university of groningen

faculty of mathematics
and natural sciences

Control of Networked Multi-Agent Systems

Master Project Applied Mathematics

July 2016

Student: H.J. van Waarde

Supervisors: prof. dr. H.L. Trentelman and prof. dr. M.K. Camlibel

Second assessor: prof. dr. J. Top

ABSTRACT

Networked multi-agent systems appear in a variety of disciplines including chemical engineering, financial networks and robotics. In this thesis we consider two control problems in the context of networked multi-agent systems, namely targeted controllability analysis and the linear quadratic regulator problem.

The first part of this thesis is concerned with the control of a prescribed subset of agents of a network, called target nodes. This specific form of output control is known under the name target control. We consider target control of a family of linear control systems associated with the network, and investigate under which graph-theoretic conditions all systems of this family are targeted controllable. As our main results, we present both a necessary and a sufficient graph-theoretic condition for targeted controllability. Furthermore, a leader selection algorithm is established to compute leader sets achieving target control.

Secondly, we study the linear quadratic regulator (LQR) problem for identical decoupled linear systems, where the quadratic cost depends on the relative states and inputs of the systems. No initial network structure is imposed. Instead, we investigate under which conditions there exists a network such that the optimal control law of each agent can be written as the weighted sum of relative neighbouring states. These so-called diffusive control laws are desirable, as in many applications only relative information is measured. In this thesis, the free endpoint and zero endpoint LQR problems are studied, and for both problems we establish necessary and sufficient conditions for the diffusiveness of the optimal control law.

ACKNOWLEDGEMENTS

I would like to express a few words of gratitude towards my supervisors, Harry Trentelman and Kanat Camlibel.

Dear Harry, thank you for your guidance during the master project. I really appreciated the meetings and the thorough comments you provided on my work. Your enthusiasm and compliments are very motivating. Besides your help during the master project, I would like to thank you for your help arranging the internship, and for your inspiring systems theory courses. You have taught me so much over the last years, for which I am very grateful.

Dear Kanat, I find it remarkable how quickly you can come up with creative new ideas. A while ago we had some email conversations during weekends, about targeted controllability. These conversations mean a lot to me, as I was honored that you wanted to share ideas, even in your free time. Thank you for your relaxed and friendly behavior towards me, the explanation of the subject of the project, and all your valuable suggestions for research.

CONTENTS

1

INTRODUCTION

Motivated by their wide range of applications, networks of dynamical agents have attracted considerable attention in recent years. Networks of dynamical agents appear in chemical engineering [3], financial networks [12] and in many mechanical engineering applications like satellite formation control [8], power grids [46] and control of robotic networks [6].

A network of dynamical agents is a dynamical system composed of multiple input/output systems, called *agents* of the network. These agents interact by exchanging information with their neighbours in the network. Networks of dynamical agents are also referred to as *networked dynamical systems*, or *networked multi-agent systems*.

It is customary to represent a network of dynamical agents by a graph, where vertices correspond to agents, and edges between vertices indicate which agents are neighbours of each other. Depending on the context, both undirected and directed graphs are used to describe respectively bidirectional or unidirectional communication in multi-agent systems. The mainstream of research has focussed on time-independent network graphs, however also results are known for time-dependent network topologies [32], [33].

One of the most fundamental questions regarding networks of dynamical systems is how *local* interactions, and control laws based on local information can induce a desired, *global* network behaviour. In the literature, the most prominent example of desired network behaviour is *consensus*, which has been extensively studied [21], [32], [34], [37], [38]. Roughly speaking, agents of a networked dynamical system reach consensus if they agree upon certain common quantities of interest.

A specific type of consensus problem is the problem of *synchronization*. Agents of a network are said to synchronize if the states of all agents asymptotically converge to the same trajectory [21], [39], [45], [47]. The more general problem of *output synchronization* is also studied [48], in which the objective is to synchronize the outputs of the agents.

Apart from the problems of consensus and synchronization, a general problem within the study of networked multi-agent systems is that many classical techniques used for 'ordinary' input/output systems are either impractical or inapplicable to networked dynamical systems. Hence, there is a demand for novel techniques suitable for networked multi-agent systems. To this extent, model reduction methods [18], [28], controllability analysis [11], [29] and the linear quadratic regulator problem [5], [30] have been investigated in the context of networked dynamical systems. In this thesis, the two main topics of interest are *targeted controllability analysis* and the *linear quadratic regulator problem* in the context of networked multi-agent systems.

Controllability of networked dynamical systems has received much attention [7], [9], [24], [27], [29], [42]. In the study of controllability of networked dynamical systems, two types of agents are distinguished: *leaders*, to which external input is applied, and *followers* whose dynamics are only influenced by the behaviour of their neighbours. In this framework, network controllability comprises the ability to drive the states of all agents to any desired state, by applying appropriate input to the leaders.

Apart from controllability analysis for networked dynamical systems whose dynamics are described by the Laplacian matrix, *structural controllability* analysis is treated in [9], [24], [29], [42]. In the structural controllability framework, a family of linear control systems is associated with a network graph, where the structure of the state matrix of each system depends on the network topology, and the input matrix is fixed by the set of leaders. In this context, a network is said to be *strongly structurally controllable* if all systems associated with the network graph are controllable. Structural controllability analysis is motivated by model uncertainties. Indeed, if a network is strongly structurally controllable, we can conclude that a multi-agent system associated with the network is controllable, even though we do not know its exact dynamics. Classical results on controllability of input/output systems are inapplicable to structural controllability analysis of networked systems. Indeed, a naive implementation of Kalman's rank condition would require the computation of the (high dimensional) controllability matrix for each member of the family of systems. Hence, *graph-theoretic* conditions for strong structural controllability of networks have been derived in terms of zero forcing sets [29] and maximum matchings [9].

However, in some applications full control over the network is not required, and it is of interest to control only a subset of nodes, called *target nodes*. This specific form of output control, where the output consists of the states of the target nodes, is called *targeted controllability*. A network is then said to be strongly targeted controllable if all systems in the the family associated with the network graph are targeted controllable. Although necessary and sufficient graph-theoretic conditions are known for strong structural controllability of networks, such conditions are still largely unexplored for strong targeted controllability. Therefore, in this thesis we provide graph-theoretic conditions for strong targeted controllability, and a leader selection algorithm to compute leader sets achieving target control. For a more extensive introduction on target control, and our main results on this subject we refer to Chapter 2.

The second subject treated in this thesis is the *linear quadratic regulator problem* in the context of multi-agent systems. The linear quadratic regulator (LQR) problem is defined for linear time-invariant systems, and concerns finding a control law that minimizes a quadratic cost functional dependent on both state and input of the system. Such a control law is referred to as an *optimal control law*. The LQR problem has been extensively studied in the 1970s, and its solution is well understood [19], [20], [44], [49].

However, direct application of these classical LQR results to multi-agent systems is in general not possible. This is due to the fact that the feedback law that solves the linear quadratic regulator problem makes use of the entire *global* state vector, while the agents of the network only receive *local* information of the states of their neighbouring agents.

As identified in [5], it is in general a difficult optimization problem to minimize the quadratic cost functional under the constraint that the control law is *distributed*, in the sense that the control input of each agent uses only states of its neighbours. Hence, so-called 'sub-optimal' controllers have been developed in [5], [10], which are distributed but do not fully minimize the cost functional.

In contrast to these papers, we follow the setup used in [30]. We consider identical decoupled linear systems, and a cost functional that depends on a quadratic function of the relative states and the inputs of the systems. No initial network structure is imposed, instead we are interested in the question whether the optimal control law can be implemented as a diffusive coupling on a network graph. That is: we investigate under which conditions there exists a network graph such that the optimal control law of each agent can be written as weighted sum of relative neighbouring states. Such a control law is called *diffusive*.

Our main results extend the the ones in [30] to the case that the agent dynamics is allowed to have eigenvalues in the open right half plane. Furthermore, our approach is attractive in its simplicity compared to [30], in the sense that it avoids the analysis of the Riccati differential equation.

The outline of this thesis is as follows. In Chapter 2 we consider targeted controllability of networks. We provide graph-theoretic conditions for targeted controllability of networked dynamical systems, and describe an algorithm to compute leader sets achieving target control. Subsequently, in Chapter 3 we consider the linear quadratic regulator problem in the context of multi-agent systems. We provide necessary and sufficient conditions for the diffusiveness of the optimal control law. Finally, Chapter 4 contains our conclusions and topics for future research.

2

A DISTANCE-BASED APPROACH TO STRONG TARGET CONTROL OF NETWORKED MULTI-AGENT SYSTEMS

2.1 INTRODUCTION

During the last two decades, networks of dynamical agents have been extensively studied. It is customary to represent the infrastructure of such networks by a graph, where nodes are identified with agents and arcs correspond to the communication between agents. In the study of controllability of networks, two types of nodes are distinguished: leaders, which are influenced by external input, and followers whose dynamics are completely determined by the behaviour of their neighbours. Network controllability comprises the ability to drive the states of all nodes of the network to any desired state, by applying appropriate input to the leaders.

Motivated by model uncertainties, the notion of structural controllability of linear control systems fully described by the pair (A, B) was introduced by Lin [22]. Here the matrices A and B are zero-nonzero patterns, i.e. each entry of A and B is either a fixed zero or a free nonzero parameter. In this framework, weak structural controllability requires almost all realizations of (A, B) to be controllable. That is: for almost all parameter settings of the entries of A and B, the pair (A, B) is controllable. Lin provided a graph-theoretic condition under which (A, B) is weakly structurally controllable in the single-input case. Many papers followed [22], amongst others we name [15] and [40] in which extensions to multiple leaders are given, and the article [25], that introduces strong structural controllability, which requires all realizations of (A, B) to be controllable.

In recent years, structural controllability gained much attention in the study of networks of dynamical agents [7], [9], [24], [29], [42]. With a given network graph, a family of linear control systems is associated, where the structure of the state matrix of each system depends on the network topology, and the input matrix is determined by the leader set. In this framework, a network is said to be weakly (strongly) structurally controllable if almost all (all) systems associated to the network are controllable. The graph-theoretic results obtained in classical papers [22], [25] lend themselves excellently in the study of structural controllability of networks. A topological condition for weak structural controllability of networks is given in terms of maximum matchings in [24], while strong structural controllability is fully characterized in terms of zero forcing sets in [29]. Such graph-theoretic conditions have a considerable advantage in their computational robustness compared to rank conditions, and aid in finding leader selection procedures.

However, in large-scale networks with high vertex degrees, a substantial amount of nodes must be chosen as leader to achieve full control in the strong sense, which is often unfeasible. Furthermore, in some applications full control over the network is unnecessary. Hence, we are interested in controlling a subset of agents, called target nodes. This specific form of output control is known under the name target control [13], [27]. Potential applications of target control within the areas of biology, chemical engineering and economic networks are identified in [13].

A network is said to be strongly targeted controllable if all systems in the family associated to the network graph are targeted controllable. In this thesis we consider strong targeted controllability for the class of state matrices called distance-information preserving matrices. The adjacency matrix and symmetric, indegree and outdegree Laplacian matrices are examples of distance-information preserving matrices. As these matrices are often used to describe network dynamics (see e.g. [11], [16], [35], [41], [50]), distance-information preserving matrices form an important class of matrices associated with network graphs.

Our main results are threefold. Firstly, we provide a sufficient topological condition for strong targeted controllability of networks, that substantially generalizes the results of [27] for the class of distance-information preserving matrices. Furthermore, we note that the 'k-walk theory' described in [13] is easily obtained as a special case of our result. Secondly, noting that our proposed sufficient condition for target control is not a one-to-one correspondence, we establish a necessary graph-theoretic condition for strong targeted controllability. Finally, we provide a two-phase leader selection algorithm consisting of a binary linear programming phase and a greedy approach to obtain leader sets achieving target control.

This chapter is organized as follows: in Section 2.2 we introduce preliminaries and notation. Subsequently, the problem is stated in Section 2.3. Our main results are presented in Section 2.4. Finally, Section 2.5 contains our conclusions.

2.2 PRELIMINARIES

Throughout this thesis, directed and undirected graphs are considered, all assumed to be simple and without self-loops. We make the distinction between a directed arc between the vertices i and j, which will be denoted by (i,j), and an undirected edge, denoted by $\{i,j\}$.

First consider a directed graph $G=(V,E)$, where V is a set of n vertices, and E is the set of directed arcs. The cardinality of a vertex set V' is denoted by $|V'|$. We define the distance $d(u,v)$ between two vertices $u,v\in V$ as the length of the shortest path from u to v. If there does not exist a path in the graph G from vertex u to v, the distance $d(u,v)$ is defined as infinite. Moreover, the distance from a vertex to itself is equal to zero.

For a nonempty subset $S\subseteq V$ and a vertex $j\in V$, the distance from S to j is defined as

$$d(S,j) := \min_{i\in S} d(i,j). \tag{2.1}$$

A directed graph $G=(V,E)$ is called bipartite if there exist disjoint sets of vertices V^- and V^+ such that $V=V^-\cup V^+$ and $(u,v)\in E$ only if $u\in V^-$ and $v\in V^+$. We denote bipartite graphs by $G=(V^-,V^+,E)$, to indicate the partition of the vertex set.

With every directed graph $G=(V,E)$, we can associate an undirected graph $G'=(V,E')$, where $\{v_i,v_j\}\in E'$ if and only if at least one of the arcs (v_i,v_j) or (v_j,v_i) is contained in E. A directed graph is called a directed tree if its associated undirected graph is a tree.

2.2.1 *Qualitative class and pattern class*

The qualitative class of a directed graph G is a family of matrices associated to the graph. Each of the matrices of this class contains a nonzero element in position i, j if and only if there is an arc (j, i) in G, for $i \neq j$. More explicitly, the qualitative class $\mathcal{Q}(G)$ of a graph G is given by

$$\mathcal{Q}(G) = \{X \in \mathbb{R}^{n \times n} \mid \text{ for } i \neq j,\ X_{ij} \neq 0 \iff (j, i) \in E\}.$$

Note that the diagonal elements of a matrix $X \in \mathcal{Q}(G)$ do not depend on the structure of G, these are 'free elements' in the sense that they can be either zero or nonzero.

Next, we look at a different class of matrices associated to a bipartite graph $G = (V^-, V^+, E)$, where the vertex sets V^- and V^+ are given by

$$\begin{aligned} V^- &= \{r_1, r_2, ..., r_s\} \\ V^+ &= \{q_1, q_2, ..., q_t\}. \end{aligned} \tag{2.2}$$

The pattern class $\mathcal{P}(G)$ of the bipartite graph G, with vertex sets V^- and V^+ given by (2.2), is defined as

$$\mathcal{P}(G) = \{M \in \mathbb{R}^{t \times s} \mid M_{ij} \neq 0 \iff (r_j, q_i) \in E\}. \tag{2.3}$$

Note that the cardinalities of V^- and V^+ can differ, hence the matrices in the pattern class $\mathcal{P}(G)$ are not necessarily square.

2.2.2 *Subclass of distance-information preserving matrices*

In this subsection we investigate properties of the powers of matrices belonging to the qualitative class $\mathcal{Q}(G)$. The relevance of these properties will become apparent later on, when we provide a graph-theoretic condition for targeted controllability of systems defined on graphs.

We first provide the following lemma, which states that if the distance between two nodes is greater than k, the corresponding element in X^k is zero.

Lemma 2.1. *Consider a directed graph $G = (V, E)$, two distinct vertices $i, j \in V$, a matrix $X \in \mathcal{Q}(G)$ and a positive integer k. If $d(j, i) > k$, then $(X^k)_{ij} = 0$.*

Proof. Note that the statement trivially holds for $k = 1$. Suppose now that the lemma holds for a $k \geq 1$. We want to prove that the lemma holds for $k + 1$ as well. Let $i, j \in V$ be two distinct vertices with $d(i, j) > k + 1$, and consider a matrix $X \in \mathcal{Q}(G)$. Note that

$$(X^{k+1})_{ji} = \sum_{l=1}^{n} (X^k)_{jl} X_{li}. \tag{2.4}$$

We will prove that all terms present in the sum on the right hand side of (2.4) are equal to zero. First consider the term $(X^k)_{jj} X_{ji}$. Note that $d(i, j) > 1$, hence there is no directed arc from i to j. This implies $X_{ji} = 0$, from which we conclude that $(X^k)_{jj} X_{ji} = 0$.
Next, consider the term $(X^k)_{ji} X_{ii}$. By assumption $d(i, j) > k + 1 > k$, which yields $(X^k)_{ji} = 0$ using the induction hypothesis. It follows that $(X^k)_{ji} X_{ii} = 0$.
Finally, consider the remaining terms $(X^k)_{jl} X_{li}$, where $l \neq i, j$. Suppose $(X^k)_{jl} \neq 0$, then by the induction hypothesis the distance between l and j is less than or equal to k. However, this

means that the entry X_{li} must be zero, as we assumed that $d(i,j) > k+1$. On the other hand, suppose that $X_{li} \neq 0$, then by the same reasoning $(X^k)_{jl} = 0$, otherwise there would exist a path of length less than or equal to $k+1$ from i to j. We conclude that all summands on the right-hand side of (2.4) are zero, hence $(X^{k+1})_{ji} = 0$. □

Subsequently, we consider the class of matrices for which $(X^k)_{ij}$ is nonzero if the distance $d(j,i)$ is exactly equal to k. Such matrices are called distance-information preserving, more precisely:

Definition 2.2. Consider a directed graph $G = (V, E)$. A matrix $X \in \mathcal{Q}(G)$ is called *distance-information preserving* if for any two distinct vertices $i, j \in V$ we have that $d(j,i) = k$ implies $(X^k)_{ij} \neq 0$.

