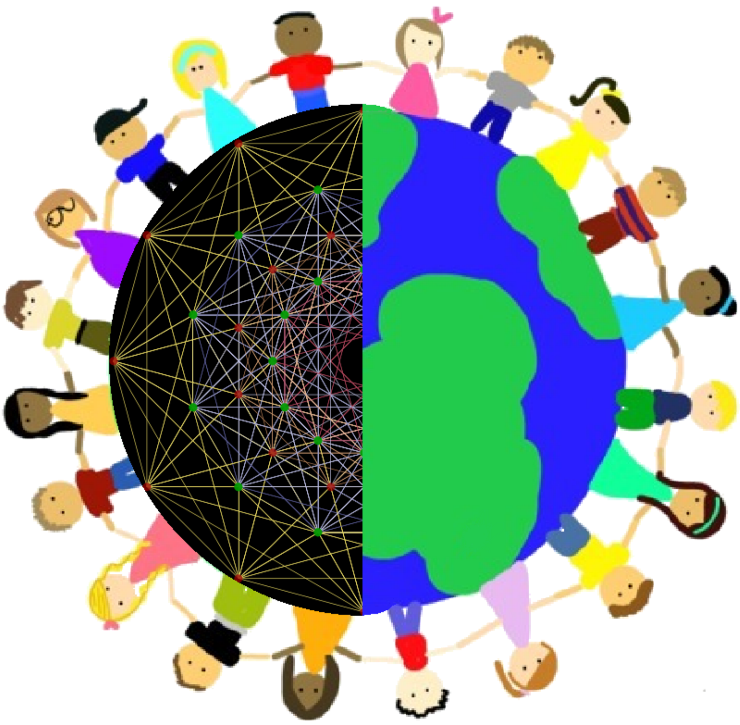
university of groningen

faculty of mathematics and natural sciences

Master research

Extending Symmetries of the Standard Model towards Grand Unification



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August 3, 2018

Abstract

In this master thesis the spacetime, global and gauge symmetries of the Standard Model are reviewed. These symmetries are used as a basis for finding possible extensions of this successful model, such as the two Higgs doublet model and Left-Right model. Methods of finding subgroups and the slitting up of representations are discussed based on Dynkin diagrams. These methods are applied to analyse the subgroups of the exceptional group E_6 as candidates for Grand Unified Theory groups. In this study $SU(5)$, $SO(10)$ and $SU(3)^3$ and $SU(4) \times SU(2) \times SU(2)$ are most important. A phenomenological comparison between these models is given focussed on the different types of leptoquarks that could be responsible for the not yet observed proton decay.

Contents

Chapter 1

Introduction

The number of objects around us without any *symmetry* is astoundingly low. Even more surprising is how hardly any of these symmetries is exact. But most remarkable of all is how these broken symmetries can be used to describe nature amazingly accurately.
Because of this large amount of approximate symmetry it is understandable that many classical natural philosophies were based on geometric shapes. Even in the first scientific texts written centuries later, when natural philosophy was evolving into natural science, explanations were often supported by symmetrical diagrams. The motion of the objects in our universe was first described using structures involving many lines and circles rather than mathematical formulas, which prove to be so useful today.
Although we have come a long way since, the importance of symmetry in physics remained. Even the basic idea of being able to describe the universe by a set of equations requires a symmetry, that nature follows simple, uniform causal patterns. Some other assumptions have a direct implication for the equations describing nature, the equivalence of physical laws over all time and space imply *spacetime symmetries*, that will be discussed at the end of the chapter 2.
Next to translations and rotations in space and time *internal symmetries* of some abstract space turn out also to be of great use in describing the world around us. Both spacetime and internal symmetries play a fundamental role in Quantum Field Theories, one of the common frameworks in modern physics. In Quantum Field Theories both *matter* particles and the *interactions* between them are described by *fields* spanned out over space and time. The mathematics describing symmetries is *group theory*, which will be reviewed in chapter 2.
The most successful Quantum Field Theory of today is the *Standard Model*, including *leptons* and *quarks* as matter particles and *gluons*, *W and Z bosons* and *photons* as interaction particles between them. The *Higgs boson* plays a special role as it is responsible for *spontaneous symmetry breaking* and the resulting masses of most of the particles. The Standard Model is based on a spacetime dependent internal symmetry, called a *gauge symmetry*, known as the *Standard Model group*,

$$SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{1.1}$$

This gauge theory consisting of three separate sectors is extremely successful in predicting and describing results for experiments. The structure and symmetries of the Standard Model are discussed in chapter 3.
Despite its great success the Standard Model it is not a complete theory. The largest limitation is its lack of describing *gravity*, it describes only three of the four fundamental interactions in nature. Also the less understood *Dark matter* is not included in the Standard Model. Both these problems are still unsolved and will not be discussed in this thesis. There are however other hints that the Standard Model is incomplete. It does, for instance, not explain the masses of *neutrinos* needed to describe their observed flavour *oscillations*. Also the asymmetry between left and right-handed particles, which is put explicitly to match the $V - A$ structure observed in experiments, could be a reason to believe there is physics beyond the Standard Model. Some basic extensions are presented in chapter 4.
The main topic of this thesis is however the possibility that the Standard Model might be the low energy effective theory of a larger *Grand Unified Theory*. The main reason for this idea is the unsatisfactory amount of free parameters in the Standard Model, such as *masses* and

coupling strengths. If the universe seems to be so symmetric, is there no symmetry relating and constraining these many parameters? The attempt is to combine the three separate components of the model into one gauge theory based on a simple symmetry group, resulting in a link between the otherwise unrelated coupling strengths. How symmetry groups can be combined into larger symmetry groups is discussed in chapter 5.
Many Grand Unification groups have been suggested during the last 40 years. Although these indeed solve some problems of the Standard Model, such as the exact *cancellation of the electric charge* of the proton and the electron, other problems emerge. For instance *new particles* are predicted, of which the absence in observations must be explained. This also includes new gauge and Higgs bosons leading to interactions, predicting the proton to be unstable. The lower bound on its lifetime found in experiments restricts the possible decay modes of the proton. A phenomenological analysis for some popular models is given in chapter 6.
At first sight the number of possibilities for unification models might be overwhelming, but in a structured framework many patterns can be understood. Some models, such as those based on $SU(5)$ and $SO(10)$ gauge groups, have been suggested based on their simplicity and the perfect way in which Standard Model fits within. Others were based on more principle reasons, such as Left-Right models and trinification. It turns out that many subgroups of E_6 are of interest for unified model building, therefore these will be used throughout this thesis as unification groups. In the end the mathematical framework needed to deal with these groups will turn out to bear more resemblance to the diagrams of circles used in classical natural sciences than formulas, showing that there is even symmetry in scientific methodologies used over centuries.

Chapter 2

Symmetry groups and Gauge theories

In this chapter some of the relevant properties of *groups*, in particular of *Lie groups* are revisited. After this the *local transformations* of fields and their derivatives are analysed, as a first step towards *gauge theories* describing the interactions between the fundamental particles in nature. At the end of this chapter the *spacetime symmetries* and corresponding *discrete symmetries* are reviewed. There are many more elaborate texts dealing with the basics of Group theory. This review is based on [1, 2, 3, 4, 5, 6, 7], to which the reader is referred for further details.

2.1 Group theory

Symmetry plays a fundamental role in physics. Many theories can be viewed as results of invariance of the system under translations and rotations, either in real or in a more abstract space. These sets of operations that leave systems invariant can be described mathematically using groups, as described here.

2.1.1 Definition of a group

The mathematical definition of a *group* G consists of a set of conditions that have to fulfilled in all the *multiplication* $\circ$ between *elements* g_i of the group.

(i) *Closure*: $g_1, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$,
(ii) *Associativity*: $g_1, g_2, g_3 \in G \Rightarrow g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$,
(iii) *Identity*: $\exists\, e \in G : \forall g \in G, g \circ e = e \circ g = g$,
(iv) *Inverse*: $\forall g \in G \quad \exists\, g^{-1} \in G : g \circ g^{-1} = g^{-1} \circ g = e$.

These can be interpreted as,

(i) *Closure*: The group multiplication of any two elements in the group results in another element in the group,
(ii) *Associativity*: It does not matter which multiplication is done first, as long as the order of the elements is unchanged,
(iii) *Identity*: There is an element that corresponds to doing nothing,
(iv) *Inverse*: For every element there is one undoing its operation.

The elements of a group can be a set of discrete transformations, such as the operations that can be done on a triangle (rotation over $2\pi/3$ and reflections) or the three roots of unity in Z_3. The elements can however also depend continuously on a set of parameters. Examples of these are the rotations of a directed circle and the complex phases in $U(1)$. In that case the group is called a *Lie group*. There exists a more formal definition, but this more intuitive description suffices for the following text.
An *Abelian group* is one in which the order of operations does not matter. For the elements of

the Abelian group G this means,

$$\forall g_1, g_2 \in G : g_1 \circ g_2 = g_2 \circ g_1. \tag{2.1}$$

2.1.2 Local properties and Lie algebras

Since the elements of a Lie group depend on a continuous set of parameters the structure of these groups can be considered by expanding in the element in these parameters. Let $g = g(\xi)$ be an element of a n-dimensional Lie group with $\xi = (\xi_1, \ldots, \xi_n)$ the set of parameters it can depend on. The identity element of the group is obtained by putting all parameters to zero, corresponding to the trivial transformation of doing nothing, $e = g(0)$. For infinitesimal transformations, connected to the identity element by a continuous path described by the parameters ξ_a, the group element can be expanded around the identity element using an exponential map as,

$$g(\xi) = \exp(i\xi^a t_a) = e + i\xi^a t_a - \frac{1}{2}(\xi^a t_a)^2 + \mathcal{O}(\xi^3), \tag{2.2}$$

where $a = 1, 2, \ldots n$ is summed over.The *generators* of the group t_a are found from this relation to be,

$$t_a = -i \left. \frac{\partial g(\xi)}{\partial \xi^a} \right|_{\xi=0}. \tag{2.3}$$

Since the group multiplication of two infinitesimal transformations in a group must also be an infinitesimal transformation in that group,

$$g(\xi_1)g(\xi_2) = \exp(i\xi_1^a\, t_a)\exp(i\xi_2^b\, t_b) = \exp(i\xi_3^c\, t_c) = g(\xi_3). \tag{2.4}$$

This is provides a restriction on the commutator between the generators, as the product of two exponentials involves this via the Campbell–Baker–Hausdorff formula. By taking the natural logarithm on both sides one finds,

$$\begin{aligned}
i\xi_3^c t_c &= \ln\left(\exp(i\xi_1^a\, t_a)\exp(i\xi_2^b\, t_b)\right), \\
&= \ln\left(e + i(\xi_1 + \xi_2)^a t_a - \frac{1}{2}\left(\xi_1^a\, t_a\right)^2 - \frac{1}{2}\left(\xi_2^b\, t_b\right)^2 - \xi_1^a\, t_a\, \xi_2^b\, t_b + \mathcal{O}(\xi^3)\right), \\
&= i(\xi_1 + \xi_2)^a t_a - \frac{1}{2}\left(\xi_1^a\, t_a\right)^2 - \frac{1}{2}\left(\xi_2^b\, t_b\right)^2 - \xi_1^a\, t_a \xi_2^b\, t_b + \frac{1}{2}\left((\xi_1 + \xi_2)^a t_a\right)^2 + \mathcal{O}(\xi^3), \\
&= i(\xi_1 + \xi_2)^a t_a - \frac{1}{2}\left[\xi_1^a\, t_a, \xi_2^b\, t_b\right] + \mathcal{O}(\xi^3),
\end{aligned} \tag{2.5}$$

where the Taylor expansion of $\ln(1 + x) = x - x^2/2 + \mathcal{O}(x^3)$ was used. It is assumed for this argument that all terms converge, for which elements close to identity must be considered. Now for the left and right side to be consistent the commutation relation must be of the form,

$$[t_a, t_b] = i f_{ab}^c\, t_c, \tag{2.6}$$

where the i is convention, resulting in,

$$\xi_3^c = \xi_1^c + \xi_2^c - \frac{1}{2}\xi_1^a\, \xi_2^b\, f_{ab}^c + \mathcal{O}(\xi^3) \tag{2.7}$$

The commutation relation in (2.6) is called the *Lie product* between two generators. The *Lie Algebra* corresponding to a Lie group with elements generated by the generators t_a is the set of these generators. The Lie algebra is determined by the *structure constants* f_{ab}^c. From the relation between the Lie algebra and the Lie group it is clear that this commutation relation determines the *local structure* of the group.

2.1.3 Representations

For application in physics the abstract concept of groups and algebras must be transferred to a concrete set of operations working on states of the system. For instance the operation corresponding to an abstract rotations must be transferred to a rotation of the system around some axis over some angle. For this interpretation of the group a mapping of the group elements to operators working on vector space describing the physical system has to be considered. This is called a *representation* of the group, denoted by D. The operators working on the vector space correspond to matrices, whose matrix multiplication follows the multiplication of the group,

$$D(g_1 \circ g_2) = D(g_1)D(g_2). \tag{2.8}$$

There must be the representation of the identity element $D(e)$ such that,

$$D(g \circ e) = D(e \circ g) = D(e)D(g) = D(g)D(e) = D(g), \tag{2.9}$$

from which it is clear that $D(e) = \mathbb{1}$ is the identity matrix. For the representation of the inverse this implies,

$$D(g)D(g^{-1}) = D(g \circ g^{-1}) = D(e) = \mathbb{1}, \tag{2.10}$$

and thus the representation of the inverse of an element must be the inverse of the representation of that element, $D(g^{-1}) = D(g)^{-1}$. From this we find that the representation matrices must be non-singular.
Every group has a trivial representation, where all elements are represented by the identity, $D(g) = \mathbb{1} \quad \forall g$. In the exponential map (2.2) this representation can be recognized by taking all parameters ξ^a zero. At first this representation might seem useless, but in practice many particles will be recognized to transform under the trivial representation of a group. Mostly this particle is then said not to transform under the group.
The *dimension of the representation* is the size of the matrices to which the group elements are mapped. An n-dimensional representation of a group corresponds to $n \times n$ non-singular matrices following the group multiplication. This must not be confused with the *dimension of the group*, which is the number of parameters in the group.
From the exponential map it is clear that in considering a representation of the group elements a representation of the generators must be found. The Lie product in the Lie algebra (2.6) is independent of the representation. The structure constants are determined by the group, and are therefore independent of the representation. Any non-trivial representation can therefore be used to find the structure constants.
In Abelian groups the order of multiplication of the elements does not matter. From (2.5) it follows that in this case the commutation of the generators must be equal to minus itself. Therefore the structure constants of an Abelian group are always zero.

Irreducible representations

There exist infinitely many representations of every group. One can always construct larger representations from combining multiple smaller ones. Consider for instance a two-dimension rotation represented by 2×2 matrices A. There exists the three-dimensional representation of the form $\mathrm{diag}(A, 1)$, which clearly follows the same multiplication. This would be the rotations around some invariant direction in three-dimensional space. These representations with an invariant subspace are called *reducible representations.* In general it turns out that in a reducible representation by a basis transformation all matrices can simultaneously (the transformation is independent on the group element) brought to the form,

$$D(g) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \tag{2.11}$$

An *irreducible representation* is one that is not reducible. The irreducible representations are often labelled by their dimension, although there exist inequivalent representations that have the same dimension, thus it is not a unique label. From the multiplication of irreducible representations new irreducible representations can be constructed, for instance by composition into symmetric and antisymmetric part or trace and traceless part. Fortunately this has been done for many groups and representations, the results can be found in [8].
Most of the groups that will be considered in gauge theories are defined by the multiplication of a specific set of matrices. It is clear that these matrices form a representation of the group,

called the *defining representation*. In the following sections some Lie groups will be discussed in terms of their defining representation, note however that there can also be other irreducible representations that follow the same multiplication. The defining representation is of great importance in physics, since mostly matter fields transform under the defining representation of (part of) the gauge group.

2.2 Unitary and Special Unitary groups

Unitary groups $U(N)$ are groups with *unitary* $N \times N$ *matrices* as elements U, meaning $U^{-1} = U^{\dagger}$. *Special unitary groups* $SU(N)$ have as an extra condition that the *determinant* of the matrices U must be one, $\det(U) = 1$. Unitary $N \times N$ matrices have N^2 real degrees of freedom, therefore the Lie group $U(N)$ depends on N^2 real parameters. In other words, the *dimension* of $U(N)$ is N^2. The condition on the determinant in $SU(N)$ reduces the number of independent real parameters by one, resulting in a dimension of $N^2 - 1$.
From the exponential map we find that the unitarity of the elements restricts the generators of these groups to be *hermitian*, $t_a = t_a^{\dagger}$, since,

$$U(\xi) = \exp(i\xi^a t_a), \tag{2.12}$$

$$U(\xi)^{-1} = \exp(-i\xi^a t_a), \tag{2.13}$$

$$U(\xi)^{\dagger} = \exp(-i\xi^a t_a^{\dagger}). \tag{2.14}$$

For the generators of a special unitary group an extra condition applies, $\mathrm{tr}(t_a) = 0$, since,

$$\exp(0) = 1 = \det(U(\xi)) = \det(\exp(i\xi^a t_a)) = \exp(i\xi^a \mathrm{tr}(t_a)) \tag{2.15}$$

2.2.1 U(1)

The unitary group $U(1)$ consists by definition of unitary 1×1 matrices, thus complex scalars U with $U^{-1} = \frac{1}{U} = U^*$. These can be represented by complex phases, $U = \exp(i\phi)$, in which ϕ is one real free parameter, as expected from the dimension of this group. This group is generated by a real scalar, consistent with the hermiticity condition. The resulting exponential map is consistent with the complex phase proposed earlier as representation. The commutation of numbers is zero, thus all structure constants for this group are zero, from which it is clear that this is an Abelian group.

2.2.2 SU(2)

The elements of $SU(2)$ are defined by their representation in unitary 2×2 matrices with unit determinant,

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}, \tag{2.16}$$

with $\det(U) = \alpha\alpha^* + \beta\beta^* = a^2 + b^2 + c^2 + d^2 = 1$, where $\alpha, \beta \in \mathbb{C}$ and $a, b, c, d \in \mathbb{R}$. Indeed we find three independent real parameters. The defining representation of the generators of $SU(2)$ must be three 2×2 hermitian traceless matrices. A common choice is $t_a = \frac{\sigma_a}{2}$ using the Pauli matrices which indeed obey these conditions,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.17}$$

An important property of these matrices is that for each of them $\sigma_a^2 = \mathbb{1}$. Using this one finds for the exponential map,

$$U = \exp(i\xi^a t_a) = \exp(i\xi n^a \sigma_a/2) = \cos(\xi/2)\mathbb{1} + i\sin(\xi/2) n^a \sigma_a, \tag{2.18}$$

$$= \begin{pmatrix} \cos(\xi/2) + i\sin(\xi/2)n^3 & i\sin(\xi/2)(n^1 - in^2) \\ i\sin(\xi/2)(n^1 + in^2) & \cos(\xi/2) - i\sin(\xi/2)n^3 \end{pmatrix}, \tag{2.19}$$

where $\xi^a = \xi n^a$ with $n^a n_a = 1$ is just a redefinition of the parameters. The obtained result is consistent with (2.16), since,

$$a = \cos(\xi/2), \quad b = \sin(\xi/2)n^3, \quad c = \sin(\xi/2)n^2, \quad d = \sin(\xi/2)n^1, \tag{2.20}$$

indeed fulfil $a^2+b^2+c^2+d^2 = 1$, from which we find that these generators indeed generate $SU(2)$. It can be checked that these generators obey the commutation relation,

$$[t_b, t_c] = i\epsilon_{abc}\, t_a, \tag{2.21}$$

where ϵ_{abc} is the fully antisymmetric Levi-Civita symbol. As always, there are many more useful irreducible representations, for instance used as higher spin representations. Since the three-dimensional (or spin-1) representation will be explicitly used in this thesis a convention for its generators is also stated here,

$$J_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.22}$$

One can check that $t_a = J_a^1$ also follow the Lie algebra (2.21).

2.2.3 U(2)

If the unit determinant condition is dropped the $U(2)$ group is found. The unitary 2×2 matrices defining $U(2)$ satisfy $U^{-1} = U^\dagger$, which for the determinant means,

$$\det(\mathbb{1}) = \det(UU^\dagger) = \det(U)\det(U^\dagger) = \det(U)\det(U)^*, \tag{2.23}$$

from which it is clear that $\det(U) = e^{i\theta}$ is a phase. Therefore,

$$U^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \frac{1}{e^{i\theta}}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = U^\dagger. \tag{2.24}$$

This can be solved for γ and δ to find that any unitary 2×2 matrix can be parametrized as,

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* e^{i\theta} & \alpha^* e^{i\theta} \end{pmatrix}, \tag{2.25}$$

which satisfies $\det(U) = e^{i\theta}$ if $\alpha\alpha^* + \beta\beta^* = 1$. Via redefinitions of α and β the place of the explicit phase can be shifted around to get a more suitable form if needed. Indeed we find 4 real free parameters in this representation, as should be for $U(2)$.

2.2.4 SU(3)

It is clear that explicit calculations on the eight-dimensional $SU(3)$ start becoming rather involved. One would expect the group to be generated by eight hermitian traceless 3×3 matrices, and indeed an explicit form of these is well known. It is common to define the generators using the Gell-Mann matrices, $t_a = \frac{\lambda_a}{2}$, where,

$$\begin{gathered}
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{gathered} \tag{2.26}$$

are indeed all hermitian and traceless.

2.2.5 SU(N)

This construction of the generators of the special unitary groups can be extended for even larger groups $SU(N)$. A basis must be found for $N\times N$ hermitian traceless matrices. The normalisation of the Pauli and Gell-Mann matrices is continued,

$$\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab}. \tag{2.27}$$

The labelling of these N^2-1 matrices becomes rather arbitrary, but the structure of these should be understood, since these matrices will be used later. There are $N(N-1)/2$ non-diagonal real matrices, of the form,

$$[t_a]_{xy} = \frac{1}{2}(\delta_{ix}\delta_{jy} + \delta_{jx}\delta_{iy}), \tag{2.28}$$

where $x, y = 1, \ldots N$ are the matrix indices, and $i, j = 1, \ldots N$ are two generator specific indices determining which two off-diagonal components are non-zero. To avoid double counting consider $i < j$. For example,

$$t_{i=1,j=2} = \frac{1}{2}\begin{pmatrix} 0 & 1 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.29}$$

The next $N(N-1)/2$ generators are non-diagonal imaginary matrices,

$$[t_a]_{xy} = \frac{-i}{2}(\delta_{ix}\delta_{jy} - \delta_{jx}\delta_{iy}), \tag{2.30}$$

with the same interpretation and range of x, y, i and j. The final $N-1$ real diagonal matrices are defined as [2],

$$[t_a]_{xy} = \frac{1}{\sqrt{2m(m+1)}}\left(\sum_{k=1}^{m}\delta_{kx}\delta_{ky} - m\delta_{m+1,x}\delta_{m+1,y}\right), \tag{2.31}$$

where x, y are again the matrix indices and $m = 1, \ldots, N-1$ is the generator specific index. One can check that the Pauli and Gell-Mann matrices indeed are of this structure, up to a factor $1/2$. The generator label a ranging from 1 to N^2-1 that fixes i, j and m can be chosen in many different ways, resulting in a reordering of the generators. Note again that this is only the defining representation.

