 0 2 1 3    -1/9
2 4 1   0 1 0 2 0 5 1 4    -1/18
2 4 1   0 1 0 1 1 2 1 3    1/18
2 4 1   0 1 0 4 1 3 1 2    1/18
2 4 1   0 3 1 2 0 5 1 4    1/72
2 4 1   0 1 0 1 0 2 2 3    16*w_4_1
2 4 1   0 1 0 1 1 2 2 3    16*w_4_1
2 4 1   0 1 0 1 0 2 3 4    16*w_4_2
2 4 1   0 1 0 1 1 2 3 4    16*w_4_2
2 4 1   0 1 0 1 2 3 2 3    4*w_4_3
2 4 1   0 1 0 1 2 3 2 4    16*w_4_4
2 4 1   0 1 0 1 2 5 3 4    8*w_4_5
2 4 1   0 1 0 2 0 2 0 2    8/3*w_4_6
2 4 1   0 1 1 2 1 2 1 2    -8/3*w_4_6
2 4 1   0 1 0 2 0 2 0 3    16*w_4_7
2 4 1   0 1 1 2 1 2 1 3    -16*w_4_7
2 4 1   0 1 0 2 0 2 1 2    8*w_4_8
2 4 1   0 1 0 2 1 2 1 2    -8*w_4_8
2 4 1   0 1 0 2 0 2 1 3    16*w_4_9
2 4 1   0 1 0 4 1 2 1 2    -16*w_4_9
2 4 1   0 1 0 2 0 2 2 3    16*w_4_10
2 4 1   0 1 1 2 1 2 2 3    -16*w_4_10
2 4 1   0 1 0 2 0 3 0 3    8*w_4_11
2 4 1   0 1 1 2 1 3 1 3    -8*w_4_11
2 4 1   0 1 0 2 0 3 0 4    16*w_4_12
2 4 1   0 1 1 2 1 3 1 4    -16*w_4_12
2 4 1   0 1 0 2 0 3 1 2    16*w_4_13
2 4 1   0 1 0 2 1 2 1 4    -16*w_4_13
2 4 1   0 1 0 2 0 3 1 3    16*w_4_14
2 4 1   0 1 0 4 1 2 1 4    -16*w_4_14
2 4 1   0 1 0 2 0 3 1 4    16*w_4_15
2 4 1   0 1 0 4 1 5 1 2    -16*w_4_15
2 4 1   0 1 0 2 0 3 2 3    16*w_4_16
2 4 1   0 1 1 2 1 3 2 3    -16*w_4_16
2 4 1   0 1 0 2 0 3 2 4    16*w_4_17
2 4 1   0 1 1 2 1 3 2 4    -16*w_4_17
2 4 1   0 1 0 2 0 3 3 4    16*w_4_18
2 4 1   0 1 1 2 1 3 3 4    -16*w_4_18
2 4 1   0 1 0 2 0 5 1 2    16*w_4_19
2 4 1   0 1 0 2 1 2 1 3    -16*w_4_19
2 4 1   0 1 0 2 0 5 1 3    16*w_4_20
2 4 1   0 1 0 4 1 2 1 3    -16*w_4_20
2 4 1   0 1 0 2 0 5 2 3    16*w_4_21
2 4 1   0 1 1 2 1 5 2 3    -16*w_4_21
2 4 1   0 1 0 2 0 5 2 4    16*w_4_22
2 4 1   0 1 1 2 1 5 2 4    -16*w_4_22
2 4 1   0 1 0 2 0 5 3 4    16*w_4_23
2 4 1   0 1 1 2 1 5 3 4    -16*w_4_23
2 4 1   0 1 0 2 1 2 2 3    16*w_4_24
2 4 1   0 1 0 2 1 2 2 4    -16*w_4_24
2 4 1   0 1 0 2 1 2 3 4    16*w_4_25
2 4 1   0 1 0 2 1 3 1 3    8*w_4_26
2 4 1   0 1 0 4 1 2 0 4    -8*w_4_26
2 4 1   0 1 0 2 1 3 1 4    16*w_4_27
2 4 1   0 1 0 4 0 5 1 2    -16*w_4_27
2 4 1   0 1 0 2 1 3 2 3    16*w_4_28
2 4 1   0 1 0 4 1 2 2 4    -16*w_4_28
2 4 1   0 1 0 2 1 3 2 4    16*w_4_29
2 4 1   0 1 0 4 1 2 2 3    -16*w_4_29
2 4 1   0 1 0 2 1 3 3 4    16*w_4_30
2 4 1   0 1 0 4 1 2 3 4    16*w_4_30
2 4 1   0 1 0 2 1 5 2 3    16*w_4_31
2 4 1   0 1 0 4 2 5 1 2    -16*w_4_31
2 4 1   0 1 0 2 1 5 2 4    16*w_4_32
2 4 1   0 1 0 4 2 3 1 2    -16*w_4_32
2 4 1   0 1 0 2 1 5 3 4    16*w_4_33
2 4 1   0 1 0 4 3 5 1 2    16*w_4_33
2 4 1   0 1 0 2 2 3 2 3    8*w_4_34
2 4 1   0 1 1 2 2 3 2 3    -8*w_4_34
2 4 1   0 1 0 2 2 3 2 4    16*w_4_35
2 4 1   0 1 1 2 2 3 2 4    -16*w_4_35
2 4 1   0 1 0 2 2 3 3 4    16*w_4_36
2 4 1   0 1 1 2 2 3 3 4    -16*w_4_36
2 4 1   0 1 0 2 2 5 2 4    8*w_4_37
2 4 1   0 1 1 2 2 5 2 4    -8*w_4_37
2 4 1   0 1 0 2 2 5 3 4    16*w_4_38
2 4 1   0 1 1 2 2 5 3 4    -16*w_4_38
2 4 1   0 1 0 2 3 5 3 4    8*w_4_39
2 4 1   0 1 1 2 3 5 3 4    -8*w_4_39
2 4 1   0 1 0 4 0 3 2 3    16*w_4_40
2 4 1   0 1 1 4 1 3 2 3    -16*w_4_40
2 4 1   0 1 0 4 0 5 2 3    16*w_4_41
2 4 1   0 1 1 4 1 5 2 3    -16*w_4_41
2 4 1   0 1 0 4 0 5 2 4    16*w_4_42
2 4 1   0 1 1 4 1 5 2 4    -16*w_4_42
2 4 1   0 1 0 4 1 3 2 3    16*w_4_43
2 4 1   0 1 0 4 1 3 2 4    -16*w_4_43
2 4 1   0 1 0 4 1 5 2 3    16*w_4_44
2 4 1   0 1 0 4 2 5 1 3    -16*w_4_44
2 4 1   0 1 0 4 1 5 2 4    16*w_4_45
2 4 1   0 1 0 4 2 3 1 3    -16*w_4_45
2 4 1   0 1 0 4 2 3 0 4    16*w_4_46
2 4 1   0 1 1 4 2 3 1 4    -16*w_4_46
2 4 1   0 1 0 4 2 3 1 4    16*w_4_47
2 4 1   0 1 0 4 2 5 1 4    -16*w_4_47
2 4 1   0 1 0 4 2 3 2 3    16*w_4_48
2 4 1   0 1 1 4 2 3 2 3    -16*w_4_48
2 4 1   0 1 0 4 2 3 2 4    16*w_4_49
2 4 1   0 1 1 4 2 3 2 4    -16*w_4_49
2 4 1   0 1 0 4 2 3 3 4    16*w_4_50
2 4 1   0 1 1 4 2 3 3 4    -16*w_4_50
2 4 1   0 1 0 4 2 5 2 3    16*w_4_51
2 4 1   0 1 1 4 2 5 2 3    -16*w_4_51
2 4 1   0 1 0 4 2 5 2 4    16*w_4_52
2 4 1   0 1 1 4 2 5 2 4    -16*w_4_52
2 4 1   0 1 0 4 2 5 3 4    16*w_4_53
2 4 1   0 1 1 4 2 5 3 4    -16*w_4_53
2 4 1   0 1 0 4 3 5 2 3    16*w_4_54
2 4 1   0 1 1 4 3 5 2 3    -16*w_4_54
2 4 1   0 1 0 4 3 5 2 4    16*w_4_55
2 4 1   0 1 1 4 3 5 2 4    -16*w_4_55
2 4 1   0 1 2 4 2 3 2 3    16*w_4_56
2 4 1   0 1 2 4 2 5 2 3    16/3*w_4_57
2 4 1   0 1 2 4 2 5 3 4    16*w_4_58
2 4 1   0 1 2 4 3 5 3 4    16*w_4_59
2 4 1   0 3 0 2 0 2 1 2    16*w_4_60
2 4 1   0 3 1 4 1 3 1 3    16*w_4_60
2 4 1   0 3 0 2 0 2 1 3    16*w_4_61
2 4 1   0 3 1 4 1 3 1 4    16*w_4_61
2 4 1   0 3 0 2 0 2 1 4    16*w_4_62
2 4 1   0 3 1 4 1 5 1 4    16*w_4_62
2 4 1   0 3 0 2 0 5 1 2    16*w_4_63
2 4 1   0 3 1 4 1 3 1 2    16*w_4_63
2 4 1   0 3 0 2 1 2 1 2    8*w_4_64
2 4 1   0 3 1 4 1 3 0 3    8*w_4_64
```

Table 5 (continued).

2 4 1	0 3 0 2 1 2 1 3	8*w_4_65
2 4 1	0 3 1 4 1 3 0 4	8*w_4_65
2 4 1	0 3 0 2 1 2 1 4	16*w_4_66
2 4 1	0 3 0 4 1 5 1 4	16*w_4_66
2 4 1	0 3 0 2 1 2 2 3	16*w_4_67
2 4 1	0 3 1 4 1 3 3 4	16*w_4_67
2 4 1	0 3 0 2 1 2 2 4	16*w_4_68
2 4 1	0 3 1 4 1 3 2 3	-16*w_4_68
2 4 1	0 3 0 2 1 2 3 4	16*w_4_69
2 4 1	0 3 1 4 1 3 2 4	-16*w_4_69
2 4 1	0 3 0 2 1 5 2 4	16*w_4_70
2 4 1	0 3 2 4 1 5 1 4	-16*w_4_70
2 4 1	0 3 0 4 0 2 1 2	16*w_4_71
2 4 1	0 3 1 4 1 5 1 3	16*w_4_71
2 4 1	0 3 0 4 0 5 1 2	16*w_4_72
2 4 1	0 3 1 4 1 5 1 2	16*w_4_72
2 4 1	0 3 0 4 0 5 1 3	16*w_4_73
2 4 1	0 3 1 4 1 2 1 3	16*w_4_73
2 4 1	0 3 0 4 0 5 1 4	16*w_4_74
2 4 1	0 3 1 2 1 3 1 4	16*w_4_74
2 4 1	0 3 0 4 1 2 0 3	16*w_4_75
2 4 1	0 3 1 4 1 2 1 4	16*w_4_75
2 4 1	0 3 0 4 1 2 0 4	16*w_4_76
2 4 1	0 3 1 4 1 2 1 2	16*w_4_76
2 4 1	0 3 0 4 1 2 1 2	16*w_4_77
2 4 1	0 3 1 4 1 2 0 3	16*w_4_77
2 4 1	0 3 0 4 1 2 1 3	16*w_4_78
2 4 1	0 3 1 4 0 5 1 3	16*w_4_78
2 4 1	0 3 0 4 1 2 1 4	16*w_4_79
2 4 1	0 3 0 4 1 5 1 3	16*w_4_79
2 4 1	0 3 0 4 1 2 2 3	16*w_4_80
2 4 1	0 3 1 4 1 2 3 4	16*w_4_80
2 4 1	0 3 0 4 1 2 2 4	16*w_4_81
2 4 1	0 3 1 4 1 2 2 3	-16*w_4_81
2 4 1	0 3 0 4 1 2 3 4	16*w_4_82
2 4 1	0 3 1 4 1 2 2 4	-16*w_4_82
2 4 1	0 3 0 4 1 3 0 3	8*w_4_83
2 4 1	0 3 1 2 1 3 1 3	8*w_4_83
2 4 1	0 3 0 4 1 3 0 4	16*w_4_84
2 4 1	0 3 1 2 1 2 1 3	16*w_4_84
2 4 1	0 3 0 4 1 3 1 2	16*w_4_85
2 4 1	0 3 1 2 0 5 1 3	16*w_4_85
2 4 1	0 3 0 4 1 3 1 3	16*w_4_86
2 4 1	0 3 1 2 0 3 1 3	16*w_4_86
2 4 1	0 3 0 4 1 3 1 4	16*w_4_87
2 4 1	0 3 0 4 1 3 2 3	16*w_4_88
2 4 1	0 3 1 2 1 3 3 4	-16*w_4_88
2 4 1	0 3 0 4 1 3 2 4	16*w_4_89
2 4 1	0 3 1 2 1 3 2 4	-16*w_4_89
2 4 1	0 3 0 4 1 3 3 4	16*w_4_90
2 4 1	0 3 1 2 1 3 2 3	-16*w_4_90
2 4 1	0 3 0 4 1 5 0 4	16*w_4_91
2 4 1	0 3 1 2 1 2 1 4	16*w_4_91
2 4 1	0 3 0 4 1 5 1 2	16*w_4_92
2 4 1	0 3 0 4 1 5 2 3	16*w_4_93
2 4 1	0 3 4 5 1 2 1 4	-16*w_4_93
2 4 1	0 3 0 4 1 5 2 4	16*w_4_94
2 4 1	0 3 2 4 1 5 1 2	-16*w_4_94
2 4 1	0 3 0 4 1 5 3 4	16*w_4_95
2 4 1	0 3 2 4 1 2 1 4	-16*w_4_95
2 4 1	0 3 0 4 2 3 1 2	16*w_4_96
2 4 1	0 3 1 4 1 5 3 4	16*w_4_96
2 4 1	0 3 0 4 2 3 1 3	16*w_4_97
2 4 1	0 3 1 4 3 5 1 3	-16*w_4_97
2 4 1	0 3 0 4 2 3 1 4	16*w_4_98
2 4 1	0 3 4 5 1 3 1 4	-16*w_4_98
2 4 1	0 3 0 4 2 5 1 2	16*w_4_99
2 4 1	0 3 1 4 1 5 2 3	-16*w_4_99
2 4 1	0 3 0 4 2 5 1 3	16*w_4_100
2 4 1	0 3 1 4 2 5 1 3	-16*w_4_100
2 4 1	0 3 0 4 2 5 1 4	16*w_4_101
2 4 1	0 3 2 4 1 5 1 3	-16*w_4_101
2 4 1	0 3 0 4 3 5 1 2	16*w_4_102
2 4 1	0 3 1 4 1 5 2 4	-16*w_4_102
2 4 1	0 3 0 4 3 5 1 3	16*w_4_103
2 4 1	0 3 1 4 2 3 1 3	-16*w_4_103
2 4 1	0 3 0 4 3 5 1 4	16*w_4_104
2 4 1	0 3 2 4 1 3 1 4	-16*w_4_104
2 4 1	0 3 1 2 0 3 0 3	8*w_4_105
2 4 1	0 3 1 2 1 2 1 2	8*w_4_105
2 4 1	0 3 1 2 0 3 1 2	16*w_4_106
2 4 1	0 3 1 2 0 3 1 4	16*w_4_107
2 4 1	0 3 1 2 0 5 1 2	16*w_4_107
2 4 1	0 3 1 2 0 3 2 3	16*w_4_108
2 4 1	0 3 1 2 1 2 2 3	-16*w_4_108
2 4 1	0 3 1 2 0 3 2 4	16*w_4_109
2 4 1	0 3 1 2 1 2 3 4	16*w_4_109
2 4 1	0 3 1 2 0 3 3 4	16*w_4_110
2 4 1	0 3 1 2 1 2 2 4	16*w_4_110
2 4 1	0 3 1 2 0 5 2 3	16*w_4_111
2 4 1	0 3 1 2 1 5 2 3	-16*w_4_111
2 4 1	0 3 1 2 0 5 2 4	16*w_4_112
2 4 1	0 3 1 2 1 5 3 4	16*w_4_112
2 4 1	0 3 1 2 0 5 3 4	16*w_4_113
2 4 1	0 3 1 2 1 5 2 4	16*w_4_113
2 4 1	0 3 1 2 2 3 2 3	8*w_4_114
2 4 1	0 3 1 2 2 3 2 4	16*w_4_115
2 4 1	0 3 1 2 2 3 3 4	-16*w_4_115
2 4 1	0 3 1 2 2 5 2 4	8*w_4_116
2 4 1	0 3 1 2 3 5 3 4	8*w_4_116
2 4 1	0 3 1 2 2 5 3 4	16*w_4_117
2 4 1	0 3 1 4 0 5 1 2	8*w_4_118
2 4 1	0 3 1 4 0 5 2 3	16*w_4_119
2 4 1	0 3 1 4 2 5 1 2	-16*w_4_119
2 4 1	0 3 1 4 0 5 2 4	16*w_4_120
2 4 1	0 3 1 4 3 5 1 2	-16*w_4_120
2 4 1	0 3 1 4 0 5 3 4	16*w_4_121
2 4 1	0 3 1 4 2 3 1 2	16*w_4_121
2 4 1	0 3 1 4 2 3 0 3	16*w_4_122
2 4 1	0 3 2 4 1 2 1 2	-16*w_4_122
2 4 1	0 3 1 4 2 3 0 4	16*w_4_123
2 4 1	0 3 2 4 1 2 1 3	-16*w_4_123
2 4 1	0 3 1 4 2 3 1 4	16*w_4_124
2 4 1	0 3 2 4 1 2 0 3	-16*w_4_124
2 4 1	0 3 1 4 2 3 2 3	16*w_4_125
2 4 1	0 3 2 4 1 2 2 4	16*w_4_125
2 4 1	0 3 1 4 2 3 2 4	16*w_4_126
2 4 1	0 3 2 4 1 2 3 4	16*w_4_126
2 4 1	0 3 1 4 2 3 3 4	16*w_4_127
2 4 1	0 3 2 4 1 2 2 3	-16*w_4_127
2 4 1	0 3 1 4 2 5 0 4	16*w_4_128
2 4 1	0 3 4 5 1 2 1 3	16*w_4_128
2 4 1	0 3 1 4 2 5 1 4	16*w_4_129
2 4 1	0 3 2 4 1 5 0 3	-16*w_4_129
2 4 1	0 3 1 4 2 5 2 3	16*w_4_130
2 4 1	0 3 4 5 1 2 2 4	-16*w_4_130
2 4 1	0 3 1 4 2 5 2 4	16*w_4_131
2 4 1	0 3 4 5 1 2 3 4	-16*w_4_131
2 4 1	0 3 1 4 2 5 3 4	16*w_4_132
2 4 1	0 3 4 5 1 2 2 3	16*w_4_132
2 4 1	0 3 1 4 3 5 0 4	16*w_4_133
2 4 1	0 3 2 4 1 3 1 2	16*w_4_133
2 4 1	0 3 1 4 3 5 1 4	16*w_4_134
2 4 1	0 3 2 4 0 3 1 2	16*w_4_134
2 4 1	0 3 1 4 3 5 2 3	16*w_4_135
2 4 1	0 3 2 4 2 5 1 2	-16*w_4_135
2 4 1	0 3 1 4 3 5 2 4	16*w_4_136
2 4 1	0 3 2 4 3 5 1 2	-16*w_4_136
2 4 1	0 3 1 4 3 5 3 4	16*w_4_137
2 4 1	0 3 2 4 2 3 1 2	16*w_4_137
2 4 1	0 3 2 4 1 3 0 3	16*w_4_138
2 4 1	0 3 2 4 1 3 1 3	-16*w_4_138
2 4 1	0 3 2 4 1 3 2 3	16*w_4_139
2 4 1	0 3 2 4 1 3 3 4	16*w_4_139
2 4 1	0 3 2 4 1 3 2 4	16*w_4_140
2 4 1	0 3 2 4 1 5 2 3	16*w_4_141
2 4 1	0 3 4 5 1 5 2 4	-16*w_4_141
2 4 1	0 3 2 4 1 5 2 4	16*w_4_142
2 4 1	0 3 2 4 1 5 3 4	16*w_4_143
2 4 1	0 3 2 4 2 5 1 4	16*w_4_143
2 4 1	0 3 2 4 2 3 1 3	16*w_4_144
2 4 1	0 3 4 5 1 3 3 4	-16*w_4_144
2 4 1	0 3 2 4 2 3 1 4	16*w_4_145
2 4 1	0 3 4 5 1 5 3 4	-16*w_4_145
2 4 1	0 3 2 4 2 5 1 3	16*w_4_146
2 4 1	0 3 4 5 1 3 2 4	-16*w_4_146
2 4 1	0 3 2 4 3 5 1 3	16*w_4_147
2 4 1	0 3 4 5 1 3 2 3	-16*w_4_147
2 4 1	0 3 2 4 3 5 1 4	16*w_4_148
2 4 1	0 3 4 5 1 5 2 3	16*w_4_149

Table 6. Relations between weights of $\hbar^4$-basic graphs: 149 via 10.

```
w_4_1==-1/144
w_4_2==-1/288
w_4_3==17/360 + 6*w_4_108
w_4_4==49/2880 - 3*w_4_104 - w_4_107 + (3*w_4_108)/2
w_4_5==-1/96 + 6*w_4_104 + 2*w_4_107
w_4_6==1/80
w_4_7==1/360
w_4_8==-1/240
w_4_9==-13/1440
w_4_10==-7/1440
w_4_11==1/240
w_4_12==-1/720
w_4_13==1/720
w_4_14==1/480
w_4_15==-1/1440
w_4_16==1/1440
w_4_17==-1/480
w_4_18==-1/360
w_4_19==-1/480
w_4_20==-1/240
w_4_21==-1/480
w_4_22==-1/720
w_4_23==1/1440
w_4_24==1/360
w_4_25==53/1440 + 3*w_4_100 + 12*w_4_103 - 15*w_4_104 - w_4_107 + 6*w_4_108 - 6*w_4_109
w_4_26==1/120
w_4_27==1/1440
w_4_28==-1/960 - (3*w_4_108)/2
w_4_29==-49/1440 - (3*w_4_100)/2 - 9*w_4_103 + (21*w_4_104)/2 + (3*w_4_107)/2 - (9*w_4_108)/2 + 3*w_4_109
w_4_30==1/72 + 6*w_4_103 - 6*w_4_104 + 3*w_4_108 - 3*w_4_109
w_4_31==61/2880 + (3*w_4_100)/2 + 6*w_4_103 - (15*w_4_104)/2 - w_4_107/2 + 3*w_4_108 - 3*w_4_109
w_4_32==1/1440
w_4_33==5/288 + 6*w_4_103 - 6*w_4_104 + 3*w_4_108 - 3*w_4_109
w_4_34==1/96 + w_4_108
w_4_35==-w_4_103
w_4_36==-13/2880 - w_4_100/2 + (3*w_4_104)/2 + w_4_107/2
w_4_37==0
w_4_38==1/1440 - w_4_100/2 + w_4_103 + (3*w_4_104)/2 + w_4_107/2 + w_4_108/2
w_4_39==0
w_4_40==0
w_4_41==1/1440
w_4_42==1/1440
w_4_43==37/1440 + 6*w_4_103 - 6*w_4_104 - w_4_107 + 3*w_4_108 - 3*w_4_109
w_4_44==17/360 + 15*w_4_103 - 18*w_4_104 - 2*w_4_107 + 6*w_4_108 - 6*w_4_109
w_4_45==7/1440 - 3*w_4_104 - w_4_107
w_4_46==-1/480
w_4_47==1/60 + 6*w_4_103 - 6*w_4_104 + 3*w_4_108 - 3*w_4_109
w_4_48==11/1440 - w_4_100/2 - w_4_103 + (5*w_4_104)/2 + w_4_107/2 + (3*w_4_108)/2
w_4_49==-w_4_104
w_4_50==-1/192 - w_4_108/2
w_4_51==-w_4_103
w_4_52==-1/1440 + w_4_100/2 - (3*w_4_104)/2 - w_4_107/2 - w_4_108/2
w_4_53==w_4_103
w_4_54==-1/576 + w_4_103 - w_4_104 - w_4_108/2
w_4_55==w_4_104
w_4_56==0
w_4_57==0
w_4_58==0
w_4_59==0
w_4_60==0
w_4_61==0
w_4_62==0
w_4_63==0
w_4_64==0
w_4_65==0
w_4_66==0
w_4_67==0
w_4_68==0
w_4_69==0
w_4_70==0
w_4_71==0
w_4_72==1/1440
w_4_73==1/1440
w_4_74==1/1440
w_4_75==-1/480
w_4_76==-1/720
w_4_77==1/180 + 3*w_4_103 - 3*w_4_104 - w_4_107
w_4_78==-1/144 - 3*w_4_103 + 3*w_4_104 + w_4_107
w_4_79==-1/1440
w_4_80==1/80 + w_4_100 - 3*w_4_104 + 3*w_4_108 - 2*w_4_109 - 2*w_4_125
w_4_81==1/480 - w_4_100/2 + 2*w_4_103 + w_4_104/2 + w_4_107/2 + w_4_108/2 + 2*w_4_125
```

Table 6 (continued).

```
w_4_82==1/2880 - w_4_103 - w_4_104 + w_4_108/2 - w_4_109 - 2*w_4_125
w_4_83==-1/480
w_4_84==-1/720
w_4_85==1/180 - w_4_107
w_4_86==1/480
w_4_87==-1/1440
w_4_88==1/96 + 4*w_4_103 - 4*w_4_104 + 2*w_4_108 - 2*w_4_109
w_4_89==1/240 + (3*w_4_100)/2 + 3*w_4_103 - (9*w_4_104)/2 + w_4_107/2 + (3*w_4_108)/2 - 2*w_4_109
w_4_90==-1/192 - w_4_108/2
w_4_91==-1/720
w_4_92==17/1440 + 6*w_4_103 - 6*w_4_104 - 2*w_4_107
w_4_93==3/320 - w_4_100/2 + w_4_103 - (5*w_4_104)/2 + w_4_107/2 + 2*w_4_108 - 2*w_4_109 + w_4_119
w_4_94==1/1440 - w_4_100/2 - w_4_102 + w_4_103 - (3*w_4_104)/2 + w_4_107/2 + w_4_108/2 - w_4_109
w_4_95==-1/576 + w_4_103 - w_4_104 - w_4_108/2
w_4_96==0
w_4_97==0
w_4_98==0
w_4_99==-7/2880 - w_4_100/2 + w_4_103 + w_4_104/2 - w_4_107/2 - w_4_108 + w_4_109 - w_4_119
w_4_105==-1/160
w_4_106==13/1440
w_4_110==-1/288 - 2*w_4_103 + 2*w_4_104 - w_4_108 + w_4_109
w_4_111==-17/2880 - w_4_100/2 - 2*w_4_103 + (5*w_4_104)/2 - w_4_107/2 - w_4_108 + w_4_109
w_4_112==-7/576 - 4*w_4_103 + 4*w_4_104 - 2*w_4_108 + 2*w_4_109
w_4_113==-1/192 - 2*w_4_103 + 2*w_4_104 - w_4_108 + w_4_109
w_4_114==1/360 + w_4_108
w_4_115==23/5760 - w_4_100/2 + w_4_103 + (3*w_4_108)/4
w_4_116==0
w_4_117==-19/2880 + w_4_100 - 2*w_4_103 - w_4_108
w_4_118==-31/1440 - 12*w_4_103 + 12*w_4_104 + 4*w_4_107
w_4_120==-1/96 - w_4_100 - w_4_102 + 2*w_4_103 - 2*w_4_108 + w_4_109
w_4_121==-1/288 + 2*w_4_103 - 2*w_4_104 - w_4_108
w_4_122==-2*w_4_103
w_4_123==-7/2880 + w_4_100/2 - w_4_103 + w_4_104/2 - w_4_107/2 - w_4_108 + w_4_109
w_4_124==1/144 + w_4_100 + w_4_103 - 2*w_4_104 + w_4_108 - w_4_109
w_4_126==29/5760 + w_4_100/2 - w_4_103 + (5*w_4_108)/4 - w_4_109 - w_4_125
w_4_127==-1/640 + w_4_103 + w_4_104/2 - (3*w_4_108)/4 + w_4_109/2 + w_4_125
w_4_128==-1/144 + w_4_101 - 2*w_4_103 + 3*w_4_104 - w_4_108 + w_4_109
w_4_129==1/144 + w_4_101 + 2*w_4_103 - 3*w_4_104 + w_4_108 - w_4_109
w_4_130==7/1920 - w_4_100/2 + w_4_103 + w_4_107/2 + (3*w_4_108)/4 - w_4_109/2 + w_4_119 + w_4_125
w_4_131==23/5760 - w_4_100/4 + w_4_101/2 - w_4_102/2 + w_4_103/2 - (5*w_4_104)/4 + w_4_107/4 + w_4_108 - w_4_109 +
         w_4_119/2 - w_4_125
w_4_132==-1/240 + w_4_101/2 + w_4_103/2 - w_4_108 + w_4_109/2 + w_4_125
w_4_133==2*w_4_104
w_4_134==0
w_4_135==-1/360 - w_4_100/4 - w_4_102/2 + w_4_104/4 + w_4_107/4 - w_4_108/4 + w_4_119/2
w_4_136==-7/1440 - w_4_100/2 - w_4_102 + w_4_103 - w_4_104/2 - w_4_108 + w_4_109/2
w_4_137==0
w_4_138==-1/144 - 2*w_4_103 + 2*w_4_104 - w_4_108 + w_4_109
w_4_139==1/1920 + w_4_103 - (3*w_4_104)/2 + w_4_108/4 - w_4_109/2
w_4_140==-1/1440 + w_4_100 + 2*w_4_103 - 5*w_4_104 - w_4_109
w_4_141==-w_4_100/4 - w_4_101/2 - w_4_102/2 - w_4_103/2 + w_4_104/4 + w_4_107/4 + w_4_108/4 - w_4_109/2 + w_4_119/2
w_4_142==1/5760 - w_4_102 + 2*w_4_103 - 3*w_4_104 - w_4_109
w_4_143==7/1440 + w_4_101/2 + (5*w_4_103)/2 - (5*w_4_104)/2 + (3*w_4_108)/4 - w_4_109
w_4_144==0
w_4_145==0
w_4_146==13/5760 + w_4_100/2 - w_4_101/2 + (3*w_4_103)/2 - 2*w_4_104 + w_4_108/4 - w_4_109/2
w_4_147==1/320 - w_4_101/2 + (3*w_4_103)/2 - w_4_104 + w_4_108/2 - w_4_109/2
w_4_148==11/1920 + 2*w_4_103 - w_4_104 + w_4_108 - w_4_109
w_4_149==-11/2880 + w_4_100/2 + w_4_104/2 - w_4_107/2 - w_4_108 + w_4_109 - w_4_119
```

Appendix D. Encoding of the associator of the $\star$-product modulo $\bar{o}(\hbar^4)$

Encodings of graphs (see Implementation 1 on p. 5) are followed by their coefficients, in the following table containing the expansion of the associator $(f \star g) \star h - f \star (g \star h)$.