Although the distance-information preserving property does not hold for all matrices $X \in \mathcal{Q}(G)$, it does hold for the adjacency and Laplacian matrices [36]. Because these matrices are often used to describe network dynamics, distance-information preserving matrices form an important subclass of $\mathcal{Q}(G)$, which from now on will be denoted by $\mathcal{Q}_d(G)$. More explicitly:

$$\mathcal{Q}_d(G) = \{X \in \mathcal{Q}(G) \mid X \text{ is distance-information preserving}\}. \tag{2.5}$$

At this point, we remark that in general, $\mathcal{Q}_d(G)$ is a strict subset of $\mathcal{Q}(G)$. However, for some types of graphs the subclass $\mathcal{Q}_d(G)$ and qualitative class $\mathcal{Q}(G)$ are identical. Examples of such graphs are complete graphs and directed trees, denoted by K and T respectively. In the case of the former, the distance between all pairs of distinct nodes is equal to one. Hence, for a matrix $X \in \mathcal{Q}(K)$ to be distance-information preserving, we require X_{ji} to be nonzero if $\{i, j\}$ is an edge in K, which holds by definition. The fact that $\mathcal{Q}_d(T) = \mathcal{Q}(T)$ in the case of a directed tree T is less trivial. We devote some time to prove this statement.

Proposition 2.3. *Consider a directed tree $T = (V, E)$. The qualitative class $\mathcal{Q}(T)$ and subclass $\mathcal{Q}_d(T)$ are identical.*

Proof. We have to prove that any matrix $X \in \mathcal{Q}(T)$ is distance-information preserving, i.e. we want to show the validity of the statement

$$d(i,j) = k \Longrightarrow (X^k)_{ji} \neq 0 \text{ for distinct } i, j \in V \text{ and } k \in \mathbb{N}. \tag{2.6}$$

For $k = 1$ statement (2.6) is true by definition of $X \in \mathcal{Q}(T)$. Suppose now that (2.6) holds for a $k \geq 1$. We want to prove that it holds for $k+1$ as well. Let $i, j \in V$ be two distinct vertices with $d(i,j) = k+1$, and consider a matrix $X \in \mathcal{Q}(T)$. We write

$$(X^{k+1})_{ji} = \sum_{l=1}^{n} (X^k)_{jl} X_{li}. \tag{2.7}$$

Let $q \in V$ be the vertex such that $d(i,q) = 1$, and $d(q,j) = k$. Note that this vertex exists, as we assumed $d(i,j) = k+1$. We can rewrite (2.7) in the following way

$$(X^{k+1})_{ji} = (X^k)_{jq} X_{qi} + \sum_{\substack{l=1 \\ l \neq q}}^{n} (X^k)_{jl} X_{li}. \tag{2.8}$$

By the induction hypothesis, $(X^k)_{jq} \neq 0$. Moreover, $X_{qi} \neq 0$ as there is an arc from i to q. This implies $(X^k)_{jq} X_{qi} \neq 0$. We proceed as follows: firstly we prove that $(X^k)_{jl} X_{li} = 0$ for $l = 1, ..., q-1, q+1, ..., n$. Subsequently, we conclude from (2.8) that $(X^{k+1})_{ji}$ is nonzero.

First consider the term $(X^k)_{jj}X_{ji}$. As $d(i,j) = k+1 \geq 2$, we have $X_{ji} = 0$, from which we conclude $(X^k)_{jj}X_{ji} = 0$.

Next, we investigate the term $(X^k)_{ji}X_{ii}$. Because $d(i,j) > k$, we conclude from Lemma 2.1 that $(X^k)_{ji} = 0$, which yields $(X^k)_{ji}X_{ii} = 0$.

Finally, we consider the remaining terms $(X^k)_{jl}X_{li}$, for $l \neq i,j,q$. Suppose $X_{li} \neq 0$, we claim that $d(l,j) > k$. This can be seen in the following way: if $d(l,j) = k$, then there would be two different paths of length $k+1$ from i to j. This is impossible as we are dealing with a directed tree. Furthermore, if $d(l,j) < k$, there would exist a path of length less than $k+1$ from i to j, which contradicts the assumption $d(i,j) = k+1$. To conclude: if $X_{li} \neq 0$, then $d(l,j) > k$. This means that $(X^k)_{jl} = 0$ by Lemma 2.1.
On the other hand, assume $(X^k)_{jl} \neq 0$. We claim that this implies $X_{li} = 0$. Indeed, as $(X^k)_{jl} \neq 0$, it follows from Lemma 2.1 that $d(l,j) \leq k$. If $X_{li} \neq 0$, then either $d(i,j) < k+1$, which is a contradiction, or there exist two different paths of length $k+1$ from i to j, which is a contradiction as well. Hence, $X_{li} = 0$. We conclude that $(X^k)_{jl}X_{li} = 0$, for $l \neq q$. Therefore it follows from (2.8) that $(X^{k+1})_{ji} \neq 0$, which proves the proposition. □

2.2.3 *Zero forcing sets*

In this section we review the notion of zero forcing. The reason for this is the correspondence between zero forcing sets and the sets of leaders rendering a system defined on a graph controllable. More on this will follow in the next subsection.

For now, let $G = (V,E)$ be a directed graph with vertices colored either black or white. The color-change rule is defined in the following way: If $u \in V$ is a black vertex and exactly one out-neighbour $v \in V$ of u is white, then change the color of v to black [17].

When the color-change rule is applied to u to change the color of v, we say u forces v, and write $u \to v$.

Given a coloring of G, that is: given a set $C \subseteq V$ containing black vertices only, and a set $V \setminus C$ consisting of only white vertices, the derived set $D(C)$ is the set of black vertices obtained by applying the color-change rule until no more changes are possible [17].

A zero forcing set for G is a subset of vertices $Z \subseteq V$ such that if initially the vertices in Z are colored black and the remaining vertices are colored white, then $D(Z) = V$.

Finally, for a given zero forcing set, we can construct the derived set, listing the forces in the order in which they were performed. This list is called a chronological list of forces. Note that such a list does not have to be unique.

Example 2.4. Consider the directed graph $G = (V,E)$ depicted in Figure 2.1 and let $C = \{2\}$.

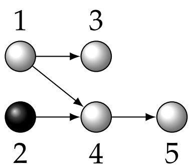

Figure 2.1: Graph G.

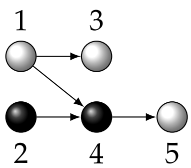

Figure 2.2: Force $2 \to 4$.

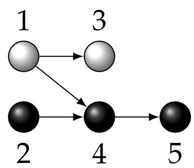

Figure 2.3: Force $4 \to 5$.

Note that vertex 2 can force 4, and subsequently node 4 can force 5. No further color changes can be made, so $D(C) = \{2,4,5\}$. As $D(C) \neq V$, C is not a zero forcing set. However, suppose we choose $C = \{1, 2\}$. In this case it is easy to see that we can color all vertices black, hence $C = \{1, 2\}$ is a zero forcing set.

2.2.4 *Systems defined on graphs*

Consider a directed graph $G = (V, E)$, where the vertex set is given by $V = \{1, 2, ..., n\}$. Furthermore, let $V' = \{v_1, v_2, ..., v_r\} \subseteq V$ be a subset. The $n \times r$ matrix $P(V; V')$ is defined by

$$P_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

We now introduce the subset $V_L \subseteq V$ consisting of so-called *leader nodes*, i.e. agents of the network to which an external control input is applied. The remaining nodes $V \setminus V_L$ are appropriately called followers. We consider finite-dimensional linear time-invariant systems of the form

$$\dot{x}(t) = Xx(t) + Uu(t), \tag{2.10}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input of the system. Here $X \in \mathcal{Q}(G)$ and $U = P(V; V_L)$, for some leader set $V_L \subseteq V$. An important notion regarding systems of the form (3.2) is the notion of strong structural controllability.

Definition 2.5. [29] A system of the form (3.2) is called *strongly structurally controllable* if the pair (X, U) is controllable for all $X \in \mathcal{Q}(G)$.

In the case that (3.2) is strongly structurally controllable we say $(G; V_L)$ is controllable, with a slight abuse of terminology. There is a one-to-one correspondence between strong structural controllability and zero forcing sets, as stated in the following theorem.

Theorem 2.6. *[29] Let $G = (V, E)$ be a directed graph and let $V_L \subseteq V$ be a leader set. Then $(G; V_L)$ is controllable if and only if V_L is a zero forcing set.*

In this thesis, we are primarily interested in cases for which $(G; V_L)$ is not controllable. In such cases, we wonder whether we can control the state of a subset $V_T \subseteq V$ of nodes, called *target nodes*. To this extent we specify an output equation $y(t) = Hx(t)$, which together with (3.2) yields the system

$$\begin{aligned} \dot{x}(t) &= Xx(t) + Uu(t) \\ y(t) &= Hx(t), \end{aligned} \tag{2.11}$$

where $y \in \mathbb{R}^p$ is the output of the system consisting of the states of the target nodes, and $H = P^T(V; V_T)$. Note that the ability to control the states of all target nodes in V_T is equivalent with the output controllability of system (2.11) [27]. As the output of system (2.11) specifically consists of the states of the target nodes, we say (2.11) is targeted controllable if it is output controllable.

Furthermore, system (2.11) is called strongly targeted controllable if (X, U, H) is targeted controllable for all $X \in \mathcal{Q}(G)$ [27]. In case (2.11) is strongly targeted controllable, we say $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}(G)$. The term 'with respect to $\mathcal{Q}(G)$' clarifies the class of state matrices under consideration. This thesis mainly considers strong targeted controllability with respect to $\mathcal{Q}_d(G)$. We conclude this section with well-known conditions for strong targeted controllability. Let $U = P(V; V_L)$ and $H = P^T(V; V_T)$ be the input and output matrices respectively, and define the reachable subspace $\langle X \mid \operatorname{im} U \rangle = \operatorname{im} U + X \operatorname{im} U + \cdots + X^{n-1} \operatorname{im} U$.

Proposition 2.7. *[27] The following statements are equivalent:*

1) $(G; V_L; V_T)$ *is targeted controllable with respect to* $\mathcal{Q}(G)$
2) $\operatorname{rank}\begin{pmatrix} HU & HXU & \cdots & HX^{n-1}U \end{pmatrix} = p$ *for all* $X \in \mathcal{Q}(G)$
3) $H\langle X \mid \operatorname{im} U \rangle = \mathbb{R}^p$ *for all* $X \in \mathcal{Q}(G)$
4) $\ker H + \langle X \mid \operatorname{im} U \rangle = \mathbb{R}^n$ *for all* $X \in \mathcal{Q}(G)$.

2.3 PROBLEM STATEMENT

Strong targeted controllability with respect to $\mathcal{Q}(G)$ was studied in [27], and a sufficient graph-theoretic condition was provided. Motivated by the fact that $\mathcal{Q}_d(G)$ contains important network-related matrices like the adjacency and Laplacian matrices, we are interested in extending the results of [27] to the class of distance-information preserving matrices $Q_d(G)$. More explicitly, the problem that we will investigate in this thesis is given as follows.

Problem 2.8. *Given a directed graph* $G = (V, E)$*, a leader set* $V_L \subseteq V$ *and target set* $V_T \subseteq V$*, provide necessary and sufficient graph-theoretic conditions under which* $(G; V_L; V_T)$ *is targeted controllable with respect to* $\mathcal{Q}_d(G)$*.*

Such graph-theoretic conditions have a considerable advantage in their computational robustness compared to rank conditions, and aid in finding leader selection procedures. In addition, we are interested in a method to compute leader sets achieving targeted controllability. More precisely:

Problem 2.9. *Given a directed graph* $G = (V, E)$ *and target set* $V_T \subseteq V$*, compute a leader set* $V_L \subseteq V$ *of minimum cardinality such that* $(G; V_L; V_T)$ *is targeted controllable with respect to* $\mathcal{Q}_d(G)$*.*

2.4 MAIN RESULTS

Our main results are presented in this section. Firstly, in Section 2.4.1 we provide a sufficient graph-theoretic condition for strong targeted controllability with respect to $Q_d(G)$. Subsequently, in Section 2.4.2 we review the notion of sufficient richness of subclasses, and prove that the subclass $\mathcal{Q}_d(G)$ is sufficiently rich. This result allows us to establish a necessary condition for strong targeted controllability, which is presented in Section 2.4.3. Finally, in Section 2.4.4 we show Problem 2.9 is NP-hard and provide a heuristic leader selection algorithm to determine leader sets achieving targeted controllability.

2.4.1 *Sufficient condition for targeted controllability*

This section discusses a sufficient graph-theoretic condition for strong targeted controllability. We first introduce some notions that will become useful later on.

Consider a directed graph $G = (V, E)$ with a leader set V_L and target set V_T. The derived set of V_L is given by $D(V_L)$. Furthermore, let $V_S \subseteq V \setminus D(V_L)$ be a subset. We partition the set V_S according to the distance of its nodes with respect to $D(V_L)$, that is

$$V_S = V_1 \cup V_2 \cup \cdots \cup V_d, \tag{2.12}$$

where for $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$ for $i = 1, 2, ..., d$. Moreover, we define $\check{V}_i$ and $\hat{V}_i$ to be the sets of vertices in V_S of distance respectively less than i and greater than i with respect to $D(V_L)$. More precisely:

$$\begin{aligned} \check{V}_i &:= V_1 \cup ... \cup V_{i-1} \text{ for } i = 2, ..., d \\ \hat{V}_i &:= V_{i+1} \cup ... \cup V_d \text{ for } i = 1, ..., d-1. \end{aligned} \tag{2.13}$$

By convention $\check{V}_1 = \varnothing$ and $\hat{V}_d = \varnothing$. With each of the sets $V_1, V_2, ..., V_d$ we associate a bipartite graph $G_i = (D(V_L), V_i, E_i)$, where for $j \in D(V_L)$ and $k \in V_i$ we have $(j,k) \in E_i$ if and only if $d(j,k) = i$ in the network graph G.

Example 2.10. We consider the network graph $G = (V, E)$ as depicted in Figure 2.4. The set of leaders is $V_L = \{1, 2\}$, which implies that $D(V_L) = \{1, 2, 3\}$.

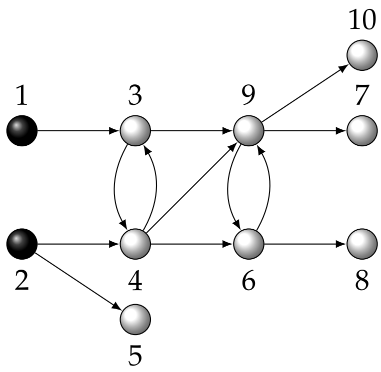

Figure 2.4: Graph G with $V_L = \{1, 2\}$.

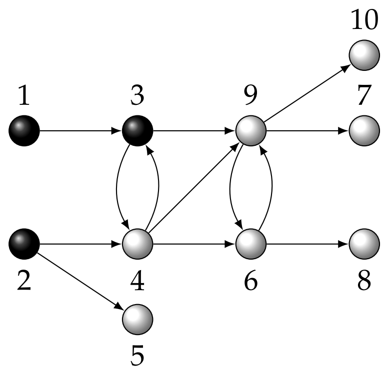

Figure 2.5: $D(V_L) = \{1, 2, 3\}$.

In this example, we define the subset $V_S \subseteq V \setminus D(V_L)$ as $V_S := \{4, 5, 6, 7, 8\}$. Note that V_S can be partitioned according to the distance of its nodes with respect to $D(V_L)$ as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4, 5\}$, $V_2 = \{6, 7\}$ and $V_3 = \{8\}$. The bipartite graphs G_1, G_2 and G_3 are given in Figures 2.6,2.7 and 2.8 respectively.

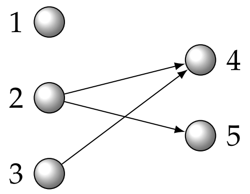

Figure 2.6: Graph G_1.

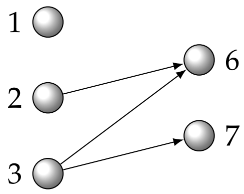

Figure 2.7: Graph G_2.

