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The Dynamics of the One-Dimensional Tent Map Family and Quadratic Family

Bachelor's Project Mathematics

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Abstract

The goal of this paper is to give an overview of the dynamics of two one dimensional discrete dynamical systems: the Tent map family T_c and the quadratic family F_μ (also known as the logistic map) are investigated. The dynamics for different values of the parameters μ and c are studied. The quadratic family has chaotic regions for $\mu > 3$ and the Tent map family has chaotic regions for $c > 1$. Furthermore, the Cantor sets that occur as non-wandering sets for $\mu > 4$ and $c > 2$ are studied.

Contents

1 Introduction

The goal of dynamical systems is to describe a state over a course of time. We will investigate the dynamics of a one-dimensional discrete dynamical system. Such a system is an iterative procedure of a continuous function with some initial value. [4]

We want to investigate the dynamics of such a system to find out what happens after a large number of iterations. These are some questions that we can ask: Does the system converge to a certain value. What is chaos? When is our dynamical system chaotic?

The mathematician Edward Lorenz first discovered chaotic behaviour in the 1950s. He discovered how a small difference in initial values can result in a large difference in long-term results. He summarized it as: "Chaos: When the present determines the future, but the approximate present does not approximately determine the future." He came across chaos by accident when he was working on weather predictions. The mathematician Robert L. Devaney was the first to formally define chaos in 1986.

We will look specifically at the following families of functions, as they illustrate many interesting phenomena that occur in dynamical systems:

- quadratic family $F_\mu : \mathbb{R} \to \mathbb{R}$

$$F_\mu = \mu x(1-x)$$

- Tent map family $T_c : \mathbb{R} \to \mathbb{R}$

$$T_c(x) = \begin{cases} cx & 0 \leq x < \frac{1}{2} \\ c(1-x) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

These families of functions show different behaviour for the different real parameters $\mu > 0$ and $c > 0$. The dynamics of these families will be investigated. Different papers will be used to do this. [2] [14]

Especially for $\mu > 4$ and $c > 2$, the functions behave similar. For these values, we will see that they each generate a Cantor set as their non-wandering. A non-wandering set consists of all points that never leave the interval. Furthermore, we will see that the functions are chaotic on these Cantor sets.

There is some more chaotic behaviour for the quadratic family and the Tent map family that will also be discussed.

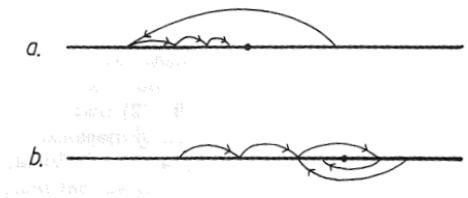
2 Basics

A discrete dynamical system is an iterative procedure of a continuous function f with some initial value. We will denote this initial value by x_0 and the sequence of iterates will look like:

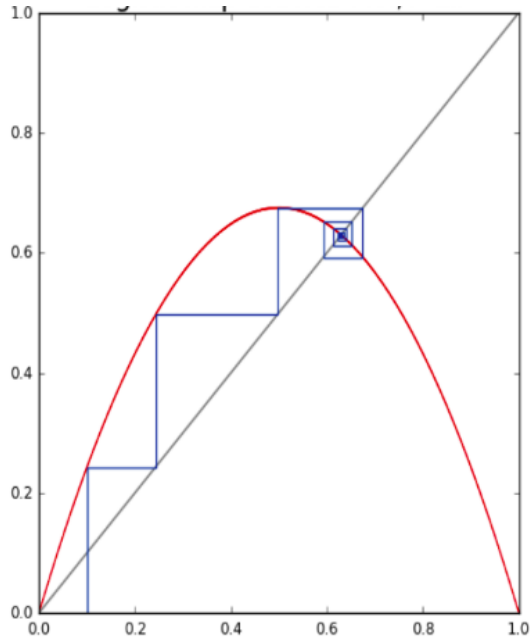
$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots.$$

Because all the interesting things happen in $I = [0, 1]$, we will limit ourselves to $x_0 \in I$.

The nth iterate can be written as $f^n(x_0) = (f \circ \cdots \circ f)(x_0)$. To avoid confusion, the nth derivative at x_0 is written as $f^{(n)}(x_0)$.



(a) Phase portrait



(b) Graphical analysis

Figure 1: Figure (a) shows the phase portrait of a: $1 < \mu < 2$ and b: $2 < \mu < 3$. Figure (b) shows the graphical analysis for $2 < \mu < 3$.

2.1 Definitions

Definition 1 (Orbits)**.** The set of points $\{x, f(x), f^2(x), \dots\}$ is called the forward orbit of x. It is denoted by $O^+(x)$. If f is a homeomorphism, we can define $O(x)$ as the full orbit of x as the set of points $f^n(x)$ for $n \in \mathbb{Z}$. The set $O^-(x)$ is defined as the backward orbit of x and consists of the set of points $\{x, f^{-1}(x), f^{-2}(x)\}$.

The goal of dynamical systems is to understand the geometric organisation of all orbits, especially in the long run. The goal is first to find the set of orbits that are periodic, eventually periodic etc. Before we do that, we have to make a few definitions.

Definition 2 (Fixed points and periodic points)**.** A point x is a fixed point for f if $f(x) = x$. A point x is called a periodic point of period n if $f^n(x) = x$. The set of periodic points of period n is denoted by $\mathrm{Per}_n(f)$.

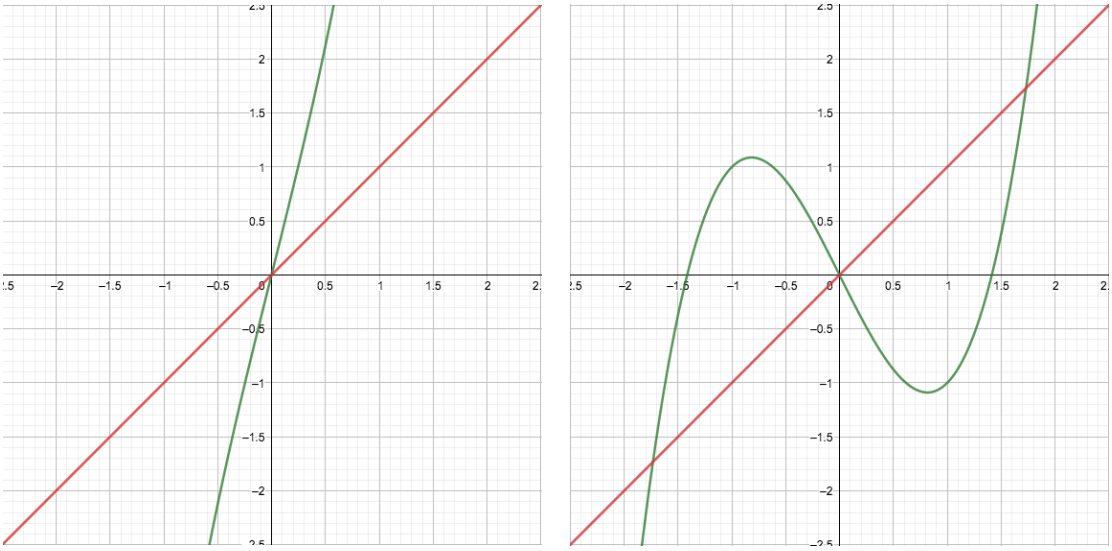
Definition 3 (Eventually periodic points.)**.** A point x is eventually periodic of period n there exists $m > 0$ such that $f^{n+i}(x) = f^i(x)$ for all $i \geq m$.

In other words, a point is eventually periodic if $f^i(x)$ is periodic for $i \geq m$.
Here are some simple examples to illustrate these definitions:

- The map $f(x) = x^3$ has three fixed points, namely $x = 0$,$x = 1$ and $x = -1$.
- The map $P(x) = x^2 - 1$ has fixed points at $x = \frac{1 \pm \sqrt{5}}{2}$. It also has two periodic points of period 2, namely $x = 0$ and $x = -1$. ($P(0) = -1$ and $P(-1) = 0$).
- The map $Q(x) = x^2$ has fixed points $x = 1$ and $x = 0$ since $Q(1) = 1$ and $Q(0) = 0$. There is one eventually fixed point at $x = -1$ since $Q(-1) = 1$ and $x = 1$ is a fixed point.

Definition 4 (Critical points)**.** A point x is called a critical point if $f'(x) = 0$. A critical point is non-degenerate if $f''(x) \neq 0$ and degenerate if $f''(x) = 0$.

Finding a set of orbits which are periodic turns out to be difficult. To find points of period n, we would have to solve the equation $f^n(x) = x$. For a function like the quadratic $F_\mu(x) = \mu x(1 - x)$ this results in a polynomial equation of degree 2^n. Another way of looking at the behaviour of an orbit is making a geometric picture of the behaviour of all orbits of such a system. We can draw



(a) $a < 0$ (b) $a > 0$

Figure 2: Graphs of $G_a(x)$ for different values of a.

the iterations of a point in a picture. Figure 1a shows such iterations for the quadratic family for different values of μ. This picture is called a phase portrait.
We can also do a graphical analysis to observe the behaviour of different orbits. In figure 1b we can see the graphical analysis of the quadratic family with a fixed point. Define the diagonal as $\Delta = \{(x, x) | x \in \mathbb{R}\}$. The orbit is given by repeatedly drawing line segments vertically from the diagonal to the graph and then horizontally from the graph back to the diagonal. We start at x_0 on the diagonal, this is point (x_0, x_0). Then we draw a line segment vertically to the graph, to the point $(x_0, f(x_0))$. Next, we draw a line horizontally back to the diagonal, to the point $(f(x_0), f(x_0))$. This can be repeated to find out how the iterations continue. Clearly, fixed points are found on the intersection with the diagonal. First we need to make some more definitions:

Definition 5 (Hyperbolic points)**.** Let p be a periodic point of prime period n. The point p is hyperbolic if $|(f^n)'(p)| \neq 1$ The number $(f^n)'(p)$ is called the multiplier of the periodic point.

Definition 6 (Attracting and repelling periodic points)**.** Let p be a hyperbolic periodic point of period n. If $|f^n(p)| < 1$, the point p is called an attracting periodic point. If $|f^n(p)| > 1$, the point p is called a repelling periodic point.