Table 7. The associator of $\star$ up to order 4 in terms of 149 parameters.

```
h^0:
h^1:
h^2:
# 1 1 1
3 2 1   0 1 2 3   -2/3
3 2 1   0 2 1 3   2/3
3 2 1   0 4 1 2   -2/3
h^3:
# 1 2 2
3 3 1   0 1 1 2 2 3   -2/3
3 3 1   0 2 1 2 1 3   2/3
3 3 1   0 4 1 2 1 2   -2/3
# 2 1 2
3 3 1   0 1 0 2 2 3   -2/3
3 3 1   0 2 0 2 1 3   2/3
3 3 1   0 2 0 5 1 2   -2/3
# 2 2 1
3 3 1   0 1 0 1 2 3   -2/3
3 3 1   0 1 0 2 1 4   2/3
3 3 1   0 1 0 5 1 2   -2/3
# 1 2 1
3 3 1   0 1 1 3 2 4   1/3
3 3 1   0 1 1 5 2 3   -1/3
3 3 1   0 4 1 5 1 2   -1/3
3 3 1   0 4 1 2 1 3   1/3
# 1 1 1
3 3 1   0 4 1 3 2 4   1/6
3 3 1   0 1 2 3 3 4   -1/6
3 3 1   0 1 2 5 3 4   -1/6
3 3 1   0 4 1 2 3 4   -1/6
3 3 1   0 4 2 3 1 4   -1/6
3 3 1   0 4 1 5 2 4   1/6
3 3 1   0 4 3 5 1 2   -1/6
3 3 1   0 4 2 3 1 3   -1/6
# 1 1 2
3 3 1   0 1 2 3 2 3   1/3
3 3 1   0 1 2 3 2 4   1/3
3 3 1   0 2 1 5 2 3   -1/6
3 3 1   0 4 2 5 1 2   1/6
3 3 1   0 2 1 3 2 4   -1/6
3 3 1   0 4 1 2 2 3   1/6
3 3 1   0 4 1 2 2 4   1/3
3 3 1   0 2 1 2 3 4   -1/3
3 3 1   0 2 1 3 2 3   -1/3
# 2 1 1
3 3 1   0 1 0 3 2 3   -1/3
3 3 1   0 1 0 3 2 4   -1/6
3 3 1   0 1 0 2 3 4   1/3
3 3 1   0 1 0 5 2 3   -1/6
3 3 1   0 2 0 3 1 3   1/3
3 3 1   0 4 1 2 0 4   -1/3
3 3 1   0 4 0 5 1 2   -1/3
3 3 1   0 2 0 5 1 3   1/6
3 3 1   0 2 0 3 1 4   1/6
h^4:
# 3 3 1
3 4 1   0 1 0 1 0 1 2 3   -1/3
3 4 1   0 1 0 1 0 2 1 5   1/3
3 4 1   0 1 0 1 0 6 1 2   -1/3
# 3 2 2
3 4 1   0 1 0 1 0 2 2 3   -2/3
3 4 1   0 1 0 2 0 2 1 4   2/3
3 4 1   0 1 0 2 0 6 1 2   -2/3
# 3 1 3
3 4 1   0 1 0 2 0 2 2 3   -1/3
3 4 1   0 2 0 2 0 2 1 3   1/3
3 4 1   0 2 0 2 0 6 1 2   -1/3
# 2 3 2
3 4 1   0 1 0 1 1 2 2 3   -2/3
3 4 1   0 1 0 2 1 2 1 4   2/3
3 4 1   0 1 0 5 1 2 1 2   -2/3
# 1 3 3
3 4 1   0 1 1 2 1 2 2 3   -1/3
3 4 1   0 2 1 2 1 2 1 3   1/3
3 4 1   0 4 1 2 1 2 1 2   -1/3
# 2 2 3
3 4 1   0 1 0 2 1 2 2 3   -2/3
3 4 1   0 2 0 2 1 2 1 3   2/3
3 4 1   0 2 0 5 1 2 1 2   -2/3
# 1 3 2
3 4 1   0 1 1 2 1 3 2 5   1/3
3 4 1   0 1 1 2 1 4 2 3   -2/9
3 4 1   0 1 1 2 1 6 2 3   -1/3
3 4 1   0 2 1 2 1 3 1 4   2/9
3 4 1   0 4 1 2 1 2 1 5   -2/9
3 4 1   0 4 1 5 1 2 1 2   -1/3
3 4 1   0 4 1 2 1 2 1 3   1/3
# 2 3 1
3 4 1   0 1 0 1 1 3 2 4   2/9
3 4 1   0 1 0 1 1 3 2 5   1/3
3 4 1   0 1 0 1 1 6 2 3   -1/3
3 4 1   0 1 0 2 1 3 1 4   -2/9
3 4 1   0 1 0 5 1 2 1 3   2/9
3 4 1   0 1 0 5 1 6 1 2   -1/3
3 4 1   0 1 0 5 1 2 1 4   1/3
# 2 1 3
3 4 1   0 1 0 2 2 3 2 3   1/3
3 4 1   0 1 0 2 2 3 2 4   2/9
3 4 1   0 1 0 2 2 3 2 5   1/3
3 4 1   0 2 0 2 1 6 2 3   -1/6
3 4 1   0 2 0 5 2 6 1 2   1/6
3 4 1   0 2 0 2 1 3 2 5   -1/6
3 4 1   0 2 0 5 1 2 2 4   1/6
3 4 1   0 2 0 2 1 3 2 4   -2/9
3 4 1   0 2 0 5 1 2 2 3   2/9
3 4 1   0 2 0 5 1 2 2 5   1/3
3 4 1   0 2 0 2 1 2 3 5   -1/3
3 4 1   0 2 0 2 1 3 2 3   -1/3
# 3 2 1
3 4 1   0 1 0 1 0 3 2 3   -1/3
3 4 1   0 1 0 1 0 3 2 4   -2/9
3 4 1   0 1 0 1 0 3 2 5   -1/6
3 4 1   0 1 0 1 0 2 3 5   1/3
3 4 1   0 1 0 1 0 6 2 3   -1/6
3 4 1   0 1 0 2 0 3 1 4   2/9
3 4 1   0 1 0 3 0 6 1 2   -2/9
3 4 1   0 1 0 2 0 4 1 4   1/3
3 4 1   0 1 0 5 1 2 0 5   -1/3
3 4 1   0 1 0 5 0 6 1 2   -1/3
3 4 1   0 1 0 2 0 6 1 4   1/6
3 4 1   0 1 0 2 0 4 1 5   1/6
# 3 1 2
3 4 1   0 1 0 2 0 3 2 3   -1/3
3 4 1   0 1 0 2 0 3 2 5   -1/6
3 4 1   0 1 0 2 0 2 3 4   1/3
3 4 1   0 1 0 2 0 4 2 3   -2/9
3 4 1   0 1 0 2 0 6 2 3   -1/6
3 4 1   0 2 0 2 0 3 1 3   1/3
3 4 1   0 2 0 5 1 2 0 5   -1/3
3 4 1   0 2 0 2 0 3 1 4   2/9
3 4 1   0 2 0 3 0 6 1 2   -2/9
3 4 1   0 2 0 5 0 6 1 2   -1/3
3 4 1   0 2 0 2 0 6 1 3   1/6
3 4 1   0 2 0 2 0 3 1 5   1/6
# 1 2 3
3 4 1   0 1 1 2 2 3 2 3   1/3
3 4 1   0 1 1 2 2 3 2 4   2/9
3 4 1   0 1 1 2 2 3 2 5   1/3
3 4 1   0 2 1 2 1 6 2 3   -1/6
3 4 1   0 4 2 5 1 2 1 2   1/6
3 4 1   0 2 1 2 1 3 2 5   -1/6
3 4 1   0 4 1 2 1 2 2 3   1/6
3 4 1   0 2 1 2 1 3 2 4   -2/9
3 4 1   0 4 1 2 1 2 2 5   2/9
3 4 1   0 4 1 2 1 2 2 4   1/3
3 4 1   0 2 1 2 1 2 3 4   -1/3
3 4 1   0 2 1 2 1 3 2 3   -1/3
# 2 2 2
3 4 1   0 1 0 1 2 3 2 3   1/3
3 4 1   0 1 0 1 2 3 2 4   4/9
3 4 1   0 1 0 3 1 2 2 3   -1/3
3 4 1   0 1 0 3 1 2 2 4   -1/6
3 4 1   0 1 0 2 1 3 2 5   1/3
3 4 1   0 1 0 1 2 3 2 5   1/3
3 4 1   0 1 0 2 1 2 3 4   1/3
3 4 1   0 1 0 2 1 4 2 3   -4/9
3 4 1   0 1 0 2 1 6 2 3   -1/3
3 4 1   0 1 0 5 2 3 1 2   -1/6
3 4 1   0 1 0 2 1 6 2 4   -1/6
3 4 1   0 1 0 5 2 6 1 2   1/6
3 4 1   0 1 0 2 1 4 2 5   -1/6
3 4 1   0 1 0 5 1 2 2 4   1/6
3 4 1   0 2 0 3 1 2 1 3   1/3
3 4 1   0 4 1 2 0 4 1 2   -1/3
3 4 1   0 2 0 5 1 2 1 3   4/9
3 4 1   0 4 1 2 0 6 1 2   -4/9
3 4 1   0 2 0 5 1 6 1 2   -1/3
3 4 1   0 1 0 5 1 2 2 5   1/3
3 4 1   0 4 0 5 1 2 1 2   -1/3
3 4 1   0 1 0 2 1 2 4 5   -1/3
3 4 1   0 2 0 5 1 2 1 4   1/3
3 4 1   0 2 0 5 1 3 1 2   1/6
3 4 1   0 2 0 3 1 2 1 4   1/6
3 4 1   0 1 0 2 1 4 2 4   -1/3
# 1 3 1
3 4 1   0 1 1 3 1 3 2 3   -1/6+8*w_4_6
3 4 1   0 1 1 3 1 3 2 4   -1/3+16*w_4_7
3 4 1   0 1 1 2 1 3 3 4   2/9
3 4 1   0 1 1 2 1 3 4 5   1/9
3 4 1   0 1 1 3 1 6 2 3   1/9+16*w_4_7
3 4 1   0 1 1 3 1 6 2 4   1/9+16*w_4_12
3 4 1   0 1 1 2 1 4 3 4   2/9
3 4 1   0 1 1 5 2 3 1 5   -1/6+8*w_4_11
3 4 1   0 2 1 3 1 3 1 3   16/3*w_4_6
3 4 1   0 4 1 2 1 4 1 4   -1/6+8*w_4_6
3 4 1   0 2 1 3 1 3 1 4   32*w_4_7
3 4 1   0 4 1 2 1 3 1 4   1/9+16*w_4_7
3 4 1   0 4 1 2 1 4 1 5   16*w_4_7
3 4 1   0 4 1 5 1 2 1 5   -1/3+16*w_4_7
3 4 1   0 2 1 3 1 4 1 4   16*w_4_11
3 4 1   0 4 1 2 1 3 1 3   -1/6+8*w_4_11
3 4 1   0 4 1 5 1 2 1 4   16*w_4_11
3 4 1   0 2 1 3 1 4 1 5   32*w_4_12
3 4 1   0 4 1 2 1 3 1 5   16*w_4_12
3 4 1   0 4 1 5 1 2 1 3   1/9+16*w_4_12
3 4 1   0 4 1 5 1 6 1 2   16*w_4_12
3 4 1   0 1 1 2 1 4 3 5   -1/9
3 4 1   0 1 1 3 1 4 2 3   16*w_4_7
3 4 1   0 1 1 3 1 4 2 4   16*w_4_11
3 4 1   0 1 1 3 1 4 2 5   16*w_4_12
3 4 1   0 1 1 5 1 6 2 3   16*w_4_12
# 3 1 1
3 4 1   0 1 0 3 0 3 2 3   -1/6-8*w_4_8
3 4 1   0 1 0 3 0 3 2 4   -1/3-16*w_4_9
3 4 1   0 1 0 2 0 3 3 4   1/9-16*w_4_1
3 4 1   0 1 0 2 0 3 4 5   -1/9-16*w_4_2
3 4 1   0 1 0 3 0 6 2 3   -1/9-16*w_4_19
3 4 1   0 1 0 3 0 6 2 4   -1/9-16*w_4_20
3 4 1   0 1 0 2 0 4 3 4   1/3+16*w_4_1
3 4 1   0 1 0 5 2 3 0 5   -1/6+8*w_4_26
3 4 1   0 2 0 3 0 3 1 3   -8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 0 4 0 4   -1/6+8/3*w_4_6
3 4 1   0 2 0 3 0 3 1 4   -16*w_4_9+16*w_4_7
3 4 1   0 2 0 3 0 4 1 3   -16*w_4_13+16*w_4_7
3 4 1   0 2 0 3 0 6 1 3   -16*w_4_19+16*w_4_7
3 4 1   0 4 0 5 1 2 0 5   -1/3+16*w_4_7
3 4 1   0 2 0 3 0 4 1 4   16*w_4_11-16*w_4_14
3 4 1   0 2 0 5 1 3 0 5   8*w_4_11+8*w_4_26
3 4 1   0 4 0 5 1 2 0 4   -1/6+8*w_4_11
```

Table 7 (part 2).

```
3 4 1   0 2 0 3 0 4 1 5   -16*w_4_15+16*w_4_12
3 4 1   0 2 0 3 0 6 1 4   16*w_4_12-16*w_4_20
3 4 1   0 2 0 5 0 6 1 3   16*w_4_12+16*w_4_27
3 4 1   0 4 0 5 0 6 1 2   16*w_4_12
3 4 1   0 1 0 2 0 4 3 5   -16*w_4_2
3 4 1   0 1 0 3 0 4 2 3   -16*w_4_13
3 4 1   0 1 0 3 0 4 2 4   -16*w_4_14
3 4 1   0 1 0 3 0 4 2 5   -16*w_4_15
3 4 1   0 1 0 5 0 6 2 3   16*w_4_27
# 1 1 3
3 4 1   0 1 2 3 2 3 2 3   -1/6+8/3*w_4_6
3 4 1   0 1 2 3 2 3 2 4   -1/3+16*w_4_7
3 4 1   0 1 2 3 2 4 2 4   -1/6+8*w_4_11
3 4 1   0 2 1 2 2 3 3 4   1/3+16*w_4_1
3 4 1   0 2 1 2 2 4 3 4   1/9-16*w_4_1
3 4 1   0 2 1 2 2 3 4 5   16*w_4_2
3 4 1   0 2 1 2 2 4 3 5   1/9+16*w_4_2
3 4 1   0 2 1 3 2 3 2 3   -8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 2 4 2 4   -1/6-8*w_4_8
3 4 1   0 2 1 5 2 3 2 3   -16*w_4_9+16*w_4_7
3 4 1   0 4 2 5 1 2 2 5   -1/3-16*w_4_9
3 4 1   0 2 1 3 2 3 2 5   -16*w_4_13+16*w_4_7
3 4 1   0 4 1 2 2 4 2 5   -16*w_4_13
3 4 1   0 2 1 5 2 3 2 5   16*w_4_11-16*w_4_14
3 4 1   0 4 2 5 1 2 2 4   -16*w_4_14
3 4 1   0 2 1 5 2 6 2 3   -16*w_4_15+16*w_4_12
3 4 1   0 4 2 5 2 6 1 2   -16*w_4_15
3 4 1   0 2 1 3 2 3 2 4   -16*w_4_19+16*w_4_7
3 4 1   0 4 1 2 2 3 2 4   -1/9-16*w_4_19
3 4 1   0 2 1 5 2 3 2 4   16*w_4_12-16*w_4_20
3 4 1   0 4 2 5 1 2 2 3   -1/9-16*w_4_20
3 4 1   0 2 1 3 2 4 2 4   8*w_4_11+8*w_4_26
3 4 1   0 4 1 2 2 3 2 3   -1/6+8*w_4_26
3 4 1   0 2 1 3 2 4 2 5   16*w_4_12+16*w_4_27
3 4 1   0 4 1 2 2 3 2 5   16*w_4_27
3 4 1   0 1 2 3 2 4 2 5   16*w_4_12
# 2 2 1
3 4 1   0 1 0 5 1 4 2 3   1/9
3 4 1   0 1 0 5 1 4 2 5   1/6
3 4 1   0 1 0 5 1 3 2 3   1/6+16*w_4_9
3 4 1   0 1 0 5 1 3 2 4   1/6+16*w_4_20
3 4 1   0 1 0 5 1 3 2 5   1/6+16*w_4_14
3 4 1   0 1 0 3 1 4 2 3   1/6+16*w_4_19
3 4 1   0 1 0 3 1 4 2 4   1/6-16*w_4_26
3 4 1   0 1 0 3 1 4 2 5   1/6-16*w_4_27
3 4 1   0 1 0 1 2 3 3 5   -1/6
3 4 1   0 1 0 1 2 3 4 5   -1/3-16*w_4_2
3 4 1   0 1 0 3 1 2 3 5   1/9
3 4 1   0 1 0 3 1 2 4 5   -1/18
3 4 1   0 1 0 3 1 6 2 3   -1/9+16*w_4_13
3 4 1   0 1 0 3 1 6 2 4   -1/9-16*w_4_27
3 4 1   0 1 0 2 1 3 3 4   -1/9-16*w_4_1
3 4 1   0 1 0 2 1 3 4 5   1/9-16*w_4_2
3 4 1   0 1 0 5 2 3 1 3   1/9+16*w_4_9
3 4 1   0 1 0 5 2 6 1 3   1/9+16*w_4_15
3 4 1   0 2 0 5 1 4 1 3   -1/9
3 4 1   0 4 1 2 0 6 1 5   1/9
3 4 1   0 1 0 1 2 6 3 5   -1/6
3 4 1   0 1 0 2 1 4 3 4   1/3+16*w_4_1
3 4 1   0 1 0 5 2 3 1 5   -1/3+16*w_4_14
3 4 1   0 1 0 5 1 2 4 5   -1/6
3 4 1   0 1 0 5 2 4 1 5   -1/6
3 4 1   0 2 0 3 1 3 1 3   8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 0 4 1 4   -1/6+8*w_4_6
3 4 1   0 2 0 3 1 3 1 4   16*w_4_19+16*w_4_7
3 4 1   0 2 0 3 1 3 1 5   16*w_4_13+16*w_4_7
3 4 1   0 2 0 5 1 3 1 3   16*w_4_9+16*w_4_7
3 4 1   0 4 1 2 0 4 1 3   1/3+16*w_4_7
3 4 1   0 4 0 5 1 2 1 5   -1/9+16*w_4_7
3 4 1   0 4 1 2 0 6 1 4   -1/3+16*w_4_7
3 4 1   0 2 0 3 1 4 1 4   8*w_4_11-8*w_4_26
3 4 1   0 2 0 5 1 3 1 5   16*w_4_11+16*w_4_14
3 4 1   0 4 0 5 1 2 1 4   16*w_4_11
3 4 1   0 4 1 5 1 2 0 4   -1/6+8*w_4_11
3 4 1   0 2 0 3 1 4 1 5   16*w_4_12-16*w_4_27
3 4 1   0 2 0 5 1 3 1 4   16*w_4_12+16*w_4_20
3 4 1   0 2 0 5 1 6 1 3   16*w_4_15+16*w_4_12
3 4 1   0 4 0 5 1 2 1 3   1/6+16*w_4_12
3 4 1   0 4 1 2 0 6 1 3   1/6+16*w_4_12
3 4 1   0 4 0 5 1 6 1 2   -1/9+16*w_4_12
3 4 1   0 1 0 5 1 6 2 5   1/6
3 4 1   0 1 0 5 4 6 1 2   -1/6
3 4 1   0 1 0 5 3 6 1 2   -1/6
3 4 1   0 1 0 5 2 4 1 4   -1/6
3 4 1   0 1 0 1 2 3 3 4   -16*w_4_1
3 4 1   0 1 0 2 1 4 3 5   -16*w_4_2
3 4 1   0 1 0 3 1 3 2 3   16*w_4_8
3 4 1   0 1 0 3 1 3 2 5   16*w_4_13
3 4 1   0 1 0 5 1 6 2 3   16*w_4_15
3 4 1   0 1 0 3 1 3 2 4   16*w_4_19
3 4 1   0 1 0 5 2 3 1 4   16*w_4_20
# 1 2 2
3 4 1   0 1 1 3 2 3 2 3   -1/6+8*w_4_6
3 4 1   0 1 1 3 2 3 2 4   -1/3+16*w_4_7
3 4 1   0 1 1 3 2 4 2 4   -1/6+8*w_4_11
3 4 1   0 4 1 3 1 2 2 4   1/6
3 4 1   0 1 1 3 2 3 2 5   -1/9+16*w_4_7
3 4 1   0 1 1 3 2 4 2 5   -1/9+16*w_4_12
3 4 1   0 1 1 2 2 3 3 5   -1/6
3 4 1   0 1 1 5 2 3 2 3   1/3+16*w_4_7
3 4 1   0 1 1 2 2 6 3 5   -1/6
3 4 1   0 1 1 5 2 4 2 3   1/9
3 4 1   0 1 1 2 2 4 3 5   -1/18
3 4 1   0 1 1 5 2 6 2 3   1/6+16*w_4_12
3 4 1   0 1 1 2 2 6 3 4   -1/6
3 4 1   0 1 1 5 2 3 2 4   1/6+16*w_4_12
3 4 1   0 4 1 2 1 2 3 4   -1/6
3 4 1   0 4 2 3 1 2 1 4   -1/6
3 4 1   0 2 1 3 1 6 2 5   -1/9
3 4 1   0 4 1 2 1 6 2 5   1/9
3 4 1   0 2 1 2 1 3 3 4   1/3+16*w_4_1
3 4 1   0 2 1 2 1 4 3 4   -1/9-16*w_4_1
3 4 1   0 4 1 2 1 2 4 5   16*w_4_1
3 4 1   0 2 1 2 1 3 4 5   16*w_4_2
3 4 1   0 2 1 2 1 4 3 5   -1/9+16*w_4_2
3 4 1   0 4 1 2 1 2 3 5   -1/3-16*w_4_2
3 4 1   0 2 1 3 1 3 2 3   8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 1 4 2 4   16*w_4_8
3 4 1   0 2 1 3 1 3 2 4   16*w_4_9+16*w_4_7
3 4 1   0 4 1 2 1 4 2 3   1/9+16*w_4_9
3 4 1   0 4 1 2 1 4 2 5   1/6+16*w_4_9
3 4 1   0 2 1 3 1 4 2 3   16*w_4_13+16*w_4_7
3 4 1   0 4 1 2 1 3 2 4   -1/9+16*w_4_13
3 4 1   0 4 1 5 1 2 2 5   16*w_4_13
3 4 1   0 2 1 3 1 4 2 4   16*w_4_11+16*w_4_14
3 4 1   0 4 1 2 1 3 2 3   -1/3+16*w_4_14
3 4 1   0 4 1 5 1 2 2 4   1/6+16*w_4_14
3 4 1   0 2 1 3 1 4 2 5   16*w_4_15+16*w_4_12
3 4 1   0 4 1 2 1 3 2 5   16*w_4_15
3 4 1   0 4 1 5 1 2 2 3   1/9+16*w_4_15
3 4 1   0 2 1 3 1 6 2 3   16*w_4_19+16*w_4_7
3 4 1   0 4 1 2 1 6 2 4   1/6+16*w_4_19
3 4 1   0 4 2 5 1 2 1 5   16*w_4_19
3 4 1   0 2 1 3 1 6 2 4   16*w_4_12+16*w_4_20
3 4 1   0 4 1 2 1 6 2 3   16*w_4_20
3 4 1   0 4 2 5 1 6 1 2   1/6+16*w_4_20
3 4 1   0 2 1 5 2 3 1 5   8*w_4_11-8*w_4_26
3 4 1   0 4 2 5 1 2 1 4   1/6-16*w_4_26
3 4 1   0 2 1 5 1 6 2 3   16*w_4_12-16*w_4_27
3 4 1   0 4 1 5 2 6 1 2   1/6-16*w_4_27
3 4 1   0 4 2 5 1 2 1 3   -1/9-16*w_4_27
3 4 1   0 4 1 5 2 4 1 2   1/6
3 4 1   0 1 1 2 2 4 3 4   1/9
3 4 1   0 4 3 5 1 2 1 2   -1/6
3 4 1   0 4 2 3 1 2 1 3   -1/6
3 4 1   0 1 1 5 2 3 2 5   16*w_4_11
# 2 1 2
3 4 1   0 1 0 3 2 3 2 3   1/6+8*w_4_8
3 4 1   0 1 0 3 2 3 2 4   1/3+16*w_4_19
3 4 1   0 1 0 3 2 4 2 4   1/6-8*w_4_26
3 4 1   0 2 0 5 1 4 2 5   1/6
3 4 1   0 1 0 3 2 3 2 5   1/9+16*w_4_13
3 4 1   0 1 0 3 2 4 2 5   1/9-16*w_4_27
3 4 1   0 1 0 2 2 3 3 4   -1/3-16*w_4_1
3 4 1   0 1 0 2 2 3 3 5   -1/6
3 4 1   0 1 0 5 2 3 2 3   1/3+16*w_4_9
3 4 1   0 1 0 2 2 6 3 5   -1/6
3 4 1   0 1 0 5 2 4 2 3   1/9
3 4 1   0 1 0 2 2 4 3 5   -1/6-16*w_4_2
3 4 1   0 1 0 5 2 6 2 3   1/6+16*w_4_15
3 4 1   0 1 0 2 2 6 3 4   -1/6
3 4 1   0 1 0 5 2 3 2 4   1/6+16*w_4_20
3 4 1   0 2 0 5 1 2 4 5   -1/6
3 4 1   0 2 0 5 2 4 1 5   -1/6
3 4 1   0 2 0 5 2 4 1 3   -1/9
3 4 1   0 4 1 2 0 6 2 5   1/9
3 4 1   0 2 0 3 1 2 3 5   16*w_4_1
3 4 1   0 2 0 5 1 2 3 5   -1/3-16*w_4_1
```

Table 7 (part 3).