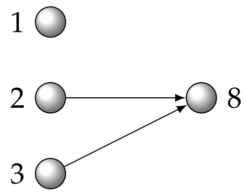

Figure 2.8: Graph G_3.

The main result presented in this section is given in Theorem 2.11. This statement provides a sufficient graph-theoretic condition for targeted controllability of $(G; V_L; V_T)$ with respect to $\mathcal{Q}_d(G)$.

Theorem 2.11. *Consider a directed graph $G = (V, E)$, with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Let $V_T \setminus D(V_L)$ be partitioned as in* (2.12)*, and assume $D(V_L)$ is a zero forcing set in $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, ..., d$. Then $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.*

The 'k-walk theory' for target control, described in [13] is just a special case of Theorem 2.11. Indeed, in the single-leader case, the condition of Theorem 2.11 reduces to the condition that no pair of target nodes has the same distance with respect to the leader. However, it is worth mentioning that Theorem 2.11 holds for general directed graphs and multiple leaders, while the results of [13] are only applicable to directed tree networks in the case that the leader set is singleton. Furthermore, note that Theorem 2.11 significantly improves the known condition for strong targeted controllability given in [27] for the class $\mathcal{Q}_d(G)$. In Theorem 2.11 target nodes with arbitrary distance with respect to the derived set are allowed, while the main result Theorem VI.6 of [27] is restricted to target nodes of distance one with respect to $D(V_L)$. Before proving Theorem 2.11, we provide an illustrative example and two auxiliary lemmas.

Example 2.12. Once again, consider the network graph depicted in Figure 2.4, with leader set $V_L = \{1,2\}$ and assume the target set is given by $V_T = \{1,2,...,8\}$. The goal of this example is to prove that $(G;V_L;V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

Note that $V_S := V_T \setminus D(V_L)$ is given by $V_S = \{4,5,6,7,8\}$, which is partitioned according to (2.12) as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4,5\}$, $V_2 = \{6,7\}$ and $V_3 = \{8\}$. The graphs G_1, G_2 and G_3 have been computed in Example 2.10. Note that $D(V_L) = \{1,2,3\}$ is a zero forcing set in all three graphs (see Figures 2.9, 2.10 and 2.11). We conclude by Theorem 2.11 that $(G;V_L;V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

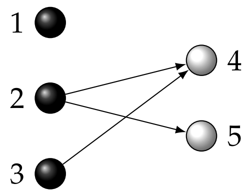

Figure 2.9: Graph G_1.

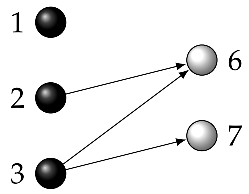

Figure 2.10: Graph G_2.

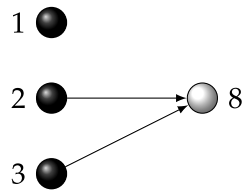

Figure 2.11: Graph G_3.

Lemma 2.13. *Consider a directed graph $G = (V,E)$ with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Let $\mathcal{Q}_s(G) \subseteq \mathcal{Q}(G)$ be any subclass. Then $(G;V_L;V_T)$ is targeted controllable with respect to $Q_s(G)$ if and only if $(G;D(V_L);V_T)$ is targeted controllable with respect to $Q_s(G)$.*

Proof. Let $U = P(V;V_L)$ index the leader set V_L and $W = P(V;D(V_L))$ index the derived set of V_L. We have that $(G;V_L;V_T)$ is targeted controllable with respect to $Q_s(G)$ if and only if

$$H\langle X \mid \operatorname{im} U\rangle = \mathbb{R}^p \text{ for all } X \in Q_s(G), \tag{2.14}$$

However, as $\langle X \mid \operatorname{im} U\rangle = \langle X \mid \operatorname{im} W\rangle$ for any $X \in \mathcal{Q}(G)$ (see Lemma VI.2 of [27]), (2.14) holds if and only if

$$H\langle X \mid \operatorname{im} W\rangle = \mathbb{R}^p \text{ for all } X \in Q_s(G). \tag{2.15}$$

We conclude that $(G;V_L;V_T)$ is targeted controllable with respect to $Q_s(G)$ if and only if $(G;D(V_L);V_T)$ is targeted controllable with respect to $Q_s(G)$. $\square$

Lemma 2.14. *Let $G = (V^-,V^+,E)$ be a bipartite graph and assume V^- is a zero forcing set in G. Then all matrices $M \in \mathcal{P}(G)$ have full row rank.*

Proof. Note that forces of the form $u \to v$, where $u,v \in V^+$ are not possible, as G is a bipartite graph. Relabel the nodes of V^- and V^+ such that a chronological list of forces is given by

$u_i \to v_i$, where $u_i \in V^-$ and $v_i \in V^+$ for $i = 1, 2, ..., |V^+|$. Let $M \in \mathcal{P}(G)$ be a matrix in the pattern class of G. Note that the element M_{ii} is nonzero, as $u_i \to v_i$. Furthermore, M_{ji} is zero for all $j > i$. The latter follows from the fact that u_i would not be able to force v_i if there was an arc $(u_i, v_j) \in E$. We conclude that the columns $1, 2, ..., |V^+|$ of M are linearly independent, hence M has full row rank. $\square$

Proof of Theorem 2.11. Let $D(V_L) = \{1, 2, ..., m\}$, and assume without loss of generality that the matrix U has the form (see Lemma 2.13):

$$U = \begin{pmatrix} I_{m\times m} & 0_{m\times(n-m)} \end{pmatrix}^T. \tag{2.16}$$

Furthermore, we let $V_S := V_T \setminus D(V_L)$ be given by $\{m+1, m+2, ..., p\}$, where the vertices are ordered in non-decreasing distance with respect to $D(V_L)$. Partition V_S according to the distance of its nodes with respect to $D(V_L)$ as

$$V_S = V_1 \cup V_2 \cup \cdots \cup V_d, \tag{2.17}$$

where for $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$ for $i = 1, 2, ..., d$. Finally, assume the target set V_T contains all nodes in the derived set $D(V_L)$. This implies that the matrix H is of the form

$$H = \begin{pmatrix} I_{p\times p} & 0_{p\times(n-p)} \end{pmatrix}. \tag{2.18}$$

Note that by the structure of H and U, the matrix HX^iU is simply the $p \times m$ upper left corner submatrix of X^i. We now claim that HX^iU can be written as follows.

$$HX^iU = \begin{pmatrix} \Lambda_i \\ M_i \\ 0_i \end{pmatrix}, \tag{2.19}$$

where $M_i \in \mathcal{P}(G_i)$ is a $|V_i| \times m$ matrix in the pattern class of G_i, Λ_i is an $\left(m + |\check{V}_i|\right) \times m$ matrix containing elements of lesser interest, and 0_i is a zero matrix of dimension $|\hat{V}_i| \times m$.

We proceed as follows: first we prove that the bottom submatrix of (2.19) contains zeros only, secondly we prove that $M_i \in \mathcal{P}(G_i)$. From this, we conclude that equation (2.19) holds.

Note that for $k \in D(V_L)$ and $j \in \hat{V}_i$, we have $d(k, j) > i$ and by Lemma 2.1 it follows that $(X^i)_{jk} = 0$. As $D(V_L) = \{1, 2, ..., m\}$, this means that the bottom $|\hat{V}_i| \times m$ submatrix of HX^iU is a zero matrix.

Subsequently, we want to prove that M_i, the middle block of (2.19), is an element of the pattern class $\in \mathcal{P}(G_i)$. Note that the jth row of M_i corresponds to the element $l := m + |\check{V}_i| + j \in V_i$.

Suppose $(M_i)_{jk} \neq 0$ for a $k \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., |V_i|\}$. As M_i is a submatrix of HX^iU, this implies $(HX^iU)_{lk} \neq 0$. Recall that HX^iU is the $p \times m$ upper left corner submatrix of X^i, therefore it holds that $(X^i)_{lk} \neq 0$. Note that for the vertices $k \in D(V_L)$ and $l \in V_i$ we have $d(k, l) \geq i$ by the partition of V_S. However, as $(X^i)_{lk} \neq 0$ it follows from Lemma 2.1 that $d(k, l) = i$. Therefore, by the definition of G_i, there is an arc $(k, l) \in E_i$.

Conversely, suppose there is an arc $(k, l) \in E_i$ for $l \in V_i$ and $k \in D(V_L)$. This implies $d(k, l) = i$ in the network graph G. By the distance-information preserving property of X we consequently have $(X^i)_{lk} \neq 0$. We conclude that $(M_i)_{jk} \neq 0$ and hence $M_i \in \mathcal{P}(G_i)$. This implies that equation (2.19) holds, We compute the first dm columns of the output controllability matrix $\begin{pmatrix} HU & HXU & HX^2U & \dots & HX^dU \end{pmatrix}$ as follows:

$$\begin{pmatrix} I & * & * & \dots & * & * \\ 0 & M_1 & * & \dots & * & * \\ 0 & 0 & M_2 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & * & * \\ \vdots & \vdots & \vdots & \ddots & M_{d-1} & * \\ 0 & 0 & 0 & \dots & 0 & M_d \end{pmatrix}, \tag{2.20}$$

where zeros denote zero matrices and asterisks denote matrices of less interest. As $D(V_L)$ is a zero forcing set in G_i for $i = 1, 2, ..., d$, the matrices $M_1, M_2, ..., M_d$ have full row rank by Lemma 2.14. We conclude that the matrix (2.20) has full row rank, and consequently $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. $\square$

Note that the condition given in Theorem 2.11 is sufficient, but not necessary. This is illustrated in the following example.

Example 2.15. Consider the directed graph $G = (V, E)$ as given in Figure 2.12, with leader set $V_L = \{1\}$ and target set $V_T = \{2, 3\}$. Note that the condition of Theorem 2.11 is not satisfied.

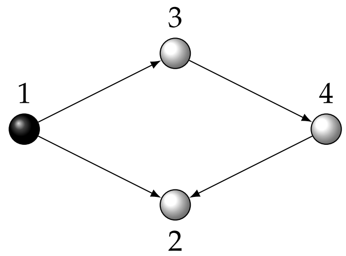

Figure 2.12: Graph $G = (V, E)$.

Matrices $X \in \mathcal{Q}(G)$ have the form

$$X = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ a_1 & d_2 & 0 & a_4 \\ a_2 & 0 & d_3 & 0 \\ 0 & 0 & a_3 & d_4 \end{pmatrix}, \tag{2.21}$$

where a_1, a_2, a_3 and a_4 are nonzero and $d_1, d_2, d_3, d_4 \in \mathbb{R}$. The output controllability matrix for this example is given by

$$\begin{pmatrix} 0 & a_1 & a_1(d_1 + d_2) & a_1(d_1^2 + d_2^2 + d_1 d_2) + a_2 a_3 a_4 \\ 0 & a_2 & a_2(d_1 + d_3) & a_2(d_1^2 + d_3^2 + d_1 d_3) \end{pmatrix}. \tag{2.22}$$

Suppose that the two rows of (2.22) are linearly dependent, then for a constant $c \in \mathbb{R}$ we have

$$\begin{aligned} ca_1 &= a_2 \\ ca_1(d_1 + d_2) &= a_2(d_1 + d_3) \\ c(a_1(d_1^2 + d_2^2 + d_1 d_2) + a_2 a_3 a_4) &= a_2(d_1^2 + d_3^2 + d_1 d_3). \end{aligned} \tag{2.23}$$

By substitution of $ca_1 = a_2$ into the second equality of (2.23) we obtain $d_2 = d_3$. However, this implies that the third equality in (2.23) can be simplified as

$$\begin{aligned} c(a_1(d_1^2 + d_2^2 + d_1 d_2) + a_2 a_3 a_4) &= ca_1(d_1^2 + d_2^2 + d_1 d_2) \\ ca_2 a_3 a_4 &= 0. \end{aligned} \tag{2.24}$$

Because a_2, a_3 and a_4 are nonzero, we obtain $c = 0$ which is a contradiction as $ca_1 = a_2$ for nonzero a_1, a_2. We conclude that the rows of (2.22) are linearly independent, which implies that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}(G)$.

2.4.2 *Sufficient richness of $Q_d(G)$*

The notion of sufficient richness of a qualitative subclass was introduced in [29]. We provide an equivalent definition as follows.

Definition 2.16. Let $G = (V, E)$ be a directed graph with leader set $V_L \subseteq V$. A subclass $Q_s(G) \subseteq \mathcal{Q}(G)$ is called *sufficiently rich* if $(G; V_L)$ is controllable with respect to $Q_s(G)$ implies $(G; V_L)$ is controllable with respect to $Q(G)$.

The following geometric characterization of sufficient richness is proven in [29].

Proposition 2.17. *A qualitative subclass $Q_s(G) \subseteq \mathcal{Q}(G)$ is sufficiently rich if for all $z \in \mathbb{R}^n$ and $X \in \mathcal{Q}(G)$ satisfying $z^T X = 0$, there exists an $X' \in Q_s(G)$ such that $z^T X' = 0$.*

The goal of this section is to prove that the qualitative subclass of distance-information preserving matrices is sufficiently rich. This result will be used later on, when we provide a necessary condition for targeted controllability with respect to $\mathcal{Q}_d(G)$. First however, we state two auxiliary lemmas which will be the building blocks to prove the sufficient richness of $\mathcal{Q}_d(G)$.

Lemma 2.18. *Consider q nonzero multivariate polynomials $p_i(x)$, where $i = 1, 2, ..., q$ and $x \in \mathbb{R}^n$. There exists an $\bar{x} \in \mathbb{R}^n$ such that $p_i(\bar{x}) \neq 0$ for $i = 1, 2, ..., q$.*

Proof. The proof follows immediately from continuity of polynomials and is omitted. □

Remark: Without loss of generality, we can assume that the point $\bar{x} \in \mathbb{R}^n$ has only nonzero coordinates. Indeed, if $p_i(\bar{x}) \neq 0$ for $i = 1, 2, ..., q$, there exists an open ball $B(\bar{x})$ around $\bar{x}$ in which $p_i(x) \neq 0$ for $i = 1, 2, ..., q$. Obviously, this open ball contains a point with the aforementioned property.

Lemma 2.19. *Let $X \in \mathcal{Q}(G)$ and $D = diag\ (d_1, d_2, ..., d_n)$ be a matrix with variable diagonal entries. If $d(i, j) = k$ for distinct vertices i and j, then $((XD)^k)_{ji}$ is a nonzero polynomial in the variables $d_1, d_2, ..., d_n$.*

Proof. Note that $((XD)^k)_{ji}$ is given by

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} (XD)_{i_1,i} (XD)_{i_2,i_1} \cdots (XD)_{j,i_{k-1}},$$

which equals

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} d_i X_{i_1,i} \cdot d_{i_1} X_{i_2,i_1} \cdots d_{i_{k-1}} X_{j,i_{k-1}}. \tag{2.25}$$

Since the distance $d(i,j)$ is equal to k, there exists at least one path of length k from i to j, which we denote by $(i,i_1),(i_1,i_2),...,(i_{k-1},j)$. It follows that the corresponding elements of the matrix X, i.e. the elements $X_{i_1,i}, X_{i_2,i_1}, X_{j,i_{k-1}}$ are nonzero. Therefore, the term

$$d_i X_{i_1,i} \cdot d_{i_1} X_{i_2,i_1} \cdots d_{i_{k-1}} X_{j,i_{k-1}} \tag{2.26}$$

is nonzero (as a function of $d_i, d_{i_1}, d_{i_2}, ..., d_{i_{k-1}}$). Furthermore, this combination of k diagonal elements is unique in the sense that there does not exist another summand on the right-hand side of (2.25) with exactly the same elements. This implies that the term (2.26) does not vanish (as a polynomial). We conclude that $((XD)^k)_{ji}$ is a nonzero polynomial function in the variables $d_1, d_2, ..., d_n$. $\square$

Theorem 2.20. *The subclass $\mathcal{Q}_d(G)$ is sufficiently rich.*

Proof. Given a matrix $X \in \mathcal{Q}(G)$, using Lemmas 2.18 and 2.19, we first prove there exists a diagonal matrix $\bar{D}$ with nonzero diagonal components such that $X\bar{D} \in \mathcal{Q}_d(G)$. From this we will conclude $\mathcal{Q}_d(G)$ is sufficiently rich.