The fixed point in figure 1b seems to be an attracting point since the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ seems to zigzag to the fixed point.

For a family of functions F_c, the phase portrait may differ for different values for c. We can draw the behaviour for all values of c in a so-called bifurcation diagram. A bifurcation diagram is a graph with the parameter value horizontally and the values of x vertically. So for each value of the parameter, you can see the attracting and repelling fixed points. One slice of the graph at some parameter value is equal to the phase portrait of that value. An example of a bifurcation diagram is given:

Example 1. Consider the family of equations: $G_a(x) = x^3 - ax + x$. We can find the fixed points:

$$\begin{aligned} x^3 - ax + x &= x \\ x^2 - a + 1 &= 1 \\ x^2 &= a \\ x &= \pm\sqrt{a} \end{aligned}$$

The fixed points are $x = 0$ for all values of a and $x = \pm\sqrt{a}$ for $a \geq 0$. We can see some graphs of this family in figure 2. For $a < 0$ there is indeed only the fixed point $x = 0$. With the derivative

$$G_a'(x) = 3x^2 - a + 1$$

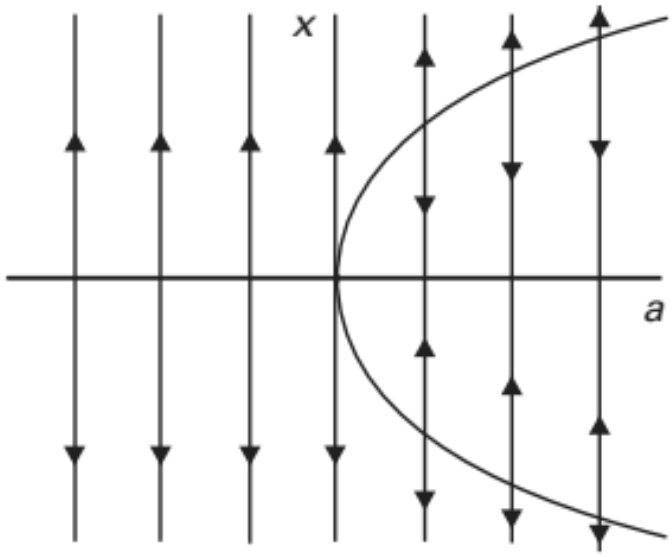


Figure 3: Bifurcation diagram for $G_a(x)$.

we can determine that $x = 0$ has $|G'_a(0)| = |-a+1|$ which is larger than one for negative values of a and smaller than one for positive values of a. This means that the fixed point $x = 0$ is repelling for $a < 0$ and attracting for $a > 0$. The other fixed points are repelling since $|G'_a(\pm\sqrt{a})| = |2a+1| > 1$ for $a > 0$. This behaviour is shown in figure 3.

Now that we have discussed the main definitions and introduced some methods to investigate the behaviour of a discrete dynamical system, we will look into some specific functions. The tent map and the quadratic family will be discussed as they demonstrate many interesting phenomena.

2.2 The Tent Map family

The Tent map family is a family of functions. Its iterations form a discrete dynamical system. This is the Tent map family for some value of c:

$$x_{n+1} = T_c(x_n) = \begin{cases} cx_n & 0 \leq x_n < \frac{1}{2} \\ c(1-x_n) & \frac{1}{2} \leq x_n \leq 1. \end{cases} \tag{1}$$

There are some functions of the tent map shown in figure 4b. The function is tent-shaped with the top at $x = \frac{1}{2}$ with a value of $\frac{1}{2}c$.

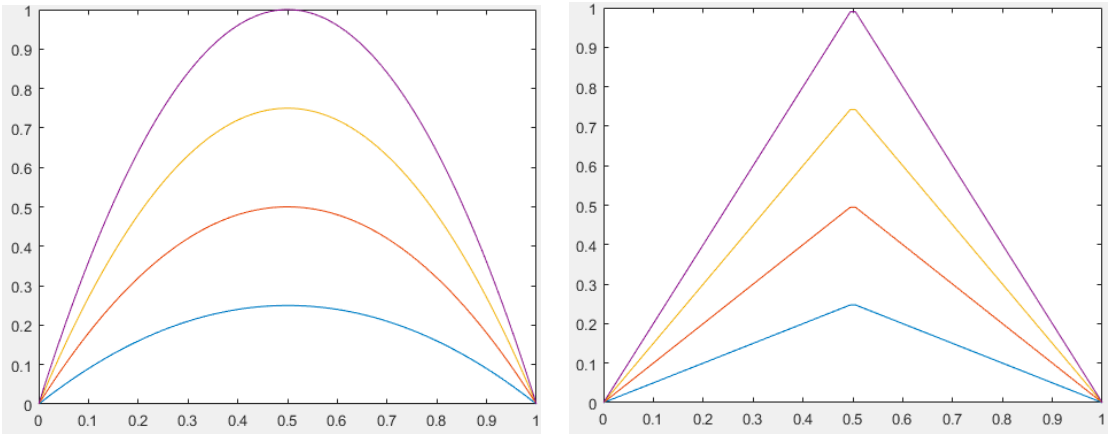
In the region $0 \leq x_n < \frac{1}{2}$ there is one fixed point at $x = 0$. In the region $\frac{1}{2} \leq x_n \leq 1$ there is also one fixed point:

$$\begin{aligned} c(1-x) &= x \\ 1-x &= \frac{x}{c} \\ 1 &= x(\frac{1}{c}+1) \\ \frac{c}{c+1} &= x_p. \end{aligned}$$

Note that this second fixed point only exists for $c \geq 1$.

2.3 The quadratic family

The family of functions $F_\mu = \mu x(1-x)$ is called the quadratic family or the logistic map. We will refer to it as the quadratic map. In figure 4a we see some functions of this family. These are the



(a) $\mu = 1, 2, 3, 4$ (b) $c = 0.5, 1, 1.5, 2$

Figure 4: Graphs of the quadratic family and the Tent map family

derivatives of the function:

$$\begin{aligned} F'_\mu(x) &= \mu - 2\mu x \\ F''_\mu(x) &= -2\mu \\ F^{(n)}(x) &= 0 \qquad \text{for } n \geq 3. \end{aligned} \tag{2}$$

From these derivatives we can determine that the top of F_μ lies at $x = \frac{1}{2}$. $F_\mu(x) = \frac{1}{4}\mu$ at this top.

We can find the fixed points of F_μ. It is evident that $x = 0$ is a fixed point. There is second fixed point:

$$\begin{aligned} F_\mu(x) &= x \\ \mu x(1-x) &= x \\ 1 - x &= \frac{1}{\mu} \\ x_p &= \frac{\mu - 1}{\mu} \end{aligned}$$

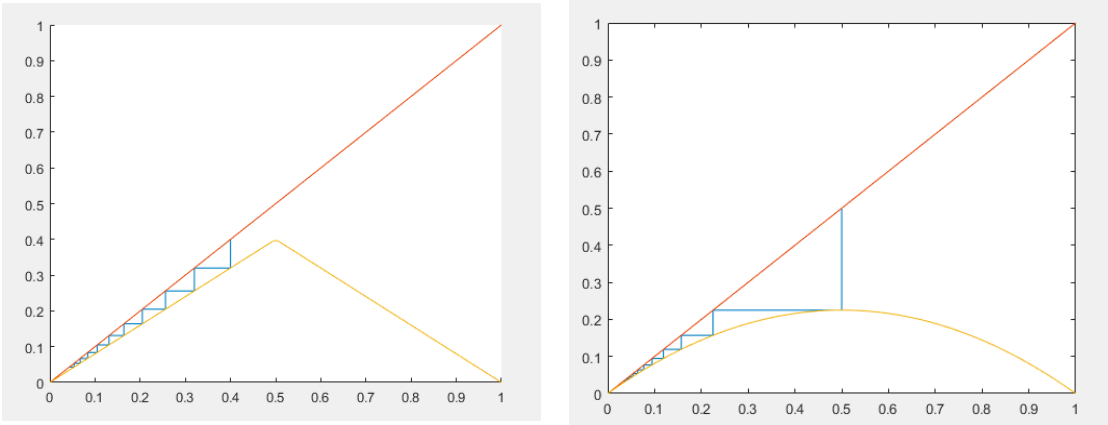
So we have one fixed point at $x = 0$ and one fixed point at $x = x_p$. We find that $|F'_\mu(0)| = \mu$ and $|F'_\mu(x_p)| = |-\mu + 2|$. So, depending on the value of μ, the fixed points can be either repelling or attracting. The dynamics of the Tent map family and the quadratic family will be investigated in the following sections.

2.4 Dynamics of the Tent map family for $0 < c < 1$

For $c < 1$ the Tent map family has one fixed point at $x = 0$. It has $T'_c(0) = c < 1$ and is therefore an attracting fixed point. We can see this in figure 5a, the iterations converge to the fixed point $x = 0$.
After $c = 1$, the Tent map family will become chaotic. This will be discussed later on.

2.5 Dynamics of the quadratic family for $0 < \mu < 3$.

For $\mu < 1$ the behaviour of the quadratic family is similar to the behaviour of the Tent map family for $c < 1$.
We have seen that the quadratic family has two fixed points. The first is $x = 0$ with $|F'_(0)| = \mu < 1$ which makes it an attracting fixed point. The second fixed point falls outside of I for $\mu < 1$ and will not be considered yet. Figure 5b shows the graphical analysis of this quadratic family for



(a) The Tent map family

(b) Quadratic family

Figure 5: The dynamics for $\mu < 1$ and $c < 1$.

$\mu < 1$. The Tent map family and the quadratic family both have one attracting fixed point at $x = 0$.
After $\mu = c = 1$ the dynamics of the Tent map family and the quadratic family differ. The quadratic family with the values $1 < \mu < 3$ have two fixed points in I. We observe the behaviour in figure 6. The first fixed point is $x = 0$ and the second is $x = x_p$. We find that $F'_\mu(0) = \mu > 1$ and $|F'_\mu(p)| = |-\mu+2| < 1$ for $1 < \mu < 3$. So $x = 0$ is a repelling point and $x = p_\mu$ is an attracting point. This is exactly what we would expect from figure 6.

