



university of
 groningen

faculty of science
 and engineering

Fractals, Lindenmayer- Systems and Dimensions

BetaWetenschappelijk Onderzoek Wiskunde

March 2017

Student: K. Lok

First supervisor: prof. dr. H. Waalkens

Second supervisor: dr. A.E. Sterk

Abstract

In this paper we try to grasp mathematically what fractals are. In particular we discuss Lindenmayer systems, which can represent a large class of fractals in a mathematical way. In doing so, this paper discusses the Cantor set and the Koch curve in detail, expressing them in terms of the usual fractal geometry as well as in terms of L-systems. Different types of fractal dimensions are introduced and discussed in detail. We consider different aspects of the fractal dimensions and give a conclusion about the applicability of the dimensions and their desirable features. The L-system dimension that was introduced by Ortega and Alfonseca in [7] is analyzed in this paper and we discuss its limitations and usefulness.

Contents

1	Introduction	3
2	Fractals	4
2.1	Examples of fractals	4
2.1.1	The Cantor set	4
2.1.2	Koch curve	7
2.2	Dimension	9
2.2.1	Similarity dimension	10
2.2.2	Box-counting dimension	11
2.2.3	Hausdorff measure and dimension	16
3	Lindenmayer-systems	20
3.1	Grammar of L-systems	20
3.2	Turtle graphics	21
3.3	Examples of Lindenmayer-systems	22
3.3.1	The Koch curve	22
3.3.2	The Cantor set	23
3.4	Dimension	25
3.4.1	Problems in the definition of L-system fractal dimension	26
4	Comparison of the dimensions	29
4.1	Fractal dimensions	29
4.2	L-system dimension	32
5	Conclusion	33
A	MuPAD codes	36

1 Introduction

What do mountains, cauliflowers, plants and coastlines have in common? “*Probably not that much*” may be your first guess. But did you notice the repeating pattern you can observe in these objects? The amount of detail? They have more in common than you may think. Their properties are best explained by the theory of fractals, the branch in geometry that describes irregularities, or ‘mathematical monsters’ as they were labeled earlier. Grasping these self-similarities in nature in a mathematical way has always been a challenge and led to this field.

In the past, scientists thought that fractals were not worthy of studying. Only 40 years ago, people started to note that fractals were not so useless after all. In the early days of the field, fractals were only used as counterexamples. They could use for example the Koch curve for the fact that it is a continuous non-smooth curve, meaning that it does not admit a tangent at any of its points. We will discuss the Koch curve excessively in this paper. Later, people realized that there was more to study in this field. Namely, the study of fractals can be used to better understand nature. A lot of fractals can be identified in nature. For example the patterns in snowflakes, or the way plants grow. Or even inside the body, fractal patterns are found in kidneys and other organs.

In this paper we will introduce the ideas of fractals. In particular we discuss Lindenmayer systems, which can represent a large class of fractals in a mathematical way. In doing so, we will evaluate two different examples of fractals in the usual way as well as in terms of L-systems. The examples are widely analyzed. We will concentrate most on one part of the field of fractal geometry: fractal dimensions. There are many different sorts of dimensions, each with different applicability, different aspects and sometimes even different outcomes. In terms of usual fractal geometry, three common used fractal dimensions will be discussed here: the similarity dimension, the box-counting dimension and the Hausdorff dimension. In terms of L-systems, one other dimension is examined. This L-system fractal dimension is mentioned in Ortega and Alfonso, *Determination of fractal dimensions from equivalent L-systems* [7]. We will review this article. Then, a discussion follows about the dimensions we considered. We will end this paper with a conclusion.

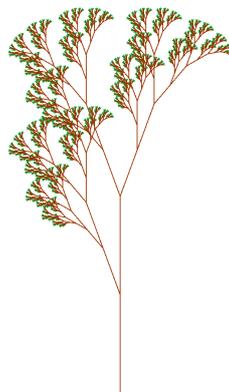


Figure 1 – A fractal representation of a plant.

2 Fractals

Fractals were not always considered to be an interesting subject. Sets or functions that were not sufficiently smooth or regular were not considered worth studying. However, slowly the attitude changed and people saw that irregular sets were often a better representation of nature than regular sets. Fractal geometry provides the general framework for those irregular sets. The word ‘fractal’ was introduced by Mandelbrot, and was derived from the Latin word *fractus* which means broken. In this way he described objects that were too irregular to fit into a traditional geometric setting.

2.1 Examples of fractals

Let us consider some simple examples of fractals. These examples are well-known since their construction idea is very simple, but the actual fractals have infinite detail.

2.1.1 The Cantor set

Georg Cantor (1845 – 1918) was a German mathematician who worked on the foundations of mathematics, now called *set theory*. He published the Cantor set in 1883 as a set with exceptional features. The Cantor set plays a role in many branches of mathematics and is therefore considered one of the most important early fractals, although it is less visually appealing than some of the other fractals which we will see later on.

The most basic Cantor set is an infinite set of points in the unit interval $[0, 1]$ and is constructed in the following way. Take E_0 as the entire interval $[0, 1]$. Then let E_1 be the set E_0 but with the middle third removed, so that we are left with $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. E_2 will then be the set of removing the middle thirds of the two combined sets of E_1 , so that E_2 comprises the four intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$. We continue like this, so in general we have that E_k is derived from removing all middle thirds of the intervals of which E_{k-1} consists. E_k consists of 2^k intervals of length 3^{-k} . The Cantor set F consists of the numbers that are in E_k for all k , i.e: $F = \bigcap_{k=0}^{\infty} E_k$ (see Figure 2). Obviously, it is impossible to draw F to its infinitesimal detail. The representations of F are therefore always pictures of one of the E_k 's. When k is large enough, this will be a good representation of F .

But what exactly is left of the Cantor set? We are removing so much of $[0, 1]$, does anything remain at all? We see that the end points of the intervals are always there (for example $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}$ and $\frac{8}{9}$). It is very tempting to believe that these are the only points that remain and therefore all points in F should be related to powers of $\frac{1}{3}$. If this was true, the set would be countable. However:

Theorem 2.1. *The Cantor set is uncountable.*

To prove that theorem 2.1 is true, we need to introduce the definition of limit points and perfect sets [1].

Definition 2.2 (Limit point). *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

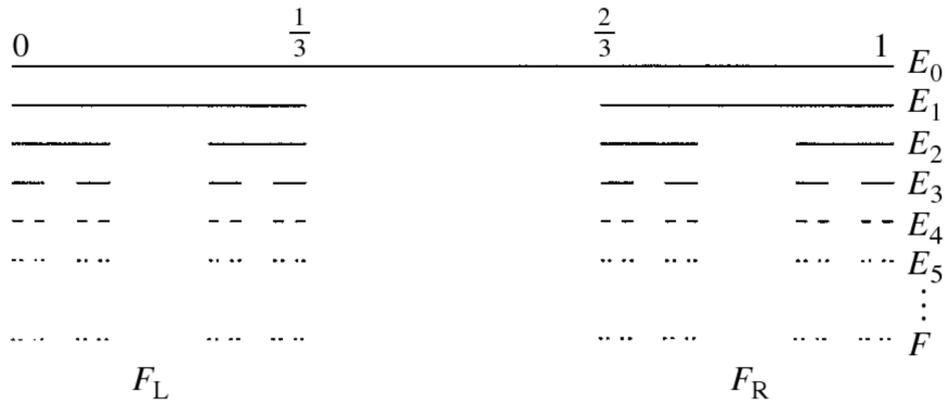


Figure 2 – The construction of the Cantor set F . Notice that F_L and F_R are in fact copies of F scaled down by the factor $\frac{1}{3}$.

We will need the definition of a limit point to state the definition of a perfect set. This will help us prove that the Cantor set is uncountable (theorem 2.1).

Definition 2.3 (Perfect sets). *A set $P \subseteq \mathbb{R}$ is called perfect if it is closed and every point in P is a limit point.*

This leads to the following theorem:

Theorem 2.4. *The Cantor set is perfect.*

Proof. We saw that the Cantor set is defined as the intersection $F = \bigcap_{k=0}^{\infty} E_k$, where each E_k is a finite union of closed intervals (2^k intervals). A union of a finite collection of closed sets is closed, so E_k is closed. F is now an intersection of closed sets, which means that F is closed. All that remains is showing that every point in F is a limit point.

To prove that, for all x in F , x is a limit point, we must construct for all x a sequence (a_k) of points in F and different from x itself that converges to x .

Take an arbitrary $x \in F$. We will first prove that for each $k \in \mathbb{N}$, there exists $a_k \in F$, $a_k \neq x$, satisfying $|x - a_k| \leq \frac{1}{3^k}$. For each $k \in \mathbb{N}$, the length of each interval that makes up E_k is $\frac{1}{3^k}$, and the endpoints are always elements of F . We have $x \in F$, so $x \in E_k$ for all $k \in \mathbb{N}$ and therefore x is in one of the intervals of which E_k consists.

For every k , let a_k be an endpoint of the interval that contains x , and if x happens to be an endpoint, take a_k to be the opposite endpoint of the interval. We now have $a_k \in F$ with $x \neq a_k$ such that $|x - a_k| \leq \frac{1}{3^k}$.

Since

$$\lim_{k \rightarrow \infty} \frac{1}{3^k} = 0,$$

it follows that $a_k \rightarrow x$. This means that x is a limit point, and since we have picked an arbitrary $x \in F$, we have now shown that all points of F are limit points. \square

The next theorem will be the last step in proving theorem 2.1 in which we stated that the Cantor set is uncountable.

Theorem 2.5. *A nonempty perfect set is uncountable.*

Proof. Let P be perfect and nonempty. Then it must be infinite. Let us assume for contradiction that P is countable. Then we can write P as a list of elements where every element of P appears on the list:

$$P = \{x_1, x_2, x_3, \dots\}.$$

Our goal is to construct a sequence of nested compact sets K_n , all contained in P , with the property that $x_1 \notin K_2$, $x_2 \notin K_3$, $x_3 \notin K_4$, \dots .