```
3 4 1   0 2 0 3 1 2 4 5   -1/6-16*w_4_2
3 4 1   0 2 0 5 1 2 3 4   16*w_4_2
3 4 1   0 2 0 3 1 3 2 3   32*w_4_8
3 4 1   0 4 1 2 0 4 2 4   1/6+8*w_4_8
3 4 1   0 2 0 3 1 3 2 4   16*w_4_9+16*w_4_19
3 4 1   0 2 0 3 1 3 2 5   16*w_4_9+16*w_4_13
3 4 1   0 4 1 2 0 4 2 3   1/3+16*w_4_9
3 4 1   0 2 0 3 1 4 2 3   16*w_4_19+16*w_4_13
3 4 1   0 2 0 5 1 3 2 3   16*w_4_9+16*w_4_13
3 4 1   0 4 0 5 1 2 2 5   1/9+16*w_4_13
3 4 1   0 2 0 3 1 4 2 4   -16*w_4_26+16*w_4_14
3 4 1   0 2 0 5 1 3 2 5   32*w_4_14
3 4 1   0 4 0 5 1 2 2 4   16*w_4_14
3 4 1   0 2 0 3 1 4 2 5   16*w_4_15-16*w_4_27
3 4 1   0 2 0 5 1 3 2 4   16*w_4_15+16*w_4_20
3 4 1   0 4 0 5 1 2 2 3   1/6+16*w_4_15
3 4 1   0 2 0 3 1 6 2 3   16*w_4_19+16*w_4_13
3 4 1   0 2 0 5 2 3 1 3   16*w_4_9+16*w_4_19
3 4 1   0 4 1 2 0 6 2 4   1/3+16*w_4_19
3 4 1   0 2 0 3 1 6 2 4   16*w_4_20-16*w_4_27
3 4 1   0 2 0 5 2 6 1 3   16*w_4_15+16*w_4_20
3 4 1   0 4 1 2 0 6 2 3   1/6+16*w_4_20
3 4 1   0 2 0 5 2 3 1 5   -16*w_4_26+16*w_4_14
3 4 1   0 4 2 5 1 2 0 4   1/6-8*w_4_26
3 4 1   0 2 0 5 1 6 2 3   16*w_4_15-16*w_4_27
3 4 1   0 2 0 5 2 3 1 4   16*w_4_20-16*w_4_27
3 4 1   0 4 0 5 2 6 1 2   1/9-16*w_4_27
3 4 1   0 2 0 5 1 6 2 5   1/6
3 4 1   0 2 0 5 4 6 1 2   -1/6
3 4 1   0 2 0 5 3 6 1 2   -1/6
3 4 1   0 2 0 5 2 4 1 4   -1/6
3 4 1   0 1 0 2 2 4 3 4   16*w_4_1
3 4 1   0 1 0 2 2 3 4 5   -16*w_4_2
3 4 1   0 1 0 5 2 3 2 5   16*w_4_14
# 1 2 1
3 4 1   0 1 1 3 2 3 3 4   1/6+16*w_4_10
3 4 1   0 1 1 3 2 4 3 4   1/6+16*w_4_16
3 4 1   0 1 1 3 2 6 3 4   1/6+16*w_4_21
3 4 1   0 4 1 3 1 3 2 3   -1/6-16*w_4_105+16*w_4_60
3 4 1   0 4 1 3 1 3 2 4   -1/6+16*w_4_61-16*w_4_84
3 4 1   0 4 1 3 1 3 2 5   -1/6-16*w_4_91+16*w_4_62
3 4 1   0 1 1 3 2 3 3 5   1/9+16*w_4_10
3 4 1   0 1 1 3 2 3 4 5   1/9
3 4 1   0 1 1 3 2 4 3 5   1/9+16*w_4_17
3 4 1   0 1 1 3 2 4 4 5   1/9+16*w_4_18
3 4 1   0 4 1 3 1 2 3 5   1/9-16*w_4_40
3 4 1   0 4 1 3 1 2 4 5   -1/18-16*w_4_40
3 4 1   0 4 1 3 1 6 2 3   1/18+16*w_4_63-16*w_4_91
3 4 1   0 4 1 3 1 6 2 4   1/18+16*w_4_63-16*w_4_74
3 4 1   0 1 1 3 2 6 3 5   1/18+16*w_4_22
3 4 1   0 1 1 3 2 6 4 5   1/18+16*w_4_23
3 4 1   0 1 1 5 2 3 3 5   -1/3+16*w_4_16
3 4 1   0 1 1 2 3 4 4 5   -1/6
3 4 1   0 1 1 5 2 3 4 5   -1/6-16*w_4_18
3 4 1   0 1 1 5 2 4 3 5   -1/9+16*w_4_40
3 4 1   0 1 1 5 2 6 3 5   -1/6+16*w_4_42
3 4 1   0 2 1 3 1 3 3 4   32*w_4_10
3 4 1   0 4 1 2 1 4 3 4   -1/9-16*w_4_10
3 4 1   0 4 1 2 1 4 4 5   1/6+16*w_4_10
3 4 1   0 2 1 3 1 4 3 4   32*w_4_16
3 4 1   0 4 1 2 1 3 3 4   1/3-16*w_4_16
3 4 1   0 4 1 5 1 2 4 5   -1/6-16*w_4_16
3 4 1   0 2 1 3 1 4 3 5   32*w_4_17
3 4 1   0 4 1 2 1 3 4 5   16*w_4_17
3 4 1   0 4 1 5 1 2 3 5   -1/9-16*w_4_17
3 4 1   0 2 1 3 1 4 4 5   32*w_4_18
3 4 1   0 4 1 2 1 3 3 5   1/6+16*w_4_18
3 4 1   0 4 1 5 1 2 3 4   -1/9-16*w_4_18
3 4 1   0 2 1 3 1 6 3 4   32*w_4_21
3 4 1   0 4 1 2 1 6 3 4   -16*w_4_21
3 4 1   0 4 5 6 1 2 1 5   1/6+16*w_4_21
3 4 1   0 2 1 3 1 6 3 5   32*w_4_22
3 4 1   0 4 1 2 1 6 4 5   16*w_4_22
3 4 1   0 4 3 5 1 2 1 5   -1/18-16*w_4_22
3 4 1   0 2 1 3 1 6 4 5   32*w_4_23
3 4 1   0 4 1 2 1 6 3 5   16*w_4_23
3 4 1   0 4 3 5 1 6 1 2   -1/18-16*w_4_23
3 4 1   0 2 1 5 1 4 3 4   32*w_4_40
3 4 1   0 2 1 5 1 6 3 4   32*w_4_41
3 4 1   0 4 1 5 3 6 1 2   -16*w_4_41
3 4 1   0 4 5 6 1 2 1 3   16*w_4_41
3 4 1   0 2 1 5 1 6 3 5   32*w_4_42
3 4 1   0 4 1 5 4 6 1 2   -16*w_4_42
3 4 1   0 4 3 5 1 2 1 3   1/6-16*w_4_42
3 4 1   0 2 1 5 3 4 1 5   32*w_4_46
3 4 1   0 4 3 5 1 2 1 4   -16*w_4_46
3 4 1   0 4 5 6 1 2 1 4   16*w_4_46
3 4 1   0 4 1 3 1 4 2 4   16*w_4_60-16*w_4_83
3 4 1   0 4 1 3 1 4 2 3   16*w_4_61-16*w_4_84
3 4 1   0 4 1 3 1 4 2 5   -16*w_4_74+16*w_4_62
3 4 1   0 4 1 5 1 4 2 3   -16*w_4_63+16*w_4_62
3 4 1   0 4 2 5 1 6 1 5   16*w_4_63-16*w_4_62
3 4 1   0 4 1 5 1 3 2 3   -16*w_4_76+16*w_4_71
3 4 1   0 4 1 5 1 3 2 5   16*w_4_71-16*w_4_75
3 4 1   0 4 1 5 1 3 2 4   -16*w_4_73+16*w_4_71
3 4 1   0 4 1 5 1 6 2 4   16*w_4_73-16*w_4_71
3 4 1   0 4 1 5 2 3 1 3   -16*w_4_76+16*w_4_73
3 4 1   0 4 2 5 1 3 1 5   16*w_4_73-16*w_4_75
3 4 1   0 4 1 5 1 6 2 5   16*w_4_74-16*w_4_62
3 4 1   0 4 1 5 2 4 1 3   -1/18-16*w_4_63+16*w_4_74
3 4 1   0 4 2 3 1 3 1 5   16*w_4_74-16*w_4_91
3 4 1   0 4 1 5 2 3 1 4   -16*w_4_73+16*w_4_75
3 4 1   0 4 2 5 1 3 1 3   -16*w_4_76+16*w_4_75
3 4 1   0 4 1 5 2 6 1 4   -16*w_4_71+16*w_4_75
3 4 1   0 4 1 5 2 3 1 5   16*w_4_76-16*w_4_75
3 4 1   0 4 2 5 1 3 1 4   16*w_4_76-16*w_4_73
3 4 1   0 4 2 5 1 6 1 4   16*w_4_76-16*w_4_71
3 4 1   0 4 1 5 2 4 1 4   -16*w_4_60+16*w_4_83
3 4 1   0 4 2 3 1 3 1 3   -8*w_4_105+8*w_4_83
3 4 1   0 4 1 5 2 4 1 5   1/6-16*w_4_61+16*w_4_84
3 4 1   0 4 2 5 1 4 1 5   -16*w_4_61+16*w_4_84
3 4 1   0 4 1 5 2 6 1 5   1/6+16*w_4_91-16*w_4_62
3 4 1   0 4 2 5 1 4 1 3   -1/18-16*w_4_63+16*w_4_91
3 4 1   0 4 2 3 1 4 1 5   -16*w_4_74+16*w_4_91
3 4 1   0 4 2 3 1 4 1 4   8*w_4_105-8*w_4_83
3 4 1   0 4 2 5 1 4 1 4   1/6+16*w_4_105-16*w_4_60
3 4 1   0 4 1 2 1 4 3 5   -1/9
3 4 1   0 1 1 5 2 4 3 4   1/18+16*w_4_40
3 4 1   0 1 1 2 3 4 3 5   -1/6
3 4 1   0 1 1 5 2 3 3 4   16*w_4_17
3 4 1   0 1 1 5 3 6 2 3   16*w_4_21
3 4 1   0 1 1 5 3 4 2 3   16*w_4_22
3 4 1   0 1 1 5 4 6 2 3   -16*w_4_23
3 4 1   0 1 1 5 2 6 3 4   16*w_4_41
3 4 1   0 1 1 5 3 6 2 4   16*w_4_41
3 4 1   0 1 1 5 3 4 2 4   16*w_4_42
3 4 1   0 1 1 5 3 4 2 5   16*w_4_46
3 4 1   0 1 1 5 3 6 2 5   16*w_4_46
# 2 1 1
3 4 1   0 1 0 3 2 3 3 4   1/6-16*w_4_24
3 4 1   0 1 0 3 2 4 3 4   1/6-16*w_4_28
3 4 1   0 1 0 3 2 6 3 4   1/6-16*w_4_31
3 4 1   0 4 1 3 0 4 2 3   1/6-16*w_4_106+16*w_4_61
3 4 1   0 4 1 3 0 4 2 4   1/6+16*w_4_60-16*w_4_86
3 4 1   0 4 1 3 0 4 2 5   1/6-16*w_4_107+16*w_4_62
3 4 1   0 1 0 3 2 3 3 5   -1/9+16*w_4_24
3 4 1   0 1 0 3 2 3 4 5   -1/9-16*w_4_25
3 4 1   0 1 0 3 2 4 3 5   -1/9-16*w_4_29
3 4 1   0 1 0 3 2 4 4 5   -1/9-16*w_4_30
3 4 1   0 2 0 5 1 4 3 4   1/18+16*w_4_40-16*w_4_43
3 4 1   0 2 0 5 1 4 3 5   1/18+16*w_4_40+16*w_4_43
3 4 1   0 4 1 3 0 6 2 3   1/18+16*w_4_63-16*w_4_107
3 4 1   0 4 1 3 0 6 2 4   1/18+16*w_4_63-16*w_4_85
3 4 1   0 1 0 3 2 6 3 5   -1/18-16*w_4_32
3 4 1   0 1 0 3 2 6 4 5   -1/18-16*w_4_33
3 4 1   0 1 0 2 3 4 3 4   1/6-8*w_4_3
3 4 1   0 1 0 5 2 3 3 5   -1/3+16*w_4_28
3 4 1   0 1 0 2 3 4 4 5   -1/6+16*w_4_4
3 4 1   0 1 0 5 2 3 4 5   -1/6-16*w_4_30
3 4 1   0 1 0 5 2 4 3 5   -1/6+16*w_4_43
3 4 1   0 1 0 5 2 6 3 5   -1/6-16*w_4_45
3 4 1   0 2 0 3 1 3 3 4   -16*w_4_24+16*w_4_10
3 4 1   0 2 0 3 1 3 3 5   16*w_4_24+16*w_4_10
3 4 1   0 4 1 2 0 4 3 4   -1/3-16*w_4_10
3 4 1   0 2 0 3 1 4 3 4   -16*w_4_28+16*w_4_16
3 4 1   0 2 0 5 1 3 3 5   16*w_4_28+16*w_4_16
3 4 1   0 4 0 5 1 2 4 5   -16*w_4_16
3 4 1   0 2 0 3 1 4 3 5   -16*w_4_29+16*w_4_17
3 4 1   0 2 0 5 1 3 3 4   16*w_4_29+16*w_4_17
3 4 1   0 4 0 5 1 2 3 5   -1/6-16*w_4_17
3 4 1   0 2 0 3 1 4 4 5   16*w_4_18-16*w_4_30
3 4 1   0 2 0 5 1 3 4 5   -16*w_4_18-16*w_4_30
3 4 1   0 4 0 5 1 2 3 4   -1/6-16*w_4_18
3 4 1   0 2 0 3 1 6 3 4   16*w_4_21-16*w_4_31
3 4 1   0 2 0 5 3 6 1 3   16*w_4_21+16*w_4_31
3 4 1   0 4 1 2 0 6 3 4   -1/6-16*w_4_21
3 4 1   0 2 0 3 1 6 3 5   16*w_4_22-16*w_4_32
3 4 1   0 2 0 5 3 4 1 3   16*w_4_22+16*w_4_32
3 4 1   0 4 1 2 0 6 4 5   1/6+16*w_4_22
```

TABLE 7 (part 4).

3 4 1	0 2 0 3 1 6 4 5	16*w_4_23-16*w_4_33
3 4 1	0 2 0 5 4 6 1 3	-16*w_4_23-16*w_4_33
3 4 1	0 4 1 2 0 6 3 5	16*w_4_23
3 4 1	0 4 0 3 1 2 3 5	-16*w_4_40
3 4 1	0 2 0 5 1 6 3 4	-16*w_4_44+16*w_4_41
3 4 1	0 2 0 5 3 6 1 4	16*w_4_44+16*w_4_41
3 4 1	0 4 0 5 3 6 1 2	-16*w_4_41
3 4 1	0 2 0 5 1 6 3 5	-16*w_4_45+16*w_4_42
3 4 1	0 2 0 5 3 4 1 4	16*w_4_45+16*w_4_42
3 4 1	0 4 0 5 4 6 1 2	-16*w_4_42
3 4 1	0 2 0 5 3 4 1 5	-16*w_4_47+16*w_4_46
3 4 1	0 2 0 5 3 6 1 5	16*w_4_47+16*w_4_46
3 4 1	0 4 3 5 1 2 0 4	-1/6-16*w_4_46
3 4 1	0 4 0 3 1 3 2 3	16*w_4_60-16*w_4_64
3 4 1	0 4 0 5 1 4 2 4	16*w_4_60-16*w_4_86
3 4 1	0 4 0 3 1 3 2 4	16*w_4_61-16*w_4_65
3 4 1	0 4 0 5 1 4 2 5	16*w_4_61-16*w_4_87
3 4 1	0 4 0 3 1 3 2 5	-16*w_4_66+16*w_4_62
3 4 1	0 4 0 5 1 4 2 3	16*w_4_62-16*w_4_85
3 4 1	0 4 0 3 1 6 2 3	-16*w_4_66+16*w_4_63
3 4 1	0 4 0 5 1 3 2 3	-16*w_4_77+16*w_4_71
3 4 1	0 4 0 5 1 3 2 4	16*w_4_71-16*w_4_78
3 4 1	0 4 0 5 1 3 2 5	-16*w_4_79+16*w_4_71
3 4 1	0 4 0 5 1 6 2 3	-16*w_4_92+16*w_4_72
3 4 1	0 4 1 5 0 6 2 3	-16*w_4_118+16*w_4_72
3 4 1	0 4 0 5 2 6 1 3	-16*w_4_92+16*w_4_72
3 4 1	0 4 0 5 1 6 2 4	-16*w_4_79+16*w_4_73
3 4 1	0 4 1 5 0 6 2 4	16*w_4_73-16*w_4_78
3 4 1	0 4 0 5 2 3 1 3	16*w_4_73-16*w_4_77
3 4 1	0 4 0 5 1 6 2 5	1/18-16*w_4_66+16*w_4_74
3 4 1	0 4 1 5 0 6 2 5	16*w_4_74-16*w_4_107
3 4 1	0 4 0 5 2 4 1 3	16*w_4_74-16*w_4_85
3 4 1	0 4 0 5 2 3 1 4	-16*w_4_78+16*w_4_75
3 4 1	0 4 1 5 2 3 0 4	-16*w_4_77+16*w_4_75
3 4 1	0 4 0 5 2 6 1 4	-16*w_4_79+16*w_4_75
3 4 1	0 4 0 5 2 3 1 5	-16*w_4_79+16*w_4_76
3 4 1	0 4 1 5 2 3 0 5	16*w_4_76-16*w_4_78
3 4 1	0 4 2 5 1 3 0 4	16*w_4_76-16*w_4_77
3 4 1	0 4 0 5 2 4 1 4	-16*w_4_86+16*w_4_83
3 4 1	0 4 1 5 2 4 0 4	1/12+8*w_4_83-8*w_4_64
3 4 1	0 4 0 5 2 4 1 5	-16*w_4_87+16*w_4_84
3 4 1	0 4 1 5 2 4 0 5	1/6+16*w_4_84-16*w_4_65
3 4 1	0 4 2 3 0 4 1 3	-16*w_4_106+16*w_4_84
3 4 1	0 4 0 5 2 6 1 5	1/18-16*w_4_66+16*w_4_91
3 4 1	0 4 1 5 2 6 0 5	16*w_4_91-16*w_4_85
3 4 1	0 4 2 3 0 4 1 5	-16*w_4_107+16*w_4_91
3 4 1	0 4 2 3 0 4 1 4	16*w_4_105-16*w_4_86
3 4 1	0 4 2 5 1 4 0 4	1/12+8*w_4_105-8*w_4_64
3 4 1	0 1 0 2 3 4 3 5	-16*w_4_4
3 4 1	0 1 0 2 3 6 4 5	-16*w_4_5
3 4 1	0 2 0 3 1 3 4 5	-16*w_4_25
3 4 1	0 1 0 5 2 3 3 4	16*w_4_29
3 4 1	0 1 0 5 3 6 2 3	16*w_4_31
3 4 1	0 1 0 5 3 4 2 3	16*w_4_32
3 4 1	0 1 0 5 4 6 2 3	-16*w_4_33
3 4 1	0 1 0 5 2 4 3 4	-16*w_4_43
3 4 1	0 1 0 5 2 6 3 4	-16*w_4_44
3 4 1	0 1 0 5 3 6 2 4	16*w_4_44
3 4 1	0 1 0 5 3 4 2 4	16*w_4_45
3 4 1	0 1 0 5 3 4 2 5	-16*w_4_47
3 4 1	0 1 0 5 3 6 2 5	16*w_4_47
# 1 1 2		
3 4 1	0 4 1 3 2 3 2 3	-1/12-8*w_4_105+8*w_4_64
3 4 1	0 4 1 3 2 3 2 4	-1/6-16*w_4_84+16*w_4_65
3 4 1	0 4 1 3 2 4 2 4	-1/12-8*w_4_83+8*w_4_64
3 4 1	0 4 1 3 2 3 2 5	-1/18+16*w_4_66-16*w_4_91
3 4 1	0 4 1 3 2 4 2 5	-1/18+16*w_4_66-16*w_4_74
3 4 1	0 1 2 3 2 3 3 4	1/3+16*w_4_10
3 4 1	0 1 2 3 2 6 3 5	1/6+16*w_4_22
3 4 1	0 1 2 3 2 4 4 5	1/6+16*w_4_18
3 4 1	0 1 2 3 2 4 3 5	1/6+16*w_4_17
3 4 1	0 1 2 3 2 6 3 4	1/6+16*w_4_21
3 4 1	0 1 2 5 3 4 2 5	1/6+16*w_4_46
3 4 1	0 2 1 2 3 4 3 4	-1/6+8*w_4_3
3 4 1	0 2 1 2 3 4 3 5	-1/6+16*w_4_4
3 4 1	0 2 1 2 3 4 4 5	-16*w_4_4
3 4 1	0 2 1 2 3 6 4 5	16*w_4_5
3 4 1	0 2 1 3 2 3 3 4	16*w_4_24+16*w_4_10
3 4 1	0 4 1 2 2 4 3 4	1/9-16*w_4_24
3 4 1	0 2 1 3 2 3 3 5	-16*w_4_24+16*w_4_10
3 4 1	0 4 1 2 2 4 4 5	1/6-16*w_4_24
3 4 1	0 2 1 3 2 3 4 5	16*w_4_25
3 4 1	0 4 1 2 2 4 3 5	1/9+16*w_4_25
3 4 1	0 2 1 3 2 4 3 4	16*w_4_28+16*w_4_16
3 4 1	0 4 1 2 2 3 3 4	1/3-16*w_4_28
3 4 1	0 2 1 5 2 3 3 5	-16*w_4_28+16*w_4_16
3 4 1	0 4 2 5 1 2 4 5	-1/6+16*w_4_28
3 4 1	0 2 1 3 2 4 3 5	16*w_4_29+16*w_4_17
3 4 1	0 4 1 2 2 3 4 5	16*w_4_29
3 4 1	0 2 1 5 2 3 3 4	-16*w_4_29+16*w_4_17
3 4 1	0 4 2 5 1 2 3 5	1/9+16*w_4_29
3 4 1	0 2 1 3 2 4 4 5	16*w_4_18+16*w_4_30
3 4 1	0 4 1 2 2 3 3 5	1/6+16*w_4_30
3 4 1	0 2 1 5 2 3 4 5	-16*w_4_18+16*w_4_30
3 4 1	0 4 2 5 1 2 3 4	1/9+16*w_4_30
3 4 1	0 2 1 3 2 6 3 4	16*w_4_21+16*w_4_31
3 4 1	0 4 1 2 2 6 3 4	-16*w_4_31
3 4 1	0 2 1 5 3 6 2 3	16*w_4_21-16*w_4_31
3 4 1	0 4 5 6 1 2 2 5	1/6-16*w_4_31
3 4 1	0 2 1 3 2 6 3 5	16*w_4_22+16*w_4_32
3 4 1	0 4 1 2 2 6 4 5	16*w_4_32
3 4 1	0 2 1 5 3 4 2 3	16*w_4_22-16*w_4_32
3 4 1	0 4 3 5 1 2 2 5	1/18+16*w_4_32
3 4 1	0 2 1 3 2 6 4 5	16*w_4_23+16*w_4_33
3 4 1	0 4 1 2 2 6 3 5	16*w_4_33
3 4 1	0 2 1 5 4 6 2 3	-16*w_4_23+16*w_4_33
3 4 1	0 4 3 5 2 6 1 2	1/18+16*w_4_33
3 4 1	0 2 1 5 2 4 3 4	1/18+16*w_4_40+16*w_4_43
3 4 1	0 4 2 3 1 2 3 5	1/6-16*w_4_43
3 4 1	0 2 1 5 2 4 3 5	1/18+16*w_4_40-16*w_4_43
3 4 1	0 4 2 3 1 2 4 5	16*w_4_43
3 4 1	0 2 1 5 2 6 3 4	16*w_4_44+16*w_4_41
3 4 1	0 4 2 5 3 6 1 2	-16*w_4_44
3 4 1	0 2 1 5 3 6 2 4	-16*w_4_44+16*w_4_41
3 4 1	0 4 5 6 1 2 2 3	-16*w_4_44
3 4 1	0 2 1 5 2 6 3 5	16*w_4_45+16*w_4_42
3 4 1	0 4 2 5 4 6 1 2	-16*w_4_45
3 4 1	0 2 1 5 3 4 2 4	-16*w_4_45+16*w_4_42
3 4 1	0 4 3 5 1 2 2 3	1/6+16*w_4_45
3 4 1	0 2 1 5 3 4 2 5	16*w_4_47+16*w_4_46
3 4 1	0 4 3 5 1 2 2 4	-16*w_4_47
3 4 1	0 2 1 5 3 6 2 5	-16*w_4_47+16*w_4_46
3 4 1	0 4 5 6 1 2 2 4	-16*w_4_47
3 4 1	0 4 2 5 2 4 1 4	-16*w_4_60+16*w_4_64
3 4 1	0 4 2 5 2 4 1 5	-16*w_4_61+16*w_4_65
3 4 1	0 4 1 5 2 6 2 5	16*w_4_66-16*w_4_62
3 4 1	0 4 2 5 2 4 1 3	16*w_4_66-16*w_4_63
3 4 1	0 4 1 5 2 3 2 3	-16*w_4_76+16*w_4_77
3 4 1	0 4 2 5 1 3 2 5	16*w_4_77-16*w_4_75
3 4 1	0 4 2 5 2 3 1 4	-16*w_4_73+16*w_4_77
3 4 1	0 4 2 5 2 6 1 4	16*w_4_77-16*w_4_71
3 4 1	0 4 1 5 2 3 2 4	-16*w_4_73+16*w_4_78
3 4 1	0 4 2 5 1 3 2 3	-16*w_4_76+16*w_4_78
3 4 1	0 4 2 5 1 6 2 4	-16*w_4_71+16*w_4_78
3 4 1	0 4 2 5 2 3 1 5	16*w_4_78-16*w_4_75
3 4 1	0 4 1 5 2 3 2 5	16*w_4_79-16*w_4_75
3 4 1	0 4 2 5 1 3 2 4	16*w_4_79-16*w_4_73
3 4 1	0 4 1 5 2 6 2 4	16*w_4_79-16*w_4_71
3 4 1	0 4 2 5 2 3 1 3	16*w_4_79-16*w_4_76
3 4 1	0 4 1 5 2 4 2 3	-1/18-16*w_4_63+16*w_4_85
3 4 1	0 4 2 3 1 3 2 5	-16*w_4_91+16*w_4_85
3 4 1	0 4 2 3 1 6 2 4	-16*w_4_74+16*w_4_85
3 4 1	0 4 2 5 2 6 1 5	-16*w_4_62+16*w_4_85
3 4 1	0 4 1 5 2 4 2 4	-1/6-16*w_4_60+16*w_4_86
3 4 1	0 4 2 3 1 3 2 3	-16*w_4_105+16*w_4_86
3 4 1	0 4 2 3 1 4 2 4	16*w_4_86-16*w_4_83
3 4 1	0 4 2 5 1 4 2 4	-16*w_4_60+16*w_4_86
3 4 1	0 4 1 5 2 4 2 5	-16*w_4_61+16*w_4_87
3 4 1	0 4 2 3 1 3 2 4	16*w_4_87-16*w_4_84
3 4 1	0 4 1 5 2 6 2 3	16*w_4_92-16*w_4_72
3 4 1	0 4 2 5 2 6 1 3	16*w_4_92-16*w_4_72
3 4 1	0 4 2 3 1 4 2 3	16*w_4_106-16*w_4_84
3 4 1	0 4 2 5 1 4 2 5	-1/6+16*w_4_106-16*w_4_61
3 4 1	0 4 2 3 1 4 2 5	-16*w_4_74+16*w_4_107
3 4 1	0 4 2 5 1 4 2 3	-1/18-16*w_4_63+16*w_4_107
3 4 1	0 4 2 3 1 6 2 3	16*w_4_107-16*w_4_91
3 4 1	0 4 2 5 1 6 2 5	-1/6+16*w_4_107-16*w_4_62
3 4 1	0 4 2 5 1 6 2 3	16*w_4_118-16*w_4_72
3 4 1	0 1 2 3 2 4 3 4	16*w_4_16
3 4 1	0 1 2 3 2 6 4 5	16*w_4_23
3 4 1	0 1 2 5 2 4 3 4	16*w_4_40
3 4 1	0 1 2 5 2 6 3 4	16*w_4_41
3 4 1	0 1 2 5 2 6 3 5	16*w_4_42
# 1 1 1		
3 4 1	0 4 1 3 2 3 3 4	1/6+16*w_4_108+16*w_4_67
3 4 1	0 4 1 3 2 4 3 4	1/6-16*w_4_67+16*w_4_90
3 4 1	0 4 1 3 2 6 3 4	1/6+16*w_4_111
3 4 1	0 4 1 3 2 3 3 5	1/18-16*w_4_110+16*w_4_68
3 4 1	0 4 1 3 2 3 4 5	1/18-16*w_4_109+16*w_4_69
3 4 1	0 4 1 3 2 4 3 5	1/18+16*w_4_89+16*w_4_69
3 4 1	0 4 1 3 2 4 4 5	1/18+16*w_4_68+16*w_4_88

Table 7 (part 5).

