The Coleman Mandula Theorem and its Nonrelativistic Limit

Bachelor Thesis

in

Theoretical Physics

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Abstract

In this paper, we will perform several theoretical analysis revolving the Coleman-Mandula theorem. Firstly, we motivate the theorem itself and argue why it is interesting to study it. Then, we present two versions of the proof for the theorem: one thought to be as detailed and conscientious as possible and another where we try to illustrate only the most essemtial arguments in order to get a general impression about its flow and structure. At the same time that we undertake this last version of the proof, we also attempt and fail to extrapolate the original proof to the nonrelativistic limit, highlighting all the conflicting points that we encounter along the way.

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1 Introduction

The Coleman-Mandula theorem is a very important theorem in the realm of quantum field theory which states that, given some reasonable assumptions of physical nature, the most general Lie algebra of symmetry generators that can commute with the S-matrix is a linear combination of the generators of the Poincaré group and those of an internal symmetry. The importance of this theorem resides in that it severely limits the symmetries that we may find in a system of interacting particles, since they will be restricted to a trivial mixing of the Poincaré group and some internal symmetry.

To illustrate why this is the case without requiring a rigorous discussion of the theorem and to put in perspective how relevant the conclusion of the theorem is, we can examine the simple case of two-body scattering. If we take this system to conserve angular momentum and four-momentum, then the only free variable for its dynamics is the scattering angle. But now, if we impose an additional symmetry which couples non-trivially the Poincaré symmetry with a given internal symmetry, we will need to include a new space-time associated generator in our analysis, which will provide us with a new conservation law that would probably further constrain the scattering, resulting in the values for our scattering angle being restricted to a discrete set. However, this is not physically sound, since one would expect that the scattering angle should be able to adopt any possible value, so, under this perspective, it does not make sense to have the scattering process depending in such a way on the angle. The only way to reconcile these two opposing propositions is to drop the possibility of the scattering altogether, that is, to set the S-matrix to zero. Therefore, for this particular case, we can conclude that, if we consider a symmetry that would be forbidden by the Coleman-Mandula theorem for the two-body system, then it is not physically possible to formulate scattering processes for it.

Now that we have provided a basic explanation of the implications of the Coleman-Mandula theorem, we are ready to introduce a problem in which the conclusion of the theorem plays a major role and which ultimately serves as the primary motivation for this report. Namely, the problem is that the appearance of only trivial combinations of the Poincaré group and transformations on the fields independent of the point in spacetime as symmetries is not exclusive to the scattering context, to the point that it is often promoted to a general assumption. For instance, in gauge theories, when looking at symmetries, it is never even considered nothing but a trivial coupling between the spacetime symmmetries and any other invariant transformation that we may propose. This issue does not pose a problem for the theorem as an independent structure, since it does not question any of the suppositions or the arguments applied to proove it, but it leads us inevitably to consider the possibility that the conclusion of the theorem is more general than the premises that we used to proove it. If that were the case, then it would not be surprising to reach the same result that the theorem proposes by working with an alternative set of assumptions.

This prospect is exactly what we are interested in exploring. To do so, during this report, we will first provide a heavily detailed version of the formal statement of the theorem and its proof, heavily inspired by a simillar effort done in Weinberg's "The Quantum Theory of Fields, vol. 3" [**1**], the original proof published by Coleman and Mandula [**2**] and a student paper that also deals with the proof of the theorem [**3**], and then we will disect its most essemtial arguments and structure to then examine it critically to see whether any of the conditions that it deals with can be relaxed and also to check if there exist some explicit limitations to it. Besides that, even though we know in advance that we will fail given experimental observations, we will attempt to adapt the proof to a non-relativistic context. We choose this non-relativistic limit to try to extend the scope of the theorem because it is one of the most simillar contexts to the one that completely supports the proof and also because we will get an idea about how rigidly the theorem is inserted inside the quantum field theory framework by pointing where exactly things go wrong.

2 Theory preliminaries

Before starting the proper discussion about the Coleman-Mandula theorem, its proof and its possible non-relativistic limit, it is convenient to briefly introduce some notions of scattering theory and about the Poincaré and galilean groups that we will need to use later on.

2.1 Poincaré group

The Poincaré group [**4**] is a ten-dimensional noncompact Lie group which consists of a semidirect product of the Lorentz group and the four-dimensional translation group. Since this group is built from the combination of Lorentz transformations Λ and translations a, its elements can always be fully specified by (Λ, a), which transform a fiven four-vector x in the following way.

$$(\Lambda, a) : (x) \rightarrow (x') = (\Lambda x + a) \tag{1}$$

In the context of quantum field theory, the Poincaré group is the spacetime symmetry group, and it is very important because it provides all the possible coordinate transformations that one can apply to the Minkowski space, which is the manifold in which the field states that the theory works with need to be defined. Due to its interpretation as coordinate transformations, whenever a relativistic quantum theory is formulated, Poincaré symmetry is sought, which means that if a given state of the field is physically possible, any state resulting from a Poincaré transformation acting on that state should still be a possible configuration of the field.

Since during this report we will work all the time with quantum field states and also due to the content of the Coleman-Mandula theorem, we will very often need to use some Poincaré group properties, the most important of which are the commutation relations of the generators of its algebra. Because of this, it is convenient to introduce those relations already here:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
\tfrac{1}{i}[M_{\mu\nu}, P_\rho] &= \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu \\
\tfrac{1}{i}[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}
\end{aligned}
\tag{2}
$$

where P_i are the translation generators, η_{ij} is the Minkowski metric and M_{ij} are the Lorentz generators, which include rotations ($J_i = \epsilon_{imn}M^{mn}$) and boosts ($K_i = M_{i0}$). All of them can be obtained by using any of their explicit representations.

Another feature of the Poincaré group that it is convenient that we know is the differential operators for Lorentz transformations, which we need to perform transformations of the Lorentz group on the quantum fields. If any element of the Lorentz group can be expressed in terms of the elements of its Lie algebra through the exponential map as:

$$\Lambda = e^{-i\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}} \tag{3}$$

then an infinitesimal transformation can be easily expressed through an approximation of the exponential function

$$\Lambda = 1 - -i\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu} \tag{4}$$

If we use this element to transform some space-time coordinates x^μ, which are the variables in which our fields will depend, knowing how each of those generators transform any four-vector, we get that:

$$x \xrightarrow{\Lambda} x' = \Lambda x \quad ; \quad \Lambda x^\mu = \delta^\mu_\nu x^\nu + \omega^\mu_\nu x^\nu \tag{5}$$

Comparing this to expression (5), we can redefine the generators in terms of the differential operators:

$$\hat{M}^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{6}$$

and express the infinitesimal Lorentz transformations as:

$$\Lambda x = x + i\frac{1}{2}[\hat{M}^{\mu\nu}, x] \tag{7}$$

These new generators will satisfy all the previous commutation relations in (2), provided that we also use the operator representation for the four-momentum, and will form a perfectly valid Lorentz algebra. Thus, the operator representation of Lorentz transformations will be:

$$\hat{D}(\Lambda) \equiv e^{-i\frac{1}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu}} \tag{8}$$

2.2 Galilean group

The galilean group [5] is a ten-parameter Lie group which is defined through the set $G = \{(R, v, a, b)\}$ along with the composition law $G \bigotimes G \to G$, where R is an orthogonal matrix, v and a are vectors transformable under R and b is a scalar. These parameters acquire physical meaning when using this group to transform the four-dimensional spacetime defined in euclidean space:

$$(R, v, a, b) : \begin{pmatrix} x \\ t \end{pmatrix} \to \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} Rx + vt + a \\ t + b \end{pmatrix} \tag{9}$$

In this context, R becomes a rotation in three-dimensional real space, v a galilean boost, and a and b a space and a time translation respectively.

The most important feature of this group in what concerns this report is that, given how it encloses all coordinate transformations possible for euclidean space, it constitutes the spacetime symmetry group of nonrelativistic quantum mechanics. That way, similarly to the role of the Poincaré group in the relativistic quantum field theory, whenever a nonrelativistic quantum theory is formulated, galilean symmetry is sought, which means that if a given quantum state is physically possible, any state resulting from a galilean transformation acting on that state should still be allowed by the physics of the system.

As we did before for the Poincaré group, we introduce a set of generators for the Lie algebra of the galilean group and their commutation relations given the use that we will give them later in this report.

$$\begin{array}{ccc} [J_i, J_j] = i\epsilon_{ijk}J_k & [J_i, K_j] = i\epsilon_{ijk}K_k & [J_i, P_j] = i\epsilon_{ijk}P_k \\ [K_i, K_j] = 0 & [P_i, P_j] = 0 & [K_i, P_j] = 0 \\ [J_i, H] = 0 & [P_i, H] = 0 & [K_i, H] = iP_i \end{array} \tag{10}$$

where J_i are rotation generators, K_i are boost generators, P_i are space translation generators and H is the time translation generator. In this case we need to necessarily separate the boosts from the rotations and the space translations from the time translations, since now the time and space coordinates no longer lie on equal footing.

A final property of this group which is convenient to introduce now is the way the different generators transform quantum states which depend on the space coordinates x and the time coordinate t, which can be found by using the definition of the generators:

$$J_i f(x,t) = i\frac{\partial}{\partial \theta}\left(\hat{U}(R_i(\theta), 0, 0, b) f\right)\big|_e (x,t) = -i\epsilon_{ijk}x^j \frac{\partial}{\partial x^k} f(x,t) \tag{11}$$

$$K_i f(x,t) = i\frac{\partial}{\partial v_i}\left(\hat{U}(I, v_i, 0, 0) f\right)\big|_e (x,t) = -it\frac{\partial}{\partial x^i} f(x,t) \tag{12}$$

$$P_i f(x,t) = i\frac{\partial}{\partial a_i}\left(\hat{U}(I, 0, a_i, 0) f\right)\big|_e (x,t) = -i\frac{\partial}{\partial x^i} f(x,t) \tag{13}$$

$$Hf(x,t) = i\frac{\partial}{\partial b}\left(\hat{U}(I,0,0,b)f\right)\big|_e(x,t) = -i\frac{\partial}{\partial t}f(x,t) \tag{14}$$

where $\hat{U}(R,v,a,b)$ is the representation of the elements of the galilean group when acting on functions $f(x,t)$ and the derivatives are evaluated for the identity element e of the group. If we define the functions to be quantum mechanical states in the Hilbert space $L^2(\mathbb{R})$, then the representation will consist of unitary operators.

2.3 Scattering theory

The objective of the scattering theory is to provide a way to study how the interactions between systems of particles are carried out in the context of the quantum field theory. In order to do that, the theory analyzes the system at times infinitely distant in the past and in the future from the instant when the interaction takes place, in which we have an 'out' state or an 'in' state respectively. At those times, the particles are supposed to be so far apart that they can be seen as non-interacting, so we can always express the states as a direct product of one-particle states that keeps the same structure and conserves individual masses and spins when transformed by the Poincaré group. These one-particle states are defined in a Hilbert space $\mathcal{H}^{(1)}$ and during the whole report we will only consider those which have their four-momentum, their spin z-component and their particle type fixed, although generally in nature we can only find continuous superpositions of these momentum and spin eigenstates. Taking this into account, the general multiparticle Hilbert space can be formulated as

$$\mathcal{H} = \bigoplus_n \mathcal{H}^{(n)} \tag{15}$$

where

$$\mathcal{H}^{(n)} = \bigotimes^n \mathcal{H}^{(1)} \tag{16}$$

In order to study interactions through these simple states, we introduce the scattering matrix, or S-matrix. The elements of this matrix are defined as:

$$S_{\beta\alpha} = (\phi_\alpha, \phi_\beta) \tag{17}$$

where ϕ_α is the wavefunction of a multi-particle 'out' state and ϕ_β is the wavefunction of a multi-particle 'in' state, and the indices α and β comprise all the information about the four-momentum, spin and particle type of every individual particle. Here we can see clearly how the 'in' and 'out' states share the same space, since 'in' states can be seen to be composed by a combination of 'out' states and vice versa, and also how the S-matrix contains the information about all the possible outcomes of a specific interaction given its initial state.