Let $D = \text{diag}\,(d_1, d_2, ..., d_n)$ be a matrix with variable diagonal entries. We define $p_{ij} := ((XD)^{d(i,j)})_{ji}$ for distinct $i,j = 1,2,...,n$. By Lemma 2.19 we have that $p_{ij}(d_1, d_2, ..., d_n)$ is a nonzero polynomial in the variables $d_1, d_2, ..., d_n$. Moreover, Lemma 2.18 states the existence of nonzero real constants $\bar{d}_1, \bar{d}_2, ..., \bar{d}_n$ such that

$$p_{ij}(\bar{d}_1, \bar{d}_2, ..., \bar{d}_n) \neq 0 \text{ for distinct } i,j = 1,2...,n. \tag{2.27}$$

Therefore, the choice $\bar{D} = \text{diag}\,(\bar{d}_1, \bar{d}_2, ..., \bar{d}_n)$ implies $X\bar{D} \in \mathcal{Q}_d(G)$. Let $z \in \mathbb{R}^n$ be a vector such that $z^T X = 0$ for an $X \in \mathcal{Q}(G)$. The choice of $X' = X\bar{D}$ yields a matrix $X' \in \mathcal{Q}_d(G)$ for which $z^T X' = 0$. By Proposition 2.17 it follows that $\mathcal{Q}_d(G)$ is sufficiently rich. $\square$

2.4.3 *Necessary condition for targeted controllability*

In addition to the previously established sufficient condition for targeted controllability, we give a necessary graph-theoretic condition for targeted controllability in Theorem 2.21.

Theorem 2.21. *Let $G = (V,E)$ be a directed graph with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. If $(G;V_L;V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$ then $V_L \cup (V \setminus V_T)$ is a zero forcing set in G.*

Proof. Assume without loss of generality that $V_L \cap V_T = \varnothing$. Hence, $V_L \cup (V \setminus V_T) = V \setminus V_T$. Partition the vertex set into V_L, $V \setminus (V_L \cup V_T)$ and V_T. Accordingly, the input and output matrices $U = P(V;V_L)$ and $H = P^T(V;V_T)$ satisfy

$$U = \begin{pmatrix} I & 0 & 0 \end{pmatrix}^T \qquad H = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \tag{2.28}$$

Note that $\ker H = \operatorname{im} R$, where $R = P(V;(V \setminus V_T))$ is given by

$$R = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}^T. \tag{2.29}$$

Since for all $X \in \mathcal{Q}_d(G)$ we have

$$\ker H + \langle X \mid \operatorname{im} U \rangle = \mathbb{R}^n, \tag{2.30}$$

equivalently,

$$\operatorname{im} R + \langle X \mid \operatorname{im} U \rangle = \mathbb{R}^n, \tag{2.31}$$

we obtain

$$\langle X \mid \operatorname{im} \begin{pmatrix} U & R \end{pmatrix} \rangle = \mathbb{R}^n. \tag{2.32}$$

As $\operatorname{im} U \subseteq \operatorname{im} R$, (2.32) implies $\langle X \mid \operatorname{im} R \rangle = \mathbb{R}^n$ for all $X \in \mathcal{Q}_d(G)$, equivalently, the pair (X, R) is controllable for all $X \in \mathcal{Q}_d(G)$. However, by sufficient richness of $\mathcal{Q}_d(G)$, it follows that (X, R) is controllable for all $X \in \mathcal{Q}(G)$. We conclude from Theorem 2.6 that $V \setminus V_T$ is a zero forcing set. □

Example 2.22. Consider the directed graph $G = (V, E)$ with leader set $V_L = \{1, 2\}$ and target set $V_T = \{1, 2, ..., 8\}$ as depicted in Figure 2.4. We know from Example 2.12 that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. The set $V_L \cup (V \setminus V_T) = \{1, 2, 9, 10\}$ is colored black in Figure 2.14. Indeed, $V_L \cup (V \setminus V_T)$ is a zero forcing set in G. A possible chronological list of forces is: $1 \to 3$, $3 \to 4$, $2 \to 5$, $4 \to 6$, $6 \to 8$ and $9 \to 7$.

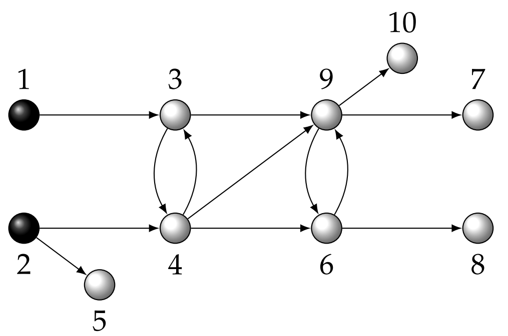

Figure 2.13: Graph G with $V_L = \{1, 2\}$.

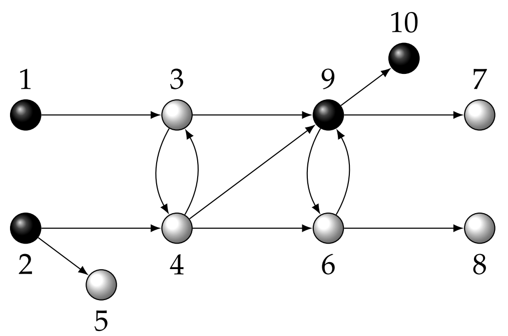

Figure 2.14: $V_L \cup (V \setminus V_T) = \{1, 2, 9, 10\}$.

The condition provided in Theorem 2.21 is necessary for targeted controllability, but not sufficient. To prove this fact, we give the following example.

Example 2.23. Consider the directed graph $G = (V, E)$ with leader set $V_L = \{1\}$ and target set $V_T = \{4, 5\}$, given in Figure 2.15. It can be shown that $(G; V_L; V_T)$ is not targeted controllable with respect to $\mathcal{Q}_d(G)$. However, $V_L \cup (V \setminus V_T)$ is a zero forcing set in G (see Figure 2.16).

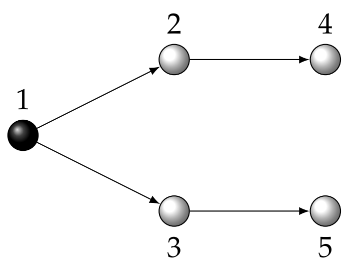

Figure 2.15: Graph $G = (V, E)$.

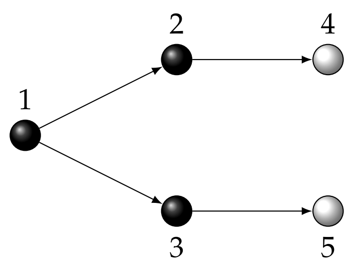

Figure 2.16: $V_L \cup (V \setminus V_T) = \{1, 2, 3\}$.

So far, we have provided a necessary and a sufficient topological condition for targeted controllability. However, given a network graph with target set, it is not clear how to choose leaders achieving target control. Hence, in the following section we focus on a leader selection algorithm.

2.4.4 *Leader selection algorithm*

The problem addressed in this section is as follows: given a directed graph $G = (V, E)$ with target set $V_T \subseteq V$, find a leader set $V_L \subseteq V$ of minimum cardinality such that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. Such a leader set is called a minimum leader set.

As the problem of finding a minimum zero forcing set is NP-hard [1], the problem of finding a minimum leader set V_L that achieves controllability of $(G; V_L)$ is NP-hard. Since this controllability problem can be seen as a special case of target control, where the entire vertex set is regarded as target set, we therefore conclude that determining a minimum leader set such that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$ is NP-hard.

It is for this reason we propose a heuristic approach to compute a (minimum) leader set that achieves targeted controllability. The algorithm consists of two phases. Firstly, we identify a set of nodes in the graph G from which all target nodes can be reached. These nodes are taken as leaders. Secondly, this set of leaders is extended to achieve targeted controllability.

To explain the first phase of the algorithm, we introduce some notation. First of all, we define the notion of *root set*.

Definition 2.24. Consider a directed graph $G = (V, E)$ and a target set $V_T \subseteq V$. A subset $V_R \subseteq V$ is called a root set of V_T if for any $v \in V_T$ there exists a vertex $u \in V_R$ such that $d(u, v) < \infty$.

A root set of V_T of minimum cardinality is called a minimum root set of V_T. Note that the cardinality of a minimum root set of V_T is a lower bound on the minimum number of leaders rendering $(G; V_L; V_T)$ targeted controllable. Indeed, it is easy to see that if there are no paths from any of the leader nodes to a target node, the graph is not targeted controllable. The first step of the proposed algorithm is to compute the minimum root set of V_T. Let the vertex and target sets be given by $V = \{1, 2, ..., n\}$ and $V_T = \{v_1, v_2, ..., v_p\} \subseteq V$ respectively. Furthermore, define a matrix $A \in \mathbb{R}^{p \times n}$ in the following way. For $j \in V$ and $v_i \in V_T$ let

$$A_{ij} := \begin{cases} 1 & \text{if } d(j, v_i) < \infty \\ 0 & \text{otherwise} \end{cases} \tag{2.33}$$

That is: the matrix A contains zeros and ones only, where coefficients with value one indicate the existence of a path between the corresponding vertices. Finding a minimum root set of V_T boils down to finding a binary vector $x \in \mathbb{R}^n$ with minimum number of ones such that $Ax \geq \mathbb{1}$, where the inequality is defined element-wise and $\mathbb{1}$ denotes the vector of all ones. In this vector x, coefficients with value one correspond to elements in the root set of V_T. It is for this reason we can formulate the minimum root set problem as a binary integer linear program

$$\begin{aligned} &\text{minimize } \mathbb{1}^T x \\ &\text{subject to } Ax \geq \mathbb{1} \\ &\quad\text{and } x \in \{0, 1\}^n. \end{aligned} \tag{2.34}$$

Linear programs of this form can be solved using software like CPLEX or Matlab. For very large-scale problems one might resort to heuristic methods (see e.g. [4]). In the following example we illustrate how the minimum root set problem can be regarded as a binary integer linear program.

Example 2.25. Consider the directed graph $G = (V, E)$ with target set $V_T = \{1, 4, 5, 6, 7\}$ depicted in Figure 2.17. The goal of this example is to find a minimum root set for V_T.

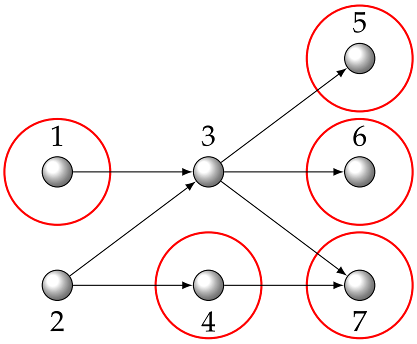

Figure 2.17: Directed graph $G = (V, E)$ with target set $V_T = \{1, 4, 5, 6, 7\}$.

The matrix A, as defined in (2.33), is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Note that $x = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T$ satisfies the inequality $Ax \geq \mathbb{1}$ and the constraint $x \in \{0, 1\}^7$. Furthermore, the vector x minimizes $\mathbb{1}^T x$ under these constraints. This can be seen in the following way: there is no column of A in which all elements equal 1, hence there is no vector x with a single one such that $Ax \geq \mathbb{1}$ is satisfied. Therefore, x solves the binary integer linear program (2.34), from which we conclude that the choice $V_R = \{1, 4\}$ yields a minimum root set for V_T. Indeed, observe in Figure 2.17 that we can reach all nodes in the target set starting from the nodes 1 and 4. It is worth mentioning that the choice of minimum root set is not unique: the set $\{1, 2\}$ is also a minimum root set for V_T.

In general, the minimum root set V_R of V_T does not guarantee targeted controllability of $(G; V_R; V_T)$ with respect to $\mathcal{Q}_d(G)$. For instance, it can be shown for the graph G and target set V_T of Example 2.25 that the leader set $V_L = \{1, 4\}$ does not render $(G; V_L; V_T)$ targeted controllable with respect to $\mathcal{Q}_d(G)$. Hence, we propose a greedy approach to extend the minimum root set of V_T to a leader set that does achieve targeted controllability.

Recall from Theorem 2.11 that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$ if $D(V_L)$ is a zero forcing set in the bipartite graphs $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, ..., d$, where $V_i \subseteq V_T$ is the set of target nodes having distance i from $D(V_L)$. Given an initial set of leaders V_L, we compute its derived set $D(V_L)$ and verify whether we can force all nodes in the bipartite graphs G_i for $i = 1, 2, ..., d$. Suppose that in the bipartite graph G_k the set V_k cannot be forced by $D(V_L)$ for a $k \in \{1, 2, ..., d\}$. Let $V_k = \{v_1, v_2, ..., v_l\}$, and suppose v_i is the first vertex in V_k that cannot be forced. Then we choose v_i as a leader. Consequently, we have extended our leader set V_L to $V_L \cup \{v_i\}$. With the extended leader set we can repeat the procedure, until the leaders render the graph targeted controllable. This idea is captured more formally in the following leader selection algorithm.

Algorithm 1: Leader Selection Procedure

Input: Directed graph $G = (V, E)$;
Target set $V_T \subseteq V$;
Output: Leader set $V_L \subseteq V$ achieving target control;
Let $V_L = \varnothing$;
Compute matrix A, given in (2.33);
Find a solution x for the linear program (2.34);
for $i = 1$ to n;
  if $x_i = 1$;
    $V_L = V_L \cup i$;
  end
end
Compute $D(V_L)$;
Set $i = 1$;
repeat
  Compute V_i and $G_i = (D(V_L), V_i, E_i)$;
  if $D(V_L)$ forces V_i in G_i;
    $i = i + 1$;
  else
    Let v be the first unforced vertex in V_i;
    Set $V_L = V_L \cup v$;
    Compute $D(V_L)$;
    $i = 1$;
  end
until $d(D(V_L), v) < i$ for all $v \in V_T$;
return V_L.

Example 2.26. Consider the directed graph given in Figure 2.18.

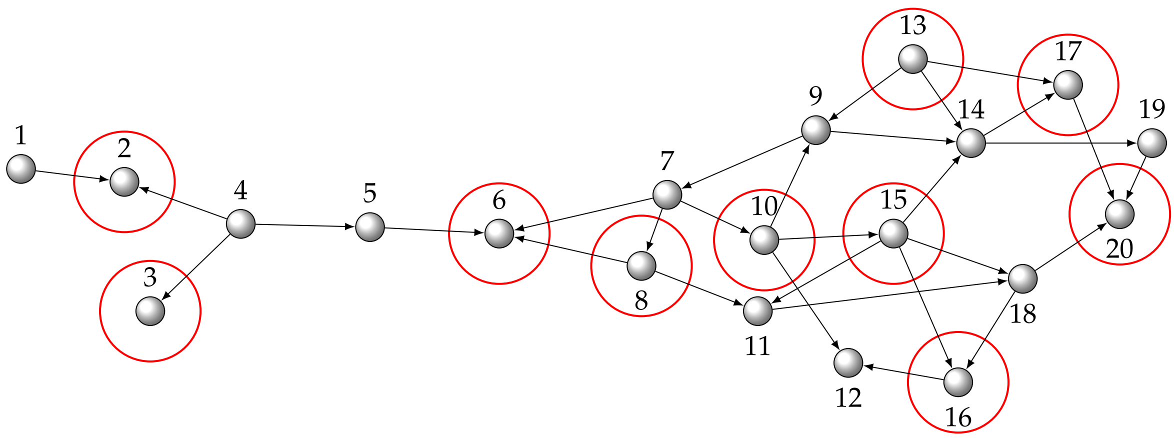

Figure 2.18: Directed graph $G = (V, E)$ with encircled target nodes.

Here, the target set V_T is given by

$$V_T = \{2, 3, 6, 8, 10, 13, 15, 16, 17, 20\}. \tag{2.36}$$

The goal of this example is to compute a leader set V_L such that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. The first step of Algorithm 1 is to compute the matrix A, defined in (2.33). For this example, A is given as follows.

$$A = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{pmatrix} \tag{2.37}$$

Using the Matlab function `intlinprog`, we find the optimal solution

$$x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{2.38}$$

to the binary linear program (2.34). Hence, a minimum root set for V_T is given by $\{4, 13\}$. Following Algorithm 1, we define our initial leader set $V_L = \{4, 13\}$. As nodes 4 and 13 both have three out-neighbours, the derived set of V_L is simply given by $D(V_L) = \{4, 13\}$. The next step of the algorithm is to compute the first bipartite graph $G_1 = (D(V_L), V_1, E_1)$, which we display in Figure 2.19.

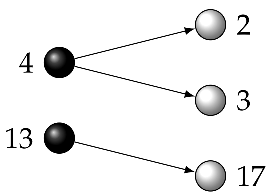

Figure 2.19: G_1 for $D(V_L) = \{4, 13\}$.

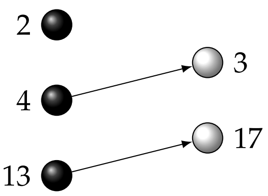

Figure 2.20: G_1 for $D(V_L) = \{2, 4, 13\}$.

Observe that the nodes 2 and 3 cannot be forced, hence we choose node 2 as additional leader. The process now repeats itself, we redefine $V_L = \{2, 4, 13\}$ and compute $D(V_L) = \{2, 4, 13\}$. Furthermore, for this leader set, the graph $G_1 = (D(V_L), V_1, E_1)$ is given in Figure 2.20. In this case, the set $V_1 = \{3, 17\}$ of nodes having distance one with respect to $D(V_L)$ is forced. Therefore, we continue with the second bipartite graph $G_2 = (D(V_L), V_2, E_2)$, given in Figure 2.21.