Let I_1 be a closed interval that contains x_1 in its interior (i.e., x_1 is not an endpoint of I_1). Now, x_1 is a limit point since it is in P and P is perfect, so there exists some other point $y_2 \in P$ that is also in the interior of I_1 . We can construct a closed interval I_2 , centered on y_2 , so that $I_2 \subseteq I_1$ but $x_1 \notin I_2$. Explicitly, if $I_1 = [a, b]$, let

$$\epsilon = \min\{y_2 - a, b - y_2, |x_1 - y_2|\}.$$

Then the interval $I_2 = [y_2 - \epsilon/2, y_2 + \epsilon/2]$ has the desired properties. This process can be continued. Because $y_2 \in P$ is a limit point, there must exist another point $y_3 \in P$ in the interior of I_2 , and we may insist that $y_3 \neq x_2$. Now we can do the same as above; we construct I_3 centered on y_3 and small enough so that $x_2 \notin I_3$ and $I_3 \subseteq I_2$. Observe that $I_3 \cap P \neq \emptyset$, because this intersection contains at least y_3 . We can do this inductively over and over again, resulting in a sequence of closed intervals I_n satisfying

- $I_{n+1} \subseteq I_n$
- $x_n \notin I_{n+1}$, and
- $I_n \cap P \neq \emptyset$

Now let $K_n = I_n \cap P$. Each K_n is a subset of P , so the intersection $\bigcap_{n=1}^{\infty} K_n$ can only contain elements of P , if it contains any elements at all. However when we take an arbitrary element $x_n \in P$, $x_n \notin I_{n+1}$ and therefore $x_n \notin K_{n+1}$, so $x_n \notin \bigcap_{n=1}^{\infty} K_n$. This leads to the conclusion that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset. \quad (1)$$

But, for each $n \in \mathbb{N}$, we have that K_n is closed because it is the intersection of closed sets, and bounded because it is contained in the bounded set I_n . Hence, K_n is compact. By construction, K_n is not empty and $K_{n+1} \subseteq K_n$. Hence

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset. \quad (2)$$

Equations 1 and 2 are contradicting each other, which means P should be uncountable. \square

With this knowledge, we can prove theorem 2.1 in just a few lines:

Proof. The Cantor set is non-empty, since it contains for example 0 and 1. It is a perfect set (see theorem 2.4). Every non-empty, perfect set is uncountable (see theorem 2.5). Hence the Cantor set is uncountable. \square

We now know that the Cantor set is uncountable, meaning we can not number all the points in the set (i.e. there exists no bijection to \mathbb{N}). Hence the set consists of many more points than just the end points of the intervals. This indicates the complexity of the Cantor set even more.

Another interesting feature of the Cantor set is its self-similarity. The notion of similarity is clear: Two objects are *similar* if they have the same shape, regardless of their size. Corresponding angles must be equal, and corresponding line segments must all have the same factor of proportionality. For example when a photo is enlarged, it is enlarged by the same factor in both horizontal and vertical directions (otherwise the enlarged photo will look distorted). The enlargement factor is called the scaling factor. The idea of self-similarity extends the term similarity. If a set is *self-similar*, the set consists of small copies of itself. Take a look at Figure 2. Here F_L is an exact copy of F , but scaled by a factor $\frac{1}{3}$. There are different degrees of self-similarity. Sets are for example also called self-similar when they contain replicas of the whole at only a few points, or when it contains replicas that are not strictly similar to the whole, but only approximately. The Cantor set is self-similar in its purest form: The set is composed only of exact copies of itself and is therefore called *strictly self-similar*.

Other features of the Cantor set are its ‘fine structure’ (it contains detail at arbitrarily small scales), its recursive pattern and the fact that its size is not easily defined. F is in some sense a very large set (it is uncountably infinite) but its length is 0 by all reasonable definitions of length. We will come back to the size of the set later.

2.1.2 Koch curve

Helge von Koch was a Swedish mathematician who introduced in 1904 what is now called the *Koch curve* (Figure 3). Fitting together three suitably rotated copies of the curve produces a snowflake curve, sometimes also called the *Koch island* (Figure 4).

The geometric construction of the Koch curve is simple. First start with a line-segment E_0 of length 1. To generate E_1 you replace the middle third by an equilateral triangle and take away the base. E_2 will then be made after applying the same procedure to each segment of E_1 , etcetera. In general, E_k comes from replacing the middle third of each segment of E_{k-1} by an equilateral triangle. When k is large, E_k and E_{k+1} differ only in fine detail. As k tends to infinity, E_k approaches the limit curve F , the Koch curve (Figure 3).

Let us look at the length of the curve. We know that E_0 is of length 1, because that is what we defined it. Then E_1 consists of four line segments, each a factor $\frac{1}{3}$ smaller than the original. So we are left with a length of $4 \cdot \frac{1}{3}$. E_2 has then $4 \cdot 4$ line-segments, each of length $\frac{1}{3} \cdot \frac{1}{3}$. So the length of E_2 is $(\frac{4}{3})^2$. Generally, the length of E_k is $(\frac{4}{3})^k$, and when k tends to infinity, the length also tends to infinity. This means that the Koch curve has infinite length. However, the curve does not occupy any area in the plane, hence length and area are not a very useful description of the size of F .

The Koch curve is also special in another sense. It is an example of a non-smooth

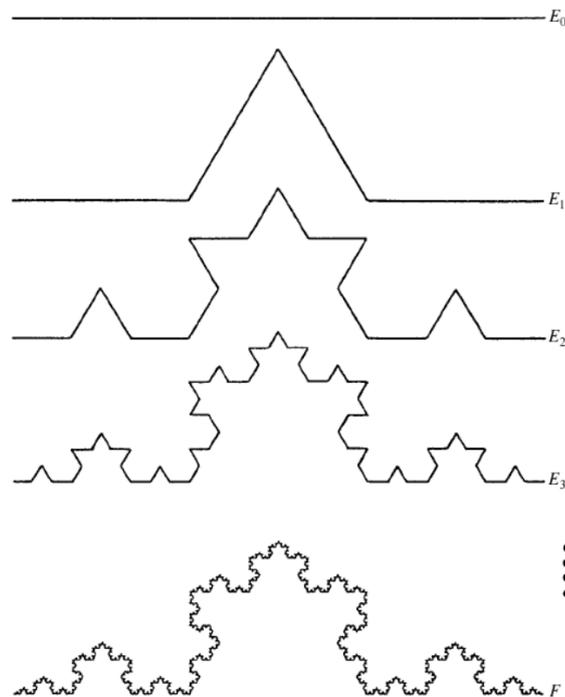


Figure 3 – The construction of the Koch curve

curve, that is: a curve which does not admit a tangent at any of its points. Intuitively, we know that when a curve has a corner, one cannot have a unique tangent line at that point. The Koch curve is actually a curve which has corners everywhere and therefore it cannot be differentiated anywhere.

We observe that the Koch curve has features that are very similar to the features of the Cantor set. The Koch curve is also self-similar (strictly self-similar to be precise: The entire curve consists for example of four copies of itself scaled by a factor $\frac{1}{3}$), it also has a fine structure, an easy to use definition but intricate detail and straightforward and its size is also not easily determined.

Although it is nearly impossible to have a precise definition of a fractal, we agree that we have those features in mind when we talk about a fractal. However it is very unsatisfactory that we cannot yet determine something like a size of the fractals. Fortunately, we can measure the size of a fractal with *dimensions*. In the following subsection we will see several definitions of dimension and at the end of that subsection we have obtained the final (and most important) feature for something to be a fractal.

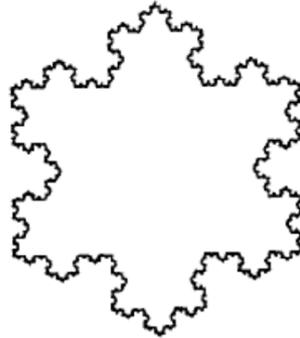


Figure 4 – Three Koch curves fitted together form a Koch snowflake.

2.2 Dimension

The concept of dimension is very old and it seems easy and clear to use. Most of the time when we refer to dimensions, we mean the number of directions in which movement is allowed. In this sense, a dimension is always an integer: 0 (a point), 1 (a line segment), 2 (a plane), 3 (a volume) and so on. Poincaré's idea of dimension was similar to this. It is inductive in nature and started with a point: a point has dimension 0. Then a line segment has dimension 1, because it can be split into two parts by a point (which has dimension 0). A square has dimension 2, because it can be split into two parts by a line (which has dimension 1) and a cube has dimension 3, because it can be split into two parts by a square (which has dimension 2) [8]. These two formulations are the intuitive translations of the definition of the *topological dimension* [3]. The formal definition for a topological dimension, i.e. the Lebesgue covering dimension, uses some definitions which we will state first. After that, the definition of the Lebesgue covering dimension is given.

Definition 2.6 (Order and refinement). *Let X be a topological space. The order of a cover is the smallest number n (if it exists) such that each point of the space belongs to at most n sets in the cover, where an (open) cover of the space X is a family of (open) sets whose union contains X .*

A refinement of a cover C is another cover, each of whose sets is a subset of a set in C . The order of the refinement can be smaller but also larger than the order of C .

Definition 2.7 (Topological dimension). *The Lebesgue covering dimension of a space X is defined to be the minimum value of n , such that every open cover C of X has an open refinement with order $n + 1$ or below. If no such minimal n exists, the space is said to be infinite dimensional.*

Fractals do have a Lebesgue covering dimension. When we consider for example the Koch curve, we can find out that each point of the curve belongs to at most 2 sets of the refinement. The Koch curve therefore has Lebesgue covering dimension 1. For the Cantor set a similar argument holds; each point of the Cantor set belongs to at most 1 set of the refinement and therefore has Lebesgue covering dimension 0.

However, it is difficult to state a fully precise proof of the topological dimensions of these sets due to the need of a topology.

The topological dimension is not satisfactory when we take fractals in consideration. Let us take the Koch curve as an example. The curve has infinite length on a bounded area as we saw earlier, which suggests that the dimension is greater than 1, but since it has zero area it should have a dimension less than 2. Therefore there are other methods to measure the dimension of a fractal. Fractal dimensions are almost always greater than their topological dimension. Fractal dimensions provide a better idea of the complexity of the set or curve. There are for example curves that are space-filling and therefore have a fractal dimension of almost 2, relating more to a plane, but also curves that are more like the Koch curve, which have a fractal dimension closer to 1, meaning that it is more related to a line than to a plane. Their topological dimension will be the same in both cases, providing nothing useful about the curves. The fractal dimension is therefore way more useful for our understanding of fractals.

Before we continue discussing some definitions of fractal dimensions, it is important to note that there are many definitions and one can be more adequate than the other. It depends on the kind of fractal which dimension fits best. The dimensions can also give different values of dimensions for the same set. Therefore, when reading about fractals, it is needed to know which definition of dimension is being used.

