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Analyzing Fredholm operators with the Adomian decomposition

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Abstract

In this paper we look at Fredholm operators in Banach spaces and their behaviour under perturbations. The Fredholm integral of the second kind is of particular interest. We analyze this inhomogeneous integral numerically by implementing Adomians decomposition in Matlab and see how it differs from the exact analytical solution.

1 Introduction

In *1903* Ivar Fredholm wrote a paper in Acta Mathematica called *Sur une Classe d'Equations Fonctionnelles* where he described an integrand with an unknown function in it [13]. This led to a couple of integrands called the Fredholm and Volterra integrals. Both integrals have two forms namely the first and second kind. Before Fredholm published this paper, Fourier wrote in 1822 his book "The analytic theory of heat" where he discussed the problem of inverting integral equations [11]. He gave a solution to this problem which is called the Fourier inversion formula. In the middle of the 19^{th} century Liouville did research in second order linear ODE's where he discussed how these equations could be written in integral forms. These integrals were of particular interest for research on boundary value problems like the Laplace equation for instance.

In this paper we discuss Fredholm's integral of the second kind. The Volterra integrands of first and second kind are left out since they can all be seen as simplifications of Fredholm integrands of the second kind. First we discuss basic definitions in order to define the Fredholm operators in general, this includes the adjoint operators and sequences of vector spaces. Fredholm operators have the property that the dimension of the kernel and the codimension of the range are finite. An example of this is the left and the right shift map. However there are numerous ways to show that an operator is Fredholm. A couple of methods which we discuss involve the use of Calkin algebras and Weyl sequences. Then we see that if a perturbation on a Fredholm operator is compact, the operators preserve the properties of having a finitely dimensional kernel and codimension. After we characterized the definitions thoroughly we use the Adomian Decomposition to solve examples of Fredholm integrals numerically and we discuss its accuracy.

In the beginning of of 1980 George Adomian wrote many papers to solve non linear functional equations [8]. In these papers he introduced the Adomian Decomposition Method [2] [1] [3]. It is a semi-analytical method to solve differential equations in general. To solve nonlinear Fredholm integrals we decompose the solution in so called Adomian polynomials. Each polynomial is computed incursive and the sum of these polynomials approximate the solution of the problem. In the last decades, the technique has been used to solve and describe various linear and nonlinear equations in various scientific branches. In both linear and nonlinear cases, the method turns out to be a reliable tool to obtain or approximate exact solutions with fast convergence. It does not rely on linearization or perturbation theory making it applicable to real case scenarios.

2 Fredholm operators

2.1 Definition of Fredholm operators

Let X, Y be a Banach spaces and let $B(X, Y)$ be the space of bounded linear operators from X to Y. Let $A \in B(X, Y)$, then we denote the constants $\alpha(A)$ and $\beta(A)$ as follows,

$$\alpha(A) = \dim(\ker(A)), \qquad \beta(A) = \dim(Y/\operatorname{ran}(A)).$$

Definition 2.1 *An operator $A \in B(X, Y)$ is called Fredholm if $\alpha(A) < \infty$ and $\beta(A) < \infty$.*

Lemma 2.2 *Let X and Y be Banach spaces and let $A \in B(X, Y)$. If $\beta(A)$ is a finite number, then* $\operatorname{ran}(A)$ *is closed.*

Proof Without loss of generality we assume that A is injective, if not we can replace A with the injective map $X/\ker(A) \to Y$ [5]. Let v_i for $i = 1, 2, \ldots, n$ be the equivalence classes of $Y/\operatorname{ran}(A)$. For each equivalence class v_i we choose a representative $w_i \in Y$. We define,

$$\operatorname{span}\{w_1, w_2, \ldots, w_n\} = Z \subset Y.$$

Since Z is a finite dimensional subspace it is closed, hence it is a Banach space [23]. We define the following map,

$$T\colon X \times Z \to Y,$$
$$T(x, z) = Ax + z.$$

If we show that T is bounded and bijective then the open mapping theorem says T is an isomorphism which maps closed subsets to closed subsets. We have that the range of the closed subset $X \times \{0\} \subset X \bigoplus Z$ under T is $\operatorname{ran}(A)$. This yields that $\operatorname{ran}(A)$ is closed.

1. *Boundedness*
   Since A is bounded we have,
   $$\|Ax + z\| \le \|Ax\| + \|z\| \le \|A\|\|x\| + \|z\| \le (1 + \|A\|)(\|x\| + \|z\|).$$
   Hence T is bounded.

2. *Surjectiveness*
   Let $y \in Y$, then there exist scalars $a_1, a_2, \ldots, a_n$ such that the equivalence class $\sum_{i=1}^{n} a_i[v_i] \in Y/\operatorname{ran}(A)$ contains y. We also have that w_j is a representative of the equivalence class containing v_i. Thus,
   $$y = \sum_{i=1}^{n} a_i w_i + \sum_{i=1}^{n} a_i k_j + k_0,$$
   where $k_0, k_1, \ldots, k_n \in \operatorname{ran}(A)$. Hence for each element y there exist elements $x \in X$ and $z \in Z$ such that $T(x, z) = y$ i.e. T is surjective.

3. *Injectiveness*
   Let,
   $$A(x_1 - x_2) + (z_1 - z_2) = 0. \tag{1}$$
   The restriction of the canonical mapping $Y \to Y/\mathrm{ran}(A)$ to Z is injective [5]. Filling this in (1) gives that $z_1 = z_2$ i.e.,
   $$A(x_1 - x_2) = 0. \tag{2}$$
   Because we assumed that A is injective it follows from (2) that $x_1 = x_2$. Hence T is injective.

□

The dual spaces of X, Y we denote respectively as X^*, Y^*. Let $g \in Y^*$ be a linear functional, applying it to Ax gives that $g(y) = g(Ax)$. For convenience we call this $f(x)$ [18]. Thus for each element $g \in Y^*$ we assign an element $f \in X^*$. The operator which maps g to f is called the adjoint operator or the pull-back,

$$A^*\colon\, Y^* \to X^*,$$

and is defined as,

$$(A^*g)(x) = g(Ax).$$

If we use the notation that $f(x) = (f, x)$ then we have that $(g, Ax) = (f, x)$ and $(g, Ax) = (A^*g, x)$, hence $(f, x) = (A^*g, x)$ again. We see three basic properties of adjoint operators which are useful to understand further theorems.

Lemma 2.3 *Let $A, B \in B(X, Y)$ where X and Y are Banach spaces, then the following equations hold:*

1. $(A + B)^* = A^* + B^*$.
2. $(\lambda A)^* = \lambda A^*$ *for any* $\lambda \in \mathbb{C}$.
3. $\|A^*\| = \|A\|$.

For the proof of this lemma we refer to [18]

Lemma 2.4 *If A is a Fredholm operator then the following holds:*

$$\beta(A^*) = \alpha(A), \qquad \alpha(A^*) = \beta(A).$$

For the proof of this lemma we refer to [17]

Definition 2.5 *Let $\alpha(A)$ and $\beta(A)$ be as above, then we define the index of A as follows:*

$$\mathrm{ind}(A) = \alpha(A) - \beta(A).$$

By this definition, if A is Fredholm then the index is a finite number. Now it is easy to see that $\mathrm{ind}(A) = -\mathrm{ind}(A^*)$ holds and that A is a Fredholm operator if and only if A^* is.

2.2 Sequences of vector spaces

Before we define more properties of Fredholm operators we discuss what exact (vector) sequences are.

Definition 2.6 *Let $X_0, X_1, \ldots, X_n$ be vector spaces and let $\phi_i \colon X_i \to X_{i+1}$ be linear transformations. Then the sequence of vector spaces,*

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_{n-1}} X_n,$$

is called exact if $\mathrm{ran}(\phi_{i-1}) = \ker(\phi_i)$ *for* $i = 1, 2, \ldots, n$ [12].

Lemma 2.7 *Let X, Z be finite dimensional vector spaces and assume that $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ is an exact sequence. Then Y is a finite dimensional vector space as well.*

Proof By the rank-nullity theorem $\dim(Y)$ can be expressed as follows,

$$\begin{aligned}\dim(Y) &= \dim(\mathrm{ran}(\psi)) + \dim(\ker(\psi)),\\ &= \dim(\mathrm{ran}(\psi)) + \dim(\mathrm{ran}(\phi)).\end{aligned}$$

Because X, Z are finite dimensional, $\dim(\mathrm{ran}(\phi))$ and $\dim(\mathrm{ran}(\psi))$ are finite. Hence Y is finite dimensional. □

Theorem 2.8 *Let $0 \to X_1 \to \ldots X_{n-1} \to 0$ be an exact sequence, where each X_i is vector space with $\dim(X_i) < \infty$. Then the following holds,*

$$\sum_{i=1}^{n} (-1)^i \dim(X_i) = 0. \tag{3}$$

Proof We prove the theorem by using induction [4]. The case where $n = 1$ gives that $0 \to X_1 \to 0$, hence $\dim(X_1) = 0$. Suppose that (3) holds for some $n \in \mathbb{N}$ and consider the following exact sequence of vector spaces,

$$0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_3} \ldots X_{n+1} \xrightarrow{\phi_{n+1}} 0.$$

Now we look at the reduced sequence,

$$0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_3} \ldots X_{n-1} \xrightarrow{\phi_{n-1}} ker(\phi_n) \to 0.$$

Because of the inductive hypothesis we get the following,

$$\sum_{i=1}^{n-1} (-1)^i \dim(X_i) + (-1)^n \dim(\ker(\phi_n)) = 0. \tag{4}$$

Using Lemma 2.7 gives,

$$\begin{aligned}\dim(X_n) &= \dim(\ker(\phi_n)) + \dim(\mathrm{ran}(\phi_n)),\\ &= \dim(\ker(\phi_n)) + \dim(X_{n+1}).\end{aligned} \tag{5}$$