3 Cantor sets

There are Cantor sets related to the Tent map family and the quadratic family. These Cantor sets exist for $c > 2$ and $\mu > 4$. In this section we will observe the dynamics of the Tent map family and the quadratic family for $c > 2$ and $\mu > 4$. Certain points 'leave' the interval I after a number of iterations. The points that never leave I form a so-called *non-wandering set*. This non-wandering set turns out to be a Cantor set. First we will define Cantor sets.

3.1 What is a Cantor set?

Definition 7 (Cantor Set)**.** A set Λ is a Cantor set if Λ is a *closed*, *totally disconnected* and *perfect* set. A set is totally disconnected if it contains no intervals. A set is perfect if every point of this set is an accumulation point or limit point of other points in the set, i.e. if it contains no isolated points.

A well-known example is given below:

Example 2 (Cantor Middle-Thirds Set)**.** The classical example of a Cantor set is the so-called *Cantor Middle-Thirds Set*. Figure 7 shows the construction of the Cantor Middle-Thirds Set. We start with $I = C_0$. Then this interval is split in three parts and the middle open part is removed. This results in the closed $C_1 = I - (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then the middle part of the remaining parts is removed again for the next set: $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Note that this set can also be written as $C_2 = \frac{1}{3}C_1 \cup (\frac{1}{3}C_1 + \frac{2}{3})$. This leads inductively to

$$C_n = \frac{1}{3}C_{n-1} \cup (\frac{1}{3}C_{n-1} + \frac{2}{3}). \tag{3}$$

The set $CM = \bigcap_{n=1}^{\infty} C_n$ is a Cantor set. The proof is given below to give some more insight in what makes a Cantor set.

Proof. 1. Every C_n is a union of closed intervals and is therefore closed. CM is an infinite intersection of closed C_n which makes CM a closed set. 2. Let $a, b \in CM$ with $a > b$ and $\epsilon = b - a$. Assume there exists an interval (a, b). We can choose n such that $\frac{1}{3^n} < \epsilon$. CM is the intersection

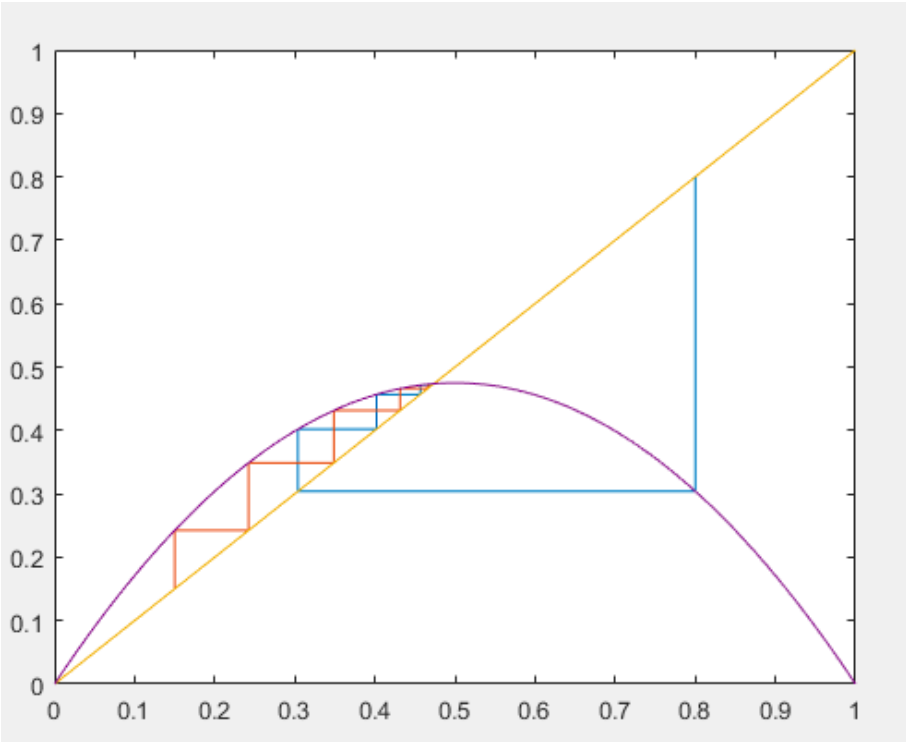


Figure 6: Graphical analysis of the quadratic family for $1 < \mu < 3$.

of closed intervals that are all smaller than ϵ for some n. Therefore the interval (a, b) cannot exist and CM is totally disconnected. 3. Take any point $p \in CM$ and fix n such that $\frac{1}{3^n} < \epsilon$. Then p needs to be in an interval of length $\frac{1}{3^n}$. This way the endpoints $0, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{9}, \frac{2}{9}, \ldots$ are all contained in an interval $(p - \epsilon, p + \epsilon)$. Therefore, each $p \in CM$ is an accumulation point which makes CM a perfect set.[8] □

3.2 The Cantor Set as non-wandering set of the Tent map family

After studying the definition and an example of a Cantor set, we now have the tools to look specifically at the Tent map family.

Example 3 (Cantor Set on Tent map family)**.** Consider the Tent map family:

$$T_c(x_n) = x_{n+1} \begin{cases} cx_n & x_n < \frac{1}{2} \\ c(1 - x_n) & x_n \geq \frac{1}{2} \end{cases} \tag{4}$$

for some value of c. For $c > 2$, the top of the Tent map family is higher than 1. This is shown in figure 8. It can be observed that a point x_0 in this top leaves the interval I after the first iteration. Furthermore, we see that the iteration of such a point will go to $-\infty$.

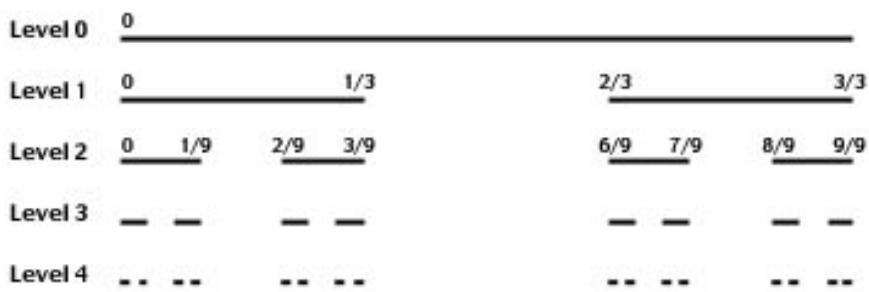


Figure 7: Cantor Middle Thirds Set.

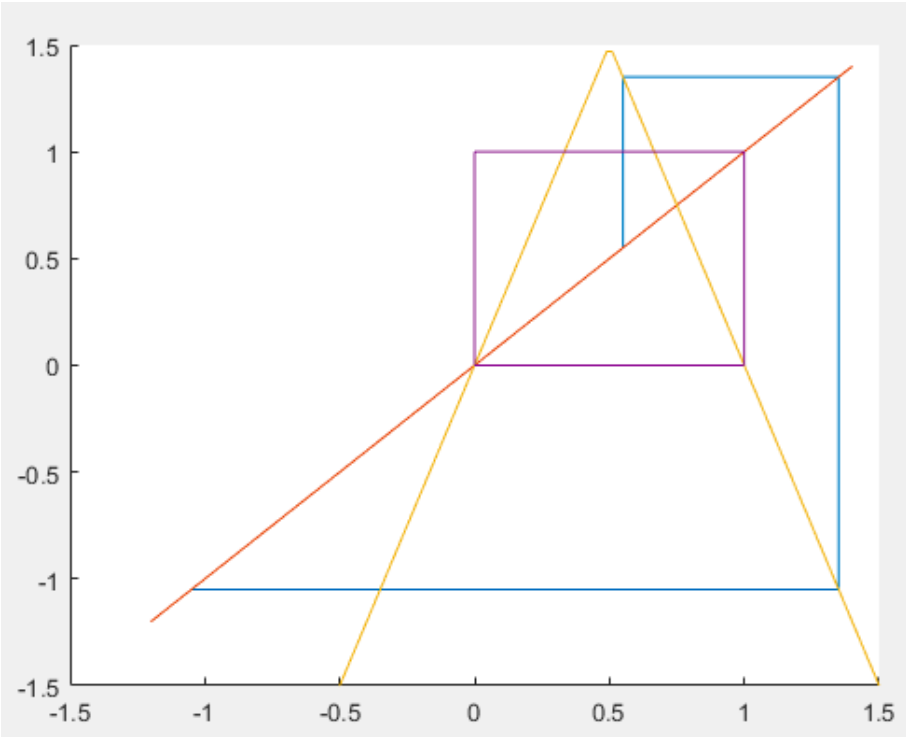


Figure 8: Dynamics of the Tent map family for $c > 2$. The purple square indicates the interval $[0, 1]$ and the horizontal $T(x) = 0$ and $T(x) = 1$.

Denote the points that leave I after the first iteration by C_0. Inductively, we denote the points that leave I after the nth iteration by C_{n-1}. The Cantor set related to the Tent map family is

$$\Lambda_T = I - (\bigcup_{n=0}^{\infty} C_n)$$

This Cantor set is formed for all $c > 2$.

Note that for $c = 3$, Λ_T is exactly the Cantor Middle-Thirds set. In figure 9 the red graph shows T_3 and the green graph shows T_3^2. The intervals that form the tops of these graphs contain all the points that are mapped out of I after one or two iterations, respectively.

We can calculate the set of points that leave I after the first iteration exactly:

$$\begin{aligned} T(x) &> 1 \\ 3x &> 1 \text{ and} \qquad & 3(1-x) &> 1 \\ x &> \frac{1}{3} & x &< \frac{2}{3}. \end{aligned}$$

So the points that do not leave I after the first iteration equals $C_0 = I - (\frac{1}{3}, \frac{2}{3})$ which is indeed the same as in the Cantor Middle-thirds set. The dotted lines in figure 9 show the 'endpoints' of the interval. A point on the dotted line is mapped exactly to 1. For example, $T_3(\frac{1}{3}) = 1$ and $T_3^2(\frac{1}{9}) = 1$. The endpoints never leave I as the next iteration maps these points to the fixed point $x = 0$. Therefore, C_0 is a closed set.