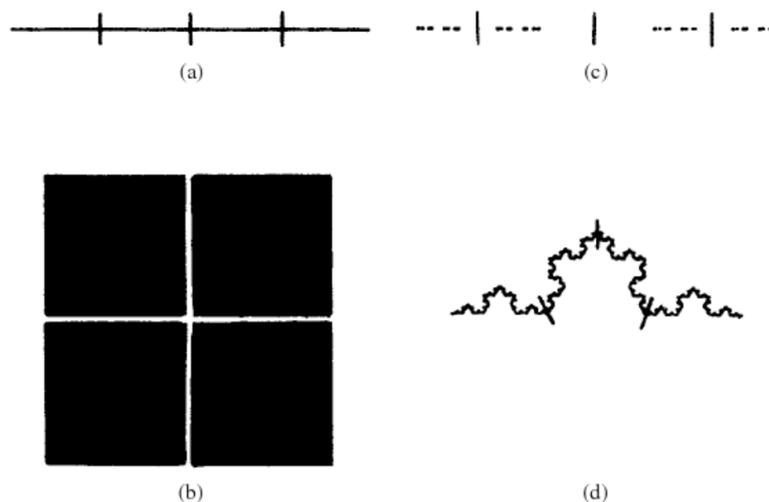


Figure 5 – Division of certain sets into four parts. The parts are similar to the entire set with ratio's (a) $\frac{1}{4}$ for a line segment, (b) $\frac{1}{2}$ for a square, (c) $\frac{1}{9}$ for the Cantor set, (d) $\frac{1}{3}$ for the Koch curve.

2.2.1 Similarity dimension

There are many different definitions of dimension. One of them is the *similarity dimension*, which uses the way in which fractals are self-similar:

Definition 2.8 (Similarity dimension). *A set A made up of m parts similar to itself scaled by a factor r has similarity dimension*

$$\dim_S(A) = -\frac{\log(m)}{\log(r)}.$$

Example 2.9. *Consider a line segment (Figure 5a). A line segment L is made up of four copies of itself scaled by a factor $\frac{1}{4}$. The dimension of a line segment is then $\dim_S(L) = -\frac{\log(4)}{\log(\frac{1}{4})} = 1$.*

A square S however (Figure 5b) is made up of four copies of itself scaled by a factor $\frac{1}{2}$. The dimension is then $\dim_S(S) = -\frac{\log(4)}{\log(\frac{1}{2})} = 2$. This corresponds to the values obtained with the topological dimension and Poincaré's inductive method.

For the fractals we discussed earlier, the Cantor set and the Koch curve, we find a more interesting value for their dimensions. The Cantor set C (Figure 5c) is made up of four copies of itself scaled down by a factor $\frac{1}{3}$. This results in a dimension of

$$\dim_S(C) = -\frac{\log(4)}{\log(\frac{1}{3})} = \frac{\log(4)}{\log(3)} = 0.631 \dots$$

and the Koch curve K (Figure 5d) has in this sense dimension

$$\dim_S(K) = -\frac{\log(4)}{\log(\frac{1}{3})} = \frac{\log(4)}{\log(3)} = 1.262 \dots$$

We observe in this example that the Koch curve has a similarity dimension less than 2, but greater than 1, as we suspected earlier. However the similarity dimension only works for a small range of fractals, namely fractals that are strictly self-similar.

2.2.2 Box-counting dimension

Fortunately there are other definitions of dimensions that are applicable to a wider range of fractals. Fundamental to most definitions of dimension is the idea of 'measurement of the set at scale δ '. For each δ , we measure a set in a way that detects irregularities of size δ , and we see how these measurements behave as $\delta \rightarrow 0$.

Let us consider the box-counting dimension, which is the most widely used dimension in the context of fractals. Given a subset F of \mathbb{R}^n , for each $\delta > 0$, we find the smallest number of sets with diameter at most δ that can cover the set F and we call this number $N_\delta(F)$. This number indicates into how many 'clumps' of size about δ the set F can be divided. This number will grow when $\delta \rightarrow 0$.

We recall that when U is any non-empty subset of an n -dimensional Euclidean space \mathbb{R}^n , the *diameter* of U is defined as

$$|U| = \sup\{|x - y| : x, y \in U\},$$

i.e. the greatest distance between any two points in U . If $\{U_i\}$ is a finite collection of sets of diameter at most δ that cover F , we call $\{U_i\}$ a δ -cover of F .

We defined $N_\delta(F)$ to be the smallest number of subsets that form a δ -cover of F . The way in which $N_\delta(F)$ grows when $\delta \rightarrow 0$ will be the dimension. This will go as follows: When $N_\delta(F)$ satisfies (at least approximately) a power law

$$N_\delta(F) \simeq c\delta^{-s}$$

for positive constants c and s , we say that F has box-counting dimension s . To solve for s , we take the logarithm to obtain

$$\log(N_\delta(F)) \simeq \log(c) - s \log(\delta),$$

so after rewriting we get

$$s \simeq \frac{\log(N_\delta(F))}{-\log(\delta)} + \frac{\log(c)}{\log(\delta)}.$$

When we now take the limit $\delta \rightarrow 0$, we will get rid of the last term and find the dimension s precisely:

$$s = \lim_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)}.$$

The *lower* and *upper box-counting dimensions* of F are respectively defined as:

$$\dim \inf_B(F) = \liminf_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)}$$

$$\dim \sup_B(F) = \limsup_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)}$$

The lower dimension is clearly smaller or equal to the upper dimension. When the dimensions are equal, we say that $\dim \inf_B(F) = \dim \sup_B(F) = \dim_B(F)$. In short we have the following definition:

Definition 2.10 (Box-counting dimension). *The box-counting dimension $\dim_B(F)$ of a subset F of the plane is given by*

$$\dim_B(F) = \lim_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)},$$

where $N_\delta(F)$ is the smallest number of sets with diameter at most δ that can cover the set F .

The box-counting dimension has the property of monotonicity, meaning that when $E \subset F$, then $\dim_B(E) \leq \dim_B(F)$. This follows directly from the definition. Another feature is that the box-counting dimension of a set $F \subset \mathbb{R}^n$ never exceeds n .

Very important for the actual computation is the following theorem, where we state that it is enough to consider limits as δ approaches 0 through a decreasing sequence δ_k :

Theorem 2.11. *Suppose the sequence of positive real numbers $\{\delta_k\}_{k=1}^\infty$ converging to zero satisfies, for some $c \in (0, 1)$, the inequalities $c \geq \delta_k > \delta_{k+1} \geq c\delta_k$, ($1 \leq k < \infty$). Then for any set $F \subset \mathbb{R}^n$ one has*

$$\dim \inf_B(F) = \liminf_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)}, \quad (3)$$

$$\dim \sup_B(F) = \limsup_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)}. \quad (4)$$

Proof. We will start with proving the first line, equation 3. In order to prove this, we need to understand what a limit inferior is. We will first make some statements about general functions $f : \mathbb{R} \rightarrow \mathbb{R}$. We will use this to say equivalent things about our function, $\delta \mapsto \frac{\log(N_\delta(F))}{-\log(\delta)}$.

In general, we know that when $A \subset B \subset \mathbb{R} \Rightarrow \inf A \geq \inf B$. We take $(x_k) \rightarrow 0$, a decreasing sequence.

By definition of the limit inferior,

$$\begin{aligned} \liminf_{x \rightarrow 0} f(x) &:= \lim_{\varepsilon \rightarrow 0} (\inf\{f(x) : x \in (-\varepsilon, \varepsilon) \setminus \{0\}\}) \\ &= \lim_{k \rightarrow \infty} (\inf\{f(x) : x \in (-x_k, x_k) \setminus \{0\}\}) \\ \liminf_{k \rightarrow \infty} f(x_k) &:= \lim_{k \rightarrow \infty} (\inf\{f(x_n) : n > k\}). \end{aligned}$$

Since (x_k) is decreasing, we have that

$$\{f(x_n) : n > k\} \subset \{f(x) : x \in (-x_k, x_k) \setminus \{0\}\}.$$

We thus know that

$$\inf\{f(x_n) : n > k\} \geq \inf\{f(x) : x \in (-x_k, x_k) \setminus \{0\}\}.$$

This leads to the fact that

$$\liminf_{x \rightarrow 0} f(x) \leq \liminf_{k \rightarrow \infty} f(x_k).$$

Since this holds for any function, the following holds:

$$\begin{aligned} \dim \inf_B(F) &= \liminf_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \\ &\leq \liminf_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)}. \end{aligned} \quad (5)$$

Given $\delta \in (0, \delta_2)$, choose k such that $\delta_{k+1} \leq \delta < \delta_k$. Then the opposite inequality can be proven using $1 > \frac{\delta_{k+1}}{\delta_k} \geq c$, $0 > \log(\frac{\delta_{k+1}}{\delta_k}) \geq \log c$ and $N_\delta(A) > N_{\delta_k}(A)$ in the following way:

$$\begin{aligned} \frac{\log(N_\delta(F))}{-\log(\delta)} &\geq \frac{\log(N_{\delta_k}(F))}{-\log(\delta_{k+1})} \\ &= \frac{\log(N_{\delta_k}(F))}{-(\log(\delta_k) + \log(\frac{\delta_{k+1}}{\delta_k}))} \\ &\geq \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k) - \log(c)} \end{aligned}$$

And so

$$\dim \inf_B(F) = \liminf_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \geq \liminf_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)}. \quad (6)$$

The two inequalities 5 and 6 combined give us equality 3.