Putting (4) and (5) together yields that

$$\begin{aligned}\sum_{i=1}^{n+1}(-1)^i \dim(X_i) &= \sum_{i=1}^{n-1}(-1)^i \dim(X_i) + (-1)^n \Big(\dim(X_n) - \dim(X_{n+1}) \Big). \\ &= 0\end{aligned}$$

□

Theorem 2.9 *Let $A, B \in B(X,X)$ be Fredholm operators, then the following properties hold:*

1. $\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B)$
2. $\operatorname{ind}(AB) = \operatorname{ind}(BA)$

Proof To see 1 we first define the codimension,

$$\operatorname{codim}(A) = \dim(X/\operatorname{ran}(A)).$$

To prove the theorem we show the following,

$$\begin{aligned}\dim(\ker(AB)) &\leq \dim(\ker(A)) + \dim(\ker(B)) < \infty, \\ \operatorname{codim}(AB) &\leq \operatorname{codim}(A) + \operatorname{codim}(B) < \infty.\end{aligned}$$

Consider the following exact sequence,

$$0 \to \ker(A) \xrightarrow{\iota} \ker(BA) \xrightarrow{A} \ker(B) \xrightarrow{q} X/\operatorname{ran}(A) \xrightarrow{B} X/\operatorname{ran}(BA) \xrightarrow{E} X/\operatorname{ran}(B) \to 0.$$

Where we used ι as the inclusion map, q: $X \to X/\operatorname{ran}(A)$ as the quotient map and E as the map which takes equivalence classes $X/\operatorname{ran}(BA)$ into equivalence classes $X/\operatorname{ran}(B)$. By definition $\ker(A)$, $\ker(B)$, $X/\operatorname{ran}(A)$ and $X/\operatorname{ran}(B)$ are all finite dimensional. Using Lemma 2.7 we conclude that $\ker(BA)$ and $X/\operatorname{ran}(BA)$ are finite dimensional, hence BA is a Fredholm operator. Theorem 2.8 gives us the following equality,

$$\begin{aligned}0 = &-\dim(\ker(A)) + \dim(\ker(BA)) - \dim(\ker(B)) + \\ &\dim(X/\operatorname{ran}(A)) - \dim(X/\operatorname{ran}(BA)) + \dim(X/\operatorname{ran}(B)), \\ = &-\operatorname{ind}(A) - \operatorname{ind}(B) + \operatorname{ind}(BA).\end{aligned} \tag{6}$$

From (6) we conclude that the second property holds as well which completes the proof. □

2.3 Examples of Fredholm operators

A well known example of the Fredholm operators are the left and right shift operator.

Example 2.10

Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for the Hilbert space X and let $R : X \to X$ be the right shift operator defined as follows,

$$R^n e_i = e_{i+n}.$$

Then it is clear that $\text{ran}(R^n) = \text{span}\{e_{i+n}, e_{i+n+1}, \dots\}$. This gives the following,

$$\begin{aligned}\beta(R^n) &= \dim(X/\text{ran}(R^n)),\\ &= \dim(\text{span}\{e_1, e_2, \dots\}/(\{e_{i+n}, e_{i+n+1}, \dots\})),\\ &= \dim(\text{span}\{e_1, e_2, \dots, e_n\}),\\ &= n.\end{aligned}$$

Because R is injective we have that,

$$R(e_{i+1}, e_{i+2}, \dots) = (0, e_{i+1}, e_{i+2}, \dots).$$

This yields that if $(e_{i+1}, e_{i+2}, \dots)$ is in the kernel then, $(0, e_{i+1}, e_{i+2}, \dots) = (0, 0, 0, \dots)$. Hence $\dim(\ker(R)) = 0$ which implies that $\text{ind}(R^n) = -n$.

Example 2.11

Assume the same orthonormal basis X for the Hilbert space and let R^* be the left shift operator defined as follows,

$$R^{n*} e_i = e_{j-n}.$$

Note that this is the adjoint operator for the right shift operator. To see the dimension of $\ker(R^{n*})$, assume $(e_i, e_{i+1}, \dots) \in \ker(R^{*n})$. Then,

$$\begin{aligned}R^{*n}(e_i, e_{i+1}, \dots) &= (e_{i-n}, e_{i+1-n}, \dots),\\ &= (0, 0, \dots).\end{aligned}$$

Because $\dim((e_{i-n}, e_{i+1-n}, \dots,)) = n$, we have that $\dim(\ker(R^{n*})) = n$. The surjectivity of R^{n*} implies that $\text{ran}(R^{*n}) = X$. This gives the following,

$$\begin{aligned}\beta(R^{n*}) &= \dim(X/\text{ran}(R^{n*})),\\ &= \dim(X) - \dim(\text{ran}(R^{n*})),\\ &= 0.\end{aligned}$$

Hence we conclude that $\text{ind}(R^{n*}) = n$.

Example 2.12

Let $A \in B(X, X)$ where $X = (C[a, b], \ \|\cdot\|_\infty)$ is the Banach space consisting of the continuous functions with the supremum norm such that,

$$Ax(s) = \int_a^b K(s, t)x(t)dt,$$

where $K\colon [a, b] \times [a, b] \to \mathbb{R}$ is a continuous map and is called the kernel of the integral operator A. Using the Arzelà–Ascoli theorem [23], the continuity of K on the compact set $[a, b] \times [a, b]$ implies that A is a compact operator. The compactness of A implies that $I - A$ is Fredholm [22, theorem 2.13] defined as,

$$(I - A)x(s) = x(s) - \int_a^b K(s, t)x(t)dt. \tag{7}$$

Equation (7) is called the linear inhomogeneous Fredholm integral of the second kind which will be discussed in chapter 4.

2.4 The essential spectrum

We call $A \in B(X, Y)$ invertible if there exists a $B \in B(Y, X)$ such that the following holds,

$$AB = I_Y, \qquad BA = I_X,$$

where I_X and I_Y are the identity operators.

Definition 2.13 *We define the spectrum of A and the essential spectrum of A respectively, just as in* [14]*, as follows,*

$$\begin{aligned} \mathrm{spec}(A) &= \{\lambda \in \mathbb{C}\colon A - \lambda I \text{ is not invertible}\}, \\ \mathrm{spec}_e(A) &= \{\lambda \in \mathbb{C}\colon A - \lambda I \text{ is not Fredholm}\}. \end{aligned}$$

We will shortly see how these two are related to each other.

Theorem 2.14 *Let X and Y be Banach spaces then $A \in B(X, Y)$ is a Fredholm operator if and only if there exists a $B_1, B_2 \in B(Y, X)$ and compact operators K_1, K_2 such that $AB_1 - I = K_1$ and $B_2A - I = K_2$, i.e. $AB_1 - I$ and $B_2A - I$ are compact operators.*

For a proof of this theorem we refer to [12], [22].

Proposition 2.15 *Let $A \in B(X, X)$ be a Fredholm operator and let $K(X, X)$ be the space consisting of compact linear operators which map X to X. Then the following holds,*

$$\mathrm{spec}_e(A) = \mathrm{spec}(A + K(X, X)).$$

Proof Let $A \in B(X,X)$ be a Fredholm operator, assume $C \in K(X,X)$ and let $\lambda \in \mathrm{spec}(A+C)$. It follows that $(A+C)-\lambda I$ is not invertible. Thus there does not exist a $B \in B(X,X)$ such that,

$$\begin{aligned}((A+C)-\lambda I)B &= I,\\ AB + CB - \lambda B &= I,\\ AB + Q - \lambda B &= I,\\ (A-\lambda I)B + Q &= I.\end{aligned}$$

We used that the composition of a bounded operator and a compact operator is compact, so CB is rewritten as a compact operator Q. Theorem 2.14 implies that $A-\lambda I$ is not Fredholm, hence $\lambda \in \mathrm{spec}_e(A)$. We showed that $\mathrm{spec}(A+K(X,X)) \subset \mathrm{spec}_e(A)$.

Let $\lambda \in \mathrm{spec}_e(A)$, then we have that,

$$A - \lambda I,$$

is not Fredholm. Theorem 2.14 implies now that there does not exist a $B \in B(X,X)$ and a compact operator K such that,

$$(A-\lambda I)B - K = I.$$

Every bounded linear operator between normed linear spaces in general is continuous, hence its inverse is well defined. Writing $K = KB^{-1}B$ gives the following,

$$\begin{aligned}(A-\lambda I)B - K &= I,\\ AB - \lambda B - BB^{-1}K &= I,\\ (A - \lambda I - KB^{-1})B &= I,\\ (A-\lambda I - Q)B &= I.\end{aligned}$$

Where again Q is defined as in the first part. We conclude that for $\lambda \in \mathrm{spec}_e(A)$, there does not exists a $B \in B(X,X)$ such that $(A-\lambda I-Q)B = I$. This yields that $A - \lambda I - Q$ is not invertible. Hence $\lambda \in \mathrm{spec}(A+K(X,X))$ and thus $\mathrm{spec}_e(A) \subset \mathrm{spec}(A+K(X,X))$. □

Definition 2.16 *Let X be an infinite dimensional Banach space, then we define the Calkin Algebra of X as the quotient space $B(X,X)/K(X,X)$.*

Lemma 2.17 *Let π: $B(X,X) \to B(X,X)/K(X,X)$ be the natural map which sends A to the Calkin Algebra. Then A is Fredholm if and only if $\pi(A)$ is invertible in the Calkin Algebra.*

Proof This follows directly from the definition of the essential spectrum. □

In the next theorem we see that a Fredholm operator maintains its Fredholm property under a compact perturbation. Also it turns out that its index is invariant under such perturbations [12, 14].