We can check that the points that leave I after the second iteration also corresponds to the Cantor Middle thirds set. Indeed, T_3^2 maps all 'endpoints' $\frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}$ to 1.

3.3 The Cantor set that is the non-wandering set of the quadratic family

In this section we will see that for $\mu > 4$ the non-wandering set of the quadratic family is a Cantor set. The proof will be provided for $\mu > 2 + \sqrt{5}$.

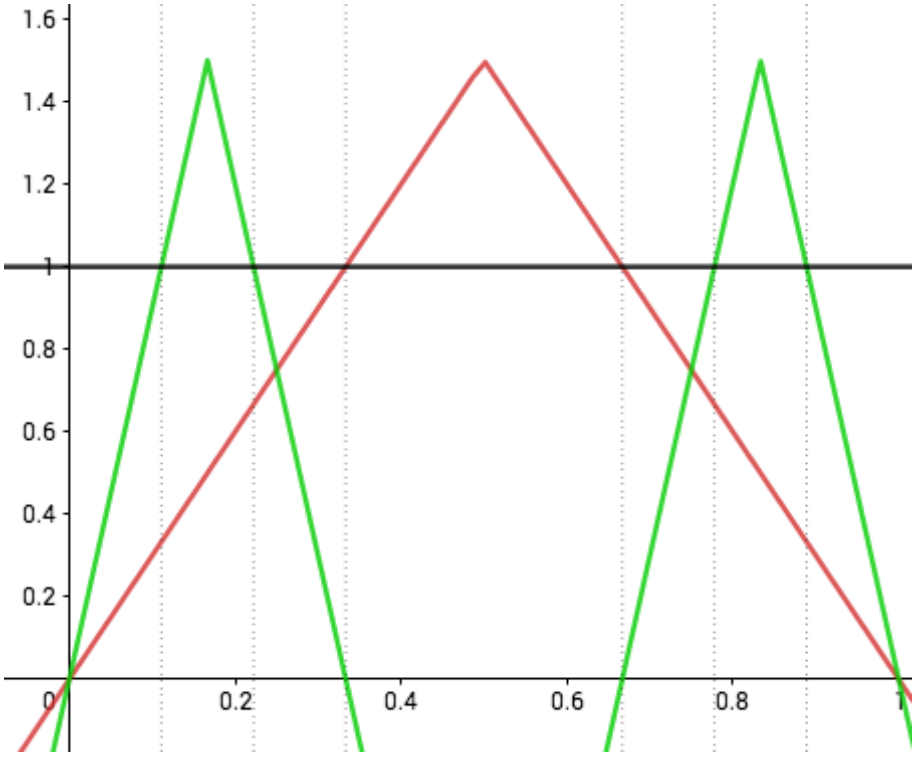


Figure 9: Visual representation of the Tent map family for $c = 3$. Red is T_3 and green is T_3^2. The dotted lines show $\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}$

$$\begin{tikzcd}
{[0,1]} \arrow[r, "T_2"'] \arrow[d, "k"'] & {[0,1]} \arrow[d, "k"] \\
{[0,1]} \arrow[r, "F_4"'] & {[0,1]}
\end{tikzcd}$$

Figure 10: Conjugacy between F_4 and the Tent map family T_2.

The Cantor set generated by the quadratic family is formed in a similar way as the Cantor set related to the Tent map family seen before in example 3. The Tent map family for $c = 2$ and quadratic family for $\mu = 4$ have a topological conjugacy.

Definition 8 (Topological conjugacy)**.** Let $f : A \to A$ and $g : B \to B$ be two maps. f and g are topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $h \circ f = g \circ h$. The homeomorphism h is called a topological conjugacy.

Proposition 1. *There is a topological conjugacy between F_4 and T_2 as in diagram 10. The proof is found in the appendix.*

Because of this topological conjugacy it is no surprise that the behaviour of both functions is similar for $\mu > 4$ and $c > 2$.

Consider $F_\mu = \mu x(1 - x)$ for $\mu > 4$ with a point x_0. Its first iteration is $F_\mu(x_0)$. For $\mu > 4$ there exists points such that $F_\mu(x) > 1$. Such a point then leaves I in the next iteration. We can calculate exactly what points leave I after the first iteration. We will denote this set by A_0.

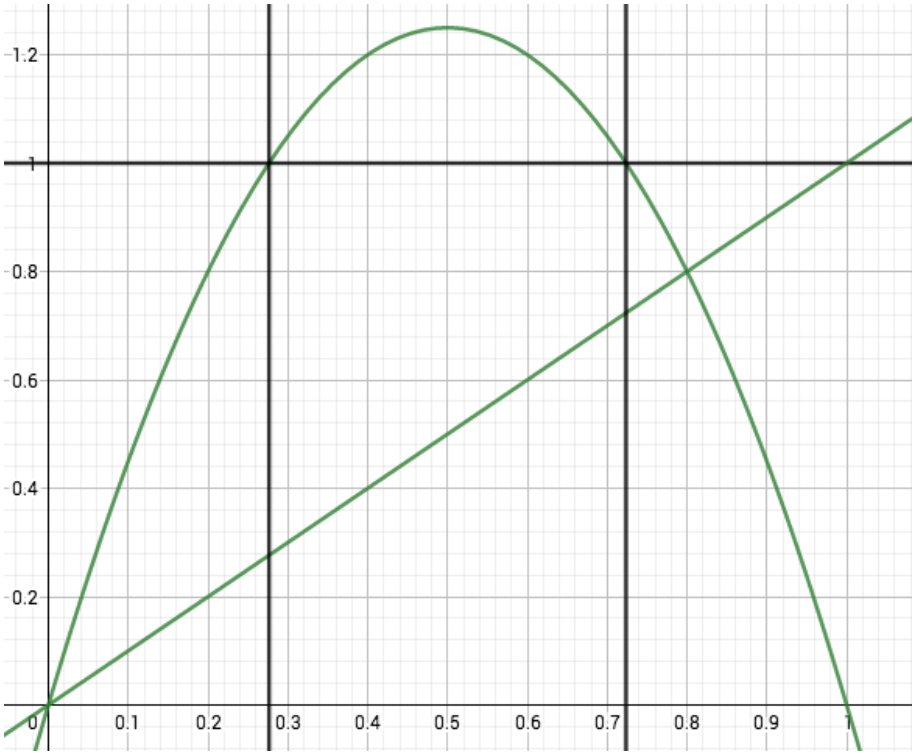


Figure 11: This is the graph of F_5. The interval between the black lines is A_0. The interval on the left is I_0 and the interval on the right is I_1.

Solving $F_\mu(x) = 1$ gives $x = \frac{1}{2} \pm \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}$. Hence, this is the open interval A_0:

$$A_0 = \left(\frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}, \frac{1}{2} + \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}\right). \tag{5}$$

In figure 11, we can see the open interval A_0 centered around $\frac{1}{2}$. As before, a point $x \in A_0$ goes to $F^n(x) \to -\infty$.
Then, let $A_1 = \{x \in I | F_\mu(x) \in A_0\} = \{x \in I | F_\mu^2(x) \notin I\}$. If $x \in A_1$, then $F_\mu^2(x) > 1$. Calculating the intervals A_n becomes increasingly difficult as for calculating A_1 we would have to solve a polynomial of degree 4. By induction, $A_n = \{x \in I | F_\mu^n(x) \in A_0\}$. This can also be written as $\{x \in I | F_\mu^n(x) \in I$ and $F_\mu^n(x) \notin I$ for $n > i\}$. This way, A_n consists of all the points that escape I after $n+1$ iterations.

Note that $F\mu$ is an increasing function in I_0 and a decreasing function in I_1. Therefore, F_μ maps I_0 and I_1 monotonically onto I. $I - A_0$ consists of two closed intervals, as we have seen. Then $I_(A_0 \cup A_1)$ consists of 4 closed intervals which are all mapped onto either I_0 or I_1 Inductively, $I - (A_0 \cup A_1 \cup \cdots \cup A_n)$ consists of 2^{n+1} closed intervals. Let $I - (A_0 \cup A_1 \cup \cdots \cup A_n) = J_n$. Then:

$$J_n = F_\mu^{-1}(J_{n-1}) = F^{-n}(J_0). \tag{6}$$

Now that we have defined these sets A_n, we can look at the following set:

$$\Lambda := I - (\bigcup_{n=1}^{\infty} A_n). \tag{7}$$

It turns out that Λ is a Cantor set for $\mu > 4$. We will prove that Λ is a Cantor set for $\mu > 2 + \sqrt{5}$. The following property shows why we choose $\mu > 2 + \sqrt(5)$:

Proposition 2. *If $\mu > 2 + \sqrt(5)$, then $|F'_\mu(x)| > 1$ for all $x \in I - A_0$.*

Proof. Split $I - A_0$ in the two closed intervals on the right and left of A_0:

$$I_0 = \left[0, \frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}\right] \text{ and } I_1 = \left[\frac{1}{2} + \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}, 1\right]. \tag{8}$$

In I_0, we have $F'_\mu(x) > 0$ and in I_1, $F'_\mu(x) < 0$. For $x \in I_0$ we know $0 < x < \frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}$. For $\mu > 2+\sqrt{5}$, this means $0 < x < \frac{1}{2}(3-\sqrt{5})$. Then $|F'_\mu(x)| = |\mu - 2\mu x| = \mu|1-2x| > (2+\sqrt{5})|1-2x|$. Since F_μ has a positive slope in I_0, we can lose the absolute signs: $|F'_\mu(x)| > (2+\sqrt{5})(1-2x)$. We know $x < \frac{1}{2}(3-\sqrt{5})$. For $x = 0$, $|F'_\mu(0)| = 2+\sqrt{5} > 1$. Then x increases monotonically to $x = \frac{1}{2}(3-\sqrt{5})$ which has $|F'_\mu(\frac{1}{2}(3-\sqrt{5})) = 1$. Therefore, for $x \in I_0$ $|F'_\mu(x)| > 1$. We can do the same reasoning for I_1 to finish the proof. □