The second equality can be proven in a similar way. The definitions of limit superior are similar to the definitions of limit inferior:

$$\begin{aligned} \limsup_{k \rightarrow \infty} f(x) &:= \lim_{\varepsilon \rightarrow 0} (\sup\{f(x) : x \in (-\varepsilon, \varepsilon) \setminus \{0\}\}) \\ &= \lim_{k \rightarrow \infty} (\sup\{f(x) : x \in (-x_k, x_k) \setminus \{0\}\}), \\ \limsup_{k \rightarrow \infty} f(x_k) &:= \lim_{k \rightarrow \infty} (\sup\{f(x_k) : n > k\}). \end{aligned}$$

When $A \subset B \subset \mathbb{R}$, then $\sup A \leq \sup B$, so

$$\sup\{f(x_k) : n > k\} \leq \sup\{f(x) : x \in (-x_k, x_k) \setminus \{0\}\},$$

so

$$\limsup_{x \rightarrow 0} f(x) \geq \limsup_{k \rightarrow \infty} f(x_k).$$

This again holds for any function $f(x)$, so consequently

$$\begin{aligned} \dim \sup_B(F) &= \limsup_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \\ &\geq \limsup_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)}. \end{aligned} \quad (7)$$

Then, given $\delta \in (0, \delta_2)$, choose k such that $\delta_{k+1} \leq \delta < \delta_k$. We observe that

$$\begin{aligned} \frac{\log(N_\delta(F))}{-\log(\delta)} &\leq \frac{\log(N_{\delta_{k+1}}(F))}{-\log(\delta_k)} \\ &= \frac{\log(N_{\delta_{k+1}}(F))}{-(\log(\delta_{k+1}) - \log(\frac{\delta_{k+1}}{\delta_k}))} \\ &\leq \frac{\log(N_{\delta_{k+1}}(F))}{-\log(\delta_{k+1}) + \log(c)}. \end{aligned}$$

Therefore,

$$\dim \sup_B(F) = \limsup_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \leq \limsup_{k \rightarrow \infty} \frac{\log(N_{\delta_{k+1}}(F))}{-\log(\delta_{k+1})}. \quad (8)$$

Inequalities 7 and 8 combined give equality 4. \square

We can use this theorem to find the box-counting dimension of the Cantor set.

Example 2.12. *Let us calculate the box-counting dimension of the Cantor set F . We know that $F = \bigcap_{k=0}^{\infty} E_k$, where E_k consist of 2^k intervals of length 3^{-k} . We can cover F by the 2^k intervals of E_k of length 3^{-k} . Hence for the cover of F we choose δ_k to be 3^{-k} . We can use this δ_k , since for $c = \frac{1}{3} \in (0, 1)$, $c \geq \delta_k > \delta_{k+1} \geq c\delta_k$, ($1 \leq k < \infty$) holds ($\frac{1}{3} \geq \frac{1}{3^k} > \frac{1}{3^{k+1}} = \frac{1}{3} \cdot \frac{1}{3^k}$). $N_{\delta_k}(F) = 2^k$, hence*

$$\dim \inf_B(F) = \liminf_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)} = \liminf_{k \rightarrow \infty} \frac{\log(2^k)}{-\log(3^{-k})} = \liminf_{k \rightarrow \infty} \frac{k \log(2)}{k \log(3)} = \frac{\log(2)}{\log(3)}$$

And

$$\dim \sup_B(F) = \limsup_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)} = \limsup_{k \rightarrow \infty} \frac{\log(2^k)}{-\log(3^{-k})} = \limsup_{k \rightarrow \infty} \frac{k \log(2)}{k \log(3)} = \frac{\log(2)}{\log(3)}.$$

We can conclude that the box-counting dimension of the Cantor set F is equal to $\dim_B(F) = \frac{\log(2)}{\log(3)}$.

The box-counting dimension of the Koch curve is calculated in the same way:

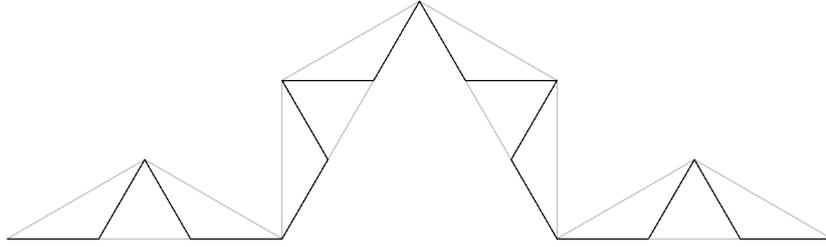


Figure 6 – E_2 of the Koch curve. The grey lines indicate the four triangles of diameter $\frac{1}{3}$ that can be used to cover the curve.

Example 2.13. *The box-counting dimension of the Koch curve F . We can cover the Koch curve by triangles. In Figure 6 we see how this is done for E_2 . In general, the diameter of such a triangle will be 3^{-k} and the number of triangles needed to cover the curve is 4^k . We can use $\delta_k = 3^{-k}$ for similar reasons as in example 2.12. We compute the dimension of the Koch curve as follows:*

$$\dim \inf_B(F) = \liminf_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)} = \liminf_{k \rightarrow \infty} \frac{\log(4^k)}{-\log(3^{-k})} = \liminf_{k \rightarrow \infty} \frac{k \log(4)}{k \log(3)} = \frac{\log(4)}{\log(3)}$$

And

$$\dim \sup_B(F) = \limsup_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(F))}{-\log(\delta_k)} = \limsup_{k \rightarrow \infty} \frac{\log(4^k)}{-\log(3^{-k})} = \limsup_{k \rightarrow \infty} \frac{k \log(4)}{k \log(3)} = \frac{\log(4)}{\log(3)}.$$

Since the two are equal, we can now state that the box-counting dimension of the Koch curve is equal to $\dim_B(F) = \frac{\log(4)}{\log(3)}$.

2.2.3 Hausdorff measure and dimension

The Hausdorff dimension is probably the oldest way to determine the dimension of a set, and has the advantage that it is defined for *any* set. However, the computation can be a lot harder than the computations of other dimensions. In order to find the Hausdorff dimension of a fractal, we first need to define a measure. The measure we are using here is the Hausdorff measure. The *Hausdorff measure* is an indication of length, area or volume. The Hausdorff measure is usually denoted as \mathcal{H}^s , where the s stands for the “dimensionality”. Roughly it measures a subset $A \subset \mathbb{R}^n$ by trying to cover A by countably many sets C_j ($1 \leq j < \infty$) with a diameter that is not exceeding an arbitrarily small $\delta > 0$ in such a way that the sum of the “ s -dimensional measures” of “ s -dimensional balls” of the same diameter as C_i gets as small as possible.

We define the Hausdorff measure $\mathcal{H}^s(F)$ as follows:

Definition 2.14 (Hausdorff measure). *The s -dimensional Hausdorff measure of a set F is defined as the limit of $\mathcal{H}_\delta^s(F)$ with $\delta \rightarrow 0$:*

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F),$$

where $\mathcal{H}_\delta^s(F)$ is defined as

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=0}^{\infty} |U_i|^s : \{U_i\} \text{ a } \delta\text{-cover of } F \text{ with } |U_i| < \delta \right\}.$$

We recall from the box-counting dimension that when U is any non-empty subset of an n -dimensional Euclidean space \mathbb{R}^n , the *diameter* of U is defined as

$$|U| = \sup\{|x - y| : x, y \in U\},$$

i.e. the greatest distance between any two points in U . In particular, we observe that the s -dimensional Hausdorff measure of the empty set is 0 and $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ if $A \subset B$. Moreover, $\mathcal{H}^1(A)$ is the length of a smooth curve A ; $\mathcal{H}^2(A)$ is the area of a smooth surface A up to a factor of $\frac{\pi}{4}$ and $\mathcal{H}^3(A)$ is the volume of a smooth three-dimensional manifold A up to a factor of $\frac{4\pi}{3}$ [8].

We can adapt our definition of the s -dimensional Hausdorff measure slightly to compensate for this factor, using

$$\alpha(s) := \frac{\pi^{s/2}}{\Gamma(s/2 + 1)},$$

where Γ denotes the gamma-function, and

$$\alpha(0) = 1 \quad \alpha(1) = 2 \quad \alpha(2) = \pi \quad \alpha(3) = \frac{4\pi}{3}.$$

This factor $\alpha(s)$ is the volume of a unit sphere in s dimensions, when s is an integer. Using this, we normalize the Hausdorff measure in such a way that in the case of Euclidean space it coincides exactly with the Lebesgue measure. The Lebesgue

measure is a standard way to assign a measure to a subset of the n -dimensional Euclidean space. For $n = 1, 2$ or 3 , it coincides with the usual measure of length, area or volume.

So we adapt our definition in the following way:

Definition 2.15 (Adapted Hausdorff measure). *The adapted s -dimensional Hausdorff measure of a set F is defined as the limit of $\mathcal{H}_\delta^s(F)$ with $\delta \rightarrow 0$:*

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F),$$

where $\mathcal{H}_\delta^s(F)$ is defined as

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \alpha(s) \cdot \sum_{i=0}^{\infty} \left(\frac{|U_i|}{2} \right)^s : \{U_i\} \text{ a } \delta\text{-cover of } F \text{ with } \text{diam}|U_i| < \delta \right\}.$$

For every set $A \subset \mathbb{R}^n$, there exists a number s_0 for which the following is true:

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{for } 0 \leq s < s_0 \\ 0 & \text{for } s > s_0 \end{cases}$$

This number s_0 is defined to be the *Hausdorff dimension*. If $s = s_0$, then $\mathcal{H}^s(F)$ may be zero, infinite or some positive real number.

Now, when A is a smooth curve, its dimension is 1 and $\mathcal{H}^1(A)$ is the length of the curve A . When A is a smooth surface, then its dimension is 2 and $\mathcal{H}^2(A)$ is the area of the surface A . Finally, when A is a smooth three-dimensional manifold, then its dimension is 3 and $\mathcal{H}^3(A)$ is the volume of the manifold A .

So when A is a smooth surface, we know that it has Hausdorff dimension of 2. The Hausdorff measure for A is then:

$$\mathcal{H}^s(A) = \begin{cases} \infty & \text{for } 0 \leq s < 2 \\ \text{Vol}(A) & \text{for } s = 2 \\ 0 & \text{for } s > 2 \end{cases}$$

Formally, we state the following definition for the Hausdorff dimension:

Definition 2.16 (Hausdorff dimension). *The Hausdorff dimension $\dim_H(F)$ of a set F is denoted by*

$$\dim_H(F) = s_0 = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

Intuitively, the number s_0 for which the Hausdorff measure of F jumps from being ∞ to being 0 is equal to the Hausdorff dimension of F . This dimension has monotonicity (i.e. when $E \subset F$, then $\dim_H(E) \leq \dim_H(F)$) and if $F \subset \mathbb{R}^n$, then $\dim_H(F) \leq n$.

There are two nice theorems on Hausdorff measures which can help us with the heuristic calculation of the Hausdorff dimension of strictly self-similar fractals, like the Cantor set and the Koch curve.