2.3 Orthogonal and Special Orthogonal groups

The *Orthogonal groups* $O(N)$ are defined as the real *orthogonal* $N \times N$ matrices. This means that a group element O has an inverse $O^{-1} = O^T$ which is also a group element. *Special Orthogonal* adds the condition that $\det(O) = 1$. The generators of the orthogonal group must be *antisymmetric*, $t_a = -t_a^T$, since,

$$\begin{aligned} O(\xi) &= \exp(i\xi^a t_a), \\ O^{-1}(\xi) &= \exp(-i\xi^a t_a), \\ O^T(\xi) &= \exp(i\xi^a t_a^T). \end{aligned} \tag{2.32}$$

To make the antisymmetry of the generators clear it is common to consider the generators in the form $M_{ab} = -M_{ba}, a, b = 1, \ldots N$ instead. In this way the group elements are,

$$O(\xi) = \exp(i\xi^{ab}M_{ab}), \tag{2.33}$$

where ξ^{ab} are again real parameters. Since the group elements O are real by definition the generators t_a must be fully imaginary. Imaginary anti-symmetric $N\times N$ matrices have $(N^2-N)/2 = N(N-1)/2$ real degrees of freedom, from which it follows that the *dimension* of $O(N)$ is $N(N-1)/2$. The dimension of $SO(N)$ is the same, since antisymmetric matrices are already traceless, in contrast to $U(N)$ and $SU(N)$ there is no local distinction between $O(N)$ and

$SO(N)$. The difference is in the global properties, since $O(N)$ consists of two connected components, with $\det(O) = \pm 1$ respectively. These are the only two options since,

$$1 = \det(\mathbb{1}) = \det(OO^T) = \det(O)^2. \tag{2.34}$$

The defining representation can be parametrized as [2],

$$[M_{ab}]_{xy} = -i\left(\delta_{ax}\delta_{by} - \delta_{bx}\delta_{ay}\right), \tag{2.35}$$

where $x, y = 1, \ldots, N$ are the matrix indices. In comparing with (2.30) one finds that the generators of $SO(N)$ are included in the generators $SU(N)$, which is as expected since real orthogonal matrices are unitary. It follows that $SO(N)$ is a subgroup of $SU(N)$. From the defining representation one finds the Lie algebra to be,

$$[M_{ab}, M_{cd}] = i\left(\delta_{ac}M_{bd} - \delta_{ad}M_{bc} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad}\right). \tag{2.36}$$

2.4 Gauge theories and fields

Both in the Standard Model and in beyond the Standard Model physics *gauge symmetries* play an important role. In this section the occurrence of *gauge fields* is studied resulting from invariance under imposed local symmetries.
A field ψ is defined to transform as follows,

$$\psi(x) \to \psi'(x) = U\,\psi(x). \tag{2.37}$$

Any object transforming in this way is said to transform *covariantly*. As long as a *global symmetry* is considered the transformation of spacetime derivative of the field is covariant,

$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = U\,\partial_\mu\psi(x), \tag{2.38}$$

but if instead a *local transformation* is considered, in which the transformation is considered to be spacetime dependent, $U(x)$, the ordinary derivative no longer transforms covariantly,

$$\psi(x) \to \psi'(x) = U(x)\,\psi(x), \tag{2.39}$$
$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = U(x)\,\partial_\mu\psi(x) + \partial_\mu U(x)\,\psi(x). \tag{2.40}$$

A new derivative is introduced that transforms covariantly by construction, therefore called the *covariant derivative*,

$$D_\mu\psi(x) \to \left(D_\mu\psi(x)\right)' = U(x)\,D_\mu\psi(x). \tag{2.41}$$

This covariant derivative is defined as,

$$D_\mu\psi(x) = \partial_\mu\psi(x) - iW_\mu(x)\,\psi(x). \tag{2.42}$$

From now on the spacetime dependence of the fields is considered understood and will no longer be written explicitly. The transformation of W_μ must be such that the covariant derivative follows the defining transformation (2.41), therefore,

$$\begin{aligned} iW_\mu\,\psi \to (iW_\mu\,\psi)' &= (\partial_\mu\psi - D_\mu\psi)', \\ &= \partial_\mu U\,\psi + U\,\partial_\mu\psi - U\,D_\mu\psi, \\ &= \partial_\mu U\,\psi + U\,\partial_\mu\psi - U\,\partial_\mu\psi + U\,iW_\mu\,\psi, \\ &= (U\,iW_\mu + \partial_\mu U)\,\psi, \\ &= (U\,iW_\mu + \partial_\mu U)\,U^{-1}\,\psi'. \end{aligned} \tag{2.43}$$

From this we find that W_μ must transform as follows,

$$W_\mu \to W'_\mu = UW_\mu U^{-1} - i\partial_\mu U\,U^{-1}. \tag{2.44}$$

Now consider the order expansion of the exponential map (2.2) for the transformation matrix $U(x)$.

$$U(x) = \exp\left(i\xi^a(x)\, t_a\right) = 1 + i\xi^a(x)\, t_a + \mathcal{O}(\xi^2), \tag{2.45}$$

W_μ can also be decomposed in a linear combination of the generators,

$$W_\mu(x) = B^a_\mu(x)\, t_a, \tag{2.46}$$

where the parameters B^a_μ are called the *gauge fields.* Here the generators can in principle still be in any representation, but if the field ϕ transforms under the defining representation, as is often the case, the generators are those of the *adjoint representation.* The first order transformation of these gauge fields can be derived from (2.44),

$$\begin{aligned} iB^a_\mu\, t_a \to iB^{a\prime}_\mu\, t_a &= (1 + i\xi^a t_a)\, iB^b_\mu\, t_b\, (1 - i\xi^c t_c) + i\partial_\mu \xi^a\, t_a \left(1 - i\xi^b t_b\right) + \mathcal{O}(\xi^2), \\ &= iB^a_\mu\, t_a - \xi^a t_a\, B^b_\mu t_b + B^b_\mu t_b\, \xi^c t_c + i\partial_\mu \xi^a\, t_a + \mathcal{O}(\xi^2), \\ &= iB^a_\mu\, t_a - \xi^b\, B^c_\mu\, [t_b,\, t_c] + i\partial_\mu \xi^a\, t_a + \mathcal{O}(\xi^2), \\ &= iB^a_\mu\, t_a - f^a_{bc}\, \xi^b\, B^c_\mu\, t_a + i\partial_\mu \xi^a\, t_a + \mathcal{O}(\xi^2). \end{aligned} \tag{2.47}$$

Resulting in the first order transformation of the gauge fields,

$$B^a_\mu \to B^{a\prime}_\mu = B^a_\mu + i f^a_{bc}\, \xi^b\, B^c_\mu + \partial_\mu \xi^a. \tag{2.48}$$

Field strength

Since the covariant derivative of the field ψ transforms covariantly,

$$D_\mu \to D'_\mu = U D_\mu U^{-1}, \tag{2.49}$$

any product of these covariant derivative also transforms in the same way. For instance the commutation between two covariant derivatives, here introduced in combination with a test function ψ,

$$\begin{aligned} G_{\mu\nu}\, \psi = i[D_\mu,\, D_\nu]\psi &= i[\partial_\mu - iW_\mu, \partial_\nu - iW_\nu]\psi, \\ &= (\partial_\mu W_\nu - \partial_\nu W_\mu - i[W_\mu,\, W_\nu])\; \psi. \end{aligned} \tag{2.50}$$

This can again be decomposed in terms of the generators, $G_{\mu\nu} = G^a_{\mu\nu} t_a$, where $G^a_{\mu\nu}$ is called the *field strength.* In terms of the gauge fields B^a_μ the field strength is therefore,

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - i f^a_{bc}\, B^b_\mu\, B^c_\nu. \tag{2.51}$$

This field strength will be relevant in the construction of Lagrangians as the kinetic term for the gauge fields.

Coupling constant

For describing physical systems based on these gauge theories a measure of strength of the interaction must be included. This is implemented by the introduction of a *coupling constant* g, via the following rescaling of the parameters,

$$B^a_\mu \to g\, B^a_\mu, \qquad G^a_{\mu\nu} \to g\, G^a_{\mu\nu}, \qquad \xi^a \to g\, \xi^a. \tag{2.52}$$

Therefore the covariant derivative (2.42) reads,

$$D_\mu = \partial_\mu - ig\, B^a_\mu t_a. \tag{2.53}$$

The transformation up to order ξ^2 of the gauge fields (2.48) is,

$$B^a_\mu \to B^{a\prime}_\mu = B^a_\mu + ig\, f^a_{bc}\, \xi^b\, B^c_\mu + \partial_\mu \xi^a. \tag{2.54}$$

The definition of the field strength (2.51) becomes,

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - ig\, f^a_{bc}\, B^b_\mu\, B^c_\nu. \tag{2.55}$$

2.5 Spacetime symmetries

One of the most important symmetries in physics is the symmetry of spacetime itself. Since from *Special Relativity* we know that the speed of light is the same in all inertial reference frames the following relation between coordinate systems x'^{μ} and x^{μ} holds [4],

$$g_{\mu\nu}dx'^{\mu}dx'^{\nu} = g_{\mu\nu}dx^{\mu}dx^{\nu}, \tag{2.56}$$

from which it follows that,

$$g_{\mu\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}g_{\alpha\beta} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}g_{\alpha\beta}, \tag{2.57}$$

the coordinate transformations Λ leave the Minkowski metric invariant. The transformation of spacetime coordinates is therefore,

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu}x^{\nu}, \tag{2.58}$$

These transformations form the *Lorentz group*, with identity element $\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu}$ and inverse $(\Lambda^{-1})^{\mu}_{\ \nu} = \Lambda_{\mu}^{\ \nu}$, since from (2.57),

$$\Lambda^{\mu}_{\ \alpha}\Lambda_{\nu}^{\ \alpha} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\beta}_{\ \gamma}g_{\beta\nu}g^{\alpha\gamma} = \frac{1}{4}g_{\alpha\gamma}g^{\mu\beta}g_{\beta\nu}g^{\alpha\gamma} = \delta^{\mu}_{\ \nu}. \tag{2.59}$$

Since physics is supposed to be the same in all space and at all time, theories are also supposed to be invariant under translations in spacetime, $x'^{\mu} = x^{\mu} + a^{\mu}$. The combination of Lorentz transformations and spacetime translations form a bigger group, the *Poincaré group* with elements $U(\Lambda, a)$, under which the spacetime coordinates transform as,

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu}x^{\nu} + a^{\mu}. \tag{2.60}$$

From performing this transformation twice it is clear that the group multiplication of the Poincaré group is,

$$U(\tilde{\Lambda}, \tilde{a})U(\Lambda, a) = U(\tilde{\Lambda}\Lambda, \tilde{\Lambda}a + \tilde{a}). \tag{2.61}$$

The identity element of this group is $U(\mathbb{1}, 0)$, from which it is clear that the inverse $U(\Lambda, a)^{-1} = U(\Lambda^{-1}, -\Lambda^{-1}a)$.
It is clear from the above that the Lorentz group is a subgroup of the Poincaré group, with $a = 0$. This Lorentz group is in itself not a connected group, but can be split in four parts [1, 4]. From equation (2.57) it is clear that $\det(\Lambda)^2 = 1$ and $(\Lambda^{0}_{\ 0})^2 = 1 + (\Lambda^{i}_{\ 0})^2$, from which it follows that,

$$\det(\Lambda) = +1 \text{ or } \det(\Lambda) = -1 \text{ and } \Lambda^{0}_{\ 0} \geq +1 \text{ or } \Lambda^{0}_{\ 0} \leq -1. \tag{2.62}$$

Therefore the Lorentz group can be split up in four disconnected subsets, related to each other via the discrete transformations,

$$\mathcal{P}^{\mu}_{\ \nu} = \mathrm{diag}(1, -1, -1, -1), \quad \mathcal{T}^{\mu}_{\ \nu} = \mathrm{diag}(-1, 1, 1, 1), \tag{2.63}$$

corresponding to space and time coordinate inversion respectively. The subset with $\det(\Lambda) = 1$ and $\Lambda^{0}_{\ 0} \geq +1$ form a subgroup $L^{\uparrow}_{+}$, called the *proper orthochronous Lorentz group*. The other subsets do not form subgroups since the identity element $\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu}$ is not in these sets. An overview of the sets and their relations is given in table 2.1

	$\det(\Lambda) = +1$		$\det(\Lambda) = -1$
$\Lambda^{0}_{\ 0} \geq +1$	$L^{\uparrow}_{+}$	$\overset{\mathcal{P}}{\leftrightarrow}$	$L^{\uparrow}_{-}$
	$\updownarrow \mathcal{T}$		$\updownarrow \mathcal{T}$
$\Lambda^{0}_{\ 0} \leq -1$	$L^{\downarrow}_{+}$	$\overset{\mathcal{P}}{\leftrightarrow}$	$L^{\downarrow}_{-}$

Table 2.1: Overview of the connected subsets of the Lorentz group and the discrete transformation between them, based on [3].

Since the Lorentz transformations correspond to rotations in (3,1)-dimensional spacetime, the corresponding Lie group is $SO(3,1)$. The corresponding Lie algebra, the Lorentz algebra, is [3],

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma}\right), \tag{2.64}$$

which is the extension from Euclidean (2.36) to Minkowski spacetime.

2.6 C, P and T transformations

Next to the continuous transformations described above there are two *discrete spacetime symmetries*, *parity* and *time-reversal*. These are the connections between the subsets of the Lorentz group, suitably labelled $\mathcal{P}$ and $\mathcal{T}$ before. Parity mirrors the space dimensions $(t,x) \to (t,-x)$, where time reversal mirrors time, $(t,x) \to (-t,x)$.
A third discrete symmetry, unrelated to spacetime symmetries, that is often discussed in the same context is *charge conjugation* $\mathcal{C}$, which brings particles to their anti-particle and vice versa. These symmetries are discussed in great detail in many other texts, such as [3, 4], here only the results are given as they will be needed later,

$$\begin{aligned}
\mathcal{P}\psi(t,x)\mathcal{P}^{-1} &= \eta\gamma^0\psi(t,-x), & \mathcal{P}\bar{\psi}(t,x)\mathcal{P}^{-1} &= \eta^*\bar{\psi}(t,-x)\gamma^0, \\
\mathcal{T}\psi(t,x)\mathcal{T}^{-1} &= \zeta\gamma^1\gamma^3\psi(-t,x), & \mathcal{T}\bar{\psi}(t,x)\mathcal{T}^{-1} &= \zeta^*\bar{\psi}(-t,x)\gamma^1\gamma^3, \\
\mathcal{C}\psi(t,x)\mathcal{C}^{-1} &= \xi i\gamma^2\gamma^0\bar{\psi}^T(t,x), & \mathcal{C}\bar{\psi}(t,x)\mathcal{C}^{-1} &= \xi^* i\psi^T(t,x)\gamma^2\gamma^0.
\end{aligned} \tag{2.65}$$

The gamma matrices here are the Dirac matrices as discussed before in appendix A. The phases η, ζ and ξ can often be all set to one to simplify computations, since they do not appear in bilinears [3]. Note that the transpose in these equations only works on the spinor indices. Often the *left* and *right-handed chirality* components are treated separately, defined as,

$$P_L\psi = \frac{1-\gamma^5}{2}\psi = \psi_L, \qquad P_R\psi = \frac{1+\gamma^5}{2}\psi = \psi_R. \tag{2.66}$$

It is interesting to consider what happens with these components under the discrete transformations. From filling in the definitions above and using properties of the Dirac matrices one finds,

$$\begin{aligned}
\mathcal{P}\psi_L(t,x)\mathcal{P}^{-1} &= \gamma^0\psi_R(t,-x), & \mathcal{P}\bar{\psi}_L(t,x)\mathcal{P}^{-1} &= \bar{\psi}_R(t,-x)\gamma^0, \\
\mathcal{T}\psi_L(t,x)\mathcal{T}^{-1} &= \gamma^1\gamma^3\psi_L(-t,x), & \mathcal{T}\bar{\psi}_L(t,x)\mathcal{T}^{-1} &= \bar{\psi}_L(-t,x)\gamma^1\gamma^3, \\
\mathcal{C}\psi_L(t,x)\mathcal{C}^{-1} &= i\gamma^2\gamma^0\bar{\psi}_R^T(t,x), & \mathcal{C}\bar{\psi}_L(t,x)\mathcal{C}^{-1} &= i\psi_R^T(t,x)\gamma^2\gamma^0.
\end{aligned} \tag{2.67}$$

where one can also swap all left and right labels. The general phase is left out for simplicity. Not that $\mathcal{P}$ and $\mathcal{C}$ relate lift and right-handed spinors, while $\mathcal{T}$ does not. A combined transformation of charge conjugation and parity will leave the chirality invariant,

$$\mathcal{CP}\psi_L(t,x)\mathcal{CP}^{-1} = -i\gamma^2\bar{\psi}_L^T(t,-x), \qquad \mathcal{CP}\bar{\psi}_L(t,x)\mathcal{CP}^{-1} = i\psi_L^T(t,-x)\gamma^2. \tag{2.68}$$

The transformation of scalar and vector fields and more straightforward to find. Since scalar fields do not transform under spacetime transformations by definition, the discrete transformations are,

$$\begin{aligned}
\mathcal{P}\phi(t,x)\mathcal{P}^{-1} &= \eta\phi(t,-x), \\
\mathcal{T}\phi(t,x)\mathcal{T}^{-1} &= \zeta\phi(-t,x), \\
\mathcal{C}\phi(t,x)\mathcal{C}^{-1} &= \xi\phi^*(t,x).
\end{aligned} \tag{2.69}$$

where the in general η, ζ and ξ are included for completeness. These can be different from the phases for spinor fields. Since the fields W_μ transform as vectors under spacetime transformations should transform accordingly. The transformations of vectors are [4],

$$\begin{aligned}
\mathcal{P}W_\mu(t,x)\mathcal{P}^{-1} &= -\eta\mathcal{P}_\mu{}^\nu W_\nu(t,-x), \\
\mathcal{T}W_\mu(t,x)\mathcal{T}^{-1} &= -\zeta\mathcal{T}_\mu{}^\nu W_\nu(-t,x), \\
\mathcal{C}W_\mu(t,x)\mathcal{C}^{-1} &= \xi W_\mu(t,x).
\end{aligned} \tag{2.70}$$

From the Coleman-Mandula theorem [9] we know that the generators of the internal symmetries, such as the gauge symmetries, are not allowed to transform in any but the trivial way. This fixes the transformation of the gauge bosons W_μ^a under parity and time reversal, but since charge conjugation is not a spacetime symmetry the transformation of the gauge bosons still has freedom. It will therefore be derived separately when it is needed.

Chapter 3

The Standard Model

In the *Standard Model* the building blocks of matter and the interactions between those via quantum mechanics are described. This *quantum field theory* includes *leptons* and *quarks*, which can interact by the exchange of *gauge bosons* corresponding to the *strong nuclear force*, the *weak nuclear force* and the *electromagnetic force*. The *Higgs bosons*, responsible for the mass of many fundamental particles, is also included in the Standard Model. The lack of describing gravity, neutrino oscillations, and the observation of dark matter are among the reasons to assume that the Standard Model is not a complete theory. Nevertheless, the Standard Model has enormous successes in both describing and predicting phenomena in particle physics. It should therefore be considered a fundamental basis for any theory beyond it, such as unification or a theory of everything. Therefore some important aspects of the Standard Model will be revisited here, to build a framework in which the possible extensions can be understood.

3.1 The Electroweak interaction

In the Standard Model the weak and electromagnetic interactions are unified according to the *Glashow-Weinberg-Salam* theory. This theory is based on a *spontaneously broken* $SU(2)_L \times U(1)_Y$ gauge symmetry. The $SU(2)_L$ *weak isospin* symmetry is generated by the Pauli matrices (2.17). The generator of $U(1)_Y$ is the real scalar Y, called the *hypercharge*. Note that the interactions have different *couplings* g and g'. An arbitrary transformation under this gauge symmetry is therefore,

$$U(x) = e^{i\xi^a(x)\sigma_a/2} e^{i\theta(x)Y}, \tag{3.1}$$

where $a = 1, 2, 3$ ensures summation over all $SU(2)_L$ generators. From (2.53) the covariant derivative is therefore,

$$D_\mu^{ij} = \partial_\mu \delta^{ij} - ig W_\mu^a \frac{\sigma_a^{ij}}{2} - ig' B_\mu Y \delta^{ij}, \tag{3.2}$$

The isospin indices $i, j = 1, 2$ are denoted explicitly. In this equation W_μ^a and B_μ are the *gauge bosons* corresponding to the $SU(2)$ and $U(1)$ symmetry respectively.

3.1.1 The Higgs mechanism

To describe the observed particles and interactions the gauge symmetry must be spontaneously broken. This means that although the Lagrangian is constructed to be invariant under $SU(2)_L \times U(1)_Y$ transformations, the vacuum is not. The theory must be constructed such that the vacuum is however invariant under a $U(1)_Q$ gauge transformation, where Q stands for the *electric charge*.

The symmetry breaking is performed by a complex scalar field ϕ, with isospin and hypercharge of 1/2, therefore transforming as a doublet under $SU(2)_L$. An example of a scalar potential that is invariant under these transformations and allows spontaneous symmetry breaking is the Mexican hat potential,

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2, \tag{3.3}$$

with μ and λ real constants. For $\lambda < 0$ this potential is not bounded from below and therefore unphysical. For $\lambda > 0$ it obtains a minimum for,

$$\phi^\dagger \phi = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}. \tag{3.4}$$

Every *vacuum expectation value* $\langle\phi\rangle$ satisfying this minimum equation is a vacuum state, resulting in equivalent physical quantities. One can therefore pick any orientation, thereby fixing the gauge. A common choice is,

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \tag{3.5}$$

in which v is the constant real scalar defined in (3.4). A gauge transformation does not leave this state invariant but takes it to another vacuum state. The $SU(2)_L \times U(1)_Y$ is thus spontaneously broken.
The kinetic term $|D_\mu\phi|^2$ at this vacuum state is,

$$|D_\mu\langle\phi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}\partial_\mu - \frac{ig}{2}W_\mu^3 - \frac{ig'}{2}B_\mu & -\frac{ig}{2}\left(W_\mu^1 - iW_\mu^2\right)\\ -\frac{ig}{2}\left(W_\mu^1 + iW_\mu^2\right) & \partial_\mu + \frac{ig}{2}W_\mu^3 - \frac{ig'}{2}B_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2, \tag{3.6}$$

$$= \frac{v^2}{8}\left(g^2\left|W_\mu^1 - iW_\mu^2\right|^2 + \left|gW_\mu^3 - g'B_\mu\right|^2\right). \tag{3.7}$$

From this we recognize three *massive vector bosons*: one particle anti-particle pair,

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \quad \text{with mass} \quad m_W = g\frac{v}{2}, \tag{3.8}$$

and one that is its own antiparticle,

$$Z_\mu^0 = \frac{1}{\sqrt{g^2 + g'^2}}\left(gW_\mu^3 - g'B_\mu\right) \quad \text{with mass} \quad m_Z = \sqrt{g^2 + g'^2}\,\frac{v}{2}. \tag{3.9}$$

Indeed one finds that only three linear combinations of the four gauge bosons W_μ^a and B_μ arise in this Lagrangian. One vector boson remains massless, the linear combination orthogonal to Z_μ^0,

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(g'W_\mu^3 + gB_\mu\right). \tag{3.10}$$

These relations can be inverted to,

$$\begin{aligned} W_\mu^1 &= \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right), & W_\mu^3 &= \frac{1}{\sqrt{g^2 + g'^2}}\left(g'A_\mu + gZ_\mu^0\right),\\ W_\mu^2 &= \frac{i}{\sqrt{2}}\left(W_\mu^+ - W_\mu^-\right), & B_\mu &= \frac{1}{\sqrt{g^2 + g'^2}}\left(gA_\mu - g'Z_\mu^0\right), \end{aligned} \tag{3.11}$$

From filling in these re-parametrizations in (3.2) one finds the covariant derivative in terms of the mass eigenstates of the gauge bosons,

$$\begin{aligned} D_\mu = \partial_\mu &- \frac{ig}{\sqrt{2}}W_\mu^+\frac{1}{2}(\sigma_1 + i\sigma_2) - \frac{ig}{\sqrt{2}}W_\mu^-\frac{1}{2}(\sigma_1 - i\sigma_2)\\ &- \frac{i}{\sqrt{g^2 + g'^2}}Z_\mu^0\left(g^2\frac{\sigma_3}{2} - g'^2 Y\right) - \frac{igg'}{\sqrt{g^2 + g'^2}}A_\mu\left(\frac{\sigma_3}{2} + Y\right). \end{aligned} \tag{3.12}$$

In the last term the electromagnetic interaction can be recognized, carried by the massless *photon*, in which the coupling is the *elementary charge* e and the generator is the *electric charge* Q,

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}, \qquad Q = \frac{\sigma_3}{2} + Y. \tag{3.13}$$

This interpretation is consistent with the previous choices. The components of the scalar doublet with hypercharge 1/2 have an electric charge of $+e$ and 0 respectively, and can therefore be suitably labelled as,

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{3.14}$$

The vacuum in the form of (3.5) is recognized to be chargeless, as it is supposed to be.