Theorem 2.18 *Let $A \in B(X,X)$ be a Fredholm operator and let $K \in K(X,X)$ be any compact operator. Then the following properties hold,*

1. *$A+K$ is a Fredholm operator*
2. $\mathrm{ind}(A+K) = \mathrm{ind}(A)$

Proof We use Theorem 2.14 for the first part of the proof. Assume $A \in B(X,Y)$ is a Fredholm operator. This implies that there exists a $B_1, B_2 \in B(Y,H)$ and compact operators K_1, K_2 such that $B_1 A = I + K_1$ and $AB_2 = I + K_2$. Using this gives that,

$$\begin{aligned} B_1(A+K) &= B_1 A + B_1 K, \\ &= I + (K_1 + B_1 K), \\ &= I + Q. \end{aligned}$$

Where Q is the sum of two compact operators which is again compact. The part involving B_2 is done analogously. Using Theorem 2.14 we conclude that $A+K$ is a Fredholm operator. Without loss of generality we assume that $I+K(X,X)$ is the identity operator inside the Calkin Algebra. Using that $\dim(\ker(I)) = 0$ and that $\beta(A) = \alpha(A^*)$ we see the following,

$$\begin{aligned} \mathrm{ind}(I+K(X,X)) &= \alpha(I+K) - \alpha((I+K)^*), \\ &= \dim(\ker(I+K)) - \dim(\ker((I+K)^*)), \\ &= 0. \end{aligned}$$

This yields that,

$$\mathrm{ind}(B_1 A) = 0.$$

A more elaborate proof of this is given in [12] and [24]. Using Theorem 2.9 implies the following,

$$\begin{aligned} 0 &= \mathrm{ind}(B_1 A), \\ &= \mathrm{ind}(B_1) + \mathrm{ind}(A). \end{aligned}$$

Hence,

$$\mathrm{ind}(A) = -\mathrm{ind}(B_1).$$

With the same argument we conclude that,

$$\mathrm{ind}(I+Q) = 0.$$

Thus the following holds,

$$\begin{aligned} 0 &= \mathrm{ind}(B_1(A+K)), \\ &= \mathrm{ind}(A+K) + \mathrm{ind}(B_1), \\ &= \mathrm{ind}(A+K) - \mathrm{ind}(A), \end{aligned}$$

which completes the proof. $\square$

2.5 Singular Weyl sequences

There is another way to see if an operator $A \in B(X, X)$ is Fredholm or not and that is to check whether it has a singular Weyl sequence.

Definition 2.19 *A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called a Weyl sequence if the following conditions hold,*

1. $\|x_n\| = 1$, $\forall n \in \mathbb{N}$,
2. $\lim_{n\to\infty} Ax_n = 0$.

If the sequence $\{x_n\}_{n\in\mathbb{N}}$ additionally has no convergent subsequence we call it a singular Weyl sequence.

Before we can prove that having a singular Weyl sequence and being Fredholm is equivalent, we need to define the definition of being bounded below.

Lemma 2.20 *Let X, Y be Banach spaces, then an operator $A \in L(X, Y)$ is bounded below i.e., $\inf_{\|x\|=1}\|Ax\| > 0$ if and only if A is injective and* $\mathrm{ran}(A)$ *is closed.*

Proof For the proof of this lemma we refer to [19, theorem 2.32]

Theorem 2.21 *An operator $A \in B(X, X)$ is Fredholm if and only if A or A^* has no singular Weyl sequence* [14].

Proof For the only if part we use contradiction. Let A be a Fredholm operator and let $\{x_n\}_{n\in\mathbb{N}}$ be a singular Weyl sequence for A. Using Theorem 2.14 we know that there exists a compact operator K and $B \in B(X, X)$ such that $BA = I + K$. So the following expression holds,

$$\begin{aligned} \lim_{n\to\infty} x_n + Kx_n &= \lim_{n\to\infty} (I + K)x_n, \\ &= \lim_{n\to\infty} BAx_n, \\ &= 0. \end{aligned} \tag{8}$$

Because $\{x_n\}_{n\in\mathbb{N}}$ is bounded and K is compact, $\{Kx_n\}_{n\in\mathbb{N}}$ is relatively compact. Hence it has a subsequence which converges to x. Then a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to the same limit with a different sign by (8) i.e. $-x$. This contradicts with our assumption that $\{x_n\}_{n\in\mathbb{N}}$ is a singular Weyl sequence. We conclude that if A is a Fredholm operator then it has no singular Weyl sequence.

For the if part we again use contradiction. Assume A is not Fredholm, this automatically implies that A^* is not Fredholm either and thus we only look at the case for A. By definition this means that either (1) $\alpha(A) = \infty$ or (2) $\beta(A) = \infty$

1. *First case*
   The first case implies that the dimension of $\ker(A)$ is infinite. Hence it is an infinite dimensional closed subspace of X and any closed subspace of a Banach space is Banach itself. A direct consequence of Riesz's compact theorem is that the closed unit balls of any infinite normed linear space are not compact. This assures the existence of a singular Weyl sequence, which leads to a contradiction.

2. *Second case* [14]
   Assume that $\operatorname{codim}(A) < \infty$. Then using [14, theorem 2.1] we conclude that $\operatorname{ran}(A)$ is not closed. Let $\tilde{A}$ be the induced operator defined as follows,

   $$\begin{aligned} \tilde{A}\colon \quad X/\ker(A) &\to X, \\ x + \ker(A) &\mapsto Ax \in X. \end{aligned}$$

   This implies that $\operatorname{ran}(\tilde{A}) = \operatorname{ran}(A)$ and we see that $\operatorname{ran}(\tilde{A})$ is not closed either. Because $\operatorname{ran}(\tilde{A})$ is not closed, Lemma 2.20 implies that $\tilde{A}$ is not bounded below. This means that there exists a sequence $\{x_n + \ker(A)\}_{n\in\mathbb{N}}$ in $X/\ker(A)$ with the property that $\left\|x_n + \ker(A)\right\| = 1$ for all $n \in \mathbb{N}$ such that $\tilde{A}(x_n + \ker(A)) \xrightarrow{n\to\infty} 0$. If the constructed sequence has a convergent subsequence which converges to $x + \ker(A)$ then $\tilde{A}(x + \ker(A)) = 0$. This gives that $x \in \ker(A)$ which contradicts with the fact that $\left\|x_n + \ker(A)\right\| = 1$ for all $n \in \mathbb{N}$. We see that the sequence $\{x_n + \ker(A)\}_{n\in\mathbb{N}}$ has no convergent subsequence and hence it is a singular Weyl sequence for the induced operator $\tilde{A}$. We will continue showing that $\{x_n\}_{n\in\mathbb{N}}$ is a singular Weyl sequence for A. Assume that $\ker(A)$ is finite dimensional, if not we can return to the first case. Then the following holds for all $n \in \mathbb{N}$,

   $$\begin{aligned} 1 &= \left\|x_n + \ker(A)\right\|, \\ &= \inf_{y\in\ker(A)} \|x_n + y\|, \\ &= \|x_n\|. \end{aligned}$$

   The construction of $\tilde{A}$ guarantees that $Ax \to 0$ when $n \to \infty$. Also $\{x_n\}_{n\in\mathbb{N}}$ can't have a convergent subsequence because $\{x_n + \ker(A)\}_{n\in\mathbb{N}}$ doesn't have one. This yields that $\{x_n\}_{n\in\mathbb{N}}$ is a singular Weyl sequence for A which leads to a contradiction. Hence A is a Fredholm operator.

The proof for A^* is done analogously and thus we conclude that A is Fredholm if and only if A nor A^* have a singular Weyl sequence. $\square$

2.6 Perturbations of semi Fredholm operators

We introduce perturbation classes of semi Fredholm operators. In this section we use in some cases one of the two properties of Fredholm operators. If either the dimension of the kernel or the codimension is finite, we call an operator semi-Fredholm.

Definition 2.22 *Let X, Y be two Banach spaces, then the set of all upper semi-Fredholm operators is defined as follows,*

$$\Phi_+(X,Y) = \{A \in B(X,Y) | \quad \alpha(T) < \infty\}.$$

the set of all lower semi-Fredholm operators is defined as follows,

$$\Phi_-(X,Y) = \{A \in B(X,Y) | \quad \beta(T) < \infty\}.$$

We define the set of all semi-Fredholm operators $\Phi_\pm(X,Y)$ and the class of all Fredholm operators $\Phi(X,Y)$ as follows,

$$\begin{aligned} \Phi_\pm(X,Y) &= \Phi_+(X,Y) \cup \Phi_-(X,Y), \\ \Phi(X,Y) &= \Phi_+(X,Y) \cap \Phi_-(X,Y). \end{aligned}$$

Lemma 2.23 *Let $A \in \Phi_+(X,Y)$, then there exists a number $\epsilon > 0$ such that for each perturbation parameter $T \in B(X,Y)$ satisfying $\|T\| < \epsilon$, the following properties hold,*

1. $A + T \in \Phi_+(X,Y)$,
2. $\alpha(A+T) \leq \alpha(A)$.

Similarly if $A \in \Phi_-(X,Y)$, the same statements hold for $\beta(A+T)$.