3 4 1	0 4 1 3 2 6 3 5	1/36+16*w_4_70-16*w_4_113
3 4 1	0 4 1 3 2 6 4 5	1/36-16*w_4_112+16*w_4_70
3 4 1	0 1 2 3 3 4 3 4	-1/6+8*w_4_34
3 4 1	0 1 2 3 3 4 4 5	1/6+16*w_4_36
3 4 1	0 1 2 5 3 4 3 4	-1/6+16*w_4_48
3 4 1	0 1 2 5 3 4 4 5	1/6+16*w_4_50
3 4 1	0 2 1 3 3 4 3 4	16*w_4_34
3 4 1	0 4 1 2 3 4 3 4	-1/6+8*w_4_34
3 4 1	0 2 1 3 3 4 3 5	32*w_4_35
3 4 1	0 4 1 2 3 4 4 5	-16*w_4_35
3 4 1	0 2 1 3 3 4 4 5	32*w_4_36
3 4 1	0 4 1 2 3 4 3 5	-1/6-16*w_4_36
3 4 1	0 2 1 3 3 6 3 5	16*w_4_37
3 4 1	0 4 1 2 4 6 4 5	8*w_4_37
3 4 1	0 2 1 3 3 6 4 5	32*w_4_38
3 4 1	0 4 1 2 3 6 4 5	16*w_4_38
3 4 1	0 2 1 3 4 6 4 5	16*w_4_39
3 4 1	0 4 1 2 3 6 3 5	8*w_4_39
3 4 1	0 2 1 5 3 4 3 4	32*w_4_48
3 4 1	0 4 3 5 1 2 3 5	-1/6+16*w_4_48
3 4 1	0 2 1 5 3 4 3 5	32*w_4_49
3 4 1	0 4 3 5 1 2 4 5	16*w_4_49
3 4 1	0 2 1 5 3 4 4 5	32*w_4_50
3 4 1	0 4 3 5 1 2 3 4	-1/6-16*w_4_50
3 4 1	0 2 1 5 3 6 3 4	32*w_4_51
3 4 1	0 4 5 6 1 2 3 5	-16*w_4_51
3 4 1	0 2 1 5 3 6 3 5	32*w_4_52
3 4 1	0 4 5 6 1 2 4 5	-16*w_4_52
3 4 1	0 2 1 5 3 6 4 5	32*w_4_53
3 4 1	0 4 5 6 1 2 3 4	16*w_4_53
3 4 1	0 2 1 5 4 6 3 4	32*w_4_54
3 4 1	0 4 3 5 3 6 1 2	-16*w_4_54
3 4 1	0 2 1 5 4 6 3 5	32*w_4_55
3 4 1	0 4 3 5 4 6 1 2	-16*w_4_55
3 4 1	0 4 1 5 2 3 3 4	16*w_4_80+16*w_4_81
3 4 1	0 4 2 5 1 3 3 5	-16*w_4_80+16*w_4_82
3 4 1	0 4 1 5 2 3 3 5	16*w_4_81+16*w_4_82
3 4 1	0 4 2 5 1 3 4 5	-16*w_4_80-16*w_4_81
3 4 1	0 4 1 5 2 3 4 5	-16*w_4_80+16*w_4_82
3 4 1	0 4 2 5 1 3 3 4	16*w_4_81+16*w_4_82
3 4 1	0 4 1 5 2 4 3 4	1/18+16*w_4_68+16*w_4_88
3 4 1	0 4 2 3 1 3 3 5	-16*w_4_110-16*w_4_88
3 4 1	0 4 1 5 2 4 3 5	1/18+16*w_4_89+16*w_4_69
3 4 1	0 4 2 3 1 3 4 5	-16*w_4_89-16*w_4_109
3 4 1	0 4 1 5 2 4 4 5	1/6-16*w_4_67+16*w_4_90
3 4 1	0 4 2 3 1 3 3 4	16*w_4_108+16*w_4_90
3 4 1	0 4 1 5 2 6 3 4	16*w_4_99+16*w_4_93
3 4 1	0 4 2 5 3 6 1 3	-16*w_4_93+16*w_4_119
3 4 1	0 4 1 5 2 6 3 5	16*w_4_102+16*w_4_94
3 4 1	0 4 2 5 4 6 1 3	-16*w_4_94+16*w_4_120
3 4 1	0 4 1 5 2 6 4 5	16*w_4_95-16*w_4_96
3 4 1	0 4 2 5 3 4 1 3	16*w_4_95-16*w_4_121
3 4 1	0 4 1 5 3 4 2 3	-16*w_4_121+16*w_4_96
3 4 1	0 4 3 5 1 3 2 5	16*w_4_95-16*w_4_96
3 4 1	0 4 1 5 3 4 2 4	16*w_4_103+16*w_4_97
3 4 1	0 4 3 5 1 3 2 3	16*w_4_122-16*w_4_97
3 4 1	0 4 1 5 3 4 2 5	16*w_4_98-16*w_4_124
3 4 1	0 4 3 5 1 3 2 4	16*w_4_123-16*w_4_98
3 4 1	0 4 1 5 3 6 2 3	16*w_4_99+16*w_4_119
3 4 1	0 4 5 6 1 3 2 5	16*w_4_99+16*w_4_93
3 4 1	0 4 1 5 3 6 2 4	32*w_4_100
3 4 1	0 4 5 6 1 3 2 3	16*w_4_100
3 4 1	0 4 1 5 3 6 2 5	-16*w_4_129+16*w_4_101
3 4 1	0 4 5 6 1 3 2 4	-16*w_4_128+16*w_4_101
3 4 1	0 4 1 5 4 6 2 3	16*w_4_102+16*w_4_120
3 4 1	0 4 3 5 2 6 1 3	16*w_4_102+16*w_4_94
3 4 1	0 4 1 5 4 6 2 4	16*w_4_103+16*w_4_97
3 4 1	0 4 3 5 2 3 1 3	16*w_4_122+16*w_4_103
3 4 1	0 4 1 5 4 6 2 5	-16*w_4_134+16*w_4_104
3 4 1	0 4 3 5 2 4 1 3	-16*w_4_133+16*w_4_104
3 4 1	0 4 2 3 1 4 3 4	16*w_4_108+16*w_4_90
3 4 1	0 4 2 5 1 4 4 5	-1/6-16*w_4_108-16*w_4_67
3 4 1	0 4 2 3 1 4 3 5	16*w_4_89+16*w_4_109
3 4 1	0 4 2 5 1 4 3 5	1/18-16*w_4_109+16*w_4_69
3 4 1	0 4 2 3 1 4 4 5	16*w_4_110+16*w_4_88
3 4 1	0 4 2 5 1 4 3 4	1/18-16*w_4_110+16*w_4_68
3 4 1	0 4 2 3 1 6 3 4	32*w_4_111
3 4 1	0 4 5 6 1 6 2 5	1/6+16*w_4_111
3 4 1	0 4 2 3 1 6 3 5	16*w_4_112-16*w_4_113
3 4 1	0 4 3 5 1 6 2 5	1/36-16*w_4_112+16*w_4_70
3 4 1	0 4 2 3 1 6 4 5	-16*w_4_112+16*w_4_113
3 4 1	0 4 3 5 2 6 1 5	1/36+16*w_4_70-16*w_4_113
3 4 1	0 4 2 5 1 6 3 4	16*w_4_99+16*w_4_119
3 4 1	0 4 5 6 1 6 2 3	-16*w_4_93+16*w_4_119
3 4 1	0 4 2 5 1 6 3 5	16*w_4_102+16*w_4_120
3 4 1	0 4 3 5 1 6 2 3	16*w_4_94-16*w_4_120
3 4 1	0 4 2 5 1 6 4 5	16*w_4_121-16*w_4_96
3 4 1	0 4 3 5 2 3 1 5	16*w_4_95-16*w_4_121
3 4 1	0 4 2 5 3 4 1 4	16*w_4_122+16*w_4_103
3 4 1	0 4 2 5 4 6 1 4	-16*w_4_122+16*w_4_97
3 4 1	0 4 2 5 3 4 1 5	16*w_4_123-16*w_4_124
3 4 1	0 4 5 6 1 6 2 4	16*w_4_123-16*w_4_98
3 4 1	0 4 3 5 2 3 1 4	16*w_4_123-16*w_4_124
3 4 1	0 4 5 6 1 4 2 5	16*w_4_98-16*w_4_124
3 4 1	0 4 2 5 3 6 1 5	16*w_4_128-16*w_4_129
3 4 1	0 4 3 5 1 6 2 4	-16*w_4_128+16*w_4_101
3 4 1	0 4 3 5 2 6 1 4	-16*w_4_129+16*w_4_101
3 4 1	0 4 5 6 1 4 2 3	16*w_4_128-16*w_4_129
3 4 1	0 4 2 5 4 6 1 5	-16*w_4_134+16*w_4_133
3 4 1	0 4 3 5 2 4 1 5	-16*w_4_133+16*w_4_104
3 4 1	0 4 3 5 1 4 2 3	16*w_4_134-16*w_4_133
3 4 1	0 4 3 5 1 4 2 5	-16*w_4_134+16*w_4_104
3 4 1	0 4 3 5 2 4 1 4	32*w_4_138
3 4 1	0 4 5 6 1 4 2 4	16*w_4_138
3 4 1	0 1 2 3 3 4 3 5	16*w_4_35
3 4 1	0 1 2 3 3 6 3 5	8*w_4_37
3 4 1	0 1 2 3 3 6 4 5	16*w_4_38
3 4 1	0 1 2 3 4 6 4 5	8*w_4_39
3 4 1	0 1 2 5 3 4 3 5	16*w_4_49
3 4 1	0 1 2 5 3 6 3 4	16*w_4_51
3 4 1	0 1 2 5 3 6 3 5	16*w_4_52
3 4 1	0 1 2 5 3 6 4 5	16*w_4_53
3 4 1	0 1 2 5 4 6 3 4	16*w_4_54
3 4 1	0 1 2 5 4 6 3 5	16*w_4_55
3 4 1	0 4 2 5 3 6 1 4	16*w_4_100
3 4 1	0 4 3 5 1 4 2 4	16*w_4_138

The table below contains the output of the command

`$ reduce_mod_jacobi associator4_intermsof10_part100.txt`

as described in Implementation 15.

TABLE 8. Sample output of `reduce_mod_jacobi`.

```
3 4 1   0 1 0 3 2 6 3 4    -24+c_1_1221_211
3 4 1   0 4 1 3 2 6 3 4    -8+c_1_1240_111
3 4 1   0 1 0 3 2 3 4 5    -48+c_1_1221_211
                             -c_1_513_211
3 4 1   0 1 0 3 2 4 3 5    24-c_1_1221_211
3 4 1   0 4 1 3 2 4 3 5    24-c_1_1240_111
                             -c_1_540_111
3 4 1   0 1 2 3 3 4 4 5    -8-c_1_1228_111
3 4 1   0 1 2 5 3 4 3 4    -8-c_1_1228_111
3 4 1   0 2 0 3 1 4 3 5    24-c_1_1005_211
3 4 1   0 2 0 5 1 3 3 4    -24-c_1_516_211
3 4 1   0 2 0 3 1 6 3 4    -24+c_1_1005_211
3 4 1   0 2 0 5 3 6 1 3    24+c_1_516_211
3 4 1   0 2 1 3 2 3 4 5    48+c_1_1230_112
                             -c_1_1008_112
3 4 1   0 4 1 2 2 4 3 5    48-c_1_525_112
                             -c_1_1239_112
3 4 1   0 2 1 3 2 4 3 5    -24-c_1_1230_112
3 4 1   0 4 1 2 2 3 4 5    -24+c_1_1239_112
3 4 1   0 2 1 5 2 3 3 4    24-c_1_1008_112
3 4 1   0 4 2 5 1 2 3 5    -24+c_1_525_112
3 4 1   0 2 1 3 2 6 3 4    24+c_1_1230_112
3 4 1   0 4 1 2 2 6 3 4    -24+c_1_1239_112
3 4 1   0 2 1 5 3 6 2 3    -24+c_1_1008_112
3 4 1   0 4 5 6 1 2 2 5    -24+c_1_525_112
3 4 1   0 2 1 3 3 4 4 5    -16-c_1_1012_111
3 4 1   0 4 1 2 3 4 3 5    8+c_1_529_111
3 4 1   0 2 1 3 3 6 4 5    -16-c_1_1012_111
3 4 1   0 4 1 2 3 6 4 5    -8-c_1_529_111
3 4 1   0 2 1 5 3 4 3 4    -16-c_1_1012_111
3 4 1   0 4 3 5 1 2 3 5    -8-c_1_529_111
3 4 1   0 2 1 5 3 6 3 5    16+c_1_1012_111
3 4 1   0 4 5 6 1 2 4 5    -8-c_1_529_111
3 4 1   0 4 1 5 2 3 3 4    8-c_1_1023_111
3 4 1   0 4 2 5 1 3 3 5    -16+c_1_540_111
3 4 1   0 4 1 5 2 3 3 5    -8-c_1_538_111
3 4 1   0 4 2 5 1 3 4 5    -8+c_1_1021_111
3 4 1   0 4 1 5 2 3 4 5    -16+c_1_1242_111
3 4 1   0 4 2 5 1 3 3 4    -8-c_1_1245_111
3 4 1   0 4 1 5 2 4 3 5    24-c_1_536_111
                             -c_1_1242_111
3 4 1   0 4 2 3 1 3 4 5    -24-c_1_1245_111
                             +c_1_1019_111
3 4 1   0 4 1 5 2 6 3 4    -16+c_1_1242_111
3 4 1   0 4 2 5 3 6 1 3    8+c_1_1245_111
3 4 1   0 4 1 5 2 6 3 5    -8-c_1_538_111
3 4 1   0 4 2 5 4 6 1 3    -8+c_1_1021_111
3 4 1   0 4 1 5 3 4 2 5    -16+c_1_1242_111
3 4 1   0 4 3 5 1 3 2 4    8+c_1_1245_111
3 4 1   0 4 1 5 3 6 2 3    -8+c_1_1023_111
3 4 1   0 4 5 6 1 3 2 5    -16+c_1_540_111
3 4 1   0 4 1 5 3 6 2 4    32-c_1_1242_111
                             -c_1_540_111
3 4 1   0 4 5 6 1 3 2 3    16-c_1_1023_111
                             +c_1_1245_111
3 4 1   0 4 1 5 4 6 2 3    -16+c_1_1242_111
3 4 1   0 4 3 5 2 6 1 3    -8-c_1_1245_111
3 4 1   0 4 2 3 1 4 3 5    24+c_1_538_111
                             -c_1_1019_111
3 4 1   0 4 2 3 1 6 3 4    -16+c_1_1019_111
3 4 1   0 4 5 6 1 6 2 5    -8+c_1_536_111
3 4 1   0 4 2 5 1 6 3 4    -8+c_1_1021_111
3 4 1   0 4 5 6 1 6 2 3    8+c_1_538_111
3 4 1   0 4 2 5 1 6 3 5    -16+c_1_540_111
3 4 1   0 4 3 5 1 6 2 3    8-c_1_1023_111
3 4 1   0 4 2 5 3 4 1 5    -8+c_1_1021_111
3 4 1   0 4 5 6 1 6 2 4    8+c_1_538_111
3 4 1   0 4 3 5 2 3 1 4    -8+c_1_1023_111
3 4 1   0 4 5 6 1 4 2 5    -16+c_1_540_111
3 4 1   0 2 0 3 1 3 4 5    -48+c_1_1005_211-c_1_516_211
3 4 1   0 1 0 5 2 3 3 4    -24-c_1_513_211
3 4 1   0 1 0 5 3 6 2 3    24+c_1_513_211
3 4 1   0 1 2 3 3 6 4 5    -8-c_1_1228_111
3 4 1   0 1 2 5 3 6 3 5    8+c_1_1228_111
3 4 1   0 4 2 5 3 6 1 4    16+c_1_538_111-c_1_1021_111
3 4 1   0 4 1 3 2 3 4 5    -c_1_1023_111+c_1_1240_111
3 4 1   0 4 2 5 1 4 3 5    c_1_536_111-c_1_1021_111

3 4 1   0 1 2 3 0 3 5 4    c_1_513_211==-24
3 4 1   0 2 1 3 0 3 5 4    c_1_516_211==-24
3 4 1   1 2 2 3 0 3 5 4    c_1_525_112==24
3 4 1   1 2 3 5 0 3 5 4    c_1_529_111==-8
3 4 1   1 4 2 3 0 3 5 4    c_1_536_111==8
3 4 1   1 4 2 5 0 3 5 4    c_1_538_111==-8
3 4 1   1 5 2 3 0 3 5 4    c_1_540_111==16
3 4 1   0 2 0 3 1 3 5 4    c_1_1005_211==24
3 4 1   0 2 2 3 1 3 5 4    c_1_1008_112==24
3 4 1   0 2 3 5 1 3 5 4    c_1_1012_111==-16
3 4 1   0 4 2 3 1 3 5 4    c_1_1019_111==16
3 4 1   0 4 2 5 1 3 5 4    c_1_1021_111==8
3 4 1   0 5 2 3 1 3 5 4    c_1_1023_111==8
3 4 1   0 1 0 3 2 3 5 4    c_1_1221_211==24
3 4 1   0 1 3 5 2 3 5 4    c_1_1228_111==-8
3 4 1   0 2 1 3 2 3 5 4    c_1_1230_112==-24
3 4 1   0 4 1 2 2 3 5 4    c_1_1239_112==24
3 4 1   0 4 1 3 2 3 5 4    c_1_1240_111==8
3 4 1   0 4 1 5 2 3 5 4    c_1_1242_111==16
3 4 1   0 5 1 3 2 3 5 4    c_1_1245_111==-8
```

The first part of the output lists the graph series $S^{(1)} - \Diamond$, reduced modulo skew-symmetry, wherein the coefficients of $\Diamond$ are still undetermined. The second part of the ouput (after the blank line) specifies the coefficients such that $S^{(1)} = \Diamond$. Every coefficient in the second part is preceded by the encoding of the graph that specifies a differential consequence of the Jacobi identity. Such a differential consequence expands into a sum of graphs that can be read in the first part of the output.

Appendix E. Gauge transformation that removes 4 master-parameters out of 10

Encodings of graphs (see Implementation 1 on p. 5) built over one sink vertex are followed by their coefficients, in the following table containing the gauge transformation which was claimed to exist in Theorem 13.

Table 9. Gauge transformation that removes 4 master-parameters out of 10.

```
h^0:
1 0 1                        1
h^4:
1 4 1   0 2 0 3 1 4 0 3    16*w_4_101
1 4 1   0 2 0 3 1 4 1 3    8*w_4_101
1 4 1   0 2 0 3 1 4 2 3    8*w_4_101
1 4 1   0 2 1 3 0 4 1 2    -8*w_4_101
1 4 1   0 2 1 3 0 4 2 3    8*w_4_101
1 4 1   0 2 1 3 1 4 0 2    -8*w_4_101
1 4 1   0 2 1 3 2 4 0 2    -8*w_4_101
1 4 1   0 2 0 3 0 4 1 3    -16*w_4_102
1 4 1   0 2 0 3 1 4 1 3    -8*w_4_102
1 4 1   0 2 0 3 2 4 1 2    -8*w_4_102
1 4 1   0 2 0 3 2 4 1 3    -16*w_4_102
1 4 1   0 2 1 3 0 4 1 2    -8*w_4_102
1 4 1   0 2 1 3 0 4 1 3    -8*w_4_102
1 4 1   0 2 0 3 0 4 1 2    16*w_4_119
1 4 1   0 2 0 3 1 4 1 2    16*w_4_119
1 4 1   0 2 0 3 1 4 1 3    8*w_4_119
1 4 1   0 2 0 3 2 4 1 2    8*w_4_119
1 4 1   0 2 1 3 0 4 1 2    8*w_4_119
1 4 1   0 2 3 4 0 4 1 2    -8*w_4_119
1 4 1   0 2 0 3 0 1 1 2    -32*w_4_125
1 4 1   0 2 0 3 1 2 1 2    16*w_4_125
1 4 1   0 2 0 3 1 2 1 3    -16*w_4_125
1 4 1   0 2 0 3 1 2 2 3    16*w_4_125
1 4 1   0 2 0 3 1 4 1 2    16*w_4_125
1 4 1   0 2 0 3 1 4 1 3    -16*w_4_125
1 4 1   0 2 0 3 1 4 2 3    16*w_4_125
```

On the Kontsevich $\star$-product associativity mechanism

R. Buring*, A. V. Kiselev*,§

Abstract

The deformation quantization by Kontsevich is a way to construct an associative noncommutative star-product $\star = \times + \hbar\,\{\,,\,\}_{\mathcal{P}} + \bar{o}(\hbar)$ in the algebra of formal power series in $\hbar$ on a given finite-dimensional affine Poisson manifold: here $\times$ is the usual multiplication, $\{\,,\,\}_{\mathcal{P}} \neq 0$ is the Poisson bracket, and $\hbar$ is the deformation parameter. The product $\star$ is assembled at all powers $\hbar^{k\geqslant 0}$ via summation over a certain set of weighted graphs with $k+2$ vertices; for each $k>0$, every such graph connects the two co-multiples of $\star$ using k copies of $\{\,,\,\}_{\mathcal{P}}$. Cattaneo and Felder interpreted these topological portraits as genuine Feynman diagrams in the Ikeda–Izawa model for quantum gravity.

By expanding the star-product up to $\bar{o}(\hbar^3)$, i.e., with respect to graphs with at most five vertices but possibly containing loops, we illustrate the mechanism $\mathsf{Assoc} = \Diamond\,(\mathsf{Poisson})$ that converts the Jacobi identity for the bracket $\{\,,\,\}_{\mathcal{P}}$ into the associativity of $\star$.

Denote by $\times$ the multiplication in the commutative associative unital algebra $C^\infty(N^n \to \mathbb{R})$ of scalar functions on a smooth n-dimensional real manifold N^n. Suppose first that a noncommutative deformation $\star = \times + O(\hbar)$ of $\times$ is still unital ($f\star 1 = f = 1\star f$) and associative, $(f\star g)\star h = f\star(g\star h)$ for $f,g,h \in C^\infty(N^n)[[\hbar]]$. By taking $3! = 6$ copies of the associativity equation for the star-product $\star$, we infer that the skew-symmetric part of the leading deformation term, $\{f,g\}_\star := \frac{1}{\hbar}\bigl(f\star g - g\star f\bigr)\bigr|_{\hbar:=0}$, is a Poisson bracket.[1]

Now the other way round: can the multiplication $\times$ on a Poisson manifold N^n be deformed using the bracket $\{\,,\,\}_{\mathcal{P}}$ such that the $\Bbbk[[\hbar]]$-linear star-product $\star = \times + \hbar\,\{\,,\,\}_{\mathcal{P}} + \bar{o}(\hbar)$ stays associative? Kontsevich proved [1] that on finite-dimensional affine[2] Poisson manifolds, this is always possible: from $\{\,,\,\}_{\mathcal{P}}$ one obtains the bi-differential terms $B_k(\cdot,\cdot)$ at all powers of $\hbar^{k\geqslant 0}$ in the formal series for $\star$. This associative unital $\star$-product was constructed in [1] using a pictorial language: the operators $B_k = \sum_{\{\Gamma\}} w(\Gamma) \times B_k^\Gamma(\cdot,\cdot)$ are encoded by the weighted oriented graphs Γ with $k+2$ vertices and $2k$ edges but without tadpoles or multiple edges; in every such Γ, there are k internal vertices (each of them is a tail for two edges) and 2 sinks (no issued edges). The Poisson bracket $\{\,,\,\}_{\mathcal{P}}$ with coefficients $\mathcal{P}^{ij}(\boldsymbol{u})$ at $\boldsymbol{u} \in N^n$ provides the "building block" $\wedge = \xleftarrow[\text{Left}]{i} \bullet \xrightarrow[\text{Right}]{j}$ in which $\sum_{i,j=1}^n$ is implicit and the vertex contains $\mathcal{P}^{ij}(\boldsymbol{u})$. To indicate the ordering of indexes in $\mathcal{P}^{ij} = -\mathcal{P}^{ji}$, the out-going edges

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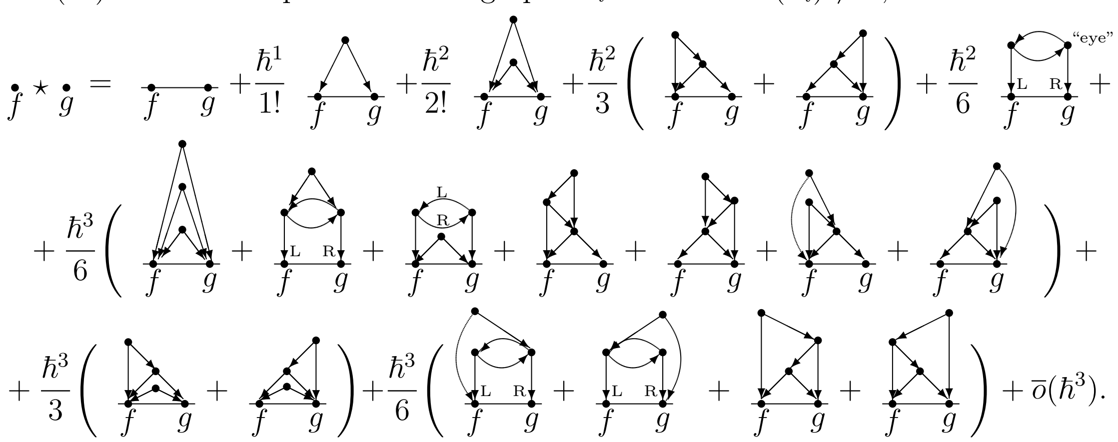
[1]The left-hand side of the Jacobi identity $\sum_{\circlearrowleft}\{\{f,g\}_\star,h\}_\star = 0$ is an obstruction to the associativity of the star-product: whenever the Jacobi identity is violated, one cannot have that $(f\star g)\star h = f\star(g\star h)$.

[2]On affine manifolds N^n, the only shape of coordinate changes is $\widetilde{\boldsymbol{u}} = A\cdot\boldsymbol{u} + \vec{\boldsymbol{c}}$. Yet no loss of generality occurs if the space N^n is the fibre in an affine bundle π of physical fields $\{\boldsymbol{u} = \phi(\boldsymbol{x})\}$ over the space-time $M^m \ni \boldsymbol{x}$; the Jacobians $\partial\widetilde{\boldsymbol{u}}/\partial\boldsymbol{u} = A(\boldsymbol{x})$ are then constant over N^n. (The arguments of $\star$ are local functionals of sections, $\phi \in \Gamma(\pi) \to \Bbbk$; the $\star$-product is marked by the variational Poisson brackets $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ on the jet space $J^\infty(\pi)$.) The deformation quantization from [1] is lifted to the gauge field set-up in [2].

are ordered by Left $\prec$ Right. The edges carry the derivatives $\partial_i \equiv \partial/\partial u^i$ and $\partial_j \equiv \partial/\partial u^j$, respectively. Every such derivation acts on the content of the vertex at the arrowhead via the Leibniz rule (and it does so independently from the other in-coming arrows, if any).[3]

The weights[4] $w(\Gamma) \in \mathbb{R}$ of such graphs Γ are given by the integrals over configuration spaces of k distinct points in the hyperbolic plane $\mathbb{H}^2$ (e.g., in its upper half-plane model).[5]

The associativity postulate for $\star$ yields the infinite system of quadratic algebraic equations for the weights $w(\Gamma)$ of graphs.[6] Kontsevich shows [1] that the left-hand side $\mathrm{Jac}_{\mathcal{P}}(\cdot,\cdot,\cdot) := \sum_{\circlearrowright}\{\{\cdot,\cdot\}_{\mathcal{P}},\cdot\}_{\mathcal{P}}$ of the Jacobi identity for $\{\,,\,\}_{\mathcal{P}}$ is the *only* obstruction to the balance $\mathsf{Assoc}\,(f,g,h) := (f\star g)\star h - f\star(g\star h) = 0$ at all powers $\hbar^k$ of the deformation parameter at once.[7] The core question that we address in this note is how the mechanism $\mathsf{Assoc} = \Diamond\,(\mathsf{Poisson})$ works explicitly, making the star-product $\star = \times + \hbar\,\{\,,\,\}_{\mathcal{P}} + \bar{o}(\hbar)$ associative by virtue of Jacobi identity for the Poisson bracket $\{\,,\,\}_{\mathcal{P}}$. Expanding the Kontsevich $\star$-product in $\hbar$ up to $\bar{o}(\hbar^3)$ and with respect to all the graphs Γ_i such that $w(\Gamma_i) \neq 0$, we obtain[8]



(1)

[3]For example, $\{f,g\}_{\mathcal{P}}(\boldsymbol{u}) = f \xleftarrow[\text{Left}]{i} \bullet \xrightarrow[\text{Right}]{j} g = (f)\overleftarrow{\partial_i}\big|_{\boldsymbol{u}} \cdot \mathcal{P}^{ij}(\boldsymbol{u}) \cdot \overrightarrow{\partial_j}\big|_{\boldsymbol{u}}(g)$, see (1) above.

[4]Willwacher and Felder (2010) conjecture that the weights can be *irrational* numbers for some graphs.

