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Infrared Triangles Everywhere

On the IR Triangular Equivalence Relation in Gravity and QED

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Abstract

This thesis gives a comprehensive review of the recently discovered infrared triangular equivalence relation between the topics of asymptotic symmetries, soft theorems and memory effect, in the context of massless QED and gravity, with applications to black holes. The Ward identity associated with the asymptotic symmetries is shown to be equal to the soft theorem. Furthermore, we discover that the asymptotic symmetries spontaneously break at the vacuum. For gravity, the asymptotic symmetry group is given by BMS supertranslations. We show that supertranslating Schwarzschild geometry yields a black hole with a lush head of infinite supertranslation 'hair'.

Contents

Chapter 1

Introduction

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

William Blake (1803).

Recently, it has been discovered by Andrew Strominger and collaborators [1–6] that the once seemingly unrelated subjects of 'asymptotic symmetries', 'soft theorems' and 'memory effect' are all in fact different faces of the same underlying physical system. This thesis gives a detailed discussion of this triangular equivalence relation (see Figure 1.1) in the context of massless quantum electrodynamics (QED) and gravity. Throughout this thesis, we will be working with metric signature $(-,+,+,+)$.

The **soft theorems** specify the relations between n and $n+1$ particle scattering amplitudes, where the massless external particle is 'soft' (i.e. its energy is taken to zero). The soft photon theorem originated in 1937, with Nordsieck's paper about the low frequency radiation of a scattered electron [7,8], and was further developed in the 1950s in [9–11]. Subsequently, in 1965, Weinberg proposed the soft graviton theorem [12].
The second corner of the triangle is given by the **asymptotic symmetries**. The asymptotic symmetry group consists of the nontrivial exact symmetries that are found at the asymptotic regions (or boundaries) of infinite spacetime. These symmetries have corresponding conserved charges, as stated by Noether's theorem. For gravity, the asymptotic symmetry group was derived in 1962, by Bondi, van der Burg, Metzner and Sachs (BMS) [13,14]. Somewhat accidentally, they discovered the infinite-dimensional

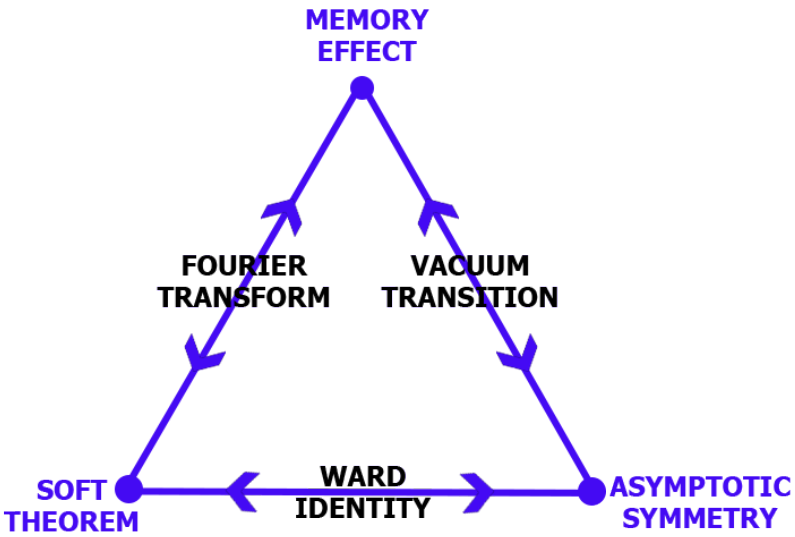


Figure 1.1: The infrared triangle.

BMS group, which consists of the Poincaré group plus an infinite set of angle dependent translations known as 'supertranslations'. For QED and non-abelian gauge theory, the asymptotic symmetries were discovered recently, in 2014 [2, 15–19]. For QED, the asymptotic symmetry group is given by large gauge transformations, which are gauge transformations that do not die off at infinity. The Ward identity associated to the asymptotic symmetries denotes the scattering amplitude constructed from charge conservation at the boundaries of spacetime. As it turns out, by constructing this Ward identity one exactly obtains Weinberg's soft theorems. This implies that the soft theorem is not a distinct theorem, but merely a disguised version of the asymptotic symmetry group.
The third corner of the infrared triangle is given by the **memory effect**, which is quite well-known in its gravitational form. When radiation passes a pair of inertial detectors, it results in a permanent displacement of their relative positions. This is known as the gravitational memory effect. It was discovered in 1974 by Zeldovich and Polnarev [20], and has been notoriously studied ever since [21–30].

The infrared triangular equivalence relation can be found throughout numerous distinct areas of physics (see Figure 1.2), and immensely speeds up the rate at which scientific discoveries in the infrared sector are made: as soon as one corner of the triangle is discovered, the other two can be determined as well. Even if all three corners are already known, the triangular relation still leads to a better conceptual understanding of physics in the infrared.

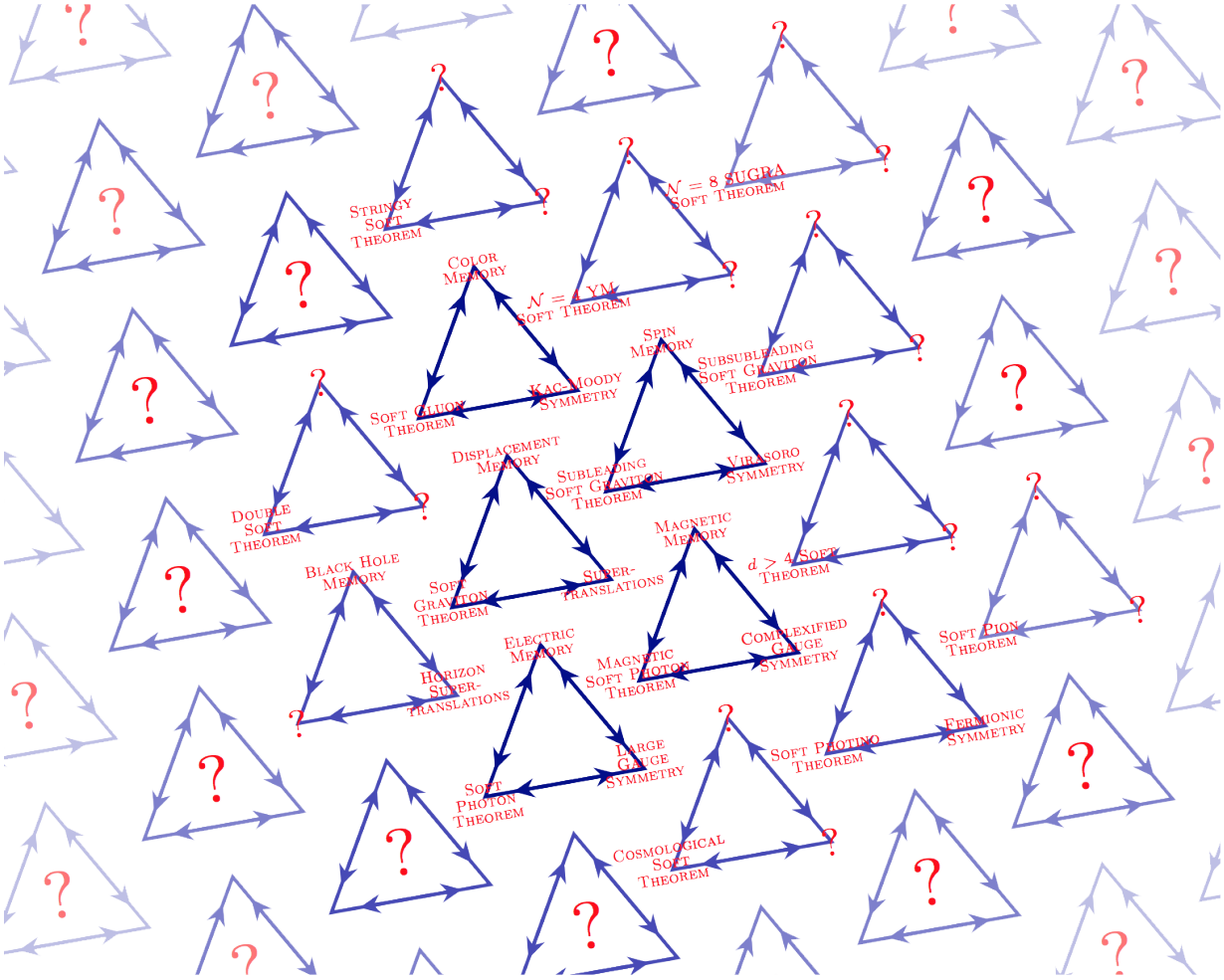


Figure 1.2: An abundance of infrared triangles within physics [1].

This thesis is organised as follows. In Chapter 2, we start off by discussing the infrared triangle in its easiest context: massless QED. By analysing the infrared triangle in its simplest form, we are able to build up conceptual understanding of the infrared triangle analysis, and construct a 'template' method that can be used to analyse the infrared structure of other, more complicated physical theories. For quantum electrodynamics (QED), the asymptotic symmetry group is given by the 'large' gauge symmetries, gauge symmetries which do not die off at infinity. Next, in Chapter 3, we move on to the more complicated case of gravity. This chapter has a similar structure to Chapter 2, albeit a bit more difficult. In Chapter 4, we apply the results found in Chapter 3 to Schwarzschild geometry, and discover that black holes have a lush head of infinite supertranslation hair.

1.1 Minkowski Space Penrose diagram

In this thesis, we will investigate the behaviour of incoming and outgoing massless fields at $r = \infty$. Working with an infinite-sized diagram of spacetime is quite the impossible task: while trying to find spatial infinity on the diagram, one would spend an infinite amount of time going to the right, were it not that the human life span is actually finite. Therefore, it is of lifesaving importance that we find a way to map infinity to the palm of our hand, as it were. A very useful diagram would be a two-dimensional diagram that captures the global properties of spacetime, while still preserving the causal structure. Luckily, such a diagram has already been constructed and is called the Penrose-Carter diagram [31]. Penrose managed to map the entire infinite spacetime to a finite region by means of a conformal transformation. The most basic of Penrose diagrams is the one for four-dimensional Minkowski space, which will be introduced below.

In polar coordinates (t, r, θ, ϕ), the Minkowski line element is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1.1}$$

with ranges $-\infty < t < \infty$, $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. In order to obtain coordinates with finite ranges, it is useful to introduce null coordinates

$$u = t - r, \quad v = t + r, \tag{1.2}$$

with corresponding ranges

$$-\infty < u < \infty, \quad -\infty < v < \infty, \quad u \leqslant v. \tag{1.3}$$

By using the arctangent, coordinates with finite ranges are obtained,

$$U = \arctan(u), \quad V = \arctan(v), \tag{1.4}$$

with ranges

$$-\pi/2 < U < \pi/2, \quad -\pi/2 < V < \pi/2, \quad U \leqslant V. \tag{1.5}$$

Now, we construct a timelike coordinate T and a radial coordinate R,

$$T = U + V, \quad R = V - U, \tag{1.6}$$

with

$$0 \leqslant R < \pi, \quad |T| + R < \pi. \tag{1.7}$$

Infinite Minkowski spacetime is now successfully mapped to a finite region, and one can draw the Minkowski space Penrose diagram (see Figure 1.3).
Conformal infinity is divided into the following regions:

$$
\begin{aligned}
i^+ &= \text{future timelike infinity} && (T=\pi,\ R=0), && (u=\infty, v=\infty, r=0)\\
i^0 &= \text{spatial infinity} && (T=0,\ R=\pi), && (u=-\infty, v=\infty, r=\infty),\\
i^- &= \text{past timelike infinity} && (T=-\pi,\ R=0), && (u=-\infty, v=-\infty, r=0),\\
\mathcal{I}^+ &= \text{future null infinity} && (T=\pi-R,\ 0<R<\pi), && (-\infty<u<\infty, r=\infty),\\
\mathcal{I}^- &= \text{past null infinity} && (T=-\pi+R,\ 0<R<\pi), && (-\infty<v<\infty, r=\infty).
\end{aligned}
$$

Topologically, future and past null infinity are $\mathbb{R}\times S^2$, while their four boundary components $\mathcal{I}^{\pm}_{\pm}$ are S^2. In the Penrose diagram, light rays (depicted by the wavy grey lines in Figure 1.3) start at past null infinity $\mathcal{I}^-$, propagate at 45° and end up at future null infinity $\mathcal{I}^+$. Massive particles (depicted by the thick grey lines) start and end up at at i^- and i^+, respectively. Slices of constant t are depicted by the red lines, while slices of constant r are depicted by the blue lines. In the left figure of Figure 1.3, every point except $r=0$ is a two-sphere S^2. By reinstating one of the repressed dimensions and rotating the image on the left, we obtain the image on the right of Figure 1.3. Now, every S^2 is represented by two points, one on the right side of the origin and one on the left side.

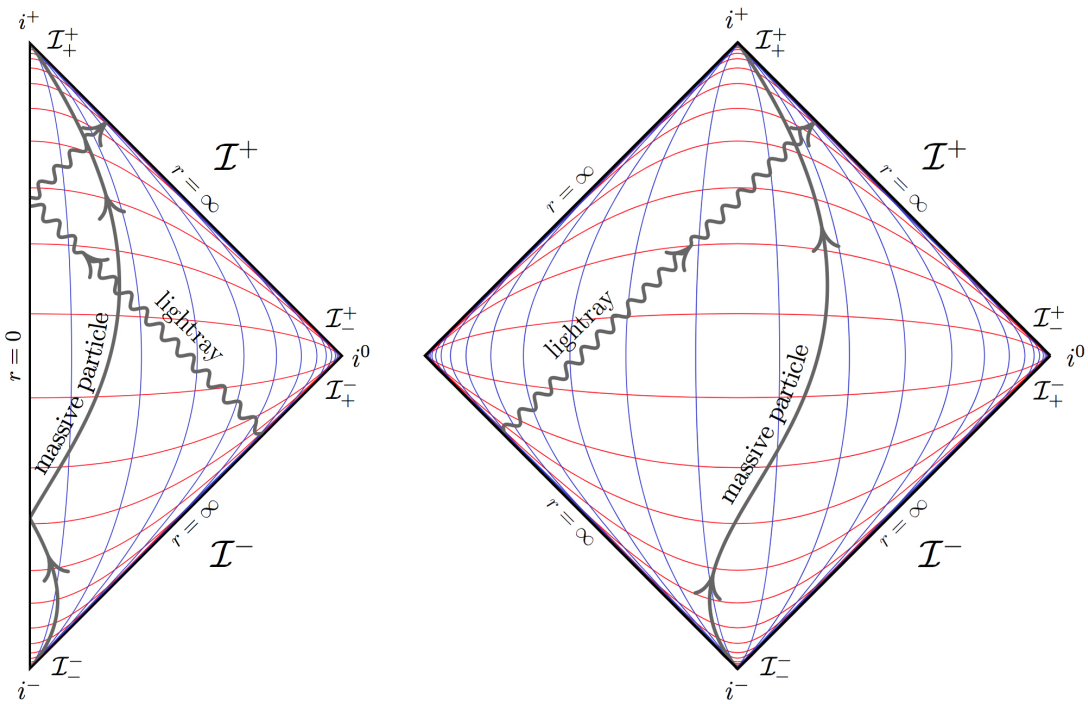


Figure 1.3: Penrose diagrams of Minkowski space. Sourced from [1].

Chapter 2

Quantum Electrodynamics

If it turns out there is a simple ultimate law which explains everything,
So be it - that would be very nice to discover.
If it turns out it's like an onion with millions of layers...
Then that's the way it is.

Richard Feynman (1918-1988).

2.1 Introduction

In this chapter, we will discuss the infrared triangle in the context of massless scalar QED (see Figure 2.1), where the content is heavily based on work presented in [1,2]. In 2014, it was shown in [2] that Weinberg's soft photon theorem [12] is equivalent to the Ward identity associated with the infinite-dimensional asymptotic symmetry group. For quantum electrodynamics (QED), the asymptotic symmetries are given by the large gauge symmetries, the gauge symmetries that do not die off at infinity.
Throughout the chapter, we will mostly focus on $\mathcal{I}^+$, but the derivations for $\mathcal{I}^-$ can be done in an analogous manner.