Proposition 3. *If $\mu > 2+\sqrt{5}$, then Λ is a Cantor set.*

Proof. In order to prove that Λ is a Cantor set, we need to show that it is a closed, totally disconnected and perfect set:
1. Λ is closed. We observed that A_0 is open. Furthermore, we know $A_{n+1} = F_\mu^{-1}(A_n)$. F_μ is a continuous function so the pre-image of an open set is open. Therefore, $A_1 = F_\mu^{-1}(0)$ is open and by induction, every A_n is open. A union of open sets is open, so $\bigcup_{n=1}^{\infty} A_n$ is open. Its complement, $I - \bigcup_{n=1}^{\infty} A_n = \Lambda$ is thus closed.
2. Λ is totally disconnected. From proposition 2 we know that $|F'_\mu(x)| > 1$ for $\mu > 2+\sqrt{5}$ and $x \in I - A_0$. There exists $\lambda > 1$ such that $|F'_\mu(x)| > \lambda > 1$ for all $x \in \Lambda$. Then $|F^n_\mu(x)| > \lambda^n$ by the chain rule. Assume there exists $x, y \in \Lambda$ such that $x \neq y$ that form a closed interval $[x, y] \subset \Lambda$. Then $|F_\mu n(\alpha)| > \lambda^n$ for all $\alpha \in [x, y]$. We can choose n in such a way that $\lambda^n|x-y| > 1$. Then we can apply the Mean Value Theorem: $|F^n_\mu(x) - F^n_\mu(y)| > \lambda|x-y| > 1$. This implies that the distance between $F^n_\mu(x)$ and $F^n_\mu(y)$ is larger than 1 and thus at least one of them must be outside of I. This is contradicting with $x, y \in \Lambda$. This implied that x and y can never leave I Therefore, there are no intervals in Λ and it must be totally disconnected.
3. Λ is perfect. A set is perfect if all its point are limit points. So, we have to prove that for all $x_0 \in \Lambda$ and all $\epsilon > 0$, there is a $y \in \Lambda$ such that $x_0 \neq y$ and $|x_0 - y| < \epsilon$. We will use I_0 and I_1 as defined before in equation 8. The restrictions

$$F_\mu|_{I_0} : I_0 \to [0,1] \text{ and } F_\mu|_{I_1} : I_1 \to [0,1]$$

are homeomorphisms and thus there exist inverse maps

$$h_0 : I_0 \to [0,1] \text{ and } h_1 : I_1 \to [0,1]$$

such that $x = F_\mu(h_1(x))$. The orbits of $x \in \Lambda$ never leave $[0,1]$ so if $x \in \Lambda$, we know that $h_0(x), h_1(x) \in \Lambda$.
We know that $x_{n+1} = F_\mu(x_n)$ so $x_n = h_0(x_{n+1})$ or $x_n = h_1(x_{n+1})$ depending on whether x_n is in I_0 or I_1. There exists $\lambda > 1$ such that, for $a, b \in \Lambda$:

$$|h_0(a) - h_0(b)| \leq \frac{1}{\lambda}|a-b| \text{ and } |h_1(a) - h_1(b)| \leq \frac{1}{\lambda}|a-b|.$$

Now we can write $x_0 = h \circ h \circ \cdots \circ h(x_)$. Now we choose $y' = 0$, which is a fixed point in Λ. For any $\epsilon > 0$ there exists an n such that $\frac{1}{\lambda^n} < \epsilon$. Then for some $y \neq x_0$ converging to y' we find that

$$|x_0 - y| \leq \frac{1}{\lambda^n}|x_n - y'| \leq \frac{1}{\lambda^n} < \epsilon$$

which completes the proof. □

3.4 Symbolic Dynamics

Symbolic dynamics is another approach to describing the orbit of a point. Recall that a point $x \in \Lambda$ never leaves the Cantor set Λ. We know that $\Lambda \subset I_0 \cup I_1$.
We denote:

$$s_j = \begin{cases} 0 & F^j(x) \in I_0 \\ 1 & F^j(x) \in I_1 \end{cases} \tag{9}$$

So every x_n is mapped by F_μ to either I_0 or I_1. We make the following definitions:

Definition 9 (Itinerary Space)**.** The itinerary space is denoted by $\Sigma_2 = \{\mathbf{s} = (s_0 s_1 s_2 \ldots) | s_j = 0 \text{ or } 1\}$

Proposition 4. *(Σ_2, d) is a metric space.*

Proof. Devaney has proven that this is a metric on Σ_2 [3]. A set together with a metric is a metric space. □

Definition 10. The itinerary of x is a sequence $S(x) = (s_0 s_1 s_2 \dots)$, where s_j is defined as in equation 9. $S(x)$ is a function $S : \Lambda \to \Sigma_2$.

Example 4. Consider the Tent map family (equation 4). These are its iterations for $x_0 = \frac{7}{9}$:

$$\begin{aligned}
x_0 &= \frac{7}{9} && \in I_1 \\
x_1 &= T_3(x_0) = \frac{6}{9} && \in I_1 \\
x_2 &= T_3(x_1) = 1 && \in I_1 \\
x_3 &= T_3(x_2) = 0 && \in I_0 \\
x_4 &= T_3(x_3) = 0 && \in I_0 \\
&\dots
\end{aligned}$$

To find $S(x_0)$, we simply look at the iterations above. Because $x = 0$ is a fixed point we know that every digit will be 0 for $j \geq 3$. So $S(x_0) = (11100000\dots)$.

We can do a similar example for the quadratic family.

Example 5. Consider one of the 'endpoints' of A_1 from equation 5:

$$x_0 = \frac{1}{2} - \frac{\sqrt{\mu - 4}}{2\sqrt{\mu}}$$

Its iterations are:

$$\begin{aligned}
x_0 &= \frac{1}{2} - \frac{\sqrt{\mu - 4}}{2\sqrt{\mu}} && \in I_0 \\
x_1 &= F_\mu(x_0) = 1 && \in I_1 \\
x_2 &= F_\mu(x_1) = 0 && \in I_0 \\
x_3 &= F_\mu(x_2) = 0 && \in I_0 \\
&\dots
\end{aligned}$$

Now we can easily see that $S(x_0) = (01000000\dots)$.

The distance between two sequences $s = (s_0 s_1 s_2 \dots)$ and $t = (t_0 t_1 t_2 \dots)$ is defined as:

$$d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \tag{10}$$

The shift map is an important map in symbolic dynamics. It shifts the entire sequence $S(x)$ one place to the left in such a way that the first entry, s_0 is "forgotten". It is formally defined as follows:

Definition 11 (Shift map)**.** The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is given by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots)$.

Proposition 5. *The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is continuous.*

Proof. Devaney has proven this[3] on page 41. □

Note that for a fixed point $S(x) = (s_0 s_0 s_0 \dots)$ and thus $\sigma(S(x)) = S(x)$. For a periodic point of period 2 we have $S(x) = (s_0 s_1 s_0 s_1 \dots)$ and $\sigma(\sigma(S(x)) = \sigma \circ \sigma \circ S(x) = S(x)$. More generally, $S(x) = (s_0 \dots s_{n-1} s_0 \dots s_{n-1} s_0 \dots s_{n-1} \dots)$ and $\sigma \circ \dots \circ \sigma(S(x)) = \sigma^n(S(x) = S(x)$ for periodic points of period n.

This leads us to the following proposition:

Proposition 6. *There are topological conjugacies between F_μ and σ and between T_c and σ. The so-called commutative diagram is shown in figure 12. So the following is true:*

- $S \circ T_c = \sigma \circ S$
- $S \circ F_\mu = \sigma \circ S$

Proof. Let $I_{s_0 s_1 \dots s_n} = \{x \in I | T_c^i(x) \in I_{s_i} \forall i = 0, \dots, n\}$. This can also be written as $\bigcap_{i=0}^n T_c^{-i}(I_{s_i})$. A point $x \in \Lambda_T$ can be defined as $\bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n}$. We can write $T_c(I_{s_0 \dots s_n}) = \bigcap_{i=0}^n T_c^{-i+1}(I_{s_i}) = I_{s_1 \dots s_n}$ since $T_c(I_{s_0}) = I$. Now we can conclude:

$$\begin{aligned} S \circ T_c(x) &= S \circ T_c(\bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n}) \\ &= S(\bigcap_{n \geq 0} I_{s_1 \dots s_n}) \\ &= (s_1 s_2 \dots) \\ &= \sigma \circ S(x) \end{aligned}$$

We can follow the same steps to prove the conjugacy between F_μ and σ. □

All topological properties are preserved by such a topological conjugacy. Because Λ_T and Λ_F are Cantor sets, we can make the following proposition:

Proposition 7. *Σ_2 is a Cantor set.*

$S : \Lambda \to \Sigma_2$ is a homeomorphism for $\mu > 2 + \sqrt{5}$. [3] So by proposition 6, there exists a conjugacy between σ and F_μ for $\mu > 2 + \sqrt{5}$.

Proposition 8. *Let σ be the shift map. Then:*

1. *The cardinality of $Per_n(\sigma)$ is 2^n*
2. *$Per(\sigma)$ is dense in Σ_2.*
3. *There exists a dense orbit for σ in Σ_2.*

Proof.

1. For any periodic point of period n there is a sequence of length n repeated over and over. there are 2^n possible sequences for a periodic point of period n so the cardinality of $\mathrm{Per}_n(\sigma)$ is 2^n.
2. A subset is dense in Σ_2 if it closure equals Σ_2. We can produce a sequence $\tau_n = (s_0 \dots s_n s_0 \dots s_n)$ such that τ_n is the repeating sequence with its entries agreeing with some arbitrary $s \in \Sigma_2$ up till the nth entry. Then the distance $d[\tau_n, s] \leq \frac{1}{2^n}$ [3] which shows that for every $s \in \Sigma_2$ there exists a τ_n converging to it. Hence the closure of $\mathrm{Per}(\sigma)$ is equal to Σ_2 and it is a dense subset.
3. Consider s^* as in equation 11. It is constructed by successively listing all blocks of each length. After enough iterations of σ, we find a sequence that agrees with any other sequence in an arbitrarily large number of places. Therefore, there exist a dense orbit for σ in Σ_2.

$$s^* = (0\,1|00\,01\,10\,11|000\,001 \dots | \dots) \tag{11}$$

□

A topological conjugacy preserves the dynamics of its functions. For example, if x_p is a fixed point for F_μ, then $S(x_p)$ is a fixed point for σ:

$$S(x_p) = S(F_\mu(x_p)) = \sigma(S(x_p)).$$

$$\begin{tikzcd}
\Lambda_T \arrow[r, "T_c"'] \arrow[d, "S"'] & \Lambda_T \arrow[d, "S"'] \\
\Sigma_2 \arrow[r, "\sigma"'] & \Sigma_2 \\
\Lambda_F \arrow[r, "F_\mu"'] \arrow[u, "S"] & \Lambda_F \arrow[u, "S"]
\end{tikzcd}$$

Figure 12: The commutative diagram of the conjugacy between F_μ and σ and the conjugacy between T_c and σ.