Symmetries after breaking

To find out what is left over of the $SU(2)_L \times U(1)_Y$ symmetry after breaking an arbitrary transformation (3.1) up to first order is considered.

$$\frac{1}{\sqrt{2}} \left(\mathbb{1} + \frac{i}{2} \begin{pmatrix} \alpha^3 + \beta & \alpha^1 + i\alpha^2 \\ \alpha^1 - i\alpha^2 & \alpha^3 - \beta \end{pmatrix} \right) \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} + \frac{iv}{2\sqrt{2}} \begin{pmatrix} \alpha^1 + i\alpha^2 \\ \alpha^3 - \beta \end{pmatrix}. \tag{3.15}$$

The vacuum is invariant under transformations if the last term is zero. This leads to the restrictions,

$$\alpha^1 = \alpha^2 = 0, \qquad \alpha^3 = \beta \equiv \alpha, \tag{3.16}$$

resulting in the transformation that leaves the vacuum invariant,

$$U = \exp\left(i \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \right) = e^{i\alpha Q}. \tag{3.17}$$

The electric charge enters here via its definition (3.13), from which it is clear that after spontaneous symmetry breaking a $U(1)_Q$ symmetry survives, the symmetry of *Quantum Electro Dynamics*. This is again consistent with the findings before. The degree of freedom corresponding to one generator survives, resulting in one massless gauge boson A_μ, while the three other degrees of freedom are absorbed by the $W_\mu^\pm$ and Z_μ^0 resulting in their mass. In group theoretical terms this Standard Model Higgs mechanism could be described as,

$$SU(2)_L \times U(1)_Y \to U(1)_Q \tag{3.18}$$

3.1.2 The Higgs boson

A general expansion of the doublet ϕ around its vacuum expectation value can be parametrized as,

$$\phi(x) = U(x) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tag{3.19}$$

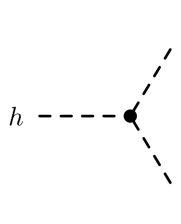
where $U(x)$ is a gauge transformation. The real scalar excitation $h(x)$ is called the *Higgs field*, corresponding to the Higgs boson. If now one again considers the Mexican-hat potential (3.3), one finds,

$$V(\phi) = -\frac{\mu^2}{2}(v + h(x))^2 + \frac{\lambda}{4}(v + h(x))^4 = -\frac{\mu^4}{4\lambda} + \mu^2 h^2 + \sqrt{\lambda}\mu h^3 + \frac{\lambda}{4} h^4. \tag{3.20}$$

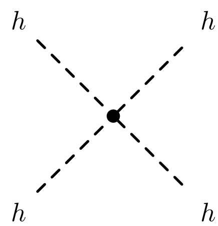
In this Lagrangian a constant term occurs that can be scaled away. Also a mass term can be recognized from which the Higgs mass is predicted to be,

$$m_h = \sqrt{2}\mu. \tag{3.21}$$

The self interactions of the Higgs field can also be found from (3.20),



$= -6i\sqrt{\lambda}\mu,$

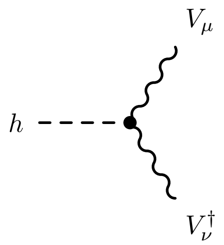


$= -6i\lambda.$

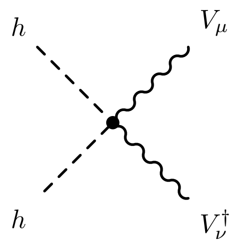
Now consider the kinetic term $|D_\mu \phi|^2$ in terms of the mass eigenstates of the gauge bosons (3.12) and for the expansion around the vacuum expectation value,

$$
\begin{aligned}
|D_\mu \phi|^2 &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_\mu - \frac{i(g^2-g'^2)}{2\sqrt{g^2+g'^2}} Z^0_\mu - ieA_\mu & -\frac{ig}{\sqrt{2}} W^+_\mu \\ -\frac{ig}{\sqrt{2}} W^-_\mu & \partial_\mu + i\sqrt{g^2+g'^2} Z^0_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \right|^2 , \\
&= \frac{1}{2} (\partial_\mu h)^2 + g^2 \frac{v^2}{4} W^+_\mu W^{-\mu} \left(1 + \frac{h}{v}\right)^2 + (g^2 - g'^2)^2 \frac{v^2}{4} Z^0_\mu Z^{0\mu} \left(1 + \frac{h}{v}\right)^2 , \\
&= \frac{1}{2} (\partial_\mu h)^2 + \left(m_W^2 W^+_\mu W^{-\mu} + \frac{m_Z^2}{2} Z^0_\mu Z^{0\mu} \right) \left(1 + \frac{2}{v} h + \frac{1}{v^2} h^2 \right).
\end{aligned}
\tag{3.22}
$$

Here we recognize the kinetic term for the Higgs field, the mass terms for the gauge bosons and the following interactions between the Higgs and massive gauge bosons ($V_\mu = W^+_\mu, W^-_\mu, Z^0_\mu$),



$$= \frac{2im_V^2}{v} g_{\mu\nu},$$



$$= \frac{2im_V^2}{v^2} g_{\mu\nu}.$$

3.2 The Strong interaction

From the group theoretical point of view the *Strong interaction* is less involved that the electroweak interaction. The $SU(3)_C$ gauge symmetry on which the theory is based is not broken. The gauge transformation is generated by the Gell-Mann matrices (2.26),

$$U(x) = e^{i\xi^a(x)\lambda_a/2}, \tag{3.23}$$

where a ensures summation over all eight generators. The corresponding covariant derivative (2.53) is,

$$D^{ij}_\mu = \partial_\mu \delta^{ij} - ig_S\, G^a_\mu \frac{\lambda^{ij}_a}{2}, \tag{3.24}$$

where $i, j = 1, 2, 3$ are now *colour* indices. The eight gauge bosons G^a_μ are the massless *gluons*. This short description of the Strong interaction skips over the many complications this non-abelian gauge theory brings. This summary is however enough for the arguments in this thesis.

3.3 Fermion content

Next to the Higgs and gauge bosons the Standard Model involves fields corresponding to the *leptons* and *quarks*. These fermions are described by the *Dirac Lagrangian*,

$$\mathcal{L}_D = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) = \bar{\psi}(x)(i\not{\partial} - m)\psi(x), \tag{3.25}$$

This Lagrangian can be generalized to a gauge invariant form by using the covariant derivative D_μ instead of the ordinary derivative ∂_μ, to describe the interactions with gauge bosons. Spinors can be decomposed in their right-handed and left-handed *chirality* component using the projection operators,

$$P_L \psi = \frac{1 - \gamma^5}{2} \psi = \psi_L, \qquad P_R \psi = \frac{1 + \gamma^5}{2} \psi = \psi_R. \tag{3.26}$$

where the projection operators follow,

$$P_L^2 = P_L, \qquad P_R^2 = P_R, \qquad P_L P_R = 0. \tag{3.27}$$

Using this and the anti-commutation of the Dirac matrices one find the decomposition in left- and right-handed spinors,

$$\mathcal{L}_D = \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L), \tag{3.28}$$

The left- and right-handed leptons and quarks behave differently in the Standard Model. Only the left-handed fermions transform under the defining representation of $SU(2)_L$, explaining the subscript. The right-handed fermions transform trivially under this symmetry and therefore have zero weak isospin. The lepton sector consists therefore of left-handed isospin doublets and right-handed isospin singlets. For the quarks this implies,

$$Q_L = \left(\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L \right), \quad u_R = (u_R, c_R, t_R), \quad d_R = (d_R, s_R, b_R), \tag{3.29}$$

where all three generations are denoted explicitly. For the lepton sector we find,

$$L_L = \left(\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \right), \quad e_R = (e^-_R, \mu^-_R, \tau^-_R). \tag{3.30}$$

No right-handed *neutrino* is considered in the Standard Model.
The upper and lower components in Q_L and in L_L differ by one in electric charge, according to the definition in (3.13). The hypercharge of the different multiplets are fixed by the experimentally known electric charge of the content. The *up-like* quarks (*up*, *charm* and *top*) have an electric charge of $+\frac{2}{3}e$, while *down-like* quarks (*down*, *strange* and *bottom*) have electric charge $-\frac{1}{3}e$. Therefore,

$$Y(Q_L) = \frac{1}{6}, \quad Y(u_R) = \frac{2}{3}, \quad Y(d_R) = -\frac{1}{3}, \tag{3.31}$$

Since the *electron*, *muon* and *tauon* (or *tau*) have electric charge $-e$, while the neutrinos are chargeless, the hypercharge is,

$$Y(L_L) = -\frac{1}{2}, \quad Y(e_R) = -1. \tag{3.32}$$

Only the quarks take part in the strong interaction, resulting in an $SU(3)_C$ colour charge for each quark, arbitrarily labelled *red, blue* and *green.* The leptons transform trivially under $SU(3)_C$ and therefore carry no colour. The colour index is not denoted in (3.29) for short.
It is common to label the transformation of the fermions by the dimension of $SU(3)_C$ and $SU(2)_L$ under which these transform, and their hypercharge. The resulting fermion content of the Standard Model and its transformation under $SU(3)_C \times SU(2)_L \times U(1)_Y$ is,

$$\begin{aligned} Q_L &= (3, 2, 1/6), \qquad u_R = (3, 1, 2/3), \qquad d_R = (3, 1, -1/3), \\ L_L &= (1, 2, -1/2), \qquad e_R = (1, 1, -1). \end{aligned} \tag{3.33}$$

3.3.1 Coupling of gauge bosons to fermions

The gauge invariant Dirac Lagrangian (3.25) for these massless fermion fields is,

$$\mathcal{L}_f = \bar{Q}^i_L i D\!\!\!\!/ Q^i_L + \bar{u}^i_R i D\!\!\!\!/ u^i_R + \bar{d}^i_R i D\!\!\!\!/ d^i_R + \bar{L}^i_L i D\!\!\!\!/ L^i_L + \bar{e}^i_R i D\!\!\!\!/ e^i_R, \tag{3.34}$$

where $i = 1, 2, 3$ is the generation index. The covariant derivatives contain contributions from the electroweak (3.2) and strong (3.24) sector.

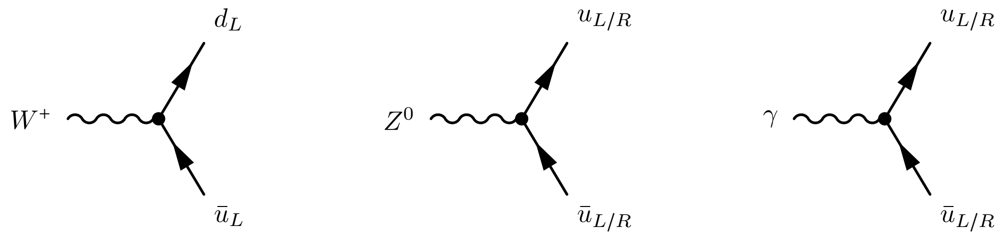
The electroweak interactions between the fermion fields and gauge bosons are,

$$\mathcal{L}_f^{EW} = \frac{g}{\sqrt{2}} W^+_\mu \left(\bar{u}^i_L \gamma^\mu d^i_L + \bar{\nu}^i_L \gamma^\mu e^i_L\right) + \frac{g}{\sqrt{2}} W^-_\mu \left(\bar{d}^i_L \gamma^\mu u^i_L + \bar{e}^i_L \gamma^\mu \nu^i_L\right) \tag{3.35}$$

$$\begin{aligned} &+ \frac{1}{\sqrt{g^2 + g'^2}} Z^0_\mu \left(\frac{3g^2 - g'^2}{6} \bar{u}^i_L \gamma^\mu u^i_L - \frac{3g^2 + g'^2}{6} \bar{d}^i_L \gamma^\mu d^i_L \right. \\ &\qquad - \frac{2g'^2}{3} \bar{u}^i_R \gamma^\mu u^i_R + \frac{g'^2}{3} \bar{d}^i_R \gamma^\mu d^i_R \\ &\qquad \left. + \frac{g^2 + g'^2}{2} \bar{\nu}^i_L \gamma^\mu \nu^i_L - \frac{g^2 - g'^2}{2} \bar{e}^i_L \gamma^\mu e^i_L + g'^2 \bar{e}^i_R \gamma^\mu e^i_R \right) \end{aligned} \tag{3.36}$$

$$+ eA_\mu \left(\frac{2}{3} \bar{u}^i \gamma^\mu u^i - \frac{1}{3} \bar{d}^i \gamma^\mu d^i - \bar{e}^i \gamma^\mu e^i \right). \tag{3.37}$$

In these equations there is summed over the generation index $i = 1, 2, 3$. It is clear that the $W^\pm$ indeed does not couple to right-handed fields, it couples the two elements of the doublets. The Z^0 couples to each particle-antiparticle pair with a slightly different coupling constant. The photon treats left- and right-handed particles equivalently, but does not couple to the chargeless neutrinos. The first interactions in (3.35), (3.36) and (3.37) are illustrated below.



3.3.2 Fermion masses and Yukawa interactions

From (3.28) it is clear that a Dirac mass term couples left to right-handed fields. Since these fields transform differently under the gauge group this term is forbidden in the Standard Model. Therefore no explicit term in the Lagrangian can ensure the mass of the fermions, the mass is also obtained via the Higgs mechanism. Consider the following Yukawa coupling of the fermion fields to the Higgs field ϕ,

$$\mathcal{L}_Y = -g^{ij}_\psi \bar{\psi}^i_L \phi \psi^j_R - g^{\dagger ij}_\psi \bar{\psi}^i_R \phi^\dagger \psi^j_L. \tag{3.38}$$

If now the expansion around the vacuum expectation value of ϕ is considered, and the lepton representations are filled in for the fermion fields the Lagrangian becomes,

$$\mathcal{L}_{Ye} = -\frac{v}{\sqrt{2}} g^{ij}_e \bar{e}^i_L e^j_R \left(1 + \frac{h}{v}\right) - \frac{v}{\sqrt{2}} g^{\dagger ij}_e \bar{e}^i_R e^j_L \left(1 + \frac{h}{v}\right). \tag{3.39}$$

Since the Yukawa coupling matrix was not required to be diagonal there is a coupling between different lepton generations. One can however transform g^{ij}_e to a real diagonal matrix M^{ij}_e by a unitary basis transformation,

$$M_e = U^\dagger_{eL} g_e U_{eR} = M^\dagger_e, \quad e_L = U_{eL} e'_L, \quad e_R = U_{eR} e'_R. \tag{3.40}$$

The Yukawa interaction term in this new basis is,

$$\mathcal{L}_{Ye} = -\frac{v}{\sqrt{2}} M^{ij}_e \left(\bar{e}'^i_L e'^j_R + \bar{e}'^i_R e'^j_L\right) \left(1 + \frac{h}{v}\right) = -\frac{v}{\sqrt{2}} M^{ij}_e \bar{e}'^i e'^j \left(1 + \frac{h}{v}\right), \tag{3.41}$$

in which we recognize mass terms and couplings to the Higgs field h. The masses of the fermions are therefore parametrized as,

$$m^i_{e'} = \frac{v}{\sqrt{2}} M^{ii}_e. \tag{3.42}$$

In the Yukawa interaction for the quarks an extra term is introduced to ensure the mass of the up-like quarks, with scalar field related to the Higgs field as,

$$\tilde{\phi} = i\sigma_2 \phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}, \tag{3.43}$$

where $\phi^- = \phi^{+*}$. The following Yukawa interaction is considered,

$$\mathcal{L}_{Yq} = -g_d^{ij}\overline{Q}_L^i \phi d_R^j - g_u^{ij}\overline{Q}_L^i \tilde{\phi} u_R^j + h.c., \tag{3.44}$$

where now $h.c.$ denotes the hermitian conjugate of the previous terms, for short. Note that in general the Yukawa couplings need not be the same, resulting in the subscript u and d. After expanding around the vacuum again a transformation to diagonal coupling matrices is considered,

$$M_q = U_{qL}^\dagger g_q U_{qR}, \quad q_L = U_{qL} q'_L, \quad q_R = U_{qR} q'_R, \quad q = d, u, \tag{3.45}$$

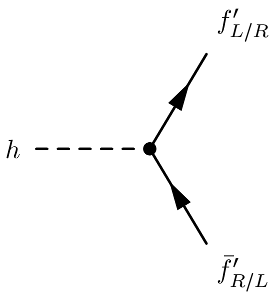
resulting in the following Yukawa coupling,

$$\mathcal{L}_{Yq} = -\frac{v}{\sqrt{2}}\left(M_d^{ij}\bar{d}'^i d'^j + M_u^{ij}\bar{u}'^i u'^j\right)\left(1 + \frac{h}{v}\right). \tag{3.46}$$

The mass of the quarks is therefore parametrized in a similar way to the leptons,

$$m_{d'}^i = \frac{v}{\sqrt{2}} M_d^{ii}, \quad m_{u'}^i = \frac{v}{\sqrt{2}} M_u^{ii}. \tag{3.47}$$

The Yukawa interaction between the fermions and the Higgs boson are found to be as illustrated below. Note that the Higgs boson couples to the mass eigenstates f' of the fermions instead of the flavour eigenstates f. In the diagram f' can be any massive fermion, e'^i, d'^i, u'^i. In contrast to the coupling via the vector bosons in this case left and right-handed chirality fields are coupled.



$$= -i\frac{m_{f'}}{v}$$

3.4 Eigenstates and the CKM-matrix

For finding the mass of the fermions a basis transformation had to be applied to the fermion fields, bringing flavour eigenstates f to mass eigenstates f'. The latter is the physical state of the particles since mass eigenstates are the one observed in experiment. It would therefore be useful to transform the results found before for the kinetic terms and the coupling to vector bosons to the new basis. In many cases the change of basis has no effect, since the effect on f and $\bar{f}$ is opposite and therefore cancels in most terms. In the *flavour changing charged current* term (3.35), up-like and down-like quarks couple. One therefore finds the following interaction in terms of the mass eigenstates of the quarks,

$$\begin{aligned}
\mathcal{L}_{CC} &= \frac{g}{\sqrt{2}}\left(W_\mu^+ \bar{u}_L^i \gamma^\mu d_L^i + W_\mu^- \bar{d}_L^i \gamma^\mu u_L^i\right), & (3.48)\\
&= \frac{g}{\sqrt{2}}\left(W_\mu^+ \bar{u}'^i_L U_{uL}^{\dagger ik} \gamma^\mu U_{dL}^{kj} d'^j_L + W_\mu^- \bar{d}'^i_L U_{dL}^{\dagger ik} \gamma^\mu U_{uL}^{kj} u'^j_L\right), \\
&= \frac{g}{\sqrt{2}}\left(W_\mu^+ \bar{u}'^i_L \gamma^\mu V^{ij} d'^j_L + W_\mu^- \bar{d}'^i_L \gamma^\mu V^{\dagger ij} u'^j_L\right), & (3.49)
\end{aligned}$$

where in the last line the constant unitary *Cabibbo–Kobayashi–Maskawa matrix* $V = U_{uL}^\dagger U_{dL}$ was introduced to describe this inequality of mass and flavour eigenstates.

3.4.1 CP violation in the CKM-matrix

The Standard Model is of course constructed to be invariant under the Poincaré group. It turns out however that it is in large parts also invariant under certain discrete spacetime related

symmetries. According to the $\mathcal{CPT}$*-theorem* [3, 4, 10] every Lorentz invariant quantum field theory with hermitian Lagrangian, such as the Standard Model, is invariant under a combined transformation of charge conjugation, parity and time reversal, which were discussed in 2.6. This implies that the transformation under $\mathcal{CP}$ and under $\mathcal{T}$ are opposite. It turns out however that most parts of the Standard Model Lagrangian are invariant under $\mathcal{CP}$ and $\mathcal{T}$ separately already. The only potentially $\mathcal{CP}$ *violating* term involves the CKM-matrix and will be discussed here in detail.
Quarks are spinors and thus transform according to (2.65). The transformation of the first bilinear in (3.48) is therefore,

$$\begin{aligned}\mathcal{CP}\bar{u}_L^i\gamma^\mu d_L^i\,\mathcal{CP}^{-1} &= u_L^{iT}\gamma^2\gamma^\mu\gamma^2\bar{d}_L^{iT},\\ &= u_L^{iT}\gamma^0\gamma^{\mu T}\gamma^0\bar{d}_L^{iT},\\ &= -\bar{d}_L^i\gamma^0\gamma^\mu\gamma^0 u_L^i,\\ &= \begin{cases}-\bar{d}_L\gamma^\mu u_L(t,-\mathbf{x}) & \text{if } \mu = 0,\\ \bar{d}_L\gamma^\mu u_L(t,-\mathbf{x}) & \text{if } \mu = 1,2,3,\end{cases}\end{aligned} \tag{3.50}$$

where many of the properties of Dirac matrices were used, as discussed in A. One finds that the result is of the form of the other bilinear in (3.48). The flavour changing charged current is therefore $\mathcal{CP}$ invariant if the charged vector bosons is imposed to transform as,

$$\mathcal{CP}\,W_\mu^\pm\,(\mathcal{CP})^{-1} = \begin{cases}-W_\mu^\mp(t,-\mathbf{x}) & \text{if } \mu = 0,\\ W_\mu^\mp(t,-\mathbf{x}) & \text{if } \mu = 1,2,3.\end{cases} \tag{3.51}$$

This is consistent with (2.70) if the generators of $SU(2)_L$ do not transform under charge conjugation. If one however changes to mass eigenstates of the quarks as in (3.49) and performs the same reasoning as for (3.50) one finds that,

$$\mathcal{CP}\bar{u}_L'^i\gamma^\mu V^{ij} d_L'^j\,\mathcal{CP}^{-1} = \begin{cases}-\bar{d}_L'^j\gamma^\mu V^{ij} u_L'^i(t,-\mathbf{x}) & \text{if } \mu = 0,\\ \bar{d}_L'^j\gamma^\mu V^{ij} u_L'^i(t,-\mathbf{x}) & \text{if } \mu = 1,2,3.\end{cases} \tag{3.52}$$

From combine these with (3.51) and comparing terms with (3.49) we find that only basis transformations are allowed by a CKM-matrix with real entries, since $V^{ji} = V^{\dagger ij}$ implies $V^{ij} = V^{*ij}$. In other words, $\mathcal{CP}$ violation only occurs in the Standard Model if the CKM-matrix is complex.

3.5 Global symmetries

Next to the local symmetries on which the Standard Model theory is build there are multiple related and unrelated global symmetries. In this section first the global symmetries related to the gauge invariance are studied, including an approximate extension. Then the other global symmetries of the theory are discussed.

3.5.1 The Higgs bi-doublet

The Higgs-like scalar field $\tilde{\phi} = i\sigma_2\phi^*$ introduced before to ensure the mass of up-like quarks (3.43), can be combined ϕ field itself into a single *Higgs bi-doublet*,

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\tilde{\phi} & \phi\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi^{0*} & \phi^+\\ -\phi^- & \phi^0\end{pmatrix}, \tag{3.53}$$

The Higgs sector of the Standard Model can also be described in terms of this bi-doublet, as described in [11] and in some more detail in [12]. In this framework the *custodial symmetry* becomes immediately clear.
It is known that ϕ transforms as an $SU(2)_L$ isospin 1/2 doublet with $U(1)_Y$ hypercharge 1/2,

$$\phi \to \phi' = e^{i\xi^a\sigma_a/2}e^{i\theta/2}\phi \tag{3.54}$$

The transformation of $\tilde{\phi}$ is therefore,

$$\begin{aligned}\tilde{\phi} \to \tilde{\phi}' &= i\sigma_2 \left(e^{i\xi^a\sigma_a/2}e^{i\theta/2}\phi\right)^*, \\ &= i\sigma_2 e^{-i\xi^a\sigma_a^*/2}e^{-i\theta/2}\phi^*, \\ &= e^{i\xi^a\sigma_a/2}e^{-i\theta/2}i\sigma_2\phi^*, \\ &= e^{i\xi^a\sigma_a/2}e^{-i\theta/2}\tilde{\phi},\end{aligned} \tag{3.55}$$

where it was used that $\sigma_2\exp(-i\xi^a\sigma_a^*/2) = \exp(i\xi^a\sigma_a/2)\sigma_2$, as can be shown using the transformation in the form of (2.18) and the anti-commutation of the Pauli matrices, $\{\sigma_a, \sigma_b\} = 2\delta_{ab}$,

$$\begin{aligned}\sigma_2 e^{-i\xi n^a\sigma_a^*/2} &= \sigma_2(\cos(\xi/2)\mathbb{1} - i\sin(\xi/2)n^a\sigma_a^*), \\ &= (\cos(\xi/2)\mathbb{1} + i\sin(\xi/2)n^a\sigma_a)\sigma_2 = e^{i\xi n^a\sigma_a/2}\sigma_2,\end{aligned} \tag{3.56}$$

since $\sigma_2^* = \sigma_2$. Apparently $\tilde{\phi}$ transforms in the same way as ϕ under $SU(2)_L$, but oppositely under $U(1)_Y$. The transformation under $SU(2)_L \times U(1)_Y$ of the bi-doublet is therefore,

$$\Phi \to \Phi' = e^{i\xi^a\sigma_a/2}\Phi e^{-i\theta\sigma_3/2} = L\Phi Y. \tag{3.57}$$

Note the appearance of $-\sigma_3$ by the opposite way ϕ and $\tilde{\phi}$ transform under $U(1)_Y$. The Higgs Lagrangian invariant under local $SU(2)_L \times U(1)_Y$ transformations,

$$\mathcal{L} = |D_\mu\phi|^2 - V(\phi^\dagger\phi), \tag{3.58}$$

can be written in terms of the bi-doublet as,

$$\mathcal{L} = \mathrm{Tr}\,|D_\mu\Phi|^2 - V(\mathrm{Tr}\,\Phi^\dagger\Phi), \tag{3.59}$$

where the covariant derivative of Φ is,

$$D_\mu\Phi = \partial_\mu\Phi - \frac{ig}{2}W_\mu^a\sigma_a\Phi + \frac{ig'}{2}B_\mu\Phi\sigma_3, \tag{3.60}$$

where in the last term $-\Phi\sigma_3 = (-\tilde{\phi} \quad \phi)/\sqrt{2}$ ensures the opposite transformation of $\tilde{\phi}$ under $U(1)_Y$.

All local symmetries have a corresponding global symmetry as well, by the trivial spacetime dependence in which the transformation is the same everywhere. The Lagrangian (3.59) is therefore invariant under global $SU(2)_L \times U(1)_Y$ transformations by construction. It had been discussed before that the Higgs field obtaining its vacuum expectation value breaks the local $SU(2)_L \times U(1)_Y$ symmetry to $U(1)_Q$. Now the breaking of the global symmetry will be considered.