The outline of this chapter is as follows. In Section 2.2, we lay some groundwork and describe the the tools needed in order to do the infrared triangle analysis: some QED preliminaries, the coordinate systems, and the boundary falloff conditions for the fields.

Subsequently, we can start with the the analysis of the infrared triangle, which follows the ensuing recipe:

1. Derive the asymptotic symmetry group (Section 2.2.2);
2. Construct its associated conserved charges (Section 2.3);
3. Derive the Ward identities associated to the asymptotic symmetries (Section 2.4);
4. Show that the Ward identity is equivalent to the soft photon theorem (Section 2.5).

The recipe appears to be pretty short, but since we are translating one physical theory into the other, it actually contains a great deal of substeps in which we re-express formulas into a more convenient form. For example, after having found the conserved charges in Section 2.3, we need to split them into a 'soft' charge and a 'hard' charge, where the soft charge annihilates and creates soft particles on the boundaries.

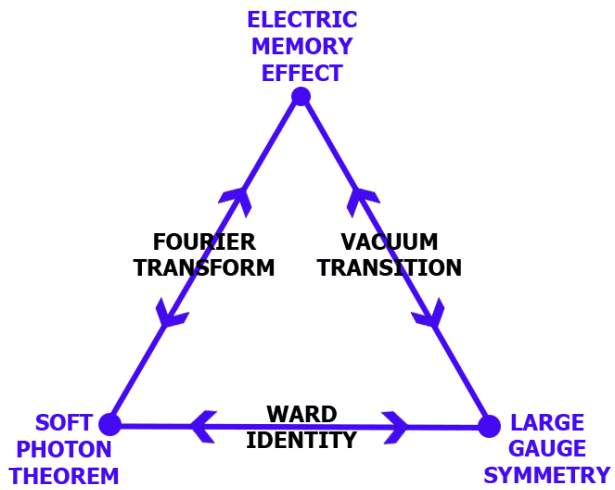


Figure 2.1: The infrared triangle in the context of QED.

2.2 Preliminaries

Before we proceed, it is necessary to lay down a concrete foundation, on which we can build our theoretical multiple-story flat block. In other words, we first need to give some QED preliminaries, describe our coordinate system (Section 2.2.1), and derive the boundary falloff conditions for A_μ and $F_{\mu\nu}$ (Section 2.2.2).
For quantum electrodynamics, the action is given by

$$S = -\frac{1}{4e^2}\int \mathrm{d}^4x\,\sqrt{-g}F_{\mu\nu}F^{\mu\nu} - \int \mathrm{d}^4x\,\sqrt{-g}A^\mu j_\mu \qquad \text{where } \mu,\nu = 0,1,2,3. \tag{2.1}$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the antisymmetric electromagnetic field strength tensor, and $j_\mu = iQ(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)$ is the conserved Noether current obeying $\nabla^\mu j_\mu = 0$. The equations of motion that follow from the action (2.1) are given by

$$\mathrm{d} \star F = e^2 \star j, \tag{2.2}$$

or in coordinate representation,

$$\nabla^\mu F_{\mu\nu} = e^2 j_\nu, \tag{2.3}$$

and they are invariant under finite gauge transformations

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \lambda. \tag{2.4}$$

A tremendous part of this chapter deals with electric charge conservation at the boundaries of spacetime, where the electric charge inside a two-sphere S^2 at infinity is given by [1]

$$Q_E = \frac{1}{e^2} \int_{S^2_\infty} \star F. \tag{2.5}$$

Here, $\star$ denotes the Hodge dual.[1] In order to adopt a convention in which Q_E is integer, a constant e^2 is introduced in the action (2.1). By using Stokes' theorem and equation (2.2), it can be seen that we do indeed end up with an integer value for the electric charge,

$$Q_E = \frac{1}{e^2} \int_{S^2_\infty} \star F = \frac{1}{e^2} \int_\Sigma \mathrm{d} \star F = \int_\Sigma \star j \in \mathbb{Z}, \tag{2.6}$$

where Σ is any surface with S^2_∞ as boundary.

2.2.1 Coordinate Systems near $\mathcal{I}^+$ and $\mathcal{I}^-$

In this section, we will describe the coordinate systems that are used throughout this thesis. In the neighbourhood of $\mathcal{I}^+$, retarded Bondi coordinates $(u, r, z, \bar{z})$ are used, where $(z, \bar{z})$ are coordinates on the unit S^2. Near $\mathcal{I}^-$, on the other hand, we will use advanced Bondi coordinates $(v, r, z, \bar{z})$. Employing these particular coordinate systems may perhaps seem a bit unusual, but we will soon see that they have a certain feature which ensures that subsequent formulae will have a simple form.

[1] For those unfamiliar with differential forms, a short introduction can be found in Appendix A.

Future Null Infinity $\mathcal{I}^+$

First, we will introduce the retarded Bondi coordinate system on $\mathcal{I}^+$. In standard coordinates, the Minkowski line element is specified as

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + (\mathrm{d}\vec{x})^2, \quad \text{with} \quad \vec{x} = (x^1, x^2, x^3). \tag{2.7}$$

The coordinate transformation from $(t, \vec{x})$ to retarded Bondi coordinates $(u, r, z, \bar{z})$ is given by

$$u = t - r, \quad r = (\vec{x})^2, \quad \text{and} \quad z = \frac{x^1 + ix^2}{x^3 + r}, \tag{2.8}$$

whereas the inverse transformation is

$$t = u + r, \quad \vec{x} = \frac{r}{1 + z\bar{z}}\Big(z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}\Big). \tag{2.9}$$

Coordinate z runs over the entire complex plane: $z = 0$ corresponds to the north pole, $z = \infty$ to the south pole and $z\bar{z} = 1$ is the equator. Furthermore, the antipodal map is given by $z \to -1/\bar{z}$. The antipodal point of z denotes the point that is diametrically opposite to z on the sphere. By using coordinate transformation (2.8), one obtains the line element (2.7) in $(u, r, z, \bar{z})$-coordinates:

$$\mathrm{d}s^2 = -\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}. \tag{2.10}$$

Here, $\gamma_{z\bar{z}}$ is the round metric on the unit S^2,

$$\gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}, \qquad \text{with} \qquad \int \mathrm{d}^2z\,\gamma_{z\bar{z}} = 4\pi. \tag{2.11}$$

Given line element (2.10), the inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\gamma^{z\bar{z}}}{r^2} \\ 0 & 0 & \dfrac{\gamma^{z\bar{z}}}{r^2} & 0 \end{pmatrix}. \tag{2.12}$$

By using

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\Big(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\Big), \tag{2.13}$$

one can determine the nonzero Christoffel symbols with respect to metric $g_{\mu\nu}$, which are given by

$$\Gamma^u_{z\bar{z}} = r\gamma_{z\bar{z}}, \qquad \Gamma^r_{z\bar{z}} = -r\gamma_{z\bar{z}}, \qquad \Gamma^z_{rz} = \frac{1}{r}, \qquad \Gamma^z_{zz} = \gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}}, \tag{2.14}$$

plus their complex conjugates. The covariant derivative with respect to $\gamma_{z\bar{z}}$ is denoted by D_z, with $D^z = \gamma^{z\bar{z}} D_{\bar{z}}$. The only nonzero Christoffel symbols with respect to the unit round S^2 metric $\gamma_{z\bar{z}}$ are specified by

$$\tilde{\Gamma}^z_{zz} = \gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}} = -\frac{2\bar{z}}{1+z\bar{z}}, \quad \text{and} \quad \tilde{\Gamma}^{\bar{z}}_{\bar{z}\bar{z}} = \gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}} = -\frac{2z}{1+z\bar{z}}. \tag{2.15}$$

Past Null Infinity $\mathcal{I}^-$

In the neighbourhood of $\mathcal{I}^-$, advanced coordinates $(v, r, z, \bar{z})$ are used, with

$$v = t + r, \quad r = (\vec{x})^2, \quad \text{and} \quad z = -\frac{x^3 + r}{x^1 - ix^2}. \tag{2.16}$$

The inverse transformation is given by

$$t = v - r, \quad \vec{x} = -\frac{r}{1+z\bar{z}}\Big(z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}\Big). \tag{2.17}$$

A noteworthy feature of the above coordinate transformation is that advanced Bondi coordinate z is identified as the antipodal point on the sphere of retarded Bondi coordinate z, i.e.

$$z|_{\mathcal{I}^-} = -\frac{1}{\bar{z}|_{\mathcal{I}^+}}. \tag{2.18}$$

Thus, a light ray passing through the interior of Minkowski space reaches the same value of $(z, \bar{z})$ at both $\mathcal{I}^+$ and $\mathcal{I}^-$. In advanced coordinates $(v, r, z, \bar{z})$, the Minkowski line element is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + 2r^2 \gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}. \tag{2.19}$$

The inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\gamma^{z\bar{z}}}{r^2} \\ 0 & 0 & \dfrac{\gamma^{z\bar{z}}}{r^2} & 0 \end{pmatrix}, \tag{2.20}$$

and the nonzero Christoffel symbols with respect to $g_{\mu\nu}$ are

$$\Gamma^v_{z\bar{z}} = -r\gamma_{z\bar{z}}, \qquad \Gamma^r_{z\bar{z}} = -r\gamma_{z\bar{z}}, \qquad \Gamma^z_{rz} = \frac{1}{r}, \qquad \Gamma^z_{zz} = \partial_z \ln(\gamma_{z\bar{z}}), \tag{2.21}$$

plus their complex conjugates.

2.2.2 Asymptotic Boundary Falloff Conditions

In this section, we will determine the boundary falloff conditions for the gauge field A_μ, from which we obtain the structure of the large gauge symmetries and the boundary falloff conditions for the field strength $F_{\mu\nu}$. These boundary conditions will be used every now and again during subsequent derivations in this chapter. We choose to work in retarded radial gauge, with gauge-fixing conditions [2]

$$A_r = 0, \qquad A_u\big|_{\mathcal{I}^+} = 0. \tag{2.22}$$

The long range electric field is defined by F_{ur}, so in order to have finite energy configurations it is necessary that $F_{ur} \sim \mathcal{O}(r^{-2})$. Since $F_{ur} = \partial_u \cancel{A_r} - \partial_r A_u$, it follows that $A_u \sim \mathcal{O}(r^{-1})$. The T_{00}-component of the stress-energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{2.23}$$

defines the energy density [32]. In $(u, r, z, \bar{z})$-coordinates, it is given by

$$T_{uu} = \frac{1}{4\pi}\left(F_{u\rho}F_{u\sigma}g^{\rho\sigma} + \mathcal{O}(r^{-3})\right) \sim F_{uz}F_{u\bar{z}}\frac{\gamma^{z\bar{z}}}{r^2} + \mathcal{O}(r^{-3}), \tag{2.24}$$

The surface area of a sphere grows like r^2. Thus, in order to have finite energy flux it is required that $T_{uu} \sim \mathcal{O}(r^{-2})$. It now follows from equation (2.24) that $F_{uz} \sim \mathcal{O}(1)$. Since $F_{uz} = \partial_u A_z - \partial_z A_u$, this in turn implies that $A_z \sim \mathcal{O}(1)$. Summarising,

$$\begin{aligned} &A_u \sim \mathcal{O}(r^{-1}), \\ &A_r = 0, \\ &A_z \sim \mathcal{O}(1). \end{aligned} \tag{2.25}$$

As previously mentioned, the equations of motion (2.3) are invariant under gauge transformations,

$$\delta A_u = \partial_u \varepsilon, \qquad \delta A_r = \partial_r \varepsilon, \qquad \delta A_z = \partial_z \varepsilon, \tag{2.26}$$

where we take $\lambda = \varepsilon$ to be infinitesimal in the gauge symmetry equation (2.4). Now, importantly, observe that the function ε that satisfies the boundary falloff conditions (2.25) is specified by

$$\varepsilon = \varepsilon(z, \bar{z}) + \mathcal{O}(r^{-1}). \tag{2.27}$$

This implies that the large gauge symmetries are of the form

$$\delta A_z = \partial_z \varepsilon(z, \bar{z}). \tag{2.28}$$

It will be shown later on that the large gauge transformations (2.28) are generated by conserved electric charges. Observe that on the vacuum (with $A_\mu = 0$), the action of a large gauge transformation is given by $A_\mu = 0 \to \partial_z \varepsilon(z, \bar{z})$, which is only zero for constant ε. This implies that the symmetry is spontaneously broken by the vacuum.

All of the gauge field components die off near $\mathcal{I}^+$ or remain constant, so they can be asymptotically expanded in powers of r^{-1}. For example, an expansion of the z-component of the gauge field gives

$$A_z(u, r, z, \bar{z}) = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u, z, \bar{z})}{r^n}, \tag{2.29}$$

where $A_z^{(n)}$ denotes the r^{-n}th order term in the expansion. It is important to keep this superscript notation in mind, because it will be used in further sections. Using (2.25), one finds

$$\begin{aligned}
F_{ur} &= -\partial_r A_u &&\cong \frac{A_u^{(1)}}{r^2} &&\sim \mathcal{O}(r^{-2}),\\
F_{uz} &= \partial_u A_z - \partial_z A_u &&\cong \partial_u A_z^{(0)} &&\sim \mathcal{O}(1),\\
F_{rz} &= \partial_r A_z &&\cong -\frac{A_z^{(1)}}{r^2} &&\sim \mathcal{O}(r^{-2}),\\
F_{z\bar{z}} &= \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z &&\cong \partial_z A_{\bar{z}}^{(0)} - \partial_{\bar{z}} A_z^{(0)} &&\sim \mathcal{O}(1),
\end{aligned} \tag{2.30}$$

which are the boundary falloff conditions for the field strength components.

2.2.3 Antipodal Matching Condition

In the following sections, our main focus will lie on what happens near spatial infinity i^0. Particularly, we explore in what manner the fields at the future of $\mathcal{I}^-$ ($\mathcal{I}^-_+$), and the past of $\mathcal{I}^+$ ($\mathcal{I}^+_-$) are related. In this section, we will describe the Liénard-Wiechert solution, and lift the veil on its vital feature regarding i^0.

Consider n particles (with charges $Q_k, \ldots, Q_n$), moving at a constant 4-velocity $U_k^\mu = \gamma_k(1, \vec{\beta}_k)$, where $k = 1, \ldots, n$. Here, $U_k^2 = -1$, and $\gamma_k^2 = \dfrac{1}{1 - \vec{\beta}_k^2}$. The current produced by the n moving particles is given by [1]

$$j_\mu(x) = \sum_{k=1}^{n} Q_k \int \mathrm{d}\tau \, U_{k\mu} \delta^4(x^\nu - U_k^\nu \tau), \tag{2.31}$$

which is related to the electromagnetic field strength $F_{\mu\nu}$ by equation (2.3). In 1898, Liénard and Wiechert found the radial component of the electric field,

$$F_{rt}(\vec{x},t) = \frac{e^2}{4\pi}\sum_{k=1}^{n}\frac{Q_k\gamma_k\left(r - t\hat{x}\cdot\vec{\beta}_k\right)}{\left|\gamma_k^2\left(t - r\hat{x}\cdot\vec{\beta}_k\right)^2 - t^2 + r^2\right|^{3/2}}, \tag{2.32}$$

where the unit 3-vector $\hat{x}$ is given by $\hat{x} = \vec{x}/r$.

Remarkably, it can be observed that the Liénard-Wiechert solution has a discontinuity at $r = \infty$. This crucial discovery will be made by approaching spatial infinity i^0 via two different paths (see Figure 2.2). We will first approach i^0 via $\mathcal{I}^+$, and subsequently go via $\mathcal{I}^-$. By doing this, one is able to see that, in general, $F_{rt}\big|_{\mathcal{I}^+_-} \neq F_{rt}\big|_{\mathcal{I}^-_+}$.

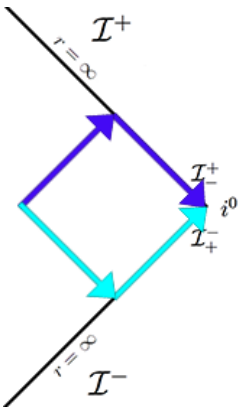


Figure 2.2: The two paths along which i^0 will be approached.