In the same way, if F_μ has a periodic point of period n x_p then σ has a periodic point $S(x_p)$ of period n:

$$S(x_p) = S(F_\mu^n(x_p)) = \sigma \circ S \circ F^{n-1}(x_p) = \sigma^2 \circ S \circ F^{n-2}(x_p) = \cdots = \sigma^n(S(x_p)).$$

Because of the conjugacy between F_μ and σ all the topological properties that hold for σ in proposition 10 also hold for F_μ. We can summarize this in the following proposition:

Proposition 9. *Let $F_\mu(x) = \mu x(1-x)$ with $\mu > 2 + \sqrt{5}$. Then:*

1. *The cardinality of $Per_n(F_\mu)$ is 2^n*
2. *$Per(F_\mu)$ is dense in Λ_F.*
3. *F_μ has a dense orbit in Λ_F.*

Note that we there is also a topological conjugacy between T_c and σ. Therefore, the same properties hold for T_c as well:

Proposition 10. *Let T_c be the Tent map family with $c > 2$. Then:*

1. *The cardinality of $Per_n(T_c)$ is 2^n*
2. *$Per(T_c)$ is dense in Λ_T.*
3. *T_c has a dense orbit in Λ_T.*

4 Chaos

4.1 Devaney's Definition of Chaos

Devaney was the first one to formally define chaos. He stated that a chaotic map has the following three ingredients:

Definition 12. Let V be a set. $f : V \to V$ is said to be chaotic on V if

1. f has sensitive dependence on initial conditions
2. f is topologically transitive
3. periodic points are dense in V

These three properties are defined below:

Definition 13. Sensitive dependence on initial conditions
A function $f : J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \leq 0$ such that $|f^(x) - f^n(y)| > \delta$.

Definition 14. Topologically transitive
A function $f : J \to J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exist $k > 0$ such that $f^k(U) \cap V \neq 0$.

If J has no isolated points, then we can say that if there is a point $x_0 \in J$ such that its orbit $x_0, f(x_0), f^2(x_0), \ldots$ is dense we can also say that f is topologically transitive. This is proven by [9]

Definition 15. A subset $U \subset S$ is dense in S if $\bar{U} = S$, where $\bar{U}$ is the closure of U.

After Devaney made this definition, there was a paper written by J. Banks [1] in which he proved that the first property of Devaney's definition follows from the last two properties. This is written in the following proposition.

Proposition 11. *If $f : X \to X$ is transitive and has dense periodic points then f has sensitive dependence on initial conditions.*

The Tent map family and the quadratic family display chaos in certain regions. This will be discussed below.

Proposition 12. *$F_\mu(x) = \mu x(1-x)$ is chaotic on the cantor set Λ_F for $\mu > 2 + \sqrt{5}$.*

Proof. To prove this we need to prove all three properties of definition 15:
1. Let A_0 be the set A_0 as in the Cantor set. Let $\delta < A_0$ and $x, y \in \Lambda_F$. If $x \neq y$, then $S(x) \neq S(y)$ which means the itineraries of x and y must differ in at least one spot. Suppose they differ in the nth spot. Then $F_\mu^n(x)$ lies in I_0 while $F_\mu^n(y)$ lies in I_1 or vice versa. Then $|F_\mu^n(x) - F_\mu^n(y)| > \delta$.
2. F_μ on the Cantor set Λ_F is a function $F_\mu(x) : \Lambda_F \to \Lambda_F$. Consider x in any open set in Λ_F. There is a topological conjugacy, as shown in proposition 6. This means that $F^k(x) = S^{-1} \circ \sigma^k \circ S(x)$. Since Λ_F is a Cantor set for $\mu > 2 + \sqrt{5}$ we know it is a perfect set. Therefore, noting that there exits a dense orbit is enough to prove topological transitivity. According to proposition 9, F_μ has a dense orbit in Λ_F
3. Proposition 9 shows that the periodic points are dense in Λ_F, finishing the proof. [5] [3] □

The quadratic family is not only chaotic for $\mu > 2 + \sqrt{5}$ but for all values $\mu > 4$ [3]. Following the same reasoning as proposition 12, the Tent map family for $c > 2$ is chaotic on the Cantor set Λ_T. More chaotic regions for the quadratic family and the Tent map family will be dealt with in the following sections.

4.2 Lyapunov exponents

A Lyapunov exponent is a measure for sensitivity to initial conditions. [7]

The lyapunov exponent λ for discrete systems is:

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log_2 \mid \frac{\mathrm{dx}_{n+1}}{\mathrm{dx}_n} \mid, \tag{12}$$

where

$$x_n + 1 = f(x_n)$$

.
The Lyapunov exponent indicates the following:

- $\lambda < 0$ indicates a periodic attractor. (no chaos)
- $\lambda = 0$ indicates a bifurcating periodic point
- $\lambda > 0$ indicates sensitivity to initial conditions. (chaos)

4.3 Chaos on the Quadratic family

We discussed chaos for the quadratic family for $\mu > 4$. For $\mu < 4$ there is also chaotic behaviour. This is a different kind of chaos. We will look at the dynamics of $\mu > 3$ and we will see that there are a number of period-doubling bifurcations that lead to chaos.

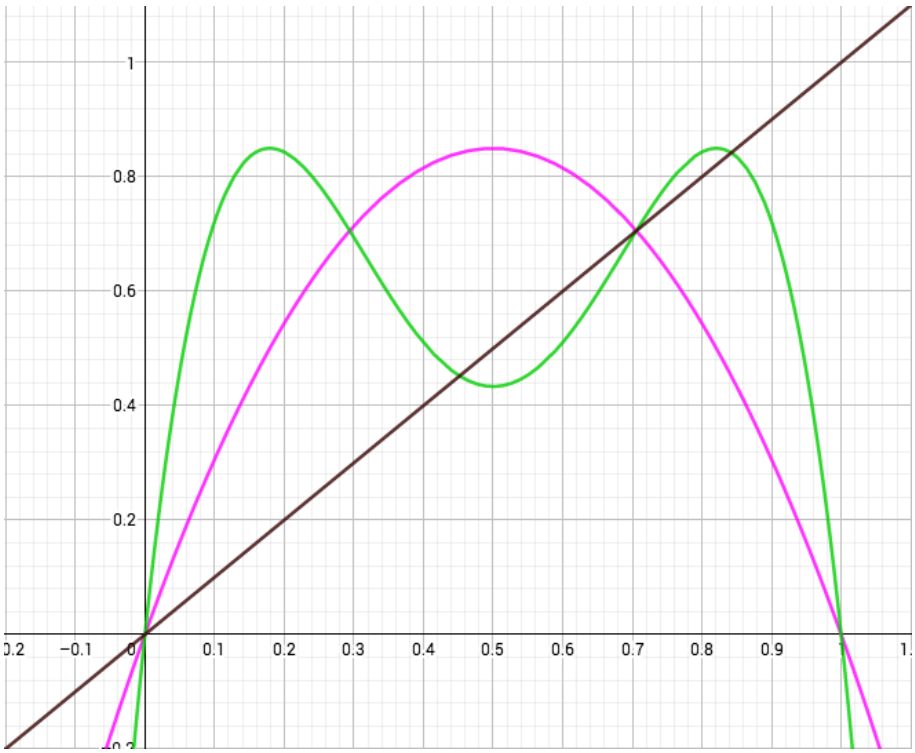


Figure 13: The graph of F_μ (purple) and $F^2\mu$ (green) for $\mu = 3.4$.

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We will start with an example:

Example 6. For $\mu = 3.4$, the fixed points are $x = 0$ and $x = \frac{\mu-1}{\mu} = \frac{2.4}{3.4}$.
We can calculate that both are repelling fixed points. When we take a look at $F^2_{3.4}(x)$, we find that $F^2_{3.4}$ has four fixed points as can be seen in figure 13. This means that $F_{3.4}$ has two repelling fixed points but also two periodic points of period two. To calculate these points we have to solve $F^2_\mu(x) = -\mu^3x^4 + 2\mu^3x^3 - (\mu^3 + \mu^2)x^2 + \mu^2x = x$. The solution to this equation is:

$$p_1 = 0, p_2 = \frac{\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}}{2\mu}, p_3 = \frac{\mu - 1}{\mu}, p_4 = \frac{\mu + 1 + \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$$

Then we take the derivative: $\frac{\mathrm{d}}{\mathrm{dx}}F^2_\mu = -4\mu^3x^3 + 6\mu^3x^2 - 2(\mu^3 + \mu^2)x + \mu^2$. Now we can determine if our new found periodic points are attracting or repelling: $|\frac{\mathrm{d}}{\mathrm{dx}}F^2_{3.4}(p_2)| = 0.76 < 1$ and $|\frac{\mathrm{d}}{\mathrm{dx}}F^2_{3.4}(p_4)| = 0.76 < 1$. So for $\mu = 3.4$ there are two repelling fixed points and two attracting periodic points of period 2.

Note that p_2 and p_4 exist only for $\mu > 3$ because for $\mu < 3$, they are imaginary and for $\mu = 3$, $p_2 = p_4 = p_3$. We say there is a period-doubling bifurcation at $\mu = 3$.

Proposition 13 (Period-doubling bifurcation)**.** *Suppose*

- $f_\mu(0) = 0$ *for all* μ *in an interval about* μ_0.
- $f'_{\mu_0}(0) = -1$
- $\frac{\partial(f^2_\mu)'}{\partial\mu}|_{\mu=\mu_0}(0) \neq 0$.