Theorem 2.17. *Let $F \subset \mathbb{R}^n$ and $f : F \rightarrow \mathbb{R}^m$ be a Hölder mapping such that*

$$|f(x) - f(y)| \leq c \cdot |x - y|^\alpha \quad x, y \in F$$

for constants $\alpha > 0$ and $c > 0$. Then for each s ,

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F). \quad (9)$$

Proof. If $\{U_i\}$ is a δ -cover of F , then

$$|f(F \cap U_i)| \leq c \cdot |F \cap U_i|^\alpha \leq c \cdot |U_i|^\alpha.$$

From that, it follows that $\{f(F \cap U_i)\}$ is a $c\delta^\alpha$ -cover of $f(F)$. Then

$$\left(\sum_i |f(F \cap U_i)| \right)^{s/\alpha} \leq \left(\sum_i c \cdot |U_i|^\alpha \right)^{s/\alpha},$$

and so

$$\sum_i |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \sum_i |U_i|^s.$$

From this and using definition 2.14, it follows that

$$\mathcal{H}_{c\delta^\alpha}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}_\delta^s(F).$$

When we take the limit of $\delta \rightarrow 0$, we obtain equation 9. □

When we enlarge a curve, plane or 3-dimensional object, we have to deal with an enlargement factor. When we have an enlargement by a factor λ , it is well-known the length of a curve has to be multiplied by λ , the length of the area of a plane is multiplied by λ^2 and the volume of a 3-dimensional object is multiplied by λ^3 . For the Hausdorff measure we can say something similar: the s -dimensional Hausdorff measure scales with a factor λ^s . This property follows directly from theorem 2.17 and is fundamental for Hausdorff measures.

Theorem 2.18 (Scaling property). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a similarity transformation with scaling factor λ . If $F \subset \mathbb{R}^n$, then*

$$\mathcal{H}^s(f(F)) = \lambda^s \mathcal{H}^s(F).$$

Proof. Since f is a similarity transformation with scaling factor λ , we know that $|f(x) - f(y)| = \lambda \cdot |x - y|$. Applying theorem 2.17 with $\alpha = 1$ and $c = \lambda$, we conclude that $\mathcal{H}^s(f(F)) = \lambda^s \mathcal{H}^s(F)$. □

A nice consequence of the scaling property is that when f is a congruence or isometry, i.e. $|f(x) - f(y)| = |x - y|$, then $\mathcal{H}^s(f(F)) = \mathcal{H}^s(F)$. Thus, Hausdorff measures are translation invariant ($\mathcal{H}^s(F + z) = \mathcal{H}^s(F)$, where $F + z = \{x + z : x \in F\}$), and rotation invariant, which is a property we would certainly expect from a measure.

Example 2.19. *Let us now calculate the Hausdorff dimension of the Cantor set F , using the previous theorem. We know that we can split F into two parts, a left part $F_L = F \cap [0, \frac{1}{3}]$ and a right part, $F_R = F \cap [\frac{2}{3}, 1]$. These parts are completely similar to F itself, but scaled by a factor $\frac{1}{3}$, as we have seen earlier in section 2.1.1. $F = F_L \cup F_R$ where the union is disjoint. Now, for any s ,*

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(F) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(F),$$

by the scaling property 2.18. We will assume that for the critical value $s = \dim_H(F)$, $0 < \mathcal{H}^s(F) < \infty$. This is a strong assumption, but it is one that can be justified [4]. We may now divide by $\mathcal{H}^s(F)$, to obtain $1 = 2\left(\frac{1}{3}\right)^s$. This naturally leads to the Hausdorff dimension of the Cantor set being $s = \frac{\log(2)}{\log(3)}$.

We can calculate the Hausdorff dimension of the Koch curve in a similar way.

Example 2.20. *The Hausdorff dimension of the Koch curve F can be calculated as follows. The Koch curve is made of four copies of itself, lets call them F_c , scaled by a factor of $\frac{1}{3}$. Now we can say that for any s ,*

$$\mathcal{H}^s(F) = 4 \cdot \mathcal{H}^s(F_c) = 4 \cdot \left(\frac{1}{3}\right)^s \mathcal{H}^s(F).$$

Again, we assume that for the critical value $s = \dim_H(F)$, $0 < \mathcal{H}^s(F) < \infty$. After dividing by $\mathcal{H}^s(F)$, we acquire $1 = 4\left(\frac{1}{3}\right)^s$. We end up with a Hausdorff dimension of the Koch curve of $\dim_H(F) = s = \frac{\log(4)}{\log(3)}$.

We can clearly only apply theorem 2.18 on strictly self-similar fractals. Calculating the Hausdorff measure and dimension of fractals without the use of the last theorem can be particularly hard and is therefore not included in this paper.

3 Lindenmayer-systems

Fractals are almost always regarded as *static*. However, in nature, a fractal is always a result of a growth process. Indeed, a Koch curve also seems to be growing after applying the same step over and over again, but our focus was the end-result. Biologist Aristid Lindenmayer invented in 1968 a language which shows more of the intermediate stages in the production of a fractal. This language of fractals was specifically created for the description of natural growth processes and therefore enables us to see in more detail how a fractal grows. The central concept of L-systems is that of re-writing.

Example 3.1. *One example of an L-system is a plant which consists of a string of cells. This plant we are considering has two types of cells: the type that does not divide and a type that does divide. The last type is responsible for the growth of the plant. We will call the type that does not divide A and the other type will be called B. B splits into four cells: a cell that can divide, then two cells that do not divide and at the end a cell that can divide. So formally we see that the production rule is: $B \rightarrow BAAB$. A does not divide, so its production rule is $A \rightarrow A$. Before the plant starts growing, it consists of only one cell: B. The next step in the growth-process of the plant is then BAAB. We now have to apply the production rules simultaneously to the four cells. This results in ten cells:*

$$BAABAABAAB.$$

The next cell division gives us 22 cells:

$$BAABAABAABAABAABAABAAB,$$

and so forth.

We will first define L-systems and continue with its graphical interpretation, which together can help to represent the Cantor set and the Koch curve in a nice and simple manner, as an L-system.

3.1 Grammar of L-systems

Definition 3.2 (L-system). *An L-system is defined by an alphabet*

$$V = \{a_1, a_2, \dots, a_n\},$$

the production map

$$\begin{aligned} P : V &\rightarrow V^* \\ a &\mapsto P(a), \end{aligned}$$

where V^ is the set of all strings formed by symbols from V , and an axiom*

$$a^{(0)} \in V^*,$$

the initial string.

The production (or rewriting) rules are applied iteratively starting from the axiom, and as many rules as possible are applied simultaneously per iteration.

In addition, an L-system is *context-free* if each production rule refers only to an individual symbol and not to its neighbors. If one rule does refer to symbols and its neighbors, the system is called *context-sensitive*.

Example 3.3. *An example of a context-sensitive system is:*

Alphabet: $\{A, B\}$
 Axiom: A
 Production rules: $A \mapsto BAB$
 $B > A \mapsto B$
 B otherwise: $\mapsto A$,

where the production rule $B > A \mapsto B$ means that when B is adjacent to the left to A , then B will stay B . When it is not left to A , B will map to A . When we work out the L-system, we will get this for the first couple of stages:

axiom: A
 stage 1: BAB
 stage 2: $BBABA$
 stage 3: $ABBABBBAB$
 stage 4: $BABABBABAABBABA$

When for all symbols of the alphabet $a \in V$ there is exactly one production rule $P(a)$, then the system is called *deterministic*. If there are several rules for just one symbol, where each rule has a probability assigned to it, the system is called *stochastic*.

Deterministic, context-free L-systems (D0L-systems) are most common: Example 3.1 is for instance a D0L-system, since for each symbol (A and B) there was exactly one production rule, meaning that it is deterministic, and each production rule only refers to an individual symbol, meaning that it is context-free.

The approach of L-systems is very useful in describing growth phenomena in a short and precise manner. The production rules are often derived from research, for example when considering a tree or an alga organism. However a string of symbols can be extremely unpleasant to work with, since it can be too complicated to be understood. This is where visualization comes in. We will see in the next subsection how one can visualize L-systems.

3.2 Turtle graphics

A nice way to visualize L-systems is by turtle graphics. The idea is simple; we let a turtle walk on a piece of white paper. It can walk with its dirty tail up, meaning that it does not leave a line on the paper, but it can also walk with its tail down. In that way, it leaves a trace on the paper when it walks. The turtle can also move in different directions. In order for this graphical interface to work well, all aspects of

the graphical interpretation have to be firmly defined. We will represent the different movements of the turtle as symbols. The first set of instructions will be:

- F move forward by a particular fixed step length l and draw a straight line from the old to the new position (tail down)
- f move exactly like F but do not draw a line (tail up)
- $+$ turn left (counterclockwise) by a fixed angle δ
- $-$ turn right (clockwise) by the same angle δ

The step length l and the angle δ must be specified before the interpretation of a system can be started. The size of the angle has an important impact on the shape of the resulting graphics. If you chose the angle to be different, you can end up with a completely different image.

The four commands we have specified can also be written down in a more mathematical way. The state of the turtle is given by two coordinates x and y , and the direction it is facing. The triplet (x, y, α) will then represent the state. The commands are then as follows:

command	state (x, y, α) changes to
F	$(x + l \cos(\alpha), y + l \sin(\alpha), \alpha)$
f	$(x + l \cos(\alpha), y + l \sin(\alpha), \alpha)$
$+$	$(x, y, \alpha + \delta)$
$-$	$(x, y, \alpha - \delta)$

In this table, we use step length l , angle δ and starting state $(0, 0, 0)$, so the turtle is headed to the right. This way of representing L-systems is particularly useful for programming.

3.3 Examples of Lindenmayer-systems

We can describe the two fractals that were introduced in the previous section also as an L-system. We start with the Koch curve, since this fractal is a little bit easier to describe as an L-system than the Cantor set.