The vacuum expectation value of the bi-doublet is found from the definition (3.53) and the vacuum expectation value of the Higgs field (3.5) found to be,

$$\langle\Phi\rangle = \frac{1}{2}\begin{pmatrix} v & 0 \\ 0 & v\end{pmatrix}. \tag{3.61}$$

This choice of the degenerate vacuum breaks the symmetry. The remaining symmetry is the one that leaves the vacuum expectation value invariant,

$$\langle\Phi\rangle \to e^{i\xi^a\sigma_a/2}\langle\Phi\rangle e^{-i\theta\sigma_3/2} = \langle\Phi\rangle, \tag{3.62}$$

where for a global symmetry the parameters ξ^a and θ are not spacetime dependent. Since $\langle\Phi\rangle$ is proportional to identity the restriction is,

$$e^{i\xi^a\sigma_a/2}e^{-i\theta\sigma_3/2} = \mathbb{1}. \tag{3.63}$$

This restricts the 4 real global $SU(2)_L \times U(1)_Y$ parameters to 1, as $\xi^1 = \xi^2 = 0$ and $\xi^3 = \theta$. The resulting global symmetry is $U(1)_{\sigma_3/2+Y} = U(1)_Q$, corresponding to the electromagnetic charge defined in (3.13).

3.5.2 Approximate custodial symmetry

The Higgs Lagrangian (3.59) almost has an extra global $SU(2)_R$, where the transformation on the bi-doublet from the right,

$$\Phi \to \Phi e^{i\zeta^a \sigma_a/2} = \Phi R^\dagger. \tag{3.64}$$

This transformation leaves the Lagrangian invariant if,

$$D_\mu \Phi = \partial_\mu \Phi - \frac{ig}{2} W_\mu^a \sigma_a \Phi + \frac{ig'}{2} B_\mu \Phi \sigma_3 \to D_\mu \Phi R^\dagger, \tag{3.65}$$

but the last term in the covariant derivative restrict the $SU(2)_R$ to its $U(1)$ subgroup. Consider the parameterization of (2.16) for the $SU(2)_R$ transformation to see that $R^\dagger \sigma_3 = \sigma_3 R^\dagger$ restricts the transformation to $R^\dagger = \exp(i\zeta^3 \sigma_3/2)$.
Consequently $SU(2)_R$ is not a global symmetry of the Lagrangian. In the limit where the coupling to $U(1)_Y$ is taken to zero, $g' \to 0$, the symmetry becomes valid and the global symmetry becomes $SU(2)_L \times SU(2)_R$,

$$\Phi \to L\Phi R^\dagger, \tag{3.66}$$

$$D_\mu \Phi = \partial_\mu \Phi - \frac{ig}{2} W_\mu^a \sigma_a \Phi \to L D_\mu \Phi R^\dagger. \tag{3.67}$$

After symmetry breaking the global symmetry is smaller,

$$\langle \Phi \rangle \to L \langle \Phi \rangle R^\dagger = \langle \Phi \rangle, \tag{3.68}$$

Since the vacuum expectation value is proportional to identity this implies $LR^\dagger = \mathbb{1} \Rightarrow L = R$, therefore the symmetry $SU(2)_L \times SU(2)_R$ is broken to $SU(2)_{L+R}$. This unbroken approximate global symmetry of the Standard Model is called the *custodial symmetry* of the Standard Model. This result can also be obtained by taking the limit $g' \to 0$ after symmetry breaking. This limit of $g' \to 0$ is reasonable to consider if $g' \ll g$. Whether this holds can be checked from the masses of the weak gauge bosons given in (3.8) and (3.9). From experiments it follows,

$$\frac{g'}{g} = \sqrt{\frac{m_Z^2}{m_W^2} - 1} \approx 0.54, \tag{3.69}$$

which is not extremely small, from which it follows that the symmetry is strongly broken. In this case the $U(1)_{L+Y}$ symmetry is increased to $SU(2)_{L+R}$, where the left and right transformation have equal parameters. A summary of the relations between these global symmetries is given in table 3.1 on page 26.

3.5.3 Special orthogonal quadruplet

An instructive way to see what happens during spontaneous symmetry breaking and how an invariant direction occurs is by relating the $SU(2)_L \times SU(2)_R$ symmetry to a $SO(4)$ symmetry. Instead of the complex doublet $\phi = (\phi^+, \phi^0)^T$ the real quadruplet $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$ can be considered, which are related via the transformation,

$$T = \begin{cases} \phi^+ = \varphi_1 + i\varphi_2, \\ \phi^0 = \varphi_3 + i\varphi_4, \end{cases} \tag{3.70}$$

The bi-doublet Φ in terms of the quadruplet φ parameters can be found from its relation to the doublet ϕ. An arbitrary $SU(2)_L \times SU(2)_R$ transformation of the bi-doublet in terms of these parameters is,

$$\begin{pmatrix} \varphi_3 - i\varphi_4 & \varphi_1 + i\varphi_2 \\ -\varphi_1 + i\varphi_2 & \varphi_3 + i\varphi_4 \end{pmatrix} \to \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \varphi_3 - i\varphi_4 & \varphi_1 + i\varphi_2 \\ -\varphi_1 + i\varphi_2 & \varphi_3 + i\varphi_4 \end{pmatrix} \begin{pmatrix} c^* & -d \\ d^* & c \end{pmatrix}, \tag{3.71}$$

where $aa^* + bb^* = 1$ and $cc^* + dd^* = 1$. From this the transformation of the real fields φ_i can be found. The resulting transformation matrix for the transformation of the quadruplet $\varphi \to O\varphi$ is,

$$O = \begin{pmatrix} \mathrm{Re}(ac+bd) & \mathrm{Im}(-ac+bd) & \mathrm{Re}(-ad+bc) & \mathrm{Im}(-ad-bc) \\ \mathrm{Im}(ac+bd) & \mathrm{Re}(ac-bd) & \mathrm{Im}(-ad+bc) & \mathrm{Re}(ad+bc) \\ \mathrm{Re}(ad^*+bc^*) & \mathrm{Im}(-ad^*-bc^*) & \mathrm{Re}(ac^*+bd^*) & \mathrm{Im}(ac^*-bd^*) \\ \mathrm{Im}(ad^*+bc^*) & \mathrm{Re}(-ad^*-bc^*) & \mathrm{Im}(ac^*+bd^*) & \mathrm{Re}(ac^*-bd^*) \end{pmatrix}. \tag{3.72}$$

It can be checked that this matrix is an arbitrary $SO(4)$ transformation matrix. One can observe that if one multiplies all parameters by a minus sign,

$$a \to -a, \quad b \to -b, \quad c \to -c, \quad d \to -d, \tag{3.73}$$

the $SO(4)$ transformation in (3.72) remains the same, where the $SU(2)_L \times SU(2)_R$ transformations in (3.71) both obtain a minus sign as well. This is exactly the $\mathbb{Z}_2$ symmetry lost in the $2:1$ mapping of $SU(2)_L \times SU(2)_R$ to $SO(4)$. In the case that the $SU(2)_L \times SU(2)_R$ symmetry is broken to $SU(2)_{L+R}$ the transformation is restricted by $a = c$ and $b = d$. In this case the corresponding transformation matrix of the quadruplet becomes,

$$O' = \begin{pmatrix} \mathrm{Re}(a^2+b^2) & -\mathrm{Im}(a^2-b^2) & 0 & -2\,\mathrm{Im}(ab) \\ \mathrm{Im}(a^2+b^2) & \mathrm{Re}(a^2-b^2) & 0 & -2\,\mathrm{Re}(ab) \\ 0 & 0 & 1 & 0 \\ -2\,\mathrm{Im}(ab^*) & -2\,\mathrm{Re}(ab^*) & 0 & aa^*-bb^* \end{pmatrix}. \tag{3.74}$$

This reduction of the symmetry results in an invariant direction, which is consistent with the vacuum expectation value as presented before, since for the quadruplet it is of the form,

$$\langle\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ v \\ 0 \end{pmatrix}, \tag{3.75}$$

which is indeed invariant under this transformation. By spontaneous symmetry breaking the $SO(4)$ symmetry is therefore broken to $SO(3)$. Table 3.1 includes the relation of this global symmetry to the global symmetries discussed before.

$$\begin{array}{cccccc} SU(2)_L \times U(1)_Y & 4 & \xrightarrow{\text{vev}} & U(1)_Q & 1 \\ \downarrow g' \to 0 & & & \downarrow g' \to 0 & \\ SU(2)_L \times SU(2)_R & 6 & \xrightarrow{\text{vev}} & SU(2)_{L+R} & 3 \\ \downarrow \text{T (2:1)} & & & \downarrow \text{T (2:1)} & \\ SO(4) & 6 & \xrightarrow{\text{vev}} & SO(3) & 3 \end{array}$$

Table 3.1: An overview of the relation between the (approximate) global symmetries of the Standard Model before and after spontaneous symmetry breaking. The number of (real) degrees of freedom is also shown.

3.5.4 Lepton and baryon number

Other global symmetries can be found that are not directly related to the gauge symmetries of the theory. In principle all matter fields in (3.29) and (3.30) could transform under separate global $U(3)$ transformations between the different generations, implying a $U(3)^5$ accidental symmetry [12],

$$U(3)_Q \times U(3)_u \times U(3)_d \times U(3)_L \times U(3)_e. \tag{3.76}$$

This is however not a global symmetry of the Standard Model, the different interaction terms restrict these transformations. Since the Yukawa interactions (3.38) couple the $SU(2)_L$ doublets and singlets these should transform in the same way under the global symmetry. Since the quarks and leptons do not mix two $U(3)$ symmetries survive, labelled *baryon* and *lepton number*,

$$U(3)_B \times U(3)_L. \tag{3.77}$$

After diagonalization into mass eigenstates the coupling between different generations is lost, resulting in a $U(1)$ transformation per generation, resulting in a $U(1)^6$ subgroup. In the coupling of the quarks with the charged gauge bosons (3.49) the CKM matrix occurs. This unitary 3×3 matrix mixes the different generations and does not commute with the general diagonal matrix corresponding to the $U(1)_B^3$ transformation. Only if the transformation is proportional to identity, resulting in an equal phase for each generation, and thereby only one $U(1)$ transformation. The accidental global symmetry corresponding to baryon and lepton number is therefore,

$$U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau. \tag{3.78}$$

For extending the Standard Model this global symmetry could be promoted to a gauge symmetry. It turns out however that only the subgroup $U(1)_{B-L}$ is *anomaly free* [12], which is the combination that is for instance used in the Left-Right models, discussed in the next chapter.

Chapter 4

Beyond the Standard Model

Despite the great success of the Standard Model there are reasons to believe that it is still incomplete, as discussed before. Many theories have been suggested to improve the model, with large variation in complexity. Numerous suggestions involve extensions of the Higgs sector, including the unified theories discussed at the end of this thesis. The *two Higgs doublet model* is a relatively simple extension of the Higgs sector and will therefore be studied here to get a feeling for the implications of these extensions.
Thereafter the *Left-Right model* will be studied as a first extension to the gauge group underlying the interactions between matter particles. This model will be also be used later as a possible first step towards a Grand Unified Theory based on an even larger gauge group.

4.1 Two Higgs doublet model

The Standard Model Higgs sector contains only one scalar Higgs doublet. Extensions can be suggested by including more doublets [11]. The minimal extension would be a second doublet,

$$\phi_1 = \begin{pmatrix} \phi_1^+ \\ \phi_1^0 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}. \tag{4.1}$$

The model is based on the same gauge symmetry as the Standard Model. The two doublets are identical copies of the Higgs doublet discussed before in chapter 3.1.1, and therefore have equal values for the quantum numbers hypercharge and weak isospin. Because of the equivalence of these fields the labelling is arbitrary. One could consider a transformation to another basis,

$$\phi_i' = U_{ij}\phi_j, \tag{4.2}$$

where the fields in the new basis are linear combinations of the fields in the old basis. The kinetic term for these scalar doublets is,

$$(D_\mu\phi_1)^\dagger D^\mu\phi_1 + (D_\mu\phi_2)^\dagger D^\mu\phi_2 = (D_\mu\phi_i)^\dagger D^\mu\phi_i, \quad i = 1,2. \tag{4.3}$$

This kinetic term is clearly independent of the basis transformation (4.2) if the transformation is unitary, $U \in U(2)$. Off-diagonal kinetic terms are not considered here.
The gauge invariant *Mexican hat potential* (3.3) includes second and fourth order terms and can be generalized to all combinations of the fields, as in [13],

$$V(\phi_1,\phi_2) = Y_{ij}\phi_i^\dagger\phi_j + Z_{ijkl}(\phi_i^\dagger\phi_j)(\phi_k^\dagger\phi_l), \qquad i,j,k,l = 1,2, \tag{4.4}$$

with $Z_{ijkl} = Z_{klij}$. To ensure the hermiticity of the Lagrangian one has to require $Y_{ij} = Y_{ji}^*$ and $Z_{ijkl} = Z_{jilk}^*$, resulting in fourteen real parameters in this potential. The potential is not basis independent, but transforms as,

$$\begin{aligned} Y_{ij}' &= U_{im}Y_{mn}U_{jn}^*, \\ Z_{ijkl}' &= U_{im}U_{ok}Z_{mnop}U_{jn}^*U_{lp}^*, \end{aligned} \tag{4.5}$$

and therefore not all parameters are physical. For the potential to be physical it must be bounded from below, which constrains the parameters even more. Nevertheless, finding the minimum of this potential is more complicated than in the case with only one doublet, where there are only two real parameters. If we however consider the interpretation where the lower component of the doublet is electrically neutral it follows from the *neutrality of the vacuum* that the vacuum expectation values are of the form,

$$\langle \phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 e^{i\zeta} \end{pmatrix}, \tag{4.6}$$

where v_1 and v_2 real parameters and ξ is the relative phase between them. In general an overall phase could be included, but by the gauge choice this phase can be shifted away, thereby breaking the gauge invariance, just as in the Standard Model. It turns out that even the *relative phase*, which could be a cause of $\mathcal{CP}$ violation, can be shifted away, with a basis transformation. After the unitary transformation,

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\xi} \end{pmatrix}, \tag{4.7}$$

the phase is shifted to the potential. In this way the relative phase can not completely removed, it will appear via the parameters of the potential instead of via the vacuum expectation values.

The Yukawa interaction

Since the two Higgs doublet model introduces another scalar field the Yukawa sector can different from to the Standard Model. The fermion content can couple to the each of the doublets, resulting in the *generalized Yukawa Lagrangian* of the form [14],

$$\mathcal{L}_Y = -\bar{\psi}^i_L (g_1^{ij} \phi_1 + g_2^{ij} \phi_2) \psi^j_R - \bar{\psi}^i_R (g_1^{\dagger ij} \phi_1^\dagger + g_2^{\dagger ij} \phi_2^\dagger) \psi^j_L. \tag{4.8}$$

Note that the coupling between the fermions and the one or the other doublet are independent, some fermions may couple much weaker or even not at all to one of the fields. Also remember that the labelling of the doublets is not fixed, one could perform a basis transformation and find another coupling to a linear combination of the Higgs doublets. The independence of the couplings also implies that g_1 and g_2 can not in general be diagonalized at the same time, which could result in another source of $\mathcal{CP}$ violation. It also results in flavour changing neutral currents via at least one of the Higgs doublets. To avoid these flavour changing neutral currents, that have not been observed in experiments, one needs to apply extra restrictions. The restriction can be that in some basis the fermions couple each to only one of the Higgs doublets. This can be implemented via the requirement of an extra global $\mathbb{Z}_2$ symmetry,

$$\phi_1 \to \phi_1, \quad \phi_2 \to -\phi_2, \tag{4.9}$$

or a more general $U(1)$ symmetry,

$$\phi_1 \to \phi_1, \quad \phi_2 \to e^{i\omega} \phi_2, \tag{4.10}$$

under which the fermions coupling to ϕ_1 do not transform, but those coupling to ϕ_2 transform to make the Lagrangian invariant under this transformation. Note that the coupling of the fermions to only one of the doublet fixes the basis, in another basis again a mixing of these couplings occurs. Also note that this extra symmetry constrains the potential (4.4), but not to the extent where coupling between these doublets is removed.
Similar to the Standard Model three different Yukawa coupling must be considered, corresponding to the coupling to up-type quarks, down-type quarks (via $\tilde{\phi}_i = i\sigma_2 \phi_i$) and leptons.

4.1.1 C, P and T transformations

In section 2.6 the $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations of scalar fields were introduced,

$$\begin{aligned} \mathcal{P}\phi(t,x)\mathcal{P}^{-1} &= \eta\phi(t,-x), \\ \mathcal{T}\phi(t,x)\mathcal{T}^{-1} &= \zeta\phi(-t,x), \\ \mathcal{C}\phi(t,x)\mathcal{C}^{-1} &= \xi\phi^*(t,x). \end{aligned} \tag{4.11}$$

Unfortunately, multiple equivalent fields complicate the matter, since charge conjugation is no longer uniquely defined in this case. Charge conjugation turns out to be *basis-dependent*, it is no longer clear what the eigenstates of charge conjugation are. The transformation under $\mathcal{C}$ can therefore be *generalized* to,

$$\mathcal{C}\phi_i(t,x)\mathcal{C}^{-1} = X_{ij}\phi_j^*(t,x), \tag{4.12}$$

where X is a 2×2 unitary transformation matrix. In literature one often finds a generalized $\mathcal{CP}$-transformation to be of this form, but since the problem is with charge conjugation, not parity, we stick to $\mathcal{C}$ here. The transformation of X under basis transformations is found from performing (4.2) before and after charge conjugation. From this we find that,

$$X'_{ij} = U_{ik}X_{kl}U_{jl} = (UXU^T)_{ij}, \tag{4.13}$$

from which it is clear that a diagonal transformation in one basis is not also diagonal in another basis. It is not even generally true that an arbitrary unitary transformation X can be diagonalized by a basis transformation [11], since it transforms with U^T not $U^\dagger$. It can be checked however that every unitary X can by a suitable choice of U be brought to the form [11, 13],

$$X' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{4.14}$$

where U can be chosen such that $0 \leq \theta \leq \pi/2$.
According to the $\mathcal{CPT}$ *theorem*, that has been discussed before in section 3.4.1, any Lorentz-invariant quantum field theory with hermitian Lagrangian is invariant under the combined transformation $\mathcal{CPT}$. The behaviour of parity and time reversal is independent on the basis, and therefore these transformations do not need to be generalized. From the corresponding transformations (4.11) it is clear that the result of performing parity and time reversal twice is only a phase. Because of the $\mathcal{CPT}$ theorem the same should hold for charge conjugation. This is not however immediately clear from the generalized transformation (4.12). Instead one finds,

$$\mathcal{C}^2\phi_i(\mathcal{C}^{-1})^2 = X_{ij}X^*_{jk}\phi_k = (XX^*)_{ik}\phi_k. \tag{4.15}$$

The eigenvalues of $X'X'^*$ are easily computed to be $e^{\pm 2i\theta}$. Since $X'X'^* = UXU^TU^*X^*U^\dagger = UXX^*U^\dagger$ the characteristic polynomial of $X'X'^*$ is the same as for XX^*, resulting in the same eigenvalues for XX^*. This can be used in the classification of X in an arbitrary basis. It is insightful to consider three cases separately, where $\theta = 0$, where $\theta = \pi/2$ and where $0 < \theta < \pi/2$, as in [15].

Case 1: $\theta = 0$

In the case where $\theta = 0$ the transformation matrix X' from (4.14) is,

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.16}$$

equivalent to the non-generalized charge conjugation,

$$\mathcal{C}_1\phi_1\mathcal{C}_1^{-1} = \phi_1^*, \quad \mathcal{C}_1\phi_2\mathcal{C}_1^{-1} = \phi_2^*. \tag{4.17}$$

It is clear from (4.16) that,

$$X'X'^* = \mathbb{1}, \tag{4.18}$$

from which it is clear that XX^* in another basis is also identity, which is consistent with the constraint of only a phase after performing charge conjugation twice.

Case 2: $\theta = \frac{\pi}{2}$

In the other special case, where $\theta = \pi/2$, the transformation matrix X' is (4.14),

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.19}$$

which results in an interchange of the doublets,

$$\mathcal{C}_2\phi_1\mathcal{C}_2^{-1} = \phi_2^*, \quad \mathcal{C}_2\phi_2\mathcal{C}_2^{-1} = -\phi_1^*. \tag{4.20}$$

For performing charge conjugation twice this implies,

$$X'X'^* = XX^* = -\mathbb{1}, \tag{4.21}$$

After performing this charge conjugation operation twice one therefore is back at itself, up to a minus sign, which is an acceptable phase.

Case 3: $0 < \theta < \frac{\pi}{2}$

After these two special cases a range of charge conjugation transformations is left over. If one would start the analysis of generalized charge conjugation from the idea that after performing it twice one should be back up to a phase, one would find that all these cases have already been dealt with in the previous cases. It turns out that,

$$XX^* = e^{i\varphi}\mathbb{1}, \tag{4.22}$$

with X any unitary 2×2 matrix, results in exactly the two cases before, with $e^{i\varphi} = \pm 1$ respectively. It is also clear from,

$$X'X'^* = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}, \tag{4.23}$$

which is only proportional to identity if $\theta = k\pi/2$. For $0 < \theta < \frac{\pi}{2}$ the product XX^* is therefore never proportional to identity, since it was shown in the discussion after (4.15) that if it is proportional to identity in one basis, it is in any. This is reason enough not to consider this case as an acceptable charge conjugation.

4.1.2 Constraints on the potential

To remove the flavour changing neutral current from the Yukawa interaction at least a global $\mathbb{Z}_2$ symmetry (4.9) had to be imposed under which,

$$\phi_1 \to \phi_1, \quad \phi_2 \to -\phi_2. \tag{4.24}$$

This symmetry also strongly constrains the generalized Mexican hat potential (4.4), since all terms with an odd number of each of the doublets violate this symmetry and must therefore be removed. The result is the following potential with only 8 instead of 14 real free parameters,

$$\begin{aligned} V(\phi_1,\phi_2) = &-\mu_1^2\phi_1^\dagger\phi_1 - \mu_2^2\phi_2^\dagger\phi_2 + \lambda_1(\phi_1^\dagger\phi_1)^2 + \lambda_2(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\ &+ \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \lambda_5\left(e^{i\delta}(\phi_1^\dagger\phi_2)^2 + e^{-i\delta}(\phi_2^\dagger\phi_1)^2\right). \end{aligned} \tag{4.25}$$

Where the new labelling of the parameters is introduced to show the structure of each term specifically. The more general global $U(1)$ symmetry of (4.10) would restrict the potential even further, removing the last term also and thereby removing two more parameters.
The general form of the vacuum expectation values is given in (4.6),

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_2 e^{i\zeta} \end{pmatrix}, \tag{4.26}$$

Since the vacuum expectation values are supposed to be the minima of the potential the contributions of all terms in (4.25) must be as low as possible at these values for the fields. The last term becomes,

$$\lambda_5\left(e^{i\delta}(\langle\phi_1\rangle^\dagger\langle\phi_2\rangle)^2 + e^{-i\delta}(\langle\phi_2\rangle^\dagger\langle\phi_1\rangle)^2\right) = \frac{\lambda_5 v_1 v_2}{2}\cos(\delta + 2\zeta), \tag{4.27}$$

which is lowest if $\cos(\delta + 2\zeta) = -1$, since $\lambda_5 > 0$ for the potential to be bounded from below. This fixes the relative phase between the vacuum expectation values in this basis in terms of the parameters of the potential,

$$\delta + 2\zeta = \pi + 2k\pi, \quad k \in \mathbb{Z}. \tag{4.28}$$

Now we can look for a $\mathcal{CP}$ transformation that leaves the potential invariant. Clearly all but the last term are invariant in themselves if the generalized charge conjugation matrix is diagonal, since they involve an equal number of ϕ_i and $\phi_i^\dagger$ fields each. Consider for instance the transformation with,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\zeta} \end{pmatrix}, \tag{4.29}$$

which corresponds to a case 1 generalized charge conjugation matrix, since $XX^* = \mathbb{1}$. Under this $\mathcal{CP}$ transformation the potential is invariant, since,

$$\begin{aligned} \mathcal{CP}\left(e^{i\delta}(\phi_1^\dagger\phi_2)^2 + e^{-i\delta}(\phi_2^\dagger\phi_1)^2\right)\mathcal{CP}^{-1} &= \left(e^{i(\delta+4\zeta)}(\phi_2^\dagger\phi_1)^2 + e^{-i(\delta+4\zeta)}(\phi_1^\dagger\phi_2)^2\right), \\ &= \left(e^{-i\delta}(\phi_2^\dagger\phi_1)^2 + e^{i\delta}(\phi_1^\dagger\phi_2)^2\right), \end{aligned} \tag{4.30}$$

where in the last line the relation between ζ and δ (4.28) was used. Apparently one can always find a $\mathcal{CP}$ transformation of case 1 under which the potential is invariant, if the $\mathbb{Z}_2$ symmetry is imposed. Since the global $U(1)$ symmetry restricts the potential even more, putting the λ_5 term to zero, one can also find a $\mathcal{CP}$ transformation that leaves this potential invariant.