Proof To prove the first statement we consider a $P \in B(X,X)$ such that $P = 0$ on a closed subspace of X. Note that for any $A \in B(X,Y)$, ran(A) is closed if and only if there exists a $C > 0$ with such a P such that the following holds [21],

$$\|x\| \leq C\|Ax\| + \|Px\|, \quad x \in X.$$

We can say that for $A + T \in B(X,Y)$,

$$\begin{aligned} \|x\| &\leq C\big\|(A+T)x\big\| + \|Px\|, \\ &\leq C\big\|(A+T)x\big\| + C\|Tx\| + \|Px\|. \end{aligned}$$

Then taking $\epsilon = 1/2C$ and using the previous yields,

$$\begin{aligned} \|x\| &\leq C\big\|(A+T)x\big\| + C\epsilon\|x\| + \|Px\|, \\ &= C\big\|(A+T)x\big\| + 1/2\|x\| + \|Px\|. \end{aligned} \tag{9}$$

Rewriting gives,

$$\|x\| \leq 2C\big\|(A+T)x\big\| + 2\|Px\|. \tag{10}$$

We make use of [21, theorem 6.2], which says that if there is a seminorm denoted as $|\cdot|$, compact relative to the norm of X such that the following holds,

$$\|x\| \leq X\|Ax\| + |x|\,, \quad x \in X,$$

then $A \in \Phi_+(X, Y)$. Set,

$$|x| = \|Px\|\,.$$

Hence we conclude that $A + T \in \Phi_+(X, Y)$. To see the second statement note that $P = 0$ on a closed subspace $X_0 \subset X$. Filling this in in (9) gives,

$$\|X\| \leq 2C\big\|(A+T)x\big\|\,, \quad x \in X_0.$$

Hence

$$\ker(A + X) \cap X_0 = \{0\}.$$

Because $X = X_0 \bigoplus \ker(A)$ [21, lemma 1.1] we conclude that $\dim(\ker(A+T)) \leq \dim(\ker(A))$ [21, lemma 3.3]. The proof for the case $A \in \phi_-(X, Y)$ is done analogously. □

Theorem 2.24 (Punctured neighbourhood theorem) *Assume $A \in \Phi_\pm(X, X)$ where X is a Banach space and let λ be a constant, then there exists $\epsilon > 0$ such that,*

1. *$\alpha(\lambda I + A)$ is constant for $|\lambda| < \epsilon$,*
2. *$\beta(\lambda I + A)$ is constant for $|\lambda| < \epsilon$.*

Proof Assume that $A \in \Phi_+(X, X)$. Let $|\lambda| < \epsilon$, then using Lemma 2.23 and [17, theorem 5.17] on A and λI we get that,

$$\begin{aligned} \beta(A + \lambda I) &\leq \beta(A), \\ &= 0. \end{aligned} \tag{11}$$

Hence $\beta(A + \lambda I) = 0$ as well and,

$$\begin{aligned} \alpha(A + \lambda I) &= \operatorname{ind}(A + \lambda I), \\ &= \operatorname{ind}(A), \\ &= \alpha(A). \end{aligned} \tag{12}$$

We conclude that both $\alpha(A + \lambda I)$ and $\beta(A + \lambda I)$ are constant for a sufficiently small λ which proves the first case. Using [17, theorem 5.13] we consider the adjoint operator of A. Then using the proof of the first case we see that the second case holds as well. □

3 Fredholm integrals

3.1 Fredholm integrals of the 2^{nd} kind

An integral equation in general is an equation with an unknown function under the integral sign. In this chapter we continue discussing Example 2.12 and study a decomposition method to numerically approximate the solution for such integral equations. Consider Fredholm's integral operator in (7) and equate it to a given function $y \in X$,

$$y(s) + \lambda \int_a^b K(s,t)x(t)dt = x(s), \qquad \lambda \in \mathbb{C} \tag{13}$$

The value of λ for which the homogeneous integral equation has a non-trivial solution is called the eigenvalue of the kernel. The solution $x \in C[a,b]$ is then called an eigenfunction of the kernel and will be of interest later.

Lemma 3.1 *The Fredholm integral equation of the second kind in* (13) *has a unique solution for a sufficiently small* $|\lambda|$.

Proof Consider the Fredholm integral equation in (13) i.e.,

$$(I - \lambda A)x = y. \tag{14}$$

Because A is a bounded map, it satisfies the Lipschitz condition,

$$\|Ax_1 - Ax_2\| \leq k\|x_1 - x_2\|, \quad k \geq 0.$$

If we rewrite the integral equation in (13) as follows,

$$x = Tx, \tag{15}$$

where T: $X \to X$ is nonlinear and is defined as follows,

$$Tx = \lambda Ax + y,$$

then it follows from (15) that x is a solution for (13) if and only if it is a fixed point for the operator T. If we show that T is a contraction then by Banach's fixed point theorem there exists a unique fixed point for (15). Hence there exists an unique solution for the Fredholm integral of the second kind. To see the condition for the contraction of T consider,

$$\begin{aligned}\|Tx_1 - Tx_2\| &= \big\|\lambda Ax_1 + y - (\lambda Ax_2 + y)\big\|, \\ &= \|\lambda Ax_1 - \lambda Ax_2\|, \\ &= |\lambda|\big\|A(x_1 - x_2)\big\|, \\ &\leq |\lambda k|\|x_1 - x_2\|.\end{aligned}$$

We conclude that if,

$$|\lambda| < 1/k,$$

then there exists a unique solution for the Fredholm integral of the second kind.

□

3.2 Degenerate kernels

Definition 3.2 *The kernel $K(s,t)$ is degenerate if it can be separated as a sum of products, where each product consists of a function of s and a function of t. Thus,*

$$K(s,t) = \sum_{j=1}^{n} u_j(s)v_j(t) = \boldsymbol{u}^T(s)\boldsymbol{v}(t) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle. \tag{16}$$

Proposition 3.3 *If $K(s,t)$ is a degenerate kernel, equation (14) can be solved by reducing it into a system of linear equations.*

Proof To see this we use (16) in (13),

$$\begin{aligned} x(s) &= \lambda \int_a^b K(s,t)x(t)dt + y(t), \\ &= \lambda \int_a^b \sum_{j=1}^{n} \Big[u_j(s)v_j(t) \Big] x(t)dt + y(s), \\ &= \lambda \sum_{j=1}^{n} \Big[u_j(s) \int_a^b v_j(t)x(t)dt \Big] + y(s). \end{aligned} \tag{17}$$

For convenience let,

$$c_j = \int_a^b v_j(t)x(t)dt = \langle v_j, x \rangle. \tag{18}$$

Then substituting (18) in (17) gives,

$$x(s) = \lambda \sum_{j=1}^{n} c_i u_j(s) + y(s). \tag{19}$$

In order to obtain the solution for the integral equation it is sufficient to find c_i. We eliminate y by substituting (17) in (18) [20],

$$\begin{aligned} c_i &= \int_a^b v_i(t) \Big[\lambda \sum_{j=1}^{n} c_j u_j(t) + y(t) \Big] dt, \\ &= \lambda \sum_{j=1}^{n} c_j \int_a^b v_i(t)u_j(t)dt + \int_a^b v_i(t)y(t)dt. \end{aligned} \tag{20}$$

For convenience let,

$$\begin{aligned} a_{ij} &= \int_a^b v_i(t)u_j(t)dt = \langle v_i, u_j \rangle, \\ y_i &= \int_a^b v_i(t)y(t)dt = \langle v_i, y \rangle. \end{aligned}$$

Using this in (20) gives,

$$c_i = \lambda \sum_{j=1}^{n} a_{ij} c_j + y_i. \tag{21}$$

If we define the following matrices,

$$A = (a_{ij}), \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

then putting these matrices in (21) yields,

$$(I - \lambda A)\mathbf{c} = \mathbf{y}.$$

□

3.3 The Adomian decomposition

The Adomian decomposition method was first introduced by George Adomian in [2]. It is applied to various fields in physics, biology and engineering to solve or obtain solutions of linear and non-linear problems. The solutions with the Adomian decomposition can be an exact or an approximation of the exact solution. It is easy in use as it makes no use of linearization or perturbation theorems [2] [25]. The method gives solutions of differential and integral equations in the following series form,

$$x(s) = \sum_{n=0}^{\infty} x_n(s). \tag{22}$$

If we substitute (22) in (13) we get,

$$\sum_{n=0}^{\infty} x_n(s) = y(s) + \lambda \int_a^b K(s,t) \left[\sum_{n=0}^{\infty} x_n(t) \right] dt.$$

Each component x_j is obtained recursively by,

$$\begin{aligned} x_0(s) &= y(s), \\ x_j(s) &= \lambda \int_a^b K(s,t) x_{j-1}(t) dt, \qquad j \geq 1. \end{aligned} \tag{23}$$

Filling the components of (23) back in (22) gives us the solution in series form. The series form of $x_n(s)$ converges to the exact solution in closed form if the solution exists [7] [6]. For more complicated cases, we use a truncated series $\sum_{n=0}^{k} x_n(s)$ to approximate the solution.

Example 3.4 Consider the following Fredholm integral equation of the second kind,

$$x(s) = \frac{9}{10}s^2 + \int_0^1 \frac{1}{2}s^2t^2x(t)dt.$$

We see that,

$$\lambda = 1, \qquad K(s,t) = \frac{1}{2}s^2t^2, \qquad y(s) = \frac{9}{10}s^2.$$

Using this in (23) gives,

$$\begin{aligned} x_0(s) &= \frac{9}{10}s^2, \\ x_1(s) &= \int_0^1 \frac{1}{2}s^2t^2x_0(t)dt = \frac{9}{100}s^2 = \frac{1}{10}x_0 \end{aligned} \tag{24}$$

We show by induction that the general case is,

$$x_k(s) = \frac{1}{10}x_{k-1}(s). \tag{25}$$

The case for $k = 1$ is given by (24). Assume it holds for some $k+1 \in \mathbb{N}$ then,

$$\begin{aligned} \frac{1}{10}x_{k+1}(s) &= \frac{1}{10}\int_0^1 \frac{1}{2}s^2t^2x_k(t)dt, \\ &= \int_0^1 \frac{1}{2}s^2t^2\frac{1}{10}x_k(t)dt, \\ &= \int_0^1 \frac{1}{2}s^2t^2x_{k+1}(t)dt, \\ &= x_{k+2}(s). \end{aligned}$$

We see that it holds for $k+2$ as well and we conclude that (25) is true. Using that $x(s) = \sum_{n=0} x_n(s)$ we get,

$$x(s) = \frac{9}{10}s^2 + \frac{9}{100}s^2 + \frac{9}{1000}s^2 + \dots,$$

hence the solution in closed form is,

$$x(s) = s^2.$$

3.4 Nonlinear integral equations

We saw that the Adomian decomposition is a useful tool to analyze linear Fredholm integrals. However it can be used to analyse nonlinear Fredholm integrals as well. Let $X = (C[a,b], \|\cdot\|_\infty)$ and consider the following integral equation,

$$x(s) = y(s) + \lambda\int_a^b K(s,t)F(x(t))dt, \tag{26}$$

where $y \in X$. Let M be a constant such that $\left|K(s,t)\right| \le M$ for all $s, t \in [a,b]$ and let $F(x(t))$ be a nonlinear term such that it is Lipschitz continuous with,

$$\left\|F(x(t)) - F(\hat{x}(t)\right\|_\infty \le L\left\|x(t) - \hat{x}(t)\right\|_\infty, \qquad L \ge 0.$$

For the integral equation in (26) we search for a solution $x \in X$ of the form $x = \sum_{n=0}^{\infty} x_n$. The nonlinear term $F(x(t))$ in (26) can be decomposed in a series form,

$$F(x(t)) = \sum_{i=0}^{\infty} A_i,$$

where each A_i is called an Adomian polynomial and is constructed such that it only depends on $x_0, x_1, \ldots, x_i$. This property assures that each x_i is obtained by using only $x_0, x_1, \ldots, x_{i-1}$ for all $i \in \{0, 1, \ldots, n\}$ Then by writing (26) in terms of these series forms, we can find x recursively. We see that the sum of these polynomials form the Taylor series of the nonlinear term.