In this formulation of this S-matrix there is a detail that we should emphasize. Even though the 'in' and 'out' states are formulated as non-interacting, this is only an approximation, otherwise we could not extract from a given 'in' state the probability for it to reach several 'out' states only through a scalar product. If there were no interaction whatsoever, the elements of the S-matrix as expressed in (6) would just be $S_{\beta\alpha} = \delta(\alpha - \beta)$. We can circunvent this issue and work with purely non-interacting states, which we will call ψ_α, ψ_β, and defining an S-operator such that its matrix elements between free-particle states are equal to the corresponding elements of the S-matrix:

$$S_{\beta\alpha} \equiv (\psi_\alpha, S\psi_\beta) \tag{18}$$

This S-operator and the purely non-interacting states it transforms will be the objects that we will work with in our discussion on the proof and the nonrelativistic limit.

Let us now go on and explain what constitutes a symmetry group of the scattering matrix, since we will work with this concept during the majority of the proof. A symmetry transformation will be a unitary operator that is able to perform variations on 'in' and 'out' states such that the S-matrix element that is built through the transformed fields is equal to the one that they defined before the action of the symmetry. Because of this condition, they will need to verify the following properties:

-It turns one-particle states into one-particle states.

-It acts on many-particle states as if they were tensor products of one-particle states.

-It commutes with S.

One of the most important groups that will constitute a symmetry of the S-matrix is the Poincaré group, since its presence will heavily condition the structure of the S-matrix and the flow of the proof overall. For instance, its existence as a symmetry implies that the scattering will preserve the total four-momentum of the system, which allows us to rewrite the S-matrix as:

$$S = 1 - i2\pi\delta^4(P_\mu - P'_\mu)T \tag{19}$$

where 1 is the identity matrix, T is the scattering amplitude, and P_μ and P'_μ are the four-momenta operators for the 'out' and 'in' states respectively. Additionally, the Poincaré symmetry partly motivates the labelling that we applied to our one-particle states, since we know well how the four-momentum and spin z-components change under these transformations.

Another type of symmetries that we will encounter as symmetries for the scattering matrix are the internal symmetries, defined in this context by the fact that they commute with the Poincaré group. The most important practical consequence of this is that these transformations will only affect the particle types when acting on the states.

Now that we have introduced the most fundamental concepts that we will work with when discussing the Coleman-Mandula theorem, we are now ready to formulate its statement and develop its subsequent proof.

3 Coleman-Mandula Theorem

3.1 Statement

Let G be a symmetry group of the S matrix, and let the following set of assumptions hold:

-Assumption 1: Particle finiteness. For any finite mass M there will be a finite number of particle types with mass less than M.

-Assumption 2: Weak elastic analyticity. For each possible set of particles, elastic scattering amplitudes are analytic functions of center-of-mass energy, s, and invariant momentum transfer, t, in some neighborhood of the physical region, except at normal thresholds.

-Assumption 3: Ocurrence of scattering. Let $|p>$ and $|p'>$ be any two one-particle momentum eigenstates, and let $|p, p'>$ be the two-particle state made from these. Then:

$$T|p, p'> \neq 0 \tag{20}$$

except perhaps for certain isolated values of s.

-Assumption 4: Lorentz invariance. G contains a subgroup locally isomorphic to the generators P of the Poincaré group.

-Assumption 5: Technical assumption. The generators of the group G when acting on states on momentum space have distributions for their kernels.

Then, the generators of the symmetry group must be locally isomorphic to a direct product of the generators of the Poincaré group and those of an internal symmetry.

3.2 Proof

3.2.1 Step 1: Proof for the subalgebra B_α consisting of symmetry generators which commute with the momentum operator P_μ.

Step 1.1. General remarks and motivation of a theorem. We will study these generators by examining how they act on multiparticle states where each particle has fixed its four-momentum p, its spin z-component and its particle type. We will label jointly these two last properties with the letters m, n, etc. Taking into account the remarks about the symmetry group, and the fact that the momentum of the states remains unchanged under the action of this algebra, we define now how a given generator B_α acts on one of these states:

$$B_\alpha|pm, qn, ... >= \sum_{m'}(b_\alpha(p))_{m'm}|pm', qn, ... > + \sum_{n'}(b_\alpha(q))_{n'n}|pm, qn', ... > +... \tag{21}$$

where every $(b_\alpha(p))$ will be a hermitian matrix. These matrices will always be finite-dimensional if we take into account Assumption 1. That way, since the mass of each of the particles $\sqrt{p^\mu p_\mu}$ is preserved in the transformation, there will always be a finite number of indices m' that we will be able to consider. In order to progress further with our proof, we want to apply the following theorem:

-Theorem 1. Any Lie algebra of finite hermitian matrices must be a direct sum of a compact semi-simple Lie algebra[1] and U(1) algebras.

which will deny the possibility of a continuous range of masses for the particles.

Step 1.2. Necessity of an isomorphism. A possible way to implement Theorem 1 may be by considering a map that is inherent to the definition that we made of the action of the generators B_α:

$$B_\alpha \to b_\alpha(p) \tag{22}$$

These $b_\alpha(p)$ will constitute a Lie algebra if the B_α form one as well, fact that we can see if we consider the Lie relation in the defining representation:

$$[B_\alpha, B_\beta] = i\sum_\gamma C^\gamma_{\alpha\beta} B_\gamma \tag{23}$$

and we apply it on a one-particle state:

$$\begin{gathered}[B_\alpha, B_\beta]|pm> = i\sum_\gamma C^\gamma_{\alpha\beta}B_\gamma|pm> = i\sum_\gamma C^\gamma_{\alpha\beta}\sum_{m'}(b_\gamma(p))_{m'm}|pm'> \Rightarrow \\ \Rightarrow [b_\alpha(p), b_\beta(p)] = i\sum_\gamma C^\gamma_{\alpha\beta} b_\gamma(p)\end{gathered} \tag{24}$$

This same reasoning applies if we consider a mapping between B_α and a representation of the group acting on a multiparticle state. It may seem that we are now ready to use our theorem, but there is still the matter about whether the map we defined is isomorphic. So far we have determined that it is a homomorphism, otherwise we would not have been able to build equation (24), but we may find instances of degeneracy. For example, we may stumble into:

[1]-Semi-simple Lie algebras: Direct sum of simple Lie algebras, i.e., non-abelian Lie algebras g whose only ideals are the identity element and the alagebra itself (an ideal of a Lie algebra is a subset $i \subseteq g$ such that $[i, g] \subseteq i$).

$$b_\alpha(p) = b_\beta(p), \alpha \neq \beta \tag{25}$$

In order to generally prevent this from happening, we need to ensure that the following restriction is held:

$$\sum_\alpha c^\alpha b_\alpha(p) = 0 \Rightarrow \sum_\alpha c^\alpha B_\alpha = 0 \Rightarrow \sum_\alpha c^\alpha b_\alpha(k) = 0 \quad \forall k \tag{26}$$

That way, we impose that, if any degeneracy appears in a given representation, it must be present too in the original representation, and by extension in every other representation that we may consider for any combination of momenta. This condition can also be interpreted so that if there exist some coefficients c^α such that the $b_\alpha(p)$ are not linearly independent, then also the B_α are not linearly independent.

Step 1.3. Solution to the degeneracy problem by a redefinition of the mapping. In order to get a valid isomorphism, we will first examine symmetry generators for two-particle states:

$$(b_\alpha(p,q))_{m'n',mn} = (b_\alpha(p))_{m'm}\delta_{n'n} + (b_\alpha(q))_{n'n}\delta_{m'm} \tag{27}$$

constructed from the structure introduced in equation (21). Now, we use the fact that the symmetry generators commute with the scattering matrix to get a concrete condition for an elastic 2-2 scattering where two particles with momenta p, q end up with momenta p' and q' respectively:

$$< p'm', q'n'|[B_\alpha, S]|pm, qn >= 0 \Rightarrow ... \Rightarrow b_\alpha(p', q')T(p', q'; p, q) = T(p', q'; p, q)b_\alpha(p, q) \tag{28}$$

where we need to impose that $p' + q' = p + q$, $p^2 = p'^2$ and $q^2 = q'^2$ due to the scattering being elastic and the commutation of B_α with the four-momentum generators, and $T(p', q'; p, q)$ is the scattering amplitude matrix that contains information about all the states of two particle with momenta p' and q' into which any state of two particles with momenta p and q can scatter to. This matrix will have the same dimensionality as $b_\alpha(p, q)$, and it was generally defined in equation (19) under the condition that the S-matrix was Lorentz invariant.

We now need to make use of Assumption 2 and Assumption 3. These two conditions together ensure that $T(p', q'; p, q)$ is invertible for almost all posssible combinations of momenta, and reveal equation (28) as a similarity transformation between the representations of symmetry generators labelled by the two different sets of momenta (p, q) and (p', q'). Then, it follows that:

$$\sum_\alpha c^\alpha b_\alpha(p, q) = 0 \Rightarrow \sum_\alpha c^\alpha b_\alpha(p', q') = 0 \tag{29}$$

for all (p', q') in the same mass shell as (p, q). This is still not sufficient to claim the mapping between B_α and $b_\alpha(p, q)$ as isomorphic, we have just proven the restriction (26) for the two-particle representations that share mass shell. In fact, the only thing that (29) implies for one-particle states is that:

$$\sum_{\alpha} c^{\alpha} b_{\alpha}(p') \ , \ \sum_{\alpha} c^{\alpha} b_{\alpha}(p') \ \propto \ 1 \tag{30}$$

In order to solve this issue, we will need to define a new set of symmetry generators. Before we do that, it is convenient to introduce a property that is a direct consequence of the existence of the similarity transformation:

$$Tr(b_{\alpha}(p,q)) = Tr(b_{\alpha}(p',q')) \tag{31}$$

This expression can be expanded if we substitute the $b_{\alpha}(p,q)$ using equation (27):

$$N(\sqrt{q^{\mu}q_{\mu}})tr(b_{\alpha}(p')) + N(\sqrt{p^{\mu}p_{\mu}})tr(b_{\alpha}(q')) = N(\sqrt{q^{\mu}q_{\mu}})tr(b_{\alpha}(p)) + N(\sqrt{p^{\mu}p_{\mu}})tr(b_{\alpha}(q)) \tag{32}$$

where $N(m)$ is the number of particle types with mass m and tr indicates a sum over one-particle rather than two-particle labels. In order for this expression to be compatbible with the momentum conservation $p+q = p'+q'$, upon performing a series expansion on this last expression, we can conclude that:

$$\frac{tr(b_{\alpha}(p))}{N(\sqrt{p^{\mu}p_{\mu}})} = a^{\mu}_{\alpha} p_{\mu} \tag{33}$$

With this in mind, we may now introduce a new set of symmetry generators:

$$B^{\#}_{\alpha} \equiv B_{\alpha} - a^{\mu}_{\alpha} P_{\mu} \tag{34}$$

which upon acting on one-particle states take the form of traceless matrices:

$$(b^{\#}_{\alpha}(p))_{n'n} = (b_{\alpha}(p))_{n'n} - \frac{tr(b_{\alpha}(p))}{N(\sqrt{p^{\mu}p_{\mu}})} \delta_{n'n} \tag{35}$$