Invariance of the vacuum

Another question that can be asked is whether the vacuum preserves $\mathcal{C}$ invariance, or that it is spontaneously broken by going to the vacuum expectation values. Vacuum invariance implies,

$$\langle\phi_i\rangle = X_{ij}\langle\phi_j\rangle^*. \tag{4.31}$$

Both the vacuum and the charge conjugation transformation matrix are not invariant under basis transformation. It turns out however that if the vacuum is invariant in one basis, it is in any,

$$\langle\phi_i'\rangle = X'_{ij}\langle\phi_j'\rangle^* = (UXU^T)_{ij}U^*_{jk}\langle\phi_k\rangle^* = U_{ij}X_{jk}\langle\phi_k\rangle^* = U_{ij}\langle\phi_j\rangle. \tag{4.32}$$

Consequently the vacuum invariance can be studied in the X' basis (4.14) without loss of generality. For case 1 charge conjugation $\mathcal{C}_1$, where $\theta = 0$, the vacuum is invariant if,

$$\begin{pmatrix} \langle\phi_1'\rangle \\ \langle\phi_2'\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \langle\phi_1'\rangle^* \\ \langle\phi_2'\rangle^* \end{pmatrix} = \begin{pmatrix} \langle\phi_1'\rangle^* \\ \langle\phi_2'\rangle^* \end{pmatrix}, \tag{4.33}$$

both $\langle\phi_1'\rangle$ and $\langle\phi_2'\rangle$ are real. It has been shown before that there always exists a basis where both vacuum expectation values are real. Clearly the vacuum is only invariant under $\mathcal{C}_1$ if the basis where the vacuum expectation values are both real coincides with the basis where the charge conjugation transformation matrix is identity. Otherwise $\mathcal{C}_1$ invariance is spontaneously broken. In another basis, where there is a relative phase between the vacuum expectation values of the form of (4.6), the vacuum is invariant under the $\mathcal{C}_1$ transformation with,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\zeta} \end{pmatrix}. \tag{4.34}$$

For case 2, $\theta = \frac{\pi}{2}$ and case 3, $0 < \theta < \frac{\pi}{2}$, one can find that there are no vacuum expectation values that satisfy $\langle\phi_i'\rangle = X'_{ij}\langle\phi_j'\rangle^*$, from which it is clear that the vacuum is never invariant under charge conjugation in these cases.

	Case 1 $\theta = 0$	**Case 2** $\theta = \frac{\pi}{2}$	**Case 3** $0 < \theta < \frac{\pi}{2}$
X'	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
$XX^* = e^{i\varphi}\mathbb{1}$	$e^{i\varphi} = +1$	$e^{i\varphi} = -1$	×
$\langle\phi_i\rangle = X_{ij}\langle\phi_j\rangle^*$	$\langle\phi_1'\rangle, \langle\phi_2'\rangle$ real	×	×

Table 4.1: An overview of the cases in generalized charge conjugation.

Since the case 1 transformation is the only one behaving as expected for charge conjugation and it is the only one that can leave the vacuum invariant, in the rest of the text charge conjugation will refer to this case unless specified differently.

4.1.3 Conclusion

Since the two Higgs fields are equivalent in the two Higgs doublet model it is inevitable that basis transformations between these fields must be considered. A generalized charge conjugation also includes such a basis transformation. It turns out that these charge conjugation transformations can be classified into three separate cases. After performing charge conjugation twice the first case gives back the initial field, the second case adds a minus sign and the third is not proportional to the original field. By the $\mathcal{CPT}$ theorem this last case can therefore be excluded as charge conjugation candidate.

One can define a $\mathcal{CP}$ transformation following case 1 that leaves the two Higgs doublet model invariant if a $\mathbb{Z}_2$ or $U(1)$ symmetry is imposed to remove flavour changing neutral currents from the theory. The vacuum expectation values can also be invariant, but the $\mathcal{CP}$ invariance can also be spontaneously broken.

Since the global $\mathbb{Z}_2$ or $U(1)$ symmetry was constructed such that in the Yukawa sector each fermion field couples to only one of the Higgs doublets, the diagonalization problem has been solved. To each fermion field a basis transformation can be applied to the mass eigenstates, thereby diagonalizing the Yukawa coupling resulting only to (real) masses. This sector can also be constructed to be $\mathcal{CP}$ invariant, leaving as the only source the a complex CKM-matrix as in the Standard Model.

It is therefore possible to construct a $\mathcal{CP}$ transformation under which the two Higgs doublet model can be invariant. It turns out that these transformations are similar to the transformation one would expect without generalization. Apparently one can add Higgs fields to a theory, as will be needed in Grand Unified Theories.

4.2 Left-Right models

In section 3.5 it was discovered that the Standard Model is invariant under a larger global symmetry than only the global version of the gauge group. It has been shown there that by putting the hypercharge coupling to zero an approximate global $SU(2)_L \times SU(2)_R$ symmetry of the Standard Model is obtained. This might be a hint that there is more symmetry between left and right-handed chirality than in the Standard Model. Remember that in the Standard Model the asymmetry is put in by construction because of the phenomenological $V - A$ behaviour observed experimentally in the weak interaction. The idea in Left-Right models is that the Standard Model is a low energy result of a spontaneously broken theory based on a group treating left and right-handed spinors in equivalent ways. For this theory the global $SU(2)_L \times SU(2)_R$ symmetry is *promoted to a gauge symmetry.* Resulting in three gauge bosons coupling to left-handed particles and three gauge bosons coupling to right-handed particles.
The $U(1)_Y$ hypercharge symmetry of the Standard Model is replaced in the Left-Right models by a $U(1)_{B-L}$ *baryon minus lepton number* symmetry, transforming the baryons and the leptons by an opposite phase. This is the anomaly free subgroup of the global baryon and lepton number symmetries of the Standard Model. In the breaking of the right-handed symmetry the $SU(2)_R \times U(1)_{B-L}$ will be broken to the Standard Model $U(1)_Y$. The simplest Left-Right models are therefore based on the gauge group,

$$SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}. \tag{4.35}$$

The $SU(3)_C$ is not changed compared to the Standard Model and therefore it is not specifically treated here again.
There are many interesting concepts in Left-Right models, that will not be treated in detail here. In this text the general structure of the models will be discussed, since a symmetry between left and right-handed particles occurs in many Grand Unified Theories, and it is insightful to discover some of the implications before going into these more involved theories. Also the breaking of the larger group to the Standard Model group will be discussed, since similar breaking mechanisms will be useful in breaking the Unification groups. A more detailed review of Left-Right models can be found in [16] and [17].

4.2.1 Particle content

In the Standard Model the right-handed spinors corresponding to the quarks are in separate representations, since these do not mix under any of the gauge symmetries. In the Left-Right models the structure of the left-handed spinors is copied for the right-handed spinors. The up-like and down-like right-handed spinors transform as a doublet under $SU(2)_R$. *Right-handed neutrinos* are introduced to form a doublet with the right-handed charged leptons. Since a baryon consists of three quarks the baryon-lepton number of a quark 1/3. The resulting transformation dimensions of the fermion content and their $B-L$ charge for the gauge group $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ are,

$$\begin{aligned} Q_L &= (3, 2, 1, 1/3), \qquad & Q_R &= (3, 1, 2, 1/3), \\ L_L &= (1, 2, 1, -1), \qquad & L_R &= (1, 1, 2, -1). \end{aligned} \tag{4.36}$$

From the comparison with the representations of the Standard Model (3.33) the elegance of the models is immediately clear, the Left-Right models restore the symmetries between right-handed and left-handed particles.
But as every extension of the Standard Model it comes at a cost. New particles, the right-handed neutrinos, were introduced that have never been observed in experiments. This does not immediately falsify the theory however. The right-handed neutrinos do not couple to any of the Standard Model gauge bosons and therefore do not play a role in measured processes. Nevertheless the mass of these neutrinos must be large to explain the lack of observation. If the neutrinos are their own anti-particles, which is possible since they are chargeless, and therefore behave as *Majorana spinors*, the *see-saw mechanism*, explained in more detail in [18], can provide an explanation for the large mass of the right-handed and small but non-zero mass of the left-handed neutrinos at the same time. The mass of the right and left-handed neutrinos are

respectively proportional and inversely proportional to the vacuum expectation value of the Higgs-like particle responsible for the breaking of the Left-Right group to the Standard Model group, thereby relating small and large masses.
Also three extra gauge bosons are predicted by the $SU(2)_R$ gauge symmetry, $W_R^{i\mu}$, similar to the three gauge bosons corresponding to $SU(2)_L$. These gauge bosons must be very heavy for the interactions involving them to be suppressed. As in the Standard Model the mass of these gauge bosons can be explained by a spontaneous symmetry breaking.

4.2.2 Spontaneous breaking

The breaking of the Left-Right symmetric model is rather complicated, since it has to occur in two steps. The details will not be treated here, but an overview of the structure is presented to give an idea of the procedure.
The last step in the breaking is the same as in the Standard Model, from $SU(2)_L \times U(1)_Y$ to $U(1)_Q$. The scalar field responsible for this breaking and the resulting (Dirac) masses of the Standard Model particles forms a doublet under $SU(2)_R$ to ensure the symmetry between left and right,

$$\phi = \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix} = (0,2,2,0). \tag{4.37}$$

The vacuum expectation value can contain a spontaneously $\mathcal{CP}$ violating phase,

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \kappa & 0 \\ 0 & \kappa' e^{i\alpha} \end{pmatrix}, \tag{4.38}$$

where κ and κ' are on the electroweak scale. The breaking of the Left-Right models to the Standard Model can be done by two doublets $\chi_{L,R}$ under the corresponding gauge groups [19, 20],

$$\chi_{L,R} = \begin{pmatrix} \chi_{L,R}^+ \\ \chi_{L,R}^0 \end{pmatrix}, \qquad \langle\chi_{L,R}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_{L,R} \end{pmatrix}, \tag{4.39}$$

or two triplets $\Delta_{L,R}$ under these groups [16, 18, 21],

$$\Delta_{L,R} = \begin{pmatrix} \delta_{L,R}^{++} \\ \delta_{L,R}^+ \\ \delta_{L,R}^0 \end{pmatrix}, \qquad \langle\Delta_{L,R}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ v_{L,R} \end{pmatrix}, \tag{4.40}$$

The advantage of the second is that in this way the neutrinos can obtain a Majoranna mass, which via the (type II) see-saw mechanism explains the high mass of the right-handed neutrino and the low mass for the left-handed neutrino.
The explicit gauge invariant potential is rather involved and will not be presented here, but can be found in for instance [17]. The relation between the quantum numbers after each step of symmetry breaking is,

$$Q = \frac{\sigma_{3L}}{2} + Y = \frac{\sigma_{3L}}{2} + \frac{\sigma_{3R}}{2} + \frac{B-L}{2}. \tag{4.41}$$

4.2.3 C, P and T transformations

In restoring the symmetry between left and right-handed spinors even more symmetry occurs. Remember that in section 2.6 it was shown that left and right-handed spinors are related via parity and charge conjugation,

$$\begin{aligned} \mathcal{P}\psi_L(t,x)\mathcal{P}^{-1} &= \gamma^0\psi_R(t,-x), \\ \mathcal{T}\psi_L(t,x)\mathcal{T}^{-1} &= \gamma^1\gamma^3\psi_L(-t,x), \\ \mathcal{C}\psi_L(t,x)\mathcal{C}^{-1} &= i\gamma^2\gamma^0\overline{\psi}_R^T(t,x). \end{aligned} \tag{4.42}$$

It is therefore possible to construct a theory that is not only invariant under $\mathcal{CP}$, but also $\mathcal{C}$ and $\mathcal{P}$ separately. In these theories the couplings and transformations of the left and right-handed spinors must be completely equivalent [16].

4.2.4 Conclusion

Although many of the details in the Left-Right models have been skipped over the elegance of the theory that appeals to so many physicists should be clear. It provides an explanation for the imposed left-handed nature of the Standard Model as a low-energy result of a larger underlying symmetry. It allows an explanation for the light left-handed neutrino mass needed to explain the observed flavour oscillations. It predicts particles that have not yet been observed, but also provides a reason for this absence in experiments by their large mass. Nevertheless it is still just a theory without confirmation. The structure could however be used as a first step towards understanding of Unified Theories.

Chapter 5

A further study of Lie Groups and Algebras

Since *Grand Unified Theories* are based on large groups containing the *Standard Model group* $SU(3)_c \times SU(2)_L \times U(1)_Y$, the dealing with explicit representations becomes more and more complicated. A new framework for studying the structure of these groups is needed. Such a framework is provided by the *root systems* as analysed by Élie Cartan and Eugene Dynkin [22]. Fortunately many of the concepts needed can be understood by considering relatively easy examples, which will be reviewed here without too much of the mathematical backgrounds and proofs. For a more extensive overview the reader is referred to the many textbooks on this topic, for instance [2, 8], on which the rest of this chapter is inspired.

5.1 Root systems

It is well known from the study of angular momentum in Quantum Mechanics that due to the commutation relation between the generators of $SU(2)$,

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \qquad i, j, k = 1, 2, 3, \tag{5.1}$$

eigenstates of one of these generators are no eigenstates of the others, corresponding to an uncertainty relation. One can however define $J_\pm = (J_1 \pm iJ_2)/\sqrt{2}$, with corresponding commutation relations,

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = J_3. \tag{5.2}$$

If one considers eigenstates of J_3, labelled by their eigenvalue, $J_3|m\rangle = m|m\rangle$, it follows from the commutation relations that $J_\pm$ behave as raising and lowering operators,

$$J_3 J_\pm |m\rangle = (m \pm 1) J_\pm |m\rangle. \tag{5.3}$$

It turns out that this procedure of finding a set of commuting generators and raising and lowering operators for the eigenstates of these commuting generators can be generalized. The commuting hermitian generators are represented by simultaneously diagonalizable matrices and are called *Cartan generators*,

$$H_i = H_i^\dagger, \quad [H_i, H_j] = 0, \quad i, j = 1, 2, \ldots, m, \tag{5.4}$$

where m, the number of these independent Cartan generators is called the *rank* of the algebra. The eigenstates of the Cartan generators are labelled $|\mu\rangle$, with eigenvalues μ_i called the *weights*,

$$H_i|\mu\rangle = \mu_i|\mu\rangle. \tag{5.5}$$

The m-component vector with elements μ_i is called the *weight vector* μ. Sometimes the weight vector is also called the weight for short. In the *adjoint representation*, where the states are described by the generators themselves, the weights are called *roots*. From the commutation of the Cartan generators it is clear that the roots of these are zero,

$$H_i|H_j\rangle = |[H_i, H_j]\rangle = 0. \tag{5.6}$$

The raising and lowering operators are denoted by the corresponding non-zero root vectors α,

$$H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle, \quad \Rightarrow \quad [H_i, E_\alpha] = \alpha_i E_\alpha. \tag{5.7}$$

Since the Cartan generators are defined to be hermitian it follows from taking the hermitian conjugate of the commutation relation that,

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger = [H_i, E_{-\alpha}], \quad \Rightarrow \quad E_\alpha^\dagger = E_{-\alpha}. \tag{5.8}$$

Clearly for every raising operator there is a lowering operator and vice versa. That these operators indeed behave as raising and lowering operators can be seen as follows,

$$H_i E_{\pm\alpha}|\mu\rangle = E_{\pm\alpha} H_i|\mu\rangle + [H_i, E_{\pm\alpha}]|\mu\rangle = (\mu_i \pm \alpha_i) E_{\pm\alpha}|\mu\rangle. \tag{5.9}$$

One can also show that the commutation relation between this raising and lowering operator is in accordance with the $SU(2)$ case,

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i. \tag{5.10}$$

In this procedure the Lie algebra $g = \{t_1, t_2, \ldots, t_n\}$ has been changed by a complex transformation to the Cartan basis,

$$\tilde{g} = \{H_1, H_2, \ldots, H_m, E_{\pm\alpha_1}, E_{\pm\alpha_2}, \ldots, E_{\pm\alpha_{(n-m)/2}}\} \tag{5.11}$$

Here n denotes the dimension, number of independent elements, of the algebra, m the rank, and α_i the i-th root vector, not the i-th element of the root vector α.
So far we have discussed raising and lowering operators, but have not explicitly stated which is which, since for this the notion of a higher and lower is needed. For this we define a *positive weight vector* as one with first non-zero element positive. By this definition it is also clear what the *higher weight vector* of two is, the higher minus the lower weight must be positive. One can also define a *simple weight vector*, one that can not be written as a linear combination of other positive weights with positive coefficients. Since the roots are the weights of the adjoint representation hereby positive, higher and simple roots are also defined.

5.1.1 Example: Root system of $SU(3)$

In (2.26) the *Gell-Mann matrices* were introduced as generators of $SU(3)$. Since λ_3 and λ_8 are already hermitian and diagonal, these would be a suitable choice for the Cartan generators. It is common to define,

$$H_1 = \lambda_3/2, \quad H_2 = \lambda_8/2. \tag{5.12}$$

Since these are already diagonal the weights are equal to the components on the diagonal, correspond to the eigenvalues of the eigenvectors $|\mu_i\rangle = e_i$, the *Cartesian unit vectors*, $(e_i)_j = \delta_{ij}$.

$$\begin{aligned} &\mu_i = (H_{1_{ii}}, H_{2_{ii}}), && \mu_1 = (1/2, \sqrt{3}/6), \\ &\mu_2 = (-1/2, \sqrt{3}/6), && \mu_3 = (0, -\sqrt{3}/3), \end{aligned} \tag{5.13}$$

Raising and lowering operators should add or subtract a root from a weight to obtain another weight (5.9). Therefore a root must be the difference between two weights,

$$\begin{aligned} &\pm\alpha_i = \pm|\mu_i - \mu_{i+1}|, && \pm\alpha_1 = \pm(1, 0), \\ &\pm\alpha_2 = \pm(1/2, -\sqrt{3}/2), && \pm\alpha_3 = \pm(1/2, \sqrt{3}/2). \end{aligned} \tag{5.14}$$

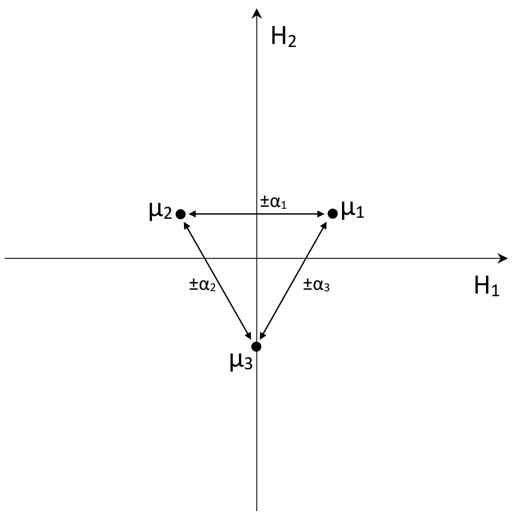
Note that this method of finding the reversing of the argument does not hold in general, although a root is always related to a raising and lowering operator, the difference between two weights is not always a root. One can check whether the correct number of roots has been found by comparing the number of generators. Since there are two generators, one for raising and one for lowering, per root the total *number of roots* must be,

$$k = \frac{n-m}{2}, \tag{5.15}$$

where n is the dimension of the algebra and m the rank. The algebra of $SU(N)$ has dimension $N^2 - 1$, so for $SU(3)$, which has rank 2 and dimension 8 there are indeed 3 roots.
The raising and lowering operators in $SU(3)$ are those with one non-zero off-diagonal component, for instance $(E_{+\alpha_1})_{ij} = \delta_{i1}\delta_{j2}$. In terms of the Gell-Mann matrices the raising and lowering operators can hereby be found to be,

$$\begin{aligned} E_{\pm\alpha_1} &= \frac{1}{2}(\lambda_1 \pm i\lambda_2), \\ E_{\pm\alpha_2} &= \frac{1}{2}(\lambda_6 \pm i\lambda_7), \\ E_{\pm\alpha_3} &= \frac{1}{2}(\lambda_4 \pm i\lambda_5). \end{aligned} \tag{5.16}$$

It becomes more intuitive if one considers the *weight* and *root diagrams* of $SU(3)$, where the weight and root corresponding to the Cartan generator is plotted along each axis.



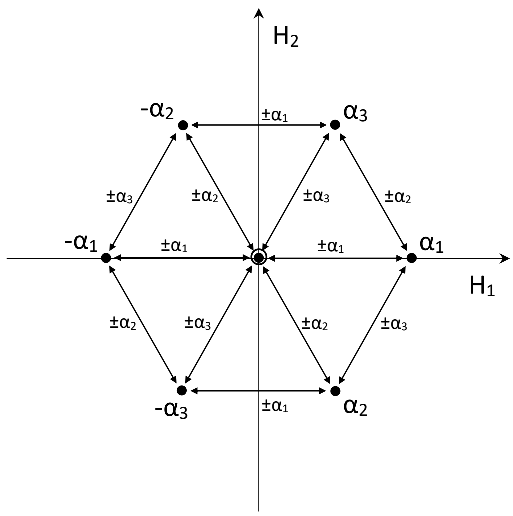


Figure 5.1: At the left the weight diagram corresponding to the defining representation of $SU(3)$ and at the right the root diagram of $SU(3)$. The root labelling of the links along the arrows refer to operators raising and lowering by these roots.

The double circle at the centre of the root diagram stands for the two zero roots corresponding to the two Cartan generators. The positive roots are all at the right of the diagram, since the H_1-axis correspond to the first component of the root vector. The simple roots are α_2 and α_3, α_1 is not a simple root, since $\alpha_1 = \alpha_2 + \alpha_3$. The *number of simple roots* is always equal to the *rank* of the algebra the root system corresponds to, since the simple roots span a basis for the space with dimension equal to the number of Cartan generators.

5.2 Dynkin Diagrams

Dynkin [22] showed that for two different simple roots α_1 and α_2,

$$\frac{2\alpha_1 \cdot \alpha_2}{\alpha_1^2} = -q, \tag{5.17}$$

where q is a non-negative integer. From this it follows that,

$$\cos^2\theta_{12} = \frac{(\alpha_1 \cdot \alpha_2)^2}{\alpha_1^2\alpha_2^2} = \frac{qq'}{4}. \tag{5.18}$$

This provides strong restrictions on the possible angles between two simple roots. From (5.17) it follows that $\cos\theta_{12} \leq 0$, and therefore $90° \leq \theta_{12} \leq 180°$. Since $0 \leq \cos^2\theta_{12} \leq 1$ the values of $0 \leq qq' \leq 4$. However $qq' = 4$ corresponds to $\theta_{12} = 180°$, which means only one can be a positive root and therefore at most one of the two can be a simple root. Because q and q' are non-negative integers the only cases of interest are listed in the table below,

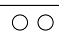
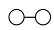
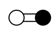
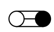
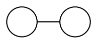
qq'	$\cos\theta_{12}$	θ_{12}	$\lvert\alpha_1\rvert/\lvert\alpha_2\rvert$	Dynkin
0	0	90°		
1	$-1/2$	120°	1	
2	$-\sqrt{2}/2$	135°	$\sqrt{2}$	
3	$-\sqrt{3}/2$	150°	$\sqrt{3}$	

Table 5.1: Possible relative configuration of simple roots. The relative length between the roots is given assuming $|\alpha_1| \geq |\alpha_2|$. The length ratio between two orthogonal roots is not fixed. The shorter of the two roots in the Dynkin diagram is denoted by a filled circle.

In the table Dynkin stands for the *Dynkin diagram*, where the two circles correspond the two simple roots and the connecting line denotes the angle between the roots. It turns out that there are only four of options in the relation between two simple roots. Nevertheless one could expect that there are still many different options for systems with multiple simple roots. It will however be shown these are also strongly restricted.
If the system of circles and lines can be divided in disconnected (orthogonal) components the system is called *decomposable*. Each separate subset corresponds to a simple Lie algebra and commutes with the others. The total decomposable system therefore corresponds to a semi-simple Lie algebra, corresponding to a *direct product* of the subgroups corresponding to the indecomposable subsets. These indecomposable systems of simple roots are by Dynkin called Π-systems.

5.2.1 Example: Dynkin diagram of $SU(3)$

It was shown that the simple roots of $SU(3)$ are $\alpha_2 = (1/2, -\sqrt{3}/2)$ and $\alpha_3 = (1/2, \sqrt{3}/2)$. From (5.18) it is found that $qq' = 1$ and therefore $\theta_{12} = 120°$. Both simple roots have unit length, which is consistent with table 5.1. The corresponding Dynkin diagram is therefore,

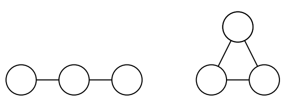


(5.19)

One could have also found this from the root diagram in figure 5.1. Indeed the simple roots α_2 and α_3 are found to have equal length and to be at an angle of 120°.