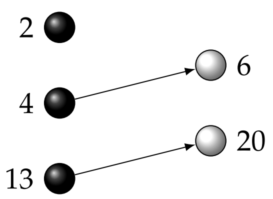

Figure 2.21: G_2 for $D(V_L) = \{2, 4, 13\}$.

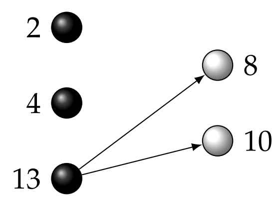

Figure 2.22: G_3 for $D(V_L) = \{2, 4, 13\}$.

The set V_2 is forced by $D(V_L)$ in the graph G_2, hence we continue to investigate the third bipartite graph consisting of nodes having distance three with respect to $D(V_L)$. This graph is displayed in Figure 2.22. As neither node 8 nor 10 can be forced, we add node 8 to the leader set. In other words, we redefine $V_L = \{2, 4, 8, 13\}$. Furthermore, the derived set of V_L is given by $D(V_L) = \{2, 4, 8, 13\}$. As we adapted the derived set, we have to recalculate the bipartite graphs G_1, G_2, G_3 and G_4 (see Figures 2.23, 2.24, 2.25 and 2.26).

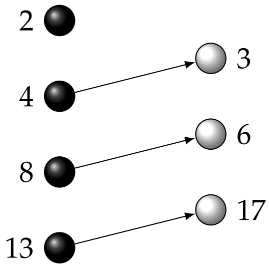

Figure 2.23: G_1 for $D(V_L) = \{2, 4, 8, 13\}$.

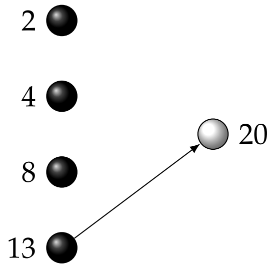

Figure 2.24: G_2 for $D(V_L) = \{2, 4, 8, 13\}$.

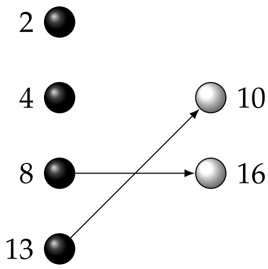

Figure 2.25: G_3 for $D(V_L) = \{2, 4, 8, 13\}$.

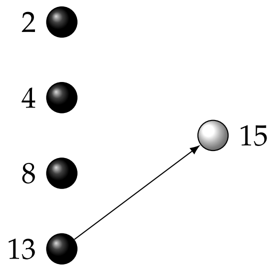

Figure 2.26: G_4 for $D(V_L) = \{2, 4, 8, 13\}$.

Note that in this case $D(V_L)$ is a zero forcing set in all four bipartite graphs. Furthermore, since $d(D(V_L), v) < 5$ for all $v \in V_T$, Algorithm 1 returns the leader set $V_L = \{2, 4, 8, 13\}$. This choice of leader set guarantees that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. For the sake of clarity, we display the network graph in Figure 2.27, where the leader nodes are colored black, and the target nodes are encircled.

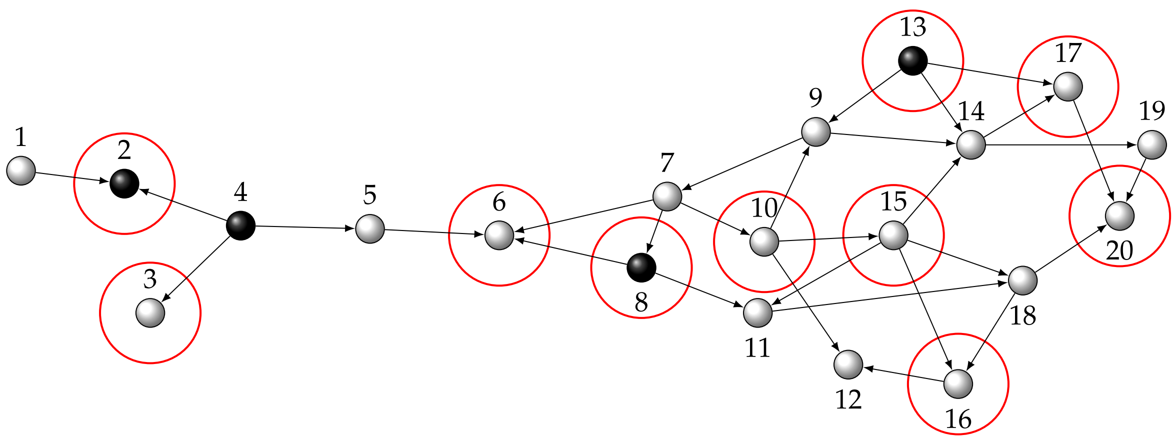

Figure 2.27: Network graph $G = (V, E)$ with encircled target nodes and black leader nodes.

It is worth mentioning that Algorithm 1 can also be applied to compute a leader set V_L such that $(G; V_L)$ is controllable. Indeed, as full control can be regarded as a special case of target control where the entire vertex set V is chosen as target set, Algorithm 1 can simply be applied to a directed graph $G = (V, E)$ with target set V. For example, we can retrieve the results on minimum leader sets found in [29] for cycle and complete graphs, using Algorithm 1 and the fact that $\mathcal{Q}_d(G)$ is sufficiently rich. That is: for cycle and complete graphs, Algorithm 1 returns leader sets of respectively 2 and $n-1$ leaders, which is in agreement with [29].

2.5 CONCLUSIONS

In this chapter, strong targeted controllability for the class of distance-information preserving matrices has been discussed. We have provided a sufficient graph-theoretic condition for strong targeted controllability, expressed in terms of zero-forcing sets of particular distance-related bipartite graphs. We have shown that this result significantly improves the known sufficient topological condition [27] for strong targeted controllability of the class of distance-information preserving matrices. Furthermore, our result contains the 'k-walk theory' [13] as a special case.

Motivated by the observation that the aforementioned sufficient condition is not a one-to-one correspondence, we provided a necessary topological condition for strong targeted controllability. This condition was proved using the fact that the subclass of distance-information preserving matrices is sufficiently rich. Finally, we showed that the problem of determining a minimum leader set achieving targeted controllability is NP-hard. Therefore, a heuristic leader selection algorithm was given to compute (minimum) leader sets achieving target control. The algorithm comprises two phases: firstly, it computes a minimum root set of the target set, i.e. a set of vertices from which all target nodes can be reached. Secondly, this minimum root set is greedily extended to a leader set achieving target control.

Both graph-theoretic conditions for strong targeted controllability provided in this thesis are not one-to-one correspondences. Hence, finding a necessary and sufficient topological condition for strong targeted controllability is still an open problem. Furthermore, investigating other system-theoretic concepts like disturbance decoupling and fault detection for the class of distance-information preserving matrices is among the possibilities for future research.

3

ON THE NECESSITY OF DIFFUSIVE CONTROL LAWS IN OPTIMAL CONTROL PROBLEMS OF DECOUPLED DYNAMICAL SYSTEMS

3.1 INTRODUCTION

The linear quadratic regulator (LQR) problem has a rich history in the literature [19], [20] [26], [43], [49]. Although the problem dates back to the 1940s, it was first rigorously formulated by Kalman in 1960 [19]. The LQR problem is defined for linear time-invariant systems, and concerns finding a control law that minimizes a quadratic cost functional dependent on both state and input of the system under consideration.

Different variants of the LQR problem exist, of which the most notable distinction is made between the so-called *finite horizon* and *infinite horizon* problem [44]. In the finite horizon problem a cost functional is considered that integrates a quadratic function over a finite time interval, while this integration is performed over an infinite time interval in the infinite horizon problem. An additional distinction within the infinite horizon problem is made between the *free endpoint* and *zero endpoint* problems [44]. The objective of the zero endpoint problem is to find a control law that minimizes the cost functional under the constraint that the state trajectory of the system converges to zero, while this constraint is not present in the free endpoint problem.

It is well known that the solutions to both versions of the infinite horizon LQR problem are related to solutions of the underlying algebraic Riccati equation (ARE). A classification of all solutions to the ARE are given in the classical paper by Willems [49] (also see [26]). Moreover, Willems [49] extends the solution to the zero endpoint LQR problem in the sense that no definiteness conditions are imposed on the cost functional, which are normally assumed in LQR problems. Such an extension to a general, indefinite cost functional is also considered by Trentelman [43] in case of the free endpoint LQR problem. More information on linear quadratic regulators can be found in the textbooks [2], [20] and [44].

In recent years, the linear quadratic regulator problem has been considered in the context of networks of dynamical agents [5], [10], [23], [30]. The authors of [5] consider identical decoupled linear systems, and a cost functional that depends on the states, relative states and inputs of the systems. It is assumed that the systems exchange states with their neighbours over a given undirected, connected network graph. The objective is to solve the zero endpoint problem in this setup, under the additional constraint that the applied control law is distributed. This additional constraint is a difficult one, hence the authors of [5] propose a 'suboptimal' controller that stabilizes the linear systems, is compatible with the network structure but does not minimize the cost functional. A similar setup is used in [10], but different control laws are proposed.

In contrast to these papers, we follow the setup used in [30]. We consider identical decoupled linear systems, and a cost functional that depends on a quadratic function of the relative states and the inputs of the systems. No initial network structure is imposed, instead we are interested in the question whether the optimal control law can be implemented as a diffusive coupling on a network graph. That is: we investigate under which conditions there exists a network graph such that the optimal control law can be written as weighted sum of relative neighbouring states. Such a control law is called *diffusive*.

The novelty of our approach is that we introduce and solve both the free endpoint and zero endpoint versions of the above problem, while only the free endpoint version is considered in [30]. Our result regarding the free endpoint problem generalizes the results in [30] to the case in which the agent dynamics is allowed to have eigenvalues in the open right half plane. Moreover, our approach is simple in comparison to [30] as it avoids the analysis of the Riccati differential equation.

This chapter is organized as follows: in Section 3.2 we introduce the notation used throughout. Furthermore, we review the classical solutions to the free and zero endpoint LQR problems, and provide properties of the solutions to the ARE, which will be used to prove our main results. Subsequently, the problem formulation is given in Section 3.3, and Section 3.4 contains our main results. Finally, our conclusions are provided in Section 3.5.

3.2 PRELIMINARIES

We first introduce the notation used throughout this chapter. Let $\mathbb{C}^-$ denote the open left half plane, and let $\mathbb{C}_0^-$ be the closed left half plane. Analogously, $\mathbb{C}^+$ denotes the open right half plane, and $\mathbb{C}_0^+$ is used for the closed right half plane. Moreover, the spectrum of a real square matrix A is denoted by $\sigma(A)$. For a pair (Q, A) of real matrices of compatible dimensions, the unobservable subspace is denoted by $\langle \ker Q \mid A \rangle$. Furthermore, for a linear time-invariant system $\dot{x}(t) = Ax(t) + Bu(t)$, the stabilizable subspace is given by

$$\mathcal{X}_{\mathsf{stab}} := \{x_0 \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } \lim_{t \to \infty} x(t, x_0, u) = 0\}. \tag{3.1}$$

3.2.1 *Classical optimal control problems*

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.2}$$

where $x \in \mathbb{R}^n$ is the state of the system and $u \in \mathbb{R}^m$ is the input. Furthermore, consider the infinite-horizon quadratic cost functional associated with system (3.2), given by

$$J(x_0, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt, \tag{3.3}$$

where $Q \geq 0$ and $R > 0$ are symmetric matrices. We recall two classical linear quadratic regulator problems from the literature. The first, free endpoint problem entails finding a control input u such that the cost functional (3.3) is minimized. The second, zero endpoint problem consists of determining an input function u such that (3.3) is minimized under the constraint $x(t) \to 0$ as $t \to \infty$. More explicitly, the two problems are as follows.

Problem 3.1. *[44] Consider system* (3.2) *equipped with the cost functional* (3.3)*. Determine for every initial state $x_0 \in \mathbb{R}^n$ an input function $u^* \in \mathbb{R}^m$ such that $J(x_0, u^*)$ is minimal, i.e. such that*

$$J(x_0, u^*) = \inf\{J(x_0, u) \mid u \in \mathbb{R}^m\}. \tag{3.4}$$

We shall denote the optimal cost by $J^*(x_0)$, that is: $J^*(x_0) := \inf\{J(x_0, u) \mid u \in \mathbb{R}^m\}$.

Problem 3.2. *[44] Consider the system* (3.2) *and the cost functional* (3.3)*. Determine for every initial state $x_0 \in \mathbb{R}^n$ an input $u^* \in \mathbb{R}^m$ such that $x(t) \to 0$ $(t \to \infty)$ and such that under this condition $J(x_0, u^*)$ is minimized. That is: find the input u^* such that*

$$J(x_0, u^*) = \inf\{J(x_0, u) \mid u \in \mathbb{R}^m, \lim_{t\to\infty} x(t) = 0\}. \tag{3.5}$$

In the case of the latter problem, the optimal cost is denoted by $J_0^*(x_0)$, more explicitly

$$J_0^*(x_0) := \inf\{J(x_0, u) \mid u \in \mathbb{R}^m, \lim_{t\to\infty} x(t) = 0\}. \tag{3.6}$$

It is well-known that both problems are solvable under mild conditions, and that their solutions can be expressed in terms of certain symmetric, positive semidefinite solutions of the so-called algebraic Ricatti equation

$$XA + A^T X - XBR^{-1}B^T X + Q = 0. \tag{3.7}$$

Conditions under which problem 3.1 is solvable, and an explicit expression for the control input are given in the following theorem.

Theorem 3.3. *[44] Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$ together with the cost functional*

$$J(x_0, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt, \tag{3.8}$$

with $Q \geq 0$ and $R > 0$. The following statements are equivalent:

- *For every $x_0 \in \mathbb{R}^n$ there exists a control input $u \in \mathbb{R}^m$ such that $J(x_0, u) < \infty$.*
- *The algebraic Ricatti equation* (3.7) *has a real symmetric positive semidefinite solution X.*
- $\langle \ker Q \mid A \rangle + \mathcal{X}_{stab} = \mathbb{R}^n$.

Assume one of the above conditions holds. Then there exists a smallest real symmetric positive semidefinite solution of (3.7)*, i.e. there exists a solution $X^- \geq 0$ such that for every real symmetric solution $X \geq 0$ we have $X^- - X \leq 0$. Furthermore, for every x_0 we have*

$$J^*(x_0) = x_0^T X^- x_0. \tag{3.9}$$

Finally, for every $x_0 \in \mathbb{R}^n$ there is exactly one optimal input function, i.e. a function $u^* \in \mathbb{R}^m$ such that $J(x_0, u^*) = J^*(x_0)$. This optimal input is generated by the time-invariant feedback law

$$u^*(t) = -R^{-1}B^T X^- x(t). \tag{3.10}$$

Subsequently, we consider Problem 3.2. The following theorem discusses conditions under which Problem 3.2 is solvable, and provides an explicit expression for the control input.

Theorem 3.4. *[44] Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$ together with the cost functional*

$$J(x_0, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt, \tag{3.11}$$

with $Q \geq 0$ and $R > 0$. Assume that (A, B) is stabilizable. Then the following statements hold.

- *There exists a largest real symmetric solution $X^+ \geq 0$ of the algebraic Riccati equation (3.7) such that for every real symmetric solution X of (3.7) we have $X^+ - X \geq 0$.*
- *For every initial state $x_0 \in \mathbb{R}^n$ we have*

$$J_0^*(x_0) = x_0^T X^+ x_0. \tag{3.12}$$

- *For every initial state $x_0 \in \mathbb{R}^n$ there exists an optimal input function, i.e. a function u^* with $\lim_{t\to\infty} x(t) = 0$ such that $J(x_0, u^*) = J_0^*(x_0)$ if and only if every eigenvalue of A on the imaginary axis is (Q, A)-observable.*

  Under this assumption we have:

- *For every initial state $x_0 \in \mathbb{R}^n$ there is exactly one optimal input function u^*, generated by the time-invariant feedback law*

$$u(t) = -R^{-1} B^T X^+ x(t). \tag{3.13}$$

- *The optimal closed-loop system $\dot{x}(t) = (A - BR^{-1}B^T X^+)x(t)$ is stable. In other words, we have $x(t) \to 0$ as $t \to \infty$.*

3.2.2 *Properties of solutions to the algebraic Riccati equation*

The goal of this section is to derive properties of the largest and smallest solutions to the algebraic Riccati equation. More explicitly, we will investigate the eigenvalues of the closed-loop system matrix $A - BR^{-1}B^T X^+$, and the kernels of X^- and X^+. These properties will be used later on, to prove our main results.