To show that they are a set of symmetry generators of their own, we formulate their commutators, using the fact that P_{μ} commutes with B_{α}:

$$[B^{\#}_{\alpha}, B^{\#}_{\beta}] = i \sum_{\gamma} C^{\gamma}_{\alpha\beta} B_{\gamma} = i \sum_{\gamma} C^{\gamma}_{\alpha\beta} [B^{\#}_{\gamma} + a^{\mu}_{\gamma} P_{\mu}] \tag{36}$$

$$[b^{\#}_{\alpha}(p), b^{\#}_{\beta}(p)] = i \sum_{\gamma} C^{\gamma}_{\alpha\beta} b_{\gamma}(p) = i \sum_{\gamma} C^{\gamma}_{\alpha\beta} [b^{\#}_{\gamma}(p) + a^{\mu}_{\gamma} p_{\mu}] \tag{37}$$

Now, we use the fact that the commutators of finite hermitian matrices must have zero trace to impose that $\sum C^{\gamma}_{\alpha\beta} a^{\mu}_{\gamma} = 0$, therefore:

$$[B^{\#}_{\alpha}, B^{\#}_{\beta}] = i \sum_{\gamma} C^{\gamma}_{\alpha\beta} B^{\#}_{\gamma} \tag{38}$$

With these new symmetry generators, we can repeat the same procedure that led us to equation (29) and obtain that:

$$\sum_\alpha c^\alpha b_\alpha^\#(p,q) = 0 \Rightarrow \sum_\alpha c^\alpha b_\alpha^\#(p',q') = 0 \tag{39}$$

for almost all p', q' on the same mass shell with $p+q=p'+q'$. It is at this point where the difference between these new symmetry generators and the old B_α becomes important, because now due to the fact that the $B_\alpha^\#$ get mapped to a set of traceless matrices when acting on one-particle states, this last equation implies that (see Appendix A):

$$\sum_\alpha c^\alpha b_\alpha^\#(p') = \sum_\alpha c^\alpha b_\alpha^\#(q') = 0 \tag{40}$$

Finally, we will argue that this condition can be extended to any value of four-momentum k in the mass shell. To do so, we first note that, given the definition of $b_\alpha^\#(p,q)$, equations (39) and (40) implies that:

$$0 = \sum_\alpha c^\alpha b_\alpha^\#(p) = \sum_\alpha c^\alpha b_\alpha^\#(q) = \sum_\alpha c^\alpha b_\alpha^\#(p') = \sum_\alpha c^\alpha b_\alpha^\#(q') \tag{41}$$

By doing this, we can build now a new representation of the symmetry generators for two-particle states $b_\alpha^\#(p,q')$, which will still be linearly dependent through the same coefficients c^α we have been using up to now. Then, considering the elastic scattering $p,q' \to k, p+q'-k$, the similarity transformation leads us to:

$$\sum_\alpha c^\alpha b_\alpha^\#(k) = 0 \tag{42}$$

for every possible value of k in the mass shells of p and q. This freedom to choose k as whatever we want while only considering this weaker mass restriction is guaranteed by the partial freedom that we have on choosing q', since we have to be able to define a p' such that $p+q=p'+q'$.

Now, to extend this condition to every possible mass shell, let us consider a four-momentum k_0 for which $\sum_\alpha c^\alpha b_\alpha^\#(k_0) \neq 0$. If this relation holds, then a scattering process in which particles with momenta k and k_0 acquire momenta k' and k'' respectively would be forbidden for almost all k, k' and k'' by the symmetry that comes out of $\sum_\alpha c^\alpha B_\alpha^\#$, in contradiction with the non-triviality assumption for the S-matrix. To see this, we make act the operator $T\sum_\alpha c^\alpha B_\alpha^\#$ on a two-particle state $|k,k_0>$ where only the four-momenta are specified, and take into account that if $c^\alpha B_\alpha^\#$ annihilates a one-particle state, then there is no non-trivial scattering possible:

$$0 = T\sum_\alpha c^\alpha B_\alpha^\# |k,k_0>= c^\alpha B_\alpha^\# T|k,k_0> \Rightarrow T|k,k_0>= 0 \tag{43}$$

This contradiction leads us inevitably to the condition that $c^\alpha b_\alpha^\#(k) = 0$ for any k in any mass shell. This condition gets trivially extended to the representations of the group B_α that act on states with more than two particles, given the way they are built using expression (21). Therefore, we can conclude that there exists an isomorphism between B_α and $b^{\#}_\alpha(p,q)$. A direct consequence of this is that, since the number of independent matrices $b^{\#}_\alpha(p,q)$ cannot exceed $N(\sqrt{p_\mu p^\mu})N(\sqrt{q_\mu q^\mu})$ there can only be a finite number of independent generators B_α.

We are now finally ready to implement Theorem 1 to study the structure of the subalgebra B_α, which we know now is a direct sum of a compact semi-simple Lie algebra and U(1) Lie algebras. We will look at these two structures separately.

Step 1.4. U(1) Lie algebras. Let us consider a Lorentz generator J that leaves upon acting on a two-particle state $|pm,qn>$ both p and q invariant. We can always find one of those by using a rotation in the center of mass frame for the interacting particles using as axis the common direction of their momenta. Then, in the basis we have been using up to now for the states we find that:

$$J|pm,qn>=\sigma(m,n)|pm,qn> \tag{44}$$

Now, we take into account the two following facts:

$$\begin{cases} [P_\mu, B^{\#}_\alpha]=0 \\ \\ [J,P_\mu] \ \propto \ P_\mu \end{cases} \tag{45}$$

together with the Jacobi identity to find that:

$$[P_\mu,[J,B^{\#}_\alpha]]=0 \Rightarrow [J,B^{\#}_\alpha] \ \propto \ B^{\#}_\beta \tag{46}$$

given the tracelessness of the commutator of finite hermitian matrices. But any U(1) generator $B^{\#}_\alpha$ has to commute with all other elements of the whole subalgebra, so:

$$[B^{\#}_i,[J,B^{\#}_i]]=0 \tag{47}$$

We now take the expectation value of this commutator in a state $|pm,qn>$ to find that:

$$0=<pm,qn|[B^{\#}_i,[J,B^{\#}_i]]|pm,qn> \Rightarrow \sum_{m',n'}(\sigma(m',n')-\sigma(m,n))\big|(b^{\#}_i(p,q))\big|^2=0 \tag{48}$$

Analyzing this expression, if we consider the lowest value of $\sigma(m,n)$ possible, then the fact that all other $\sigma(m',n')$ are greater than this eigenvalue would imply that the sum is positive. The only remedy for this is to conclude that $(b^{\#}_i(p,q))_{m'n',mn}=0$ for all m',n' for which $\sigma(m',n')\neq\sigma(m,n)$. Then, if we go on to the second lowest $\sigma(m'',n'')$ and consider the same sum again, we notice that the negative terms get terminated due to the matrix element of the symmetry generator being zero, so we get to the same conclusion for this pair of indices m'', n''. Doing this repeatedly, we get to the conclusion that $(b_i^{\#}(p,q))_{m'n',mn} = 0$ for all m', n', m, n for which $\sigma(m', n') \neq \sigma(m, n)$. Now, if we calculate the action of the commutator on a two-particle state, we find that:

$$\begin{aligned}
[B_i^{\#}, J]|pm, qn> &= \sum_{m'n'}((b_i^{\#}(p,q))_{m'n',mn}\sigma(m,n) - J(b_i^{\#}(p,q))_{m'n',mn})|pm', qn'> = \\
&= \sum_{m'n'}((b_i^{\#}(p,q))_{m'n',mn}\sigma(m,n) - (b_i^{\#}(p,q))_{m'n',mn}\sigma(m',n'))|pm', qn'> = 0
\end{aligned} \tag{49}$$

Then, since $B_i^{\#}$ is isomorphic to $b_i^{\#}(p,q)$, we find that:

$$[B_i^{\#}, J] = 0 \tag{50}$$

We can extend this result to every generator J_μ of rotations in the Lorentz group, since it is always possible for any element of the algebra to find a pair of momenta p, q such that they remain invariant before the action of said generator. Then, we have that:

$$[B_i^{\#}, J^1] = [B_i^{\#}, J^2] = [B_i^{\#}, J^3] = 0 \tag{51}$$

From this result we can now deduce that the boost generators K_μ also commute with the symmetry generators $B_i^{\#}$. To do so, we start from the following Lorentz algebra property:

$$i\epsilon_{ijk}J^k = [K_i, K_j] \tag{52}$$

which we can insert in equation (50) to get that

$$i\epsilon_{ijk}[B_\alpha^{\#}, J^k] = [B_\alpha^{\#}, [K_i, K_j]] = 0 \tag{53}$$

This expression, with the help of the Jacobi identity, leads us to:

$$[K_i, [K_j, B_\alpha^{\#}]] - [K_j, [K_i, B_\alpha^{\#}]] = 0 \tag{54}$$

We can now expand these commutation relations and, through cancelation of terms, get that:

$$[B_\alpha^{\#}, K_iK_j] + [B_\alpha^{\#}, K_jK_i] = 0 \tag{55}$$

But, given equation (53), we can argue that:

$$[B_\alpha^{\#}, K_iK_j] = 0 \tag{56}$$

We now work with the quantity $B_\alpha^{\#}K_iK_j$:

$$\begin{aligned} B_\alpha^\# K_i K_j = K_i B_\alpha^\# K_j - [K_i, B_\alpha^\#] K_j = K_i B_\alpha^\# K_j - [K_i, B_\alpha^\#] K_j = \\ = K_i K_j B_\alpha^\# - K_i [B_\alpha^\#, K_j] - [K_i, B_\alpha^\#] K_j = K_i K_j B_\alpha^\# \end{aligned} \tag{57}$$

where we used equation (56) in the last equality. Consequently:

$$K_i [B_\alpha^\#, K_j] = -[K_i, B_\alpha^\#] K_j \Rightarrow [K_j, B_\alpha^\#] = -K_i^{-1} [K_i, B_\alpha^\#] K_j \tag{58}$$

Let us know consider this last expression and the same but with the indices i and j exchanged. Substituting one into the other, we get that:

$$[K_i, B_\alpha^\#] = K_i K_j [K_i, B_\alpha^\#] K_i^{-1} K_j^{-1} \tag{59}$$

The right-hand side of this equation is generally not traceless, but the left-side must have zero trace. The only solution to this is that the commutator goes to zero, that is:

$$[K_1, B_i^\#] = [K_2, B_i^\#] = [K_3, B_i^\#] = 0 \tag{60}$$

Therefore, we conclude that all generators $B_i^\#$ commute with the whole Lorentz group, and hence with the entire Poincaré group. As a consequence of this, the subalgebra $\{B_\alpha^\#\}$ must consist only of internal symmetries.