5.2.2 Π-systems of three roots

The next step would be to consider Π-systems consisting of three simple roots. The possibilities using only 90° and 120° angles between the simple roots are,

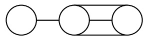


(5.20)

The first corresponds to a system with an angle of 120° between the first and second, and second an third, and of 90° between the first and third simple root, which is perfectly possible in three dimensions. The second corresponds to a system of angles of 120° between all the roots, which means that these are coplanar. These roots can therefore never be all positive or linearly independent and thus this system is not a valid Π-system. Any closed loops of three simple roots with angles of more than 120° can thereby also be excluded, it is not even possible to find three roots with sum over all angles between them larger than 360°. If one now also allows systems with angles of 135° between the roots, one finds,

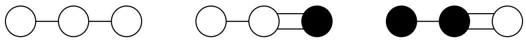
(5.21)

where the notation for shorter roots is left out for short, in general there are two options for each of the diagrams above. Again the first is perfectly possible, but the second encounters the same problem as before, since the sum over the angles is again 360°. Following the same reasoning systems with multiple angles of more than 120° can be excluded. This leaves for the three root systems with angles of 150° only the following options,



(5.22)

which can again be excluded since the sum over the three angles is 360°, from which it follows that the three roots can not be all simple. The result for Π-systems with three simple roots is that there are only three possibilities,



(5.23)

5.2.3 Π-systems of more roots

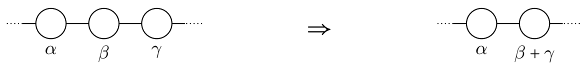
Consider a set of three connected circles in a Π-system, corresponding to three simple roots. From the discussion after (5.20) it is understood that these three simple roots can be all at angles larger than 90°, because of the exclusion of closed loops of three roots in the Dynkin diagram. If the roots at an angle of 90° are labelled α and γ, both at an angle larger than 90° to the root β, one finds,

$$\alpha \cdot \beta \neq 0, \qquad \beta \cdot \gamma \neq 0, \qquad \alpha \cdot \gamma = 0. \tag{5.24}$$

The Π-system with one root $\beta + \gamma$ instead of the two roots β and γ is also allowed. The angle of this root with α is the same as the angle between α and β in the original diagram,

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta \neq 0 \tag{5.25}$$

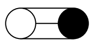
In this way a Π-system with rank one less can be constructed by taking two roots together,



(5.26)

Here the angles between the roots are both taken to be 120°, but in general this trick is possible for every angle, as long as the angle between α and β in the left and α and $\beta + \gamma$ in the right diagram are the same.

In this way one can exclude all diagrams that can be contracted to one of the previously excluded diagrams. One concludes that there are no Π-systems with multiple double lined connections between two simple roots, closed loops or with triple lined connections other than,



(5.27)

One can continue this and similar reasoning [2] and find that there exist only a restricted set of possible Π-systems, which have been classified by Dynkin [22] as found in table 5.2. The first four diagrams are infinite sets, corresponding to the *Special Unitary* and *Special Orthogonal* groups as discussed before and the *Symplectic* groups. The other five are called *exceptional groups*. The first exceptional group, E_6, will play an important role in our discussion of Grand Unified Theories.

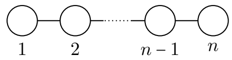
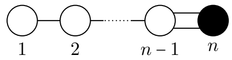
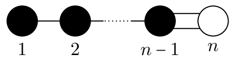
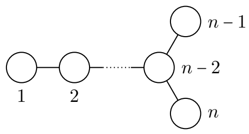
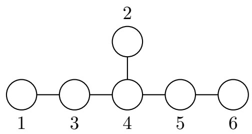
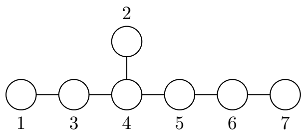
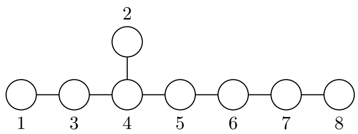
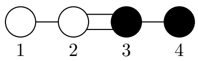
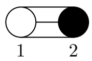
Dynkin diagram	Π-System	Lie group
	A_n	$SU(n+1)$
	B_n	$SO(2n+1)$
	C_n	$SP(2n)$
	D_n	$SO(2n)$
	E_6	E_6
	E_7	E_7
	E_8	E_8
	F_4	F_4
	G_2	G_2

Table 5.2: All possible Dynkin diagrams and their corresponding Lie groups. Note that the first four are infinite sets. Remember that a filled circle corresponds to a shorter root.

5.3 Subalgebras

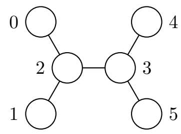
For unification one needs a (product of) algebras that contains the algebra corresponding to the Standard Model. The algebras contained in other algebras are called *subalgebras*. A subalgebra L' of an algebra $L' \subset L$ is formally defined as the set of elements in L, where for each pair of elements $a, b \in L'$ their Lie product is also in the subalgebra [1],

$$[a, b] \in L' \tag{5.28}$$

It turns out that there are two classes of subalgebras, the *regular* and *special subalgebras*, which will be treated here separately.

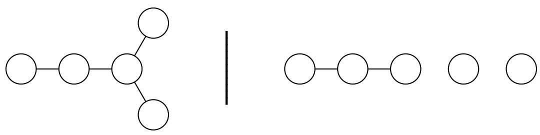
5.3.1 Regular subalgebras

The regular subalgebras can be easily obtained by considering the corresponding Dynkin diagram. To find subalgebras of the same rank as the algebra itself, called *maximal subalgebras*, one first has to find the extended diagram. The extended Dynkin diagram is one in which next to all simple roots also the lowest root, labelled as root zero, and the angles it makes with the simple roots are denoted. This lowest root is uniquily defined for every group and can for instance be found in table 16 of [8], where it is called the extended root. The extended diagram corresponding to $SO(10)$ is,



(5.29)

To find the maximal regular subalgebras one has to take out one of the circles [2], corresponding to taking out one of the roots.

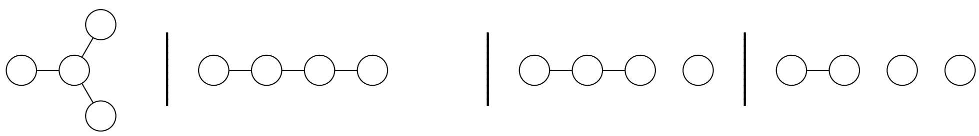


(5.30)

One finds in this case two inequivalent results for either taking out one of the roots $0, 1, 4, 5$ or $2, 3$ respectively. The first corresponds to $SO(10)$ itself, the algebra itself is of course a maximal subalgebra. From now on only *proper* subgroups will be considered, excluding the group itself as a subgroup. The second is the more interesting result,

$$SU(4) \times SU(2) \times SU(2) \subset SO(10). \tag{5.31}$$

Instead of the maximal subalgebras one can also find the subalgebras of lower rank by removing circles from the Dynkin diagram without extending it. The results obtained by taking out one simple root of $SO(10)$ are,



corresponding to $SO(8) \times U(1)$, $SU(5) \times U(1)$, $SU(4) \times SU(2) \times U(1)$ and $SU(3) \times SU(2) \times SU(2) \times U(1)$ respectively.

In taking out one simple root all non-simple roots including a multiple of this simple root are also excluded from the root system. The result is an invariant direction in the root space, which has the rank of the original algebra as dimension. But as the rank is the number of commuting Cartan generators there is a basis in which only one acts on this invariant direction. This Cartan generator thereby generates a $U(1)$ symmetry that occurs for every root taken out of the Dynkin diagram to find the regular subalgebra.

The regular subalgebras of E_6 containing the Standard Model algebra are shown in figure 5.2, obtained using the method described before. The $U(1)$ symmetries obtained in every step towards lower rank are left out for short.

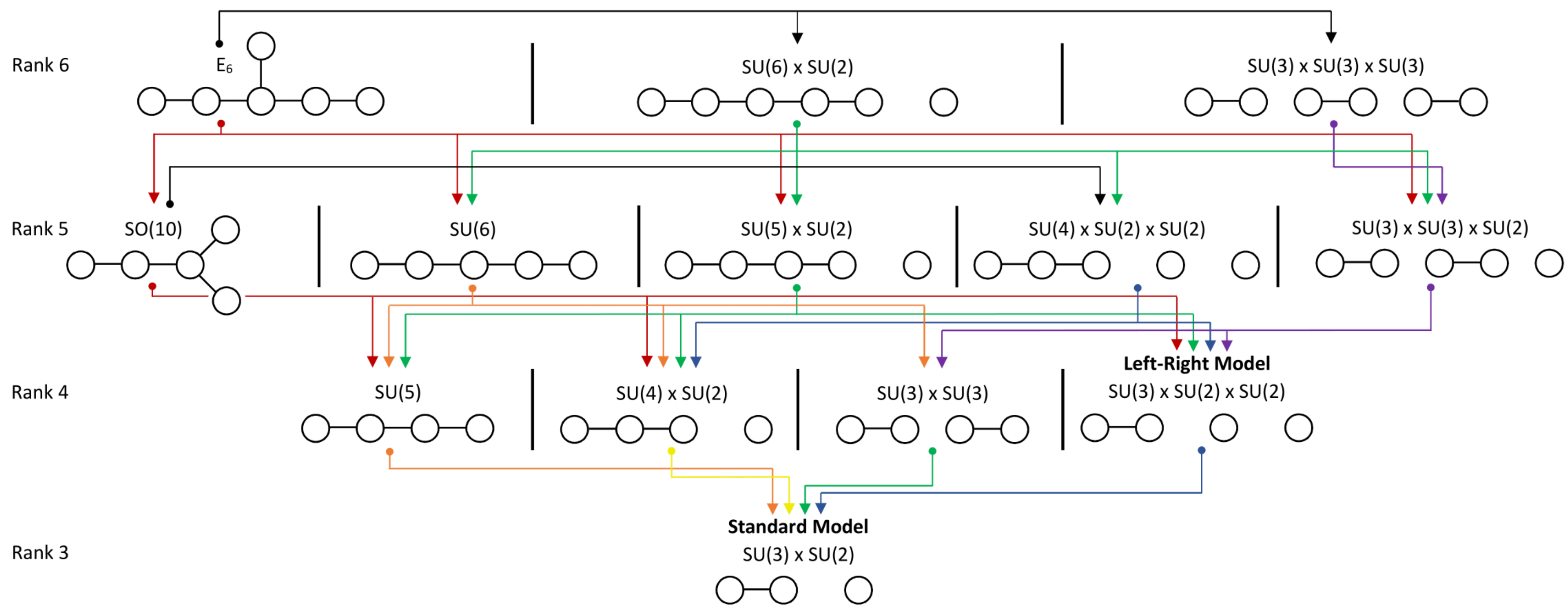


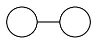
Figure 5.2: All groups corresponding to the regular subalgebras of E_6 that include the Standard Model. All $U(1)$ terms are left out for clarity, but for every rank lowered an extra $U(1)$ appears. Black arrows connect diagrams to their proper maximal subalgebras. Note that $SU(N)$ has no proper maximal subalgebras, since the corresponding extended Dynkin diagram is a ring [8]. The Left-Right model was labelled explicitly since it has been studies before. Many of the groups will be discussed in the following chapter as unification group candidates.

5.3.2 Special subalgebras

For finding the special subalgebras unfortunately there is no simple trick know based on the root diagram and corresponding Dynkin diagram. For finding the special subalgebras irreducible representations of candidate subgroups and of the group itself must be compared. In the next section the procedure will be described based on the example of $SU(2)$ as a special subalgebra of $SU(3)$, but this procedure can not easily be generalized.
Fortunately the special subalgebra of the classical Lie groups of rank 8 and lower and all exceptional Lie algebras are given in the appendices of [8].
After the illustrating example of the next section the special subalgebras will not be considered. The procedure of breaking a symmetry to one of its subgroups via the exclusion of certain generators and corresponding gauge bosons via the Higgs mechanism seems more physical than the mathematical construction of irreducible representations that fit nicely together. Nevertheless it has been claimed in [23] that special subalgebras can also be used for unification.

5.3.3 Example: $SU(2)\times U(1)$ as regular and $SU(2)$ as special subalgebra of $SU(3)$

As found in section 5.2.1 the Dynkin diagram corresponding to $SU(3)$ is,



(5.32)

To find the regular subalgebras one has to take out one of the simple roots, resulting in the the diagram with only one simple root, corresponding to $SU(2)$. By taking away this circle from the Dynkin diagram a $U(1)$ is obtained, resulting in a total regular subalgebra corresponding to $SU(2)\times U(1)$. In terms of the generators this taking away of the simple root corresponds to taking away the generators corresponding to raising and lowering by a root including a multiple of this simple root. If one takes out the simple root α_3, only the Cartan generators H_1, H_2 and the raising and lowering operators corresponding to $\alpha_2, E_{\pm\alpha_2}$ survive. One could have chosen a different definition of positivity, or changed the basis in a way that any other root survived, but the result would always be the two Cartan generator and one raising and one lowering operator in the subalgebra. If we stick to the root α_2 the subalgebra is generated by,

$$H_1 = \frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad E_{+\alpha_2} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \tag{5.33}$$
$$H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}, \qquad E_{-\alpha_2} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

The block structure of these matrices makes the invariant direction already visible, but to make the $SU(2)\times U(1)$ structure even more clear a basis transformation is performed, such that,

$$t_1 = \frac{1}{2}(E_{+\alpha_2} + E_{-\alpha_2}), \qquad t_3 = -\frac{1}{2}H_1 + \frac{\sqrt{3}}{2}H_2, \tag{5.34}$$
$$t_2 = \frac{-i}{2}(E_{+\alpha_2} - E_{-\alpha_2}), \qquad t_4 = \frac{\sqrt{3}}{2}H_1 + \frac{1}{2}H_2.$$

In this basis the generators are,

$$t_1 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \qquad t_3 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \tag{5.35}$$
$$t_2 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}, \qquad t_4 = \frac{1}{2\sqrt{3}}\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix},$$

in which one recognizes the $SU(2)$ generators t_1, t_2 and t_3, leaving the first component of a vector invariant, and a $U(1)$ generator t_4 commuting with the others. In terms of the root diagrams this corresponds to,

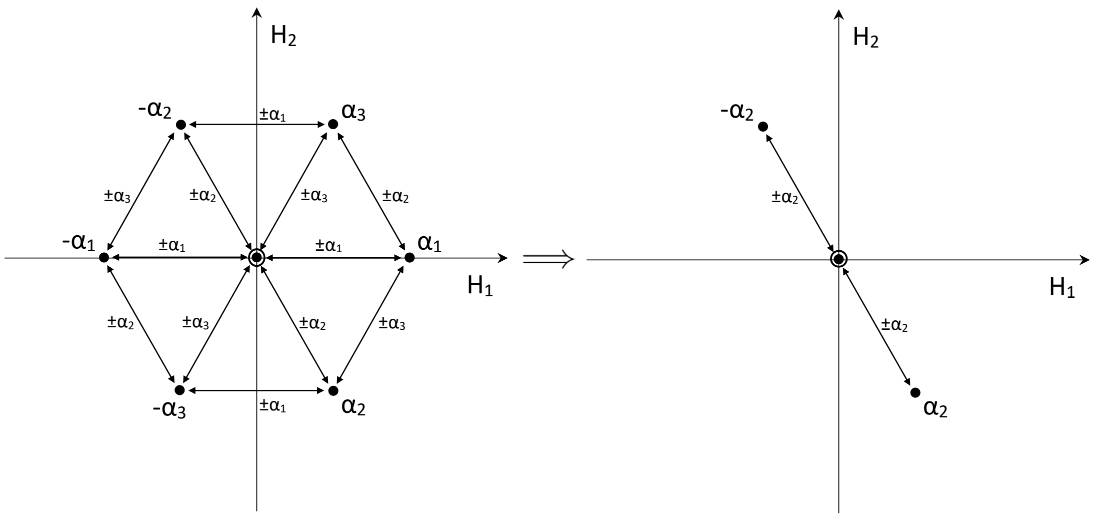


Figure 5.3: At the left the root diagram of $SU(3)$, at the right what remains for the regular subalgebra $SU(2) \times U(1)$. Note the two roots at the origin corresponding to the $SU(2)$ and $U(1)$ Cartan generator.

For the special subalgebra the three-dimensional representation of $SU(3)$ from (2.22) has to be considered. This representation can be written in terms of raising and lowering operators as,

$$J_+^1 = \frac{J_1^1 + iJ_2^1}{\sqrt{2}}, \quad J_-^1 = \frac{J_1^1 - iJ_2^1}{\sqrt{2}}, \quad H = J_3^1, \tag{5.36}$$

which in explicit matrix form corresponds to,

$$J_+^1 = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_-^1 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.37}$$

This clearly corresponds to a subalgebra of $SU(3)$, since,

$$J_+^1 = 2(E_{+\alpha_1} + E_{+\alpha_2}), \quad J_-^1 = 2(E_{-\alpha_1} + E_{-\alpha_2}), \quad H = \frac{1}{2}(\lambda_3 + \sqrt{3}\lambda_8). \tag{5.38}$$

In this case there is no room for a $U(1)$ generator. Since the three-dimensional representation of $SU(2)$ is irreducible according to *Schur's Lemma*, which states that the only matrices commuting with all generators in an irreducible representation are proportional to identity, the only generator of the $U(1)$ would be identity, which is not part of the algebra. Apparently $SU(2)$ is also a special subalgebra of $SU(3)$, which is different from the regular subalgebra, since that includes $U(1)$.

5.4 Constructing representations

The framework of roots and weights is also useful for the construction of representations of the groups. If one knows the highest weight of a representation all other weights of that representation can be found by acting with lowering operators on this weight. From this the total structure of the weight can be derived. It will turn out that this is also useful in finding the decomposition of the representation into representations of its regular subalgebras.
The *fundamental weights* are defined as the weight vectors μ_j that satisfy [2],

$$\frac{2\alpha_i \cdot \mu_j}{(\alpha_i)^2} = \delta_{ij}. \tag{5.39}$$

The fundamental weights are the highest weights of the *fundamental representations*. Since there is one fundamental weight for every simple root the number of fundamental representations is equal to the rank. However, if the fundamental representations are known larger representation can be constructed by multiplication of these.
Once the highest weight of a representation is known the other weights of the representation can be found by subtraction of the roots from the weight, as this is what the lowering operators does. Whether a weight μ_j can be lowered by a root α_i to find a lower weight is found from,

$$\frac{2\alpha_i \cdot \mu_j}{\alpha_i^2} = p - q, \tag{5.40}$$

where the integer p and q stand for the number of times μ_j can be raised or lowered respectively by α_i. This procedure can be repeated from the highest weight downwards, until a weight is found that can not be lowered by any simple root, which is the lowest root of the representation. It can happen that a weight can be lowered by multiple simple roots, in which case the representation branches. In this way all weights and their relating roots can be found, determining the complete structure of the representation. The method is illustrated in appendix B for the 27-dimensional representation of E_6.

5.4.1 Splitting representations for subalgebras

Now that the weight structure of the representation is known it can be used to find how the representation is divided under the regular subalgebras. A regular subalgebra was found by taking out a circle corresponding to a simple root from the Dynkin diagram. If the links between weights in the weight structure corresponding to that simple root are taken out in most cases the representation splits up in multiple disconnected components. The disconnected components correspond to representations of the subalgebras, which can be recognized from the simple roots relating the weights. In the following section a simple example is presented to illustrate the procedure. For a better picture of the application the reader is referred to the next chapter.

5.4.2 Example: Splitting the fundamental representations of $SU(3)$ in representations of $SU(2) \times U(1)$

The two simple roots of $SU(3)$ were found to be $\alpha_2 = (1/2, -\sqrt{3}/2)$ and $\alpha_3 = (1/2, \sqrt{3}/2)$. The fundamental weights corresponding to these are,

$$\mu_2 = (1/2, -\sqrt{3}/6), \quad \mu_3 = (1/2, \sqrt{3}/6), \tag{5.41}$$

labelled in this strange way without μ_1 to be consistent with (5.39). The corresponding representations can be found from these by lowering with α_2 and α_3, resulting in the following weight structures of the representations,

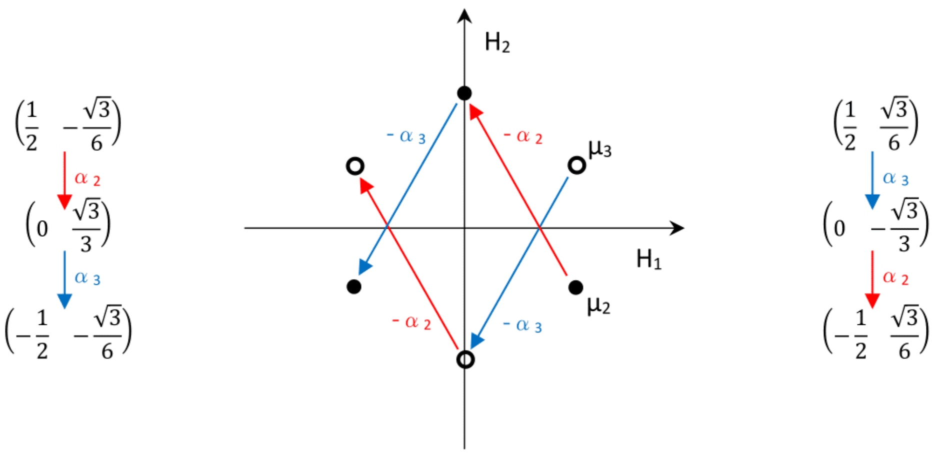


Figure 5.4: On the left the weight structure of the fundamental representation corresponding to μ_2, at the left for μ_3. In the middle the weight diagram including both representations is shown. Note that μ_2 and μ_3 here do not denote the second and third weight of the defining representation, but the second and third fundamental weight.

In the representation corresponding to μ_3 the structure of the defining representation is recognized, as shown in figure 5.1. The other fundamental representation is also three-dimensional, but corresponds to the conjugate representation, denoted by $\bar{3}$.
If the simple root α_3 is removed from the system the regular subgroup $SU(2)\times U(1)$ is obtained. It becomes clear that both representations split up in a two and a one-dimensional representation of $SU(2)$, the $U(1)$ charge of these representations must be assigned explicitly. To make the total representation chargeless under this $U(1)$ the charge of the singlet must be double and opposite to the doublet. For instance one could choose,

$$3 \to (2,-1/2)+(1,1), \qquad \bar{3} \to (2,1/2)+(1,-1). \tag{5.42}$$

The basis transformation for the Cartan generators in (5.34) correspond to a rotation of the H_1 and H_2 axis, from which it is also clear that t_4 does not mix with the $SU(2)$ but correspond to a $U(1)$. From the rotated axes in 5.5 it is observed that this assignment of the $U(1)$ charge is consistent.

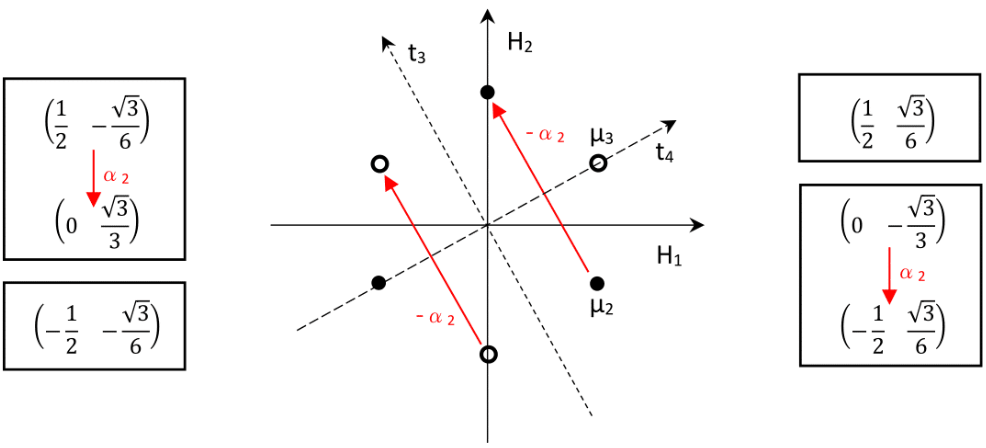


Figure 5.5: The regular subgroup of $SU(3)$ is $SU(2)\times U(1)$. The 3 and $\bar{3}$ both split up in a 2 and 1 representation. The $U(1)$ charge of these representations is not immediately clear from this construction.

Chapter 6

Grand Unified Theories

In figure 5.2 on page 44 the regular subgroups of the exceptional group E_6 that include the Standard Model group $SU(3)_C \times SU(2)_L \times U(1)_Y$ have been presented. Many of these have been suggested as candidates for *Grand Unification groups*, including E_6 itself. The $SU(5)$ group was the first suggestion in the famous paper from 1974 by H. Georgi and S. L. Glashow [24], which initiated this field of research. Not long after the larger group $SO(10)$ was suggested as a candidate by H. Fritzsch and P. Minkowski [25]. A different path both in conceptual approach and in the diagram with the subgroups involves the Left-Right models, as discussed before in section 4.2. J. Pati and A. Salam took this a step further and suggested in [26] not only a symmetry between left and right-handed particles, but also combined leptons to the quarks with lepton number as fourth colour, resulting in not a simple gauge group, but $SU(4)_C \times SU(2)_L \times SU(2)_R$ as a step towards unification. The $SU(3)_C \times SU(3)_L \times SU(3)_R$ maximal subgroup of E_6 has also been suggested for unification [27], under the name *trinification*. As this group is not a simple gauge group, it does not in itself unify the interactions, but if it is considered as a subgroup of E_6 or if a $\mathbb{Z}_3$ symmetry relating the three $SU(3)$ terms is added the coupling must be equal, resulting in a Grand Unified Theory.
Some of the implications of these subgroups of E_6 will be discussed in this chapter. The goal is to provide a general overview of the phenomenologies of the multiple unification group candidates.