Lemma 3.5. *Let (A, B) be stabilizable and consider the largest symmetric positive semidefinite solution X^+ to (3.7). The matrix X^+ is the only solution to (3.7) satisfying $\sigma(A - BR^{-1}B^T X^+) \subset \mathbb{C}_0^-$.*

Proof. Define the shorthand notation $A_+ = A - BR^{-1}B^T X^+$. We will first prove that $\sigma(A_+) \subset \mathbb{C}_0^-$. Suppose there is at least one eigenvalue of A_+ in $\mathbb{C}^+$. Without loss of generality, we can write the matrices A_+, and B in the form

$$A_+ = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{3.14}$$

where $\sigma(A_1) \subset \mathbb{C}_0^-$ and $\sigma(A_2) \subset \mathbb{C}^+$. Next, consider the Lyapunov equation with the state matrix A_2^T, given as follows.

$$SA_2^T + A_2 S + B_2 R^{-1} B_2^T = 0. \tag{3.15}$$

The stabilizability of the pair (A, B) implies that (A_+, B) is stabilizable, and consequently (A_2, B_2) is stabilizable. Recall that the eigenvalues of A_2 have positive real part, hence (A_2, B_2)

is in fact controllable. Its dual (B_2^T, A_2^T) is therefore observable. Consequently, $(B_2 R^{-1} B_2^T, A_2^T)$ is observable. As $\sigma(A_2^T) \subset \mathbb{C}^+$ we conclude that (3.15) has a unique solution $S < 0$. Define the matrix

$$D = \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}. \tag{3.16}$$

The idea of the remainder of the proof will be to prove that $X^+ - D$ is a solution of the algebraic Riccati equation. Note that (3.15) implies that

$$A_2^T S^{-1} + S^{-1} A_2 + S^{-1} B_2 R^{-1} B_2^T S^{-1} = 0. \tag{3.17}$$

Furthermore, the expression

$$A_+^T D + D A_+ + D B R^{-1} B^T D \tag{3.18}$$

can be written as

$$\begin{pmatrix} A_1^T & 0 \\ 0 & A_2^T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}, \tag{3.19}$$

which simplifies to

$$\begin{pmatrix} 0 & 0 \\ 0 & A_2^T S^{-1} + S^{-1} A_2 + S^{-1} B_2 R^{-1} B_2^T S^{-1} \end{pmatrix}. \tag{3.20}$$

By (3.17) we therefore have the following equation for D:

$$A_+^T D + D A_+ + D B R^{-1} B^T D = 0. \tag{3.21}$$

The latter equation allows us to conclude that $X^+ - D$ is a solution to the algebraic Riccati equation. Indeed, we have

$$\begin{aligned} A^T(X^+ - D) + (X^+ - D)A - (X^+ - D)BR^{-1}B^T(X^+ - D) + Q &= \\ -A^T D - DA + X^+ BR^{-1}B^T D + DBR^{-1}B^T X^+ - DBR^{-1}B^T D &= \\ -\Big((A^T - X^+ BR^{-1}B^T)D + D(A - BR^{-1}B^T X^+) + DBR^{-1}B^T D\Big) &= \\ -\Big(A_+^T D + DA_+ + DBR^{-1}B^T D\Big) &= 0. \end{aligned} \tag{3.22}$$

We now simply reach a contradiction. As $D \leq 0$, and $D \neq 0$, take any $x \notin \ker D$. Then $x^T(X^+ - D)x = x^T X^+ x - x^T D x > x^T X^+ x$. This contradicts the fact that X^+ is the largest solution to the algebraic Riccati equation. Hence, we conclude that $\sigma(A - BR^{-1}B^T X^+) \subset \mathbb{C}_0^-$.

Next, we want to show that X^+ is the only matrix with this property, i.e. for all other symmetric solutions X to the algebraic Riccati equation, the matrix $A - BR^{-1}B^T X$ has eigenvalues in $\mathbb{C}^+$. Let $X \neq X^+$ be a symmetric solution to (3.7). Consider the difference $D = X - X^+ \leq 0$, and define the shorthand notation $A_X = A - BR^{-1}B^T X$. Note that

$$A_X^T D + D A_X + D B R^{-1} B^T D \tag{3.23}$$

can be written as

$$-Q + Q + XBR^{-1}B^T X^+ + X^+ BR^{-1}B^T X - X^+ BR^{-1}B^T X - XBR^{-1}B^T X^+, \tag{3.24}$$

which is equal to zero. We hence have the following equation for D.

$$A_X^T D + DA_X + DBR^{-1}B^T D = 0. \tag{3.25}$$

Next, consider a basis in which that matrix D attains the form

$$D = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}, \tag{3.26}$$

where $S < 0$. Then for any $x \in \ker D$ it holds that

$$DA_X x = 0, \tag{3.27}$$

from which we conclude that A_X is of the form

$$A_X = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}. \tag{3.28}$$

By writing $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, we obtain the following equation from (3.25).

$$A_{22}^T S + SA_{22} + SB_2 R^{-1} B_2^T S = 0. \tag{3.29}$$

Let $\lambda \in \sigma(A_{22})$ and let v be an associated eigenvector. Then multiplication of (3.29) from left by v^* and right by v yields

$$2\,\mathrm{Re}(\lambda) v^* S v = -v^* S B_2 R^{-1} B_2^T S v \tag{3.30}$$

As S is negative definite, this implies $\mathrm{Re}(\lambda) \geq 0$. The stabilizability of the pair (A_{22}, B_2) thus yields that (A_{22}, B_2) is controllable. Consequently, the dual (B_2^T, A_{22}^T), and hence the pair $(B_2 R^{-1} B_2^T, A_{22}^T)$ is observable. We see that $S^{-1} < 0$ is a solution to the Lyapunov equation

$$S^{-1} A_{22}^T + A_{22} S^{-1} + B_2 R^{-1} B_2^T = 0, \tag{3.31}$$

and as $(B_2 R^{-1} B_2^T, A_{22}^T)$ is observable we obtain $\sigma(A_{22}) \subset \mathbb{C}^+$. Therefore, for any solution $X \neq X^+$ to the algebraic Riccati equation the closed-loop system has unstable eigenvalues. We conclude that indeed X^+ is the only solution to (3.7) satisfying $\sigma(A - BR^{-1}B^T X^+) \subset \mathbb{C}_0^-$. $\square$

Subsequently, we investigate the kernels of X^- and X^+. More explicitly, we will show that the kernel of X^- equals the unobservable subspace of the pair (Q, A). Under a condition on the eigenvalues of A, a similar statement holds for the kernel of the largest solution X^+.

Theorem 3.6. *Assume the pair (A, B) is stabilizable. Let $X^- \geq 0$ be the smallest symmetric solution to* (3.7)*. We have* $\ker X^- = \langle \ker Q \mid A \rangle$.

Proof. Let $x \in \ker X^-$. By (3.7) we have $x^T Q x = 0$, and as $Q^T = Q \geq 0$ the equality $Qx = 0$ holds. Furthermore, by multiplication of (3.7) from the right by x, we obtain

$$X^- A x = 0, \tag{3.32}$$

which implies that $\ker X^-$ is A-invariant. To conclude, the kernel of X^- is A-invariant, and contained in $\ker Q$. As $\langle \ker Q \mid A \rangle$ is the largest A-invariant subspace contained in $\ker Q$, it immediately follows that $\ker X^- \subseteq \langle \ker Q \mid A \rangle$.

Conversely, let $x_0 \in \langle \ker Q \mid A \rangle$. As the unobservable subspace is A-invariant, the optimal control input that minimizes the cost functional

$$J(x_0, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt \tag{3.33}$$

is given by $u = 0$. Furthermore, the optimal cost $J^*(x_0)$ is equal to zero. That is: $x_0^T X^- x_0 = 0$, and consequently $x_0 \in \ker X^-$. □

Theorem 3.7. *Consider system* (3.2) *and assume the pair (A, B) is stabilizable. Let $X^+ \geq 0$ be the largest symmetric solution of the algebraic Riccati equation* (3.7)*. Then we have* $\ker X^+ = \langle \ker Q \mid A \rangle$ *if and only if all (Q, A)-unobservable eigenvalues are contained in $\mathbb{C}_0^-$.*

Proof. Assume all (Q, A)-unobservable eigenvalues are contained in $\mathbb{C}_0^-$. By the exact same reasoning of the proof of Theorem 3.6 we conclude $\ker X^+ \subseteq \langle \ker Q \mid A \rangle$.

Conversely, suppose $x \in \langle \ker Q \mid A \rangle$. We want to prove that $x \in \ker X^+$. We assume $x \neq 0$, as in the case $\langle \ker Q \mid A \rangle = \{0\}$, the identity $\ker X^+ = \langle \ker Q \mid A \rangle$ trivially holds. The vector x can be written as linear combination of generalized eigenvectors associated to the (Q, A)-unobservable eigenvalues of A. Hence, if we prove that the generalized eigenvectors associated to the (Q, A)-unobservable eigenvalues are in the kernel of X^+, it immediately follows that $x \in \ker X^+$. First, suppose $v \neq 0$ is an unobservable eigenvector of A, i.e. $Av = \lambda v$ and $Qv = 0$. Multiplication of (3.7) from right and left by v and v^* respectively yields

$$v^* X^+ A v + v^* A^T X^+ v - v^* X^+ B R^{-1} B^T X^+ v = 0, \tag{3.34}$$

which simplifies to

$$2 \operatorname{Re}(\lambda)\, v^* X^+ v = v^* X^+ B R^{-1} B^T X^+ v, \tag{3.35}$$

where $\operatorname{Re}(\lambda)$ denotes the real part of λ. By assumption $\operatorname{Re}(\lambda) \leq 0$. We consider the two cases $\operatorname{Re}(\lambda) < 0$ and $\operatorname{Re}(\lambda) = 0$ separately. First consider the case that $\operatorname{Re}(\lambda) < 0$. As $R > 0$, we have $R^{-1} > 0$, which implies $v^* X^+ B R^{-1} B^T X^+ v \geq 0$. However, as $\operatorname{Re}(\lambda) < 0$ and $v^* X^+ v \geq 0$ we conclude that $v^* X^+ v = 0$. Since X^+ is symmetric positive semidefinite, it follows that $X^+ v = 0$. That is: the unobservable eigenvectors of A corresponding to eigenvalues in the open left complex plane are in the kernel of X^+.

Next, let $v_1, v_2, ..., v_k$ be a Jordan chain of unobservable generalized eigenvectors associated with the eigenvalue λ. Here v_1 is an unobservable eigenvector associated with λ, and $(A - \lambda I)\, v_{i+1} = v_i$ for $i = 1, 2, ..., k-1$. By induction we will show that $X^+ v_i = 0$ for $i = 1, 2, ..., k$. We have already shown that $X^+ v_1 = 0$, as v_1 is an eigenvector. Next suppose $X^+ v_i = 0$ for some $1 \leq i < k$. We want to show that $X^+ v_{i+1} = 0$. Multiplication of (3.7) from right and left by v_{i+1} and v_{i+1}^* respectively, yields

$$v_{i+1}^* X^+ A v_{i+1} + v_{i+1}^* A^T X^+ v_{i+1} - v_{i+1}^* X^+ B R^{-1} B^T X^+ v_{i+1} = 0. \tag{3.36}$$

Since $A v_{i+1} = \lambda v_{i+1} + v_i$, we obtain

$$v_{i+1}^* X^+ \left(\lambda v_{i+1} + v_i\right) + \left(\lambda^* v_{i+1}^* + v_i^*\right) X^+ v_{i+1} - v_{i+1}^* X^+ B R^{-1} B^T X^+ v_{i+1} = 0. \tag{3.37}$$

Subsequently, as $X^+ v_i = 0$, (3.37) simplifies to

$$2 \operatorname{Re}(\lambda)\, v_{i+1}^* X^+ v_{i+1} = v_{i+1}^* X^+ B R^{-1} B^T X^+ v_{i+1}. \tag{3.38}$$

Applying the same arguments as before, we obtain $X^+v_{i+1} = 0$. It follows that all unobservable generalized eigenvectors associated with (Q, A)-unobservable eigenvalues $\lambda \in \mathbb{C}^-$ are in the kernel of X^+.

Next, consider the case $\mathrm{Re}(\lambda) = 0$. In this case, (3.35) yields $v^* X^+ BR^{-1}B^T X^+ v = 0$ and consequently $B^T X^+ v = 0$. Multiplication of (3.7) from right by v yields

$$\lambda X^+ v + A^T X^+ v = 0, \tag{3.39}$$

in other words

$$v^* X^+ (A - \lambda I) = 0. \tag{3.40}$$

We previously found that $B^T X^+ v = 0$. If $X^+ v \neq 0$, this would imply that λ is an (A, B)-uncontrollable eigenvalue with eigenvector $X^+ v$. As $\mathrm{Re}(\lambda) = 0$ this contradicts the stabilizability of the pair (A, B). Henceforth, we obtain $X^+ v = 0$. Once again, we consider a Jordan chain $v_1, v_2, ..., v_k$ of unobservable generalized eigenvectors associated with the unobservable eigenvalue λ. Here $Av_1 = \lambda v_1$ and $(A - \lambda I)\, v_{i+1} = v_i$ for $i = 1, 2, ..., k-1$. We proceed by induction. We already know that $X^+ v_1 = 0$, hence suppose $X^+ v_i = 0$ for some $1 \leq i < k$. Multiplication of (3.7) from right and left by v_{i+1} and v_{i+1}^* respectively, yields

$$v_{i+1}^* X^+ A v_{i+1} + v_{i+1}^* A^T X^+ v_{i+1} - v_{i+1}^* X^+ BR^{-1} B^T X^+ v_{i+1} = 0. \tag{3.41}$$

By substitution of $(A - \lambda I)\, v_{i+1} = v_i$ we obtain

$$v_{i+1}^* X^+ \left(\lambda v_{i+1} + v_i\right) + \left(\lambda^* v_{i+1}^* + v_i^*\right) X^+ v_{i+1} - v_{i+1}^* X^+ BR^{-1} B^T X^+ v_{i+1} = 0, \tag{3.42}$$

which by the induction hypothesis leads to

$$v_{i+1}^* X^+ BR^{-1} B^T X^+ v_{i+1} = 0, \tag{3.43}$$

and consequently $B^T X^+ v_{i+1} = 0$. We can now apply the exact same arguments as before: the statement $X^+ v_{i+1} \neq 0$ would contradict the stabilizability of the pair (A, B). Therefore, $X^+ v_{i+1} = 0$. We conclude that $X^+ v_i = 0$ for $i = 1, 2, ..., k$. In other words, if v_i is a generalized eigenvector of A, corresponding to a (Q, A)-unobservable eigenvalue of A in $\mathbb{C}_0^-$, then $X^+ v_i = 0$. As $\langle \ker Q \mid A \rangle$ is spanned by these unobservable generalized eigenvectors, we conclude that $\langle \ker Q \mid A \rangle \subseteq \ker X^+$, and consequently the equality $\ker X^+ = \langle \ker Q \mid A \rangle$ holds.

Subsequently, we prove the converse implication, i.e. we assume $\ker X^+ = \langle \ker Q \mid A \rangle$, and want to prove that the (Q, A)-unobservable eigenvalues of A are contained in $\mathbb{C}_0^-$.

To this extent, suppose there exists a (Q, A)-unobservable eigenvalue $\lambda \in \sigma(A)$ satisfying $\mathrm{Re}(\lambda) > 0$. We have $Av = \lambda v$ and $Qv = 0$ for some nonzero vector v. Clearly, $v \in \langle \ker Q \mid A \rangle$ and therefore we have $v \in \ker X^+$. This means that $(A - BR^{-1}B^T X^+)v = \lambda v$, i.e. λ is an eigenvalue of the closed-loop system matrix $A - BR^{-1}B^T X^+$. This however is a contradiction, as the eigenvalues of $A - BR^{-1}B^T X^+$ are contained in $\mathbb{C}_0^-$ (see Lemma 3.5). We conclude that the (Q, A)-unobservable eigenvalues of A are contained in $\mathbb{C}_0^-$, which proves the statement. $\square$

We have seen that the kernels of X^- and X^+ are identical if and only if A has no (Q, A)-unobservable eigenvalues in $\mathbb{C}^+$. Next, we wonder if the solutions X^- and X^+ themselves are identical under this condition. It turns out that this is indeed the case, as stated in the following theorem.