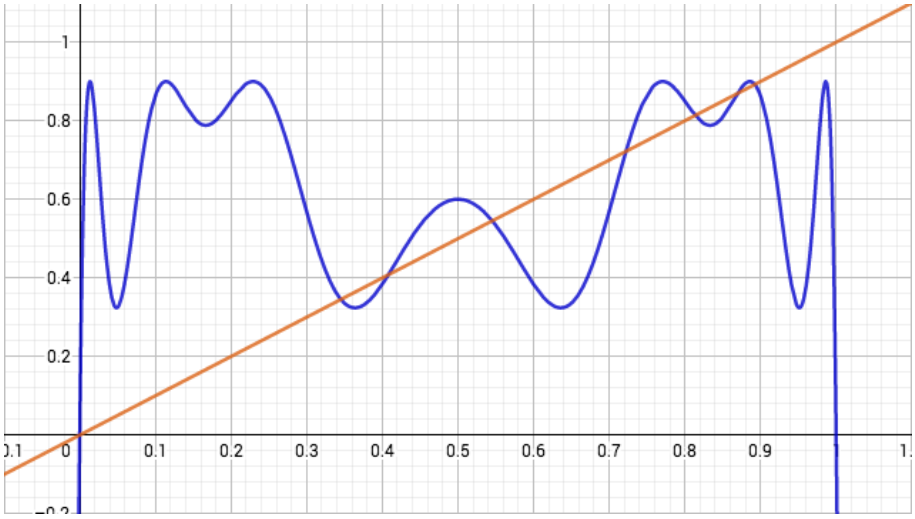

Figure 14: Graph of F_μ^4 for $\mu = 3.5$.

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Then there is an interval J about 0 and a function $p : J \to \mathbb{R}$ such that

$$f_{p(x)}(x) \neq x$$

but

$$f^2_{p(x)}(x) = x.$$

Proof. Devaney has proven this. (page 90) [3] □

This proposition shows the necessary qualities for a bifurcation at $x = 0$. We can check the properties for $\mu = 3$. For $\mu = 3$, there is a bifurcation at the fixed point $\frac{2}{3}$:

- $F_\mu(\frac{2}{3}) = \frac{2}{3}$.
- $F_3'(\frac{2}{3}) = -1$
- $\frac{\partial (F^2)'}{\partial \mu}|_{\mu=3} = 44 \neq 0$.

In the example above, the periodic points of period 2 were attracting. As μ increases, there will be another bifurcation and they will change to repelling periodic points. We can calculate when these points swap from attracting to repelling points:

$$|\frac{\mathrm{d}}{\mathrm{dx}} F_\mu^2(p_2)| = \frac{\mathrm{d}}{\mathrm{dx}} F_\mu^2(p_4)| = |-\mu^2 + 2\mu + 4| = 1$$

We find that the period-doubling bifurcation occurs at $\mu = 1 + \sqrt{6} \approx 3.449$. This means that the periodic points of period 2 are attracting for $3 < \mu \leq 1 + \sqrt{6} \approx 3.449$ and repelling for $\mu \geq 1 + \sqrt{6} \approx 3.449$.

So after $\mu = 1 + \sqrt{6}$ there are 8 periodic points (2 fixed points, 2 periodic points of period 2 and 4 periodic points of period 4). Figure 14 shows that indeed, $F_{3.5}^4(x)$ has 8 fixed points.
Figure 15 shows the first period-doubling bifurcations. Calculating all these points is getting more and more complicated so we not calculate them exactly. The Feigenbaum diagram (figure 17) shows only the attracting periodic points. More period-doubling bifurcations occur as μ increases. There are bifurcations at $\mu \approx 3.544$ (8 periodic points), at $\mu \approx 3.564$ (16 periodic points) and at $\mu \approx 3.569$ (32 periodic points). As μ increases even further, its behaviour becomes chaotic. This is seen in the bifurcation diagram in figure 17 The critical value after which there is chaos is $\mu \approx 3.56994$ [13].
In figure 16 we see the Feigenbaum diagram a little closer up. There are 'windows' in the diagram. We clearly see there are only 3 periodic points in such a window. One such a window is at $\mu = 3.8284$. Li and Yorke proved in 1975 that "period 3 implies chaos". They proved that once a period-3 orbit is established there are orbits of all other periods. [6]

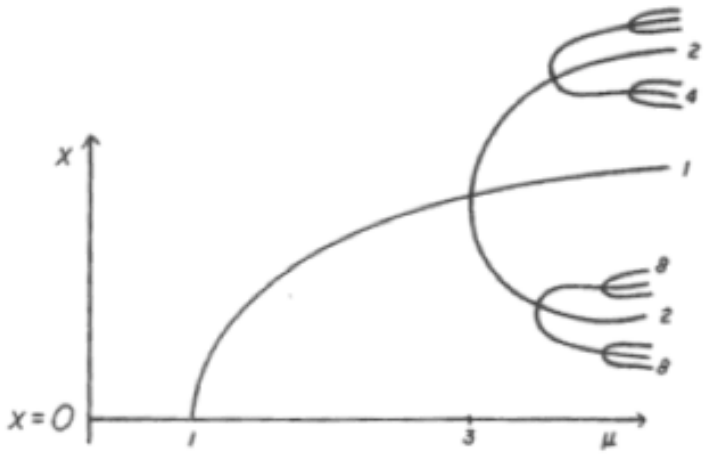

Figure 15: This figure shows the bifurcation diagram for F_μ. The integers on the right of the picture represent the periods.

Yet another way of looking at the chaotic regions is by looking at the Lyapunov exponents. For the quadratic family the Lyapunov exponent is determined by:

$$\lambda \approx \frac{1}{N} \sum_{n=1}^{N} \log_2 \mid \mu - 2\mu x_n \mid$$

Figure 17 shows the Lyapunov exponents for $3 < \mu < 4$. It shows that this corresponds to the Feigenbaum diagram. At the non-chaotic region of the quadratic family, the Lyapunov exponent is indeed negative. At the period-doubling bifurcations the Lyapunov exponent is zero. Furthermore, we see that at the 'windows' in the Feigenbaum diagram, the Lyapunov exponent has peaks in the negative side of the graph.

4.3.1 The Feigenbaum constant

period	μ	δ_k
2	3	4.7514...
4	3.449489...	4.6562...
8	3.544090...	4.6683...
16	3.564407...	4.6687...
32	3.568759...	4.6692...

Table 1: The values of δ_k calculated to show how the Feigenbaum constant is arrived at.

The mathematical physicist M.J. Feigenbaum calculated the Feigenbaum constant in 1979. The Feigenbaum constant characterizes the geometric approach of the bifurcation parameter to its limit as the parameter μ increases. He calculated it to be $\delta = 4.6692016091029\ldots$ for the quadratic family.
This constant is given by the limit [12]:

$$\delta = \lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \frac{\mu_{k+1} - \mu_k}{\mu_{k+2} - \mu_{k+1}}.$$

where μ_k is the value of μ at which a period 2^n cycle appears. With the values of μ where period-doubling bifurcations occurred, we can calculate the first values of δ_k. As we can see in table 1, the value of δ_k already agrees with the Feigenbaum constant δ for the first 4 digits. [14]

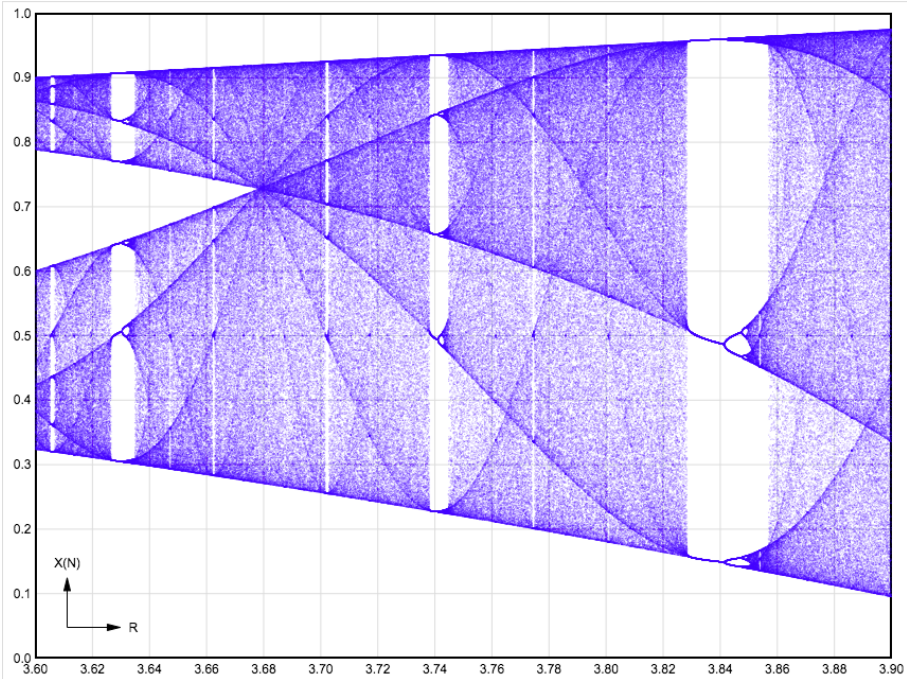

Figure 16: Feigenbaum diagram for $3.6 < \mu < 3.9$. Note that $R = \mu$.

4.3.2 Cantor set in the Feigenbaum diagram

In the Feigenbaum diagram the regions with many periodic points are the dark regions. These regions are chaotic. In between these dark regions there are light parts with only a few periodic points. These parts are called 'windows'. In figure 16, we can see some of these windows. The 'windows' in the bifurcation diagram are open. The chaotic parts in the Feigenbaum diagram form a Cantor Set of positive measure. [11].