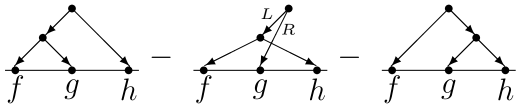
[5]The wedge factors within the integrand in the formula for $w(\Gamma)$ are copies of the kernel of the singular linear integral operator $(\mathrm{d} * \mathrm{d})^{-1}$ in the hyperbolic geometry of $\mathbb{H}^2$, see [3]. Cattaneo and Felder also showed that the $\star$-product of two functions $f, g \in C^\infty(N^n \to \mathbb{C})$ amounts to the Feynman path integral calculation of the correlation function, $\big(f\star g\big)(\boldsymbol{u}) = \int_{\mathcal{X}(\infty)=\boldsymbol{u}} \mathrm{D}\mathcal{X}\,\mathrm{D}\eta\, f\big(\mathcal{X}(0)\big) \times g\big(\mathcal{X}(1)\big) \times \exp\big(\tfrac{i}{\hbar}S(\mathcal{P},[\mathcal{X},\eta])\big)$, in the Ikeda–Izawa topological open string model on a disk $D \simeq \mathbb{H}^2$ with boundary $\partial D \ni 0, 1, \infty$; here $\mathcal{X}\colon D \to N^n$ and $\eta\colon D \to T^*D \otimes \mathcal{X}^*(T^*N^n)$. All details and further references are found in [3, 4]; still let us remember that within the Ikeda–Izawa model, the perturbative expansions in $\hbar$ run, in particular, over the graphs with tadpoles (which must be regularized by hand) but at the same time, those path integral calculations reproduce only the weighted oriented graphs without “eyes” (e.g., as in $\leftarrow \cdot\rightleftarrows\cdot \rightarrow$, see Eq. (1) above). Because, to the best of our knowledge, the eye-containing graphs Γ_i such that $w(\Gamma_i) \neq 0$ cannot all at once be eliminated from the star-product $\star$ via gauge transformations of its arguments and of its output, see Remark 1 on p. 4 and [1], many graphs in the original construction of $\star$ were not recovered in [3]. Hence there is an open problem to extend or modify the Ikeda–Izawa Poisson σ-model such that in the new set-up, the correlation functions would expand with respect to all the Kontsevich graphs Γ_i with $w(\Gamma_i) \neq 0$.

[6]That system solution is not claimed unique: one is provided by the Kontsevich integrals. Number-theoretic properties of those weights were explored by Kontsevich in the context of motives and by Willwacher–Felder in the context of Riemann ζ-function.

[7]Ensuring the associativity $\mathsf{Assoc}\,(f,g,h) = 0$, the tri-vector $\mathrm{Jac}_{\mathcal{P}}(\cdot,\cdot,\cdot)$ is not necessarily (indeed, far not always!) evaluated at the three arguments f, g, h of the associator for $\star$.

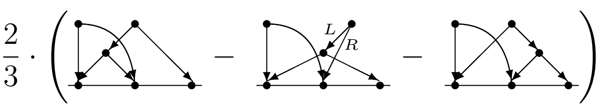
[8]Balancing the associativity of a star-product order-by-order up to $\bar{o}(\hbar^3)$, Penkava and Vanhaecke (1998) derived a set of weights for the $(k+2)$-vertex Kontsevich graphs *without* loops. Yet no loops are destroyed in either of the copies of $\star$ when the composition $\star\circ\star$ is taken; the associativity of loopless star-products is only a part of the full claim for $\star$. So, we integrate over the configuration spaces of $k \leqslant 3$ points in $\mathbb{H}^2$ for *all* the Kontsevich graphs (e.g., with loops).

In every composition $\star \circ \star$ the sums of graphs act on sums of graphs by linearity; each incoming edge acts via the Leibniz rule (see above). The mechanism for $\mathsf{Assoc}(f,g,h)$ to vanish is two-step: first, the sums in $\star\circ\star$ are reduced using the antisymmetry of the Poisson bi-vector $\mathcal{P}$. The output is then reduced modulo the (consequences of) Jacobi identity,[9]

$$\mathrm{Jac}_{\mathcal{P}}(f,g,h) =$$


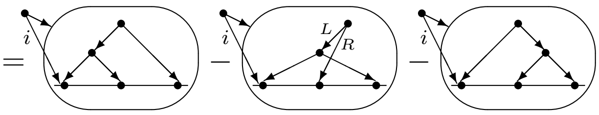
$$= 0. \qquad (2)$$

For $\star$ given by (1), the associator contains 6 terms at $\hbar$, 38 terms $\sim \hbar^2$, and 218 terms $\sim \hbar^3$. After the use of $\mathcal{P}^{ij} = -\mathcal{P}^{ji}$, we infer that $\mathsf{Assoc}(f,g,h)$ starts at $\hbar^2$ with 2/3 times (2). Next, there are 39 terms at $\hbar^3$; we now examine how their sum A vanishes by virtue of (2) and its differential consequences.[10] Of them, three which are the easiest to recognize are[11]

$$\tfrac{2}{3}\mathcal{P}^{ij}\,\mathrm{Jac}_{\mathcal{P}}(\partial_i f, \partial_j g, h) = \frac{2}{3}\cdot$$


$$= 0, \qquad (3)$$

as well as $\frac{2}{3}\mathcal{P}^{ij}\,\mathrm{Jac}_{\mathcal{P}}(f, \partial_i g, \partial_j h) = 0$ and $\frac{2}{3}\mathcal{P}^{ij}\,\mathrm{Jac}_{\mathcal{P}}(\partial_i f, g, \partial_j h) = 0$. So, there remain 30 terms which vanish via (2) in a way more intricate than (3). It is clear that

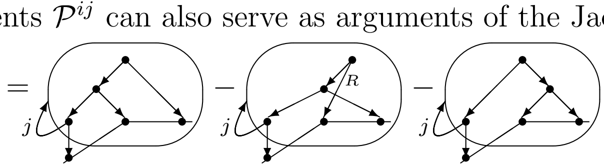
$$S_f := \mathcal{P}^{ij}\partial_j\,\mathrm{Jac}_{\mathcal{P}}(\partial_i f, g, h) =$$


$$= 0. \qquad (4)$$

Working out the Leibniz rule in (4), we collect the graphs according to the number of derivatives falling on each of (f,g,h). The edge $\xrightarrow{j}$ provides the differential orders[12] $(3,1,1)$, $(2,2,1)$, $(2,1,2)$, and $(2,1,1)$ twice. Likewise, we see $(1,1,1)$ in (2) and $(2,2,1)$ in (3).

Lemma. A tri-differential operator $\sum_{|I|,|J|,|K|\geqslant 0} c^{IJK}\ \partial_I \otimes \partial_J \otimes \partial_K$ vanishes identically iff all its homogeneous components vanish: $c^{IJK} = 0$ for every triple (I,J,K) of multi-indices; here $\partial_L = \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}$ for a multi-index $L = (\alpha_1, \ldots, \alpha_n)$. Moreover, the sums $\sum_{|I|=i,|J|=j,|K|=k} c^{IJK}\ \partial_I \otimes \partial_J \otimes \partial_K$ are then zero for all (i,j,k); in a vanishing sum X of graphs, we denote by X_{ijk} its vanishing restriction[13] to a fixed differential order (i,j,k).

The Poisson bi-vector components $\mathcal{P}^{ij}$ can also serve as arguments of the Jacobiator:[14]

$$I_f := \partial_j\bigl(\mathrm{Jac}_{\mathcal{P}}(\mathcal{P}^{ij}, g, h)\bigr)\,\partial_i f =$$


$$= 0.$$

Likewise, $I_g := \partial_i\bigl(\mathrm{Jac}_{\mathcal{P}}(f, \mathcal{P}^{ij}, h)\bigr)\,\partial_j g = 0$ and $I_h := \partial_i\bigl(\mathrm{Jac}_{\mathcal{P}}(f, g, \mathcal{P}^{ij})\bigr)\,\partial_j h = 0$. It is the expansion of I_f, I_g, I_h via the Leibniz rule that produces the graphs with "eyes". It also yields an order $(1,1,1)$ differential operator on (f,g,h) which cannot be obtained from (4).

---

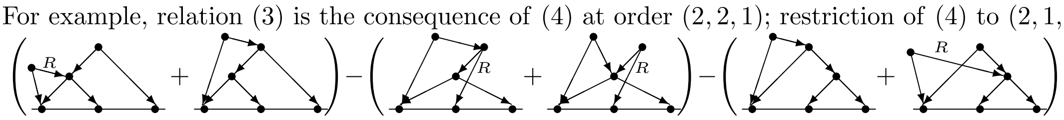
[9] By default, the $L \prec R$ edge ordering equals the left $\prec$ right direction in which edges start on these pages.

[10] Within the variational geometry of Poisson field models (cf. [2]), a tiny leak of the associativity for $\star$ may occur, if it does at all, only at orders $\hbar^{\geqslant 4}$ because at most *one* arrow falls on $\mathrm{Jac}_{\boldsymbol{\mathcal{P}}}(\cdot,\cdot,\cdot)$ in the balance $\mathsf{Assoc}(f,g,h) = \bar{o}(\hbar^3)$. But unlike the always vanishing first variation of a homologically trivial functional $\mathrm{Jac}_{\boldsymbol{\mathcal{P}}}(\cdot,\cdot,\cdot) \cong 0$, its higher-order variations can be nonzero.

[11] We use the Einstein summation convention; a sum over all indices is also implicit in the graph notation.

[12] In fact, the double edge to f contributes with zero at $(3,1,1)$ due to the skew-symmetry $\mathcal{P}^{ij} = -\mathcal{P}^{ji}$.

[13] For example, relation (3) is the consequence of (4) at order $(2,2,1)$; restriction of (4) to $(2,1,1)$ yields



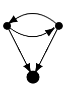
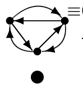
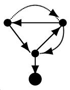
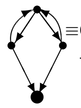
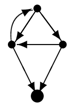
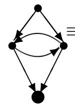
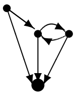
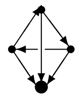
$$= 0.$$

Similarly, we have $S_g := \mathcal{P}^{ij}\partial_j\,\mathrm{Jac}_{\mathcal{P}}(f, \partial_i g, h) = 0$ and $S_h := \mathcal{P}^{ij}\partial_j\,\mathrm{Jac}_{\mathcal{P}}(f, g, \partial_i h) = 0$.

[14] The three tadpoles produce $\mathrm{Jac}_{\mathcal{P}}(\partial_i \mathcal{P}^{ij}, g, h)\,\partial_j f = 0$, which plays its rôle in A_{111} (see the claim below).

Claim. The sum A of 39 terms at $\hbar^3$ in $\mathsf{Assoc}\,(f,g,h)$ vanishes by virtue of restriction of S_f, S_g, S_h and I_f, I_g, I_h to the orders (i,j,k) that are present in A. Indeed, we have[15] $\mathsf{A}_{221} \overset{[3]}{=} \frac{2}{3}(S_f)_{221}$, $\mathsf{A}_{122} \overset{[3]}{=} \frac{2}{3}(S_g)_{122}$, and $\mathsf{A}_{212} \overset{[3]}{=} -\frac{2}{3}(S_h)_{212}$, see (3). Finally, we deduce that $\mathsf{A}_{111} \overset{[8]}{=} \frac{1}{6}(I_f - I_h)_{111}$, $\mathsf{A}_{112} \overset{[9]}{=} \big(\frac{1}{6}I_f + \frac{1}{6}I_g - \frac{1}{3}S_h\big)_{112}$, $\mathsf{A}_{121} \overset{[4]}{=} \frac{1}{3}(I_f - I_h)_{121}$, and $\mathsf{A}_{211} \overset{[9]}{=} \big(\frac{1}{3}S_f - \frac{1}{6}I_g - \frac{1}{6}I_h\big)_{211}$. The total number of terms which we thus eliminate equals $(3+3+3)+8+9+4+9 = 39$. □

Remark 1. The deformation quantization is a gauge theory: each argument $\bullet$ of $\star$ marks its gauge class $[\bullet]$ under the linear maps $\mathsf{t}\colon \bullet \mapsto [\bullet] = \bullet + \hbar\big(I^{\emptyset}\,\partial_i\partial_j(\mathcal{P}^{ij})^{\equiv 0} \times \bullet + I^{\circlearrowleft}\,\partial_i\mathcal{P}^{ij}\,\partial_j(\bullet)\big) +$

$\hbar^2 \overset{(0)}{I}$  $+\hbar^3 \Big[\overset{(1)}{I}$  $+ \overset{(2)}{I}$  $+ \overset{(3)}{I}$  $+ \overset{(4)}{I}$  $+ \overset{(5)}{I}$  $+ \overset{(6)}{I}$  $+ \overset{(7)}{I}$  $\Big]$

$+\bar{o}(\hbar^3)$, where the constants $\overset{(\alpha)}{I} \in \Bbbk$ can be arbitrary[16] and t is formally invertible over $\Bbbk[[\hbar]]$. In turn, the star-products are gauged[17] by using t: $f \star' g := \mathsf{t}^{-1}\big(\mathsf{t}(f) \star \mathsf{t}(g)\big)$. This degree of freedom extends the uniqueness problem for Kontsevich's solution $\star$ of $\mathsf{Assoc}\,(f,g,h) = 0$. Namely, not the exact balance of power series but an equivalence $[=]$ of gauge classes (up to unrelated transformations at all steps) can be sought in $\big[[f] \star [g]\big] \star [h] \,[=]\, [f] \star \big[[g] \star [h]\big]$.

Remark 2. Each graph Γ in (1) encodes the polydifferential operator of scalar arguments in a coordinate-free way. The Jacobians $\partial\boldsymbol{u}/\partial\tilde{\boldsymbol{u}}$ of *affine* mappings appear on the edges but then they join the content $\hbar\mathcal{P}^{ij}$ of internal vertices at the arrowtails,[18] forming $\tilde{\mathcal{P}}^{\alpha\beta}$ from $\mathcal{P}^{ij}$. Independent from $\boldsymbol{u} \in N^n$, these Jacobians stay invisible to all in-coming arrows (if any). So, the operator given by a graph Γ with $\hbar\mathcal{P}(\boldsymbol{u})$ in its vertices is equal to the one for $\hbar\tilde{\mathcal{P}}\big(\tilde{\boldsymbol{u}}(\boldsymbol{u})\big)$ there. This reasoning works for the variational Poisson brackets $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ on $J^\infty(\pi)$ for affine bundles π with fibre N^n over points $\boldsymbol{x} \in M^m$, see [2]. The graphs Γ then yield local variational polydifferential operators yet the pictorial language of [1] is the same.[19]

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[15] By using the symbol $\overset{[m]}{=}$ we indicate the number m of terms that are eliminated at each step.

[16] The view [3] on $\star$-products as $\hbar$-expansions of path integrals shows that the graphs Γ_i in (1) are genuine Feynman diagrams for the channel marked by $\mathcal{P}$. The weights $w(\Gamma_i)$ integrate over the energy of each intermediate vertex. Quite naturally, a particle $\bullet$ shares its energy-mass with the interaction carriers $\mathcal{P}$ as it gets coated by them. But no object $\bullet$ can spend more energy on growing its gauge tail than the amount it actually has; hence every set $[\bullet]$ is bounded in the space of parameters $\boldsymbol{I}$.

[17] For example, the loop graph at $\hbar^2/6$ in (1) is gauged out by $\mathsf{t}(\bullet) = \bullet + \frac{\hbar^2}{12}$  , see [1] for further details.

[18] E.g., $\overleftarrow{\partial'_\alpha}\tilde{\mathcal{P}}^{\alpha\beta}\big|_{\tilde{\boldsymbol{u}}}\overrightarrow{\partial'_\beta} = \overleftarrow{\partial_i}\,\frac{\partial u^i}{\partial\tilde{u}^\alpha}\,\tilde{\mathcal{P}}^{\alpha\beta}\big|_{\tilde{\boldsymbol{u}}(\boldsymbol{u})}\,\frac{\partial u^j}{\partial\tilde{u}^\beta}\,\overrightarrow{\partial_j} = \overleftarrow{\partial_i}\cdot\mathcal{P}^{ij}\big|_{\boldsymbol{u}}\cdot\overrightarrow{\partial_j}$ so that $\{f,g\}_{\mathcal{P}(\boldsymbol{u})}(\boldsymbol{u}) = \{f,g\}_{\tilde{\mathcal{P}}(\tilde{\boldsymbol{u}}(\boldsymbol{u}))}\big(\tilde{\boldsymbol{u}}(\boldsymbol{u})\big)$.

[19] A sought-for extension of the Ikeda–Izawa topological open string geometry − namely, its lift from the Poisson manifolds $\big(N^n, \{\,,\,\}_{\mathcal{P}}\big)$ in [3, 4] to the variational set-up $\big(J^\infty(\pi), \{\,,\,\}_{\boldsymbol{\mathcal{P}}}\big)$ of jet spaces in [2] − is a mechanism to quantize Poisson field models. This will be the object of another paper.

THE KONTSEVICH TETRAHEDRAL FLOWS REVISITED

A. BOUISAGHOUANE, R. BURING, AND A. KISELEV*

ABSTRACT. We prove that the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$, the right-hand side of which is a linear combination of two differential monomials of degree four in a bi-vector $\mathcal{P}$ on an affine real Poisson manifold N^n, does infinitesimally preserve the space of Poisson bi-vectors on N^n if and only if the two monomials in $\mathcal{Q}_{a:b}(\mathcal{P})$ are balanced by the ratio $a : b = 1 : 6$. The proof is explicit; it is written in the language of Kontsevich graphs.

Introduction. The main question which we address in this paper is how Poisson structures can be deformed in such a way that they stay Poisson. We reveal one such method that works for all Poisson structures on affine real manifolds; the construction of that flow on the space of bi-vectors was proposed in [14]: the formula is derived from two differently oriented tetrahedral graphs over four vertices. The flow is a linear combination of two terms, each quartic-nonlinear in the Poisson structure. By using several examples of Poisson brackets with high polynomial degree coefficients, we demonstrated in [1] that the ratio $1 : 6$ is the only possible balance at which the tetrahedral flow can preserve the property of the Cauchy datum to be Poisson. But does the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ with ratio $1 : 6$ actually preserve the space of *all* Poisson bi-vectors? In dimension 3 the description of Poisson brackets with smooth coefficients is known from [6]; a brute force calculation then verifies the claim. In this paper we prove the claim in full generality, namely, for all Poisson structures on all affine manifolds of arbitrary finite dimension. The proof is graphical: namely, to prove that equation (1) holds, we find an operator $\Diamond$, encoded by using the Kontsevich graphs, that solves the equation (8). (As soon as solution (9) is obtained, verifying that it does satisfy the determining equation (8) is elementary though tedious.[1]) The first by-product of our proof is that there is no universal mechanism (that would involve the language of Kontsevich graphs) for the tetrahedral flow to be trivial in the respective Poisson cohomology. Secondly, the factorization mechanism, on which the proof of Theorem 3 is based, explains in hindsight why the proven property of tetrahedral flows is false for the variational Poisson brackets. (This was observed empirically in [1]; the geometry of Poisson structures over jet bundles is known from [19].)

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[1]Having a solution $\Diamond$ to equation (8) is analogous to having a rational point on an elliptic curve: finding either is hard, though verifying that it does satisfy the equation at hand is almost immediate.

The text is structured as follows. In section 1 we recall how oriented graphs can be used to encode differential operators acting on the space of multivectors. In particular, differential polynomials in a given Poisson structure are obtained in the frames of this approach as soon as a copy of that Poisson bi-vector is placed in every internal vertex of a graph. Specifically, the right-hand side $\mathcal{Q}_{a:b} = a \cdot \Gamma_1 + b \cdot \Gamma_2$ of the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ on the space of bi-vectors on an affine Poisson manifold $\big(N^n, \mathcal{P}\big)$ is a linear combination of two differential monomials, $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$, of degree four in the bi-vector $\mathcal{P}$ that evolves.

In this paper we find out at which balance $a : b$ the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ infinitesimally preserves the space of Poisson bi-vectors, that is, the bi-vector $\mathcal{P} + \epsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\epsilon)$ satisfies the equation

$$[\![\mathcal{P} + \epsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\epsilon), \mathcal{P} + \epsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\epsilon)]\!] = \bar{o}(\epsilon) \qquad \text{via } [\![\mathcal{P}, \mathcal{P}]\!] = 0;$$

here we denote by $[\![\cdot,\cdot]\!]$ the Schouten bracket (see formula (11) on page 17; relevant cohomological techniques are reviewed in Appendix A). Expanding, we obtain the cocycle condition,

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] \doteq 0 \qquad \text{via } [\![\mathcal{P}, \mathcal{P}]\!] = 0, \tag{1}$$

with respect to the Poisson differential $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$. Viewed as an equation with respect to the ratio $a : b$, condition (1) is the main object of our study.

Recent counterexamples [1] show that the bi-vector $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ stays infinitesimally Poisson *only if* the balance $a : b$ in $\mathcal{Q}_{a:b}$ is equal to $1 : 6$. (Without extra assumptions, the infinitesimal deformation $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ can be completed to a finite deformation $\mathcal{P}(\varepsilon)$ at $\varepsilon > 0$ if the third Poisson cohomology group $H^3_{\mathcal{P}}(N^n)$ with respect to the differential $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ vanishes for the Poisson manifold $\big(N^n, \mathcal{P}\big)$. Therefore, unlike the Kontsevich formula for the flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ which is universal for all N^n and $\mathcal{P}$, the integration issue is Poisson model-dependent.)

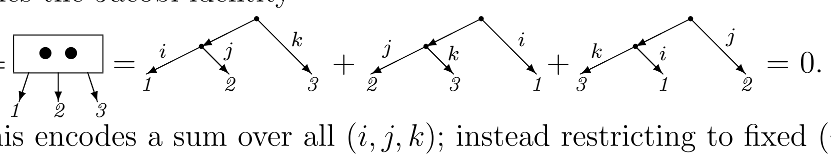
We now prove that the balance $a : b = 1 : 6$ in the Kontsevich tetrahedral flow is universal in the above sense: for all Poisson bi-vectors $\mathcal{P}$ on every affine manifold N^n, the deformation $\mathcal{P} + \varepsilon\mathcal{Q}_{1:6}(\mathcal{P}) + \bar{o}(\varepsilon)$ is infinitesimally Poisson. The proof is explicit: in section 2 we reveal the mechanism of factorization – via the Jacobi identity – in (1) at $a : b = 1 : 6$. Specifically, we find a linear polydifferential operator $\Diamond(\mathcal{P}, \cdot)$ that acts on the filtered components (see below) of the Jacobiator $\text{Jac}\,(\mathcal{P}) := [\![\mathcal{P}, \mathcal{P}]\!] = 0$ for the bi-vector $\mathcal{P}$; the operator $\Diamond$ provides the factorization $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond(\mathcal{P}, \text{Jac}\,(\mathcal{P}))$ of the $\boldsymbol{\partial}_{\mathcal{P}}$-cocycle condition, see (1), through the Jacobiator $\text{Jac}\,(\mathcal{P}) = 0$. On the one side of factorization problem (1) we expand the Poisson differential of the Kontsevich tetrahedral flow at the balance $1 : 6$ into the sum of 39 graphs (see Figure 3 on page 6 and Table 1 in Appendix D). On the other side of that factorization, we take the sum that runs with undetermined coefficients over all those fragments of differential consequences of the Jacobi identity $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ which are known to vanish independently. In our reasoning the differential consequences of the identity $\text{Jac}\,(\mathcal{P}) := [\![\mathcal{P}, \mathcal{P}]\!] = 0$ for Poisson bi-vectors are filtered up to order three according to the differential orders (k, ℓ, m), $k + \ell + m \geqslant 3$, with respect to the arguments of the tri-vector $[\![\mathcal{P}, \mathcal{P}]\!]$. We recall that every differential consequence of order (k, ℓ, m) for the Jacobi identity $\text{Jac}\,(\mathcal{P}) = 0$ then vanishes. To describe the differential operators that produce such consequences of the Jacobi identity, we use the pictorial language of graphs: every internal vertex contains a copy of the bi-vector $\mathcal{P}$ and the operators are reduced by using its skew-symmetry.

The resulting sum of graphs is reduced modulo the skew-symmetry of the bi-vector at hand; there remain $7,025$ graphs, the coefficients of which are linear in the unknowns. We now solve the arising inhomogeneous linear algebraic system. Its solution yields the polydifferential operator $\Diamond$, encoded using graphs (see p. 11), that provides the sought-for factorization $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$. It is readily seen from formula (9) that the operator $\Diamond$ is completely determined by only 8 nonzero coefficients (out of 1132, see their count in Appendix C). Therefore, although finding the operator $\Diamond$ was hard, verifying that it does solve the factorization problem has become almost immediate. This completes the proof of Theorem 3 and establishes the main result (namely, Corollary 4 on page 4). In section 3 we analyze the properties of the solution at hand. (The maximally detailed description of that solution $\Diamond$ is contained in Appendix D.) The paper concludes with a list of open problems and five appendices.

In Appendix E we outline a different method to tackle the factorization problem, namely, by making the Jacobi identity visible in (1) by perturbing the original structure $\mathcal{P}$ so that it stops being Poisson. Hence it contributes to the right-hand side of (1) such that the respectively perturbed bi-vector $\mathcal{Q}_{1:6}(\mathcal{P})$ stops being compatible (in the sense of (1)) with the perturbed Poisson structure. The first-order balance of both sides of perturbed equation (1) then suggests the coefficients of those differential consequences of the Jacobiatior which are actually involved in the factorization mechanism. The coefficients of operators realized by graphs which were found by following this scheme are reproduced in the full run-through that gave us the solution $\Diamond$ in section 2.

1. The main problem: From graphs to multivectors

1.1. **The language of graphs.** Let us formalise a way to encode polydifferential operators – in particular multivectors – using oriented graphs. In an affine real manifold N^n (here $2 \leqslant n < \infty$), consider a chart $U_\alpha \hookrightarrow \mathbb{R}^n$ and denote the Cartesian coordinates by $\boldsymbol{x} = (x^1, \ldots, x^n)$. By definition, the decorated edge $\bullet \xrightarrow{i} \bullet$ denotes at once the derivation $\partial/\partial x^i \equiv \partial_i$ (that acts on the content of the arrowhead vertex) and the summation $\sum_{i=1}^n$ (over the index i in the object which is contained within the arrowtail vertex). As it has been explained in [8, 10, 15], the operator which every graph encodes is equal to the sum (running over all the indexes) of products (running over all the vertices) of those vertices content (differentiated by the in-coming arrows, if any). For example, the graph $(\mathit{1}) \xleftarrow[L]{i} \mathcal{P}^{ij}(\boldsymbol{x}) \xrightarrow[R]{j} (\mathit{2})$ encodes the bi-differential operator $\sum_{i,j=1}^n (\mathit{1}) \overleftarrow{\partial_i} \cdot \mathcal{P}^{ij}(\boldsymbol{x}) \cdot \overrightarrow{\partial_j} (\mathit{2})$. It then specifies the Poisson bracket on the chart $U_\alpha \subset N^n$. The bracket satisfies the Jacobi identity

$\mathrm{Jac}(\mathcal{P})(\mathit{1},\mathit{2},\mathit{3}) =$  $= 0.$ (2)

In our notation this encodes a sum over all (i, j, k); instead restricting to fixed (i, j, k) corresponds to taking a coefficient of the differential operator (cf. Lemma 5), which yields the respective component of the Jacobiator. Clearly, the Jacobiator $\mathrm{Jac}(\mathcal{P})$ is totally skew-symmetric with respect to its arguments *1, 2, 3*.

From now on, let us consider only the oriented graphs whose vertices are either sinks (with no issued edges) like the vertices *1, 2, 3* in (2) or tails for an ordered pair of arrows, each decorated with its own index. (We refer to Jacobi identity (2), Figure 2 on p. 5, and to the graphical formulae on pp. 4, 10, 11, and 14). By definition, the arrowtail vertices are called *internal.* For each internal vertex $\bullet$, the pair of out-going edges is ordered L $\prec$ R. Next, every internal vertex $\bullet$ carries a copy of a given Poisson bi-vector $\mathcal{P} = \mathcal{P}^{ij}(\boldsymbol{x})\,\partial_i \wedge \partial_j$ with its own pair of indices. The ordering L $\prec$ R of decorated out-going edges coincides with the ordering "first $\prec$ second" of the indexes in the coefficients of $\mathcal{P}$. Namely, the left edge (L) carries the first index and the other edge (R) carries the second index. Moreover, we let the sinks be ordered (like *1, 2, 3* above), so that every such graph defines a polydifferential operator: its arguments are thrown into the respective sinks.

Remark 1. In principle, we allow the presence of both the tadpoles and cycles over two vertices (or "eyes"); see Figure 1. However, in hindsight there will be neither tadpoles nor eyes in the solution which we shall have found in section 2 below.

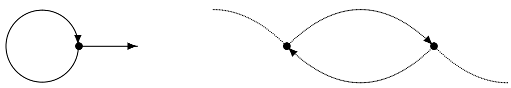
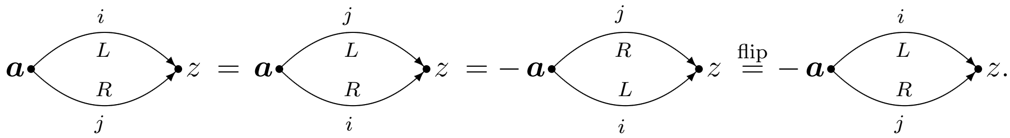
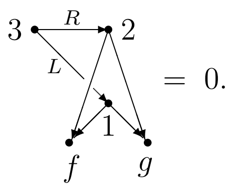


FIGURE 1. A tadpole and an "eye".