Step 1.5. Semi-simple compact Lie algebras. Our objective in this step is to proove that every possible semi-simple compact Lie algebra that we may include among the symmetry generators must commute with the elements of the Poincaré group, so that it can only be an internal symmetry. To do so, let us return to the previous generators B_α, composed in this case by a linear combination of momentum operators P_μ and some finite-parameter semi-simple compact Lie algebra. Considering the representation of these generators when they act on the Hilbert space for an arbitrary number n of particles, let us represent a Lorentz transformation acting on that same space using the unitary operator $U(\Lambda)$. With it, we can define a new set of hermitian operators $\{U(\Lambda) B_\alpha U^{-1}(\Lambda)\}$, which can be seen to commute with the operator $\Lambda^\nu{}_\mu P_\nu$ if we apply a Lorentz transformation to the commutation relation:

$$\Lambda : [B_\alpha, P_\mu] = 0 \to [U(\Lambda) B_\alpha U^{-1}(\Lambda), U(\Lambda) P_\mu U^{-1}(\Lambda)] = [U(\Lambda) B_\alpha U^{-1}(\Lambda), \Lambda^\nu{}_\mu P_\nu] = 0 \tag{61}$$

where the last step can be understood if we note that the Lorentz transformation physically only imply a change in the frame of reference, so the momentum operator can only change to be a different combination of its components. These $\Lambda^\nu{}_\mu P_\nu$ commute themselves with P_μ, given that $\Lambda^\nu{}_\mu$ is non-singular. Therefore, $U(\Lambda) B_\alpha U^{-1}(\Lambda)$ will commute with the four-momentum operators, and thus:

$$U(\Lambda)B_\alpha U^{-1}(\Lambda) = \sum_\beta D^\beta{}_\alpha(\Lambda)B_\beta \tag{62}$$

where $D^\beta{}_\alpha$ are a set of coefficients that yield a representation of the Lorentz group. We will now proove that this representation can only be the trivial one, implying the commutation relation we are looking for. In order to do that, we contract the structure constants $C^\gamma_{\alpha\beta}$ introduced in equation (12) with $C^\alpha_{\gamma\delta}$ to find the Lie algebra metric $g_{\beta\gamma}$:

$$g_{\beta\gamma} = \sum_{\alpha\gamma} C^\gamma_{\alpha\beta} C^\alpha_{\gamma\delta} \tag{63}$$

Since all of these generators commute with P_μ, we have that $C^\alpha_{\mu\beta} = -C^\alpha_{\beta\mu} = 0$, so $g_{\mu\alpha} = g_{\alpha\mu}$. Therefore, we can omit the contribution of the momenta operators to the subalgebra when working with its metric, recovering our previous operators $B^\#_\alpha$. Now, since we are working with a semi-simple compact hermitian Lie algebra, this metric has to be positive-definite, which allows us to define a finite-dimensional, real, orthogonal, and therefore unitary, representation of the Lorentz group, namely $g^{1/2}D(\Lambda)g^{-1/2}$. The finite-dimensional property stems from the consequence from step 1.3 that there are a finite number of independent generators B_α, which ensures that both $g_{\beta\gamma}$ and D^β_α run over a finite set of values for their indices. However, the Lorentz group is simple and non-compact, so the only unitary and finite-dimensional representation for it must be the trivial one, $D(\Lambda) = 1$. Then:

$$[B_\alpha, U(\Lambda)] = 0 \tag{64}$$

Therefore, since $B^\#_\alpha$ by definition also commutes with the translation generators, it must be an internal symmetry. Bringing this together with what we learned in step 1.4, we conclude that every generator B_α must be an internal symmetry, and so the generators of the symmetry group G which commute with the momentum operator must always be a linear combination of internal symmetries and the same momentum operators P_μ.

3.2.2 Step 2: Proof for the general set of generators A_α.

Step 2.1. Delimitation on the form of the symmetry group through the use of the scattering assumptions. We first formulate how a general generator acts on a one-particle state $|pn>$:

$$A_\alpha|pm> = \sum_{m'} \int d^4p(\mathcal{A}_\alpha(p',p))_{m'm}|p'm'> \tag{65}$$

We will now impose a heavy delimitation on the kernels $\mathcal{A}_\alpha(p',p)$. To do that, we will need Assumption 4. If we use it, we can argue that if A_α is a generator of our symmetry then so is $U(\Lambda,a)^\dagger A_\alpha U(\Lambda,a)$, where $U(\Lambda,a)$ is the unitary operator representing an element of the Poincaré group, being Λ a label for the Lorentz transformation and a a label for the translation. To see why this is true, it is easy to check that $U(\Lambda,a)^\dagger exp(c^\alpha A_\alpha)U(\Lambda,a) = exp(c^\alpha U(\Lambda,a)^\dagger A_\alpha U(\Lambda,a))$. This is not the case because we are using in particular the Poincaré group to build the generator, we can build them this way using any element of the symmetry group. Then, we can build a new generator as:

$$\int d^4 a U(1,a)^\dagger A_\alpha U(1,a)\tilde{f}(a) = f A_\alpha \tag{66}$$

where f is a test function with support in a region R in momentum space and $\tilde{f}$ is its Fourier transform. The right-hand side comes out naturally if we consider the action of the generator on one-particle states:

$$U(1,a)|pm> = e^{-ipa}|pm> \tag{67}$$

$$\begin{aligned} <p'm'|\int d^4 a U(1,a)^\dagger A_\alpha U(1,a)\tilde{f}(a)|pm> &= <p'm'|\int d^4 a e^{-ia(p-p')}A_\alpha \tilde{f}(a)|pm> = \\ &= f(p-p')A(p'm',pm) \end{aligned} \tag{68}$$

This implies that this new generator will only be able to connect states whose momenta can be connected by a vector in R. Now, we introduce a consequence of the particle finiteness assumption, which is that the support of our states is limited to a countable set of hyperboloids in the momentum space. Then, if we define R to be sufficiently small, there will be physically possible one-particle states with such a p that, after being acted upon by fA_α, will acquire a momentum which lies outside of any of the hyperboloids. Therefore, fA_α must annihilate all the states of this kind.

Now let us choose a momentum p whose associated state would not be annhilated by fA_α and three other physical momenta q, p', q' that would, with the only restriction that $p + q = p' + q'$. These four variables will fix some values for the center of mass energy and the invariant momentum transfer:

$$s = (p+q)^2 \ ; \ t = (p-p')^2 \tag{69}$$

which are the variables that we use for the scattering amplitudes. Taking into account that fA_α will commute with the S-matrix, we do the following calculation:

$$0 = S^\dagger f A_\alpha |p',q'> = fA_\alpha S^\dagger |p',q'> = fA_\alpha |p,q> \neq 0 \tag{70}$$

where we consider only the scattering between two states $|p,q>$ and $|p',q'>$ where only the momenta are specified. The only way to solve this contradiction is to conclude that the element $S(p,q;p',q')$ of the scatttering matrix has to be zero. This means that the symmetry fA_α forbids a scattering process to have the kinematics $p,q \rightarrow p',q'$.

But now we can choose an infinitesimally different configuration and change continuously the different values for the momenta, arriving at the same conclusion using this previous argument. Then, we conclude that the scattering amplitude vanishes over a range of its variables, so according to the analyticity assumption this implies that the scattering amplitude needs to be trivially zero. However, this is in conflict with the nontriviality assumption, so we are apparently facing a contradiction. We can repeat this argument for every possible pair of particle types α' and β', even in the case $\alpha' = \beta'$.

In order to circunvent this problem, we may think that the generators need to commute with the momentum operators P_μ. Indeed, if we impose this restriction:

$$\begin{aligned}\int d^4a\, U(1,a)^\dagger A_\alpha U(1,a)\tilde{f}(a)|pm> &= \int d^4a\, exp(iPa)A_\alpha exp(-iPa)\tilde{f}(a)|pm> \\ &= \left(\int d^4a\tilde{f}(a)\right)A_\alpha|pm> \propto A_\alpha|pm>\end{aligned} \tag{71}$$

We lose the capacity to define a symmetry relation between a given kinematic configuration of the scattering and another where we are forced to consider momenta outside of the mass hyperboloids. However, by doing this we just recover the previous generators $\{B_\alpha\}$ we have already studied. We can define a much looser condition by considering for instance the following form for the kernels of our generators:

$$\left(\mathcal{A}^0_\alpha(p,p')\right)_{m',m} = \delta^4(p-p')\left(a^0_\alpha(p,p')\right)_{m',m} \tag{72}$$

Then, if we act on a one-particle state with the associated operator fA^0_α:

$$\begin{aligned}fA^0_\alpha|pm> &= \sum_{m'}\int d^4p'\tilde{f}(p-p')\mathcal{A}^0(p',p)_{m'm}|p'm'> \\ &= \sum_{m'}\int d^4p'\tilde{f}(p-p')\delta^4(p-p')\left(a^0_\alpha(p,p')\right)_{m',m}|p'm'> = \tilde{f}(0)\sum_{m'}\left(a^0_\alpha(p)\right)_{m',m}|pm'>\end{aligned} \tag{73}$$

we observe that the function f no longer depends on the momentum, so through this kind of dependence for the kernels we lose again the capacity of changing the momentum of the states through the symmetry transformation fA_α, so we avert the contradiction using this kind of kernels. However, since these symmetry generators effectively do not change the four-momentum of the states they act on, they will also be included in the subalgebra $\{B_\alpha\}$. Despite this recurring conclusion, the delta dependence proposed for the kernels may have given us a hint for the most general functional dependence of the generators A_α. In order to explore this possibility, let us consider the following generator:

$$\left(\mathcal{A}^1_\alpha(p,p')\right)_{m',m} = \left(a^1_\alpha(p,p')\right)^\mu_{m',m}\frac{\partial}{\partial p^\mu}\delta^4(p-p') \tag{74}$$

and take into account the existence of this property for the deltas:

$$\int d^4p' f^\mu(p)\frac{\partial}{\partial p'^\mu}\delta^4(p-p') = -\int d^4p'\delta^4(p-p')\frac{\partial}{\partial p'^\mu}f^\mu(p) \tag{75}$$

Then, upon computing the action on a one-particle state of its associated operator fA^1_α, we obtain that:

$$\begin{aligned} fA^1_\alpha\,|pm> &= \sum_{m'} \int d^4p' \tilde{f}(p-p')\left(a^1(p,p')\right)^{\mu_1}_{m',m} \frac{\partial}{\partial p'^{\mu_1}} \delta^4(p-p')|p'm'> = \\ &= -\sum_{m'} \int d^4p' \frac{\partial}{\partial p'^{\mu_1}} \left(\tilde{f}(p-p')\left(a^1(p,p')\right)^{\mu_1}_{m',m}\right) \delta^4(p-p')|p'm'> = \\ &= -\sum_{m'} \frac{\partial}{\partial p^{\mu_1}} \left(\tilde{f}(0)\left(a^1_\alpha(p)\right)^{\mu_1}_{m',m} |pm'>\right) = -\tilde{f}(0) \sum_{m'} \frac{\partial}{\partial p^{\mu_1}} \left(\left(a^1_\alpha(p)\right)^{\mu_1}_{m',m} |pm'>\right) \end{aligned} \tag{76}$$

As we see, the function f again has stopped depending on the four-momentum, but this time we are not forbiding the change of the kinematic variables, so we have found a generator which is outside the subalgebra $\{B_\alpha\}$ which also dodges the contradiction instigated by the relation of the operator fA_α with the scattering assumptions for a suitable choice of f. We can check that this condition is maintained if we formulate the kernels to depend on higher order derivatives of the deltas so the most general generators for the symmetry group of the S-matrix will be of the form:

$$\begin{aligned} (\mathcal{A}_\alpha(p,p'))_{n'n} &= (a^0_\alpha(p,p'))_{n'n}\delta^4(p-p') + (a^1_\alpha(p,p'))^{\mu_1}_{n'n} \frac{\partial}{\partial p'^{\mu_1}} \delta^4(p-p') + \\ &+ (a^2_\alpha(p,p'))^{\mu_1\mu_2}_{n'n} \frac{\partial^2}{\partial p'^{\mu_1} p'^{\mu_2}} \delta^4(p-p') + ... + (a^{D_\alpha}_\alpha(p,p'))^{\mu_1\mu_2...\mu_{D_\alpha}}_{n'n} \frac{\partial^{D_\alpha}}{\partial p'^{\mu_1} p'^{\mu_2} ... p'^{\mu_{D_\alpha}}} \delta^4(p-p') \end{aligned} \tag{77}$$

which, if we make them act on one-particle states, result in:

$$A_\alpha|pn> = \sum_{n'} \left((a^0_\alpha(p))_{n'n} + (a^1_\alpha(p))^{\mu_1}_{n'n} \frac{\partial}{\partial p^{\mu_1}} + ... + (a^{D_\alpha}_\alpha(p))^{\mu_1...\mu_{D_\alpha}}_{n'n} \frac{\partial^{D_\alpha}}{\partial p^{\mu_1} ... p^{\mu_{D_\alpha}}} \right) |pn'> \tag{78}$$

where the coefficients $a^i_\alpha(p)$ are different from those in equation (77) due to the expansion of the derivatives of products. The claim that these generators are the most general ones for our symmetries is only true under Assumption 5, which states that the kernels $\mathcal{A}_\alpha(p,p')$ are distributions and also imposes that the maximum order of the derivatives D_α can never go to infinity.