Future Null Infinity $\mathcal{I}^+$

The Liénard-Wiechert solution (2.32) can be rewritten in terms of the retarded time by making the substitution $t = u + r$, which gives

$$F_{rt} = F_{ru} = \frac{e^2}{4\pi}\sum_{k=1}^{n}\frac{Q_k\gamma_k\left(r - (u+r)\hat{x}\cdot\vec{\beta}_k\right)}{\left|\gamma_k^2\left((u+r) - r\hat{x}\cdot\vec{\beta}_k\right)^2 - (u+r)^2 + r^2\right|^{3/2}}. \tag{2.33}$$

In order to reach $\mathcal{I}^+$, one needs to take the limit $r \to \infty$, while keeping u fixed,

$$\begin{aligned} F_{rt}\big|_{\mathcal{I}^+} &= \frac{e^2}{4\pi}\sum_{k=1}^{n} \frac{rQ_k\gamma_k(1-\hat{x}\cdot\vec{\beta}_k)+\mathcal{O}(1)}{\left|\gamma_k^2\left((u+r)^2-2r(u+r)\hat{x}\cdot\vec{\beta}_k+r^2(\hat{x}\cdot\vec{\beta}_k)^2\right)-u^2-2ur\right|^{3/2}} \\ &= \frac{e^2}{4\pi}\sum_{k=1}^{n} \frac{rQ_k\gamma_k(1-\hat{x}\cdot\vec{\beta}_k)+\mathcal{O}(1)}{\left|r^2\gamma_k^2\left(1-\hat{x}\cdot\vec{\beta}_k\right)^2+\mathcal{O}(r)\right|^{3/2}}. \end{aligned}$$

Since r goes to infinity, all non-leading terms can be dropped, which yields

$$F_{rt}\big|_{\mathcal{I}^+} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{n} \frac{Q_k}{\gamma_k^2\left(1-\hat{x}\cdot\vec{\beta}_k\right)^2}. \tag{2.34}$$

Subsequently taking $u \to -\infty$ ensures that $\mathcal{I}^+_-$ is reached. However, since equation (2.34) does not depend on u, it is also the expression for the Liénard-Wiechert solution at $\mathcal{I}^+_-$.

Past Null Infinity $\mathcal{I}^-$

By making the substitution $t = v - r$, the Liénard-Wiechert solution (2.32) is expressed in terms of advanced time:

$$F_{rt} = F_{rv} = \frac{e^2}{4\pi}\sum_{k=1}^{n} \frac{Q_k\gamma_k\left(r-(v-r)\hat{x}\cdot\vec{\beta}_k\right)}{\left|\gamma_k^2\left((v-r)-r\hat{x}\cdot\vec{\beta}_k\right)^2-(v-r)^2+r^2\right|^{3/2}}. \tag{2.35}$$

In order to reach $\mathcal{I}^-$, one takes the limit $r \to \infty$, while keeping v fixed. In a similar fashion as before, all non-leading terms are dropped, which results in

$$F_{rt}\big|_{\mathcal{I}^-} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{n} \frac{Q_k}{\gamma_k^2\left(1+\hat{x}\cdot\vec{\beta}_k\right)^2}. \tag{2.36}$$

Next, one takes $v \to \infty$ to reach $\mathcal{I}^-_+$, but since the above expression is independent of v, the Liénard-Wiechert solution at $\mathcal{I}^-_+$ is given by equation (2.36) as well.

By comparing equation (2.34) with equation (2.36), it is easy to see that for $\vec{\beta}_k \neq 0$, $F_{rt}\big|_{\mathcal{I}^+_-} \neq F_{rt}\big|_{\mathcal{I}^-_+}$. Crucially, however, observe that the fields are equal under the transformation $\hat{x} \to -\hat{x}$. In other words, $F_{rt}\big|_{\mathcal{I}^+_-}$ and $F_{rt}\big|_{\mathcal{I}^-_+}$ are antipodally related to each other, and

$$\lim_{r\to\infty} r^2 F_{ru}(\hat{x})|_{\mathcal{I}^+_-} = \lim_{r\to\infty} r^2 F_{rv}(-\hat{x})|_{\mathcal{I}^-_+}, \tag{2.37}$$

where $F_{ru} = F_{rv} = F_{rt}$ and $\hat{x} = \hat{x}(z, \bar{z})$. Since $F_{ru} \sim \mathcal{O}(r^{-2})$, its expansion about $r = \infty$ is given by

$$F_{ru}(u, r, z, \bar{z}) = \sum_{n=2}^{\infty} \frac{F_{ru}^{(n)}(u, z, \bar{z})}{r^n}. \tag{2.38}$$

Using the above expansion, (2.37) can be rewritten as

$$F_{ru}^{(2)}(z, \bar{z})|_{\mathcal{I}_-^+} = F_{rv}^{(2)}(z, \bar{z})|_{\mathcal{I}_+^-}, \tag{2.39}$$

which is the final form of our antipodal matching condition.

2.3 Large Gauge Symmetries

Having discovered how the fields at $\mathcal{I}_+^-$ and $\mathcal{I}_-^+$ are matched, we are ready to take things to the next level. In this section, we will show that the antipodal matching condition (2.39) implies an infinity of conserved charges. Subsequently, it will be shown that the conserved charges generate the large gauge symmetries.
A Lorentz-invariant matching condition for the gauge field is given by [16]

$$A_z(z, \bar{z})|_{\mathcal{I}_-^+} = A_z(z, \bar{z})|_{\mathcal{I}_+^-}. \tag{2.40}$$

Referring to (2.28), one can observe that above condition is satisfied for a function ε with matching condition

$$\varepsilon(z, \bar{z})|_{\mathcal{I}_-^+} = \varepsilon(z, \bar{z})|_{\mathcal{I}_+^-}. \tag{2.41}$$

The future and past charges are given by [1]

$$Q_\varepsilon^+ = \frac{1}{e^2} \int_{\mathcal{I}_-^+} \varepsilon \star F, \tag{2.42}$$

$$Q_\varepsilon^- = \frac{1}{e^2} \int_{\mathcal{I}_+^-} \varepsilon \star F. \tag{2.43}$$

Now, lo and behold, observe that boundary conditions (2.39) and (2.41) yield a conservation of charges,

$$Q_\varepsilon^+ = Q_\varepsilon^-. \tag{2.44}$$

This conservation law holds for every function $\varepsilon(z, \bar{z})$ that satisfies condition (2.41). However, since one can come up with infinitely many of such functions, this implies

that there exist an infinite number of conserved charges near spatial infinity i^0.

Now consider the future charge Q_ε^+ (2.42). By using Stokes' theorem, one can re-express the surface integral over the boundaries $\mathcal{I}_-^+$ and $\mathcal{I}_+^+$ as a volume integral over $\mathcal{I}^+$,

$$\begin{aligned}\frac{1}{e^2}\int_{\mathcal{I}_-^+}\varepsilon\star F+\frac{1}{e^2}\cancelto{0}{\int_{\mathcal{I}_+^+}\varepsilon\star F}&=\frac{1}{e^2}\int_{\mathcal{I}^+}\mathrm{d}\left(\varepsilon\star F\right)\\&=\frac{1}{e^2}\int_{\mathcal{I}^+}\mathrm{d}\varepsilon\wedge\star F+\frac{1}{e^2}\int_{\mathcal{I}^+}\varepsilon\left(\mathrm{d}\star F\right)\\&=\frac{1}{e^2}\int_{\mathcal{I}^+}\mathrm{d}\varepsilon\wedge\star F+\int_{\mathcal{I}^+}\varepsilon\star j.\end{aligned}\tag{2.45}$$

Since we assumed that all particles are massless, the electric field vanishes at $\mathcal{I}_+^+$, which causes the disappearance of the second term in the first line. Thus, equation (2.45) denotes the future charge Q_ε^+. In the last line, equation (2.2) was used. Likewise,

$$Q_\varepsilon^-=\frac{1}{e^2}\int_{\mathcal{I}^-}\mathrm{d}\varepsilon\wedge\star F+\int_{\mathcal{I}^-}\varepsilon\star j.\tag{2.46}$$

In order to simplify matters (or in this case non-matters), we will restrict ourselves to functions ε that satisfy

$$\partial_u\varepsilon=\partial_v\varepsilon=0.\tag{2.47}$$

For the special case $\varepsilon=1$, the future charge becomes

$$Q_\varepsilon^+=\frac{1}{e^2}\int_{\mathcal{I}^+}\cancel{\mathrm{d}\varepsilon}\wedge\star F+\int_{\mathcal{I}^+}\star j=\sum_{k=1}^m Q_k^{\text{out}},\tag{2.48}$$

for m outgoing particles. Thus, for $\varepsilon=1$, equation (2.44) becomes the statement that the sum of all incoming charges is equal to the sum of all outgoing charges,

$$\sum_{k=1}^n Q_k^{\text{in}}=\sum_{k=1}^m Q_k^{\text{out}}.\tag{2.49}$$

The importance of enforcing Lorentz invariance must be emphasised, since the conservation of charges follows directly from Lorentz invariant conditions.
For future derivations, it is advantageous to express the future charge (2.42) in terms of a coordinate representation. In order to do so, one needs the explicit components of $\star F$. Since $\mathcal{I}_-^+$ has coordinates $(u,r,z,\bar{z})=(-\infty,\infty,z,\bar{z})$, only the $(z,\bar{z})$-component of $\star F$ is of interest. This component is derived in Appendix C.1, and is given by

$$(\star F)_{z\bar{z}}\cong i\gamma_{z\bar{z}}F_{ru}^{(2)}.\tag{2.50}$$

Thus, the future charge (2.42) becomes

$$\begin{aligned} Q_\varepsilon^+ &= -\frac{i}{e^2}\int_{\mathcal{I}_-^+} \mathrm{d}^2z\, \varepsilon(\star F)_{z\bar{z}} \\ &= \frac{1}{e^2}\int_{\mathcal{I}_-^+} \mathrm{d}^2z\, \gamma_{z\bar{z}}\varepsilon F_{ru}^{(2)}. \end{aligned} \tag{2.51}$$

Integrating Q_ε^+ by parts over u yields

$$\begin{aligned} Q_\varepsilon^+ &= -\frac{1}{e^2}\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2z\, \gamma_{z\bar{z}}\varepsilon \nabla_u F_{ru}^{(2)} \\ &= -\frac{1}{e^2}\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2z\, \gamma_{z\bar{z}}\varepsilon \partial_u F_{ru}^{(2)}, \end{aligned} \tag{2.52}$$

where the boundary term vanishes, and both $\varepsilon(z,\bar{z})$ and $\gamma_{z\bar{z}}$ do not depend on u. Furthermore, the integral over u goes from $-\infty$ to $+\infty$, so one obtains an integral over $\mathcal{I}^+$. Lastly,

$$\nabla_u F_{ru}^{(2)} = \partial_u F_{ru}^{(2)} - \cancel{\Gamma_{ru}^\alpha} F_{\alpha u}^{(2)} - \cancel{\Gamma_{uu}^\alpha} F_{r\alpha}^{(2)} = \partial_u F_{ru}^{(2)}. \tag{2.53}$$

To re-express $\partial_u F_{ru}^{(2)}$, one requires the u-component of the equation of motion (2.3),

$$\nabla^\mu F_{\mu u} = \nabla^u \cancel{F_{uu}} + \nabla^r F_{ru} + \nabla^z F_{zu} + \nabla^{\bar{z}} F_{\bar{z}u} = e^2 j_u. \tag{2.54}$$

Note that

$$\begin{aligned} \nabla^z F_{zu} &= g^{z\bar{z}}\nabla_{\bar{z}} F_{zu} \\ &= g^{z\bar{z}}(\partial_{\bar{z}} F_{zu} - \Gamma_{\bar{z}z}^\alpha F_{\alpha u} - \cancel{\Gamma_{\bar{z}u}^\alpha} F_{z\alpha}) \\ &= \frac{\gamma^{z\bar{z}}}{r^2}(D_{\bar{z}} F_{zu} + r\gamma_{z\bar{z}} F_{ru}) \\ &= \frac{1}{r^2} D^z F_{zu}^{(0)} + \mathcal{O}(r^{-3}) \\ &= -\frac{1}{r^2} D^z F_{uz}^{(0)} + \mathcal{O}(r^{-3}), \end{aligned} \qquad \begin{aligned} \nabla^r F_{ru} &= g^{r\alpha}\nabla_\alpha F_{ru} \\ &= \nabla_r F_{ru} - \nabla_u F_{ru} \\ &= \left(\partial_r F_{ru} - \cancel{\Gamma_{rr}^\alpha} F_{\alpha u} - \cancel{\Gamma_{ru}^\alpha} F_{r\alpha}\right) \\ &\quad - \left(\partial_u F_{ru} - \cancel{\Gamma_{ur}^\alpha} F_{\alpha u} - \cancel{\Gamma_{uu}^\alpha} F_{r\alpha}\right) \\ &= -\partial_u \frac{F_{ru}^{(2)}}{r^2} + \mathcal{O}(r^{-3}), \end{aligned} \tag{2.55}$$

where $F_{zu} \sim \mathcal{O}(1)$, and $F_{ru} \sim \mathcal{O}(r^{-2})$ implies that $\partial_r F_{ru} \sim \mathcal{O}(r^{-3})$. As previously mentioned, D_z is the covariant derivative with respect to the metric $\gamma_{z\bar{z}}$. Moreover, $D_{\bar{z}} F_{zu} = \partial_{\bar{z}} F_{zu}$. In the same manner, one finds the last component of equation (2.54),

$$\nabla^{\bar{z}} F_{\bar{z}u} = -\frac{1}{r^2} D^{\bar{z}} F_{u\bar{z}}^{(0)} + \mathcal{O}(r^{-3}). \tag{2.56}$$

By substituting the above into equation (2.54), we obtain up to leading order

$$\partial_u F_{ru}^{(2)} = -D^z F_{uz}^{(0)} - D^{\bar{z}} F_{u\bar{z}}^{(0)} - e^2 j_u^{(2)}. \tag{2.57}$$

Using (2.57), the future charge (2.52) can be rewritten as

$$\begin{aligned} Q_\varepsilon^+ &= \frac{1}{e^2}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,\varepsilon\left(D_{\bar z}F^{(0)}_{uz} + D_z F^{(0)}_{u\bar z}\right) + \int_{\mathcal{I}^+}\mathrm{d}u\,\mathrm{d}^2z\,\gamma_{z\bar z}\,\varepsilon j_u^{(2)} \\ &= \underbrace{-\frac{1}{e^2}\int_{\mathcal{I}^+}\mathrm{d}u\,\mathrm{d}^2z\,\left((\partial_{\bar z}\varepsilon)F^{(0)}_{uz} + (\partial_z\varepsilon)F^{(0)}_{u\bar z}\right)}_{Q_\varepsilon^{S+}} + \underbrace{\int_{\mathcal{I}^+}\mathrm{d}u\,\mathrm{d}^2z\,\gamma_{z\bar z}\,\varepsilon j_u^{(2)}}_{Q_\varepsilon^{H+}}, \end{aligned} \tag{2.58}$$

where the first term of equation (2.58) was integrated by parts. The future charge Q_ε^+ is now defined as the sum of two terms, a 'soft' charge Q_ε^{S+} and a 'hard' charge Q_ε^{H+}. The hard charge received its name because it contains j, the matter current for energy-carrying matter fields. For m outgoing hard particles that leave from point $(z_k^{\text{out}}, \bar z_k^{\text{out}})$ on the sphere, with $k = 1, \ldots, m$, the hard charge is given by

$$Q_\varepsilon^{H+} = \int_{\mathcal{I}^+}\mathrm{d}u\,\mathrm{d}^2z\,\gamma_{z\bar z}\,\varepsilon j_u^{(2)} = \sum_{k=1}^{m} Q_k^{\text{out}}\varepsilon(z_k^{\text{out}}, \bar z_k^{\text{out}}). \tag{2.59}$$

In absence of a magnetic field, i.e. for $B = 0$, the hard charge Q_ε^{H+} equals the electric multipole moments [19]. Multipole moments are not conserved: a point charge at the origin has a monopole moment, but will obtain dipole and higher moments if it starts to move away from the origin. Thus, the hard charge Q_ε^{H+} is not conserved by itself, but the addition of the soft charge Q_ε^{S+} results in a conserved quantity Q_ε^+. The soft charge Q_ε^{S+} contains a term of the form

$$N_z^+ \equiv \int_{-\infty}^{\infty}\mathrm{d}u\,F^{(0)}_{uz} = \frac{1}{2}\lim_{\omega\to 0}\int_{-\infty}^{\infty}\mathrm{d}u\,(e^{i\omega u} + e^{-i\omega u})F^{(0)}_{uz}. \tag{2.60}$$