Theorem 3.8. *Assume (A, B) is stabilizable and let $X^- \geq 0$ and $X^+ \geq 0$ be respectively the smallest and largest symmetric solutions to the ARE (3.7). The equality $X^- = X^+$ holds if and only if all (Q, A)-unobservable eigenvalues of A are contained in C_0^-.*

Proof. First assume all (Q, A)-unobservable eigenvalues of A are contained in C_0^-. Let $\lambda \in \sigma(A - BR^{-1}B^T X^-)$, and let v be an associated eigenvector. We have

$$X^-(A - BR^{-1}B^T X^-) + (A - BR^{-1}B^T X^-)^T X^- + X^- BR^{-1}B^T X^- + Q = 0, \tag{3.44}$$

and consequently

$$2\,\mathrm{Re}(\lambda) v^* X^- v = -v^* X^- BR^{-1}B^T X^- v - v^* Q v. \tag{3.45}$$

If $X^- v \neq 0$ we conclude from (3.45) that $\mathrm{Re}(\lambda) \leq 0$. If $X^- v = 0$ then $Av = \lambda v$, $Qv = 0$ and by our assumption on the unobservable eigenvalues of A we find $\mathrm{Re}(\lambda) \leq 0$. This means that $\sigma(A - BR^{-1}B^T X^-) \subset \mathbb{C}_0^-$. However, as X^+ is the only symmetric positive semidefinite solution to (3.7) with the property $\sigma(A - BR^{-1}B^T X^+) \subset \mathbb{C}_0^-$, we conclude that $X^- = X^+$.

Conversely, assume $X^- = X^+$. This means that $\ker X^- = \ker X^+$, and thus $\ker X^+ = \langle \ker Q \mid A \rangle$ by Theorem 3.6. Finally, we conclude by Theorem 3.7 that all (Q, A)-unobservable eigenvalues of A are contained in C_0^-. This completes the proof. $\square$

3.3 PROBLEM FORMULATION

Consider N identical linear systems of the form

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \tag{3.46}$$

where $x_i \in \mathbb{R}^n$ is the state of the ith system and $u_i \in \mathbb{R}^m$ is its input, for $i = 1, 2, ..., N$. We write $x \in \mathbb{R}^{Nn}$ and $u \in \mathbb{R}^{Nm}$ for the stacked state $x(t) = \mathrm{col}(x_1(t), x_2(t), ..., x_N(t))$ and control input $u(t) = \mathrm{col}(u_1(t), u_2(t), ..., u_N(t))$ respectively, which yields

$$\dot{x}(t) = (I_N \otimes A)x(t) + (I_N \otimes B)u(t), \tag{3.47}$$

where $\otimes$ denotes the Kronecker product. The problem of synchronization concerns finding control inputs u_i such that the solution $x(t, x_0)$ of system (3.47) approaches the so-called synchronization manifold

$$\mathcal{S} = \{x \in \mathbb{R}^{Nn} \mid \exists z \in \mathbb{R}^n : \; x = \mathbb{1}_N \otimes z\} \subseteq \mathbb{R}^{Nn}, \tag{3.48}$$

as $t \to \infty$. This thesis concerns the implicit synchronization of system (3.47) by control laws that minimize a quadratic cost functional. To this extent we consider the matrix that defines the orthogonal projection onto $\mathcal{S}$, which is given by

$$\frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T \otimes I_n. \tag{3.49}$$

Furthermore, the matrix P that defines the orthogonal projection onto $\mathcal{S}^\perp$ equals

$$P = I_{Nn} - \frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T \otimes I_n = \left(I_N - \frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T\right) \otimes I_n. \tag{3.50}$$

Accordingly, we consider the cost functional

$$J(x_0, u) = \int_0^\infty x(t)^T PQPx(t) + u(t)^T Ru(t) dt, \tag{3.51}$$

where the matrices Q and R are symmetric positive definite. This means that large input vector components are penalized through the term $u(t)^T Ru(t)$, while a large distance of $x(t)$ with respect to the synchronization manifold $\mathcal{S}$ is penalized through the term $x(t)^T PQPx(t)$. The control law that minimizes (3.51) implicitly synchronizes the systems (3.46), in the sense that it makes the Q-weighted L_2-norm of the distance of the global state trajectory $x(t)$ to the synchronization manifold small.

We are interested in the question whether the optimal control law $u(t)$ that minimizes the cost functional (3.51) can be expressed as a diffusive coupling. That is: we investigate under which conditions there exists a directed graph $G = (V, E)$ with $|V| = N$ vertices, such that each $u_i(t)$ can be written as

$$u_i(t) = \sum_{(i,j)\in E} K_{ij}(x_j(t) - x_i(t)), \tag{3.52}$$

where $K_{ij} \in \mathbb{R}^{m\times n}$ for $i = 1, 2, ..., N$. In other words, the question is under which conditions the optimal control laws $u_i(t)$ can be written as the weighted sum of relative states. Such controllers are desirable, as they require only relative information, and no absolute information about the global state $x(t)$. Furthermore, the control inputs $u_i(t)$ given in (3.52) are distributed in the sense that they only require relative information of the agents in the neighbourhood of agent i. If the control input $u(t) = \mathrm{col}(u_1(t), u_2(t), ..., u_N(t))$ can be written in the form (3.52) we refer to it as being diffusive. In what follows, we consider the algebraic Riccati equation

$$X(I_N \otimes A) + (I_N \otimes A^T)X - X(I_N \otimes B)R^{-1}(I_N \otimes B^T)X + PQP = 0, \tag{3.53}$$

and denote its smallest and largest positive semidefinite solutions by X^- and X^+ respectively. The associated feedback matrices are given by $K^- = -R^{-1}(I_N \otimes B^T)X^-$ and $K^+ = -R^{-1}(I_N \otimes B^T)X^+$. With this in mind, we introduce the following two main problems.

Problem 3.9. *Consider system* (3.47) *equipped with the cost functional* (3.51). *Assume that the optimal control law $u(t) = K^-x(t)$ that minimizes* (3.51) *exists. Determine necessary and sufficient conditions under which $u(t) = K^-x(t)$ is diffusive.*

Problem 3.10. *Consider the system* (3.47) *and the cost functional* (3.51). *Assume that the optimal control law $u(t) = K^+x(t)$ that minimizes* (3.51) *under the constraint $x(t) \to 0$ $(t \to \infty)$ exists. Determine necessary and sufficient conditions under which $u(t) = K^+x(t)$ is diffusive.*

3.4 MAIN RESULTS

In this section we provide solutions to Problems 3.9 and 3.10, as introduced in the previous section. We first make some general observations, after which the solution to Problem 3.9 is given in Section 3.4.1. Subsequently, the solution to Problem 3.10 is provided in Section 3.4.2.

The first important observation is as follows. Note that the control law $u(t) = Kx(t)$ is diffusive if and only if $K(\mathbb{1}_N \otimes I_n) = 0$. To this extent we define the vector space of matrices given by

$$\mathcal{M}^N_{m\times n} = \{M \in \mathbb{R}^{Nm\times Nn} \mid M(\mathbb{1}_N \otimes I_n) = 0\}. \tag{3.54}$$

Note that this means that Problems 3.9 and 3.10 are equivalent with providing necessary and sufficient conditions under which the matrices K^- and K^+ are elements of $\mathcal{M}^N_{m\times n}$.

Secondly, we have the relation

$$\ker P = \operatorname{im}\left(\mathbb{1}_N \otimes I_n\right). \tag{3.55}$$

Indeed, we easily obtain

$$\begin{aligned} P\left(\mathbb{1}_N \otimes I_n\right) &= \left(\left(I_N - \frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T\right) \otimes I_n\right)\left(\mathbb{1}_N \otimes I_n\right) \\ &= \left(\left(I_N - \frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T\right)\mathbb{1}_N\right) \otimes I_n \\ &= 0. \end{aligned} \tag{3.56}$$

Moreover, as $\operatorname{rank} P = \operatorname{rank}\left(I_N - \frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T\right) \otimes I_n = (N-1)n$ we conclude that (3.55) holds. The final observation concerns the unobservable subspace of the pair $(PQP, I_N \otimes A)$. For ease of notation we will work with the shorthand notations $A_e = I_N \otimes A$, $B_e = I_N \otimes B$ and $Q_e = PQP$. We have

$$\begin{aligned} PA_e &= P\left(I_N \otimes A\right) \\ &= \left(\left(I_N - \frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T\right) \otimes I_n\right)\left(I_N \otimes A\right) \\ &= \left(I_N - \frac{1}{N}\mathbb{1}_N \otimes \mathbb{1}_N^T\right) \otimes A, \end{aligned} \tag{3.57}$$

and consequently

$$PA_e\left(\mathbb{1}_N \otimes I_n\right) = 0. \tag{3.58}$$

Hence, $\ker P$ is A_e-invariant. Furthermore, note that $\ker Q_e = \ker P$ by the symmetry of P and positive definiteness of Q. This immediately allows us to write

$$\left\langle \ker Q_e \mid A_e \right\rangle = \left\langle \ker P \mid A_e \right\rangle = \ker P = \operatorname{im}\left(\mathbb{1}_N \otimes I_n\right), \tag{3.59}$$

where the second identity follows from A_e-invariance of $\ker P$ and the final equality is a consequence of (3.55). With these remarks in place we are ready to tackle the first problem.

3.4.1 *Diffusiveness of the control law in the free endpoint problem*

In this section we will consider the optimal control law $u(t) = K^- x(t)$ that minimizes (3.51). We will see that this control law is always diffusive. Moreover, the proof of this statement follows easily from our preliminaries.

Theorem 3.11. *Consider system* (3.47)*, and assume* (A, B) *is stabilizable. The optimal control law* $u(t) = K^- x(t)$ *that minimizes* (3.51) *exists, and is diffusive.*

Proof. The stabilizability of (A, B) implies (A_e, B_e) is stabilizable. Therefore, X^- exists and $u(t) = K^- x(t)$ minimizes (3.51) by Theorem 3.3. It remains to be shown that the inclusion $K^- \in \mathcal{M}^N_{m\times n}$ holds. By Theorem 3.6 we have $\ker X^- = \left\langle \ker Q_e \mid A_e \right\rangle$. Subsequently, by (3.59) we conclude that $\ker X^- = \operatorname{im}\left(\mathbb{1}_N \otimes I_n\right)$ and consequently

$$K^-\left(\mathbb{1}_N \otimes I_n\right) = -R^{-1}(I_N \otimes B^T)X^-\left(\mathbb{1}_N \otimes I_n\right) = 0. \tag{3.60}$$

Thus $K^- \in \mathcal{M}^N_{m\times n}$, which completes the proof. □

Theorems 1 and 2 of [30] immediately follow from Theorem 3.11 of this thesis. Indeed, in the case that the matrix A has only eigenvalues in the closed left half plane, the identity $X^- = X^+$ holds by Theorem 3.8. Hence, in this case K^+ is indeed diffusive. However, note that Theorem 3.11 generalizes Theorems 1 and 2 of [30] to the case A has eigenvalues in the open right half plane.

Secondly, note that the proof of Theorem 3.11 is only based on the relation (3.59), and the fact that $\ker X^- = \langle \ker \bar{Q} \mid \bar{A} \rangle$. Hence, our proof simplifies the one provided in [30], and does not use any information about the Riccati differential equation.

Thirdly, our approach can also be applied to prove that the optimal control law is diffusive if we deal with N nonidentical systems, which would benefit the proof of Corollary 1 of [30].

Finally, we want to remark that the stabilizability assumption in Theorem 3.11 is not even needed for the existence of X^- (a weaker, necessary and sufficient condition is given in Theorem 3.3, also see [44]).

Example 3.12. Consider $N = 4$ linear time-invariant systems $x_i(t) = Ax_i(t) + Bu_i(t)$ for $i = 1, 2, 3, 4$, where the matrices A and B are given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.61}$$

As always, $x(t)$ denotes the global state vector, obtained by aggregation of $x_i(t)$ for $i = 1, 2, 3, 4$. In this example we choose the initial condition

$$x(0) = \begin{pmatrix} -2 & 0.5 & -1 & 0 & 0 & 2 & 2 & -1.5 \end{pmatrix}^T. \tag{3.62}$$

As $N = 4$ and $n = 2$, the projection matrix P is given by $P = \left(I_4 - \frac{1}{4}\mathbb{1}_4 \otimes \mathbb{1}_4^T\right) \otimes I_2$. For this simple example, we choose $R = I_4$ and $Q = I_8$. Using Matlab, we compute the smallest positive semidefinite solution X^- to the algebraic Riccati equation (3.53), and find the feedback gain matrix

$$K^- = \begin{pmatrix} -1.0140 & 0.3108 & 0.3380 & -0.1036 & 0.3380 & -0.1036 & 0.3380 & -0.1036 \\ 0.3380 & -0.1036 & -1.0140 & 0.3108 & 0.3380 & -0.1036 & 0.3380 & -0.1036 \\ 0.3380 & -0.1036 & 0.3380 & -0.1036 & -1.0140 & 0.3108 & 0.3380 & -0.1036 \\ 0.3380 & -0.1036 & 0.3380 & -0.1036 & 0.3380 & -0.1036 & -1.0140 & 0.3108 \end{pmatrix}$$

Here we have rounded the components of K^- to four digits, for ease of presentation. We denote the jth component of x_i by x_{ij}, for $j = 1, 2$. In Figure 3.1 we depict the state components x_{11}, x_{21}, x_{31} and x_{41} of the closed-loop system obtained by applying the control law $u(t) = K^-x(t)$. We clearly see that these states synchronize. Similarly, we display the state components x_{12}, x_{22}, x_{32} and x_{42} of the closed-loop system in Figure 3.2, and observe that also these states synchronize.

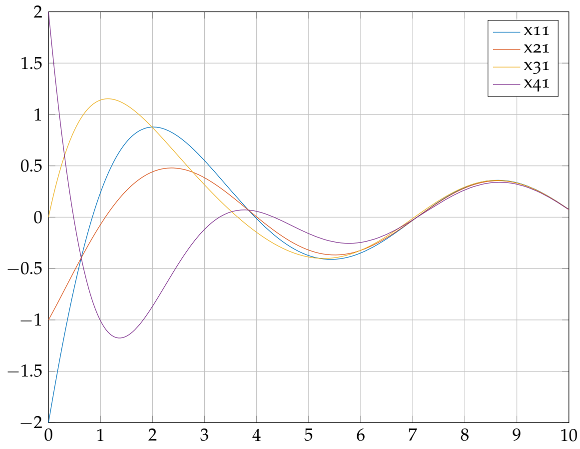

Figure 3.1: Plot of the states x_{11}, x_{21}, x_{31} and x_{41}, while applying the control law $u(t) = K^{-}x(t)$ for $t \in [0, 10]$.

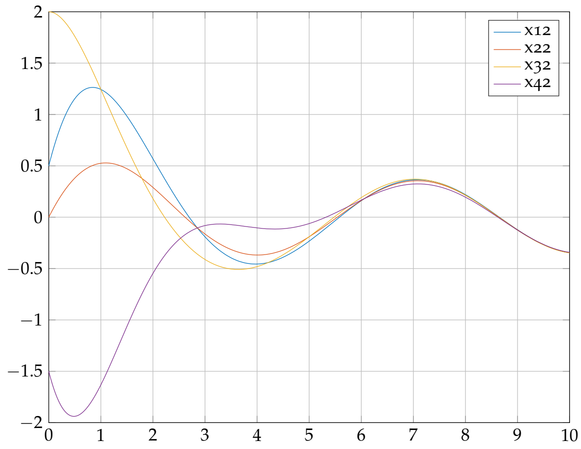

Figure 3.2: Plot of the states x_{12}, x_{22}, x_{32} and x_{42}, while applying the control law $u(t) = K^{-}x(t)$ for $t \in [0, 10]$.

Furthermore, we observe that the ith component of the control input $u(t) = K^{-}x(t)$ can be written as

$$u_i(t) = K \sum_{j=1, j\neq i}^{4} \left(x_j(t) - x_i(t)\right), \tag{3.63}$$

where $K = \begin{pmatrix} 0.3380 & -0.1036 \end{pmatrix}$. Therefore, the control law is diffusive, and can be implemented on a complete network graph as displayed in Figure 3.3.

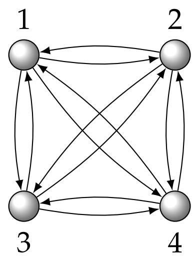

Figure 3.3: Network graph $G = (V, E)$ on which the optimal control law can be diffusively implemented.

3.4.2 *Diffusiveness of the control law in the zero endpoint problem*

In this section we will consider the optimal control law $u(t) = K^{+}x(t)$ that minimizes (3.51) under the constraint $x(t) \to 0$ as $t \to \infty$. We will see that this control law is diffusive under a condition on the eigenvalues of A. More explicitly, we have the following result.

Theorem 3.13. *Consider system (3.47). The optimal control law $u(t) = K^{+}x(t)$ that minimizes (3.51) under the constraint $x(t) \to 0$ $(t \to \infty)$ exists and is diffusive if and only if the matrix A is Hurwitz.*

Proof. We first assume A is Hurwitz. This implies the stabilizability of (A, B), and thus (A_e, B_e) is stabilizable. Furthermore, A_e is Hurwitz, so A_e has no (Q_e, A_e)-unobservable eigenvalues on the imaginary axis. We conclude by Theorem 3.4 that the optimal stabilizing control law $u(t) = K^+ x(t)$ exists. Subsequently, we want to prove that K^+ is diffusive. As A_e is Hurwitz, by Theorem 3.7 the equality $\ker X^+ = \langle \ker Q_e \mid A_e \rangle$ holds. Consequently, we conclude by (3.59) that $\ker X^+ = \operatorname{im}(\mathbb{1}_N \otimes I_n)$, and thus $K^+ \in \mathcal{M}^N_{m\times n}$.