Step 2.2. Final analysis on the generators A_α. Now that we have delimited quite a bit the form of the A_α, in order to conclude our proof let us first formulate from them some generators which commute with the momentum operators. In order to do that, we take into account that we want to remove the terms with derivatives from this last expression and upon some commutations (see Appendix 2) we can present:

$$B^{\mu_1...\mu_{D_\alpha}}_\alpha \equiv [P^{\mu_1}, [P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...] \tag{79}$$

When these generators act on one-particle states, we can express them by:

$$b^{\mu_1...\mu_{D_\alpha}}_\alpha(p) = b^{\#\mu_1...\mu_{D_\alpha}}_\alpha(p) + a^{\mu\mu_1...\mu_{D_\alpha}}_\alpha p_\mu 1 \tag{80}$$

with both $b_\alpha^{\#\mu_1\ldots\mu_{D_\alpha}}(p)$ and $a_\alpha^{\mu_1\ldots\mu_{D_\alpha}}$ symmetric under exchange of their upper indices (This property is clear if we calculate $< p'm'|B_\alpha^{\mu_1\ldots\mu_{D_\alpha}}|pm >$).

Now, we can argue that any symmetry transformation that we may define will be unable to change the mass of a particle. To do it, we recall the definition that we gave for a symmetry transformation, which is supposed to perform changes on the sets of 'in' and 'out' states such that they leave unaltered the S-matrix that they originally defined. However, if we allowed the mass to change, by particle finiteness we would very likely lose or gain access to some states, so it would be impossible to maintain the same S-matrix just by looking at its dimensionality. Hence, if we define a mass operator $P^\mu P_\mu$, which will have discrete eigenvalues due again to the particle finiteness assumption, we can easily see that

$$[A_\alpha, P^\mu P_\mu] = 0 \tag{81}$$

So it will constitute a Casimir invariant for the symmeytry group. Now, using a particular A_α, namely $A'_\alpha = [P^{\mu_2}, P^{\mu_3}...[P^{\mu_{D_\alpha}}, A_\alpha]]...]$, it follows that for $D_\alpha \geq 1$:

$$\begin{aligned}
[P^{\mu_1}P_{\mu_1}, P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...] &= P_{\mu_1}[P^{\mu_1}, P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...] + [P^{\mu_1}, P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...]P_{\mu_1} \\
&= P_{\mu_1}B_\alpha^{\mu_1\ldots\mu_{D_\alpha}} + B_\alpha^{\mu_1\ldots\mu_{D_\alpha}}P_{\mu_1} = \\
&= 2P_{\mu_1}B_\alpha^{\mu_1\ldots\mu_{D_\alpha}} = 0 \Rightarrow p_{\mu_1}b_\alpha^{\mu_1\ldots\mu_{D_\alpha}}(p) = 0
\end{aligned} \tag{82}$$

Then, given that this is satisfied for every possible p_{μ_1} and p_μ in a timelike direction, we have that:

$$\begin{cases} b_\alpha^{\#\mu_1\ldots\mu_{D_\alpha}} = 0 \\ \\ a_\alpha^{\mu\mu_1\ldots\mu_{D_\alpha}} = -a_\alpha^{\mu_1\mu\ldots\mu_{D_\alpha}} \end{cases} \tag{83}$$

But for $D_\alpha \geq 2$ the symmetry of the indices $\mu, \mu_1, ..., \mu_{D_\alpha}$ requires that $a_\alpha^{\mu\mu_1\ldots\mu_{D_\alpha}} = 0$. Therefore, we have as much that $D_\alpha = 1$, or equivalently:

$$(\mathcal{A}_\alpha(p,p'))_{n'n} = (a_\alpha^0(p,p'))_{n'n}\delta^4(p-p') + (a_\alpha^1(p,p'))_{n'n}^{\mu_1}\frac{\partial}{\partial p'^{\mu_1}}\delta^4(p-p') \tag{84}$$

We will now consider the two only types of generators that we can infer given this feature.

-$D_\alpha = 0$. In this case we recover the symmetry generators that commute with the momentum operators P_μ, which we already know that are restricted to a linear combination of P_μ and internal symmetry generators.

-$D_\alpha = 1$. In this case, we need to take into account that, given equation (84):

$$[P^\nu, A_\alpha] = B_\alpha^\nu = a_\alpha^{\mu\nu}P_\mu \tag{85}$$

Then, considering this property of the Poincaré algebra:

$$[P^{\mu}, J^{\rho\sigma}] = -i\eta^{\nu\rho}P^{\sigma} + i\eta^{\nu\sigma}P^{\rho} \tag{86}$$

and the operator algebra for the Lorentz group (6), where our spacetime coordinates are now the components of the four-momentum, we can identify the first order derivative terms as elements of the Lorentz algebra, and so:

$$A_{\alpha} = -\frac{1}{2}ia_{\alpha}^{\mu\nu}J_{\mu\nu} + B_{\alpha} \tag{87}$$

Therefore, the most general symmetry generator that we can define for the scattering matrix is a linear combination of generators of the Poincaré group and internal symmetries.

4 Non-relativistic limit

Once we have laid out the proof for the Coleman-Mandula theorem on full detail, we will begin trying to adapt it for the non-relativistic limit. To do so, we will present the proof again but in a much more summarized fashion, focusing separately on each step as they were presented in the previous part, disecting where the different arguments and tools that are used in it come from and checking if they can also be used in a non-relativistic context applying some minor modifications if necessary. On our way, we will also point at which conditions or properties can be relaxed or even dropped altogether to leave this new proof and the old one down to its most essential terms.

Before doing that, let us first define some preliminary clarifications for the nonrelativistic proof. First, we will presume that almost all previous assumptions can be translated to this case, but one change we will have to do for sure, given that we want to adapt a theorem formulated for states in Minkowski space to a context where states need to be immersed in euclidean space, is to substitute the Poincaré group for the galilean group everywhere in the discussion. That way, whenever a property of the Poincaré group is implemented, we will need to examine whether there is an analogous property in the galilean group that allows us to proceed with the proof the same way as before. Apart from that, all the properties that by hypothesis we atributed to the symmetry groups in the original proof will also be tranfered to this non-relativistic formulation of the theorem. Finally, the S-matrix defined here will essentially be the same that we defined in the quantum fields framework but it will relate now quantum mechanical multiparticle states that are approximately non-interacting, which are analogous to the 'in' and 'out' states that we formulated in the theory preliminaries looking for that same absence of interaction at infinite times before and after the scattering. Therefore, the statement for the new version of the theorem that we propose is:

Proposal of the statement of the hypothetical nonrelativistic Coleman-Mandula theorem: Let G be a symmetry group of the S-matrix, and let the following set of assumptions hold:

-Assumption 1: Particle finiteness. For any finite mass M there will be a finite number of particle types with mass less than M.

-Assumption 2: Weak elastic analyticity. For each possible set of particles, elastic scattering amplitudes are analytic functions of center-of-mass energy, s, and invariant momentum transfer, t, in some neighborhood of the physical region, except at normal thresholds.

-Assumption 3: Ocurrence of scattering. Let $|p>$ and $|p'>$ be any two one-particle momentum and energy eigenstates, and let $|p, p'>$ be the two-particle state made from these. Then:

$$T|p, p'> \neq 0 \tag{88}$$

except perhaps for certain isolated values of s.

-Assumption 4: Galilean invariance. G contains a subgroup locally isomorphic to the generators of the galilean group.

-Assumption 5: Technical assumption. The generators of the group G when acting on states on momentum space have distributions for their kernels.

Then, the generators of the symmetry group must be locally isomorphic to a direct product of the generators of the Poincaré group and those of an internal symmetry.

With all this in mind, let us begin with the nonrelativistic adaptation.

-Step 1.1. In this step we are just stablishing how the generators of the symmetry group which commute with the momentum operators act on the field eigenstates of momentum, spin and particle type and hinting at a mapping between the defining representation B_α of the symmetry group and that which consists of finite hermitian matrices acting on one-particle states $b_\alpha(p)$. Here there are several matters that we need to address in order to justify that this step can be fully reproduced in a non-relativistic context. First of all, we need to note that, strictly, the states formulated inside the quantum field theory framework and the ones in ordinary quantum mechanics are fundamentally different as mathematical structures. Perhaps the most notable difference between the two is that in quantum field theory the states are typically defined to contain the entire spacetime history of a system of particles, while in non-relativistic quantum mechanics they only provide information about the system in one instant in time. However, this and every other distinguishing properties are of no relevance in what concerns the proof, as we will see as we move forward, so if we just substitute field states for quantum mechanical states all over the proof we will not run into any inconsistency in the original reasonings because of it.

Some other obvious changes that we need to highlight which come as a consequence of this substitution are that the symmetry group will now need to be defined to transform states in a new

Hilbert space where our non-relativistic states reside and that the subgroup that we worked with during the whole step 1 will now need to commute with the momentum operators and the hamiltonian instead; but then again, these details do not force us to reconsider any argument from the original proof. Besides that, in order to continue using equation (21) despite it relating different structures now, we need to ensure that we can define the new states to have their energy, momentum, spin and particle type fixed. This should not be a problem if we focus on states with non-interacting free particles, as we will given the definition that we introduced for the scattering matrix.

A final concern regarding this step that is a little more pressing is that we need to ensure that the matrices that transform the one-particle states are finite-dimensional in order to even wonder to implement Theorem 1. This in the original proof is justified by making use of the particle-finiteness assumption and taking into account that the symmetry generator conserves the mass of the state it acts on. For the non-relativistic limit, Assumption 1 can equivalently be used, and given that the operators B_α now commute with the momentum operators and the hamiltonian, they too won't change the mass of the individual particles, so we can then be sure that $b_\alpha(p)$ have a finite number of dimensions. Therefore, we have succesfully adapted step 1.1 to the non-relativistic context.

-Step 1.2. In this step, with our sights set in Theorem 1, we justify that if the B_α conform a Lie algebra then the $b_\alpha(p)$ will do so too given the defining relation between the two sets and then we reason why the mapping (22) is a homomorphism but not an isomorphism due to the possible ocurrence of degeneracies in the one-particle representation. All these arguments rely entirely on the Lie algebra properties that by hypothesis our set of symmetry generators has, so they can be fully extrapolated to our potential non-relativistic proof.

-Step 1.3. Here we build a new set of symmetry generators $b^{\#}_\alpha(p,q)$ that provides a suitable isomorphism between the defining representation B_α and a set of finite hermitian matrices. To arrive at it, we first define the representation of the operators $b_\alpha(p,q)$ that act on two-particle states. Then, using the assumptions about the analiticity and non-triviality of the scattering matrix we manipulate the null commutation relation between the S-matrix and the symmetry generators to show that two representations $b_\alpha(p,q)$ and $b_\alpha(p',q')$ are connected through a similarity transformation, provided that $p'+q'=p+q$, $p^2=p'^2$ and $q^2=q'^2$, and with the scattering amplitude $T(p,q;p',q')$ as the change of basis matrix. The conditions $p^2=p'^2$ and $q^2=q'^2$ get imposed if we consider the scattering to be elastic.