If (2.60) is promoted to a quantum operator, it creates and annihilates outgoing soft photons (with zero energy). This is exactly the reason why one refers to Q_ε^{S+} as the soft charge. Since $F_{uz} \cong \partial_u A_z^{(0)}$, it follows that

$$N_z^+ = \int_{-\infty}^{\infty}\mathrm{d}u\,F^{(0)}_{uz} = A_z^{(0)}\Big|_{\mathcal{I}_+^+} - A_z^{(0)}\Big|_{\mathcal{I}_-^+}. \tag{2.61}$$

Thus, N_z is completely characterised by the z-component of the gauge fields existing at the boundaries of $\mathcal{I}^+$. Now assume that there are no magnetic charges at large r. Since the magnetic field is given by $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$, this implies that the $\mathcal{O}(1)$ term of $F_{z\bar z}$ vanishes, and

$$F_{z\bar z}|_{\mathcal{I}_\pm^+} = 0. \tag{2.62}$$

Moreover,

$$\begin{aligned}\partial_{\bar{z}} N_z^+ - \partial_z N_{\bar{z}}^+ &= \int_{-\infty}^{\infty} \mathrm{d}u \left[\partial_{\bar{z}} F_{uz}^{(0)} - \partial_z F_{u\bar{z}}^{(0)}\right] \\ &= -\int_{-\infty}^{\infty} \mathrm{d}u\, \partial_u F_{z\bar{z}}^{(0)} = -F_{z\bar{z}}^{(0)}\Big|_{\mathcal{I}_-^+}^{\mathcal{I}_+^+} = 0,\end{aligned} \tag{2.63}$$

where we used the Bianchi identity: $\partial_u F_{z\bar{z}} + \partial_z F_{\bar{z}u} + \partial_{\bar{z}} F_{uz} = 0$. Note that

$$N_z^+ \equiv e^2 \partial_z N^+(z, \bar{z}), \tag{2.64}$$

is a solution of (2.63), where N^+ is a real scalar. By using (2.61) and (2.64), the future soft charge Q_ε^{S+} is rephrased as

$$\begin{aligned}Q_\varepsilon^{S+} &= -\int_{\mathcal{I}_-^+} \mathrm{d}^2 z \left((\partial_{\bar{z}}\varepsilon)\partial_z N^+ + (\partial_z \varepsilon)\partial_{\bar{z}} N^+\right) \\ &= 2\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2 z\, N^+ \partial_z \partial_{\bar{z}} \varepsilon,\end{aligned} \tag{2.65}$$

where both terms were integrated by parts, with vanishing boundary terms. The coordinate representation of the past charge Q_ε^- can be obtained in an analogous manner. Thus, one finally has conserved charges $Q_\varepsilon^+ = Q_\varepsilon^-$ with

$$Q_\varepsilon^+ = 2\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2 z\, N^+ \partial_z \partial_{\bar{z}} \varepsilon + \int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2 z\, \gamma_{z\bar{z}}\, \varepsilon j_u^{(2)}, \tag{2.66}$$

$$Q_\varepsilon^- = \underbrace{2\int_{\mathcal{I}^-} \mathrm{d}v\, \mathrm{d}^2 z\, N^- \partial_z \partial_{\bar{z}} \varepsilon}_{Q_\varepsilon^{S-}} + \underbrace{\int_{\mathcal{I}^-} \mathrm{d}v\, \mathrm{d}^2 z\, \gamma_{z\bar{z}}\, \varepsilon j_v^{(2)}}_{Q_\varepsilon^{H-}}, \tag{2.67}$$

with N^- defined on $\mathcal{I}^-$ by

$$N_z^- \equiv e^2 \partial_z N^-, \quad \text{and} \quad N_z^- \equiv \int_{-\infty}^{\infty} \mathrm{d}v\, F_{vz}^{(0)}. \tag{2.68}$$

We will now show that the infinite number of conserved charges Q_ε^+ (2.66) results in an an infinite number of large gauge symmetries. The infinitesimal symmetry associated to a conserved charge Q on the fields Φ is given by Dirac bracket [33]

$$\delta\Phi = \Big\{Q, \Phi\Big\}. \tag{2.69}$$

Thus, it is necessary to switch to canonical Hamiltonian formalism.[2] The Dirac brackets are derived from the symplectic form Ω, where Ω is constructed from the phase space. The symplectic form Ω at $\mathcal{I}^+$ is derived in Appendix C.1, and is given by

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2}\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2 z \Big(\delta F_{uz}^{(0)} \wedge \delta A_{\bar{z}}^{(0)} + \delta F_{u\bar{z}}^{(0)} \wedge \delta A_z^{(0)}\Big). \tag{2.70}$$

[2]See Appendix B for a short refresher on Hamiltonian formalism.

If one proceeds naively with symplectic form (2.70), the final result will be off: the conserved charges will not generate the proper symmetries. Instead, one will obtain

$$\left\{Q^+_\varepsilon,\ A^{(0)}_z(u,z,\bar{z})\right\} = \frac{1}{2}\partial_z\varepsilon, \tag{2.71}$$

which differs by a factor 2 from the gauge symmetry (2.26). This shows that one needs to tread carefully at the boundaries of $\mathcal{I}^+$. This issue is solved by dividing $A^{(0)}_z$ in a u-dependent part $\hat{A}^{(0)}_z$ and a u-independent part $\partial_z\phi$ [1],

$$A^{(0)}_z(u,z,\bar{z}) = \hat{A}_z(u,z,\bar{z}) + \partial_z\phi(z,\bar{z}), \tag{2.72}$$

where (2.62) implies that $\hat{A}_z\big|_{\mathcal{I}^+_\pm} = 0$ and

$$\partial_z\phi \equiv \frac{1}{2}\Big[A^{(0)}_z\big|_{\mathcal{I}^+_+} + A^{(0)}_z\big|_{\mathcal{I}^+_-}\Big]. \tag{2.73}$$

Using condition (2.72) gives rise to symplectic form

$$\Omega_{\mathcal{I}^+} = \frac{2}{e^2}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\left(\delta\partial_u\hat{A}^{(0)}_z \wedge \delta\hat{A}^{(0)}_{\bar{z}}\right) - 2\int \mathrm{d}^2z\ \left(\delta\partial_z\phi \wedge \delta\partial_{\bar{z}}N^+\right), \tag{2.74}$$

where the details regarding this metamorphosis can also be found in Appendix C.1. Using symplectic form (2.74), the Dirac brackets are constructed in Appendix C.2. Here, the curious reader discovers that

$$\left\{\partial_{\bar{w}}N^+(w,\bar{w})\ ,\ \partial_z\phi(z,\bar{z})\right\} = -\frac{1}{2}\delta^2(z-w). \tag{2.75}$$

Note that $\partial_{\bar{z}}N^+$ is not coupled to $\hat{A}^{(0)}_z$ in the symplectic form (2.74), which implies that $\{\partial_{\bar{w}}N^+, \hat{A}^{(0)}_z\} = 0$. It is relevant to note that for a similar reason, $\{Q^{H+}_\varepsilon, A^{(0)}_z\} = 0$.

Now - lo and behold - one is ready to make the crucial observation that the conserved charges (2.66) do indeed generate large gauge transformations,

$$\begin{aligned}
\left\{Q^+_\varepsilon,\ A^{(0)}_z(u,z,\bar{z})\right\} =& \left\{Q^{S+}_\varepsilon,\ A^{(0)}_z(u,z,\bar{z})\right\} + \cancel{\left\{Q^{H+}_\varepsilon,\ A^{(0)}_z(u,z,\bar{z})\right\}} \\
=& 2\int \mathrm{d}^2w\ \partial_{\bar{w}}\partial_w\varepsilon(w,\bar{w})\left\{N^+(w,\bar{w})\ ,\ A^{(0)}_z(u,z,\bar{z})\right\} \\
=& -2\int \mathrm{d}^2w\ \partial_w\varepsilon(w,\bar{w})\left\{\partial_{\bar{w}}N^+(w,\bar{w})\ ,\ \partial_z\phi(z,\bar{z})\right\} \\
=& \int \mathrm{d}^2w\ \partial_w\varepsilon(w,\bar{w})\delta^2(z-w) \\
=& \partial_z\varepsilon(z,\bar{z}),
\end{aligned} \tag{2.76}$$

where one first integrates the soft charge Q^{S+}_ε by parts with respect to $\bar{w}$, in order to use bracket relation (2.75). Subsequently integrating over the delta function yields the remarkable result given above.

2.4 Ward Identity

In this section, we will derive the Ward identity that is associated with the large gauge symmetries. Quantum scattering amplitudes are denoted by $\langle \text{out}|\mathcal{S}|\text{in}\rangle$, where $\mathcal{S}$ evolves incoming states defined on $\mathcal{I}^-$ to outgoing states defined on $\mathcal{I}^+$. The conserved charges (2.66) commute with the Hamiltonian H. But since $\mathcal{S} \sim \exp(iHT)$ for $T \to \infty$ [34, 35], this implies that

$$Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^- = 0. \tag{2.77}$$

where $Q_\varepsilon^+ = Q_\varepsilon^-$. Thus, the charge conservation can be expressed as

$$\langle \text{out}|Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^-|\text{in}\rangle = 0. \tag{2.78}$$

The action of Q_ε^- on the in-state is given by

$$\begin{aligned} Q_\varepsilon^-|\text{in}\rangle &= Q_\varepsilon^{S-}|\text{in}\rangle + Q_\varepsilon^{H-}|\text{in}\rangle \\ &= 2\int \mathrm{d}^2z\, N^- \partial_z \partial_{\bar{z}} \varepsilon |\text{in}\rangle + \int \mathrm{d}v\, \mathrm{d}^2z\, \gamma_{z\bar{z}}\, \varepsilon j_v^{(2)} |\text{in}\rangle \\ &= 2\int \mathrm{d}^2z\, N^- \partial_z \partial_{\bar{z}} \varepsilon |\text{in}\rangle + \sum_{k=1}^{n} Q_k^{\text{in}} \varepsilon(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) |\text{in}\rangle. \end{aligned} \tag{2.79}$$

In a similar fashion,

$$\langle \text{out}|Q_\varepsilon^+ = 2\int \mathrm{d}^2z\, \partial_z \partial_{\bar{z}} \varepsilon \langle \text{out}|N^+(z,\bar{z}) + \sum_{k=1}^{m} Q_k^{\text{out}} \varepsilon(z_k^{\text{out}}, \bar{z}_k^{\text{out}}) \langle \text{out}|. \tag{2.80}$$

Thus, equation (2.78) yields Ward identity

$$\begin{aligned} &2\int \mathrm{d}^2z\, \partial_z \partial_{\bar{z}} \varepsilon(z,\bar{z}) \langle \text{out}|\big(N^+(z,\bar{z})\mathcal{S} - \mathcal{S} N^-(z,\bar{z})\big)|\text{in}\rangle \\ &\qquad = \left[\sum_{k=1}^{n} Q_k^{\text{in}} \varepsilon(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) - \sum_{k=1}^{m} Q_k^{\text{out}} \varepsilon(z_k^{\text{out}}, \bar{z}_k^{\text{out}})\right] \langle \text{out}|\mathcal{S}|\text{in}\rangle. \end{aligned} \tag{2.81}$$

There exists such a Ward identity for every function ε that satisfies (2.41), which results in an infinite number of Ward identities.
Now consider the easiest non-trivial choice for ε,

$$\varepsilon(w,\bar{w}) = \frac{1}{z-w}, \tag{2.82}$$

with

$$\partial_{\bar{z}} \varepsilon(w,\bar{w}) = 2\pi\delta^2(z-w). \tag{2.83}$$

It is incredibly sensible to pick (2.82) as our function ε, because the delta function in (2.83) will simplify ensuing formulae immensely. By integrating the LHS of equation (2.81) by parts with respect to z, one finds

$$-2\int \mathrm{d}^2z\, \partial_{\bar{z}}\varepsilon(z,\bar{z})\langle \mathrm{out}|\big(\partial_z N^+(z,\bar{z})\mathcal{S} - \mathcal{S}\partial_z N^-(z,\bar{z})\big)|\mathrm{in}\rangle$$
$$= \left[\sum_{k=1}^{n}\frac{Q_k^{\mathrm{in}}}{z-z_k^{\mathrm{in}}} - \sum_{k=1}^{m}\frac{Q_k^{\mathrm{out}}}{z-z_k^{\mathrm{out}}}\right]\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle. \tag{2.84}$$

Subsequently plugging in (2.83) yields

$$4\pi\langle \mathrm{out}|\partial_z N^+\mathcal{S} - \mathcal{S}\partial_z N^-|\mathrm{in}\rangle = \left[\sum_{k=1}^{n}\frac{Q_k^{\mathrm{in}}}{z-z_k^{\mathrm{in}}} - \sum_{k=1}^{m}\frac{Q_k^{\mathrm{out}}}{z-z_k^{\mathrm{out}}}\right]\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle. \tag{2.85}$$

Note that the sign change has its origin in

$$\partial_{\bar{z}}\varepsilon(z,\bar{z}) = \partial_{\bar{z}}\frac{1}{w-z} = -\partial_{\bar{z}}\frac{1}{z-w} = -2\pi\delta^2(z-w). \tag{2.86}$$

In order to move forward, the Ward identity (2.85) needs to be expressed in terms of a plane wave expansion. Referring back to equations (2.60) and (2.64), while noting that $F_{uz}^{(0)} = \partial_u A_z^{(0)}$, one has

$$\partial_z N^+ = \frac{1}{2e^2}\lim_{\omega\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}u\,(e^{i\omega u} + e^{-i\omega u})\partial_u A_z^{(0)}. \tag{2.87}$$

Thus, in order to find the plane wave expansion of $\partial_z N^+$, one needs the plane wave expansion of $A_z^{(0)}$. Near $\mathcal{I}^+$, the outgoing gauge field A_μ has the following plane wave expansion in coordinates $(t,\vec{x})$,

$$A_\mu(x) = e\sum_{\alpha=\pm}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega}\left[\varepsilon_\mu^{\alpha *}(\vec{q})a_\alpha^{\mathrm{out}}(\vec{q})e^{iq\cdot x} + \varepsilon_\mu^{\alpha}(\vec{q})a_\alpha^{\mathrm{out}\dagger}(\vec{q})e^{-iq\cdot x}\right], \tag{2.88}$$

where $\alpha = \pm$ are the two helicities of the photon, and the photon 4-momentum q is given by $q^\mu = \omega(1,\hat{x})$. In other symbols,

$$q^\mu = \frac{\omega}{1+z\bar{z}}\Big(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\Big). \tag{2.89}$$

Here, $q^2 = 0$ since the photon has zero mass. Furthermore, the polarisation tensors are

$$\varepsilon^{+\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(\bar{z},1,-i,-\bar{z}), \quad \text{and} \quad \varepsilon^{-\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(z,1,i,-z). \tag{2.90}$$

It can easily be checked that $q_\mu \varepsilon^{\pm\mu}(\vec{q}) = 0$ and $\varepsilon^{\alpha\mu}\varepsilon_\mu^{\beta *} = \delta_{\alpha\beta}$. By using equation (2.88) as a starting point, one acquires

$$A_z^{(0)}(u,z,\bar{z}) = -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty \mathrm{d}\omega \left[a_+^{\text{out}}(\omega\hat{x})e^{-i\omega u} - a_-^{\text{out}\dagger}(\omega\hat{x})e^{i\omega u}\right]. \tag{2.91}$$

Substituting the above in (2.87) results in

$$\partial_z N^+ = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{out}}(\omega\hat{x}) + \omega a_-^{\text{out}\dagger}(\omega\hat{x})\right]. \tag{2.92}$$

The entire derivation of the plane wave expansions of $A_z^{(0)}$ and $\partial_z N^+$ can be found in Appendix C.3. In a similar manner, one finds

$$\partial_z N^- = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{in}}(\omega\hat{x}) + \omega a_-^{\text{in}\dagger}(\omega\hat{x})\right]. \tag{2.93}$$

By plugging equations (2.92) and (2.93) into equation (2.85), the Ward identity is rephrased as

$$\begin{aligned}\lim_{\omega\to 0^+}&\left[\omega\langle\text{out}|\left(a_+^{\text{out}}(\omega\hat{x})\mathcal{S} - \mathcal{S}a_-^{\text{in}\dagger}(\omega\hat{x})\right)|\text{in}\rangle\right]\\ &= \sqrt{2}e(1+z\bar{z})\left[\sum_{k=1}^m \frac{Q_k^{\text{out}}}{z-z_k^{\text{out}}} - \sum_{k=1}^n \frac{Q_k^{\text{in}}}{z-z_k^{\text{in}}}\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle,\end{aligned} \tag{2.94}$$

where $a_+^{\text{in}}|\text{in}\rangle = \langle\text{out}|a_-^{\text{out}\dagger} = 0$.