Remark 2. Under the above assumptions, there exist inhabited graphs that encode zero differential operators. Namely, consider the graph with a double edge:



By first relabelling the summation indices and then swapping L $\rightleftarrows$ R (and redrawing) we evaluate the operator acting at z to $\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j(z) = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j(z)$; whence the operator is zero. In the same way, any graph containing a double edge encodes a zero operator. Graphs can also encode zero differential operators in a more subtle way. For example consider the wedge on two wedges:



(3)

Swapping the labels 1 $\rightleftarrows$ 2 of the lower wedges does not change the operator. On the other hand, doing the same in a different way, namely, by swapping 'left' and 'right' in the top wedge introduces a minus sign. Hence the graph encodes a differential operator equal to minus itself, i.e. zero. Proving that a graph which contains the left-hand side of (3) as a subgraph equals zero is an elementary exercise (cf. Example 2 on p. 22).

Besides the trivial vanishing mechanism in Remark 2, there is Jacobi identity (2) together with its differential consequences, which will play a key role in what follows.

1.2. **The Kontsevich tetrahedral flow.** In the paper [14], Kontsevich proposed the construction of flows $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ on the spaces of Poisson structures on affine real manifolds. In particular, he associated one such flow on the space of Poisson bi-vectors $\mathcal{P}$ with the full graph over four vertices, that is, the tetrahedron. Up to symmetry, there are two essentially different ways, resulting in Γ_1 and Γ_2', to orient its edges, provided that every vertex is a source for two arrows and, as an elementary count suggests, there are two arrows leaving the tetrahedron that act on the arguments of the bi-differential operator which the tetrahedral graph encodes. The two oriented tetrahedral graphs are shown in Fig. 2. Unlike the operator encoded by Γ_1, that of Γ_2'

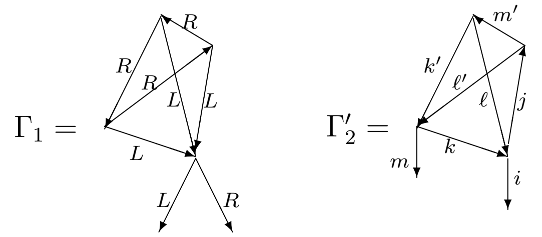


FIGURE 2. The Kontsevich tetrahedral graphs encode two bi-linear bi-differential operators on the product $C^\infty(N^n) \times C^\infty(N^n)$.

is generally speaking not skew-symmetric with respect to its arguments. By definition, put $\Gamma_2 := \frac{1}{2}(\Gamma_2'(1,2) - \Gamma_2'(2,1))$ to extract the antisymmetric part, that is, the bi-vector encoded by Γ_2'. Explicitly, the quartic-nonlinear differential polynomials $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$, depending on a Poisson bi-vector $\mathcal{P}$, are given by the formulae

$$\Gamma_1(\mathcal{P}) = \sum_{i,j=1}^{n} \Bigg(\sum_{k,\ell,m,k',\ell',m'=1}^{n} \frac{\partial^3 \mathcal{P}^{ij}}{\partial x^k \partial x^\ell \partial x^m} \frac{\partial \mathcal{P}^{kk'}}{\partial x^{\ell'}} \frac{\partial \mathcal{P}^{\ell\ell'}}{\partial x^{m'}} \frac{\partial \mathcal{P}^{mm'}}{\partial x^{k'}} \Bigg) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{4a}$$

and

$$\Gamma_2(\mathcal{P}) = \sum_{i,m=1}^{n} \Bigg(\sum_{j,k,\ell,k',\ell',m'=1}^{n} \frac{\partial^2 \mathcal{P}^{ij}}{\partial x^k \partial x^\ell} \frac{\partial^2 \mathcal{P}^{km}}{\partial x^{k'} \partial x^{\ell'}} \frac{\partial \mathcal{P}^{k'\ell}}{\partial x^{m'}} \frac{\partial \mathcal{P}^{m'\ell'}}{\partial x^j} \Bigg) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^m}, \tag{4b}$$

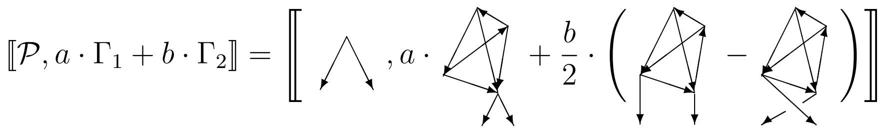
respectively. To construct a class of flows on the space of bi-vectors, Kontsevich suggested to consider linear combinations, balanced by using the ratio $a : b$, of the bi-vectors Γ_1 and Γ_2. We recall from section 1.1 that every internal vertex of each graph is inhabited by a copy of a given Poisson bi-vector $\mathcal{P}$, so that the linear combination of two graphs encodes the bi-vector $\mathcal{Q}_{a:b}(\mathcal{P}) = a \cdot \Gamma_1(\mathcal{P}) + b \cdot \Gamma_2(\mathcal{P})$, quartic in $\mathcal{P}$ and balanced using $a : b$. We now inspect at which ratio $a : b$ the bi-vector $\mathcal{P} + \varepsilon \mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ stays infinitesimally Poisson for $\varepsilon > 0$, that is (cf. Appendix A),

$$[\![\mathcal{P} + \varepsilon \mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon), \mathcal{P} + \varepsilon \mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)]\!] = \bar{o}(\varepsilon). \tag{5}$$

Expanding the left-hand side of equation (5), using the shifted-graded skew-symmetry of the Schouten bracket $[\![\cdot,\cdot]\!]$, and taking into account that $[\![\mathcal{P},\mathcal{P}]\!] = 0$ if and only if $\mathcal{P}$ is Poisson, we extract the equation

$$[\![\mathcal{P}, \mathcal{Q}_{a:b}(\mathcal{P})]\!] \doteq 0 \qquad \text{via } [\![\mathcal{P},\mathcal{P}]\!] = 0. \tag{1}$$

The left-hand side of equation (1) can be seen in terms of graphs:



(6)

Remark 3. The graphical calculation of the Schouten bracket $[\![\ ,\ ]\!]$ of two arguments amounts to the action –via the Leibniz rule– of every out-going edge in an argument on all the internal vertices in the other argument. For the Schouten bracket of a k-vector with an ℓ-vector, the rule of signs is this. For the sake of definition, enumerate the sinks in the first and second arguments by using $0, \ldots, k-1$ and $0, \ldots, \ell-1$, respectively. Then the arrow into the jth sink in the second argument acts on the internal vertices of the first argument, acquiring the sign factor $(-)^j$; here $0 \leqslant j < \ell$. On the other hand, the arrow to the ith sink in the first argument acts on the second argument's internal vertices with the sign factor $-(-)^{(k-1)-i}$ for $0 \leqslant i \leqslant k-1$. We finally recall that having a totally antisymmetric tri-vector in (6) means that a full skew-symmetrization over the three sinks' content is taken by using $\frac{1}{3!}\sum_{\sigma\in S_3}(-)^\sigma$.

For example, let $a : b = 1 : 6$ (specifically, $a = 1$ and $b = 6$). Then the left-hand side of (1) takes the shape depicted in Figure 3. After the skew-symmetrization and

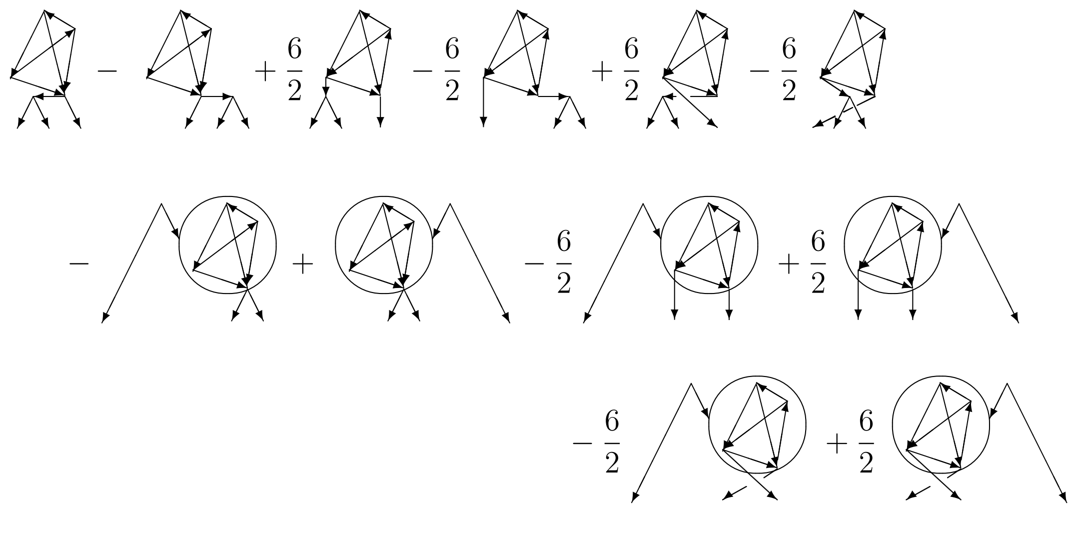


FIGURE 3. Incoming arrows act on the content of boxes via the Leibniz rule; to obtain the tri-vector, the entire picture must be skew-symmetrized over the content of three sinks.

expansion of all Leibniz rules, the sum in Figure 3 simplifies to 39 graphs; they are listed in Table 1 on p. 21 below.

The reason why we are particularly concerned with the ratio $a : b = 1 : 6$ is that this condition is *necessary* for equation (1) to hold.

Proposition 1 ([1])**.** The tetrahedral flow $\dot{P} = \mathcal{Q}_{a:b}(\mathcal{P})$ preserves the property of $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ to be infinitesimally Poisson for all Poisson bi-vectors $\mathcal{P}$ on all affine real manifolds N^n *only if* the ratio is $a : b = 1 : 6$.

Our proof amounts to producing at least one counterexample when any ratio other than $1 : 6$ violates equation (1) for a given Poisson bi-vector $\mathcal{P}$.

Proof. Let x, y, z be the Cartesian coordinates on $\mathbb{R}^3$. Consider the Poisson bracket $\{u, v\}_{\mathcal{P}} = x \cdot \det\bigl(\partial(xyz + y, u, v)/\partial(x, y, z)\bigr)$ given by the Jacobian, so that the coefficient matrix is

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & x^2y & -x(xz+1) \\ -x^2y & 0 & xyz \\ -x(xz+1) & -xyz & 0 \end{pmatrix}.$$

The coefficient matrices of both bi-vectors are

$$\Gamma_1(\mathcal{P}) = 6 \cdot \begin{pmatrix} 0 & -x^5y & -x^4(xz+1) \\ x^5y & 0 & -x^3y \\ x^4(xz+1) & x^3y & 0 \end{pmatrix}, \qquad \Gamma_2(\mathcal{P}) = \begin{pmatrix} 0 & x^5y & x^4(xz+2) \\ -x^5y & 0 & -2x^3y \\ -x^4(xz+2) & 2x^3y & 0 \end{pmatrix}.$$

It is readily seen that no non-trivial linear combination $a \cdot \Gamma_1(\mathcal{P}) + b \cdot \Gamma_2(\mathcal{P})$ of the two flows vanishes everywhere on $\mathbb{R}^3 \ni (x, y, z)$ for this example. Acting on the bi-vectors Γ_1 and Γ_2 by the Poisson differential $[\![\mathcal{P}, \cdot]\!]$, we obtain two tri-vectors which are completely determined by one component each. Namely, we have that

$$[\![\mathcal{P}, \Gamma_1(\mathcal{P})]\!]^{123} = 36x^6yz + 48x^5y, \qquad [\![\mathcal{P}, \Gamma_2(\mathcal{P})]\!]^{123} = -6x^6yz - 8x^5y.$$

Clearly, the balance $a : b = 1 : 6$ is the only ratio at which the non-trivial linear combination $\mathcal{Q}_{a:b}(\mathcal{P}) = a \cdot \Gamma_1(\mathcal{P}) + b \cdot \Gamma_2(\mathcal{P})$ solves the equation $[\![\mathcal{P}, \mathcal{Q}_{a:b}(\mathcal{P})]\!] \equiv 0$. $\square$

1.3. **Main result.** In fact, more is known — this time, about the sufficiency of the condition $a : b = 1 : 6$. First, let us recall from [6] that on $\mathbb{R}^3$ with coordinates x, y, and z (almost) all Poisson brackets amount to

$$\{u, v\}_{\mathcal{P}} = f \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x, y, z)}\right) \quad \text{for } u, v \in C^\infty(\mathbb{R}^3), \tag{7}$$

where the free parameter g is a function and the parameter f is a density so that

$$f(x, y, z) \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x, y, z)}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(x, y, z)\Big|_{\substack{x=x(x',y',z') \\ y=y(x',y',z') \\ z=z(x',y',z')}} \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x', y', z')}\right) \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'.$$

In any given coordinate system the parameter f can be chosen freely; then it is recalculated as shown above.

Proposition 2 ($\mathbb{R}^3$,$\{\cdot, \cdot\}_{\mathcal{P}}$)**.** The tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ does preserve the property of $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ to be infinitesimally Poisson for all Poisson structures (7) on $\mathbb{R}^3$.

We use Proposition 2 merely as an heuristic motivation to our main Theorem 3 (see below) in which the claim from Proposition 2 is extended to *all* Poisson structures on all finite-dimensional affine real manifolds. Therefore, in hindsight, Proposition 2 above will have been proven rigorously as soon as Theorem 3 is established by the end of the next section. (In the meantime, a computer-assisted proof by direct calculation is provided for Proposition 2 in Appendix B.)

So, let us no longer restrict the tetrahedral flow $\mathcal{Q}_{1:6}(\mathcal{P})$ to any specific class of Poisson bi-vectors $\mathcal{P}$ but let us work in the full generality. We now examine the mechanism for the tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!]$ in (1) to vanish by virtue of the Jacobi identity $\operatorname{Jac}(\mathcal{P}) := [\![\mathcal{P}, \mathcal{P}]\!] = 0$ for a given Poisson bi-vector $\mathcal{P}$ on N^n of any dimension $n \geqslant 3$. The task is to factorize the content of Figure 3 through the Jacobi identity in (2).

Theorem 3. *There exists a polydifferential operator*

$$\Diamond \in \operatorname{PolyDiff}\bigl(\Gamma(\textstyle\bigwedge^2 TN^n) \times \Gamma(\bigwedge^3 TN^n) \to \Gamma(\bigwedge^3 TN^n)\bigr)$$

which solves the factorization problem

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond\bigl(\mathcal{P}, \operatorname{Jac}(\mathcal{P})\bigr). \tag{8}$$

The polydifferential operator $\Diamond$ is realised using graphs in formula (9)*, see p.* 11 *below.*

Remark 4. Whenever a solution $\Diamond$ of (8) is found – and if it contains a reasonably small number of Leibniz-rule graphs as, e.g., our solution (9), see page 11 below – one can verify the factorization in (8) by a straightforward calculation. Indeed, by expanding the Leibniz rules and collecting similar terms, one obtains 39 graphs from the left-hand side (see Figure 3 and the encoding of those graphs in Table 1 on page 21).

Corollary 4 (Main result)**.** Whenever a bi-vector $\mathcal{P}$ on an affine real manifold N^n is Poisson, the deformation $\mathcal{P} + \varepsilon\mathcal{Q}_{1:6}(\mathcal{P}) + \bar{o}(\varepsilon)$ using the Kontsevich tetrahedral flow is infinitesimally Poisson.

Remark 5. It is readily seen that the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ is well defined on the space of Poisson bi-vectors on a given affine manifold N^n. Indeed, it does not depend on a choice of coordinates up to their arbitrary affine reparametrisations. In other words, the velocity $\dot{\mathcal{P}}\big|_{\boldsymbol{u}\in N^n}$ does not depend on the choice of a chart $\mathcal{U} \ni \boldsymbol{u}$ from an atlas in which only *affine* changes of variables are allowed. (Let us remember that affine manifolds can of course be topologically nontrivial.)

Suppose however that a given affine structure on the manifold N^n is extended to a larger atlas on it; for the sake of definition let that atlas be a smooth one. Assume that the smooth structure is now reduced – by discarding a number of charts – to another affine structure on the same manifold. The tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ which one initially had can be contrasted with the tetrahedral flow $\dot{\tilde{\mathcal{P}}} = \mathcal{Q}_{1:6}(\tilde{\mathcal{P}})$ which one finally obtains for the Poisson bi-vector $\tilde{\mathcal{P}}\big|_{\tilde{\boldsymbol{u}}(\boldsymbol{u})} = \mathcal{P}\big|_{\boldsymbol{u}}$ in the course of a nonlinear change of coordinates on N^n. Indeed, the respective velocities $\dot{\mathcal{P}}$ and $\dot{\tilde{\mathcal{P}}}$ can be different whenever they are expressed by using essentially different parametrisations of a neighbourhood of a point $\boldsymbol{u}$ in N^n. For example, the tetrahedral flow vanishes identically when expressed in the Darboux canonical variables on a chart in a symplectic manifold. But after a nonlinear canonical transformation, the right-hand side $\mathcal{Q}_{1:6}(\tilde{\mathcal{P}})$ can become nonzero at the same points of that Darboux chart.

This shows that an affine structure on the manifold N^n is a necessary part of the input data for construction of the Kontsevich tetrahedral flows $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$.

2. Solution: From graphs to polydifferential operators

Expanding the Leibniz rules in $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!]$, we obtain the sum of 39 graphs with 5 internal vertices and 3 sinks (so that from Figure 3 we produce Table 1, see page 21 below). By construction, the Schouten bracket $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] \in \Gamma(\bigwedge^3 TN^n)$ is a tri-vector on the underlying manifold N^n, that is, it is a totally antisymmetric tri-linear polyderivation $C^\infty(N^n)\times C^\infty(N^n)\times C^\infty(N^n) \to C^\infty(N^n)$. At the same time, we seek to recognize the tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!]$ as the result of application of the (poly)differential operator $\lozenge$ (see (8) in Theorem 3) to the Jacobiator $\operatorname{Jac}(\mathcal{P})$ (see (2) on p. 3).

We now explain how the operator $\lozenge$ is found by using the method of undetermined coefficients in an expansion of all relevant graphical differential consequences of the Jacobi identity.[2] By construction, the left-hand side of every such differential consequence is a sum of graphs with 5 internal vertices, of which 2 belong to the Jacobiator $\operatorname{Jac}(\mathcal{P})$. We recall that for strictly positive differential order consequences of the Jacobi identity $\operatorname{Jac}(\mathcal{P}) = 0$, the mechanism for operator $\lozenge$ to attain zero value at $\operatorname{Jac}(\mathcal{P}) = 0$ is nontrivial. In fact, it refers to a (possibility of) splitting of every such consequence into the fragments which vanish independently from each other.

Lemma 5 ([2])**.** A tri-differential operator $C = \sum_{|I|,|J|,|K|\geqslant 0} c^{IJK}\, \partial_I \otimes \partial_J \otimes \partial_K$ with coefficients $c^{IJK} \in C^\infty(N^n)$ vanishes identically iff all its homogeneous components $C_{ijk} = \sum_{|I|=i,|J|=j,|K|=k} c^{IJK}\, \partial_I \otimes \partial_J \otimes \partial_K$ vanish for all differential orders (i,j,k) of the respective multi-indices (I,J,K); here $\partial_L = \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}$ for a multi-index $L = (\alpha_1,\ldots,\alpha_n)$.

In practice, Lemma 5 states that for every arrow falling on the Jacobiator (for which, in turn, a triple of arguments is specified), the expansion of the Leibniz rule yields four fragments which vanish separately. Namely, there is the fragment such that the derivation acts on the content $\mathcal{P}$ of the Jacobiator's two internal vertices, and there are three fragments such that the arrow falls on the first, second, or third argument of the Jacobiator. It is readily seen that the action of a derivative on an argument of the Jacobiator effectively amounts to an appropriate redefinition of its respective argument. Therefore, a restriction to the order $(1,1,1)$ is enough in the run-through over all the graphs which contain Jacobiator (2) and which stand on the three arguments f, g, h of the tri-vector $\lozenge(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$.

Remark 6. In all the above reasoning, the set $\{\mathit{1}, \mathit{2}, \mathit{3}\}$ of three arguments of the Jacobiator need not coincide with the set $\{f, g, h\}$ of the arguments of the tri-vector $\lozenge(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$. Of course, the two sets can intersect; this will provide a natural filtration for the components of solution (9). Namely, the number of elements in the intersection runs from three for the first term to zero in the second or third graph.

In fact, Remark 6 reveals a highly nontrivial role of the operator $\lozenge$ in (8). Indeed, some of the three internal vertices of its graphs can be arguments of $\operatorname{Jac}(\mathcal{P})$ whereas some of the other such vertices (if any) can be tails for the arrows falling on $\operatorname{Jac}(\mathcal{P})$. In retrospect, the two subsets of such vertices of $\lozenge$ do not intersect; every vertex in the intersection, if it were nonempty, would produce a two-cycle, but there are no "eyes" in (9).

[2]Another method for solving the factorization problem is outlined in Appendix E.

By ordering the Leibniz-rule graphs in the operator $\Diamond$ according to the number of Jacobiator's arguments which simultaneously are the arguments of (totally skew-symmetric) tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$, we count the number of variants in the run-through over all the admissible graphs. (With reference to Fig. 4 below, this is done in Appendix C, see p. 19.) In total, there are 1132 variants.

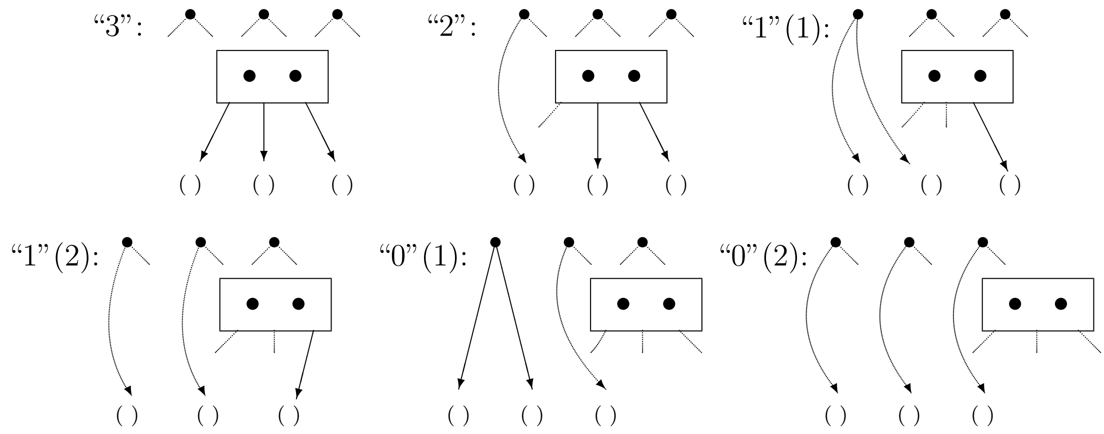


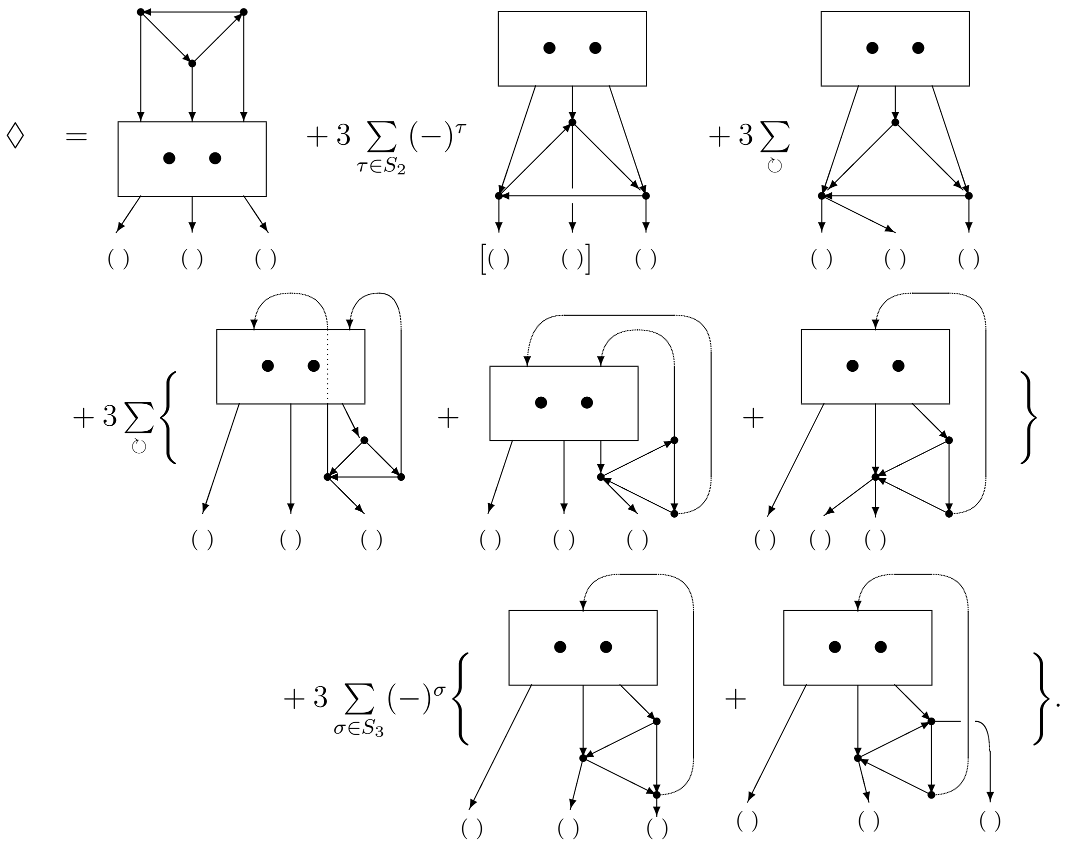
FIGURE 4. This is the list of all different types of differential consequences of the Jacobi identity which are linear in the Jacobiator and which are totally skew-symmetric with respect to the sinks. The list is ordered by the number of ground vertices on which the Jacobiator stands. The number of graphs for each type is deduced in Appendix C: namely, from top-left to bottom-right, there are 216, 432, 108, 288, 24, and 64 Leibniz-rule graphs. The total number of differential consequences is 1132.

We now split all these differential consequences of the Jacobi identity $\mathrm{Jac}(\mathcal{P}) = 0$ by using Lemma 5 (with respect to the *total* differential order (i, j, k) for arguments of $\mathrm{Jac}(\mathcal{P})$ if more than one arrow falls on it), ascribing an undetermined coefficient to every such separately vanishing fragment. That is, we do not restrict only to the differential order (1,1,1) with respect to the arguments of $\mathrm{Jac}(\mathcal{P})$ for every number of derivations acting on the Jacobiator; we agree that this way to introduce the undetermined coefficients is not minimal. However, we always restrict to the order $(1, 1, 1)$ with respect to (f, g, h). We thus have $28,202$ unknowns introduced (counted with possible repetitions of graphs which they refer to).[3] Now we expand all the Leibniz rules that run over the internal vertices in every Jacobiator; simultaneously, the object $\mathrm{Jac}(\mathcal{P})$ is expanded using formula (2). As soon as we take into account the order $L \prec R$ and the antisymmetry of graphs under the reversion of that ordering at an internal vertex, the graphs that encode zero differential operators are eliminated. There remain $7,025$ admissible graphs with 5 internal vertices and 3 sinks; the coefficient of every such graph is a linear combination of the undetermined coefficients of the splinters which the

[3]The relevant algebra of sums of graphs modulo skew-symmetry and the Jacobi identity has been realized in software by the second author. An implementation of those tools in the problem of high-order expansion of the Kontsevich $\star$-product will be explained in a separate paper [3].

Leibniz-rule graphs (see Figure 4) produced from $\mathrm{Jac}(\mathcal{P})$. In conclusion, we view (8) as the system of 7025 linear inhomogeneous equations for the coefficients of graphs in the operator $\Diamond$. Solving this linear system is a way towards a proof of our main results (which is expressed in Corollary 4); The process of finding a solution $\Diamond$ itself does not constitute that proof. Therefore, the justification of the claim in Theorem 3 will be performed separately.

In the meantime, using software tools, we solve this linear algebraic system at hand. The duplications of graph labellings are conveniently eliminated by our request for the program to find a solution with a minimal number of nonzero components. Totally antisymmetric in tri-vector's arguments, the solution consists of 27 Leibniz-rule graphs, which are assimilated into the sum of 8 manifestly skew-symmetric terms as follows:



(9)

To display the $L \prec R$ ordering at every internal vertex and to make possible the arithmetic and algebra of graphs, we use the notation which is explained in Appendix D below.

Proof of Theorem 3. So far, we have constructed operator (9); We emphasize that it is completely determined by as few as only eight integer coefficients. This permits a rigorous proof of our main claim: namely, let us show that operator (9) does satisfy equation (8).

First expand the sums in (9), which gives us 27 Leibniz graphs. Now expand all the Leibniz rules; this yields the sum of 201 Kontsevich graphs with 3 sinks and 5 internal vertices: together with their coefficients, they are listed in Table 3 in Appendix D, see page 20. Clearly, manipulating that number of graphs is still possible for man.