After that, taking the trace on the equality in (28) that illustrates that same simillarity transformation, we infer that we can turn the matrices $b_\alpha(p,q)$ traceless through substraction of a specific linear combination of the four-momentum components, for which we only need to ensure that four-momentum conservation is held, which is backed by the commutation of B_α and P_μ, and take into account that the number of particles of a given mass is finite due to Assumption 1. This is of interest to us because the set of symmetry generators that can be built with these traceless parts, which we call $b^{\#}_\alpha(p,q)$, will satisfy the condition (15), so it will be connected to the B_α through an isomorphism. In order to affirm with security this final claim, we must go through some aclarations to make sure that equation (26) is satisfied for all values of four-momenta k and not just those inside the mass shell defined by a given p

and q. Besides that, we must also show explicitly that all $b^{\#}_{\alpha}(p,q)$ conform a Lie algebra in order to be able to utilize Theorem 1. However, these two tasks are very technical in nature, and they involve only defining properties of the symmetry group and some algebra notions, so we will skip them here. An interesting consequence of this is that, since the matrices $b^{\#}_{\alpha}(p,q)$ are finite-dimensional, the existence of the isomorphism demands that the number of independent symmetry generators must be equal or lower to that same number of dimensions. Therefore, the subalgebra $\{B_\alpha\}$ is necessarily finite-dimensional.

In what concerns the non-relativistic analysis, we take into account that here the tools that we use come from the properties of the symmetry generators and of an elastic scattering, general algebra and Assumptions 1, 2, and 3. First of all, the algebra tools that are used do not depend in any way in a physical context, hence there is no problem in the parts of the reasoning where they come into play either. Besides that, as we discussed previously, all the assumptions used here are transfered unaltered to the non-relativistic formulation, so in the parts of the reasoning where the first three are used no correction is needed. This same argument is valid for every point where the symmetry generators properties are employed, given how we adapted their definition to the non-relativistic context.

In order to support this last claim, we will analyze two apparent points of conflict that seem to impede us from fully implementing step 1.3 in the non-relativistic proof. The first one is the four-momentum conservation, which we use to construct our traceless symmetry operators by substraction of a specific combination of four-momentum generators. In the non-relativistic case, what we would need to reproduce this argument is the symmetry generator to conserve both energy and momentum, but this is fully ensured due to its assumed commutation with P_i and H, so by doing this small change in the reasoning this issue is immediately solved. In that same substep, it is required that the number of particle types for a given mass is a function of only the energy and the momentum in order to expand the quantity in the left-hand side of equation (33) in terms of those four variables. In the relativistic case, this is comes as true immediately due to the mass being just $m = \sqrt{p^\mu p_\mu}$, and this is also the case for the non-relativistic case if the particles are free and non-interactive, since then $m = p^2/2E$.

The last tool that needs to be adapted is the elastic scattering. In the original proof, the elastic scattering guarantees that the equalities $p^2 = p'^2$ and $q^2 = q'^2$, are satisfied, which is useful because it allows us to keep the masses of the individual particles invariant to the scattering. This attribute can be recovered in our non-relativistic proof by external imposition on the scattering matrix characteristics, in a way that makes perfect sense in a physical way, since individual mass conservation is something that is also observed in classical elastic collisions, so we might as well consider classical elastic scattering in the global context of the non-relativistic proof. Therefore, step 1.3 can be fully reproduced in this framework thanks to the cautionary changes that we made to the theorem upon formulating it.

-Step 1.4. This step starts from the fact that the symmetry generators that commute with the four-momentum are either U(1) Lie algebras or semi-simple compact Lie algebras, and it is dedicated to prooving that the ones that belong to the first category are trivial couplings of momentum generators and internal symmetries. In order to do that, we first consider the action of a Lorentz generator J on a state $|pm, qn>$ such that it is an eigenstate for that operator. We can always define such a pair of state and operator due to the existence of the center of mass frame by using a rotation. Once we have J, by a combination of the Poincaré algebra properties, some general algebra notions and the nature of the symmetry generators $B^{\#}_{\alpha}$, which are the representation of the operators $b^{\#}_{\alpha}(p,q)$ when they act on an unspecified number of particles, we arrive at the conclusion $[B^{\#}_{\alpha}, J] = 0$ for that particular Lorentz generator. However, this same argument can be repeated for any rotation generator if we make it act on the appropriate two-particle state, so this result can be extended for the entire subgroup of rotations. More than that, through a feature of the Lorentz algebra and some algebra reasonings, boosts can also be prooved to commute with the symmetry generators. Therefore, we can conclude that the operators $B^{\#}_{\alpha}$ commute with the whole Poincaré group, hence they are internal symmetries, which leaves the general four-momentum commutating U(1) Lie algebra $\{B_{\alpha}\}$ as a linear combination of translation generators and internal symmetries.

This step cannot be fully implemented in our non-relativistic proof if we use the generators of rotations and boosts that are included in the galilean group instead of the Lorentz generators. In order to see this, let us go one by one through the points where the Lorentz generators and their properties are used. First, we need to check if a two-particle state $|pm, qn>$ in the non-relativistic framework can be defined to be invariant before the action of a boost or a rotation. It is easy to see that we can indeed, we only have to choose the coordinate frame where the two particles are set to collide head on and then perform a rotation with the common direction of their momenta as their axis. Then, in order to conclude that this rotation generator commutes with the $B^{\#}_{\alpha}$, the only property used in the original proof which will have to change for our nonrelativistic analysis is equation (2), but the only relevant information which that expression provides to the proof is that the commutator is equal to a linear combination of linear four-momenta. However, in the galilean algebra we have also the commutation relations given in equations (10), which are too a linear combination of the equivalent of the four-momentum operators in this context, albeit different ones depending if we are looking at the energy or at the momentum operators. Then, given that the other algebra notions and the properties of the $B^{\#}_{\alpha}$ are completely transferable, in the nonrelativistic case we can also conclude that the members of that algebra commute with the rotation generators.

Nevertheless, the problem comes when trying to transfer this conclusion to the galilean boosts too. The reasoning used for this in the original proof relies on the commutator presented in equation (52) being dependent on the rotation generators, but the same commutator for galilean boosts is exactly zero for every combination of generators (see equation (10)), and without such a dependence between the boosts and the rotations we cannot extract any additional information about the commutation relation between the K_i and the $B^{\#}_{\alpha}$, so using this procedure we cannot possibly conclude that the commuting subalgebra when being also a U(1) consists of internal symmetries and energy and momentum operators.

-Step 1.5. Even though we have already concluded that this proof is invalid, let us continue and go on now to discuss step 1.5, which focuses on showing that four-momentum commutating symmetry generators which are also semi-simple compact Lie algebras must yield internal symmetries. To do so, we first set ourselves in the Hilbert space where an arbitrary number of particles are defined and formulate inside of it the operator $U(\Lambda)B_\alpha U^{-1}(\Lambda)$, which will commute with the four-momentum operators P_μ given the properties of B_α. This property allows us to express the operator as a real linear combination of the symmetry generators B_α, where the coefficients will depend on the element Λ of the Lorentz group that we use to define it and will define a new representation $D(\Lambda)$ for the general Lorentz group. This representation needs to be the trivial one, and to support this claim we define a new representation $g^{1/2}D(\Lambda)g^{-1/2}$ of the Lorentz group, where g is the Lie algebra metric of the subalgebra $\{B_\alpha\}$, which turns out to be finite-dimensional, real and orthogonal, and hence unitary. This last statement can be backed by some simple checkings involving elemental algebra, the final conseguence of step 1.3 regarding the number of independent symmetry generators and the status of real matrix of g due to B_α defining a semi-simple compact Lie algebra. However, the Lorentz group is simple and noncompact, so it cannot have any finite unitary representations other than the trivial one. This implies that $D(\Lambda) = 1$, which in turn tells us that the generators B_α commute with the Lorentz group. Therefore, the symmetry generators that commute with the four-momentum and that conform a semi-simple compact Lie algebra must commute with the whole Poincaré group, and hence they must be internal symmetries.

When trying to adapt this step to our non-relativistic proof, we run into an insurmountable obstacle. If we try to follow the same path as before but using as the initial operator $U(L)B_\alpha U^{-1}(L)$, where L is an element of the subgroup of rotations and boosts inside of the galilean group, everything goes fine wherever we need to implement properties of the symmetry generators or general algebra, but we run into a wall when trying to proove that the analogous representation $D(L)$ of the galilean subgroup is the trivial one. This is the case because in order to ensure that we rely on the fact that the Lorentz group cannot have non-trivial finite unitary representations, but this is not the case for the galilean subgroup, given that it is compact. Therefore, an explicit hurdle for the existence of the Coleman-Mandula theorem in the non-relativistic limit is the difference in compactness of the Lorentz group and the galilean subgroup of rotations and boosts.

-Step 2.1. At this point we delimit the structure of a general symmetry generator A_α. To do so, we first define how A_α transforms a one-particle state $|pm>$ and build from it a convenient symmetry generator fA_α inside the algebra with the help of Assumption 4 and a Fourier transform, where f is a test function with a finite domain R in momentum space. The existence of this limited domain implies that if we make it act on a given state with four-momentum p, the transformed state will have a four-momentum p' such that the difference with the original one must fall within R. Then, if we make this operator act on four states with different momenta p, q, p' and q' such that three of them get annihilated and one of them is not, we can argue that we cannot define a two-particle scattering that relates p and q with p' and q' based on the commutation relation between A_α and the S-matrix. This is possible since we can define R to be as small as we want and we have a set of fully separated mass hyperboloids which contain all the physically possible four-momenta. This conclusion can be extended to an infinitesimally small continuous range of momenta, so by the analyticity assumption the scattering amplitude needs to be zero for the specific type of particles α and β whose four-momenta we arbitrarily fixed as p and q. However, this shows a contradiction of the non-triviality assumption when connecting these two particle types. Again, all this reasoning is equivalent for any two particle types, so the inconsistency is present in the most general level. In order to solve it, we need to use Assumption 5 and restrict the functional dependence of the generators when acting on one-particle states so that we cannot use the function f to build symmetries that can only move the four-momentum of the particles out of the mass shells.

This step could be possible to adapt to the non-relativistic context if we ignored the problems that arose when analyzing steps 1.4 and 1.5. To do so, the definition of the generator fA_α can be fully extrapolated to this case, because it involves only translation elements of the Poincaré group, which are shared with the galilean group. The only difference is that the limited range of four-momentum is now a limited interval of classical energy and momentum, as has been typical across all points of this discussion. Now, in order to infer the circunstance where fA_α annihilates three states with a specific energy and momentum but another one is not, we need to be able to limit the range of quantum states to a set of separated surfaces in energy+momentum space. We can do that thanks to Assumption 1, which we apply to this context as well, and the existence of the dependence $E = p^2/2m$, which will not yield hyperboloids, but will provide a set of disjointed surfaces in four-dimension space, which is enough for what the proof requires.

The following arguments, which involve only Assumptions 2, 3, can be fully extrapolated to this new context, so we finally reach the conclusion that the symmetry generators have an analogous structure to the one that we found in the original proof at this point, completing the adaptation of step 2.1. We can think about a minor concern in this last reasoning, related to the fact that the weak elastic analyticity assumption uses as variables the center-of-mass energy s and the invariant momentum transfer t. These two quantities have slightly different dependences in the framework of special relativity and outside of it, but in this case the explicit dependence does not matter, the only relevant aspect in the original proof is that, for a set of four-momenta p, q, p' and q', the subsequent s and t will have associated a null scattering amplitude, and this connection can be done as long as these two dynamical variables are functions of only the energy and the momentum, which is the case in both contexts. The rest of the step only involves the properties of the δ-functions and Assumption 5, so it does not pose any problem for the adaptation.