2.5 Soft Photon Theorem

Surprisingly, the Ward identity (2.94) is actually equivalent to Weinberg's soft photon theorem. In this section, we will exhibit this extraordinary feature of the theory, but in order to do so we first need to describe the soft theorem.

Consider a scattering process with n incoming (with momenta $p_1^{\text{in}},\ldots,p_n^{\text{in}}$) and m outgoing complex scalars (with momenta $p_1^{\text{out}},\ldots,p_m^{\text{out}}$), represented by Figure 2.3. This scattering process is on-shell, i.e. $p^2 = -m^2$.

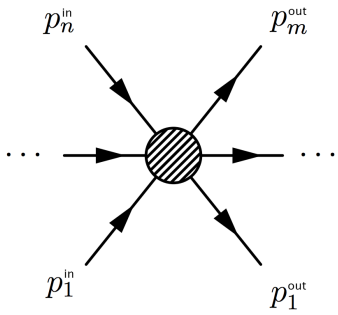


Figure 2.3: Feynman diagram representing $n \to m$ scattering of complex scalars.

Now consider the same on-shell scattering process, but then with an additional outgoing soft photon with momentum q and polarisation ε, where $\varepsilon^\mu q_\mu = 0$. In the soft $q \to 0$ limit, we obtain the leading order diagrams shown in Figure 2.4 [1].

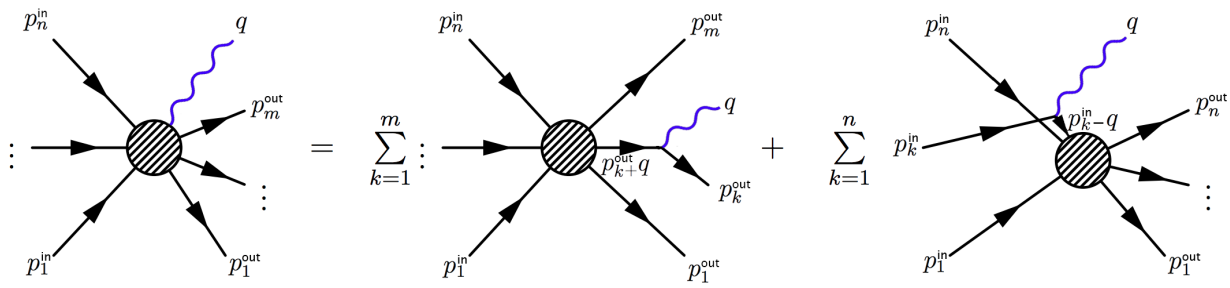


Figure 2.4: Leading order Feynman diagrams for $n \to m$ scattering, with an additional outgoing soft photon.

For a scalar particle of mass m, the Feynman rule for the propagator is given by

$$\xrightarrow{\quad p \quad} = \frac{-i}{p^2 + m^2 - i\epsilon}. \tag{2.95}$$

The interaction term of the Lagrangian has the form [1]

$$\mathcal{L}_{\text{int}} = -eA^\mu j_\mu, \tag{2.96}$$

where, for a scalar field of charge Q, we have charge current

$$j_\mu = iQ(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi), \tag{2.97}$$

with quantized scalar fields

$$\begin{aligned}\phi(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x}\right),\\ \phi^*(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p^\dagger e^{ip\cdot x} + b_p e^{-ip\cdot x}\right).\end{aligned} \tag{2.98}$$

By acting with a derivative on ϕ and ϕ^*, one obtains a factor of ip_μ which enters the Feynman vertex rule. An extra factor i comes from the expansion of $\exp(i\mathcal{L}_{\text{int}})$. Hence, in total, the vertex rule contains $2(-e)i^3Qp_\mu = 2ieQp_\mu$. For each external photon, a factor of the photon polarisation ε^μ is added, which gets contracted with the momentum [36]. Thus, up to $\mathcal{O}(q)$ corrections, the vertex factor is given by

$$2ieQp\cdot\varepsilon. \tag{2.99}$$

Now consider a photon that is attached to an outgoing particle with momentum p_k^{out}. External lines are on-shell, so $(p_k^{\text{out}})^2 = -m^2$ and $q^2 = 0$. The propagator is given by

$$\frac{-i}{(p_k^{\text{out}}+q)^2+m^2-i\epsilon} = \frac{-i}{\cancel{(p_k^{\text{out}})^2}+2p_k^{\text{out}}\cdot q+\cancel{q^2}+\cancel{m^2}-i\epsilon} = \frac{-i}{2p_k^{\text{out}}\cdot q-i\epsilon}. \tag{2.100}$$

Likewise, for an ingoing particle one has

$$\frac{-i}{(p_k^{\text{in}}-q)^2+m^2-i\epsilon} = \frac{-i}{\cancel{(p_k^{\text{in}})^2}-2p_k^{\text{in}}\cdot q+\cancel{q^2}+\cancel{m^2}-i\epsilon} = \frac{i}{2p_k^{\text{in}}\cdot q+i\epsilon}. \tag{2.101}$$

Note that Figure 2.4 does not contain any diagrams in which the soft photon attaches to an internal propagator. These diagrams do not contribute because internal propagators are never on-shell, which means that $p^2 \neq -m^2$. This implies that they lack the pole $p\cdot q$ and are thus negligible for $q\to 0$ [12].
Using above propagators, and vertex factor (2.99), one obtains the near-soft amplitude

$$\begin{aligned}&\mathcal{M}(q,p_1^{\text{out}},\dots,p_m^{\text{out}},p_1^{\text{in}},\dots,p_n^{\text{in}})\\ &\quad= \sum_{k=1}^{m}\mathcal{M}(p_1^{\text{out}},\dots,p_k^{\text{out}}+q,\dots,p_m^{\text{out}},p_1^{\text{in}},\dots,p_n^{\text{in}})\frac{2eQ_k^{\text{out}}p_k^{\text{out}}\cdot\varepsilon}{2p_k^{\text{out}}\cdot q-i\epsilon}\\ &\qquad- \sum_{k=1}^{n}\mathcal{M}(p_1^{\text{out}},\dots,p_m^{\text{out}},p_1^{\text{in}},\dots,p_k^{\text{in}}+q,\dots,p_n^{\text{in}})\frac{2eQ_k^{\text{in}}p_k^{\text{in}}\cdot\varepsilon}{2p_k^{\text{in}}\cdot q+i\epsilon}.\end{aligned} \tag{2.102}$$

Hence, for $q\to 0$,

$$\begin{aligned}&\mathcal{M}(q,p_1^{\text{out}},\dots,p_m^{\text{out}},p_1^{\text{in}},\dots,p_n^{\text{in}})\\ &\quad= e\left[\sum_{k=1}^{m}\frac{Q_k^{\text{out}}p_k^{\text{out}}\cdot\varepsilon}{p_k^{\text{out}}\cdot q}-\sum_{k=1}^{n}\frac{Q_k^{\text{in}}p_k^{\text{in}}\cdot\varepsilon}{p_k^{\text{in}}\cdot q}\right]\mathcal{M}(p_1^{\text{out}},\dots,p_m^{\text{out}},p_1^{\text{in}},\dots,p_n^{\text{in}}).\end{aligned} \tag{2.103}$$

This is known as Weinberg's soft photon theorem. After changing notation and multiplying both sides of (2.103) by $\omega \to 0$, one acquires

$$\begin{aligned}
&\lim_{\omega\to 0}\left[\omega\langle\text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S}|\text{in}\rangle\right] \\
&\qquad = e\lim_{\omega\to 0}\left[\sum_{k=1}^{m}\frac{\omega Q_k^{\text{out}}p_k^{\text{out}}\cdot\varepsilon^+}{p_k^{\text{out}}\cdot q}-\sum_{k=1}^{n}\frac{\omega Q_k^{\text{in}}p_k^{\text{in}}\cdot\varepsilon^+}{p_k^{\text{in}}\cdot q}\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle,
\end{aligned} \tag{2.104}$$

where the in- and out-state are given by

$$|\text{in}\rangle = |p_1^{\text{in}},\dots,p_n^{\text{in}}\rangle, \qquad \langle\text{out}| = \langle p_1^{\text{out}},\dots,p_m^{\text{out}}|. \tag{2.105}$$

Since the momentum of the soft photon is given by $q^\mu = \omega(1,\hat{x})$, the multiplication by ω yields a finite limit on the RHS of (2.104) in the soft limit.
We could also have considered the same scattering process, but then with an incoming soft photon instead of an outgoing one. This would have resulted in

$$\lim_{\omega\to 0}\left[\omega\langle\text{out}|\mathcal{S}a_-^{\text{in}\dagger}(\vec{q})|\text{in}\rangle\right] = -\lim_{\omega\to 0}\left[\omega\langle\text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S}|\text{in}\rangle\right]. \tag{2.106}$$

In order to show that the soft theorem is equivalent to the Ward identity (2.94), the momentum of each outgoing hard particle needs to be be defined in terms of its energy E_k and a point on the sphere $(z_k,\bar{z}_k)$,

$$(p_k^{\text{out}})^\mu = \frac{E_k^{\text{out}}}{1+z_k^{\text{out}}\bar{z}_k^{\text{out}}}\left(1, z_k^{\text{out}}+\bar{z}_k^{\text{out}}, -i(z_k^{\text{out}}-\bar{z}_k^{\text{out}}), 1-z_k^{\text{out}}\bar{z}_k^{\text{out}}\right), \tag{2.107}$$

whereas the momentum of the incoming hard particle is

$$(p_k^{\text{in}})^\mu = \frac{E_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}\left(1, z_k^{\text{in}}+\bar{z}_k^{\text{in}}, -i(z_k^{\text{in}}-\bar{z}_k^{\text{in}}), 1-z_k^{\text{in}}\bar{z}_k^{\text{in}}\right). \tag{2.108}$$

From the definitions of ε^+ (2.90) and q^μ (2.89), one can determine

$$\begin{aligned}
\varepsilon^{+\mu}(\vec{q}) &= \frac{1}{\sqrt{2}\omega}\,\partial_z\left[(1+z\bar{z})q^\mu\right], \\
p_k^{\text{out}}\cdot q &= -\frac{2\omega E_k^{\text{out}}}{(1+z\bar{z})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})}\left|z-z_k^{\text{out}}\right|^2.
\end{aligned} \tag{2.109}$$

This implies

$$\begin{aligned}
\frac{p_k^{\text{out}}\cdot\varepsilon^+}{p_k^{\text{out}}\cdot q} &= \frac{1}{\sqrt{2}\omega}\frac{\partial_z\left[(1+z\bar{z})p_k^{\text{out}}\cdot q\right]}{p_k^{\text{out}}\cdot q} \\
&= \frac{1}{\sqrt{2}\omega}(1+z\bar{z})\,\partial_z\log\left[(1+z\bar{z})p_k^{\text{out}}\cdot q\right] \\
&= \frac{1}{\sqrt{2}\omega}(1+z\bar{z})\left[\cancel{\partial_z\log\left(-\frac{2\omega E_k^{\text{out}}}{(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})}\right)}+\partial_z\log\left|z-z_k^{\text{out}}\right|^2\right] \\
&= \frac{1}{\sqrt{2}\omega}\frac{1+z\bar{z}}{z-z_k^{\text{out}}}.
\end{aligned} \tag{2.110}$$

Likewise,

$$\frac{p_k^{\text{in}} \cdot \varepsilon^+}{p_k^{\text{in}} \cdot q} = \frac{1}{\sqrt{2}\omega} \frac{1 + z\bar{z}}{z - z_k^{\text{in}}}. \tag{2.111}$$

Now - lo and behold - by using (2.110) and (2.111), one can finally express the soft theorem (2.104) as

$$\lim_{\omega \to 0} \left[\omega \langle \text{out} | a_+^{\text{out}}(\vec{q}) \mathcal{S} | \text{in} \rangle\right] = \frac{e}{\sqrt{2}}(1 + z\bar{z}) \left[\sum_{k=1}^{m} \frac{Q_k^{\text{out}}}{z - z_k^{\text{out}}} - \sum_{k=1}^{n} \frac{Q_k^{\text{in}}}{z - z_k^{\text{in}}} \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle. \tag{2.112}$$

The result above is absolutely brilliant: referring back to (2.106), it is exactly equivalent to the Ward identity (2.94). It implies that Weinberg's soft photon theorem can be understood as a consequence of the large gauge symmetries.

By running the argument backwards, we could also have shown that the soft theorem (2.104) yields the Ward identity with

$$\varepsilon(w, \bar{w}) = \frac{1}{z - w}. \tag{2.113}$$

However, note that the soft charge (2.65) is linear in ε and that any function $\varepsilon(z, \bar{z})$ can be expressed as

$$\varepsilon(w, \bar{w}) = \frac{1}{2\pi} \int \mathrm{d}^2 z \, \varepsilon(z, \bar{z}) \partial_{\bar{z}} \frac{1}{z - w}. \tag{2.114}$$

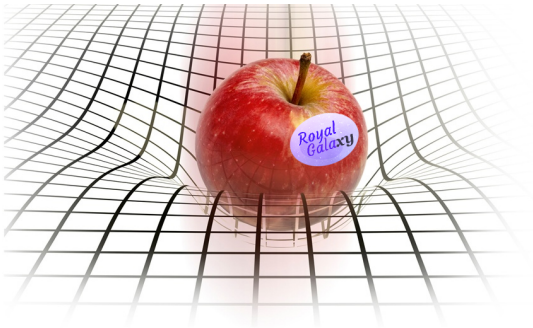
Thus, the Ward identity for the asymptotic symmetries and Weinberg's soft photon theorem are equivalent for any function $\varepsilon(z, \bar{z})$ [2].

The third corner of the triangle is given by the memory effect. An electromagnetic analogue of the gravitational memory effect, and a method of measuring it, are presented in [37–39]. The electromagnetic memory effect is manifested as an electromagnetic wave kick, a change in relative phases between two test particles.

Chapter 3

Gravity

Even if I knew that tomorrow the world would go to pieces,
I would still plant my apple tree.



Karl Lotz (1944).

3.1 Introduction

Having found the triangular equivalence relation for QED in the previous chapter, we will now discuss the infrared triangle in the context of gravity (see Figure 3.1), where we still make the assumption that we are only dealing with massless fields. For gravity, the asymptotic symmetry group is given by supertranslations, which are angle dependent translations. Most of this chapter is based on work presented in [4–6].

This chapter is organised as follows. First, in Section 3.2, we describe the asymptotically flat geometry, and present the asymptotic boundary conditions. Subsequently, in Section 3.3 we discuss the supertranslations and their associated conserved charges. In Section 3.4, we construct the Ward identities associated to the asymptotic symmetries. Finally, we show in Section 3.5, that the Ward identity is equivalent to Weinberg's soft graviton theorem. Lastly, in Section 3.7, we extend the asymptotic symmetry group to superrotations for a bit and introduce the conserved superrotation charges.

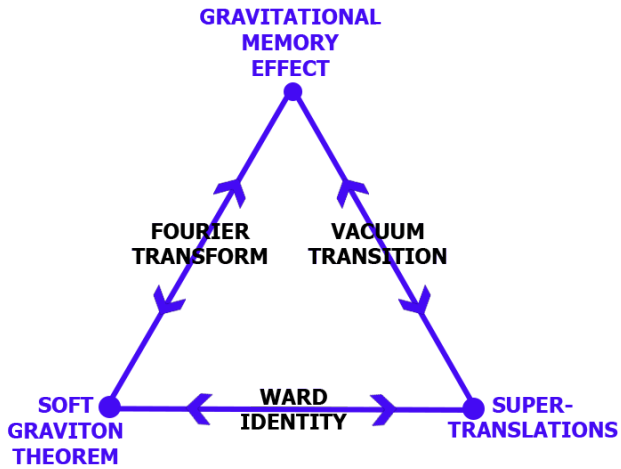


Figure 3.1: The infrared triangle in the context of gravity.