Definition 3.5 *Let $x \in X = (C[a,b],\ \|\cdot\|_\infty)$ and $F(x(t))$ be as in* (26)*. Then the Adomian polynomials A_n are defined as follows* [25] [2]*,*

$$A_n(x_0, x_1, \ldots, x_n) = \frac{1}{n!}\frac{d^n}{dp^n}\Big[F(\sum_{i=0}^{n} p^i x_i)\Big]_{p=0}, \qquad n \in \mathbb{N}.$$

Example 3.6 Let $F(x(t)) = x^2(t)$, then the Adomian polynomials up to order 2 are,

$$\begin{aligned}
A_0 &= F(x_0),\\
&= x_0^2,\\
A_1 &= \frac{d}{dp}F(x_0 + x_1 p)\Big|_{p=0},\\
&= x_1 F'(x_0 + x_1 p)\Big|_{p=0},\\
&= x_1 2(x_0 + x_1 p)\Big|_{p=0},\\
&= 2x_1 x_0,\\
A_2 &= \frac{1}{2}\frac{d^2}{dp^2}F(x_0 + x_1 p + x_2 p^2)\Big|_{p=0},\\
&= \frac{1}{2}\frac{d}{dp}\Big[(x_1 + 2px_2)F'(x_0 + x_1 p + x_2 p^2)\Big]_{p=0},\\
&= \frac{1}{2}\Big[(2x_2)F'(x_0 + x_1 p + x_2 p^2) + (x_1 + 2px_2)^2 F''(x_0 + x_1 p + x_2 p^2)\Big]_{p=0},\\
&= x_2 F'(x_0) + \frac{x_1}{2}F''(x_0),\\
&= 2x_0 x_2 + x_1^2,
\end{aligned}$$

where F' and F'' are the first and second order derivatives with respect to x.

Proposition 3.7 *Let $x \in X = (C[a,b], \ \|\cdot\|_\infty)$ and let $F(x(t))$ be as in (26). Then the nonlinear part $F(x(t))$ can be decomposed in a series form,*

$$F(x(t)) = \sum_{n=0}^{\infty} A_n, \tag{27}$$

such that the sum of all A_n is the Taylor expansion of $F(x(t))$ around x_0.

Proof Using Definition 3.5 we obtain the following polynomials,

$$\begin{aligned}
A_0 &= F(x_0),\\
A_1 &= x_1 F'(x_0),\\
A_2 &= x_2 F'(x_0) + \frac{x_1^2}{2!} F''(x_0),\\
A_3 &= x_3 F'(x_0) + x_1 x_2 F''(x_0) + \frac{x_1^3}{3!} F^{(3)}(x_0),\\
A_4 &= x_4 F'(x_0) + \left(\frac{x_2^2}{2!} + x_1 x_3\right) F''(x_0)\\
&\quad + \frac{x_1^2 x_2}{2!} F^{(3)}(x_0) + \frac{x_1^4}{4!} F^{(4)}(x_0),\\
&\ \vdots\\
A_n &= \frac{1}{n!} \frac{d^n}{dp^n} \left[F(\sum_{i=0}^{n} p^i x_i) \right]_{p=0},
\end{aligned} \tag{28}$$

where $F^{(n)}$ is the n-th order derivative with respect to x. Summing these polynomials up we get,

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n &= F(x_0) + (x_1 + x_2 + \ldots) F'(x_0) +\\
&\quad \frac{1}{2}(x_1^2 + 2x_1 x_2 + 2x_1 x_3 + x_2^2 + \ldots) F''(x_0) + \ldots\\
&= F(x_0) + (x - x_0) F'(x_0) + \frac{1}{2}(x - x_0)^2 F''(x_0) + \ldots.
\end{aligned}$$

Hence we see that the sum of the Adomian polynomials forms the Taylor expansion of the nonlinear term $F(x(t))$ around x_0. $\square$

Proposition 3.8 *Let $x \in X = (C[a,b], \ \|\cdot\|_\infty)$ be a solution for (26). If x is decomposed in the following series form,*

$$x = x_0 + \sum_{n=1}^{\infty} x_n, \tag{29}$$

with $x_0 = y(s)$, then each term of the series form (29) can be read off recursively.

Proof Filling (29) in (26) and using Proposition 3.7 gives the following [15] [25],

$$\sum_{n=0}^{\infty} x_n(s) = y(s) + \lambda \int_a^b K(s,t) \sum_{i=0}^{\infty} A_i(t)dt, \tag{30}$$

From (30) we can read of the components x_i where $i = 0, 1, 2, \ldots,$

$$\begin{aligned} x_0(s) &= y(s), \\ x_{i+1}(s) &= \lambda \int_a^b K(s,t) A_i(t) dt. \end{aligned} \tag{31}$$

Hence, each component of the serie $\sum_{n=0}^{\infty} x_n$ can be read off recursively. □

Remark 3.9 *Note that in Definition 3.5* $x(s) = \sum_{i=0}^{\infty} x_i(s)$ *is parameterized with a scalar variable* p *such that* $x(s,p) = \sum_{i=0}^{\infty} p^i x_i(s)$. *This is necessary to guarantee that* A_n *only depends on* $x_0, x_1, \ldots, x_n$ *so that* (31) *can be constructed accordingly.*

Definition 3.10 *Using* (31) *we define the n-term approximation of the solution of* (26) *by,*

$$\Psi_n(s) = \sum_{i=0}^{n} x_i(s). \tag{32}$$

Theorem 3.11 *The integral equation in* (26) *has a unique solution whenever* $0 < \alpha < 1$ *where,*

$$\alpha = \lambda LM(b-a).$$

Proof Let $x, \hat{x}$ be two different solutions of (26) then,

$$\begin{aligned} \|x - \hat{x}\|_\infty &= \left\| y - y + \lambda \int_a^b K(s,t)(F(x) - F(\hat{x})dt \right\|_\infty, \\ &\leq |\lambda| \int_a^b \big\|K(s,t)\big\|_\infty \big\|F(x) - F(\hat{x})\big\|_\infty dt, \\ &= |\lambda| \, LM(b-a) \|x - \hat{x}\|_\infty . \end{aligned}$$

This yields that,

$$\Big[1 - |\lambda| \, LM(b-a)\Big] \|x - \hat{x}\|_\infty \leq 0.$$

Since $0 < \alpha < 1$ we have that $x = \hat{x}$. □

Theorem 3.12 *The series solution of* (26) *using the Adomian decomposition converges if* $0 < \alpha < 1$ *and* $x_1(s) < \infty$ *[10, theorem 2].*

Proof Assume (26) and its properties. Let $\Psi_n(s)$ be a sequence of partial sums defined as in Definition 3.10. If $\Psi_n(s)$ is a Cauchy sequence in the Banach space X the proof is complete. Consider $\Psi_m(s)$ such that $n \geq m$ then,

$$\begin{aligned}\|\Psi_n - \Psi_m\|_\infty &= \max_{s\in[a,b]} \left|\Psi_n(s) - \Psi_m(s)\right|, \\ &= \max_{s\in[a,b]} \left|\sum_{i=m+1}^{n} x_i(s)\right|, \\ &= \max_{s\in[a,b]} \left|\lambda \int_a^b K(s,t) \sum_{i=m+1}^{n} A_{i-1}(t)dt\right|, \\ &= \max_{s\in[a,b]} \left|\lambda \int_a^b K(s,t) \sum_{i=m}^{n-1} A_i(t)dt\right|.\end{aligned}$$

If we use the fact that $\sum_{i=m}^{n-1} A_i = F(\Psi_{n-1}) - F(\Psi_{m-1})$ [9] we have,

$$\begin{aligned}\|\Psi_n - \Psi_m\|_\infty &= \max_{s\in[a,b]} \left|\lambda \int_a^b K(s,t)\Big[F(\Psi_{n-1}) - F(\Psi_{m-1})\Big]dt\right|, \\ &\leq \alpha \|\Psi_{n-1} - \Psi_{m-1}\|_\infty .\end{aligned}$$

Let $n = m+1$ then,

$$\begin{aligned}\|\Psi_{m+1} - \Psi_m\|_\infty &\leq \alpha \|\Psi_m - \Psi_{m-1}\|_\infty , \\ &\leq \alpha^2 \|\Psi_{m-1} - \Psi_{m-2}\|_\infty , \\ &\ \ \vdots \\ &\leq \alpha^m \|\Psi_1 - \Psi_0\|_\infty .\end{aligned} \tag{33}$$