Conversely, suppose the optimal control law $u(t) = K^+ x(t)$ that minimizes (3.51) under the constraint $x(t) \to 0$ $(t \to \infty)$ exists, and is diffusive. This implies that the closed-loop system matrix $A_e + B_e K^+$ is Hurwitz, and $K^+ (\mathbb{1}_N \otimes I_n) = 0$. Let λ be an eigenvalue of A with eigenvector v. This yields

$$(I_N \otimes A)(\mathbb{1}_N \otimes v) = \mathbb{1}_N \otimes Av = \lambda (\mathbb{1}_N \otimes v), \tag{3.64}$$

and thus λ is an eigenvalue of A_e with eigenvector $(\mathbb{1}_N \otimes v)$. Moreover, we have

$$\left(A_e + B_e K^+\right)(\mathbb{1}_N \otimes v) = A_e (\mathbb{1}_N \otimes v) = \lambda (\mathbb{1}_N \otimes v). \tag{3.65}$$

Therefore, λ is an eigenvalue of the closed-loop system matrix $(A_e + B_e K^+)$, and as $(A_e + B_e K^+)$ is Hurwitz, λ has negative real part. We conclude that the matrix A is Hurwitz, which proves the theorem. □

Hence, the optimal stabilizing control law is only diffusive if we have a N stable systems to begin with.

Example 3.14. Consider $N = 8$ identical dynamical systems of the form

$$\dot{x}_i(t) = -x_i(t) + u_i(t), \tag{3.66}$$

where $x_i, u_i \in \mathbb{R}$ for $i = 1, 2, ..., 8$. As usual, we write $x(t)$ and $u(t)$ for the stacked state vector and input vector respectively. In this example we initialize the systems at

$$x(0) = \begin{pmatrix} -10 & 23 & 17 & -4 & 27 & -1 & 10 & 21 \end{pmatrix}^T. \tag{3.67}$$

Furthermore, The projection matrix P is given by $P = \left(I_8 - \frac{1}{8}\mathbb{1}_8 \otimes \mathbb{1}_8^T\right)$. We choose $R = I_8$ and $Q = 50 \cdot I_8$, that is, we associate a large cost to asynchronous states. The optimal stabilizing feedback is computed using matlab, and given by

$$K^+ = \begin{pmatrix} -5.3739 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 \\ 0.7677 & -5.3739 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 \\ 0.7677 & 0.7677 & -5.3739 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 \\ 0.7677 & 0.7677 & 0.7677 & -5.3739 & 0.7677 & 0.7677 & 0.7677 & 0.7677 \\ 0.7677 & 0.7677 & 0.7677 & 0.7677 & -5.3739 & 0.7677 & 0.7677 & 0.7677 \\ 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & -5.3739 & 0.7677 & 0.7677 \\ 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & -5.3739 & 0.7677 \\ 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & 0.7677 & -5.3739 \end{pmatrix}$$

Hence, the optimal control laws $u_i(t)$ for $i = 1, 2, ..., 8$ can be expressed as diffusive couplings of the form

$$u_i(t) = 0.7677 \sum_{j=1, j\neq i}^{8} \left(x_j(t) - x_i(t)\right). \tag{3.68}$$

Note that controller i requires the relative state $x_j(t) - x_i(t)$ of each agent $j = 1, 2, ..., 8$ for $j \neq i$. Hence, the optimal controller can be implemented on a complete graph network, where the relative states are exchanged with all neighbours.

Moreover, we display the state trajectories in Figure 3.4. Note that the states first synchronize before converging to zero. This is due to the fact that a large cost is associated with asynchronous states.

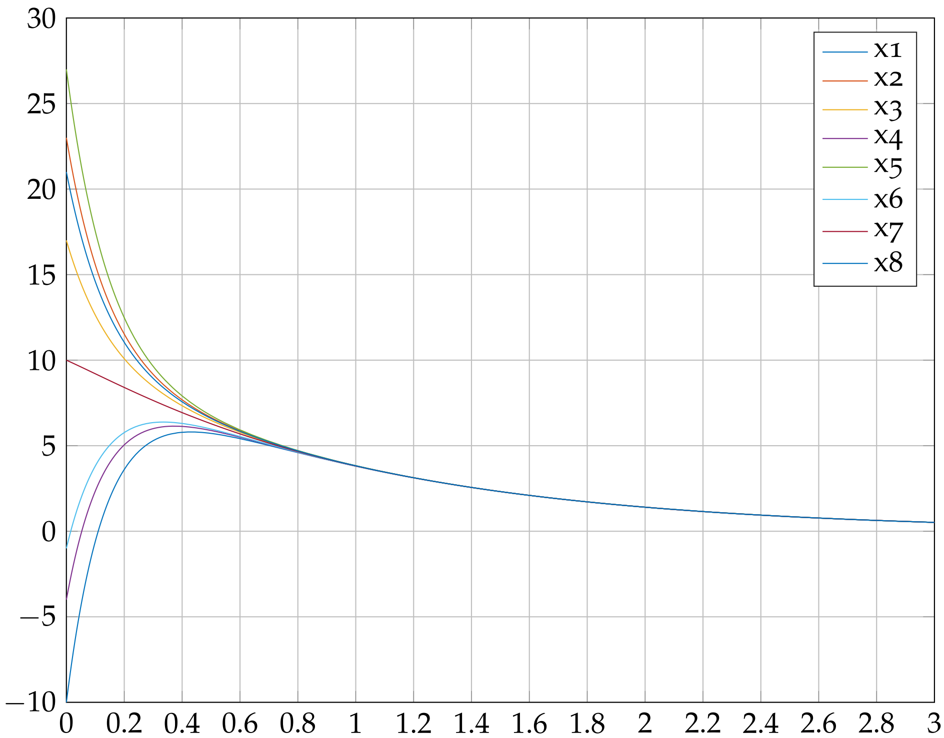

Figure 3.4: Plot of the states $x_1, x_2, ..., x_8$, while applying the control law $u(t) = K^+ x(t)$ for $t \in [0, 3]$.

3.5 CONCLUSIONS

In this chapter we have considered identical, decoupled linear systems and a quadratic cost functional that depends on the relative states and inputs of the systems. We have investigated under which conditions the optimal control law can be expressed as a diffusive coupling. Both free and zero endpoint versions of this problem were considered. We have proven that the optimal control law that solves the free endpoint problem is diffusive, extending previous results found in [30]. Our approach considerably simplifies the proofs in [30], as it avoids the analysis of the Riccati differential equation. Furthermore, we have provided a necessary and sufficient condition for the diffusiveness of the optimal control law that solves the zero endpoint problem.

Note that in both examples considered in this chapter, a complete network graph was required to implement the optimal control law (see examples 3.12 and 3.14). However, in many applications the communication required to achieve an underlying complete network topology is impractical or even impossible. Therefore, future research should focus on control laws that are compatible with a given network structure. In general, finding a feedback law that minimizes a cost functional under the constraint that it has the required sparsity structure is a difficult optimization problem [5]. Hence, effort has been put into finding so-called 'suboptimal controllers', that are compatible with a given network structure but do not fully minimize the cost functional (see for instance [5] and [10]). However, a framework to compare these suboptimal controllers is lacking, and no indication of their performance in comparison to the optimal control law is given. Hence, a relevant topic for future work is to develop a suboptimal control law, compatible with a given network topology, and to compare its performance with the optimal control law.

4

CONCLUSIONS

In this thesis we have studied control problems for networked multi-agent systems. More explicitly, we have investigated graph-theoretic conditions for targeted controllability, and the diffusiveness of optimal control laws for networked multi-agent systems. Our main results can be summarized as follows.

- We have established a sufficient condition for strong targeted controllability of networks, in terms of zero forcing sets of particular distance-related bipartite graphs. Based on this result we have developed an algorithm that, given a network with target nodes, computes a leader set that renders the network targeted controllable. The algorithm applies integer linear programming to obtain an initial set of nodes that is subsequently extended to a leader set achieving target control.
- We have proven that the subclass of distance-information preserving matrices is sufficiently rich. Consequently, strong structural controllability with respect to the subclass of distance-information preserving matrices is equivalent with strong structural controllability with respect to the entire qualitative class. Based on this result, we have established a necessary graph-theoretic condition for strong targeted controllability.
- We have considered the zero and free endpoint LQR problems for identical decoupled linear systems. Based on simple properties of the algebraic Riccati equation, it was shown that the optimal control law that solves the free endpoint LQR problem is diffusive. Furthermore, a necessary and sufficient condition was established for the diffusiveness of the optimal control law that solves the zero endpoint problem.

The following topics can be considered for future research.

- In Chapter 2 it was shown that both graph-theoretic conditions for targeted controllability are not one-to-one correspondences. Hence, providing a necessary *and* sufficient topological condition for targeted controllability is still an open problem.
- It is of interest to investigate other control problems like disturbance decoupling for the class of distance-information preserving matrices.
- As illustrated in Chapter 3 the implementation of the optimal control law typically requires a complete network topology. A relevant problem is to develop a distributed, suboptimal control law and a framework to compare its performance with respect to the optimal control law.

5

BIBLIOGRAPHY

[1] A. Aazami. Hardness results and approximation algorithms for some problems on graphs. *PhD thesis, University of Waterloo*, 2008.

[2] B. D. O. Anderson and J. B. Moore. *Linear optimal control*. Prentice-Hall Englewood Cliffs, N.J, 1971.

[3] M. Baldea and P. Daoutidis. Model reduction and control of reactor-heat exchanger networks. *Journal of Process Control*, 16(3):265 – 274, 2006.

[4] D. Bergman, A. A. Cire, W. Hoeve, and T. Yunes. Bdd-based heuristics for binary optimization. *Journal of Heuristics*, 20(2):211–234, 2014.

[5] F. Borrelli and T. Keviczky. Distributed lqr design for identical dynamically decoupled systems. *IEEE Transactions on Automatic Control*, 53(8):1901–1912, Sept 2008.

[6] F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Applied Mathematics Series. Princeton University Press, 2009.

[7] D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Transactions on Automatic Control*, 58:2349–2354, 2013.

[8] J. R. Carpenter. Decentralized control of satellite formations. *International Journal of Robust and Nonlinear Control*, 12(2-3):141–161, 2002.

[9] A. Chapman and M. Mesbahi. On strong structural controllability of networked systems: A constrained matching approach. In *American Control Conference (ACC)*, pages 6126–6131, June 2013.

[10] P. Deshpande, P. P. Menon, C. Edwards, and I. Postlethwaite. A distributed control law with guaranteed lqr cost for identical dynamically coupled linear systems. pages 5342–5347, June 2011.

[11] M. Egerstedt, S. Martini, M. Cao, K. Camlibel, and A. Bicchi. Interacting with networks: How does structure relate to controllability in single-leader, consensus networks? *IEEE Control Systems*, 32(4):66–73, Aug 2012.

[12] M. Galbiati, D. Dephini, and S. Battiston. The power to control. *Nat. Phys.*, 9:126–128, 2013.

[13] J. Gao, Y. Y. Liu, R. M. D'Souza, and A. L. Barabási. Target control of complex networks. *Nature Communications*, 5, November 2014.

[14] K. Glover. All optimal hankel-norm approximations of linear multivariable systems and their l_∞-error bounds. *International Journal of Control*, 39(6):1115–1193, 1984.

[15] K. Glover and L. Silverman. Characterization of structural controllability. *IEEE Transactions on Automatic Control*, 21(4):534–537, Aug 1976.

[16] C. D. Godsil. Control by quantum dynamics on graphs. *Physical Review A*, 81(5), 2010.

[17] L. Hogben. Minimum rank problems. *Linear Algebra and its Applications*, 432(8):1961–1974, 2010.

[18] T. Ishizaki, K. Kashima, J. i. Imura, and K. Aihara. Model reduction of multi-input dynamical networks based on clusterwise controllability. In *American Control Conference (ACC)*, pages 2301–2306, June 2012.

[19] R.E. Kalman. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana*, 5:102–119, 1960.

[20] H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*. John Wiley & Sons, Inc., New York, NY, USA, 1972.

[21] Z. Li, Z. Duan, G. Chen, and L. Huang. Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 57(1):213–224, Jan 2010.

[22] C.T. Lin. Structural controllability. *IEEE Transactions on Automatic Control*, 19(3):201–208, Jun 1974.

[23] F. Lin, M. Fardad, and M. R. Jovanovi. Design of optimal sparse feedback gains via the alternating direction method of multipliers. *IEEE Transactions on Automatic Control*, 58(9):2426–2431, Sept 2013.

[24] Y. Y. Liu, J. J. Slotine, and A. L. Barabasi. Controllability of complex networks. *Nature*, 473(7346):167–173, May 2011.

[25] H. Mayeda and T. Yamada. Strong structural controllability. *SIAM Journal on Control and Optimization*, 17(1):123–138, 1979.

[26] B.P. Molinari. The time-invariant linear-quadratic optimal control problem. *Automatica*, 13(4):347 – 357, 1977.

[27] N. Monshizadeh, M. K. Camlibel, and H.L. Trentelman. Strong targeted controllability of dynamical networks. In *54th IEEE Conference on Decision and Control (CDC), Osaka, Japan, December 15-18, 2015*, pages 4782–4787.

[28] N. Monshizadeh, H. L. Trentelman, and M. K. Camlibel. Projection-based model reduction of multi-agent systems using graph partitions. *IEEE Transactions on Control of Network Systems*, 1(2):145–154, June 2014.

[29] N. Monshizadeh, S. Zhang, and M. K. Camlibel. Zero forcing sets and controllability of dynamical systems defined on graphs. *IEEE Transactions on Automatic Control*, 59(9):2562–2567, 2014.

[30] J. M. Montenbruck, G. S. Schmidt, G. S. Seyboth, and F. Allgöwer. On the necessity of diffusive couplings in linear synchronization problems with quadratic cost. *IEEE Transactions on Automatic Control*, 60(11):3029–3034, Nov 2015.

[31] B. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, 26(1):17–32, Feb 1981.

[32] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, Feb 2005.

[33] S. Mou, J. Liu, and A. S. Morse. A distributed algorithm for solving a linear algebraic equation. *IEEE Transactions on Automatic Control*, 60(11):2863–2878, Nov 2015.

[34] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, Sept 2004.

[35] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, 2009.

[36] P. Rapisarda, A. R. F. Everts, and M. K. Camlibel. Fault detection and isolation for systems defined over graphs. In *54th IEEE Conference on Decision and Control (CDC)*, pages 3816–3821, Dec 2015.

[37] Wei Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, May 2005.

[38] R. O. Saber and R. M. Murray. Consensus protocols for networks of dynamic agents. In *Proceedings of the American Control Conference*, volume 2, pages 951–956, June 2003.

[39] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557 – 2562, 2009.

[40] R. Shields and J. Pearson. Structural controllability of multiinput linear systems. *IEEE Transactions on Automatic Control*, 21(2):203–212, Apr 1976.

[41] H. G. Tanner. On the controllability of nearest neighbor interconnections. In *43rd IEEE Conference on Decision and Control (CDC)*, volume 3, pages 2467–2472, Dec 2004.

[42] M. Trefois and J. C. Delvenne. Zero forcing number, constrained matchings and strong structural controllability. *Linear Algebra and its Applications*, 484:199–218, 2015.

[43] H. L. Trentelman. The regular free-endpoint linear quadratic problem with indefinite cost. *SIAM Journal on Control and Optimization*, 27(1):27–42, 1989.

[44] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus. *Control Theory for Linear Systems*. Springer Verlag, London, UK, 2001.

[45] H. L. Trentelman, K. Takaba, and N. Monshizadeh. Robust synchronization of uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, 58(6):1511–1523, June 2013.

[46] S. Trip, M. Bürger, and C. De Persis. An internal model approach to (optimal) frequency regulation in power grids with time-varying voltages. *Automatica*, 64:240 – 253, 2016.

[47] S. Emre Tuna. Synchronizing linear systems via partial-state coupling. *Automatica*, 44(8):2179 – 2184, 2008.

[48] P. Wieland, R. Sepulchre, and F. Allgöwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5):1068 – 1074, 2011.

[49] J. Willems. Least squares stationary optimal control and the algebraic riccati equation. *IEEE Transactions on Automatic Control*, 16(6):621–634, Dec 1971.

[50] S. Zhang, M. Cao, and M. K. Camlibel. Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents. *IEEE Transactions on Automatic Control*, 59(3):745–750, 2014.