3.2 Preliminaries

In the previous chapter, we worked in flat Minkowski space, where the retarded line element near $\mathcal{I}^+$ was given by

$$\mathrm{d}s^2 = -\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}. \tag{3.1}$$

In order to study gravity, we need to introduce a line element that is asymptotically flat. In other words, it it necessary to add subleading corrections to the flat metric that allow for all interesting solutions such as gravity waves, while excluding solutions with infinite energy. Bondi, van der Burg, Metzner and Sachs (BMS) derived in [13, 14] that in retarded Bondi coordinates $(u, r, z, \bar{z})$, with $u = t - r + \ldots$, the asymptotic form of the metric has to be of the following form:

$$\begin{aligned}\mathrm{d}s^2 = &-\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}\\ &+\frac{2m_B}{r}\,\mathrm{d}u^2 + rC_{zz}\,\mathrm{d}z^2 + rC_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z}^2 + D^zC_{zz}\,\mathrm{d}u\,\mathrm{d}z + D^{\bar{z}}C_{\bar{z}\bar{z}}\,\mathrm{d}u\,\mathrm{d}\bar{z}\\ &+\frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)\mathrm{d}u\,\mathrm{d}z + \text{c.c.} + \ldots,\end{aligned} \tag{3.2}$$

where 'c.c.' stands for 'complex conjugates', the horizontal dots denotes further subleading terms, and where m_B, C_{AB}, and N_A are dependent on $(u, z, \bar{z})$, but not on r. While deriving line element (3.2), one imposed gauge conditions $g_{rr} = 0$, $g_{rA} = 0$ and

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0, \tag{3.3}$$

where $A, B = (z, \bar{z})$. From the line element (3.2), one can read off the large r falloffs for the metric components,

$$\begin{aligned} g_{uu} &= -1 + \frac{2m_B}{r} + \mathcal{O}(r^{-2}), & g_{rr} &= g_{rz} = 0, \\ g_{ur} &= -1 + \mathcal{O}(r^{-2}), & g_{zz} &= rC_{zz} + \mathcal{O}(1), \\ g_{uz} &= \frac{1}{2}D^z C_{zz} + \mathcal{O}(r^{-1}), & g_{z\bar{z}} &= r^2\gamma_{z\bar{z}} + \mathcal{O}(1). \end{aligned} \tag{3.4}$$

The quantity m_B is called the Bondi mass aspect, where the B stands for 'Bondi' and is not an index. The total Bondi mass is obtained by integrating the Bondi mass aspect m_B over S^2. For a black hole, $m_B = GM$. The quantity N_z is known as the angular momentum aspect. If one first contracts N_z with a rotational vector field, and then integrates it over S^2, the total angular momentum is obtained. The quantity C_{AB} is the gauge potential of a gravitational wave. The 'Bondi news tensor' is defined as

$$N_{AB} = \partial_u C_{AB}, \tag{3.5}$$

and is quite similar to the electromagnetic field strength $F_{\mu\nu}$; by squaring N_{zz}, one obtains a quantity that is proportional to the energy flux at $\mathcal{I}^+$. Changes in the field directly affect N_{AB}, and vice versa. If $N_{AB} = 0$, the field remains static. Since N_{AB} contains all news there is about upcoming changes in the field, it is very appropriately named [13]. In Appendix D.1, additional information regarding the metric is given. Furthermore, Appendix D.2 contains the Christoffel symbols on $\mathcal{I}^+$.
On past null infinity $\mathcal{I}^-$, the advanced line element is given by

$$\begin{aligned} \mathrm{d}s^2 = &- \mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} \\ &+ \frac{2m_B}{r}\,\mathrm{d}v^2 + rC_{zz}\,\mathrm{d}z^2 + rC_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z}^2 - D^z C_{zz}\,\mathrm{d}v\,\mathrm{d}z - D^{\bar{z}} C_{\bar{z}\bar{z}}\,\mathrm{d}v\,\mathrm{d}\bar{z} \\ &- \frac{1}{r}\left(\frac{4}{3}(N_z - v\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)\mathrm{d}v\,\mathrm{d}z + c.c. + \ldots, \end{aligned} \tag{3.6}$$

where $v = t + r + \ldots$, and m_B, C_{AB} and N_A depend on $(v, z, \bar{z})$.

3.2.1 Asymptotic Boundary Conditions

Since it is necessary to tread lightly at the boundaries of $\mathcal{I}^+$ and $\mathcal{I}^-$, we need to impose some asymptotic boundary conditions. We will make use of the ensuing boundary conditions and fields in subsequent sections. Since $N_{zz}N^{zz}$ is proportional to the energy flux, it follows that in order to have a finite energy configuration, one has $N_{zz}\big|_{\mathcal{I}^+_\pm} = 0$.

We are working in asymptotically flat space, so the curvature vanishes at the boundaries. This implies boundary condition [4,5]

$$\left[D_z^2 C_{\bar{z}\bar{z}} - D_{\bar{z}}^2 C_{zz}\right]_{\mathcal{I}_\pm^+} = 0. \tag{3.7}$$

For $\mathcal{I}_-^+$, one can easily check that (3.7) is solved by

$$C_{zz}\Big|_{\mathcal{I}_-^+} = D_z^2 C(z,\bar{z}), \tag{3.8}$$

where $C(z,\bar{z})$ is a real boundary field. Now define another real boundary field N^+ as follows

$$\begin{aligned} D_z^2 N^+(z,\bar{z}) &\equiv \int_{-\infty}^{\infty} \mathrm{d}u\, N_{zz} = \int_{-\infty}^{\infty} \mathrm{d}u\, \partial_u C_{zz} \\ &= C_{zz}\Big|_{\mathcal{I}_+^+} - C_{zz}\Big|_{\mathcal{I}_-^+}, \end{aligned} \tag{3.9}$$

which can be re-expressed as

$$D_z^2 N^+ = \frac{1}{2} \lim_{\omega\to 0^+} \int_{-\infty}^{\infty} \mathrm{d}u (e^{i\omega u} + e^{-i\omega u}) N_{zz}. \tag{3.10}$$

Thus, when promoted to a quantum operator, $D_z^2 N^+$ creates and annihilates outgoing zero energy gravitons. Finally, the antipodal matching conditions are given by [4,5,40]

$$\begin{aligned} m_B(z,\bar{z})|_{\mathcal{I}_-^+} &= m_B(z,\bar{z})|_{\mathcal{I}_+^-}, \\ C(z,\bar{z})|_{\mathcal{I}_-^+} &= C(z,\bar{z})|_{\mathcal{I}_+^-}, \\ N_A(z,\bar{z})|_{\mathcal{I}_-^+} &= N_A(z,\bar{z})|_{\mathcal{I}_+^-}. \end{aligned} \tag{3.11}$$

It is expected that these matching conditions are the only Lorentz and CPT invariant choice [40].

3.3 Supertranslations

In asymptotically flat spacetimes, symmetries are generated by diffeomorphisms ζ that satisfy both the gauge conditions (3.3) and the large r falloffs for the metric components (3.4). Since we are working with a metric that is flat up to leading order, one would naively expect to obtain the Poincaré group, which is made up of four translations in each of the spacetime directions, plus three boosts and three spatial rotations.
Surprisingly, though, at the boundary of asymptotically flat spacetime, one does not only obtain the finite-dimensional Poincaré group, but also an additional infinite set

of (mutually commuting) angle dependent translations along null infinity, the so-called 'supertranslations' [13,14]. Together, these transformations are known as the BMS group, which consists of all asymptotic symmetries.
In this section, we will determine the action of supertranslations on the asymptotic data C_{zz}, N_{zz} and m_B. Subsequently, we will show that the antipodal matching conditions (3.11) imply an infinite number of conserved charges Q_f^+. Finally, we will check that the conserved charges generate supertranslations by deriving the Dirac bracket action of Q_f^+.

Since the focus lies on supertranslations, we make the simplifying restriction that the diffeomorphisms ζ that generate the asymptotic symmetries have the following large r falloffs

$$\zeta^u, \zeta^r \sim \mathcal{O}(1), \qquad \zeta^z, \zeta^{\bar{z}} \sim \mathcal{O}(r^{-1}), \tag{3.12}$$

thereby eliminating boosts and rotations which grow with r at infinity. Up to leading order, the structure of the diffeomorphisms ζ is given by (see Appendix D.5.1)

$$\zeta = f\partial_u + D^z D_z f \partial_r - \frac{1}{r}(D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + \ldots, \qquad f = f(z, \bar{z}). \tag{3.13}$$

This implies that the asymptotic line element (3.2) is preserved under infinitesimal supertranslations

$$\begin{aligned} u &\to u - f, & r &\to r - D^z D_z f, \\ z &\to z + \frac{1}{r} D^z f, & \bar{z} &\to \bar{z} + \frac{1}{r} D^{\bar{z}} f. \end{aligned} \tag{3.14}$$

The infinitesimal supertranslation of the metric is given by $\delta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu}$, where the Lie derivative of the metric with respect to the vector ζ is specified by [41]

$$\mathcal{L}_\zeta g_{\mu\nu} = \zeta^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \zeta^\rho + g_{\nu\rho}\partial_\mu \zeta^\rho. \tag{3.15}$$

All one needs to do in order to determine the asymptotic symmetries of C_{zz}, N_{zz} and m_B, is compute the Lie derivative of the relevant metric component, and subsequently extract the coefficient that has the appropriate order. As an example, we will determine $\mathcal{L}_\zeta C_{zz}$. Note that $\mathcal{L}_\zeta C_{zz} = \frac{1}{r} \cdot \mathcal{L}_\zeta g_{zz}\Big|_{\mathcal{O}(r)}$, with

$$\begin{aligned} \mathcal{L}_\zeta g_{zz} =& \zeta^\rho \partial_\rho g_{zz} + 2 g_{z\rho}\partial_z \zeta^\rho \\ =& \zeta^u \partial_u g_{zz} + 2 g_{z\bar{z}} \partial_z \zeta^{\bar{z}} + \mathcal{O}(1) \\ =& r\left[f\partial_u C_{zz} - 2\gamma_{z\bar{z}}\partial_z(\gamma^{z\bar{z}} D_z f)\right] + \mathcal{O}(1) \\ =& r\left[f\partial_u C_{zz} - 2(\partial_z D_z f + (\gamma_{z\bar{z}}\partial_z \gamma^{z\bar{z}}) D_z f)\right] + \mathcal{O}(1) \\ =& r\left(f\partial_u C_{zz} - 2D_z^2 f\right) + \mathcal{O}(1), \end{aligned} \tag{3.16}$$

where it was used that $D_z^2 f = \partial_z \partial_z f - (\gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}}) \partial_z f$ and $(\gamma_{z\bar{z}} \partial_z \gamma^{z\bar{z}}) = -(\gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}})$. Thus, one finds

$$\mathcal{L}_\zeta C_{zz} = f N_{zz} - 2D_z^2 f. \tag{3.17}$$

Differentiating (3.17) with respect to u yields

$$\mathcal{L}_\zeta N_{zz} = f \partial_u N_{zz}. \tag{3.18}$$

In a similar way, one can derive $\mathcal{L}_\zeta m_B$ (see Appendix D.5.1), which gives

$$\mathcal{L}_\zeta m_B = f \partial_u m_B + \frac{1}{4} \left(N^{zz} D_z^2 f + 2 D_z N^{zz} D_z f + \text{c.c.} \right). \tag{3.19}$$

Note that for $m_B = N_{zz} = C_{zz} = 0$, the asymptotically flat metric (3.2) transforms into the metric for flat Minkowski space (3.1). Now observe from transformations (3.18) and (3.19) that supertranslated flat Minkowski spacetime still has zero m_B and N_{zz}. This is exactly what is expected, since supertranslating a spacetime should not result in the creation of physical mass or gravitational waves.
However, peculiarly enough, it follows from (3.17) that, for a non-constant f, supertranslating a flat spacetime (with $C_{zz} = 0$) does yield a spacetime with non-zero C_{zz}. This implies that by supertranslating flat spacetime, one obtains a spacetime that is physically inequivalent. In other words, supertranslation symmetry is spontaneously broken by the vacuum (flat spacetime), and the vacuum is degenerate. Goldstone's theorem states that a spontaneously broken vacuum implies the existence of soft particles with zero mass [42]. It follows from (3.8) and (3.17) that, at $\mathcal{I}_-^+$,

$$\mathcal{L}_\zeta C = -2f, \tag{3.20}$$

for a vacuum with $N_{zz} = 0$. This implies that different flat spacetimes (or vacua) are related by supertranslations for which $C \to C + 2f$.
Of course, one could have done the same analysis on $\mathcal{I}^-$. Up to leading order, the diffeomorphism is given by

$$\zeta = f \partial_v - \frac{1}{2} \left(D_z D^z f + D_{\bar{z}} D^{\bar{z}} f \right) \partial_r + \frac{1}{r} (D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + \ldots, \qquad f = f(z, \bar{z}), \tag{3.21}$$

which, on $\mathcal{I}^-$, generates asymptotic symmetries

$$\begin{aligned} \mathcal{L}_\zeta C_{zz} &= f \partial_v C_{zz} + 2D_z^2 f, \\ \mathcal{L}_\zeta N_{zz} &= f \partial_v N_{zz}, \end{aligned} \tag{3.22}$$

where the Lorentz invariant matching condition for f is given by [4,5,40]

$$f(z, \bar{z})|_{\mathcal{I}_-^+} = f(z, \bar{z})|_{\mathcal{I}_+^-}. \tag{3.23}$$

Having found the action of the supertranslations, we will now construct the associated conserved charges. The supertranslation charges are given by [4,5]

$$Q_f^+ = \frac{1}{4\pi G}\int_{\mathcal{I}_-^+} \mathrm{d}^2z\, \gamma_{z\bar{z}} f m_B, \tag{3.24}$$
$$Q_f^- = \frac{1}{4\pi G}\int_{\mathcal{I}_+^-} \mathrm{d}^2z\, \gamma_{z\bar{z}} f m_B.$$

Due to matching conditions (3.11) and (3.23), one finds

$$Q_f^+ = Q_f^-. \tag{3.25}$$

Since there exist infinitely many supertranslations, there is also an infinite number of conserved charges. In order to rewrite the future charge as a sum of a hard charge Q_f^{H+} and a soft charge Q_f^{S+}, one integrates the future charge (3.24) by parts with respect to u, which gives

$$Q_f^+ = -\frac{1}{4\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2z\, \gamma_{z\bar{z}} f \partial_u m_B, \tag{3.26}$$

where both $\gamma_{z\bar{z}}$ and f do not depend on u, and the boundary term vanishes. By computing the leading order uu-component of the Einstein's equation, one obtains a constraint on m_B (see Appendix D.3), which is given by

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu}, \tag{3.27}$$

with

$$T_{uu} \equiv \frac{1}{4} N_{zz} N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T_{uu}^M\right]. \tag{3.28}$$

Here, T_{uu} denotes the total outgoing radiation energy flux at $\mathcal{I}^+$. Plugging constraint (3.27) into future charge (3.26) yields

$$Q_f^+ = \frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2z \gamma_{z\bar{z}} f\Big[N_{zz}N^{zz} - (D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}})\Big], \tag{3.29}$$

where $T_{uu}^M = 0$, since we assumed that there were no matter fields.
The soft charge Q_f^{S+}, which is given by the term linear in N_{zz}, can be re-expressed as

$$\begin{aligned} Q_f^{S+} &= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\, \mathrm{d}^2z \gamma_{z\bar{z}} f\left(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right) \\ &= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2z \gamma^{z\bar{z}} f\left(D_z^2 \int_{-\infty}^{\infty} \mathrm{d}u N_{\bar{z}\bar{z}} + D_{\bar{z}}^2 \int_{-\infty}^{\infty} \mathrm{d}u N_{zz}\right) \\ &= -\frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2z \gamma^{z\bar{z}} f D_z^2 D_{\bar{z}}^2 N^+. \end{aligned} \tag{3.30}$$