6.1 Implications of Grand Unified Theories

Unification of the fundamental forces is a high energy phenomenon. The unification group does not replace the Standard Model group, but is predicted to be a larger group that is broken to the Standard Model at some energy scale, called the *Grand Unification energy* or *GUT scale*. This scale is approximately at [17],

$$E_{GUT} \approx 10^{16} \text{ GeV}. \tag{6.1}$$

This is still some orders of magnitude under the *Planck scale*, $E_{Pl} \approx 10^{19}$ GeV, at which *quantum gravity* will start playing a role and therefore all theories not implementing this, such as Unification, will break down anyway. Nevertheless it is way above the largest energies achieved by particle accelerators such as the LHC at CERN, $E_{LHC} \approx 10^4$ GeV. One can therefore not directly study the interactions at the unification energy level. Still some predictions of unified theories can be tested, of which *proton stability* is the best constrained, already resulting in the exclusion of simple $SU(5)$ models. The most powerful prediction of unified theories of course lies in the combined description of three fundamental interactions in one gauge theory. Related to this is the at the unification scale fixed ratio between the three *coupling constants* that are independent parameters in the Standard Model.

6.1.1 Couplings and the Weinberg angle

In section 3.1.1 a basis transformation was introduced by spontaneous symmetry breaking from $SU(2)_L \times U(1)_Y$ generators W^3_μ and B_μ to a massive and massless Standard Model gauge boson

Z_μ^0 and A_μ. It is common to describe this basis transformation in terms of the *Weinberg angle*,

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \qquad \tan\theta_W = \frac{g'}{g}. \tag{6.2}$$

From (3.8) and (3.9) it is clear that this Weinberg angle is also the relation between the masses of the massive vector bosons,

$$\cos\theta_W = \frac{m_W}{m_Z}, \qquad \sin^2\theta_W = 1 - \left(\frac{m_W}{m_Z}\right)^2. \tag{6.3}$$

In unified theories the generators corresponding to W_μ^3 and B_μ are combined in the same Lie algebra. The group structure determines the way these generators combine into generators of the larger unification group. The relation between the coupling constants is therefore given by the relation between the corresponding generators of the unification group, resulting in [17],

$$\sin^2\theta_W = \frac{\Sigma_i T_{3i}^2}{\Sigma_i Q_i^2}, \tag{6.4}$$

where i labels all fermions, T_{3i} is the weak isospin and $Q_i = T_{3i} + Y_i$ is the electromagnetic charge. The Weinberg angle can be determined experimentally in many different ways from many different processes. All results are around [28],

$$\sin^2\theta_W \approx 0.23. \tag{6.5}$$

Note however that a prediction of a different value for the Weinberg angle is not immediately inconsistent, since the coupling constants are scale dependent. The predictions for the Weinberg angle from unification will be at the unification scale, where all coupling should come together. To draw any conclusions from this prediction the *running of the coupling constants* up to the unification scale has to be considered, resulting in involved calculations, as introduced in [17].

6.1.2 Proton decay

Extending the gauge group to one including the Standard Model group and combining fermions in larger representations result in an increase in possibilities for interactions between different matter sectors. These new couplings result in the allowance of processes excluded in the Standard Model. Since the leptons and quarks inhabit different representations in the Standard Model these do not mix under gauge interactions. The Standard Model Higgs bosons does also not couple the leptons and the quarks, since they are contained in separate Yukawa interactions. The lightest baryon, the proton, is therefore practically stable. A possible decay channel is through an instanton coupling [29], which is strongly suppressed by the number of particles involved in the interactions up to lifetimes far beyond the age of the universe. But in beyond the Standard Model theories coupling between leptons and quarks can occur, resulting in proton decay processes such as illustrated in figure 6.1. The bosons mediating these interactions are called *leptoquarks*.

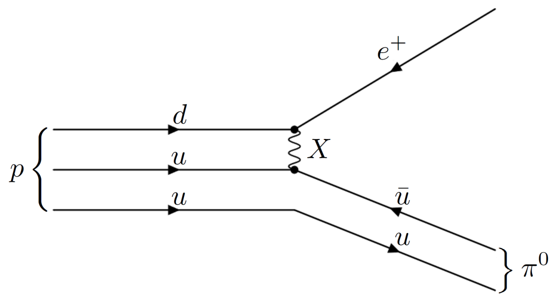


Figure 6.1: One of the proton decay modes possible in some unified theories. Since leptons and quarks are combined in representations decays violating baryon and lepton number are allowed, resulting in the instability of the proton. The X boson is a leptoquark.

The problem with theories predicting these processes is that there are strong experimental constraints on these decay channels. For instance the proton p is found by the *Super-Kamiokande* [30] not to decay into a pion π^0 and positron e^+ up to a lifetime of [31],

$$\tau(p \to e^+ + \pi^0) > 8.2 \times 10^{33} \text{ yr}, \tag{6.6}$$

at a 90% confidence level.
The boson coupling the leptons and quarks can be both in the gauge and in the Higgs sector of the theory. The predicted lifetime of the proton via a gauge interaction is [30],

$$\tau_p \propto \frac{M_X^4}{g_{GUT}^4 m_p^5}, \tag{6.7}$$

where the proportionality is given by the flavour structure of the theory. For this lifetime to be long clearly the gauge boson mediating proton decay must be very heavy. For the leptoquarks in the Higgs sector the reasoning is the same, but instead of the gauge coupling the Yukawa couplings appear in (6.7).

6.2 Candidates for Grand Unification

In this section some of the candidates from figure 5.2 are presented in series of increasing rank, corresponding to paths upwards from the Standard Model group in the diagram. The focus in this analysis will be on the general structure, more than the detailed calculations needed for a further study of these models. The presentation of series with increasing rank not only illustrates the relation between different unification models, but it is possible that unification occurs in *steps* at different energy levels [32]. One could for instance first encounter an energy scale where interactions follow the Left-Right model before unification occurs based on a larger group in which the Left-Right group is contained.

6.2.1 Smallest representation unification

The first suggestions for unified models were based on small gauge groups with representations in which the Standard Model fermions fit perfectly. As motivated in the original paper on $SU(5)$ unification "the uniqueness and simplicity of our scheme are reasons enough that it be taken seriously" [24].

SU(5)

The 15 fermions in one generation of the Standard Model (3.33) can be perfectly fitted in a $\bar{5}$ and 10 representation of $SU(5)$, which in terms of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ representations decompose as [6],

$$\begin{aligned} \bar{5} &= (\bar{3}, 1, 1/3) + (1, 2, -1/2) = d_R + \bar{L}_L, \\ 10 &= (\bar{3}, 1, -2/3) + (1, 1, 1) + (3, 2, 1/6) = \bar{u}_R + \bar{e}_R + Q_L. \end{aligned} \tag{6.8}$$

From section 2.6 we know that the conjugate of a right-handed field behaves as a left-handed field and vice versa, therefore the $\bar{5}$ representation behaves as a right-handed and the 10 as a left-handed spinor. Often only the particle corresponding to the representation is denoted, not whether the field itself or its conjugate is used.
This perfect fit comes at a cost however. The gauge bosons of the Standard Model can be contained in the adjoint representation of $SU(5)$ [17],

$$24 = (8, 1, 0) + (1, 3, 0) + (1, 1, 0) + (3, 2, 5/3) + (\bar{3}, 2, -5/3), \tag{6.9}$$

where in the first three terms the transformation properties of the gluons G_μ^a, the $SU(2)_L$ bosons W_μ^a and $U(1)_Y$ boson B_μ are recognized. There appear two other sets of 6 gauge bosons each, often called X and Y *bosons*, carrying colour, weak isospin and hypercharge, which are the leptoquarks discussed before. As leptons and quarks are mixed in the $SU(5)$ representations

one should not be surprised that these will couple in this theory. The resulting proton decay is so strong that the simplest $SU(5)$ models are excluded by experiment [17].
For breaking the $SU(5)$ gauge group to the Standard Model group a Higgs mechanism must be introduced. A general procedure for such a breaking scheme is discussed in great detail in [33], from which we find that an adjoint representation 24 is needed for the breaking of $SU(5) \to SU(3)_C \times SU(2)_L \times U(1)_Y$. The Standard Model Higgs responsible for the breaking of $SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_Q$, as discussed in section 3.1.1, is contained in the $SU(5)$ model in a 5 representation.

SO(10)

A next step in the procedure of fitting the Standard Model fermions in representations of small gauge groups could be $SO(10)$. This model is appealing as the complete fermion content of the Standard Model can be fitted in one 16 representation. In terms of $SU(5)$ representations this representation decomposes as [8],

$$16 = 1 + \bar{5} + 10, \tag{6.10}$$

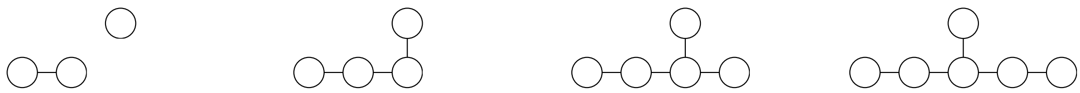
with one singlet more than needed to fit in the Standard Model. One such fermion that transforms trivially under the gauge groups is well known, this singlet might well correspond to a hypothetical *right-handed neutrino.*
Again the gauge sector can be contained in the adjoint representation, which for $SO(10)$ corresponds to the 45, which contains all $SU(5)$ gauge bosons and 21 more gauge bosons,

$$45 = 1 + 10 + \overline{10} + 24. \tag{6.11}$$

In breaking the $SO(10)$ symmetry the many extra gauge bosons and the one extra fermion (per generation) must obtain a large mass. This breaking can be performed by a 45 Higgs containing the 24 Higgs of $SU(5)$ and a 10, containing the Standard Model Higgs.
This is however not the only way in which $SO(10)$ can be used as a Grand Unified Theory. From figure 5.2 it is clear that there are many multiple other subgroups of $SO(10)$ suggesting other breaking patterns to the Standard Model. There is a clear link with the Left-Right symmetric models that will be discussed later in this chapter. This could be expected since the last fermion representation needed for these models has appeared in the 16 representation of $SO(10)$.

E_6

At first sight the relation between $SU(5)$ and $SO(10)$ seems to be rather arbitrary. If one however considers the Dynkin diagrams corresponding to these groups it is clear that these can be seen as logical extensions of the Standard Model,



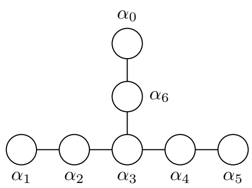
$$SU(3) \times SU(2) \quad \Rightarrow \quad SU(5) \quad \Rightarrow \quad SO(10) \quad \Rightarrow \quad E_6$$

In this framework the next step to E_6 also looks feasible, it is just the next one in the series. There have been many studies of E_6 as Grand Unification candidate, on which more can be found in [34, 35] or in the more recent [36]. In this text the main interest for E_6 is its many subgroups and related breaking schemes. It could however be that E_6 itself will turn out to be the Grand Unification group.
The fermion representation in E_6 is its smallest non-trivial representation 27, the smallest *fundamental* representation. For the gauge bosons the 78 *adjoint* representation is used. These divide into representations of $SO(10)$ as [8],

$$27 = 1 + 10 + 16, \tag{6.12}$$

$$78 = 1 + 16 + \overline{16} + 45. \tag{6.13}$$

It is clear that all Standard Model fermions, and twelve more, fit in the smallest fundamental representation of E_6. This is therefore a good representation to illustrate the transformation under its subgroups with. As discussed in section 5.4 the fundamental representations can be constructed by the method of the *highest weight*, which is applied in appendix B. The resulting structure of the 27 representation can be found in figure 6.2. The splitting up of this representation in representations of the regular subgroups of E_6 is done by taking out the links between the weights corresponding to the simple roots taken out of the (extended) Dynkin diagram. Label the roots of the extended Dynkin diagram of E_6 as [2],



(6.14)

For finding the $SO(10)$ representations one has to take out α_0 and α_5, resulting in the blocks with black borders in figure 6.2. To split these up in $SU(5)$ representations α_6 (or equivalently α_4) has to be removed as well, resulting in large grey blocks. The Standard Model representations can be found by taking out α_3 (or equivalently α_2), resulting in the diamond patterned blocks. The representations of the subgroups can be recognized by the structure of the roots between the weights in the different blocks. Note that in a representation the order of the roots is exactly opposite to its conjugate representation. Clearly all $SU(2)$ representations are real since they contain only one root. The representations of the same structure as the Standard Model fermions are labelled by the fermion content, as well as the hypothesized right-handed neutrino singlet. The splitting up of representations found in this way is consistent with the results presented before.

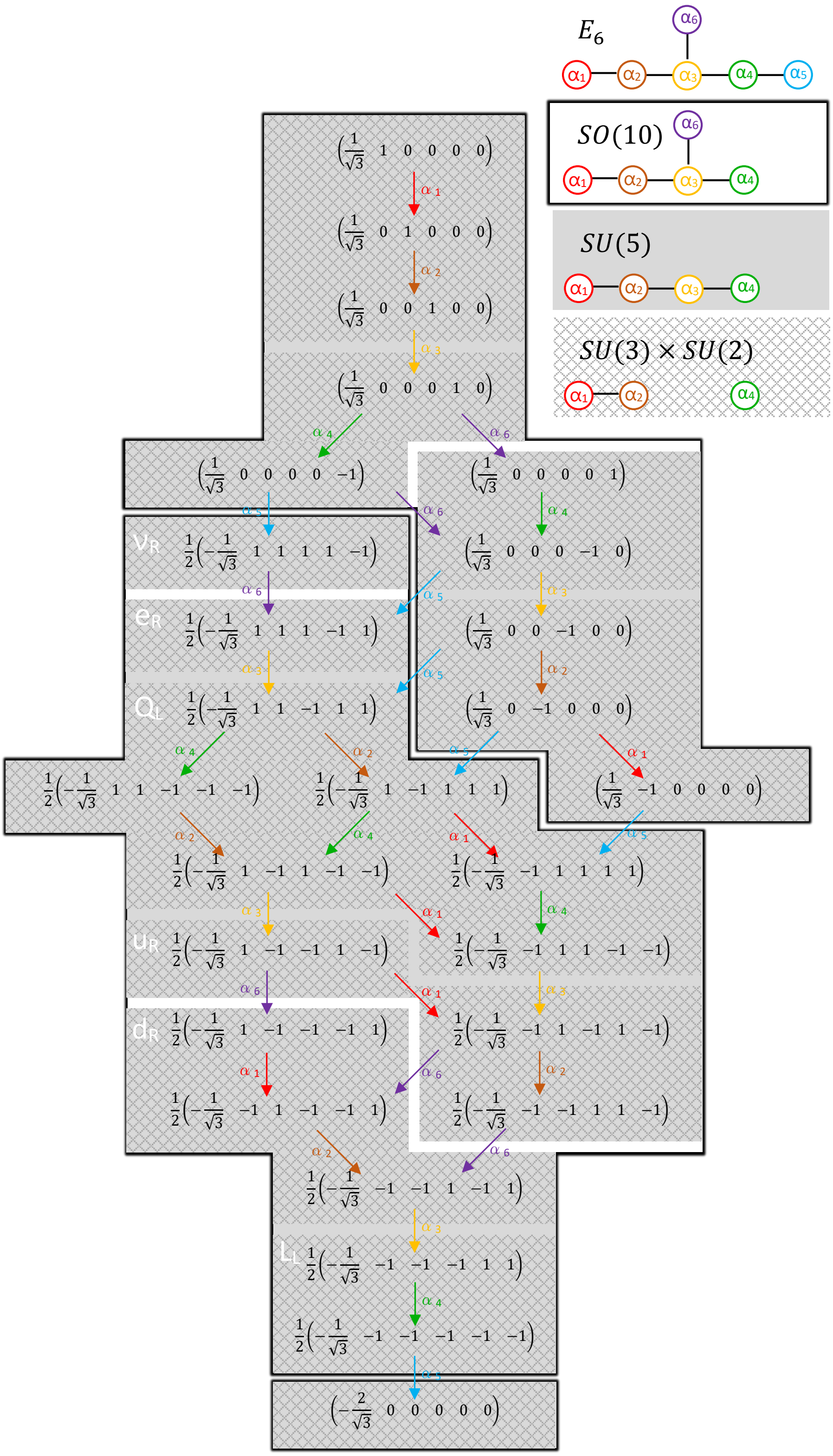


Figure 6.2: The 15 Standard Model fermions of one generation fit nicely in the $\bar{5}$ and 10 representations of $SU(5)$, marked in solid grey blocks. These in turn fit in the 16 representation of $SO(10)$ if a singlet, the right-handed neutrino, is added. The 16 representation of $SO(10)$, marked by black borders, can be combined with a 1 and 10 to form a 27 representation of E_6. In diamond pattern markings the representation blocks of the Standard Model group are shown, with the interpretation of the Standard Model fermions in the top-left corner.

6.2.2 Left-Right symmetric unification

As seen from figure 5.2, $SU(5)$ and $SO(10)$ are far from the only subgroups of E_6. Another approach to unification is along the path started in section 4.2, in which the asymmetry between left and right-handed chirality spinors in the Standard Model has been removed by the introduction of Left-Right models. From the diagram it is clear that the Left-Right model group is contained in $SO(10)$, but one could also consider this as a subgroup of E_6 via $SU(3)_C \times SU(3)_L \times SU(3)_R$. Grand Unification based on this gauge group is often called *trinification.* Another option would be not to extend the Left and Right gauge groups, but extend the colour gauge group to $SU(4)$ and implement thereby lepton number as a fourth colour, as in the *Pati-Salam model* [26], based on $SU(4)_C \times SU(2)_L \times SU(2)_R$.

$\mathbf{SU(3)_C \times SU(3)_L \times SU(3)_R}$

Although *trinification* is clearly based on a larger gauge group containing the Standard Model it is not really a Grand Unified Theory, since the it is still based on three unrelated couplings for each of the $SU(3)$ subgroups. This is often solved manually by imposing a discrete $\mathbb{Z}_3$ symmetry, symmetrizing the $SU(3)_C$, $SU(3)_L$ and $SU(3)_R$ [27]. If one however considers trinification as a subgroup of an E_6 gauge group the equality of the couplings is intrinsic.
The fermion content of one family can be contained in the 27 representation of E_6, which splits up for $SU(3)_C \times SU(3)_L \times SU(3)_R$ as [8],

$$27 = (\bar{3}, 3, 1) + (3, 1, 3) + (1, \bar{3}, \bar{3}), \tag{6.15}$$

where parts of all three components are needed to contain the Standard Model fermions. This division of representations can also be seen from figure 6.3. For the gauge section one does not need the complete 78 dimensional representation of E_6 [8],

$$78 = (8, 1, 1) + (1, 8, 1) + (1, 1, 8) + (3, 3, \bar{3}) + (\bar{3}, \bar{3}, 3). \tag{6.16}$$

Only the first three components suffice to contain the Standard Model gauge section, resulting in a total of 24 gauge bosons. Since the fermion representations of $SU(3)_C \times SU(3)_L \times SU(3)_R$ do not contain both leptons and quarks the decay of the proton via leptoquarks in the gauge sector are not expected to be present.
The Higgs mechanism responsible for the breaking in this model can be based on two 27 representations [27], which is rather different from the other unification models where an adjoint representation was used. These representations can also be used for the Standard Model breaking of the electroweak sector.
The Higgs bosons can couple in many ways to the fermions in trinification models, also including couplings between leptons and quarks. In this way, despite the absence of leptoquarks in the gauge sector, the proton is also unstable in these models, with decay modes through *scalar leptoquarks.* Since these decays are mediated by the Higgs bosons it is expected that decay channels into more massive particles are preferred. Although the exact lifetime is strongly dependent on the mass of the Higgs bosons and therefore hard to calculate [27] predicts the decay rate for $p \to \mu^+ + \pi^0$ to be 14 times larger than $p \to e^+ + \pi^0$, which could provide a method of distinguishing this model from others experimentally in the future. The current lower bound on the proton lifetime through this channel is however in the same order of magnitude [31],

$$\tau(p \to \mu^+ + \pi^0) > 6.6 \times 10^{33} \text{ yr}. \tag{6.17}$$

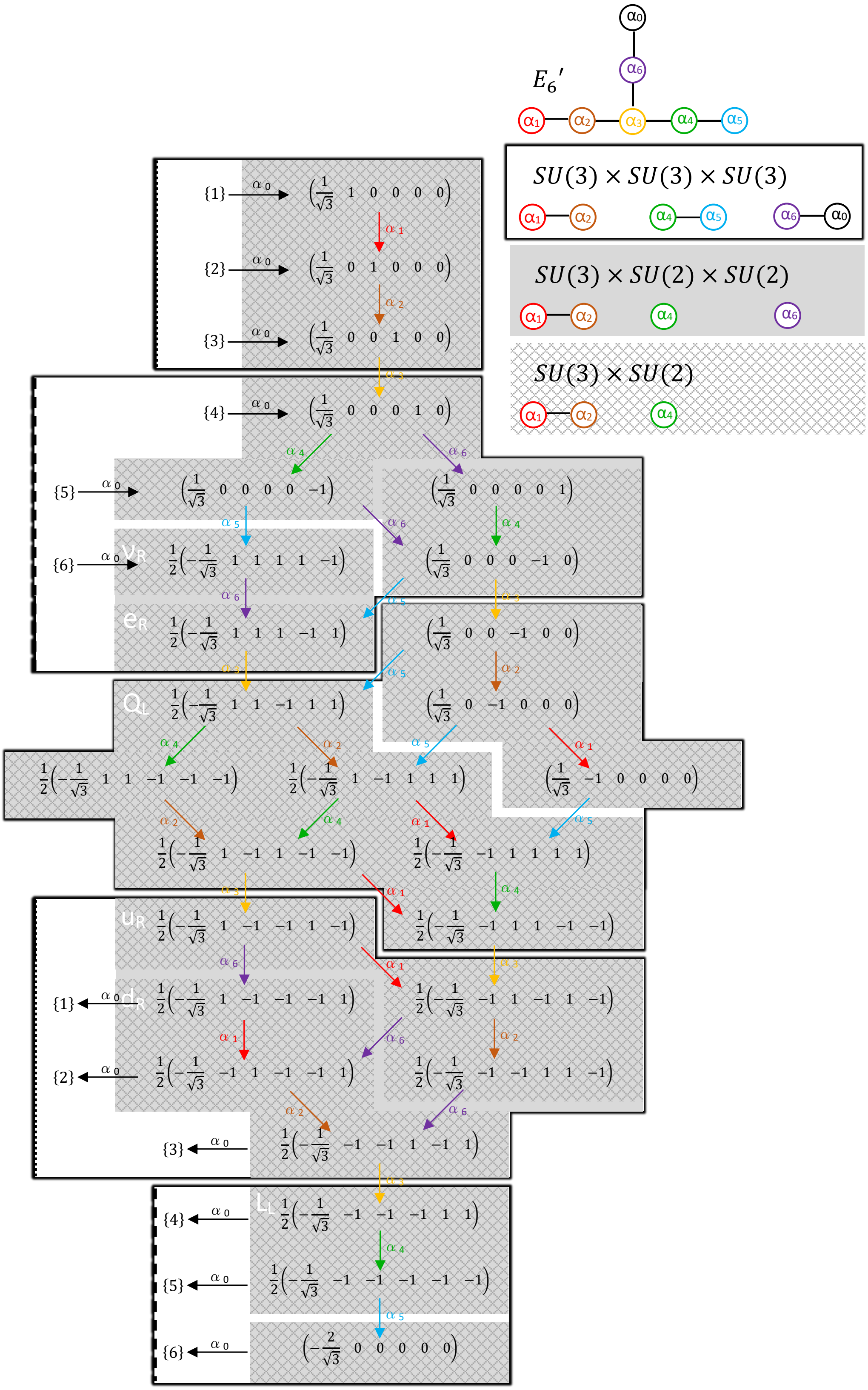


Figure 6.3: For splitting up the 27 representation of E_6 in representations of $SU(3)_C \times SU(3)_L \times SU(3)_R$ the lowest root has also to be taken into account. Note that there are two cutting lines which should be linked together resulting in three representations of $SU(3)_C \times SU(3)_L \times SU(3)_R$ marked by a black border. From the Left-Right model representations, marked in grey, it is clear that the right-handed fermions of the Standard Model also combine into doublets in this model.

$\mathbf{SU(4)_C \times SU(2)_L \times SU(2)_R}$

Another possible path from E_6 to the Left-Right model is via an extension of the colour sector. J. Pati and A. Salam [26] introduced a model based on a $SU(4)_C \times SU(2)_L \times SU(2)_R$ gauge group, in which lepton number is considered as a fourth colour for strong interactions. The observed asymmetry between left and right-handed chirality particles and between quarks and leptons are in this way both low energy phenomena. This strong relation between leptons and quarks suggests that this theory involves leptoquarks.
From figure 5.2 it is clear that this model is embedded in E_6 via $SU(6) \times SU(2)$, which will not be treated in detail in this thesis, and via $SO(10)$ as a maximal subgroup. To see this the extended Dynkin diagram of $SO(10)$ has to be considered, as shown in (5.29). In figure 6.4 this lowest root of $SO(10)$ is included as α_0, which should not be confused with the lowest root of E_6 used in the discussion of trinification. From this diagram or [8] one finds the 16 representations of $SO(10)$ splits up into representations of $SU(4)_C \times SU(2)_L \times SU(2)_R$ as,

$$16 = (4,2,1) + (\bar{4},1,2). \tag{6.18}$$

The neutrino couples to the up-quarks and the electron to down-quarks, which in turn form a left-handed and a right-handed doublet. This is exactly the structure expected and clearly can be split up even further to find the Left-Right model representations (4.36) and Standard Model representations (3.33). The gauge sector can be included in the adjoint 45 representation of $SO(10)$ [8],

$$45 = (1,3,1) + (1,1,3) + (15,1,1) + (6,2,2). \tag{6.19}$$

The first two terms can be recognized as the gauge bosons of the left and right gauge group. The third term is the extension of the gluon sector, which includes couplings between leptons and quarks. This theory therefore indeed involves leptoquarks in the gauge sector. It turns out however that these leptoquarks do not result in proton decay [26]. To analyse the leptoquarks further consider the breaking of $SU(4)_C$ to $SU(3)_C \times U(1)_{B-L}$ as in the Left-Right model studied before in section 4.2. The 15 representation splits as [8],

$$15 = (8,0) + (3,-4/3) + (\bar{3},4/3) + (1,0), \tag{6.20}$$

where the last component is the $B-L$ charge of the gauge bosons. The second and third term here are clearly leptoquarks. But since interactions via these gauge bosons conserve $B-L$ and fermion number $B+L$ [26], baryon and lepton number should also be conserved separately.
The gauge bosons in the last term of (6.19) also include leptoquarks, but since this representation does not include any Standard Model gauge bosons it is not needed in the $SU(4)_C \times SU(2)_L \times SU(2)_R$ model. It appeared because the gauge sector of the $SO(10)$, which is known to include leptoquarks responsible for proton decay, was used as starting point. Any proton decay in the Pati-Salam model must include at least three decay products and is mediated through the Higgs bosons [26], making this theory hard to exclude via constraints on proton decay.