Using (33) and the triangle equality we have,

$$\begin{aligned}\|\Psi_n - \Psi_m\|_\infty &\leq \|\Psi_{m+1} - \Psi_m\|_\infty + \|\Psi_{m+2} - \Psi_{m+1}\|_\infty + \cdots + \|\Psi_n - \Psi_{n-1}\|_\infty , \\ &\leq (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1})\|\Psi_1 - \Psi_0\|_\infty , \\ &\leq \alpha^m (1 + \alpha + \cdots + \alpha^{n-m-1})\|\Psi_1 - \Psi_0\|_\infty , \\ &\leq \alpha^m \Big(\frac{1-\alpha^{n-m}}{1-\alpha}\Big)\big\|x_1(s)\big\|_\infty , \\ &\leq \Big(\frac{\alpha^m}{1-\alpha}\Big)\big\|x_1(s)\big\|_\infty .\end{aligned} \tag{34}$$

Because $x_1(s)$ was assumed to be bounded we have $\|\Psi_n - \Psi_m\|_\infty \xrightarrow{m\to\infty} 0$. Hence Ψ_n is a Cauchy sequence in $C[a,b]$. □

Theorem 3.13 *Let x_{sol} be the exact solution of* (26) *and let the maximum absolute error of the truncated solution be denoted as* $\mathrm{Er}_n = \|\Psi_n - x_{sol}\|_\infty$. *Then [10, theorem 3],*

$$\mathrm{Er}_n \leq \frac{\alpha^{n+1}}{L(1-\alpha)} \left\|F(x_0)\right\|_\infty .$$

Proof Consider the last inequality in (34). Letting $n \to \infty$ yields,

$$\begin{aligned}
\left\|x_{sol}(s) - \Psi_m\right\|_\infty &\leq \left(\frac{\alpha^m}{1-\alpha}\right)\left\|x_1(s)\right\|_\infty , \\
&= \left(\frac{\alpha^m}{1-\alpha}\right)\left\|\lambda \int_a^b K(s,t)F(x_0)dt\right\|_\infty , \\
&\leq \left(\frac{\alpha^m}{1-\alpha}\right)|\lambda|\,(b-a)M\left\|F(x_0)\right\|_\infty , \quad . \\
&= \left(\frac{\alpha^m}{1-\alpha}\right)\frac{\alpha}{L}\left\|F(x_0)\right\|_\infty , \\
&= \frac{\alpha^{m+1}}{L(1-\alpha)}\left\|F(x_0)\right\|_\infty .
\end{aligned}$$

□

4 Numerical analysis of Fredholm integrals

4.1 Example 1

In this section we will see numerical approximations of Fredholm's integrals of the second kind described in (26). Consider the following integral,

$$x(s) = 2 + \frac{1}{8}\int_0^1 x^2(t)dt. \tag{35}$$

Where,

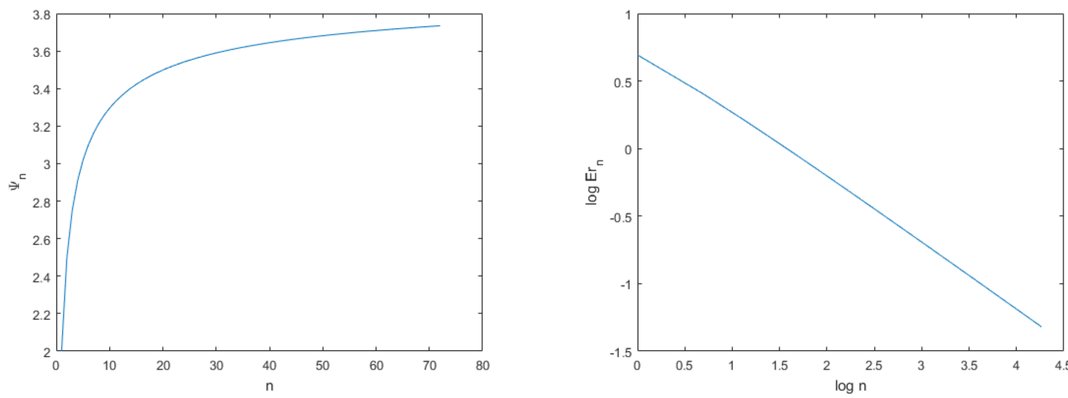
$$F(x(t)) = x^2(t), \qquad \lambda = \frac{1}{8}, \qquad K(s,t) = 1,$$

in this case. It is easy to see that we have one solutions $x_{sol} = 4$. Because both $K(s,t)$ and x_0 are constant, each $x_i(s)$ of the series form (29) will assign a constant value. Thus the truncated solution $\Psi_n(s)$ will have a constant value as well.

Remark 4.1 *Note that because $x_{sol} = 4$ we have,*

$$L = 8, \qquad \lambda = \frac{1}{8}, \qquad M = 1, \qquad b - a = 1,$$

which gives that $\alpha = 1$. Theorem 3.12 does not guarantee convergence to the exact solution with the Adomian decomposition in this case.



(a) (b)

Figure 1: (a) The behaviour of the truncated solution $\Psi_n(s)$ of (35). (b) The behaviour of Er_n in the logarithmic scale.

To compute the Adomian polynomials up to order n we use the function `adom(n,2)`, see Appendix A.1,

```
Input  adom(10,2)

Output                                                          x0^2
                                                              2*x0*x1
                                                     x1^2 + 2*x0*x2
                                                  2*x0*x3 + 2*x1*x2
                                         x2^2 + 2*x0*x4 + 2*x1*x3
                                      2*x0*x5 + 2*x1*x4 + 2*x2*x3
                             x3^2 + 2*x0*x6 + 2*x1*x5 + 2*x2*x4
                          2*x0*x7 + 2*x1*x6 + 2*x2*x5 + 2*x3*x4
                 x4^2 + 2*x0*x8 + 2*x1*x7 + 2*x2*x6 + 2*x3*x5
              2*x0*x9 + 2*x1*x8 + 2*x2*x7 + 2*x3*x6 + 2*x4*x5
 x5^2 + 2*x0*x10 + 2*x1*x9 + 2*x2*x8 + 2*x3*x7 + 2*x4*x6
```

Hence the terms of the series form are,

$$
\begin{aligned}
x_0(s) &= 2,\\
x_1(s) &= \frac{1}{8}\int_0^1 A_0(t)dt = 1/4,\\
x_2(s) &= \frac{1}{8}\int_0^1 A_1(t)dt = 1/8,\\
&\vdots
\end{aligned}
$$

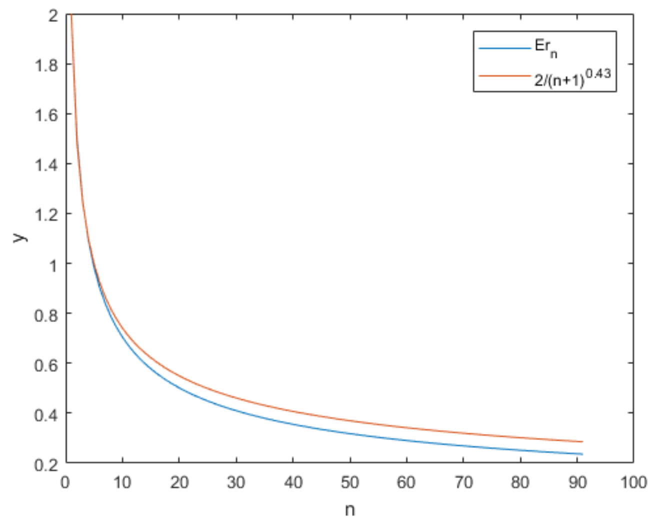


Figure 2: Er_n and $2/(n+1)^{0.4313}$ plotted against n.

The values of x_n, Ψ_n and Er_n are given in table 1. For the numerical implementation we refer to Appendix A.1. In figure 1a the values of the truncated solution are plotted and intuitively we see that $\Psi_n \xrightarrow{n\to\infty} 4$. However $\alpha = 1$ thus Theorem 3.12 does not guarantee the convergence of Ψ_n to 4. Theorem 3.13 assures that if $0 < \alpha < 1$, $\mathrm{Er}_n \xrightarrow{n\to\infty} 0$. Hence in our case this is not guaranteed. If we plot $\log(\mathrm{Er}_n)$ against $\log(n)$, we see that the error decreases almost linearly in the logarithmic scale, see figure 1b. The slope of the graph in figure 1b is calculated by using the least square regression method which is -0.4313 for $n = 120$, see Appendix A.1. Because $\mathrm{Er}_0 = 2$ we have,

$$\log(\mathrm{Er}_n) \approx -0.4313 \log(n+1) + \log(2).$$

Hence $\mathrm{Er}_n \approx \dfrac{2}{(n+1)^{0.4313}}$ which is also seen in figure 2.