Because we are free to enumerate the five internal vertices in every graph in a way we like, and because the ordering of every pair of outgoing edges is also under our control, at once do we bring all the graphs to their normal form.[4]

It is readily seen that there are many repetitions in Table 3. Now, collect the similar terms. There remain only 39 terms with nonzero coefficients. One verifies that those 39 terms are none other than the entries of Table 1, that is, realizations of the 39 graphs in the left-hand side of (8). This shows that equation (8) holds for the operator $\Diamond$ contained in (9). The proof is complete. □

3. Properties of the found solution

Remark 7. Let us recall that equation (1) yields the linear system of 7,025 inhomogeneous equations for the coefficients of 1132 patterns from Fig. 4. This shows that the algebraic system at hand is extremely overdetermined. Moreover, out of those 1132 admissible totally antisymmetric graphs, solution (9) involves only 8 of them. In this sense, the factorising operator $\Diamond$ in (1) is special; for it expands via (9) over a very low dimensional affine subspace in the affine space of unknowns in that inhomogeneous linear algebraic system.

Property 1. The relevant Leibniz-rule graphs, with respect to which the solution $\Diamond(\mathcal{P}, \cdot\,)$ expands, do not contain tadpoles nor two-cycles (or "eyes", see Fig. 1 on p. 4).
- None of the arrows that act back on the Jacobiator is issued from any of its arguments.
- In all the graphs the source vertices (if any), on which no arrows fall after all the Leibniz rules are expanded, belong to the Jacobiator (cf. (2) on p. 3).

Property 2. The found solution $\Diamond$ does contain the graphs in which two or three arrows fall on the Jacobiator.[5]

It has been explained in [8, 10] that the existence of two or more such arrows falling on the equation $[\![\mathcal{P},\mathcal{P}]\!] = 0$ is an obstruction to an extension of the main claim,

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] \doteq 0 \quad \text{via } [\![\mathcal{P},\mathcal{P}]\!] = 0, \tag{1}$$

to the infinite-dimensional geometry of jet spaces $J^\infty(\pi)$ for affine bundles over a manifold M^m or jet spaces $J^\infty(M^m \to N^n)$ of maps from M^m, and of variational Poisson brackets $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ for functionals on such jet spaces (see [19, 7] and [9, 10]). Namely, it can then be that

$$[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}_{1:6}(\boldsymbol{\mathcal{P}})]\!] \not\cong 0 \quad \text{through } [\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!] \cong 0. \tag{10}$$

[4]The normal form of a graph is obtained by running over the group $S_5 \times (\mathbb{Z}_2)^5$ of all the relabellings of internal vertices and swaps $L \rightleftarrows R$ of orderings at each vertex. (We recall that every swap negates the coefficient of a graph; the permutations from S_5 are responsible for encoding a given topological profile in seemingly "different" ways.) By definition, the normal form of a graph is the sign (times coefficient) followed by the minimal sequence of five ordered pairs of target vertices viewed as 10-digit base-$(3 + 5)$ numbers. (By convention, the three ordered sinks are enumerated $0, 1, 2$ and the internal vertices are the octonary digits $3, \ldots, 7$.)

[5]For instance, the first term in $\Diamond$ is the tripod standing on $\mathrm{Jac}(\mathcal{P})$.

We denote here by $[\![\,,\,]\!]$ the variational Schouten bracket; the variational bi-vector $\boldsymbol{\mathcal{Q}}_{1:6}$ is constructed from the variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ by using techniques from the geometry of iterated variations of functionals (see [8, 9, 10]). An explicit counterexample of (10) is known from [1] for the variational Poisson structure of the Harry Dym partial differential equation.

The reason why the obstruction arises is that in the variational setting, the second and higher order variations of a trivial integral functional $\mathrm{Jac}(\boldsymbol{\mathcal{P}}) \cong 0$ in the horizontal cohomology can still be nonzero (although its first variation would of course vanish).[6]

Remark 8. Uniqueness is currently not claimed for the found solution $\Diamond(\mathcal{P}, \cdot\,)$. The eight graphs in (9) represent a *linear* differential operator with respect to the Jacobiator $\mathrm{Jac}(\mathcal{P})$. However, a quadratic nonlinearity with respect to the two-vertex argument $\mathrm{Jac}(\mathcal{P})$ could be hidden in the five-vertex graphs in formula (9), so that it would in fact encode a bi-differential operator $\Diamond(\mathcal{P}, \cdot\,, \cdot\,)$. If this be the case, expansion of one or the other copy of the Jacobiator using (2) in such a polydifferential operator $\Diamond(\mathcal{P}, \cdot\,, \cdot\,)$ would produce two seemingly distinct linear differential operators $\Diamond(\mathcal{P}, \cdot\,)$.

The scenarios to build the bi-linear, bi-differential terms in the operator $\Diamond$ are drawn in Figure 5 below. We consider –in fact, without any loss of generality– only those 8 Leibniz-rule graphs in which

- the three arguments of each copy of Jacobiator (2) are different; in particular,
- neither of the Jacobiators acts on the other copy by two or three arrows (so that only none or one such arrow is possible).

We recall that known solution (9) is the sum of 39 graphs from which a linear dependence on the Jacobiator $\mathrm{Jac}(\mathcal{P})$ is retrieved by using the 27 Leibniz-graphs (see Table 2 on p. 22). Let us inspect whether solution (9) is just linear in $\mathrm{Jac}(\mathcal{P})$ or there is a bi-linear dependence in $\mathrm{Jac}(\mathcal{P})$ hidden in (9).

To this end, we took (with undetermined coefficients) the 27 Leibniz-graphs from (9), which are linear in $\mathrm{Jac}(\mathcal{P})$, and the 8 skew-symmetrized new patters from Fig. 5 (resp., quadratic in $\mathrm{Jac}(\mathcal{P})$). By equating their sum to zero and expanding all the Leibniz rules using the tool [3], we examined the arising system of linear algebraic equations. Due to the presence of homogeneous equations which involve only one unknown, specifically, the coefficient of a new Leibniz-graph from Fig. 5, and by noting that such is the case for every graph from that set, we conclude that the general solution of the homogeneous problem is necessarily linear in the Jacobiator, whence the non-existence of a quadratic part in (9) is manifest. Our computer-assisted reasoning motivates the following claim.

Conjecture 6. There is no quadratic part in all the solutions of equation

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond\left(\mathcal{P}, \mathrm{Jac}(\mathcal{P}), \mathrm{Jac}(\mathcal{P})\right) \tag{8$'$}$$

that expand with respect to the 39 graphs in (9).

[6]The same effect has been foreseen for a variational lift of deformation quantisation [15]: it has been argued in [10] why the associativity of noncommutative star-product $\star = \times + \hbar\{\cdot\,,\cdot\}_{\boldsymbol{\mathcal{P}}} + \bar{o}(\hbar)$ can leak and it has been shown in [2] that if it actually does at $O(\hbar^k)$, the order k at which this leak of associativity can occur is high: $k \geqslant 4$.

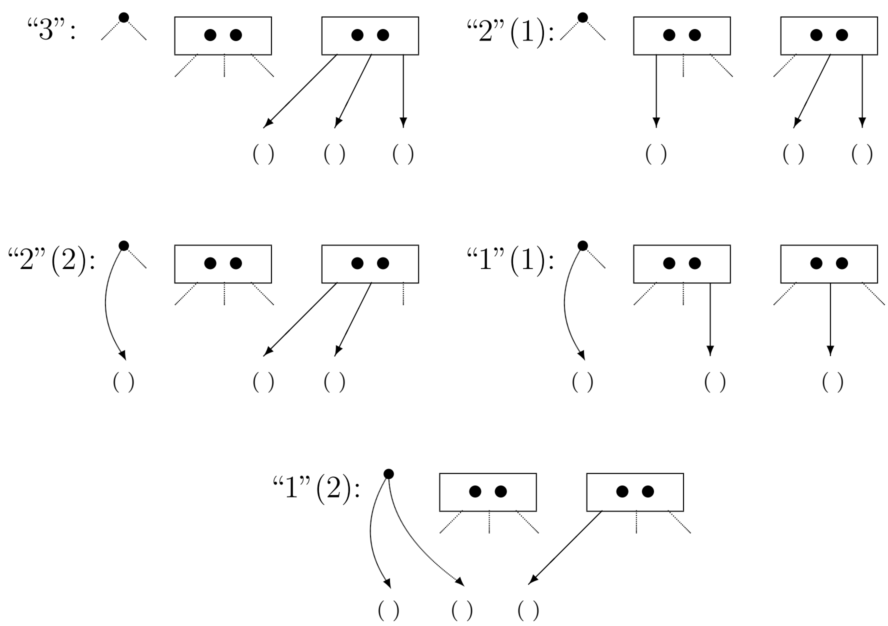


FIGURE 5. The Leibniz-graphs by using which a quadratic – with respect to the Jacobiator – part $\Diamond(\mathcal{P},\cdot,\cdot)$ of the factorizing operator could be sought for in (8); such quadratic part (if any) itself is necessarily totally skew-symmetric with respect to the three sinks ( ).

Still it could be for equation (8′) that a quadratic dependence of $\Diamond$ on $\mathrm{Jac}(\mathcal{P})$ is established for a solution $\Diamond$ which differs from any operator $\Diamond(\mathcal{P},\cdot)$ that expands only with respect to the graphs contained in (9).

4. Discussion

4.1. For the factorisation $[\![\mathcal{P},\mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond\bigl(\mathcal{P},\mathrm{Jac}(\mathcal{P})\bigr)$ to guarantee that the equality $\boldsymbol{\partial}_{\mathcal{P}}\bigl(\mathcal{Q}_{1:6}(\mathcal{P})\bigr) = 0$ holds if $\mathrm{Jac}(\mathcal{P}) = 0$, its mechanism is nontrivial. Relying on Lemma 5 (see [2]), it tells us how the differential consequences of Jacobi identity are split into separately vanishing expressions. This mechanism works not only in the construction of flows that satisfy (1) but also in the associativity,

$$\mathrm{Assoc}_{\mathcal{P}}(f,g,h) := (f\star g)\star h - f\star(g\star h) \doteq 0 \quad \text{via } [\![\mathcal{P},\mathcal{P}]\!] = 0,$$

of the non-commutative unital star-product $\star = \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + o(\hbar)$. The formula for $\star$-products was given in [15], establishing the deformation quantisation $\times \mapsto \star$ of the usual product $\times$ in the algebra $C^\infty(N^n) \ni f,g,h$ on a finite-dimentional affine Poisson manifold $(N^n,\mathcal{P})$, see also [2, 10]. In fact, the construction of graph complex and the pictorial language of graphs [14, 15] that encode polydifferential operators is common to all these deformation procedures (cf. [3], also [18]).

Open problem 1. Consider the Kontsevich star-product $\star = \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + o(\hbar)$ in the algebra $C^\infty(N^n)[[\hbar]]$ on a finite-dimensional affine Poisson manifold $(N^n,\mathcal{P})$. Given by the tetrahedra Γ_1 and Γ_2' (see Fig. 2 on p. 5), the infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon \mathcal{Q}_{1:6}(\mathcal{P}) + o(\varepsilon)$ induces the infinitesimal deformation $\star \mapsto \star + \hbar\varepsilon \,[\![[\![\mathcal{Q}_{1:6}(\mathcal{P}), \cdot]\!], \cdot]\!] + o(\varepsilon)$ of the star-product. What are the properties of this infinitesimally deformed $\star(\varepsilon)$-product? In particular, is the condition that $\mathcal{Q}_{1:6}(\mathcal{P})$ be $\boldsymbol{\partial}_{\mathcal{P}}$-trivial necessary for the $\star(\varepsilon)$-product to be gauge-equivalent to the unperturbed $\star$-product at $\varepsilon = 0$?

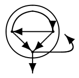
We recall that the theory of (infinitesimal) deformations of associative algebra structures is very well studied in the broadest context (e.g., of the Yang–Baxter equation, Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation, Frobenius manifolds and F-structures, etc.), see [17]. We expect that in that theory's part which is specific to the deformation of associative structures on finite-dimensional affine Poisson manifolds N^n, there must be a dictionary between the construction of Kontsevich flows for spaces of Poisson bi-vectors and other instruments to deform the associative product in the algebra $C^\infty(N^n)$.

4.2. The Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ is a universal procedure to deform a given Poisson bi-vector $\mathcal{P}$ on any finite-dimensional affine real manifold N^n (i. e. not necessarily topologically trivial). The infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon\mathcal{Q}_{1:6}(\mathcal{P}) + o(\varepsilon)$ can be completed to the construction of Poisson bi-vector $\mathcal{P}(\varepsilon)$ such that $\mathcal{P}(\varepsilon = 0) = \mathcal{P}$ and $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} \mathcal{P}(\varepsilon) = \mathcal{Q}_{1:6}(\mathcal{P})$ if the third Poisson cohomology group $H^3_{\mathcal{P}}(N^n)$ with respect to the Poisson differential $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ vanishes for the manifold N^n (see Appendix A below). In the symplectic case, i. e. for n even and bracket $\{\cdot, \cdot\}_{\mathcal{P}}$ nondegenerate, the Poisson complex is known to be isomorphic to the de Rham complex for N^n (see [16]). We are not yet aware of any way to constrain the Poisson cohomology groups $H^k_{\mathcal{P}}(N^n)$ for *degenerate* Poisson brackets $\{\cdot, \cdot\}_{\mathcal{P}}$ on real manifolds N^n of not necessarily even dimension $n < \infty$. (E.g., the algorithm for construction of cubic Poisson brackets on the basis of a class of R-matrices, which is explained in [16], yields a rank-six bracket on $N^9 \subset \mathbb{R}^9$.)

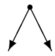
4.3. The second Poisson cohomology group $H^2_{\mathcal{P}}(N^n)$ of the manifold N^n, if nonzero, provides room for the $\boldsymbol{\partial}_{\mathcal{P}}$-nontrivial deformations of $\mathcal{P}$ using $\mathcal{Q}_{1:6}(\mathcal{P})$ such that $\mathcal{Q}_{1:6}(\mathcal{P}) \neq [\![\mathcal{P}, \mathcal{X}]\!]$ for all globally defined 1-vectors $\mathcal{X}$ on N^n. In particular, this implies that there are no $\boldsymbol{\partial}_{\mathcal{P}}$-nontrivial tetrahedral graph flows on even-dimensional star-shaped domains equipped with nondegenerate Poisson brackets.

A possibility for the right-hand side $\mathcal{Q}_{1:6}(\mathcal{P})$ of the tetrahedral flow to be $\boldsymbol{\partial}_{\mathcal{P}}$-trivial is thus a global, topological effect; it cannot always be seen within a single chart in N^n. Moreover, it is not universal with respect to the calculus of graphs.

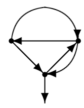
Claim 7. *In contrast with Theorem* 3, *there is no dimension-independent $\boldsymbol{\partial}_{\mathcal{P}}$-triviality mechanism which would be expressed for the tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ in terms of the Kontsevich graphs* (*see* §1.1 *and* [14, 15]) *and hence, which would be universal*[7] *with respect to all Poisson structures $\mathcal{P}$ on all finite-dimensional affine manifolds N^n.*

[7]Kontsevich notes [14] that if $n = 2$ so that every bi-vector $\mathcal{P}$ on N^2 is Poisson and every flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ preserves this property, the tetrahedron Γ_1 (or, equivalently, the velocity $\mathcal{Q}_{1:0}(\mathcal{P})$) is always $\partial_{\mathcal{P}}$-exact. The required 1-vector field $\mathcal{X}(\mathcal{P})$ in the coboundary statement $\mathcal{Q}_{1:0}(\mathcal{P}) = [\![\mathcal{P}, \mathcal{X}]\!]$ can be expressed in terms of the bi-vector $\mathcal{P}$, e.g., by the Leibniz-rule graph $\mathcal{X} =$ . (This is a particular, not general solution.) We recall that after the dimension n is fixed (here $n = 2$), a given

Proof. Indeed, consider the $\boldsymbol{\partial}_{\mathcal{P}}$-coboundary equation,

$$\Gamma_1(\mathcal{P}) + 6\,\Gamma_2(\mathcal{P}) = [\![$$  $$, \mathcal{X}]\!],$$

where the graphs Γ_1 and Γ_2', inhabited by a copy of the Poisson bi-vector $\mathcal{P}$ in every internal vertex, are shown in Fig. 2 on p. 5. Because there are four copies of $\mathcal{P}$ in each tetrahedron and the Λ-graph in $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot\,]\!]$ contains one copy of the bi-vector $\mathcal{P}$, the number of internal vertices in the 1-vector $\mathcal{X}$ must be equal to 3. Likewise, we recall that neither there are tadpoles in both $\mathcal{P}$ and $\mathcal{Q}(\mathcal{P})$ nor does the Poisson differential $[\![\mathcal{P}, \cdot]\!]$ destroy any tadpoles (cf. Remark 3 on page 6); Therefore, the graph that encodes $\mathcal{X}$ may not contain any tadpoles. The only such Kontsevich graph with three internal vertices but without tadpoles is

$$\mathcal{X} = \text{const} \cdot$$  .

Now it is readily seen that the Schouten bracket of $\mathcal{X}$ with the Poisson bi-vector $\mathcal{P}$ does contain a source vertex (to which no arrows arrive). But there is no such vertex in either of the tetrahedra within the bi-vector $\mathcal{Q}_{1:6}(\mathcal{P})$ in the left-hand side of the $\boldsymbol{\partial}_{\mathcal{P}}$-cocycle equation $\mathcal{Q}_{1:6}(\mathcal{P}) = \boldsymbol{\partial}_{\mathcal{P}}(\mathcal{X})$. This shows that there is no universal solution $\mathcal{X}(\mathcal{P})$ expressed for all $\mathcal{P}$ in terms of graphs. ☐

Remark 9. The same reasoning works for all the Kontsevich graph flows such that none of the graphs besides the bi-vector $\mathcal{P}$ itself contains a source vertex (that is, neither any graph in the flow nor the 1-vector $\mathcal{X}$).

Open problem 2. The formalism developed in [14] suggests that there are, most likely, infinitely many Kontsevich graph flows on the spaces of Poisson bi-vectors on finite-dimensional affine Poisson manifolds. Forming an example $\mathcal{Q}_{1:6}(\mathcal{P})$ of such a cocycle in the graph complex, the tetrahedra Γ_1 and Γ_2' in Fig. 2 are built over four internal vertices. What is or are the next – with respect to the ordering of natural numbers – Kontsevich graph cocycle(s) built over five or more internal vertices ?

4.4. The tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ preserves the space $\{\mathcal{P} \in \Gamma(\bigwedge^2 TN^n) \mid [\![\mathcal{P}, \mathcal{P}]\!] = 0\}$ of Poisson bi-vectors; this is guaranteed by Theorem 3 that asserts $\boldsymbol{\partial}_{\mathcal{P}}(\mathcal{Q}_{1:6}) \doteq 0$ within the (graded-)commutative geometry of finite-dimensional affine real manifolds N^n.

Open problem 3. Does the proven property,

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] \doteq 0 \quad \text{via } [\![\mathcal{P}, \mathcal{P}]\!] = 0, \tag{1}$$

generalize to the formal noncommutative symplectic supergeometry [11], to the calculus of multivectors performed by using their necklace brackets (see [9] and references therein), and to Poisson structures on the commutative non-associative unital algebras of cyclic words (e. g., see [20]) ?

differential polynomial in $\mathcal{P}$ can be encoded by the Kontsevich graphs in non-unique way. Details will be discussed elsewhere.

Appendix A. The Poisson cohomology

Let us recall several necessary facts from the deformation theory; this material is standard [5]. Denote by ξ_i the parity-odd canonical conjugate of the variable x^i for every $i = 1, \ldots, n$ (see [9] for discussion about the reverse parity symplectic duals). Every bi-vector is then realised in terms of the local coordinates x^i and ξ_i on $\Pi T^* N^n$ by using $\mathcal{P} = \frac{1}{2}\langle \xi_i \, \mathcal{P}^{ij}(\boldsymbol{x}) \, \xi_j \rangle$. We denote by $[\![\cdot,\cdot]\!]$ the Schouten bracket, i.e. the parity-odd Poisson bracket which is locally determined on $\Pi T^*\mathbb{R}^n$ by the canonical symplectic structure $\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\boldsymbol{\xi}$ (see [8] for details). Our working formula is[8]

$$[\![\mathcal{P}, \mathcal{Q}]\!] = (\mathcal{P}) \frac{\overleftarrow{\partial}}{\partial x^i} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_i}(\mathcal{Q}) - (\mathcal{P}) \frac{\overleftarrow{\partial}}{\partial \xi_i} \cdot \frac{\overrightarrow{\partial}}{\partial x^i}(\mathcal{Q}). \tag{11}$$

To be Poisson, a bi-vector $\mathcal{P}$ must satisfy the master-equation $[\![\mathcal{P}, \mathcal{P}]\!] = 0$, of which formula (2) is the component expansion with respect to the indices (i, j, k) in the tri-vector $[\![\mathcal{P}, \mathcal{P}]\!](\boldsymbol{x}, \boldsymbol{\xi})$.

Under an infinitesimal deformation $\mathcal{P}(\varepsilon) = \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ of the bi-vector $\mathcal{P}$ satisfying $[\![\mathcal{P}, \mathcal{P}]\!] = 0$, the bi-vector $\mathcal{P}(\varepsilon)$ remains Poisson only if $[\![\mathcal{P}(\varepsilon), \mathcal{P}(\varepsilon)]\!] = \bar{o}(\varepsilon)$, whence $[\![\mathcal{P}, \mathcal{Q}]\!] = 0$.

Remark 10. For a Poisson bi-vector $\mathcal{P}$, the operator $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ is readily seen to be a differential: by virtue of the Jacobi identity for the Schouten bracket $[\![\cdot,\cdot]\!]$ we have that $\boldsymbol{\partial}_{\mathcal{P}}^2 = 0$. Therefore, the leading order terms $\mathcal{Q}$ in the deformations $\mathcal{P}(\varepsilon) = \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ can be trivial in the second $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology, meaning that $\mathcal{Q} = [\![\mathcal{P}, \mathcal{X}]\!]$ for some one-vector $\mathcal{X}$ (whence $[\![\mathcal{P}, [\![\mathcal{P}, \mathcal{X}]\!]]\!] \equiv 0$). Alternatively, for the $\boldsymbol{\partial}_{\mathcal{P}}$-cocycles $\mathcal{Q}$ which are not $\boldsymbol{\partial}_{\mathcal{P}}$-coboundaries, the flows $\mathcal{P}(\varepsilon)$ stay infinitesimally Poisson but leave the $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology class of the Poisson bi-vector $\mathcal{P}$ at $\varepsilon = 0$.

For consistency, let us recall that generally speaking, not every infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ of a Poisson bi-vector $\mathcal{P}$ can be completed to a Poisson deformation $\mathcal{P} \mapsto \mathcal{P} + \mathcal{Q}(\varepsilon)$ at all orders in ε. The obstructions are contained in the third $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology group $\mathrm{H}^3_{\mathcal{P}} = \bigl\{\mathrm{T} \in \Gamma\bigl(\bigwedge^3 TN\bigr) \mid \boldsymbol{\partial}_{\mathcal{P}}(\mathrm{T}) = 0\bigr\} \bigm/ \bigl\{\mathrm{T} = \boldsymbol{\partial}_{\mathcal{P}}(\mathrm{R}), \mathrm{R} \in \Gamma\bigl(\bigwedge^2 TN\bigr)\bigr\}$. Indeed, cast the master-equation $[\![\mathcal{P} + \mathcal{Q}(\varepsilon), \mathcal{P} + \mathcal{Q}(\varepsilon)]\!] = 0$ for the Poisson deformation to the coboundary statement $[\![\mathcal{Q}(\varepsilon), \mathcal{Q}(\varepsilon)]\!] = \boldsymbol{\partial}_{\mathcal{P}}(-\mathcal{P} - 2\mathcal{Q}(\varepsilon))$, whence $\boldsymbol{\partial}_{\mathcal{P}}([\![\mathcal{Q}(\varepsilon), \mathcal{Q}(\varepsilon)]\!] \equiv 0$ by $\boldsymbol{\partial}_{\mathcal{P}}^2 = 0$. Therefore, the vanishing of the third $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology group guarantees the existence of a power series solution $\mathcal{Q}(\varepsilon)$ to the cocycle-coboundary equation $[\![\mathcal{Q}(\varepsilon), \mathcal{Q}(\varepsilon)]\!] = -2\boldsymbol{\partial}_{\mathcal{P}}(\mathcal{Q}(\varepsilon))$: known to be a cocycle, the left-hand side has been proven to be a coboundary as well.

Remark 11. Nowhere above should one expect that the leading deformation term $\mathcal{Q}$ in $\mathcal{P}(\varepsilon) = \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ itself would be a Poisson bi-vector. This may happen for $\mathcal{Q}$ only incidentally.

[8] In the set-up of infinite jet spaces $J^\infty(\pi)$ (see [19] and [8, 9, 10]) the four partial derivatives in the formula for $[\![\cdot,\cdot]\!]$ become the variational derivatives with respect to the same variables, which now parametrise the fibres in the Whitney sum $\pi \times_{M^m} \Pi\widehat{\pi}$ of (super-)bundles over the m-dimensional base M^m.

Appendix B. Computer-assisted proof of Proposition 2

To verify the claim in Proposition 2 by direct calculation, it would take years for man still only a few seconds for a computer.[9] A computer-assisted proof of Proposition 2 is realized through running the script in Maple (see below). (All computations are done with the coefficient matrices of bi-vectors at hand. The bi-vectors are computed by using working formulas (4a) and (4b).) For the balanced flow we have:

```
FlowQ := proc (P, y, a, b)
description "Eval flow Q_a:b of q-dim bi-vector P.";
local i, j, q, A, F, G, B, T, C;
q := op(P)[1];
F := proc (i, j, k, l, m, n, p, r) options operator, arrow;
a*(diff(P[i, j], y[k], y[l], y[m]))*(diff(P[k, n], y[p]))
*(diff(P[l, p], y[r]))*(diff(P[m, r], y[n])) end proc;
G := proc (i, j, k, l, m, n, p, r) options operator, arrow;
b*(diff(P[i, j], y[k], y[l]))*(diff(P[k, m], y[n], y[p]))
*(diff(P[n, l], y[r]))*(diff(P[r, p], y[j])) end proc;
B := Array(1 .. q, 1 .. q);
T := combinat:-cartprod([seq([seq(1 .. q)], i = 1 .. 8)]);
while not T[finished] do
C := op(T[nextvalue]());
B[C[1], C[2]] := B[C[1], C[2]]+F(C);
B[C[1], C[5]] := B[C[1], C[5]]+G(C);
end do;
A := Array(1 .. q, 1 .. q);
for i from 1 to q do
for j from 1 to q do
A[i, j] := simplify((1/2)*B[i, j]-(1/2)*B[j, i]);
end do;
end do;
Matrix(A);
end proc:
```

To implement the Schouten bracket of two bi-vectors A and B, we use a component expansion (cf. [4]):

$$[\![A, B]\!]^{ijk} = \sum_{s=1}^{n} A^{sk} B_s^{ij} + B^{sk} A_s^{ij} + A^{sj} B_s^{ki} + B^{sj} A_s^{ki} + A^{si} B_s^{jk} + B^{si} A_s^{jk},$$

where superscripts and subscripts denote the bi-vector components and partial derivatives with respect to the coordinates y_s, respectively.

```
SchoutenBracket := proc (A, B, y)
description "Evaluate the Schouten-bracket of A and B.";
local T, t, F, n, res, cnt;
n := op(A)[1];
F := proc (i, j, k) options operator, arrow;
```

[9]Running the script below took us approximately 5 seconds.

```
A[s, k]*(diff(B[i, j], y[s]))+B[s, k]*(diff(A[i, j], y[s]))+
A[s, j]*(diff(B[k, i], y[s]))+B[s, j]*(diff(A[k, i], y[s]))+
A[s, i]*(diff(B[j, k], y[s]))+B[s, i]*(diff(A[j, k], y[s])) end proc;
T := combinat:-choose(n, 3);
for t in T do
print([[t[1], t[2], t[3]],simplify(add(F(t[1], t[2], t[3]), s = 1 .. n))]);
end do;
end proc:
```

Finally, the following script provides a computer-assisted proof of Proposision 2.

```
# All 3-dimensional Poisson bi-vectors are of the following form.
> P:=<<0,-f(x,y,z)*(diff(g(x,y,z),z)),f(x,y,z)*(diff(g(x,y,z),y))>|
      <f(x,y,z)*(diff(g(x,y,z),z)),0,-f(x,y,z)*(diff(g(x,y,z),x))>|
      <-f(x,y,z)*(diff(g(x,y,z),y)),f(x,y,z)*(diff(g(x,y,z),x)),0>>:
# We evaluate the balanced flow Q_{1:6} on the above bi-vector.
> Q:=FlowQ(P,{x,y,z},1,6)
                [Length of output exceeds limit of 1000000]
# If so, let us inspect whether the flow Q_{1:6} vanishes.
> LinearAlgebra:-Equal(Q,Matrix(1..3,1..3,0))
                                  false
# Still, let us act on this Q_{1:6} by the Poisson differential.
> SchoutenBracket(P,Q,{x,y,z})
                               [[1,2,3], 0]
```

This reasoning hints us that the condition $a:b=1:6$ could be sufficient for equation (1) to hold for all Poisson structures on all finite dimensional affine real manifolds. A rigorous proof of the respective claim in Theorem 3 is provided in section 2.