In order to obtain the last line of (3.30), we used the definition of N^+ (3.9) and the boundary constraint on C_{zz} (3.7) in order to find that

$$D_{\bar{z}}^2(D_z^2 N^+) = D_{\bar{z}}^2 C_{zz}\Big|_{\mathcal{I}_+^+} - D_{\bar{z}}^2 C_{zz}\Big|_{\mathcal{I}_-^+} = D_z^2 C_{\bar{z}\bar{z}}\Big|_{\mathcal{I}_+^+} - D_z^2 C_{\bar{z}\bar{z}}\Big|_{\mathcal{I}_-^+} = D_z^2(D_{\bar{z}}^2 N^+). \tag{3.31}$$

Thus, the future charge can be rewritten as

$$Q_f^+ = \underbrace{\frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,\gamma_{z\bar{z}} f N_{zz} N^{zz}}_{Q_f^{H+}} - \underbrace{\frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2z\,\gamma^{z\bar{z}} f D_z^2 D_{\bar{z}}^2 N^+}_{Q_f^{S+}}. \tag{3.32}$$

The future charge is now once again divided into a soft charge Q_f^{S+} and a hard charge Q_f^{H+}. When promoted to a quantum operator, the soft charge creates and annihilates outgoing zero energy gravitons [5]. For m outgoing hard particles that leave from point $(z_k^{\text{out}}, \bar{z}_k^{\text{out}})$ on the sphere, with $k = 1, \ldots, m$, the hard charge is given by

$$Q_f^{H+} = \sum_{k=1}^{m} E_k^{\text{out}} f(z_k^{\text{out}}, \bar{z}_k^{\text{out}}). \tag{3.33}$$

For the past charge, one similarly finds

$$Q_f^- = \underbrace{\frac{1}{16\pi G}\int_{\mathcal{I}^-} \mathrm{d}v\,\mathrm{d}^2z\,\gamma_{z\bar{z}} f N_{zz} N^{zz}}_{Q_f^{H-}} + \underbrace{\frac{1}{8\pi G}\int_{\mathcal{I}^-} \mathrm{d}^2z\,\gamma^{z\bar{z}} f D_z^2 D_{\bar{z}}^2 N^-}_{Q_f^{S-}}. \tag{3.34}$$

We will now determine the bracket action of the conserved charge, $\{Q_f^+, \ \} = \delta_f$, and show that it generates the supertranslations. In order to determine the Dirac bracket action of Q_f^+, we first need to invert the symplectic form

$$\Omega_{\mathcal{I}^+} = -\frac{1}{16\pi G}\int \mathrm{d}u\,\mathrm{d}^2z\gamma^{z\bar{z}}\left(\delta N_{\bar{z}\bar{z}} \wedge \delta C_{zz} + \delta N_{zz} \wedge \delta C_{\bar{z}\bar{z}}\right) \tag{3.35}$$

and compute a variety of Dirac brackets. This is quite a lengthy process, so all the details are placed in Appendix D.4. The results are given by

$$\begin{aligned}
\left\{N_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} &= 16\pi G\gamma_{z\bar{z}}\delta(u-u')\delta^2(z-w),\\
\left\{N_{\bar{z}\bar{z}}(u,z,\bar{z}), N_{ww}(u',w,\bar{w})\right\} &= -16\pi G\gamma_{z\bar{z}}\partial_u\delta(u-u')\delta^2(z-w),\\
\left\{N_{zz}(u,z,\bar{z}), N_{ww}(u',w,\bar{w})\right\} &= 0,\\
\left\{C_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} &= 8\pi G\gamma_{z\bar{z}}\Theta(u-u')\delta^2(z-w),\\
\left\{C(z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} &= -8GD_w^2\left(S\ln|z-w|^2\right),\\
\left\{C(z,\bar{z}), N_{ww}(u',w,\bar{w})\right\} &= 0,\\
\left\{N^+(z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} &= 16GD_w^2\left(S\ln|z-w|^2\right),\\
\left\{N^+(z,\bar{z}), C(w,\bar{w})\right\} &= 16G\left(S\ln|z-w|^2\right),\\
\left\{N^+(z,\bar{z}), N_{ww}(u',w,\bar{w})\right\} &= 0,
\end{aligned} \tag{3.36}$$

where the function S is defined by

$$S \equiv \sin^2 \frac{\Delta\Theta}{2} = \frac{(z-w)(\bar{z}-\bar{w})}{(1+z\bar{z})(1+w\bar{w})}, \tag{3.37}$$

with property

$$D^2_{\bar{w}} D^2_w \left(S \ln |z-w|^2\right) = \pi \gamma_{w\bar{w}} \delta^2(z-w). \tag{3.38}$$

Using Dirac bracket relations (3.36), one finds

$$\begin{aligned}
\{Q^+_f, C_{zz}(u,z,\bar{z})\} =& \frac{1}{16\pi G} \int_{\mathcal{I}^+} \mathrm{d}u' \, \mathrm{d}^2 w \, \gamma^{w\bar{w}} f N_{ww} \{N_{\bar{w}\bar{w}}(u',w,\bar{w}), C_{zz}(u,z,\bar{z})\} \\
& - \frac{1}{8\pi G} \int_{\mathcal{I}^+} \mathrm{d}^2 w \, \gamma^{w\bar{w}} f D^2_w D^2_{\bar{w}} \{N^+(w,\bar{w}), C_{zz}(u,z,\bar{z})\} \\
=& \int_{\mathcal{I}^+} \mathrm{d}u' \, \mathrm{d}^2 w \, \gamma^{w\bar{w}} \gamma_{z\bar{z}} f \partial_{u'} C_{ww} \delta(u-u') \delta^2(z-w) \\
& - 2 \int_{\mathcal{I}^+} \mathrm{d}^2 w \, f D^2_w \delta^2(z-w) \\
=& f \partial_u C_{zz} - 2 \int_{\mathcal{I}^+} \mathrm{d}^2 w \, (D^2_w f) \delta^2(z-w) \\
=& f \partial_u C_{zz} - 2 D^2_z f,
\end{aligned} \tag{3.39}$$

where the property of S (3.38) was used, and the integral was integrated by parts twice, with vanishing boundary terms.
In a similar fashion, one can compute the bracket action of Q^+_f on N_{zz}, N^+ and C (see Appendix D.5.3), which yields

$$\begin{aligned}
\{Q^+_f, N_{zz}\} &= f \partial_u N_{zz}, \\
\{Q^+_f, N^+\} &= 0, \\
\{Q^+_f, C\} &= -2f.
\end{aligned} \tag{3.40}$$

Lo and behold, results (3.39) and (3.40) correspond exactly to the asymptotic symmetries (3.17), (3.18) and (3.20) that we derived earlier.

As a remark, if we would have computed the Dirac bracket relations without having composed condition (3.7) on C_{zz}, the final result would have been off: the conserved charges would have failed to generate the supertranslation symmetries. For example, one would have obtained

$$\{Q^+_f, C_{zz}\} = f N_{zz} - D^2_z f \neq \mathcal{L}_f C_{zz}. \tag{3.41}$$

This shows that it is extremely important to take meticulous care at the boundaries of spacetime.

3.4 Ward Identity

In this section, we will derive the Ward identity associated to the asymptotic symmetries. The conserved charges obey

$$\left\{Q_f^+, Q_{f'}^+\right\} = 0. \tag{3.42}$$

Since the Hamiltonian is given by Q_f^+ (3.32) with $f = 1$, and $\mathcal{S} \sim \exp(iHT)$ for $T \to \infty$, it now follows that the conserved charges commute with the $\mathcal{S}$-matrix [4]. Thus, the charge conservation (3.25) can be expressed as

$$\langle \text{out}|Q_f^+\mathcal{S} - \mathcal{S}Q_f^-|\text{in}\rangle = 0. \tag{3.43}$$

Since function $f(z,\bar{z})$ can be taken to be any function that satisfies boundary condition (3.11), we choose to work with the easiest non-trivial function:

$$f(w,\bar{w}) = \frac{1}{z-w}, \quad \text{with} \quad \partial_{\bar{z}} f(w,\bar{w}) = 2\pi\delta^2(z-w). \tag{3.44}$$

With this particular choice for f, the soft charge (3.30) becomes

$$\begin{aligned} Q_f^{S+} &= -\frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2z\gamma^{z\bar{z}} D_{\bar{z}}^2 D_z^2 N^+ \\ &= \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2z\gamma^{z\bar{z}} (\partial_{\bar{z}} f) D_{\bar{z}} D_z^2 N^+ \\ &= -\frac{1}{4G}\gamma^{z\bar{z}}\partial_{\bar{z}} D_z^2 N^+, \end{aligned} \tag{3.45}$$

where we integrated by parts with respect to $\bar{z}$, and subsequently plugged in (3.44). Likewise, for the past soft charge we have,

$$Q_f^{S-} = \frac{1}{4G}\gamma^{z\bar{z}}\partial_{\bar{z}} D_z^2 N^-. \tag{3.46}$$

The action of Q_f^- on the in-state is given by

$$\begin{aligned} Q_f^-|\text{in}\rangle &= Q_f^{S-}|\text{in}\rangle + Q_f^{H-}|\text{in}\rangle \\ &= \frac{1}{4G}\gamma^{z\bar{z}}\partial_{\bar{z}} D_z^2 N^-|\text{in}\rangle + \sum_{k=1}^{m} E_k^{\text{in}} f(z_k^{\text{in}}, \bar{z}_k^{\text{in}})|\text{in}\rangle \\ &= \frac{1}{4G}\gamma^{z\bar{z}}\partial_{\bar{z}} D_z^2 N^-|\text{in}\rangle + \sum_{k=1}^{m} \frac{E_k^{\text{in}}}{z - z_k^{\text{in}}}|\text{in}\rangle. \end{aligned} \tag{3.47}$$

In a similar fashion,

$$\langle \text{out}|Q_f^+ = -\frac{1}{4G}\gamma^{z\bar{z}}\langle \text{out}|\partial_{\bar{z}} D_z^2 N^+ + \sum_{k=1}^{m} \frac{E_k^{\text{out}}}{z - z_k^{\text{out}}}\langle \text{out}|. \tag{3.48}$$

Hence, (3.43) yields Ward identity

$$\langle \text{out}|\partial_{\bar{z}} D_z^2 N^+ \mathcal{S} + \mathcal{S}\partial_{\bar{z}} D_z^2 N^- |\text{in}\rangle = 4G\gamma_{z\bar{z}} \left[\sum_{k=1}^{m} \frac{E_k^{\text{out}}}{z - z_k^{\text{out}}} - \sum_{k=1}^{m} \frac{E_k^{\text{in}}}{z - z_k^{\text{in}}} \right] \langle \text{out}|\mathcal{S}|\text{in}\rangle. \quad (3.49)$$

In order to show that the Ward identity (3.49) is equal to Weinberg's soft graviton theorem, it is necessary to rephrase the Ward identity in terms of a plane wave expansion. As was mentioned before in equation (3.10), one has

$$D_z^2 N^+ = \frac{1}{2} \lim_{\omega \to 0^+} \int_{-\infty}^{\infty} \mathrm{d}u (e^{i\omega u} + e^{-i\omega u}) \partial_u C_{zz}, \quad (3.50)$$

where, on $\mathcal{I}^+$, the plane wave expansion for C_{zz} is given by [5]

$$C_{zz} = -\frac{i\sqrt{2G}}{\sqrt{\pi^3}(1 + z\bar{z})^2} \int_0^{\infty} \mathrm{d}\omega \left[a_+^{\text{out}}(\omega\hat{x}) e^{-i\omega u} - a_-^{\text{out}\dagger}(\omega\hat{x}) e^{i\omega u} \right]. \quad (3.51)$$

Plugging (3.51) into (3.50) directly yields

$$D_z^2 N^+ = -\gamma_{z\bar{z}} \sqrt{\frac{G}{2\pi}} \lim_{\omega \to 0^+} \left[\omega a_+^{\text{out}}(\omega_q \hat{x}) + \omega a_-^{\text{out}\dagger}(\omega_q \hat{x}) \right], \quad (3.52)$$

where the details can be found in Appendix D.5.2. In a similar manner, one obtains

$$D_z^2 N^- = -\gamma_{z\bar{z}} \sqrt{\frac{G}{2\pi}} \lim_{\omega \to 0^+} \left[\omega a_+^{\text{in}}(\omega\hat{x}) + \omega a_-^{\text{in}\dagger}(\omega\hat{x}) \right]. \quad (3.53)$$

Thus, Ward identity (3.49) can be re-expressed as

$$\begin{aligned} &\lim_{\omega \to 0^+} \left[\omega \partial_{\bar{z}} \left(\gamma_{z\bar{z}} \langle \text{out}| a_+^{\text{out}}(\omega\hat{x}) \mathcal{S} + \mathcal{S} a_-^{\text{in}\dagger}(\omega\hat{x}) |\text{in}\rangle \right) \right] \\ &\qquad = -\gamma_{z\bar{z}} \sqrt{32\pi G} \left[\sum_{k=1}^{m} \frac{E_k^{\text{out}}}{z - z_k^{\text{out}}} - \sum_{k=1}^{m} \frac{E_k^{\text{in}}}{z - z_k^{\text{in}}} \right] \langle \text{out}|\mathcal{S}|\text{in}\rangle. \end{aligned} \quad (3.54)$$

3.5 Soft Graviton Theorem

In this section, we will introduce Weinberg's soft graviton theorem. Subsequently, we will show the extraordinary fact that the theorem is equivalent to the Ward identity (3.54). For gravity, the action is specified by

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{2} g^{\mu\nu} \partial_u \phi \partial_v \phi \right). \quad (3.55)$$

In the weak field, the perturbation expansion is given by $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, with $\kappa^2 = 32\pi G$, and harmonic gauge $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$. Once more, we consider an on-shell scattering process with n incoming (with momenta $p_1^{\text{in}}, \ldots, p_n^{\text{in}}$) and m outgoing scalars (with momenta $p_1^{\text{out}}, \ldots, p_m^{\text{out}}$), represented by Figure 2.3. This time, we add an outgoing soft graviton to the process, the result of which is shown on the LHS of Figure 3.2. The graviton has momentum q and polarisation tensor $\varepsilon_{\mu\nu}(q)$, with $q^\mu \varepsilon_{\mu\nu} = \frac{1}{2} q_\nu \varepsilon^\mu{}_\mu$.
In the soft $q \to 0$ limit, the leading order diagrams are given on the RHS of Figure 3.2. There is again no contribution from diagrams in which the external graviton is attached to internal lines, for the same reason as in the previous chapter.

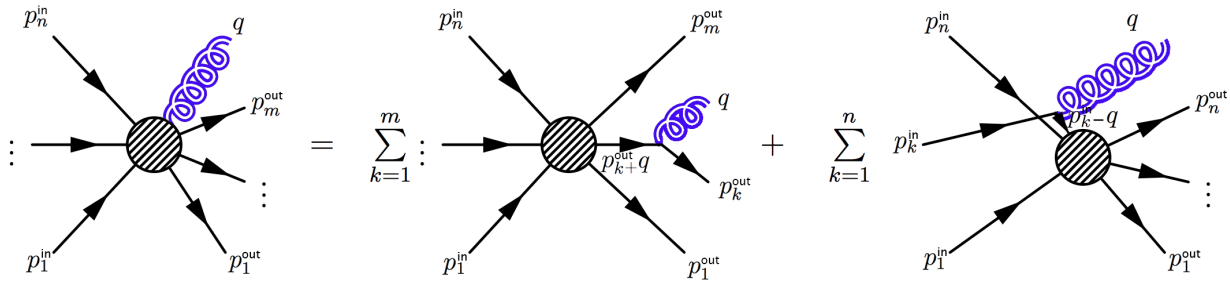
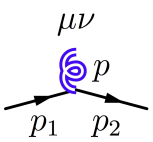


Figure 3.2: Leading order Feynman diagrams for $n \to m$ scattering, with an additional outgoing soft graviton.