There is one additional comment that we can do about this step involving equation (55), which expresses how a translation transforms $|pm>$. The fact that the one-particle states transform in this way is of essential importance to define fA_α with the necessary properties to carry out the proof as it stands. However, equation (55) is only valid when working on a flat spacetime, so if we were to consider notions of general relativity we would not be able to ensure that the Coleman-Mandula theorem is true anymore. This is another example of a context where the proof cannot be adapted due to its strong dependence of the quantum field theory framework where it is defined.

-Step 2.2. This step starts with a manipulation of the symmetry generators whose character we have just pinned down. Given their form, we can build from any general generator A_α another one which commutes with the four-momentum operators by making it commute a number D_α of times with any combination of four-momentum operators P_μ. Knowing that this structure is available to us, using the fact that the mass operator $P^\mu P_\mu$ is a Casimir invariant for the symmetry group, which is necessary to maintain the consistency A_α as symmetry generators of the S-matrix, we can deduce that the order D_α of the polynomial that constitutes A_α is at most one. In order to show this we only need to take advantage of the structure of the four-momentum commuting generators that we studied all across step 1. Then, if we consider only the polynomials with $D_\alpha = 0$, we recover the subalgebra $\{B_\alpha\}$, which we already consists of a linear combination of momentum generators and internal symmetries. On the other hand, for the case where $D_\alpha = 1$, we can easily identify the first-order derivative terms with generators of the Lorentz group by looking at some properties of the Poincaré algebra, so in the end we conclude that the elements of the algebra of any symmetry group for the scattering matrix must be a linear combination of the generators of the Poincaré group and those of an internal symmetry.

In what concerns the non-relativistic analysis, here we find yet another difference that stops us from adapting our proof to the new context. The obstacle comes from the fact that in the original proof it is necessary that the Casimir operator is a quadratic combination of the four-momentum components in order to reach the conclusion of equation (82), which we need to claim that D_α is either zero or one. However, the mass Casimir invariant that we have in the non-relativistic environment is $ME - \frac{P^2}{2}$, where M is the mass operator, which does not allow us to reach that same conclusion at all. In order to ascertain this claim, let us try to reproduce the development undertaken in equations (82) and (83) with this operator:

$$\begin{aligned}
[ME - \frac{P^2}{2}, P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...] &= [ME, -\frac{P^2}{2} P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...] - [\frac{P^2}{2}, P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...] \\
&= MB_\alpha^{\mu_0\mu_2...\mu_{D_\alpha}} - P_i B_\alpha^{\mu_i\mu_2...\mu_{D_\alpha}} = 0 \Rightarrow \\
&\Rightarrow m b_\alpha^{\mu_0\mu_2...\mu_{D_\alpha}}(p) - p_i b_\alpha^{\mu_i\mu_2...\mu_{D_\alpha}}(p) = 0
\end{aligned} \tag{89}$$

where the 0 index refers to the energy operator and the i to the momentum operators. If we now expand the $b_\alpha^{\mu_0\mu_2...\mu_{D_\alpha}}(p)$ according to equation (80):

$$m b_\alpha^{\#\mu_0\mu_2...\mu_{D_\alpha}} - p_i b_\alpha^{\#\mu_i\mu_2...\mu_{D_\alpha}} - m p_\mu a^{\mu\mu_0\mu_2...\mu_{D_\alpha}} + p_i p_\mu a^{\mu\mu_0\mu_2...\mu_{D_\alpha}} = 0 \tag{90}$$

where we use implicitly a modified version of the Einstein sum convention, where we sum over the energy and the momentum, without taking into account the change of sign that we would have needed in the relativistic case. Looking at this expression, we can argue that the quantities $b_\alpha^{\#\mu_0\mu_2...\mu_{D_\alpha}}$ are all zero if we particularize the expression for a particle at rest, so we recover half of the conclusion collected in expression (83). However, if we expand the elements of the sum for the a's, where we only write for compactness the first two indices,:

$$\begin{aligned}
&mEa^{00} + mp_1 a^{10} + mp_2 a^{20} + mp_3 a^{30} - p_1^2 a^{11} - p_2^2 a^{22} - p_3^2 a^{33} - p_1 p_2 (a^{12} + a^{21}) - \\
&- p_2 p_3 (a^{23} + a^{32}) - p_1 p_3 (a^{13} + a^{31}) - p_1 E a^{01} - p_2 E a^{02} - p_3 E a^{03} = 0
\end{aligned} \tag{91}$$

if we consider that this equality needs to hold for every possible value of the energy and the momentum and also the dependence $m = p^2/2E$ on those four variables we end up with:

$$\begin{cases} a^{00} = 2a^{11} = 2a^{22} = 2a^{33} \\ a^{03} = a^{30} = a^{02} = a^{20} = a^{01} = a^{10} = 0 \\ a^{31} = a^{13} = 0 \\ a^{21} = a^{12} = 0 \\ a^{23} = a^{32} = 0 \end{cases} \tag{92}$$

where the last six coefficients are set to zero by the same argument about the symmetry of the indices that we used before in the relativistic proof. Looking at this, using the same line of reasoning as in the original proof we conclude that we can only restrict the set $\{B_\alpha^{\mu_1\mu_2\ldots\mu_{D_\alpha}}\}$ to a subset of the generators of the form $B_\alpha^{\mu_j\mu_j\ldots\mu_j}$, which, of course, is not restrictive enough to justify that in the expansion in a sum of derivatives of the generators A_α when transforming one-particle states the only nopn-zero terms are the ones corresponding to the zero-th and first order derivatives. That way, the difference between the mass operators in the two frameworks is another factor that makes it impossible to adapt the original proof to the non-relativistic context.

As for the rest of this step, if we could have found an adequate Casimir invariant then the step would be perfectly extrapolable, since the other tools in which it supports itself are quantum field theory notions, the conclusions from previous steps and some properties of the Poincaré algebra whose use can be satisfied by some analogous relations in the galilean group. Namely, we first need that the commutation relations between the energy+momentum operators and the generators of boosts and translations are linear combinations of the dynamical operators, which is true for the galilean algebra:

$$[P^\mu, M^{\rho\sigma}] = -i\eta^{\nu\rho}P^\sigma + i\eta^{\nu\sigma}P^\rho \rightarrow \begin{cases} [J_i, P_j] = i\epsilon_{ijk}P_k \\ [K_i, P_j] = 0 \\ [J_i, H] = 0 \\ [K_i, H] = iP_i \end{cases} \tag{93}$$

and we also need that the action of the generators of rotations and boosts are linear combinations of the quantities $a_{\mu\nu}p^\mu \frac{\partial}{\partial p^\nu}$, where p_μ are the variables in four-momentum space or, in this context, the variables of the momentum + energy space, which is also the case if we look at equations (11) and (12).

Therefore, we can conclude that, following the same steps and arguments as in the original proof and taking a set of equivalent assumptions to the ones required in the first formulation of the Coleman-Mandula theorem, it is impossible to reach the conclusion that in a nonrelativistic environment the symmetry groups for the scattering must be a direct product of the galilean group and some internal symmetry group. The essential differences between the two environments that prevent the adaptations are the differences between the Lie algebras of the galilean and Poincaré group, the discrepancies in the compactness of that same two groups and the existence of two different mass operators for systems of free particles in the relativistic and nonrelativistic environments. Besides this, there is an additional comment that we can make about a decision we made implicitly when attempting to perform the adaptation. This decision concerns the use of translations in space and time and their generators the energy and momentum operators in an interchangeable way during all the steps of the proof except when some necessary comment about their physical separation is necessary, like when discussing the mass operator. This attempt to mimic the structure and relations that the energy and the momentum operators present in the relativistic context has not led us to any obstacle, but it still comes out as somewhat unnantural since in this context the space and time variables are formulated in fundamentally disjointed spaces.

5 Conclusions

This thesis aimed to perform two concrete tasks. The first one was to formulate the Coleman-Mandula theorem and explain its proof in the clearest and most detailed way possible and also focusing only on its most fundamental points. As a secondary prospect related to the development of the proof, we were interested in performing a critical analysis of the theorem in order to determine whether any of the assumptions could be relaxed and if there could be any situations in which it could be impossible to implement the Coleman-Mandula theorem given the limitations of the proof. The second goal was to attempt to adapt the proof to one of these contexts where the theorem does not apply, namely the nonrelativistic limit, and extract from the conflicting points an impression about the most important features of the quantum field theory environment which support the current version of the proof.

Upon presenting the proof of the theorem, there are several conclusions that can be extracted from it just by looking at its structure and flow. The first observation is that most of the reasonings are heavily dependent on how the different symmetry generators transform specific field states, so it is indispensable that we are able to formulate the representations of the symmetry groups that are able to transform directly those same objects. Apart from that, this speaks about the low level of abstraction that it is needed to proove this theorem.

In line with this topic, the proof only ever worked with one- and two-particle states to reach all its conclusions. The reason for this is the heavy reliance of the proof with the most simple scattering system: the 2-2 elastic scattering of point particles, and it also does not pose any problem at all towards the generalization of the results of the proof, given how the symmetry transformations are built for a system with an arbitrary number of particles as a direct product of one-particle transformations.

Another idea that comes out from the examination of the proof is that all the assumptions proposed are essemtial for its correct achievement, and from their absence we may find some clear examples of physical scenarios where the Coleman-Mandula theorem is not valid. For instance, there is also cases of scattering systems where only forward and backward scattering is allowed, in which case the analyticity and nontriviality assumptions cannot apply, where there have been observed symmetries which are not direct products of the Poincaré group and internal symmetries. Besides that, the most clear of these is the nonrelativistic limit, where we already saw that substituting the Poincaré invariance for the galilean invariance was not possible if we wanted to reach the conclusion of the theorem. This complication to apply the Coleman-Mandula theorem without relying on the Poincaré symmetry as it is formulated in flat spaces can also be seen if we try to extend the proof to a curved spacetime, which is not possible given the discussion that we did on the topic in the subsubsection 3.2.2.

An additional comment that we can extract from the proof is that all internal symmetry groups that we may formulate for the S-matrix need to be compact if we consider that there are a finite number of particles, which is not surprising given that we are attempting to find transformations that commute with an operator that in this case by definition must yield a finite and unitary matrix. This can be seen clearly if we consider a system of infinite particles, to each of which we can associate a U(1) symmetry. Then, if we consider the direct product of all those individual symmetries, the resulting internal symmetry could never be compact given its infinite dimension. However, this argument leaves the Poincaré group unchecked, but we can argue that this situation is completely justified. The reason for this is that, in the development of the proof, we inserted this group by assumption given its special status of the group as the container of all the possible coordinate transformations that we can apply to the space where the field states are formulated, so it cannot be regarded as a natural symmetry transformation for a general finite and unitary matrix.

Finally, one last conclusion that we can do about the discussion of the proof as a whole is that, despite our efforts to summarize all the intricate and obscure arguments that the proof makes use of in section 3, the proof still relies in a very big number of properties and arguments that come from very disparate contexts and in the end comes off as very unintuitive considering the generality that is presumed for the conclusion of the theorem, which does nothing but reinforce the notion that there may exist a much more simple demonstration of the Coleman-Mandula theorem. The most relevant argument that we can use to back up this statement is the need that we have of Assumption 5 to complete the proof. This assumption is not motivated by any physical argument unlike the other four, and it is fundamentally technical because its only role is to save us the trouble of looking for more possible structures that may fit with the restriction that we derived for the generators of the symmetry group in step 2.1.