Appendix C. The count of Leibniz-rule graphs in Fig. 4

We count all possible differential consequences of the Jacobi identity, that is, we consider the differential operators acting on the Jacobiator. We do this by constructing all possible graphs that encode trivector-valued differential consequences (see Lemma 5 on p. 9). The graphs that encode such differential consequences have 3 ground vertices. The Schouten bracket $[\![\mathcal{P},\mathcal{Q}_{1:6}(\mathcal{P})]\!]$ consists of graphs with 5 internal vertices. Since two of these internal vertices are accounted for by the Jacobi identity, there remain 3 spare internal vertices.

First, let the Jacobiator stand, with all its three edges, on the 3 ground vertices. The only freedom that remains is how the 3 free internal vertices act on each other and on the Jacobiator. With its first edge, every free internal vertex can act on itself, on its 2 neighbouring free vertices, or on the Jacobiator; there are 4 possible targets. No second edge can meet the first edge at the same target (as this would yield no contribution due to the anti-symmetry, which is explained in Remark 2). Hence there are only 3 possible targets for this second edge. Finally, again due to anti-symmetry, every possibility is constructed exactly twice this way. Swapping the targets of the first and second edge only contributes to the sign of the graph. The total number of this type of differential consequence is therefore $\left(\frac{4\cdot 3}{2}\right)^3 = 216$ graphs. This type of graph is drawn first from the top-left in Figure 4.

Now let the Jacobiator stand on only 2 of the ground vertices. The remaining edge of the Jacobiator has only 3 possible targets, as the third edge cannot fall back onto the Jacobiator itself. One of the free internal vertices acts with an edge on the remaining ground vertex. The other edge has 4 candidates as its target, namely the vertex itself, the neighbouring 2 free internal vertices, and the Jacobiator. The 2 internal vertices not falling on a ground vertex have each $\frac{4\cdot 3}{2}$ possible targets. The total number of graphs is therefore equal to $3 \cdot 4 \cdot \left(\frac{4\cdot 3}{2}\right)^2 = 432$. This type of graph is the second from the top-left in Figure 4.

Next, let the Jacobiator stand on only 1 ground vertex. We distinguish between two cases: namely, the case where 1 free internal vertex stands on both the remaining ground vertices and the case where two different internal vertices act by one edge each on the remaining two ground vertices. These are the third and fourth graphs from the top-left in Figure 4, respectively.

- In the first case, the remaining 2 internal vertices each have $\frac{4\cdot 3}{2}$ possible targets. The Jacobiator must act with its two remaining free edges on two different targets out of the 3 available, yielding 3 possibilities. The number of graphs in the first case is $3 \cdot \left(\frac{4\cdot 3}{2}\right)^2 = 108$.
- For the second case, two internal vertices can each act on themselves, on the neighbouring 2 internal vertices, or on the Jacobiator. With two of its edges, the Jacobiator can act in 3 different ways on the 3 internal vertices. The third internal vertex has $\frac{4\cdot 3}{2}$ possible targets. This brings the total number of graphs for the second case to $4 \cdot 4 \cdot \frac{4\cdot 3}{2} \cdot 3 = 288$.

The last case to consider is where the Jacobiator does not act on any of the ground vertices. Again, since the outgoing edges of the Jacobiator must have different targets, it is clear that the Jacobiator acts in a unique way on all 3 internal vertices. We now distinguish two cases: namely, the case where 1 free internal vertex stands on 2 ground vertices, 1 free internal vertex acts on 1 ground vertex, and 1 free internal vertex falling on no ground vertex, and the second case where each internal vertex acts with one edge on one ground vertex. These two cases are represented by the last 2 graphs in Figure 4, respectively.

- In the first case, there is a free internal vertex with one free edge, which has 4 possible targets. The remaining free internal vertex with two free edges has $\frac{4\cdot 3}{2}$ possible targets. The total number of graphs for this case is $4 \cdot \frac{4\cdot 3}{2} = 24$.
- In the second case, each internal vertex can act on itself, on its 2 neighbouring internal vertices, and on the Jacobiator. This results in a total of $4^3 = 64$ graphs.

Summarizing, the total number of all trivector-valued Leibniz-rule graphs, linear in the Jacobiator and containing five internal vertices, is 1132.

Appendix D. Encoding of the solution

Let Γ be a labelled Kontsevich graph with n internal and m external vertices. We assume the ground vertices of Γ are labelled $[0, \ldots, m-1]$ and the internal vertices are labelled $[m, \ldots, m+n-1]$. We define the *encoding* of Γ to be the *prefix* (n, m), followed by a list of *targets*. The list of targets consists of ordered pairs where the kth pair ($k \geqslant 0$) contains the two targets of the internal vertex number $m+k$.

The expansion of the Schouten bracket $[\![\mathcal{P}, \mathcal{Q}_{a:b}]\!]$ for the ratio $a : b = 1 : 6$ depicted in Figure 3 simplifies to a sum of 39 graphs with coefficients $\pm 1, \pm 3$. The encodings of these graphs, followed by their respective coefficients, are listed in Table 1. The graphs

TABLE 1. Machine-readable encoding of Figure 3 on p. 6.

1.1	3	5	4 2 0 1 4 6 4 7 4 5	1
1.2	3	5	4 0 1 2 4 6 4 7 4 5	1
1.3	3	5	4 1 2 0 4 6 4 7 4 5	1
2.1	3	5	7 0 3 5 3 6 3 4 1 2	1
2.2	3	5	7 1 3 5 3 6 3 4 2 0	1
2.3	3	5	7 2 3 5 3 6 3 4 0 1	1
3.1	3	5	5 2 0 1 4 6 4 7 4 5	3
3.2	3	5	5 0 1 2 4 6 4 7 4 5	3
3.3	3	5	5 1 2 0 4 6 4 7 4 5	3
4.1	3	5	6 7 0 3 3 4 4 5 1 2	3
4.2	3	5	6 7 1 3 3 4 4 5 2 0	3
4.3	3	5	6 7 2 3 3 4 4 5 0 1	3
5.1	3	5	4 2 7 0 1 4 4 5 5 6	3
5.2	3	5	4 0 7 1 2 4 4 5 5 6	3
5.3	3	5	4 1 7 2 0 4 4 5 5 6	3
6.1	3	5	5 2 7 0 1 4 4 5 5 6	3
6.2	3	5	5 0 7 1 2 4 4 5 5 6	3
6.3	3	5	5 1 7 2 0 4 4 5 5 6	3
7.1	3	5	6 2 7 0 1 4 4 5 5 6	3
7.2	3	5	6 0 7 1 2 4 4 5 5 6	3
7.3	3	5	6 1 7 2 0 4 4 5 5 6	3
8.1	3	5	7 2 7 0 1 4 4 5 5 6	3
8.2	3	5	7 0 7 1 2 4 4 5 5 6	3
8.3	3	5	7 1 7 2 0 4 4 5 5 6	3
9.1	3	5	4 2 7 1 0 4 4 5 5 6	−3
9.2	3	5	4 0 7 2 1 4 4 5 5 6	−3
9.3	3	5	4 1 7 0 2 4 4 5 5 6	−3
10.1	3	5	5 2 7 1 0 4 4 5 5 6	−3
10.2	3	5	5 0 7 2 1 4 4 5 5 6	−3
10.3	3	5	5 1 7 0 2 4 4 5 5 6	−3
11.1	3	5	6 2 7 1 0 4 4 5 5 6	−3
11.2	3	5	6 0 7 2 1 4 4 5 5 6	−3
11.3	3	5	6 1 7 0 2 4 4 5 5 6	−3
12.1	3	5	7 2 7 1 0 4 4 5 5 6	−3
12.2	3	5	7 0 7 2 1 4 4 5 5 6	−3
12.3	3	5	7 1 7 0 2 4 4 5 5 6	−3
13.1	3	5	6 0 7 3 3 4 4 5 1 2	−3
13.2	3	5	6 1 7 3 3 4 4 5 2 0	−3
13.3	3	5	6 2 7 3 3 4 4 5 0 1	−3

are collected into groups of three, consisting of the skew-symmetrization – by a sum over cyclic permutations – of a single graph. Within the encodings in the groups of three, the lists of targets only differ by a cyclic permutation of the target vertices $0, 1, 2$.

Consisting of 8 skew-symmetric terms, the solution (see (9) on p. 11) is encoded in Table 2: the sought-for values of coefficients are written after the encoding of the respective 27 Leibniz-rule graphs. Here the sums over permutations of the ground vertices are expanded (thus making the 27 Leibniz-rule graphs out of the 8 skew-symmetric groups). In every entry of Table 2, the sum of three graphs in Jacobiator (2) is represented by its first term. For all the in-coming arrows, the vertex 6 is the placeholder for

Table 2. Machine-readable encoding of solution (9) on p. 11.

1.1	3	5	4 6 5 6 3 6 0 1 6 2	−1
2.1	3	5	0 4 1 5 2 3 3 4 6 5	−3
2.2	3	5	0 4 2 5 1 3 3 4 6 5	3
3.1	3	5	0 4 1 2 3 4 3 4 6 5	−3
3.2	3	5	0 1 2 3 3 4 3 4 6 5	−3
3.3	3	5	0 2 1 3 3 4 3 4 6 5	3
4.1	3	5	4 5 1 6 4 6 0 2 6 3	−3
4.2	3	5	4 5 0 6 4 6 1 2 6 3	3
4.3	3	5	5 6 3 5 2 6 0 1 6 4	−3
5.1	3	5	1 4 5 6 3 6 0 2 6 3	3
5.2	3	5	0 4 5 6 3 6 1 2 6 3	−3
5.3	3	5	5 6 2 3 4 6 0 1 6 4	−3
6.1	3	5	1 2 3 5 3 6 0 3 6 4	3
6.2	3	5	0 2 3 5 3 6 1 3 6 4	−3
6.3	3	5	4 6 0 1 3 4 2 4 6 5	−3
7.1	3	5	1 5 3 5 2 6 0 3 6 4	−3
7.2	3	5	1 5 3 5 0 6 2 3 6 4	3
7.3	3	5	0 5 3 5 2 6 1 3 6 4	3
7.4	3	5	2 5 3 5 1 6 0 3 6 4	3
7.5	3	5	2 5 3 5 0 6 1 3 6 4	−3
7.6	3	5	0 5 3 5 1 6 2 3 6 4	−3
8.1	3	5	1 4 2 5 3 6 0 3 6 4	−3
8.2	3	5	1 5 2 3 4 6 0 3 6 4	−3
8.3	3	5	0 4 2 5 3 6 1 3 6 4	3
8.4	3	5	0 5 2 3 4 6 1 3 6 4	3
8.5	3	5	4 6 0 5 1 3 2 4 6 5	−3
8.6	3	5	4 6 1 5 0 3 2 4 6 5	3

the Jacobiator (again, see (2) on p. 3); in earnest, the Jacobiator contains the internal vertices 6 and 7. This convention is helpful: for every set of derivations acting on the Jacobiator with internal vertices 6 and 7, only the first term is listed, namely the one where each edge lands on 6.

Example 1. The first entry of Table 2 encodes a graph containing a three-cycle over internal vertices $3, 4, 5$. Issued from each of these three, the other edge lands on the vertex 6: the placeholder for the Jacobiator. This entry is the first term in (9) on p. 11.

Example 2. The entry 3.1 is one of three terms produced by the third graph in solution (9); the Jacobiator in this entry is expanded using formula (2), resulting in three terms (by definition). It is easy to see that the first term contains picture (3) from Remark 2 as a subgraph. Hence the polydifferential operator encoded by this graph vanishes due to skew-symmetry. However, the other two terms produced in the entry 3.1 by formula (2) do not vanish by skew-symmetry. Likewise, there is one term vanishing by the same mechanism in the entry 3.2 and in 3.3.

The proof of Theorem 3 amounts to expanding the Leibniz rules on Jacobiators in Table 2 according to the rules above (resulting in Table 3 on pp. 23–24, where the prefix "3 5" of each graph has been omitted for brevity), simplifying by collecting terms, and seeing that one obtains Table 1.

Appendix E. Perturbation method

In section 2 above, the run-through method gave all the terms at once in the operator $\Diamond$ that establishes the factorization $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$. At the same time, there is another method to find $\Diamond$; the operator $\Diamond$ is then constructed gradually, term after

Table 3. Expansion of Leibniz rules on Jacobiators in Table 2.

0 1 2 3 3 6 3 7 3 5	−1	0 4 2 5 6 7 1 3 3 6	−3	0 4 2 5 3 6 3 4 1 6	−3
0 1 2 3 3 6 3 7 4 5	−1	0 4 5 6 1 3 3 5 2 3	3	0 4 2 5 3 6 1 7 3 4	−3
0 1 2 3 3 6 3 7 4 5	−1	0 4 5 6 1 3 5 7 2 3	−3	0 4 2 5 3 6 1 4 3 6	3
0 1 2 3 3 6 4 7 4 5	−1	0 4 5 6 1 7 3 5 2 3	3	0 4 2 5 3 6 3 7 1 4	3
0 1 2 3 3 6 3 7 4 5	−1	0 4 5 6 1 7 5 7 2 3	−3	0 4 5 6 1 3 2 3 5 6	−3
0 1 2 3 3 6 4 7 4 5	−1	0 4 1 2 3 4 3 7 4 5	−3	0 4 5 6 2 3 5 7 1 3	−3
0 1 2 3 3 6 4 7 4 5	−1	0 4 1 2 3 6 4 7 4 5	3	0 4 5 6 2 3 3 5 1 6	−3
0 1 2 3 4 6 4 7 4 5	−1	0 4 5 6 1 2 5 7 4 5	3	0 4 5 6 1 7 2 3 3 6	3
0 4 1 2 4 6 4 7 4 5	−1	0 4 5 6 1 2 5 7 3 5	3	0 4 5 6 1 6 2 3 3 5	−3
0 4 1 2 3 6 4 7 4 5	−1	0 4 5 6 1 2 3 5 3 5	3	0 4 5 6 2 3 3 7 1 5	3
0 4 1 2 3 6 4 7 4 5	−1	0 4 5 6 1 2 5 7 3 5	−3	0 4 2 5 1 3 3 4 5 6	3
0 4 1 2 3 6 3 7 4 5	−1	0 4 1 5 2 6 3 4 3 5	3	0 4 2 5 6 7 1 3 3 4	3
0 4 1 2 3 6 4 7 4 5	−1	0 4 1 5 2 6 4 7 3 5	−3	0 4 2 5 3 6 3 4 1 5	−3
0 4 1 2 3 6 3 7 4 5	−1	0 4 5 6 1 6 2 7 4 5	−3	0 4 2 5 1 6 3 7 3 4	−3
0 4 1 2 3 6 3 7 4 5	−1	0 4 5 6 1 6 2 7 3 5	−3	0 4 2 5 1 6 3 4 3 5	3
0 4 1 2 3 6 3 7 3 5	−1	0 4 2 5 3 6 1 4 3 6	−3	0 4 2 5 3 6 1 7 3 4	3
0 2 1 3 3 6 3 7 3 5	1	0 4 2 5 6 7 1 4 3 6	3	0 4 1 5 2 3 3 5 4 6	3
0 2 1 3 3 6 3 7 4 5	1	0 4 1 5 3 6 2 4 3 6	3	0 4 5 6 1 7 3 7 2 3	3
0 2 1 3 3 6 3 7 4 5	1	0 4 1 5 6 7 2 4 3 6	−3	0 4 5 6 2 3 3 5 1 4	−3
0 2 1 3 3 6 4 7 4 5	1	0 4 5 6 1 7 2 5 4 6	−3	0 4 1 5 6 7 2 3 3 6	−3
0 2 1 3 3 6 3 7 4 5	1	0 4 5 6 1 7 2 5 3 6	−3	0 4 1 5 3 6 2 3 4 6	−3
0 2 1 3 3 6 4 7 4 5	1	0 4 2 5 1 6 3 4 3 5	−3	0 4 5 6 1 7 2 3 3 6	−3
0 2 1 3 3 6 4 7 4 5	1	0 4 2 5 1 6 4 7 3 5	3	0 4 1 5 2 3 3 4 5 6	−3
0 2 1 3 4 6 4 7 4 5	1	0 4 1 5 2 3 3 7 4 5	−3	0 4 1 5 6 7 2 3 3 4	−3
0 1 2 5 3 6 3 4 3 4	−3	0 4 1 5 2 6 3 7 4 5	−3	0 4 1 5 3 6 3 4 2 5	3
0 1 2 5 3 6 4 7 3 4	3	0 4 5 6 1 7 5 7 2 4	3	0 4 1 5 2 6 3 7 3 4	3
0 1 2 5 6 7 3 4 3 4	−3	0 4 5 6 1 7 5 7 2 3	3	0 4 1 5 2 6 3 4 3 5	−3
0 1 2 5 6 7 3 4 4 6	3	0 4 5 6 1 6 2 3 3 5	3	0 4 1 5 3 6 2 7 3 4	−3
0 4 1 5 2 6 4 7 4 5	3	0 4 5 6 1 6 2 7 3 5	3	0 4 2 5 1 3 3 5 4 6	−3
0 4 1 5 2 6 4 7 3 5	3	0 4 2 5 1 3 3 7 4 5	3	0 4 5 6 2 7 3 7 1 3	−3
0 4 1 5 2 6 3 7 4 5	3	0 4 2 5 1 6 3 7 4 5	3	0 4 5 6 1 3 3 5 2 4	3
0 4 1 5 2 6 3 7 3 5	3	0 4 5 6 2 7 5 7 1 4	−3	0 4 2 5 6 7 1 3 3 6	3
0 4 2 5 3 6 3 4 1 3	−3	0 4 5 6 2 7 5 7 1 3	−3	0 4 2 5 3 6 1 3 4 6	3
0 4 2 5 3 6 4 7 1 3	3	0 4 5 6 1 3 2 5 3 6	3	0 4 5 6 1 3 2 7 3 5	−3
0 4 2 5 6 7 1 3 3 4	−3	0 4 5 6 1 7 2 5 3 6	3	0 1 2 3 3 4 3 5 4 6	3
0 4 2 5 6 7 1 3 4 6	3	0 4 5 6 1 2 3 5 3 5	−3	0 1 2 3 3 6 3 7 4 5	3
0 1 2 3 3 4 3 7 4 5	−3	0 4 5 6 1 2 3 7 3 5	−3	0 1 2 5 3 6 3 7 3 5	−3
0 1 2 5 3 6 4 7 3 4	−3	0 4 5 6 3 7 3 7 1 2	−3	0 1 2 5 3 6 3 7 3 4	−3
0 1 2 3 3 6 4 7 4 5	3	0 4 5 6 3 6 3 7 1 2	3	0 1 2 5 3 6 3 4 3 4	3
0 1 2 5 3 6 4 7 4 5	3	0 4 5 6 2 3 3 5 1 5	−3	0 1 2 5 3 6 3 7 3 4	3
0 4 1 5 6 7 2 4 4 6	3	0 4 5 6 2 3 3 7 1 5	−3	0 4 1 5 3 6 2 3 4 6	3
0 4 1 5 6 7 2 3 4 6	3	0 4 5 6 1 7 3 7 2 3	−3	0 4 1 5 3 6 4 7 2 3	3
0 4 1 5 6 7 2 4 3 6	3	0 4 5 6 1 7 3 5 2 3	−3	0 4 1 5 3 6 3 4 2 6	3
0 4 1 5 6 7 2 3 3 6	3	0 4 5 6 1 3 3 5 2 5	3	0 4 1 5 3 6 2 7 3 4	3
0 4 5 6 2 3 3 5 1 3	−3	0 4 5 6 1 3 3 7 2 5	3	0 4 1 5 3 6 2 4 3 6	−3
0 4 5 6 2 7 3 5 1 3	−3	0 4 5 6 2 7 3 7 1 3	3	0 4 1 5 3 6 3 7 2 4	−3
0 4 5 6 2 3 5 7 1 3	3	0 4 5 6 2 7 3 5 1 3	3	0 4 5 6 1 3 2 5 3 6	−3
0 4 5 6 2 7 5 7 1 3	3	0 4 1 2 3 4 3 5 4 6	3	0 4 5 6 1 3 3 7 2 5	−3
0 2 1 5 3 6 3 4 3 4	3	0 4 1 2 3 6 3 7 4 5	3	0 4 5 6 1 3 3 5 2 6	3
0 2 1 5 3 6 4 7 3 4	−3	0 4 5 6 1 2 3 7 3 5	3	0 4 5 6 1 3 2 7 3 5	3
0 2 1 5 6 7 3 4 3 4	3	0 4 5 6 1 2 3 7 3 4	3	0 4 5 6 1 3 2 3 5 6	3
0 2 1 5 6 7 3 4 4 6	−3	0 4 2 5 3 6 3 4 1 4	−3	0 4 5 6 1 3 5 7 2 3	3
0 4 2 5 1 6 4 7 4 5	−3	0 4 2 5 3 6 3 7 1 4	−3	0 1 2 3 3 4 3 4 5 6	−3
0 4 2 5 1 6 4 7 3 5	−3	0 4 1 5 2 6 3 7 3 5	−3	0 1 2 3 3 4 3 7 4 5	3
0 4 2 5 1 6 3 7 4 5	−3	0 4 1 5 2 6 3 7 3 4	−3	0 1 2 3 3 4 3 5 4 6	−3
0 4 2 5 1 6 3 7 3 5	−3	0 4 1 5 3 6 3 4 2 4	3	0 2 1 3 3 4 3 4 5 6	3
0 4 1 5 3 6 3 4 2 3	3	0 4 1 5 3 6 3 7 2 4	3	0 2 1 3 3 4 3 7 4 5	−3
0 4 1 5 3 6 4 7 2 3	−3	0 4 2 5 1 6 3 7 3 5	3	0 2 1 3 3 4 3 5 4 6	3
0 4 1 5 6 7 2 3 3 4	3	0 4 2 5 1 6 3 7 3 4	3	0 4 1 2 3 4 3 4 5 6	−3
0 4 1 5 6 7 2 3 4 6	−3	0 2 1 3 3 4 3 5 4 6	−3	0 4 1 2 3 4 3 7 4 5	3

TABLE 3 (continued).

0 2 1 3 3 4 3 7 4 5	3	0 2 1 3 3 6 3 7 4 5	−3	0 4 1 2 3 4 3 5 4 6	−3
0 2 1 3 3 6 4 7 4 5	−3	0 2 1 5 3 6 3 7 3 5	3	0 4 1 5 2 3 3 4 5 6	3
0 2 1 5 3 6 4 7 3 4	3	0 2 1 5 3 6 3 7 3 4	3	0 4 1 5 2 3 3 7 4 5	3
0 2 1 5 3 6 4 7 4 5	−3	0 2 1 5 3 6 3 4 3 4	−3	0 4 1 5 2 3 3 5 4 6	−3
0 4 2 5 6 7 1 4 4 6	−3	0 2 1 5 3 6 3 7 3 4	−3	0 4 2 5 1 3 3 4 5 6	−3
0 4 2 5 6 7 1 4 3 6	−3	0 4 2 5 3 6 1 3 4 6	−3	0 4 2 5 1 3 3 7 4 5	−3
0 4 2 5 6 7 1 3 4 6	−3	0 4 2 5 3 6 4 7 1 3	−3	0 4 2 5 1 3 3 5 4 6	3

term in (9), by starting with a zero initial approximation for $\Diamond$. This is the perturbation scheme which we now outline.

In fact, the perturbation method was tried first, revealing the typical graph patterns and their topological complexity. From Proposition 2 we already know that $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = 0$ for Poisson brackets on $\mathbb{R}^3$. The difficulty is that because the condition $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = 0$ and the Jacobi identity $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ are valid, it is impossible to factorize one through the other; both are invisible. So, we first make both expressions visible by perturbing the Poisson bi-vector $\mathcal{P} \mapsto \mathcal{P}_\epsilon = \mathcal{P} + \epsilon\Delta$ in such a way that the tri-vector $[\![\mathcal{P}_\epsilon, \mathcal{Q}_{1:6}(\mathcal{P}_\epsilon)]\!]$ and the Jacobiator $[\![\mathcal{P}_\epsilon, \mathcal{P}_\epsilon]\!]$ stop vanishing identically:

$$[\![\mathcal{P}_\epsilon, \mathcal{Q}_{1:6}(\mathcal{P}_\epsilon)]\!] \neq 0 \quad \text{and} \quad [\![\mathcal{P}_\epsilon, \mathcal{P}_\epsilon]\!] \neq 0.$$

To begin with, put $\Diamond := 0$. Now consider the description [6] of Poisson brackets on $\mathbb{R}^3$ by using the pre-factor $f(x, y, z)$ and arbitrary function $g(x, y, z)$ in the formula

$$\{u, v\}_\mathcal{P} = f \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x, y, z)}\right);$$

it is helpful to start with some very degenerate dependencies of f and g of their arguments (see [1] and [21]). The next step is to perturb the coefficients of the Poisson bracket $\{\cdot, \cdot\}_\mathcal{P}$ at hand; in a similar way, one starts with degenerate dependency of the perturbation Δ. The idea is to take perturbations which destroy the validity of Jacobi identity for $\mathcal{P}_\epsilon$ in the linear approximation in the deformation parameter ϵ. It is readily seen that the expansion of (8) in ϵ yields the equality

$$[\![\mathcal{P}_\epsilon, \mathcal{Q}_{1:6}]\!](\epsilon) = (\Diamond + \bar{o}(1))\,([\![\mathcal{P}_\epsilon, \mathcal{P}_\epsilon]\!]) = 2\epsilon \cdot (\Diamond + \bar{o}(1))\,([\![\mathcal{P}, \Delta]\!]) + (\Diamond + \bar{o}(1))\,([\![\mathcal{P}, \mathcal{P}]\!]) + \bar{o}(\epsilon).$$

Knowing the left-hand side at first order in ϵ and taking into account that $[\![\mathcal{P}, \mathcal{P}]\!] \equiv 0$ for the Poisson bi-vector $\mathcal{P}$ which we perturb by Δ, we reconstruct the operator $\Diamond$ that now acts on the known tri-vector $2[\![\mathcal{P}, \Delta]\!]$. In this sense, the Jacobiator $[\![\mathcal{P}, \mathcal{P}]\!]$ shows up through the term $[\![\mathcal{P}, \Delta]\!]$.

For each pair $(\mathcal{P}, \Delta)$, the above balance at ϵ^1 contains sums over indexes that mark the derivatives falling on the Jacobiator. By taking those formulae, we guess the candidates for graphs that form the next, yet unknown, part of the operator $\Diamond$. Specifically, we inspect which differential operator(s), acting on the Jacobi identity, become visible and we list the graphs that provide such differential operators via the Leibniz rule(s). For a while we keep every such candidate with an undetermined coefficient. By repeating the iteration, now for a different Poisson bi-vector $\mathcal{P}$ or its new, less degenerate perturbation Δ, we obtain linear constraints for the already introduced undetermined coefficients. Simultaneously, we continue listing the new candidates and introducing new coefficients for them.

Remark 12. By translating formulae into graphs, we convert the dimension-dependent expressions into the dimension-independent operators which are encoded by the graphs. An obvious drawback of the method which is outlined here is that, presumably, some parts of the operator $\Diamond$ could always stay invisible for all Poisson structures over $\mathbb{R}^3$ if they show up only in the higher dimensions. Secondly, the number of variants to consider and in practice, the number of irrelevant terms, each having its own undetermined coefficient, grows exponentially at the initial stage of the reasoning.

By following the loops of iterations of this algorithm, we managed to find two non-zero coefficients and five zero coefficients in solution (9). Namely, we identified the coefficient ± 1 for the tripod, which is the first term in (9), and we also recognized the coefficient ± 3 of the sum of 'elephant' graphs, which is the second to last term in (9).

Remark 13. Because of the known skew-symmetry of the tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!]$ with respect to its arguments f, g, h, finding one term in a sum within formula (9) for $\Diamond$ means that the entire such sum is reconstructed. Indeed, one then takes the sum over a subgroup of S_3 acting on f, g, h, depending on the actual skew-symmetry of the term which has been found.

For instance, the first term in (9), itself making a sum running over $\{\text{id}\} \prec S_3$, is obviously totally antisymmetric with respect to its arguments. The other graph which we found by using the perturbation method (see the last graph in the second line of formula (9) on p. 11) is skew-symmetric with respect to its second and third arguments but it is not yet totally skew-symmetric with respect to the full set of its arguments. This shows that is suffices to take the sum over the group $\circlearrowright = A_3 \prec S_3$ of cyclic permutations of f, g, h, thus reconstructing the sixth term in solution (9).

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