3.3.1 The Koch curve

The Koch curve, as introduced in subsection 2.1.2, is basically produced by repeatedly replacing a straight line segment by a sequence of four lines. We will use turtle graphics to find the matching L-system. Our axiom is a straight line, F (the *draw*-symbol). The generator for the Koch curve is described by a straight line, then a left turn of 60 degrees, then a line forward, then a turn of 2×60 degrees to the right, a line forward, a left turn of 60 degrees and finally a line forward again (see Figure 7). The string that represents this is:

$$F + F - -F + F.$$

Thus, the L-system that belongs to the Koch curve is described by the following:

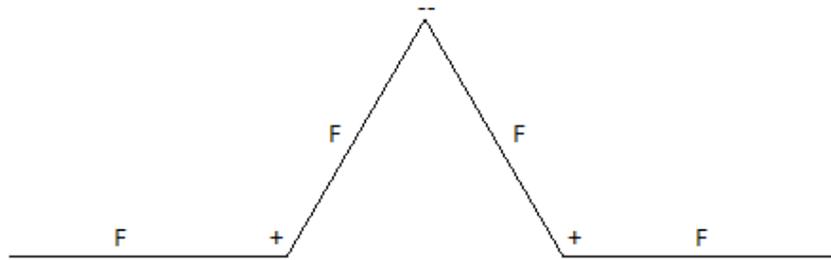


Figure 7 – The generator, i.e. the axiom, of the Koch curve in terms of turtle graphics

L-system:	Koch-curve
Axiom:	F
Production rules:	$F \rightarrow F + F - -F + F$ $+ \rightarrow +$ $- \rightarrow -$
Parameter:	$\delta = 60$ degrees

Elaborating the L-system, we will obtain the following strings for the first two stages:

axiom:	F
stage 1:	$F + F - -F + F$
stage 2:	$F + F - -F + F + F + F - -F + F - -F + F - -F + F +$ $F + F - -F + F$

This indeed produces the same image as the usual method of finding the Koch curve (Figure 8).

Finding the L-system for the Koch curve was quite simple, since the scaling turned out to be perfect, directly in the first attempt. This is unfortunately not always the case, as we will see in a moment. It is just a matter of trial and error to see what will work.

3.3.2 The Cantor set

The Cantor set is a set for which it is not trivial to find the L-system. We will demonstrate this. The axiom for the Cantor set is again a straight line, F . The generator for the set is a line divided into three parts where the middle third is removed. This can be represented by FfF , where F is the *draw*-symbol and f the *move*-symbol. This suggest that the following L-system is the one we are looking for:

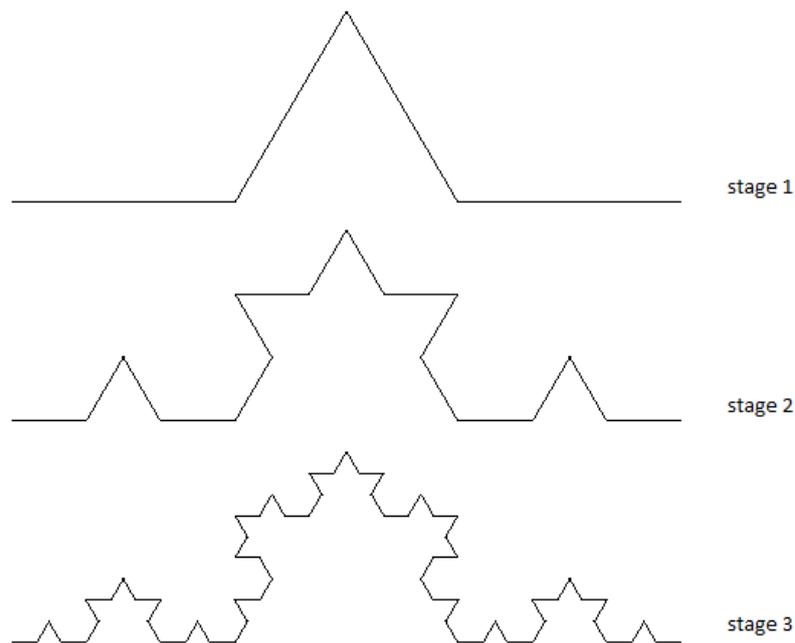


Figure 8 – The first three stages of the Koch curve

L-system: Cantor set (first try)
 Axiom: F
 Production rules: $F \rightarrow FfF$
 $f \rightarrow f$

This results in the following strings for the first two stages:

axiom: F
 stage 1: FfF
 stage 2: $FfFfFfF$
 stage 3: $FfFfFfFfFfFfFfF$

In Figure 9 we see the result of this L-system.



Figure 9 – The first three stages for the wrong L-system of the Cantor set.

We notice immediately that this is not the desired result. The gap in the middle

of the string appears to be too small. It should have been three times as large to become the correct L-system. This can be arranged by replacing the production rule $f \rightarrow f$ by $f \rightarrow fff$. The L-system for the Cantor set therefore should be:

L-system:	Cantor set
Axiom:	F
Production rules:	$F \rightarrow FfF$
	$f \rightarrow fff$

This indeed gives us the correct results, as we can see in Figure 10. The dots show exactly how many move-symbols there are in between the lines. We see that even for simple fractals, there can still be some pitfalls in the derivation of an appropriate L-system. However, when turning it into the graphical interpretation, one can observe very quickly where things go wrong.



Figure 10 – The first three stages for the correct L-system of the Cantor set.

3.4 Dimension

The techniques of finding the dimension of fractals as we have seen in the previous section attempt to measure the ratio between how much the curve or set grows in length and how much it advances. In [7], Alfonseca and Ortega are using the same approach to find the dimension when only considering the L-system without performing any graphical interpretations. In this subsection we will discuss their method.

For their method, they use the interpretation of L-systems by turtle graphics, as explained in the previous section. In terms of turtle graphics, we recall that we have *draw*-letters, *non-graphic*-letters, *move*-letters, angles and stacking symbols. For this method, we restrict ourselves to AID0L-systems (angle-invariant D0L-systems), where angle-invariant means that the direction at the beginning and the end of the fractal are the same. We notice that the Cantor set and the Koch curve are AID0L-systems.

Definition 3.4 (L-system fractal dimension). *The L-system fractal dimension derived from the L-system of a given fractal is given by*

$$D = \frac{\log(N)}{\log(d)},$$

where N is the length of the visible walk of the production rule. When there is no overlap in the generator or in the fractal itself, N is equal to the number of

draw symbols in the generator string. In section 3.4.1 it is explained what happens otherwise. The value d is the distance in a straight line from the start to the endpoint of the walk, measured in turtle step units (this number can also be deduced from the string).

We remind ourselves that there is a scale reduction: the distance between the beginning point and the end point of the graphical interpretation remains the same after the production rules are applied.

Example 3.5. *Let us consider the Koch curve. The scheme of the Koch curve can be found in subsection 3.3. The only string we need to consider is:*

$$F + F - -F + F.$$

This string describes the fractal generator. The length of the visible walk, N , is the number of draw symbols in this string, 4 in this case. The distance between the starting point and the end point, d , is 3 here. Therefore, the dimension of the Koch curve is

$$D = \frac{\log(4)}{\log(3)},$$

exactly the same value as obtained from the other methods.

Example 3.6. *Take the Cantor set. The L-system of this set is given in subsection 3.3. Here, the string we need to consider is:*

$$FfF.$$

The number of draw symbols is 2, which will be the value of N . The distance d is in this case equal to 3. The dimension of the Cantor set is hence equal to:

$$D = \frac{\log(2)}{\log(3)},$$

again the same as computed by other methods.

We see that for these examples, this method is very easy to use and seems widely applicable. However, some problems can arise.

3.4.1 Problems in the definition of L-system fractal dimension

Several things can go wrong with this definition. In [7], three cases are identified.

- Case 1: The distance d in the denominator can be zero.
When the distance d is zero, the L-system fractal dimension D will have $\log(0)$ in its denominator. However, the logarithm of 0 is undefined. We ask ourselves what kinds of set will have a distance d of 0. We will have for example the

system:

Axiom: F
 Production rules: $F \rightarrow F + F + F + F +$
 $+ \rightarrow +$
 $- \rightarrow -$
 Parameter: $\delta = 90$ degrees

Cases like this generate normally the same figure indefinitely repeated. In this example, we end up with a lot of squares. These are not identified as fractals, hence we can exclude this case.

- Case 2: The distance d in the denominator can be 1. This will also give us some problems, since we will now have $\log(1)$ in the denominator which gives a dimension D of size infinity. These L-systems also do not give rise to fractals, since, according to [7], they expand after each iteration and are hence not restricted to a limited space. Because of that, they are not considered fractals.
- Case 3: The length N of the visible walk may not be equal to the number of draw-symbols in the string. This can actually happen in two ways. It can be the case that we observe an overlap already in the generator. Take for example the following L-system:

Axiom: $F + +F + +F + +F$
 Production rules: $F \rightarrow F + FF + + + F + +F - FF + + + F + +F - -F$
 $+ \rightarrow +$
 $- \rightarrow -$
 Parameter: $\delta = 45$ degrees

Figure 11 shows the generator and the second iteration for this L-system. In Figure 11a, you cannot immediately see that there is some overlap; however when you follow the path of the turtle, you will immediately see that in the center we have overlap. When we take N to be 10, i.e. the number of draw symbols in the string, we are counting that part twice which is not desirable. Therefore the suggestion is to compute by hand how much the turtle effectively ‘paints’ on the piece of paper. In this example, we find N to be $8 + \sqrt{2} \approx 9.4142$. It can also happen that in the generator we do not encounter any overlap, however in the resulting fractal we do. In this case, we take the limit on the string of derivations from the axiom F . That is done in the following way: First you compute the effective length of the curve after each iteration, and you compute the distance d after each generation. Using the obtained N and d , find for each iteration the corresponding dimension. Once this converges, you’ll obtain a more reasonable dimension compared to using only the production rule to get N and d .

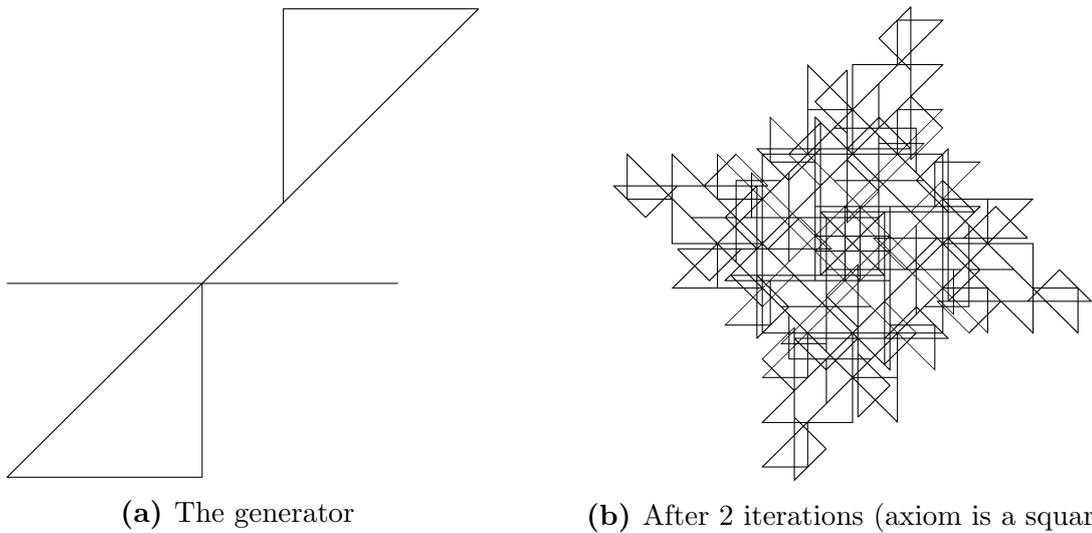


Figure 11 – The curve obtained from a generator that passes twice through the same set of points

We see that although these cases have a fairly straightforward solution, the actual computation of the effective length by hand is not something you can do quickly. Ortega en Alfonseca proposed an algorithm in APL2 ([2]) and in PROLOG ([7]) in which the effective length is computed. In this way, when the effective length is too hard to compute by hand, you can use the algorithm which does it for you. We conclude that for this dimension to work properly for every L-system, an algorithm is needed.