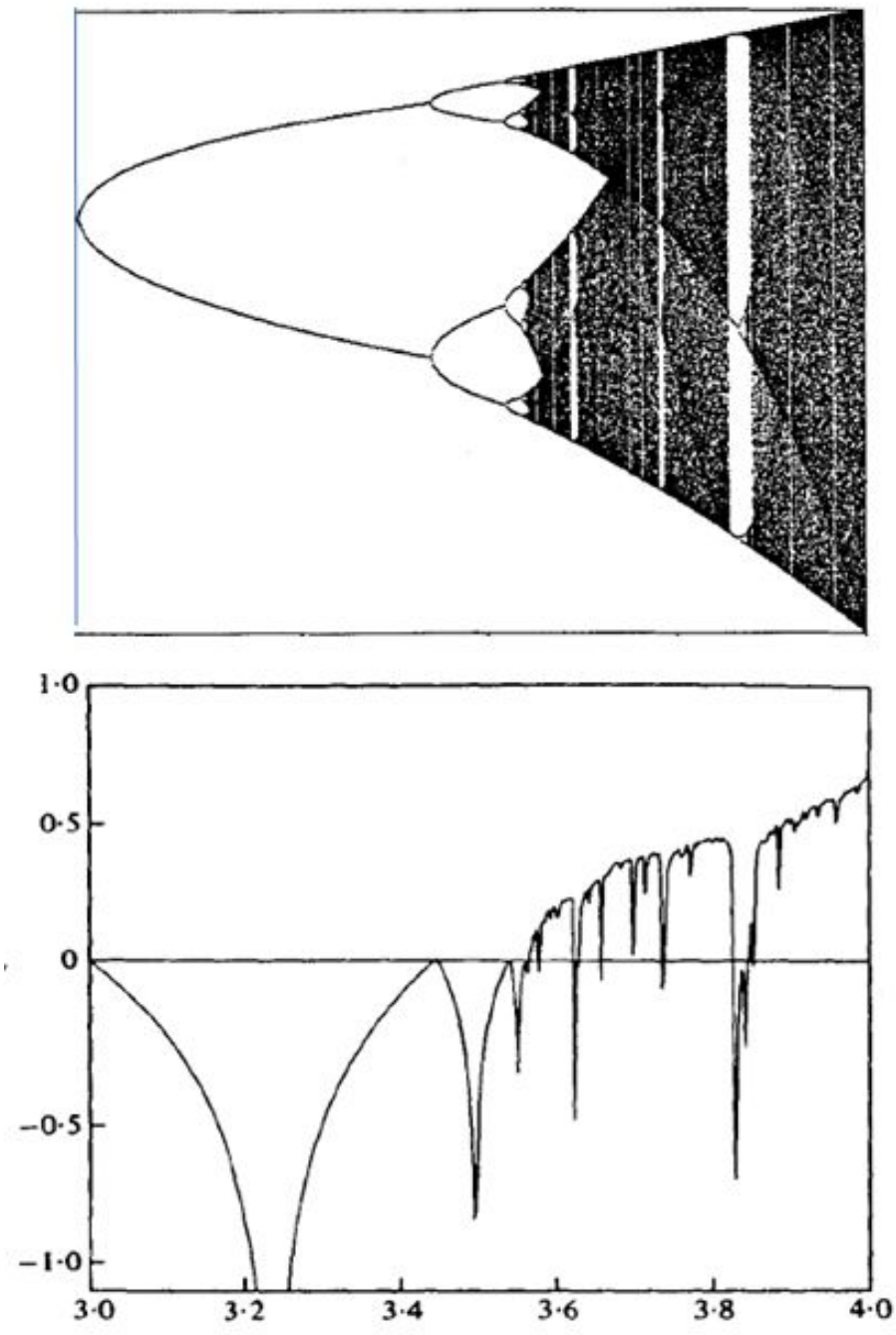

Figure 17: The top graph is the Feigenbaum diagram and the graph below shows Lyapunov exponent for values of $3 < \mu < 4$.

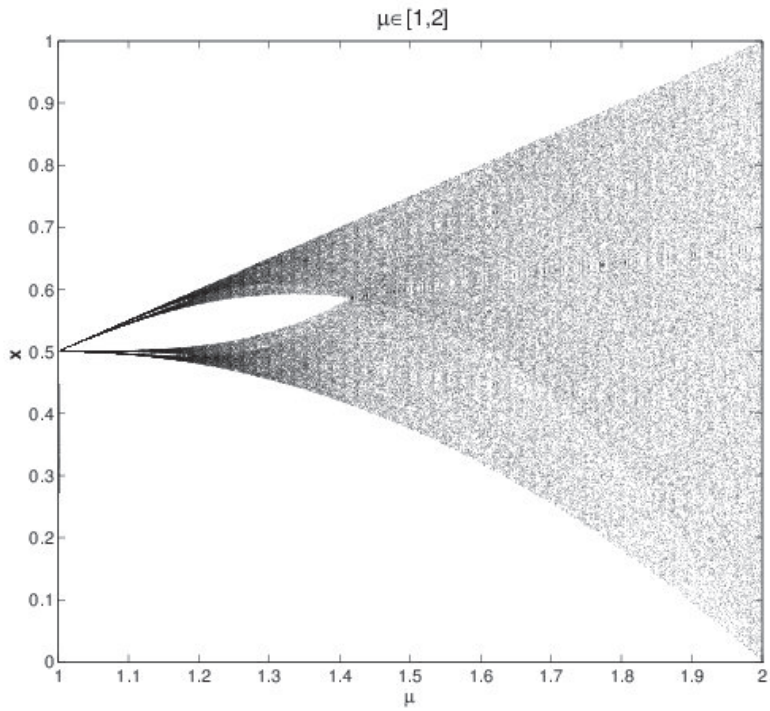

Figure 18: Bifurcation diagram for the Tent map family for $1 \leq c \leq 2$.

4.4 Chaos in the Tent Map family

The bifurcation diagram of the Tent map family for $1 \leq c \leq 2$ is shown in figure 18. For $1 \leq c \leq 2$, the Tent map family shows chaotic behaviour. Unlike the quadratic family, the Tent map family does not follow the period-doubling route to chaos. Figure 18 shows that there are many periodic points.
The Lyapunov exponent can be easily calculated for the Tent map family:

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log_2 \mid \frac{\mathrm{dx}_{n+1}}{\mathrm{dx}_n} \mid = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log_2 c = \log_2 c.$$

For $c < 1$ the Lyapunov exponent is negative and for $c > 1$ the Lyapunov exponent is positive. This indicates that the Tent map family displays chaos for $c > 1$.

5 Conclusion

In this project we have been looking at the dynamics of one-dimensional discrete dynamical systems, specifically the quadratic family F_μ and the Tent map family T_c. We have looked at the dynamics of these families as well as the role of Cantor sets and chaos.

Devaney's book [3] has been a large part of this project. His book was reviewed by the dutch mathematician F. Takens [10]. He stated that it was only recent that it was realized that one-dimensional dynamical systems have interesting properties. Takens states that Devaney uses this low-dimensional context to introduce a number of notions which are also of importance for higher dimensions. So the work done in this paper gives firstly an overview of the one-dimensional discrete dynamical systems discussed. Secondly, it gives the reader insight into one-dimensional dynamical systems which is useful for investigating dynamical systems of higher dimensions.

We have found that the quadratic family becomes chaotic for increasing values of μ, following the period-doubling route to chaos. For $\mu < 3$, there are no more than two fixed points. For $\mu > 3$, there are period-doubling bifurcations as μ increases. These period-doubling bifurcations are shown in a so-called Feigenbaum diagram, which is the bifurcation diagram for the quadratic family. Some interesting aspects of this diagram were mentioned. There is a Feigenbaum constant and the complement of the windows in the Feigenbaum diagram form a Cantor set.

The Tent map family has only one fixed point for $c < 1$. For $c > 1$ we observe chaotic behaviour. It does not follow the period-doubling route to chaos. We have calculated the Lyapunov exponents which indicated that there was indeed chaos for $c > 1$.

For $\mu > 4$ and $c > 2$ we observed that certain points of I leave the interval after the first iteration. The points that never leave are called the non-wandering set. This non-wandering set is a Cantor Set. Furthermore, we have looked at symbolic dynamics. There is a topological conjugacy between the shift map and both F_μ and T_c. We used symbolic dynamics to see that the Tent map family and quadratic family are chaotic for $c > 2$ and $\mu > 4$ on their Cantor sets.

A Appendix

A.1 Conjugation between Tent map and Logistic map

Proposition 14. *This is proposition 1.There is a topological conjugacy between F_4 and T_2 as in diagram 10. The proof is found in the appendix.*

Proof. Recall the tent map:

$$x_{n+1} = T_2(x_n) = \begin{cases} 2x_n & 0 \leq x_n < \frac{1}{2} \\ 2(1-x_n) & \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

and the quadratic equation $F_4 = 4x(1-x)$. Define

$$k(x) = \frac{2}{\pi} \arcsin \sqrt{x}.$$

The function k is a homeomorphism on $I = [0,1]$. Then it follows that $k^{-1}(x) = (\sin \frac{\pi}{2} x)^2$. We need to prove that $F_4(x) = k^{-1} \circ T_2 \circ k(x)$.
First case, $0 \leq x_n < \frac{1}{2}$:

$$\begin{aligned} F_4(x) &= k^{-1} \circ T_2 \circ k(x) \\ &= (\sin(\frac{\pi}{2}(\frac{4}{\pi} \arcsin \sqrt{x})))^2 \\ &= (\sin(2 \arcsin \sqrt{x}))^2 \\ &= (2 \sin(\arcsin \sqrt{x}) \cos(\arcsin \sqrt{x})^2 \\ &= (2\sqrt{x} \cos(\arcsin \sqrt{x}))^2 \\ &= 4x(1 - \sin^2(\arcsin \sqrt{x}))^2 \\ &= 4x(1-x) \end{aligned}$$

The second case, $\frac{1}{2} \leq x_n \leq 1$:

$$\begin{aligned} F_4(x) &= k^{-1} \circ T_2 \circ k(x) \\ &= (\sin(\frac{\pi}{2}(-\frac{4}{\pi} \arcsin \sqrt{x} + 2)))^2 \\ &= (\sin(-2 \arcsin \sqrt{x} + \pi))^2 \\ &= (2 \sin(-\arcsin \sqrt{x} + \frac{\pi}{2}) \cos(-\arcsin \sqrt{x} + \frac{\pi}{2})^2 \\ &= (2 \cos(\arcsin \sqrt{x}) \sin(\arcsin \sqrt{x})^2 \\ &= 4x(\cos(\arcsin \sqrt{x}))^2 \\ &= 4x(1-x) \end{aligned}$$

Which completes the proof. □

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