Table 1

n	x_n	ψ_n	Er_n	n	x_n	ψ_n	Er_n
1	0.5000	2.5000	1.5000	35	0.0053	3.6252	0.3748
2	0.2500	2.7500	1.2500	40	0.0043	3.6486	0.3514
3	0.1563	2.9063	1.0937	45	0.0036	3.6682	0.3318
10	0.0320	3.3272	0.6728	50	0.0031	3.6848	0.3152
15	0.0181	3.4402	0.5598	55	0.0027	3.6991	0.3009
20	0.0119	3.5105	0.4895	60	0.0024	3.7116	0.2884
25	0.0086	3.5595	0.4405	65	0.0021	3.7227	0.2773
30	0.0066	3.5963	0.4037	70	0.0019	3.7326	0.2674

4.2 Example 2

Consider the following Fredholm integral [16],

$$x(s) = \cos(s) + \sin(s) - \frac{\pi + 2}{8} + \frac{1}{4}\int_0^{\pi/2} x^2(t)dt. \tag{36}$$

From [16], [25] we know that the exact solution for (36) is,

$$x_{sol}(s) = \cos(s) + \sin(s). \tag{37}$$

Remark 4.2 *Note that because* $x_{sol} = \cos(s) + \sin(s)$ *we have,*

$$L = 2\sqrt{2}, \qquad \lambda = \frac{1}{4}, \qquad M = 1, \qquad b - a = \frac{\pi}{2},$$

which gives that $\alpha = \pi/\sqrt{8} > 1$*. Theorem 3.12 does not guarantee convergence to the exact solution with the Adomian decomposition in this case.*

Because we again assume that $K(s,t) = 1$, all the $x_i(s)$ terms of the series form will have a constant value except,

$$x_0(s) = \cos(s) + \sin(s) - \frac{\pi + 2}{8}.$$

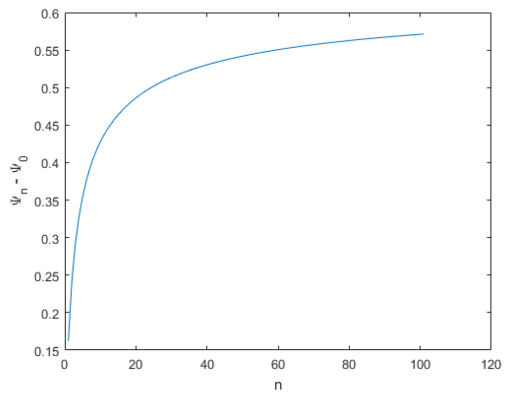
This means that the truncated solution $\Psi_n(s)$ is not constant. The Adomian polynomials are given in the same way as in Example 1. Hence the terms of the series form are,

$$\begin{aligned} x_0(s) &= \cos(s) + \sin(s) - \frac{\pi + 2}{8}, \\ x_1(s) &= \frac{1}{4}\int_0^{\pi/2} A_0(t)dt = 0.1622, \\ x_2(s) &= \frac{1}{4}\int_0^{\pi/2} A_1(t)dt = 0.0803, \\ &\vdots \end{aligned}$$

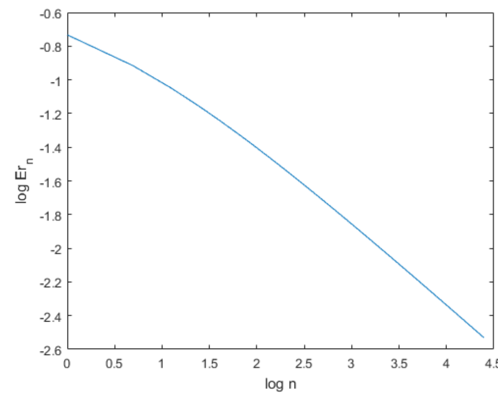
The rest of the values of x_n are given in table 2 together with $\Psi_n^* = \Psi_n - \Psi_0$ and Er_n. Leaving out $\Psi_0(s)$ gives only constant terms in which the behaviour is more explicitly seen. For the numerical implementation we refer to Appendix A.2. In figure 3a the values of the truncated solution are plotted and intuitively we see that,

$$\lim_{n\to\infty} \Psi_n - \Psi_0 = \frac{\pi + 2}{8}.$$

Because $\alpha > 1$, Theorem 3.12 does not guarantee the convergence to the exact solution hence Theorem 3.13 does not assure that $\mathrm{Er}_n \xrightarrow{n\to\infty} 0$. If we plot $\log(\mathrm{Er}_n)$ against $\log(n)$ we see that as $\log(n)$ grows, the decrease of $\log(\mathrm{Er}_n)$



(a)



(b)

Figure 3: (a) The behaviour of the truncated solution $\Psi_n^*(s)$ of (36). (b) The behaviour of Er_n in the logarithmic scale.

tends to a linear behaviour in the logarithmic scale, see figure 3b. We see that the slope is approximately $-1/2$ and because $\mathrm{Er}_0 = (\pi+2)/8$ we have,

$$\log(\mathrm{Er}_n) \approx -\frac{1}{2}\log(n+1) + \log(\frac{\pi+2}{8}).$$

Hence $\mathrm{Er}_n \approx \dfrac{\pi+2}{8\sqrt{n+1}}$ which is also seen in figure 4.

Table 2

n	x_n	ψ_n^*	Er_n	n	x_n	ψ_n^*	Er_n
1	0.1622	0.1622	0.4805	35	0.0018	0.5228	0.1199
2	0.0803	0.2425	0.4002	40	0.0014	0.5303	0.1124
3	0.0501	0.2927	0.3500	45	0.0012	0.5366	0.1061
10	0.0103	0.4275	0.2152	50	0.0010	0.5419	0.1008
15	0.0064	0.4636	0.1791	55	0.0009	0.5465	0.0962
20	0.0041	0.4861	0.1566	60	0.0008	0.5505	0.0922
25	0.0029	0.5018	0.1409	65	0.0007	0.5540	0.0887
30	0.0022	0.5136	0.1291	70	0.0006	0.5572	0.0855

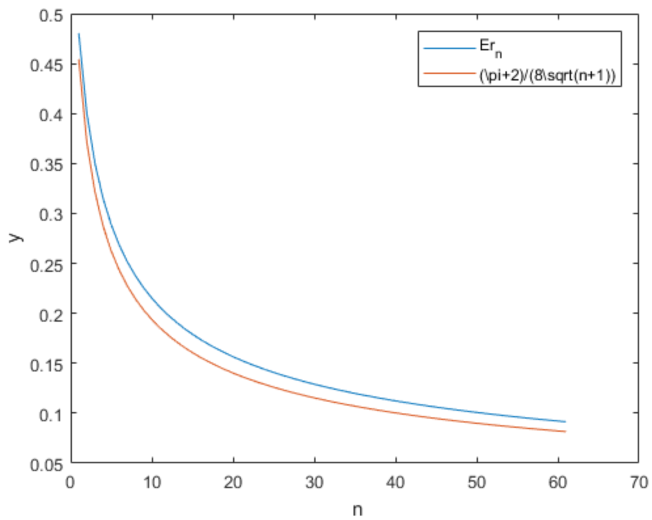


Figure 4: Er_n and $(\pi + 2)/(8\sqrt{n + 1})$ plotted against n

4.3 Example 3

Consider the following Fredholm integral,

$$x(s) = s + \frac{1}{10}\int_0^1 te^{x(t)}dt, \tag{38}$$

with the following exact solution,

$$x_{sol}(s) = s + \frac{8058385148097033}{72057594037927936} \approx s + 0.111832559158934. \tag{39}$$

The Adomian polynomials up to order 5 are as follows,

$$\begin{aligned}
A_0 &= e^{x_0},\\
A_1 &= x_1 e^{x_0},\\
A_2 &= \frac{e^{x_0 x_1}}{2} + x_2 e^{x_0},\\
A_3 &= \frac{e^{x_0} x_1^3}{6} + x_2 x_1 e^{x_0} + x_3 e^{x_0},\\
A_4 &= \frac{e^{x_0} x_1^4}{24} + \frac{e^{x_0} x_1^2 x_2}{2} + x_3 x_1 e^{x_0} + \frac{e^{x_0} x_2^2}{2} + x_4 e^{x_0},\\
A_5 &= \frac{e^{x_0} x_1^5}{120} + \frac{e^{x_0} x_1^3 x_2}{6} + \frac{x_3 x_1^2 e^{x_0}}{2} + \frac{e^{x_0} x_1 x_2^2}{2} + x_4 x_1 e^{x_0} + x_3 x_2 e^{x_0} + x_5 e^{x_0},\\
&\vdots
\end{aligned}$$

For higher order Adomian polynomials we use the function `adom(n,1)`, see Appendix A.3.

Remark 4.3 *Note that because $x_{sol} = s + 0.11183$ we have,*

$$L = 3.0399, \qquad \lambda = \frac{1}{10}, \qquad M = 1, \qquad b - a = 1,$$

which gives that $\alpha = 3.0399/10 < 1$. Theorem 3.12 does guarantee convergence to the exact solution with the Adomian decomposition in this case.

Because we assume that $K(s,t) = t$, all the $x_i(s)$ terms of the series form will have a constant value except,

$$x_0(s) = s$$

This means that the truncated solution $\Psi_n(s)$ is not constant. The terms of the series form are,

$$\begin{aligned} x_0(s) &= s, \\ x_1(s) &= \frac{1}{10}\int_0^1 tA_0(t)dt = 0.1, \\ x_2(s) &= \frac{1}{10}\int_0^1 tA_1(t)dt = 0.01, \\ &\vdots \end{aligned}$$

The rest of the values of x_n are given in table 3 together with $\Psi_n^* = \Psi_n - \Psi_0$ and Er_n. Leaving out $\Psi_0(s)$ gives only constant terms in which the behaviour is more explicitly seen. For the numerical implementation we refer to Appendix A.3. Theorem 3.13 gives an upper bound Max Er_n for the maximum truncation error which is shown in table 3. In figure 5 a log-lin plot of the error and maximum error is given. We see that the error decreases faster than the maximum error and is below the upper bound given by Theorem 3.13.