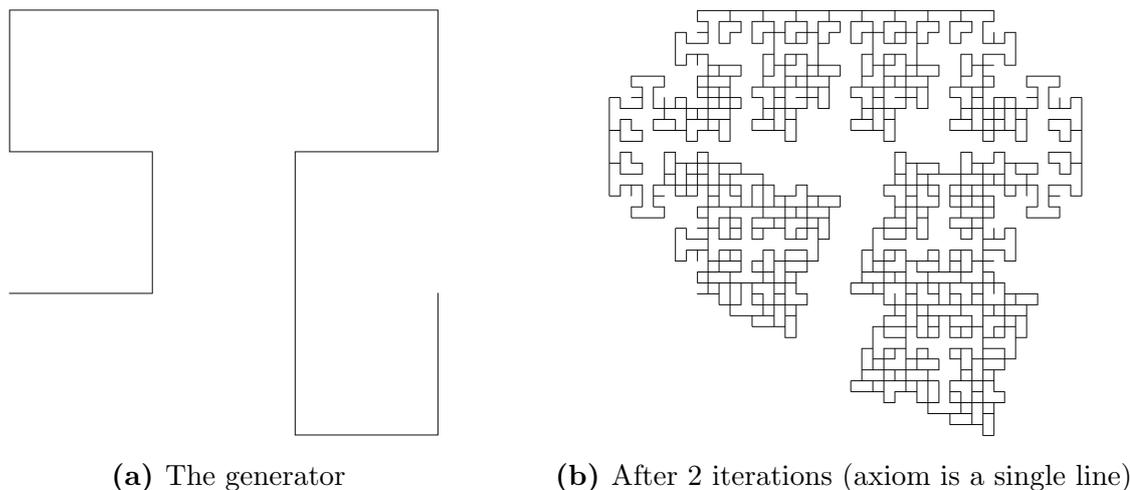


Figure 12 – A fractal that consists of 13 parts of itself scaled by a factor $\frac{1}{3}$, with no overlap in its generator, but with overlap in the curve that is generated by it.

4 Comparison of the dimensions

In this section, we take a look at the different dimensions we mentioned in this paper and compare them. We will divide this section in two parts: first we consider the ‘usual’ fractal dimensions, discussed in section 2, and compare them with each other. In the second part, we will discuss the use of L-systems and try to find the link between the L-system fractal dimension and the other dimensions.

4.1 Fractal dimensions

The first fractal dimension we discussed was the similarity dimension. The definition was straightforward: when a set F is made up of m parts similar to itself scaled by a factor r , then it has similarity dimension $\dim_S(F) = -\frac{\log(m)}{\log(r)}$. This dimension is easy to use, but can only be applied on strictly self-similar fractals. An important fact why fractals are so important is that they resemble nature. In nature, the fractals we observe are never perfectly self-similar. This definition is therefore very limited and is not useful for determining the dimension of natural phenomena, such as the coastline of an island.

The box-counting dimension goes a step further than the similarity dimension. This dimension does not limit the fractals to be strictly self-similar; it can even be used on wild structures with some scaling properties that are not at all self-similar.

The similarity dimension and the box-counting dimension are for most strictly self-similar fractals equal, since you use the similarity of the fractal when trying to find a cover for it.

However, the box-counting and the similarity dimension are not always equal. To prove this, we take a strictly self-similar fractal, because then we can find the similarity dimension as well as the box-counting dimension. This fractal will be in \mathbb{R}^2 and is made up of 13 parts similar to itself scaled by a factor $\frac{1}{3}$, precisely as in

Figure 12. The similarity dimension will thus be

$$\dim_S(F) = \frac{\log(13)}{\log(3)} \approx 2.335.$$

Notice that the box-counting dimension of a set in \mathbb{R}^2 will never exceed 2, so it can never be equal to $\frac{\log(13)}{\log(3)}$. The reason for this difference is that the curve has overlapping parts, which will only be counted once in the box-counting method, but with corresponding multiplicities in the computation of the similarity dimension. The similarity dimension of such a self-overlapping fractal can therefore be higher than its box-counting dimension. This point is another part where the similarity dimension is different from the box-counting dimension.

The box-counting dimension is a better representation of the ‘size’ of a fractal, but also lets us down in some cases. We will see these shortcomings when we compare it to the Hausdorff dimension.

The following theorem shows the relation between the box-counting dimension and the Hausdorff dimension.

Theorem 4.1. *For every non-empty bounded $F \subset \mathbb{R}^n$,*

$$\dim_H(F) \leq \dim \inf_B(F) \leq \dim \sup_B(F)$$

Proof. Suppose that $1 < \mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F)$ for some $s \geq 0$. We observe that

$$N_\delta(F)\delta^s = \inf \left\{ \sum_i \delta^s : \{U_i\} \text{ a } \delta\text{-cover of } F \text{ with } |U_i| \leq \delta \right\},$$

where $N_\delta(F)$ is the least number of subsets with diameter at most δ needed to cover F . Then for sufficiently small δ ,

$$1 < \mathcal{H}_\delta^s(F) \leq N_\delta(F)\delta^s.$$

After taking logarithms, we get

$$0 < \log(N_\delta(F)) + s \log(\delta),$$

and after rewriting:

$$\dim_H(F) = s \leq \liminf_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \leq \limsup_{\delta \rightarrow 0} \frac{\log(N_\delta(F))}{-\log(\delta)},$$

i.e.

$$\dim_H(F) \leq \dim \inf_B(F) \leq \dim \sup_B(F) \tag{10}$$

□

We do not in general have an equality here in theorem 4.1. In order to give an example, we will prove the countable stability and the countable sets theorems for the Hausdorff dimension first. After this, we can give an example for which the inequality in equation 10 is strict.

Theorem 4.2 (Countable stability). *If F_1, F_2, \dots is a countable sequence of sets, then*

$$\dim_H(\cup_{i=1}^{\infty} F_i) = \sup_{1 \leq i < \infty} \{\dim_H(F_i)\}.$$

Proof. We notice that $\dim_H(\cup_{i=1}^{\infty} F_i) \geq \sup_{1 \leq i < \infty} \{\dim_H(F_i)\}$ by the monotonicity of the Hausdorff dimension (i.e. if $E \subset F$, then $\dim_H(E) \leq \dim_H(F)$).

To prove the opposite inequality, we take $s > \dim_H(F_i)$ for all i . Then $\mathcal{H}^s(F_i) = 0$ for all i . So we know that

$$\mathcal{H}^s(\cup_{i=1}^{\infty} F_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(F_i) = 0.$$

In fact, what we just stated is that when $s > \dim_H(F_i)$, then $\mathcal{H}^s(\cup_{i=1}^{\infty} F_i) = 0$. This results in the fact that $\dim_H(\cup_{i=1}^{\infty} F_i) \leq \dim_H(F_i)$ for all i , so

$$\dim_H(\cup_{i=1}^{\infty} F_i) \leq \sup_{1 \leq i < \infty} \{\dim_H(F_i)\}.$$

□

This theorem is the foundation of the following theorem:

Theorem 4.3 (Countable sets). *If F is countable, then $\dim_H(F) = 0$.*

Proof. The set F is countable, hence F is a collection of single points, $F = \cup_{i=0}^{\infty} F_i$ where F_i is a single point for all i .

Since F_i is a single point, $\mathcal{H}^0(F_i) = 1$ and $\dim_H(F_i) = 0$. Hence by countable stability, $\dim_H \cup_{i=0}^{\infty} F_i = 0$. □

By these theorems, we now know that when A is a countable set, the Hausdorff dimension will always be 0. However, it can be shown that for any dense subset of \mathbb{R}^n , the box-counting dimension is equal to n . So, the box-counting dimension of the set of rational numbers in $[0, 1]$ is 1, whereas the Hausdorff dimension of that (countable) set is 0.

Another striking example is the following:

Theorem 4.4. *The set*

$$A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$$

has box-counting dimension $\frac{1}{2}$.

Proof. We can cover the k elements of $A = \{0, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}\}$ by intervals of length $\frac{1}{(k-1)k}$, since it is equal to the distance between two consecutive points: $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{(k-1)k}$. So if we take δ_k to be $\frac{1}{(k-1)k}$ for $k \geq 2$, then U_i for which hold that $|U_i| \leq \delta < \delta_k$ can cover at most one of the points. We can use this δ_k , since we can take $c = \frac{1}{k}$ to obtain $c = \frac{1}{k} \in (0, 1)$, $c \geq \delta_k > \delta_{k+1} \geq c\delta_k$, ($2 \leq k < \infty$). Then,

$$\begin{aligned} \dim \inf_B(A) &= \liminf_{k \rightarrow \infty} \frac{\log(N_{\delta_k}(A))}{-\log(\delta_k)} = \liminf_{k \rightarrow \infty} \frac{\log(k)}{-\log(\frac{1}{(k-1)k})} = \liminf_{k \rightarrow \infty} \frac{\log(k)}{\log((k-1)k)} \\ &= \liminf_{k \rightarrow \infty} \frac{\log(k)}{\log(k^2(1 + 1/k))} = \liminf_{k \rightarrow \infty} \frac{\log(k)}{2 \log(k) + \log(1 + 1/k)} = \frac{1}{2}. \end{aligned}$$

A similar argument holds for the upper dimension. This shows that the box-counting dimension of A is equal to $\frac{1}{2}$. \square

This set is not considered to be a fractal, however it does have this large box-counting dimension. The Hausdorff dimension of this set will be 0, which is an answer that fits better. We see that the box-counting dimension does not satisfy countable stability, which in some cases may be unsatisfactory.