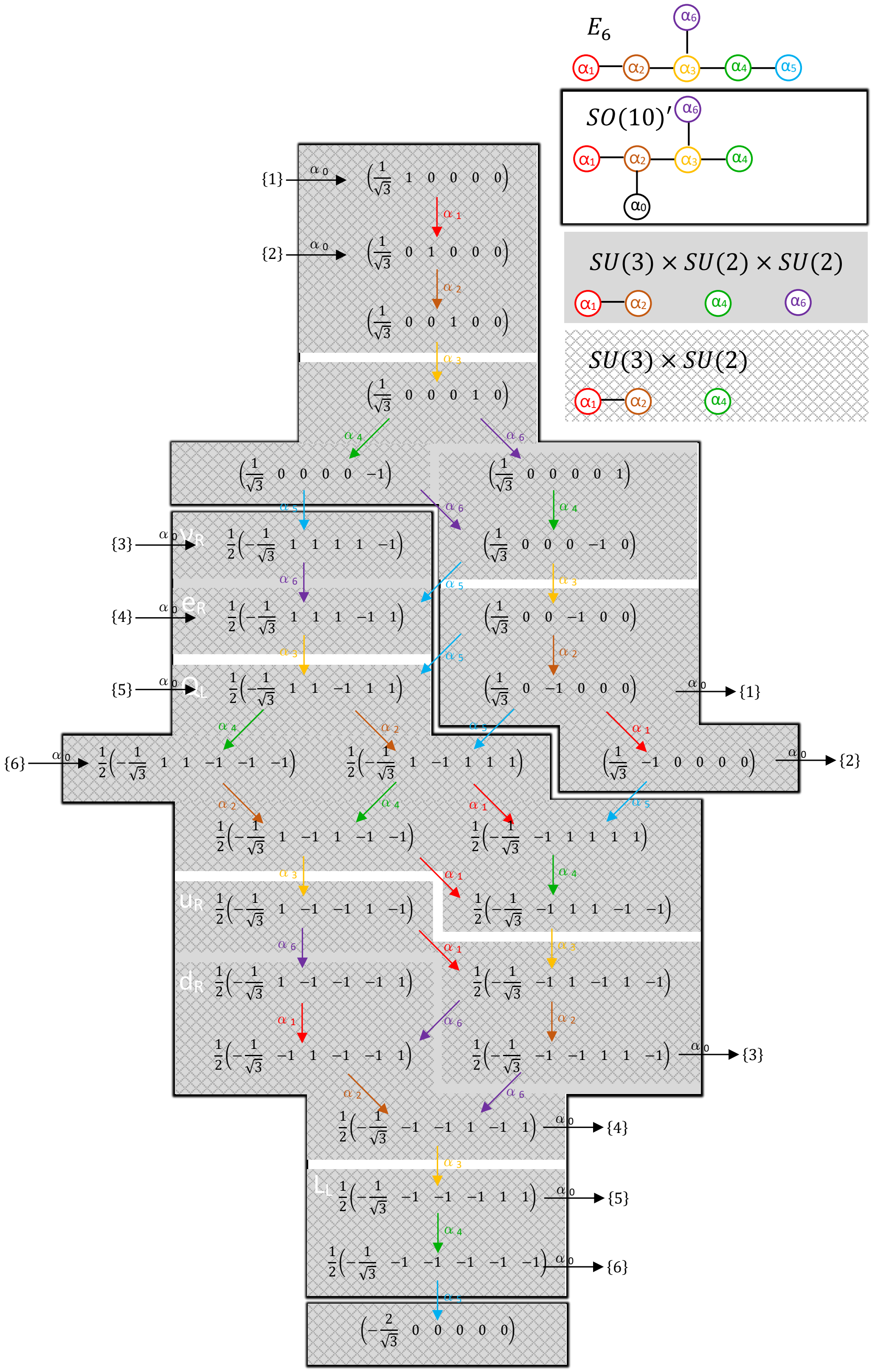


Figure 6.4: The $SU(4)_C \times SU(2)_L \times SU(2)_R$ model can be included in E_6 in multiple ways. In this diagram it is illustrated as a maximal subgroup of $SO(10)$, via its lowest root α_0. Note that this is another root than α_0 in figure 6.3, which is the lowest root of E_6.

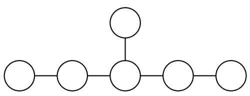
Chapter 7

Conclusion

Despite the enormous success of the *Standard Model* there are some indications that there must exist physics beyond it. An interesting extension is the *Left-Right model*, in which the asymmetry between left and right-handed chirality spinors, put in the Standard Model to fit the $V - A$ behaviour observed in experiments, is lifted at high energy scales.
This was the first encounter with an extension of the *gauge sector* in this thesis. This procedure for finding possible beyond the Standard Model theories can be generalized by considering groups that include the Standard Model group,

$$SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{7.1}$$

The mathematical framework for dealing with groups is *group theory*. A useful method for finding *subgroups* has been developed by E. Dynkin and is based on *Dynkin diagrams*, which relates the *simple roots* in a system of circles and connecting lines. Using this method the regular subgroups can be found by removing simple roots from the system.
If the group structure is determined using Dynkin diagrams specific representations following this structure can be found from the method of the *highest weight*. A fundamental representation can be constructed from the *fundamental weight*, which is the highest weight of that representation. The structure of the representation is found by lowering weights with simple roots. The decomposition of representations into representations of their regular subalgebras can be found by removing connections via the simple roots that are not in the subalgebra.
These general methods have been applied to the *exceptional group* E_6 and its smallest non-trivial representation, which is 27-dimensional. The regular subgroups are found by removing circles from the Dynkin diagram of E_6,



(7.2)

Many subgroups correspond to gauge groups that have been suggested for *Grand Unified Theories*. The 27 representation has been shown to contain representations with the same transformation properties under the Standard Model subgroup of E_6 as the 15 fermions of one generation, the building block of ordinary matter.
The concept of a larger gauge group containing the Standard Model group as a low energy effect solves some mysteries of the Standard Model. The unification group contains only one *coupling strength*, which constrains the running of the Standard Model coupling constants, by relating them at the unification scale. It also provides a reason for the exact match between quarks and leptons, since these particles are often combined in representations, which ensures related quantum numbers. Unification can provide a natural explanation for the relations between parameters that are free in the Standard Model.
These solutions for Standard Model problems come at a cost however. Grand Unified Theories predict the existence of multiple not yet observed particles, both in the fermion sector, such as the *right-handed neutrino*, and in the extended gauge and Higgs sectors. These particles can be constructed to have a high mass, so that these will only be present at the *unification scale*, many

orders of magnitude above the reach of particle accelerators,

$$E_{GUT} \approx 10^{16} \text{ GeV}. \tag{7.3}$$

But even these heavy particles could have implications for experiments. The extended gauge and Higgs sectors result in interactions that are not possible in the Standard Model. The most interesting ones are the coupling between quarks and leptons through *leptoquarks*. The processes involving these can violate baryon and lepton number, resulting in the unobserved *proton decay*. There are strong constraints on this decay for different channels, providing a lower bound on the proton lifetime of,

$$\tau_p > 10^{33} \text{ yr}. \tag{7.4}$$

This bound provides the strongest experimental constraint on Grand Unified Theories. Although most models involve some form of leptoquarks the phenomenologies can be different. Unification models based on small representations in which the fermion content of the Standard Model fits nicely, such as $SU(5)$ and $SO(10)$ involve leptoquarks in the gauge sectors, which results in a preferable proton decay mode,

$$p \to e^+ + \pi^0. \tag{7.5}$$

Unification models based on the extensions of the Left-Right model, $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ behave rather differently. *Trinification* models, $SU(3)_C \times SU(3)_L \times SU(3)_R$, include the left and right gauge groups into larger groups without combining quarks and leptons in the same representations. The gauge sector does therefore not contain leptoquarks, but the complicated Higgs sector of this model does. Since these scalar leptoquarks couple to mass the most probable decay channel for the proton in this model is,

$$p \to \mu^+ + \pi^0. \tag{7.6}$$

There exist also models that contain leptons and quarks in the same representation, but still do not contain two body proton decay modes. An example is the Pati-Salam model, $SU(4)_C \times SU(2)_L \times SU(2)_R$, which extends the Left-Right model via the colour sector rather than the left and right sectors.
Clearly there are many different models possible within the field of Grand Unification. Despite the strong relations between most common models there are ways to distinguish them phenomenologically. This suggests that a possible future experimental confirmation of proton decay would not only make Grand Unification more probable, but could also give an idea of the underlying gauge group via the decay products.

Acknowledgements

At the beginning of my Master research Sybrand Zeinstra and I worked together a lot. The many discussions helped a great deal in the development of a fundamental basis for this thesis. Ruud Peeters was also a great help, always ready to share his thoughts in weekly meetings during the first half of the year, but also later when our research paths had already split up. Through coffee breaks Alexander Boeschoten contributed to this project in many deepening discussions on fundamental topics others would have skipped over without much thought. During lunch breaks Bas van der Zee satisfied my broader interest in physics by his entertaining stories on his totally different field of research. In the second half of the year Jelle Thole helped in the exploration of the beautiful concepts in group theory, in which his skill in finding articles on every detail was of great use. But most support came from my supervisor Daniël Boer, who during meeting almost every week helped me a lot, both in project related problems and in others. My thanks go out to all these people and many others that contributed both in direct and indirect ways to my project.

Appendix A

Dirac matrices

The Dirac matrices are defined by their anti-commutation relation,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{A.1}$$

from which it is clear that the interchange of two different Dirac matrices results in a minus sign, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{if} \quad \mu \neq \nu$. If one follows the mostly minus convention $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ one finds that the square of a Dirac matrix is, $\left(\gamma^0\right)^2 = \mathbb{1}, \quad \left(\gamma^i\right)^2 = -\mathbb{1}$.
There are multiple conventions for the Dirac matrices. In this text the chiral basis is used, in which the Dirac matrices are of the form,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{A.2}$$

where $\sigma^\mu = (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3)$ include the Pauli matrices.
The explicit form of the Dirac matrices in the chiral basis is therefore,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.3}$$

resulting in the following Dirac matrices,

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{A.4}$$

One related matrix is introduced, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which anti-commutes with all others,

$$\{\gamma^\mu, \gamma^5\} = 0 \tag{A.5}$$

The explicit form in the chiral basis is,

$$\gamma^5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.6}$$

Sometimes the transpose, conjugate and hermitian conjugate of the Dirac matrices are needed.

	$\gamma^{\mu T}$	$\gamma^{\mu *}$	$\gamma^{\mu \dagger}$
γ^0	γ^0	γ^0	γ^0
γ^1	$-\gamma^1$	γ^1	$-\gamma^1$
γ^2	γ^2	$-\gamma^2$	$-\gamma^2$
γ^3	$-\gamma^3$	γ^3	$-\gamma^3$
γ^5	γ^5	γ^5	γ^5

For the hermitian conjugate one finds that,

$$\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu. \tag{A.7}$$

For the transpose another useful relation is obtained,

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^2\gamma^0 = \gamma^\mu. \tag{A.8}$$

These are useful in the computation ensuring unitarity or $\mathcal{CPT}$-invariance of spinor field combinations.

Appendix B

Constructing the fundamental 27 representation of E_6

In this appendix the structure of the smallest fundamental 27 representation of E_6 is derived explicitly by the method of the highest weight as described in section 5.4. In this method a fundamental representation is constructed from its highest weight, called the *fundamental weight*. For finding the *fundamental weights* one has to find a concrete form for the *simple roots* first. Since the relation between these simple roots is fixed by the Dynkin diagram, these can in principle be found from solving a large system of equations. But as E_6 can be seen as an extension of $SO(10)$, for which a specific representation is known, (2.35), it might be easier to find the simple roots of $SO(10)$ first. For this a set of five *Cartan generators* has to be found, which commute with each other and can therefore be diagonalized simultaneously. A suitable choice is the set of generators $M_{2j-1,2j}$ where $j = 1, 2, \ldots, 5$ [2]. These are the matrices with the second Pauli matrix σ_2 on the diagonal and all other terms zero,

$$M_{12} = \begin{pmatrix} 0 & -i & & & \\ i & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}, \qquad M_{34} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & -i & \\ & & i & 0 & \\ & & & & \ddots \end{pmatrix}, \qquad \ldots \tag{B.1}$$

These are clearly hermitian and commuting matrices. The eigenvalues of these matrices are the *weights* of the defining 10 representation, given by the eigenvalues of the Pauli matrix σ_2, which are ± 1. The 10 *weight vectors* are therefore,

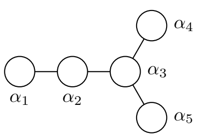
$$\mu_{2j-1} = e_j, \qquad \mu_{2j} = -e_j. \tag{B.2}$$

The *root vectors* can be found from the differences between the weight vectors. Note however that in this way one finds too many candidates, the vectors with one entry equal to ± 2 and all others zero are not roots of $SO(10)$. The *simple roots* of $SO(10)$ can therefore be found to be,

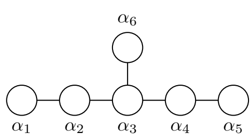
$$\begin{aligned} \alpha_1 &= (1, -1, 0, 0, 0), & \alpha_4 &= (0, 0, 0, 1, -1), \\ \alpha_2 &= (0, 1, -1, 0, 0), & \alpha_5 &= (0, 0, 0, 1, 1), \\ \alpha_3 &= (0, 0, 1, -1, 0), & & \end{aligned} \tag{B.3}$$

which indeed have all the same lengths and are at angles in accordance with the Dynkin diagram of $SO(10)$,

$SO(10)$



E_6



To extend this to all simple roots of E_6 a sixth root must be added (and some roots must be relabelled). Also a sixth Cartan generator will be added, since E_6 has rank 6, resulting in a sixth component for all root vectors. This component is added in front, and chosen to be zero for the five known root vectors, so that these are still positive and simple. The inner product with all five simple roots of $SO(10)$ following (5.18) and the length provide six equations to solve for the six component root vector α_5. The first component must be chosen positive, to make the root positive, the other components must be negative to make the root simple, resulting in the following *simple roots* for E_6,

$$\begin{aligned} \alpha_1 &= (0,1,-1,0,0,0), & \alpha_4 &= (0,0,0,0,1,-1), \\ \alpha_2 &= (0,0,1,-1,0,0), & \alpha_5 &= (\sqrt{3},-1,-1,-1,-1,-1)/2, \\ \alpha_3 &= (0,0,0,1,-1,0), & \alpha_6 &= (0,0,0,0,1,1), \end{aligned} \tag{B.4}$$

One can find the *lowest root*, needed for finding maximal subgroups, from the extended Dynkin diagram. For E_6 the extended diagram includes the lowest root at 120° with α_6 and 90° with the others. With similar reasoning as for α_5 the lowest root can be found to be,

$$\alpha_0 = (-\sqrt{3},-1,-1,-1,-1,1)/2. \tag{B.5}$$

The first fundamental weight follows (5.39) as,

$$\alpha_i \cdot \mu_1 = \delta_{i1}, \tag{B.6}$$

where α_i are the six simple roots. The first fundamental weight μ_1 can be found again by solving a system of equations, resulting in,

$$\mu_1 = (1/\sqrt{3},1,0,0,0,0). \tag{B.7}$$

The structure of the first fundamental representation of E_6, which will turn out to be the 27 representation, can be found from lowering this highest weight by subtracting simple roots. Whether a weight μ_j can be lowered by a simple root α_i is given by (5.40),

$$\frac{2\alpha_i \cdot \mu_j}{\alpha_i^2} = k, \tag{B.8}$$

where k is the number of times μ_j can be lowered by α_i. In this way the 27 representation of E_6 was found to be as in figure 6.2 on page 54.

Bibliography

[1] D. Boer. *Lie Groups in Physics*. 2016-2017.

[2] H. Georgi. “Lie Algebras in Particle Physics. From Isospin to Unified Theories”. In: *Front. Phys.* 54 (1982), pp. 1–255.

[3] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995. ISBN: 9780201503975, 0201503972.

[4] Steven Weinberg. *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 2005. ISBN: 9780521670531, 9780511252044.

[5] Steven Weinberg. *The quantum theory of fields. Vol. 2: Modern applications*. Cambridge University Press, 2013. ISBN: 9781139632478, 9780521670548, 9780521550024.

[6] A. Zee. *Unity of Forces in the Universe*. Singapore: World Scientific Pub Co Inc, 1982. ISBN: 9971950383, 9789971950385.

[7] A. Zee. *Quantum field theory in a nutshell*. 2003. ISBN: 0691140340, 9780691140346.

[8] R. Slansky. “Group Theory for Unified Model Building”. In: *Phys. Rept.* 79 (1981), pp. 1–128. DOI: `10.1016/0370-1573(81)90092-2`.

[9] Sidney Coleman and Jeffrey Mandula. “All Possible Symmetries of the S Matrix”. In: *Phys. Rev.* 159 (5 July 1967), pp. 1251–1256. DOI: `10.1103/PhysRev.159.1251`. URL: `https://link.aps.org/doi/10.1103/PhysRev.159.1251`.

[10] C. Itzykson and J. B. Zuber. *Quantum Field Theory*. International Series In Pure and Applied Physics. New York: McGraw-Hill, 1980. ISBN: 9780486445687, 0486445682. URL: `http://dx.doi.org/10.1063/1.2916419`.

[11] Igor P. Ivanov. “Building and testing models with extended Higgs sectors”. In: *Prog. Part. Nucl. Phys.* 95 (2017), pp. 160–208. DOI: `10.1016/j.ppnp.2017.03.001`. arXiv: `1702.03776 [hep-ph]`.

[12] Scott Willenbrock. “Symmetries of the standard model”. In: *Physics in D = 4. Proceedings, Theoretical Advanced Study Institute in elementary particle physics, TASI 2004, Boulder, USA, June 6-July 2, 2004*. 2004, pp. 3–38. arXiv: `hep-ph/0410370 [hep-ph]`.

[13] P. M. Ferreira, Howard E. Haber, and Joao P. Silva. “Generalized CP symmetries and special regions of parameter space in the two-Higgs-doublet model”. In: *Phys. Rev.* D79 (2009), p. 116004. DOI: `10.1103/PhysRevD.79.116004`. arXiv: `0902.1537 [hep-ph]`.

[14] Gustavo C. Branco, Luis Lavoura, and Joao P. Silva. “CP Violation”. In: *Int. Ser. Monogr. Phys.* 103 (1999), pp. 1–536.

[15] P. M. Ferreira et al. “CP4 miracle: shaping Yukawa sector with CP symmetry of order four”. In: (2017). arXiv: `1711.02042 [hep-ph]`.

[16] Wouter Gerard Dekens. “Discrete symmetry breaking beyond the standard model”. English. PhD thesis. University of Groningen, 2015. ISBN: 978-90-367-8118-3.

[17] R. N. Mohapatra. *UNIFICATION AND SUPERSYMMETRY. THE FRONTIERS OF QUARK - LEPTON PHYSICS*. Berlin: Springer, 1986. ISBN: 9781475719307, 9781475719284. DOI: `10.1007/978-1-4757-1928-4`.

[18] Rabindra N. Mohapatra and Goran Senjanovi ć. “Neutrino Mass and Spontaneous Parity Nonconservation”. In: *Phys. Rev. Lett.* 44 (14 Apr. 1980), pp. 912–915. DOI: `10.1103/PhysRevLett.44.912`.

[19] G. Senjanovic and Rabindra N. Mohapatra. "Exact Left-Right Symmetry and Spontaneous Violation of Parity". In: *Phys. Rev.* D12 (1975), p. 1502. DOI: 10.1103/PhysRevD.12.1502.

[20] Goran Senjanovic. "Spontaneous Breakdown of Parity in a Class of Gauge Theories". In: *Nucl. Phys.* B153 (1979), pp. 334–364. DOI: 10.1016/0550-3213(79)90604-7.

[21] Yue Zhang et al. "General CP Violation in Minimal Left-Right Symmetric Model and Constraints on the Right-Handed Scale". In: *Nucl. Phys.* B802 (2008), pp. 247–279. DOI: 10.1016/j.nuclphysb.2008.05.019. arXiv: 0712.4218 [hep-ph].

[22] A. A. Yushkevich, G. M. Seitz, and A. L. Onishchik. *Selected Papers of E. B. Dynkin with Commentary.* AMS and International Press of Boston, 2000. ISBN: 9780821810651.

[23] Naoki Yamatsu. "Special Grand Unification". In: *PTEP* 2017.6 (2017), 061B01. DOI: 10.1093/ptep/ptx088. arXiv: 1704.08827 [hep-ph].

[24] H. Georgi and S. L. Glashow. "Unity of All Elementary Particle Forces". In: *Phys. Rev. Lett.* 32 (1974), pp. 438–441. DOI: 10.1103/PhysRevLett.32.438.

[25] Harald Fritzsch and Peter Minkowski. "Unified interactions of leptons and hadrons". In: *Annals of Physics* 93.1 (1975), pp. 193–266. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(75)90211-0. URL: http://www.sciencedirect.com/science/article/pii/0003491675902110.

[26] Jogesh C. Pati and Abdus Salam. "Lepton Number as the Fourth Color". In: *Phys. Rev.* D10 (1974). [Erratum: Phys. Rev.D11,703(1975)], pp. 275–289. DOI: 10.1103/PhysRevD.10.275,10.1103/PhysRevD.11.703.2.

[27] K. S. Babu, Xiao-Gang He, and Sandip Pakvasa. "Neutrino masses and proton decay modes in SU(3)×SU(3)×SU(3) trinification". In: *Phys. Rev. D* 33 (3 Feb. 1986), pp. 763–772. DOI: 10.1103/PhysRevD.33.763.

[28] C. Amsler et al. "Review of Particle Physics". In: *Physics Letters B* 667.1 (2008). Review of Particle Physics. ISSN: 0370-2693. DOI: https://doi.org/10.1016/j.physletb.2008.07.018.

[29] G. 't Hooft. "Symmetry Breaking through Bell-Jackiw Anomalies". In: *Phys. Rev. Lett.* 37 (1 1976), pp. 8–11. DOI: 10.1103/PhysRevLett.37.8.

[30] Pran Nath and Pavel Fileviez Perez. "Proton stability in grand unified theories, in strings and in branes". In: *Phys. Rept.* 441 (2007), pp. 191–317. DOI: 10.1016/j.physrep.2007.02.010. arXiv: hep-ph/0601023 [hep-ph].

[31] H. Nishino et al. "Search for Proton Decay via p —¿ e+ pi0 and p —¿ mu+ pi0 in a Large Water Cherenkov Detector". In: *Phys. Rev. Lett.* 102 (2009), p. 141801. DOI: 10.1103/PhysRevLett.102.141801. arXiv: 0903.0676 [hep-ex].

[32] Thomas G. Rizzo and Goran Senjanovic. "Can There Be Low Intermediate Mass Scales in Grand Unified Theories?" In: *Phys. Rev. Lett.* 46 (1981), p. 1315. DOI: 10.1103/PhysRevLett.46.1315.

[33] Ling-Fong Li. "Group Theory of the Spontaneously Broken Gauge Symmetries". In: *Phys. Rev.* D9 (1974), pp. 1723–1739. DOI: 10.1103/PhysRevD.9.1723.

[34] F. Gursey, Pierre Ramond, and P. Sikivie. "A Universal Gauge Theory Model Based on E6". In: *Phys. Lett.* 60B (1976), pp. 177–180. DOI: 10.1016/0370-2693(76)90417-2.

[35] P. K. Mohapatra, R. N. Mohapatra, and P. B. Pal. "Implications of E6 Grand Unification". In: *Phys. Rev.* D33 (1986), p. 2010. DOI: 10.1103/PhysRevD.33.2010.

[36] Richard H. Benavides et al. "Electroweak couplings and LHC constraints on alternative Z' models in E_6". In: (2018). arXiv: 1801.10595 [hep-ph].