Then, about the second endeavour we set out to do on the nonrelativistic limit, we come out of it convinced that it is not possible to adapt the conclusion of the theorem to a context where the quantum theory of fields cannot be formulated, at least by means of the proof that was valid for the original theorem. After identifying with the biggest degree of concretion possible all the conflicting points in the failed adaptation, we can observe that all of them stem from the discrepancy between the physical character of the spacetime geometries in which each of them is defined and the natural symmetries that we can define for them given the coordinate systems that we can define for them, namely the Poincaré group for the relativistic case and the galilean group for the nonrelativistic context. This ties directly to a comment that we made before about the necessity of all the assumptions: once we alter any of them in order to try to expand the scope of the theorem we are bound to failure. Apart from that, this result is perfectly coherent with what is known experimentally, since no consequences of the Coleman-Mandula theorem have ever been observed in nonrelativistic environments.

To finalize the conclusions for the thesis, let us mention some possible outlooks for future research that could follow from the work that we have done here. Mainly, the biggest prospect that we can have is a much more extensive effort towards examining the proof and the scope of the Coleman-Mandula theorem, given its obtuse proof and the seemingly unnecessary limitation of its conclusion to the scattering matrix. Some possible starting points for this would be by trying to correct the shakiest points of the currently accepted proof or maybe trying to reach the conclusion for the Coleman-Mandula theorem in a context where it is typically applied without major concerns. To this last approach there is a slight possibility that the effort performed on our nonrelativistic proof may be of some use, since, despite its major value being just didactic, perhaps some of the conflicting points determined across the adaptation may be relevant when considering a truly sensible extension of the theorem, even though their value strictly shows when making the proof collide with the nonrelativistic limit.

A Appendix: Computations on $B_\alpha^{\#}$

In order to reach equation (40) from equation (39), we expand the null linear combination of $b_\alpha^{\#}(p', q')$ for each of the matrix elements and express it in terms of the one-particle representation matrices:

$$\sum_\alpha c^\alpha b_\alpha^{\#}(p', q')_{m'n',mn} = \sum_\alpha c^\alpha \left(\left(b_\alpha^{\#}(p')\right)_{m',m} \delta_{n',n} + \left(b_\alpha^{\#}(q')\right)_{n',n} \delta_{m',m} \right) = 0 \tag{94}$$

Now, taking into account that the matrices $b_\alpha^{\#}(p')$ and $b_\alpha^{\#}(q')$, we will take the trace of this expression over the two sets of indices for the one-particle matrices separately and see where that leads us. Starting with the m's:

$$tr_m \left(\sum_\alpha c^\alpha b_\alpha^{\#}(p', q') \right) = \sum_\alpha N(\sqrt{q'^\mu q_\mu}) c^\alpha \left(b_\alpha^{\#}(p')\right) = 0 \Rightarrow \sum_\alpha c^\alpha b_\alpha^{\#}(p') = 0 \tag{95}$$

Doing the same for the n's:

$$tr_n \left(\sum_\alpha c^\alpha b_\alpha^{\#}(p', q') \right) = \sum_\alpha N(\sqrt{p'^\mu p_\mu}) c^\alpha \left(b_\alpha^{\#}(q')\right) = 0 \Rightarrow \sum_\alpha c^\alpha b_\alpha^{\#}(q') = 0 \tag{96}$$

Therefore, the following relation is now proven:

$$\sum_\alpha c^\alpha b_\alpha^{\#}(p', q') = 0 \Rightarrow \sum_\alpha c^\alpha b_\alpha^{\#}(p') = \sum_\alpha c^\alpha b_\alpha^{\#}(q') = 0 \tag{97}$$

B Appendix: Discussion on the properties of $B_\alpha^{\mu_1 \dots \mu_{D_\alpha}}$

In order to proove that the $B_\alpha^{\mu_1 \dots \mu_{D_\alpha}}$ commute with the momentum operators, let us first consider the generators with $D_\alpha = 1$ and formulate how their associated $B_\alpha^{\mu_1}$ transforms one-particle states given the structure introduced in equation (84):

$$
\begin{aligned}
B_\alpha^{\mu_1}|pm> &= [P^{\mu_1}, A_\alpha]|pm> = \sum_{m'}[P^{\mu_1}, \left(a^0(p)\right)_{m',m} + \left(a^1(p)\right)^{\nu}_{m',m}\frac{\partial}{\partial p^\nu}]|pm'> = \\
&= \sum_{m'}[P^{\mu_1}, \left(a^1(p)\right)^{\nu}_{m',m}\frac{\partial}{\partial p^\nu}]|pm'> = \\
&= \sum_{m'}\left(P^{\mu_1}\left(a^1(p)\right)^{\nu}_{m',m}\frac{\partial}{\partial p^\nu} - \frac{\partial}{\partial p^\nu}P^{\mu_1}\left(a^1(p)\right)^{\nu}_{m',m} - P^{\mu_1}\left(a^1(p)\right)^{\nu}_{m',m}\frac{\partial}{\partial p^\nu}\right)|pm'> = \\
&= -\sum_{m'}\left(\frac{\partial}{\partial p^\nu}P^{\mu_1}\left(a^1(p)\right)^{\nu}_{m',m}\right)|pm'> = \sum_{m'}\left(\frac{\partial}{\partial p^\nu}p^{\mu_1}\left(a^1(p)\right)^{\nu}_{m',m}\right)|pm'> = \\
&= \sum_{m'}\left(\delta^{\mu_1}_{\nu}\left(a^1(p)\right)^{\nu}_{m',m}\right)|pm'> = \sum_{m'}\left(\left(a^1(p)\right)^{\mu_1}_{m',m}\right)|pm'>
\end{aligned}
\tag{98}
$$

Indeed, we see that the $B_\alpha^{\mu_1}$ commute with the momentum operators. In order to extend this conclusion to the general $B_\alpha^{\mu_1 \dots \mu_{D_\alpha}}$, we will first calculate for $D_\alpha = 1$ the value of the null commutator between states with momenta p and q:

$$
\begin{aligned}
< q|[B_\alpha^{\mu_1}, P^\mu]|p> &= < q|[[P^{\mu_1}, A_\alpha], P^\mu]|p> = \\
&= < q|\left(P^{\mu_1}A_\alpha P^\mu - A_\alpha P^{\mu_1}P^\mu - P^\mu P^{\mu_1}A_\alpha + P^\mu A_\alpha P^{\mu_1}\right)|p> = \\
&= < q|\left(q^{\mu_1}A_\alpha p^\mu - A_\alpha p^{\mu_1}p^\mu - q^\mu q^{\mu_1}A_\alpha + q^\mu A_\alpha p^{\mu_1}\right)|p> = \\
&= -(q-p)^\mu(q-p)^{\mu_1} < q|A_\alpha|p> = \\
&= -(q-p)^\mu(q-p)^{\mu_1}\left(a'^0_\alpha(p) + a^1_\alpha(p)^\nu\frac{\partial}{\partial p^\nu}\right) < q|p>
\end{aligned}
\tag{99}
$$

where

$$
a'^0_\alpha(p) = a^0_\alpha(p) + \frac{\partial}{\partial p^\nu}a^1_\alpha(p)^\nu \tag{100}
$$

Now, using the one-particle relativistic normalization:

$$
< p|q> = 2E_p(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \tag{101}
$$

We end up with

$$
< q|[B_\alpha^{\mu_1}, P^\mu]|p> \propto (q-p)^\mu(q-p)^{\mu_1}\left(a'^0_\alpha(p) + a^1_\alpha(p)^\nu\frac{\partial}{\partial p^\nu}\right)\delta^4(p-q) = 0 \tag{102}
$$

where we used that, given that the symmetry cannot connect two states with four-momenta in different mass shells, in this case $\delta^3(\mathbf{p}-\mathbf{q})$ will be proportional $\delta^4(p-q)$.

Now, using this as a reference, we will proove that generally the matrix elements of the commutators $B_\alpha^{\mu_1 \dots \mu_{D_\alpha}}$ with a momentum operator between states with momentum p and q are proportional to $D_\alpha + 1$ factors of $p-q$ times a polynomial of order D_α in derivatives of the momentum on $\delta^4(p-q)$. We will first obtain the $p-q$ factors:

$$\begin{aligned} < q|[P^{\mu_1}, P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]..., P^\mu]|p> &= -(p-q)^\mu < q|[P^{\mu_1}, P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...]|p> = \\ &= -(p-q)^\mu (p-q)^{\mu_1} < q|[P^{\mu_2}, ...[P^{\mu_{D_\alpha}}, A_\alpha]]...]|p> = \\ &= -(p-q)^\mu (p-q)^{\mu_1} ... (p-q)^{\mu_{D_\alpha}} < q|A_\alpha|p> \end{aligned} \tag{103}$$

We proceed to evaluate $< q|A_\alpha|p >$ separately:

$$\begin{aligned} < q|A_\alpha|p> = \int d^4p' \Bigg(&(a_\alpha^0(p',p))\delta^4(p-p') + (a_\alpha^1(p',p))^{\mu_1} \frac{\partial}{\partial p^{\mu_1}} \delta^4(p-p') + ... \\ &... + (a_\alpha^{D_\alpha}(p',p))^{\mu_1 \cdots \mu_{D_\alpha}} \frac{\partial^{D_\alpha}}{\partial p^{\mu_1} ... p^{\mu_{D_\alpha}}} \delta^4(p-p') \Bigg) < q|p'> \end{aligned} \tag{104}$$

Using another property of the delta-functions:

$$\int dx \delta^n(x) f(x) = (-1)^n \int dx \delta(x) f^n(x) \tag{105}$$

We get that:

$$\begin{aligned} < q|A_\alpha|p >= \int d^4p' \delta^4(p-p') \Bigg(&(a_\alpha'^0(p',p)) + (a_\alpha'^1(p',p))^{\mu_1} \frac{\partial}{\partial p^{\mu_1}} + ... \\ &... + (a_\alpha'^{D_\alpha}(p',p))^{\mu_1 \cdots \mu_{D_\alpha}} \frac{\partial^{D_\alpha}}{\partial p^{\mu_1} ... p^{\mu_{D_\alpha}}} \Bigg) < q|p'> \end{aligned} \tag{106}$$

where the -1 factors have been absorbed by the primed coefficients. Solving the integral and taking advantage of the relativistic normalization and the mass shell restriction for the symmetry transformations, we finally conclude that:

$$< q|A_\alpha|p> \propto \left((a_\alpha'^0(p)) + (a_\alpha'^1(p))^{\mu_1} \frac{\partial}{\partial p^{\mu_1}} + ... + (a_\alpha'^{D_\alpha}(p))^{\mu_1 \cdots \mu_{D_\alpha}} \frac{\partial^{D_\alpha}}{\partial p^{\mu_1} ... p^{\mu_{D_\alpha}}} \right) \delta^4(p-q) \tag{107}$$

Substituting this into expression (103)

$$\begin{aligned} < q|[B_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}, P^\mu]]..., P^\mu]|p> \propto (p-q)^\mu (p-q)^{\mu_1} ... (p-q)^{\mu_{D_\alpha}} \Bigg(&(a_\alpha'^0(p)) + +(a_\alpha'^1(p))^{\mu_1} \frac{\partial}{\partial p^{\mu_1}} + \\ &+ ... + (a_\alpha'^{D_\alpha}(p))^{\mu_1 \cdots \mu_{D_\alpha}} \frac{\partial^{D_\alpha}}{\partial p^{\mu_1} ... p^{\mu_{D_\alpha}}} \Bigg) \delta^4(p-q) = 0 \end{aligned} \tag{108}$$

Therefore, we conclude that the generators $B_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}$ must commute with the momentum operators, and hence they must belong to the subalgebra $\{B_\alpha\}$.

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