4.2 L-system dimension

The L-system dimension appears to be of a very different kind than the box-counting dimension and the Hausdorff dimension, since it is mathematically not as appealing. Where the box-counting dimension and the Hausdorff dimension can be a real challenge to compute, the L-system dimension does give an easy and straightforward method to calculate the dimension directly from the production string in the L-system. In its simplicity, it resembles the similarity dimension the most. However, L-systems can also represent fractals that are not strictly self-similar, so it can handle more fractals than the similarity dimension can. We did not see these kinds of L-systems, but with more symbols and extra letters that have different production rules we can obtain a fractal that is not strictly self-similar. This L-system dimension can handle these different structures when implemented in PROLOG or APL2 the way Ortega and Alfonseca do [2][7].

It is also different from the similarity dimension in another way. We observed that when a fractal has some points where it overlaps itself, the similarity dimension measures these overlapping points with corresponding multiplicities. The L-system dimension however, when implemented in PROLOG or APL2, does take the overlap into account and eliminates the addition. The box-counting dimension and the Hausdorff dimension also allow for fractals to have overlap and only count the double parts once. In this respect, the L-system dimension is more like these two dimensions.

The L-system dimension makes use of the fact that the growth of the 'words' corresponds to the growth of the fractal. All dimension techniques, among which the similarity dimension, the box-counting dimension and the Hausdorff dimension, try to measure the fractal dimension as a ratio between how much the fractal grows in length and how much it advances. This is also precisely the base of the L-system dimension.

It is therefore fair to say that the L-system dimension is a useful tool to measure the length of fractals when represented as L-systems. For most simple L-systems, the L-system dimension is easy to use and is able to give a useful value. However, when the fractal is a little bit more complicated, an algorithm is needed to compute the dimension. The authors Ortega and Alfonseca promise at the end of their article to expand this L-system dimension even further to make it applicable to a wider range of L-systems and make it mathematically more appealing [7].

5 Conclusion

In this paper, we tried to grasp mathematically what fractals are, both using usual fractal geometry as well as in terms of Lindenmayer systems. To do this, we examined the Cantor set and the Koch curve. These fractals have a very clear definition, but the fractals appear to be extremely detailed and complex. We proved that the Cantor set is perfect: it is closed and every point in it is a limit point. We discussed the similarities we observe in both the Cantor set and the Koch curve and concluded that the two fractals are strictly self-similar, meaning that they are composed of exact copies of themselves. Similarity is an important property of fractals.

We then introduced three kinds of dimensions to analyze and compare: the similarity dimension, the box-counting dimension and the Hausdorff dimension. The similarity dimension is only applicable on strictly self-similar fractals. For fractals of this kind, this dimension is very easy to compute. However, it does not take into account when a fractal contains overlapping points. The similarity dimension counts the overlap with corresponding multiplicity, which can lead to counterintuitive results.

The box-counting dimension is based on a δ -cover of the fractal. The way in which the number of sets to cover the fractal grows as the diameters of the sets go to zero will be the dimension. This dimension does take overlap into account. The box-counting dimension can be hard to compute, but with a little effort it gives you a very good idea of the ‘size’ of the fractal. However it is not always a good representation: it does not have countable stability and therefore countable sets can have a dimension bigger than 0, although we would expect that countable sets are not fractals and therefore have dimension 0.

The Hausdorff dimension is, of the dimensions we considered here, the one that has the best features. For the Hausdorff dimension, we defined a Hausdorff measure. Intuitively, the number s_0 for which the Hausdorff measure jumps from being ∞ to 0 is defined to be the Hausdorff dimension. For this value s_0 , the Hausdorff measure represents the length, volume or area. The scaling property holds for the Hausdorff measure, which can be used for the computation of several fractals. The Hausdorff dimension takes overlap into account, and in addition has countable stability. The Hausdorff dimension of countable sets is thus always 0, which is desirable.

We conclude that for a definition of dimension the following properties are desirable:

- *Monotonicity.* If $E \subset F$, then $\dim(E) \leq \dim(F)$.
- *Range of values.* If $F \subset \mathbb{R}^n$, then $\dim(F) \leq n$.
- *Countable stability.* $\dim(\cup_{i=1}^{\infty} F_i) = \sup_i \dim(F_i)$.
- *Countable sets.* If F is countable, $\dim(F) = 0$.

See Falconer [4] for additional properties. The only dimension we discussed in this paper which has all of the properties we noted here, is the Hausdorff dimension.

After this, we examined L-systems and their use. L-systems are a very nice way to represent a large class of fractals. We defined an L-system by an alphabet

$$V = \{a_1, a_2, \dots, a_n\},$$

the production map

$$P : V \rightarrow V^* \\ a \mapsto P(a),$$

where V^* is the set of all strings formed by symbols from V , and an axiom

$$a^{(0)} \in V^*,$$

the initial string.

L-systems are particularly useful to see how fractals grow. They can be constructed using turtle graphics and this can be extended even further than we did, but that was beyond the scope of this paper. Pictures of fractals are easily created once you know the corresponding L-system (see Appendix A for the codes of the pictures in this paper that were created using L-systems). In [7], Alfonseca and Ortega proposed a definition of a dimension that is applicable to L-systems, without the use of the graphical interpretation. It works directly on the string of the production rule. It is based on how much the curve grows in length and how much it advances. The dimension appears to give a good idea of the ‘size’ of a fractal, although it is not as mathematically satisfying as the box-counting dimension and the Hausdorff dimension. It is clear that this dimension needs to be examined and expanded further and further. The authors of the article conclude their article with the notion that they will elaborate on L-system dimension in order to make it mathematically more appealing and make it applicable to a wider range of L-systems.

References

- [1] Stephen Abbott. *Understanding analysis*. Springer, 2001.
- [2] Manuel Alfonseca and Alfonso Ortega. Using apl2 to compute the dimension of a fractal represented as a grammar. In *ACM SIGAPL APL Quote Quad*, volume 30, pages 13–23. ACM, 2000.
- [3] Manuel Cebrian, Manuel Alfonseca, and Alfonso Ortega. Grammatical evolution with restarts for fast fractal generation. *arXiv preprint arXiv:0805.1696*, 2008.
- [4] Kenneth Falconer. *Fractal geometry: mathematical foundations and applications*. John Wiley & Sons, 2004.
- [5] Gilbert Helmbert. *Getting acquainted with fractals*. Walter de Gruyter, 2007.
- [6] Encyclopedia of Mathematics. *Upper and lower limits*.
- [7] Alfonso Ortega and Manuel Alfonseca. Determination of fractal dimensions from equivalent l systems. *IBM Journal of Research and Development*, 2001.
- [8] Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe. *Chaos and fractals: new frontiers of science*. Springer Science & Business Media, 2006.

A MuPAD codes

MuPAD is included in the Symbolic Math Toolbox from MATLAB. The function `[plot::Lsys]` in MuPAD allows us to create L-systems directly. The function uses the same symbolics as we have seen in subsection 3.2. In addition we can specify color. The defaults are: “F” = Line, “f” = Move, “+” = Left, “-” = Right. The initial direction is “up” and the default color is “blue”. The following codes have been used to generate the figures for the introduction as well as section 3.

For Figure 1, the fractal representation of a plant, the following command was used:

```
plot(plot::Lsys(PI/9, "BL", "L" = "BR[+HL]BR[-GL]+HL",
                "R" = "RR", "L" = Line, "R" = Line,
                "B" = RGB::Brown, "H" = RGB::ForestGreen,
                "G" = RGB::SpringGreen, Generations = 8))
```

For Figure 7, the generator of the Koch curve, the following command was used:

```
plot(plot::Lsys(PI/6, "---ZF", "F" = "F++F----F++F",
                "Z" = RGB::Black, Generations=1))
```

For Figure 8, the first three stages of the Koch curve, the following commands were used:

```
plot(plot::Lsys(PI/6, "---ZF", "F" = "F++F----F++F",
                "Z" = RGB::Black, Generations=1))

plot(plot::Lsys(PI/6, "---ZF", "F" = "F++F----F++F",
                "Z" = RGB::Black, Generations=2))

plot(plot::Lsys(PI/6, "---ZF", "F" = "F++F----F++F",
                "Z" = RGB::Black, Generations=3))
```

For Figure 9, the first three stages for the wrong L-system of the Cantor set, the following commands were used:

```
plot(plot::Lsys(PI/2, "-ZF", "F" = "ZFfZF",
                "Z" = RGB::Black, Generations=1))

plot(plot::Lsys(PI/2, "-ZF", "F" = "ZFfZF",
                "Z" = RGB::Black, Generations=2))

plot(plot::Lsys(PI/2, "-ZF", "F" = "ZFfZF",
                "Z" = RGB::Black, Generations=3))
```

For Figure 10, the first three stages for the correct L-system of the Cantor set, the following commands were used:

```
plot(plot::Lsys(PI/2, "-ZF", "F" = "ZFfZF", "f" = "fff",
               "Z" = RGB::Black, Generations=1))

plot(plot::Lsys(PI/2, "-ZF", "F" = "ZFfZF", "f" = "fff",
               "Z" = RGB::Black, Generations=2))

plot(plot::Lsys(PI/2, "-ZF", "F" = "ZFfZF", "f" = "fff",
               "Z" = RGB::Black, Generations=3))
```

For Figure 11, the curve obtained from a generator that passes twice through the same set of points, the following commands were used:

```
plot(plot::Lsys(PI/4, "--ZF", "F" = "F+FF+++F++F-FF+++F++F
--F", "Z" = RGB::Black, Generations = 1))

plot(plot::Lsys(PI/4, "--ZF++ZF++ZF++ZF", "F" = "F+FF+++F
++F-FF+++F++F--F", "Z" = RGB::Black,
Generations = 2))
```

For Figure 12, the fractal that consists of 13 parts of itself scaled by a factor $\frac{1}{3}$, the following commands were used:

```
plot(plot::Lsys(PI/2, "-ZF", "F" = "F+F+F-F-FFF-F-F+FF+F+F
-", "Z" = RGB::Black, Generations=1))

plot(plot::Lsys(PI/2, "-ZF", "F" = "F+F+F-F-FFF-F-F+FF+F+F
-", "Z" = RGB::Black, Generations=3))
```