Table 3

n	x_n	ψ_n^*	Er_n	Max Er_n
1	0.1000	0.1000	0.0118	0.1188
2	0.0100	0.1100	0.0018	0.0361
3	$1.5 \cdot 10^{-3}$	0.1115	$3.3 \cdot 10^{-4}$	$1.1 \cdot 10^{-2}$
4	$2.7 \cdot 10^{-4}$	0.1117	$6.5 \cdot 10^{-5}$	$3.3 \cdot 10^{-3}$
5	$5.2 \cdot 10^{-5}$	0.1118	$1.3 \cdot 10^{-5}$	$1.0 \cdot 10^{-3}$
8	$5.2 \cdot 10^{-7}$	0.1118	$1.5 \cdot 10^{-7}$	$2.8 \cdot 10^{-5}$
10	$2.7 \cdot 10^{-8}$	0.1118	$8.5 \cdot 10^{-9}$	$2.6 \cdot 10^{-6}$
12	$1.6 \cdot 10^{-9}$	0.1118	$4.9 \cdot 10^{-10}$	$2.4 \cdot 10^{-7}$
14	$9.1 \cdot 10^{-11}$	0.1118	$2.9 \cdot 10^{-11}$	$2.2 \cdot 10^{-8}$
15	$2.2 \cdot 10^{-11}$	0.1118	$1.8 \cdot 10^{-12}$	$6.8 \cdot 10^{-9}$

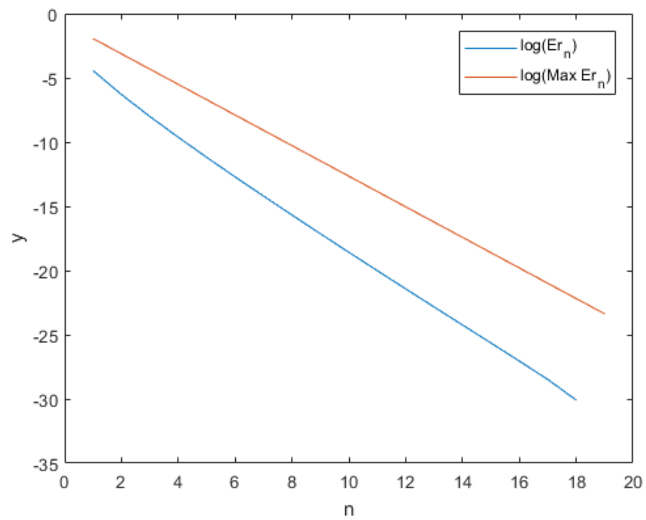


Figure 5: Log-lin plot of Er_n and Max Er_n.

5 Conclusion

We described what Fredholm operators are and how they preserve their properties under certain perturbations. We discussed how the Fredholmess of operators can be described by using Weyl sequences and Calkin algebras. Then we looked at integral operators with a continuous kernel and we saw that this implies the compactness of the operator. Using the fact that the summation of the identity and a compact operator is in the Calkin algebra we concluded that $(I - A)$ is a Fredholm operator. This gave rise to Fredholm's integral of the second kind.

We discussed the Adomian decomposition method which gives solutions for these kind of integrals and we looked in the convergence and error of this method. We numerically computed solutions for these integrals with the help of the Adomian decomposition. The Adomian decomposition turned out to be an useful tool as we saw in chapter 4. It describes a successive approach to obtain a solution for integral operators. In the first two examples, we saw that the error is still significant. This was expected because the convergence criteria of Theorem 3.13 did not hold. We analyzed the convergence by looking at the behaviour in the log-log scale. The third example however agreed with the convergence criteria and we saw that the truncation error agreed with the maximum error.

We used Matlab to implement the numerical simulations in chapter 4. However this turned out to be inconvenient as the use of symbolic variances for the elements of the series form complicates integration in Matlab. It also complicates the variable change from s to t if $K(s,t)$ is non constant. The `eval` function in

Matlab which we used to compute has an inefficient storage system. The consequence of this is that the computation time of each iteration increases rapidly. Hence for higher order terms of n, `eval` is not an efficient tool to use. The use of Mathematica would be more convenient in this case.

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A Appendix

A.1 Example 1

Nonlinear function $F(x(t))$.

```
function F = f(x,k)
 F = x^k;
end
```

Defining the Adomian polynomials.

```
function A_j = po(j,k)                   % Adomian polynomials
 if j > 0
  x =  sym('x',[1 j]);
  syms p;                                % Symbolic variable for p
  syms x0;
  S = x0 + sum(p.^(1:j) .* x);          % Sum of p*x up to order j
  Q = f(S,k);                            % k th power of S
  A_nc = diff(Q,p,j)/factorial(j);       % j th order derivative
  A_j  = subs(A_nc,p,0);                 % Filling in p=0
 else
  syms x0;
  S = x0;
  A_j = f(S,k);                          % k th power of S,
 end
end
```

Defining Adomian polynomials up to order n.

```
function A = adom(n,k)          % n*1 vector with the Adomian polynomials
 for j = 0:n
  A(j+1,:) = po(j,k);
 end
end
```

Defining the components of the series form up to order n.

```
function E = solcon1(n,k,o)
 x0 = @(t) 2*ones(size(t));
 j = 0;
 V = @(t) t^(o);
 E = zeros(n,1);
   while j < n+1;
    K = matlabFunction((1/10)*subs(po(j,k)*V));
     fstr = func2str(K);
      if fstr(3) == ')'
       eval(sprintf('x\%d=K()*(1 0);;',j+1));
      else eval(sprintf('x\%d=integral(K,0,1);',j+1));
      end
    E(j+1,1) = subs(sprintf('x\%d',j+1))
    j = j+1;
   end
end
```

Computing the truncated solution for (35)

```
function sigma = approx(n,k)
 m = 0;
 E = solcon1(n,k);
 sigma = 0;
   while m < n;
    sigma = sigma + E(m+1);
    m = m+1;
   end
 x0 = @(t) 2*ones(size(t));
 sigma = sigma +subs(x0);
end
```

Plotting the values for the truncated solutions of (35).

```
function U = plotapp(n,k)
 m  = 0;
 E  = solcon1(n,k);
 U  = zeros(n,1);
 x0 = @(t)  2*ones(size(t));
 U(1) = subs(x0)
   while m < n  ;
    U(m+2) =   E(m+1,1)+U(m+1);
    m = m + 1;
   end
  plot(U)
  xlabel('n')
  ylabel('\Psi')
end
```

Calculating the least square regression.

```
function m =least(n,k)
U = log(plotapp(n,k));
x= zeros(n,1);
      for m=1:n;
      x(m) = log(m);
      xy(m) = U(m)*x(m);
      end
sumxy = sum(xy);
sumy  = sum(U);
sumx  = sum(x);
sum2x = x*transpose(x);
m = (n*sumxy  sumx*sumy)/(n*sum2x  sumx^2);
end
```

A.2 Example 2

Computing the components of the series form for (36).

```
function E = solcon(n,k)
 x0 = @(t) cos(t) + sin(t) - (pi+2)/8;
 V  = @(t) 1/4;
 j = 0;
 E = zeros(n,1);
   while j < n + 1;
    K = matlabFunction(subs(po(j,k))*V);
    eval(sprintf('x\%d = integral(K,0,pi/2);',j+1));
    E(j+1,1) = subs(sprintf('x\%d',j+1));
    j = j+1;
   end
end
```

Computing the truncated solution for (36).

```
function sigma =  approx1(n,k)
 m = 0;
 E = solcon(n,k);
 sigma = 0;
   while m < n;
    sigma = sigma + E(m+1,1);
    m = m + 1;
   end
   x0 = @(t) cos(t) + sin(t)    (pi+2)/8;
   sigma = sigma + subs(x0);
end
```

Plotting the values for the truncated solutions of (36).

```
function U =  plotapp1(n,k)
 m = 0;
 E = solcon(n,k);
 U = zeros(n,1);
 U(1) = E(1,1);
   while m < n;
    U(m+2) =  E(m+2,1) + U(m+1);
    m = m + 1;
   end
  plot(U)
  xlabel('n')
  ylabel('\Psi_n \ Psi_0')
end
```

Computing the error of (36) with the truncated solution.

```
function W = err(n,k)
 W = zeros(n,1);
 b = 0;
   while b < n + 1;
    W(b+1) = errt(b+1,k);
     b = b + 1;
   end
end

function sigma =  errt(n,k)
 m = 0;
 E = solcon(n,k);
 sum = 0;
   while m < n  ;
     sigma = sigma + E(m+1,1);
    m = m + 1;
   end
   sigma = sigma    ((pi+2)/8);
end
```

A.3 Example 3

Nonlinear function $F(x(t))$.

```
function F = f(x,k)
 F = exp(x*k);
end
```

Defining Adomian polynomials up to order n.

```
function A = adom(n,k)        % n*1 vector with the Adomian polynomials
 for j = 0:n
  A(j+1,:) = po(j,k);
 end
end
```

Computing the components of the series form for (37).

```
function E = solcon(n,k)
 x0 = @(t) t ;
 V  = @(t) t/10;
 j = 0;
 E = zeros(n,1);
   while j < n + 1;
    K = matlabFunction(subs(po(j,k))*V);
    eval(sprintf('x\%d = integral(K,0,1);',j+1));
    E(j+1,1) = subs(sprintf('x\%d',j+1));
    j = j+1;
   end
end
```

Computing the truncated solution for (38).

```
function sigma =  approx1(n,k)
 m = 0;
 E = solcon(n,k);
 sigma = 0;
   while m < n;
    sigma = sigma + E(m+1,1);
    m = m + 1;
   end
   x0 = @(t) t ;
   sigma = sigma + subs(x0);
end
```

Plotting the values for the truncated solutions of (38).

```
function U =  plotapp1(n,k)
 m = 0;
 E = solcon(n,k);
 U = zeros(n,1);
 U(1) = E(1,1);
   while m < n;
    U(m+2) =  E(m+2,1) + U(m+1);
    m = m + 1;
   end
  plot(U)
  xlabel('n')
  ylabel('\Psi_n \ Psi_0')
end
```

Computing the error of (38) with the truncated solution.

```
function W = err(n,k)
 W = zeros(n,1);
  b = 0;
   while b < n + 1;
    W(b+1) = errt(b+1,k);
     b = b + 1;
   end
end

function sigma =  errt(n,k)
 m = 0;
 E = solcon(n,k);
 sum = 0;
   while m < n  ;
     sigma = sigma + E(m+1,1);
    m = m + 1;
   end
   sigma = 8058385148097033/72057594037927936  - sigma;
end
```