Whilst considering the leading order diagrams, note that we saw in Section 2.5 that the propagator for the outgoing and ingoing particle is given by

$$\frac{-i}{2p_k^{\text{out}} \cdot q - i\epsilon}, \quad \text{and} \quad \frac{i}{2p_k^{\text{in}} \cdot q + i\epsilon}, \quad \text{respectively.} \tag{3.56}$$

For an external graviton, $\varepsilon^{\mu\nu}$ is added to the vertex factor, which is contracted with the momenta. The Feynman rule for the vertex factor is [1]



$$= i\kappa \varepsilon^{\mu\nu} p_\mu p_\nu, \quad \text{with} \quad \kappa = \sqrt{32\pi G}. \tag{3.57}$$

By using the above, and vertex factor (3.57), we obtain the near-soft amplitude

$$\begin{aligned}
&\mathcal{M}(q, p_1^{\text{out}}, \ldots, p_m^{\text{out}}, p_1^{\text{in}}, \ldots, p_n^{\text{in}}) \\
&\quad = \sum_{k=1}^{m} \mathcal{M}(p_1^{\text{out}}, \ldots, p_k^{\text{out}} + q, \ldots, p_m^{\text{out}}, p_1^{\text{in}}, \ldots, p_n^{\text{in}}) \frac{\sqrt{32\pi G}\varepsilon^{\mu\nu} p_{k\mu}^{\text{out}} p_{k\nu}^{\text{out}}}{2p_k^{\text{out}} \cdot q - i\epsilon} \\
&\qquad - \sum_{k=1}^{n} \mathcal{M}(p_1^{\text{out}}, \ldots, p_m^{\text{out}}, p_1^{\text{in}}, \ldots, p_k^{\text{in}} + q, \ldots, p_n^{\text{in}}) \frac{\sqrt{32\pi G}\varepsilon^{\mu\nu} p_{k\mu}^{\text{in}} p_{k\nu}^{\text{in}}}{2p_k^{\text{in}} \cdot q + i\epsilon}.
\end{aligned} \tag{3.58}$$

Taking the soft limit $q \to 0$ yields obtain Weinberg's soft graviton theorem,

$$\begin{aligned}&\mathcal{M}(q, p_1^{\text{out}}, \ldots, p_m^{\text{out}}, p_1^{\text{in}}, \ldots, p_n^{\text{in}})\\ &= \sqrt{8\pi G}\left[\sum_{k=1}^{m} \frac{\varepsilon^{\mu\nu} p_{k\mu}^{\text{out}} p_{k\nu}^{\text{out}}}{p_k^{\text{out}} \cdot q} - \sum_{k=1}^{n} \frac{\varepsilon^{\mu\nu} p_{k\mu}^{\text{in}} p_{k\nu}^{\text{in}}}{p_k^{\text{in}} \cdot q}\right] \mathcal{M}(p_1^{\text{out}}, \ldots, p_m^{\text{out}}, p_1^{\text{in}}, \ldots, p_n^{\text{in}}).\end{aligned} \tag{3.59}$$

Note that $\varepsilon^{\mu\nu} = \varepsilon^{\mu}\varepsilon^{\nu}$. By changing notation, and by multiplying both sides by $\omega \to 0$, we obtain

$$\begin{aligned}&\lim_{\omega\to 0}\left[\omega\langle \text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S}|\text{in}\rangle\right]\\ &\qquad = \sqrt{8\pi G}\lim_{\omega\to 0}\left[\sum_{k=1}^{m} \frac{\omega(p_k^{\text{out}} \cdot \varepsilon^+(q))^2}{p_k^{\text{out}} \cdot q} - \sum_{k=1}^{n} \frac{\omega(p_k^{\text{in}} \cdot \varepsilon^+(q))^2}{p_k^{\text{in}} \cdot q}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{3.60}$$

Considering the same scattering process, but then instead with an additional incoming soft photon, would have yielded the same result, i.e.

$$\lim_{\omega\to 0}\left[\omega\langle \text{out}|\mathcal{S}a_-^{\text{in}\dagger}(\vec{q})|\text{in}\rangle\right] = \lim_{\omega\to 0}\left[\omega\langle \text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S}|\text{in}\rangle\right]. \tag{3.61}$$

In order to show that the soft graviton theorem (3.60) is equal to Ward identity (3.54), it is yet again necessary to define the momentum of each in- and outgoing particle in terms of its energy E_k and a point on the sphere $(z_k, \bar{z}_k)$. The formulas for q, ε^+, p_k^{out} and p_k^{in} are given by (2.89), (2.90), (2.107) and (2.108), respectively. Using these formulas, we obtain

$$\begin{aligned}\frac{\omega(p_k^{\text{out}} \cdot \varepsilon^+(q))^2}{p_k^{\text{out}} \cdot q} &= -\frac{E_k^{\text{out}}(1+z\bar{z})(\bar{z}-\bar{z}_k^{\text{out}})}{(z-z_k^{\text{out}})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})},\\ \frac{\omega(p_k^{\text{in}} \cdot \varepsilon^+(q))^2}{p_k^{\text{in}} \cdot q} &= -\frac{E_k^{\text{in}}(1+z\bar{z})(\bar{z}-\bar{z}_k^{\text{in}})}{(z-z_k^{\text{in}})(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}.\end{aligned} \tag{3.62}$$

By using (3.61) and (3.62), one can re-express the soft theorem (3.60) as

$$\begin{aligned}&\lim_{\omega\to 0}[\omega\langle \text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S} + \mathcal{S}a_-^{\text{in}\dagger}(\vec{q})|\text{in}\rangle]\\ &\quad = -\sqrt{32\pi G}(1+z\bar{z})\left[\sum_{k=1}^{m} \frac{E_k^{\text{out}}(\bar{z}-\bar{z}_k^{\text{out}})}{(z-z_k^{\text{out}})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})}\right.\\ &\qquad\qquad \left. - \sum_{k=1}^{n} \frac{E_k^{\text{in}}(\bar{z}-\bar{z}_k^{\text{in}})}{(z-z_k^{\text{in}})(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{3.63}$$

All that is left in order to show that (3.63) is equivalent to Ward identity (3.54), is putting

it in the same format. Calculating the partial derivative of (3.63) with respect to $\bar{z}$ yields

$$\begin{aligned}
&\lim_{\omega\to 0}\partial_{\bar{z}}[\omega\langle\text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S}+\mathcal{S}a_-^{\text{in}\dagger}(\vec{q})|\text{in}\rangle] \\
&\quad = -\sqrt{32\pi G}\Bigg[\sum_{k=1}^{m}\frac{zE_k^{\text{out}}(\bar{z}-\bar{z}_k^{\text{out}})+E_k^{\text{out}}(1+z\bar{z})}{(z-z_k^{\text{out}})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})} \\
&\qquad\qquad -\sum_{k=1}^{n}\frac{zE_k^{\text{in}}(\bar{z}-\bar{z}_k^{\text{in}})+E_k^{\text{in}}(1+z\bar{z})}{(z-z_k^{\text{in}})(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}\Bigg]\langle\text{out}|\mathcal{S}|\text{in}\rangle.
\end{aligned} \tag{3.64}$$

Finally, using (3.64) gives

$$\begin{aligned}
&\lim_{\omega\to 0}\left[\omega\partial_{\bar{z}}\Big(\gamma_{z\bar{z}}\langle\text{out}|a_+^{\text{out}}(\omega\hat{x})\mathcal{S}+\mathcal{S}a_-^{\text{in}}(\omega\hat{x})^\dagger|\text{in}\rangle\Big)\right] \\
&\quad =\gamma_{z\bar{z}}\lim_{\omega\to 0}\partial_{\bar{z}}\left[\omega\langle\text{out}|a_+^{\text{out}}(\omega\hat{x})\mathcal{S}+\mathcal{S}a_-^{\text{in}}(\omega\hat{x})^\dagger|\text{in}\rangle\right] \\
&\qquad -\frac{4z}{(1+z\bar{z})^3}\lim_{\omega\to 0}\left[\omega\langle\text{out}|a_+^{\text{out}}(\omega\hat{x})\mathcal{S}+\mathcal{S}a_-^{\text{in}}(\omega\hat{x})^\dagger|\text{in}\rangle\right] \\
&\quad = -\gamma_{z\bar{z}}\sqrt{32\pi G}\left[\sum_{k=1}^{m}\frac{E_k^{\text{out}}(z\bar{z}_k^{\text{out}}+1)}{(z-z_k^{\text{out}})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})}-\sum_{k=1}^{n}\frac{E_k^{\text{in}}(z\bar{z}_k^{\text{in}}+1)}{(z-z_k^{\text{in}})(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle \\
&\quad = -\gamma_{z\bar{z}}\sqrt{32\pi G}\Bigg[\sum_{k=1}^{m}\frac{E_k^{\text{out}}\big[(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})+(z-z_k^{\text{out}})\bar{z}_k^{\text{out}}\big]}{(z-z_k^{\text{out}})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})} \\
&\qquad\qquad -\sum_{k=1}^{n}\frac{E_k^{\text{in}}\big[(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})+(z-z_k^{\text{in}})\bar{z}_k^{\text{in}}\big]}{(z-z_k^{\text{in}})(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}\Bigg]\langle\text{out}|\mathcal{S}|\text{in}\rangle \\
&\quad = -\gamma_{z\bar{z}}\sqrt{32\pi G}\Bigg[\sum_{k=1}^{m}\frac{E_k^{\text{out}}}{z-z_k^{\text{out}}}-\sum_{k=1}^{n}\frac{E_k^{\text{in}}}{(z-z_k^{\text{in}})} \\
&\qquad\qquad +\cancel{\sum_{k=1}^{m}\frac{E_k^{\text{out}}\bar{z}_k^{\text{out}}}{(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})}-\sum_{k=1}^{n}\frac{E_k^{\text{in}}\bar{z}_k^{\text{in}}}{(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}}\Bigg]\langle\text{out}|\mathcal{S}|\text{in}\rangle \\
&\quad = -\gamma_{z\bar{z}}\sqrt{32\pi G}\left[\sum_{k=1}^{m}\frac{E_k^{\text{out}}}{z-z_k^{\text{out}}}-\sum_{k=1}^{n}\frac{E_k^{\text{in}}}{(z-z_k^{\text{in}})}\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle,
\end{aligned} \tag{3.65}$$

where two terms vanish due to total momentum conservation. Lo and behold, since this result equals (3.54), we have now obtained the remarkable result that Weinberg's soft graviton theorem is equivalent to the Ward identity for the asymptotic symmetries.

3.6 The Gravitational Memory Effect

Having shown the equivalence relation between two of the corners of the infrared triangle, the supertranslations and the soft graviton theorem, we still have to complete the infrared triangle by adding the third corner: the gravitational memory effect. The gravitational memory effect is known as the permanent relative displacement in the position of two nearby inertial detectors, caused by the passage of gravitational radiation. In this section, we will discuss the relation between the gravitational memory effect and supertranslation symmetry.

Consider spacetimes for which $N_{zz} = 0$ at retarded time intervals $u < u_i$ and $u > u_f$, while $N_{zz} \neq 0$ for $u_i < u < u_f$. For the vacua (with $N_{zz} = 0$), the curvature vanishes, so boundary condition (3.7) generalises to

$$D_z^2 C_{\bar{z}\bar{z}} - D_{\bar{z}}^2 C_{zz} = 0, \tag{3.66}$$

which is solved by

$$C_{zz} = -2D_z^2 C(z, \bar{z}). \tag{3.67}$$

Note that we saw earlier that the vacua before and after the passage are related by supertranslations for which $C \to C - 2f$. Now define

$$\Delta C_{zz} \equiv C_{zz}\big|_{u=u_f} - C_{zz}\big|_{u=u_i}, \qquad \Delta m_B = m_B\big|_{u=u_f} - m_B\big|_{u=u_i}. \tag{3.68}$$

Using (3.66) and integrating the constraint on m_B (3.27) over the interval $u_i < u < u_f$, results in

$$\begin{aligned}
\Delta m_B &= \frac{1}{4}\int_{u_i}^{u_f} \mathrm{d}u\, \partial_u \left[D_z^2 C^{zz} + D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\right] - \int_{u_i}^{u_f} \mathrm{d}u\, T_{uu} \\
&= \frac{1}{2}\left(D_z^2 C^{zz}\big|_{u=u_f} - D_z^2 C^{zz}\big|_{u=u_i}\right) - \int_{u_i}^{u_f} \mathrm{d}u\, T_{uu} \\
&= \frac{1}{2} D_z^2 \Delta C^{zz} - \int_{u_i}^{u_f} \mathrm{d}u\, T_{uu},
\end{aligned} \tag{3.69}$$

or

$$D_z^2 \Delta C^{zz} = 2\Delta m_B + 2\int_{u_i}^{u_f} \mathrm{d}u T_{uu}. \tag{3.70}$$

Earlier, we defined a function S (3.37), with property $D_{\bar{z}}^2 D_z^2 (S \ln |z-w|^2) = \pi \gamma_{z\bar{z}} \delta^2(z-w)$. From this, it follows that the Green's function

$$G(z, \bar{z}; w, \bar{w}) = \frac{1}{\pi} S \ln S, \tag{3.71}$$

has the property

$$D_z^2 D_{\bar{z}}^2 G(z,\bar{z};w,\bar{w}) = \gamma_{z\bar{z}}\delta^2(z-w) + \ldots, \tag{3.72}$$

with $D_z^2 D_{\bar{z}}^2 G = D_{\bar{z}}^2 D_z^2 G$, and where '...' denote remaining terms which integrate to zero when integrated over $\mathcal{C}S^2$ [6]. Using (3.67), (3.70) and the Green's function (3.71), one can construct the formula for the radiation induced supertranslation as

$$\Delta C(z,\bar{z}) \equiv 2\int \mathrm{d}^2 w \gamma_{w\bar{w}}\, G(z,\bar{z};w,\bar{w}) \left(\Delta m_B + \int_{u_i}^{u_f} \mathrm{d}u T_{uu}(w,\bar{w})\right). \tag{3.73}$$

The equivalence of (3.70) and (3.73) can be showed by acting with $(\gamma^{z\bar{z}})^2 D_z^2 D_{\bar{z}}^2$ on ΔC as

$$\begin{aligned} D_z^2 \Delta C^{zz} &= (\gamma^{z\bar{z}})^2 D_z^2 D_{\bar{z}}^2 \Delta C \\ &= 2(\gamma^{z\bar{z}})^2 \int \mathrm{d}^2 w \gamma_{w\bar{w}}\, D_z^2 D_{\bar{z}}^2 G(z,\bar{z};w,\bar{w}) \left(\Delta m_B + \int_{u_i}^{u_f} \mathrm{d}u T_{uu}(w,\bar{w})\right) \\ &= 2\gamma^{z\bar{z}} \int \mathrm{d}^2 w \gamma_{w\bar{w}}\, \delta^2(z-w) \left(\Delta m_B + \int_{u_i}^{u_f} \mathrm{d}u T_{uu}(w,\bar{w})\right) \\ &= 2\Delta m_B + 2\int_{u_i}^{u_f} \mathrm{d}u T_{uu}. \end{aligned} \tag{3.74}$$

Now consider two inertial detectors, both located at $u = u_0$ and $r = r_0$, with large r_0. Detector 1's location on the asymptotic sphere is given by $(z_1, \bar{z}_1)$, while detector 2 is at $(z_2, \bar{z}_2)$. Given metric (3.1), the leading order initial distance between the detectors is given by [6]

$$L_0 \cong \sqrt{2r_0^2 \gamma_{z_0\bar{z}_0}\delta z \delta\bar{z}} = \frac{2r_0|\delta z|}{1+z_1\bar{z}_1}, \quad \text{with} \quad \delta z \equiv z_2 - z_1. \tag{3.75}$$

Due to the passage of radiation, the metric changes according to (3.68). The new distance between the detector is given by

$$\begin{aligned} L &\cong \sqrt{2r_0^2\gamma_{z_0\bar{z}_0}\delta z\delta\bar{z} + r_1 C_{zz}(u,z_1,\bar{z}_1)\delta z^2 + r_1 C_{\bar{z}\bar{z}}(u,z_1,\bar{z}_1)\delta\bar{z}^2} \\ &= L_0\sqrt{1 + \frac{r_1}{L_0^2}\Big(C_{zz}(u,z_1,\bar{z}_1)\delta z^2 + C_{\bar{z}\bar{z}}(u,z_1,\bar{z}_1)\delta\bar{z}^2\Big)} \\ &\cong L_0 + \frac{r_1}{2L_0^2}\Big(C_{zz}(u,z_1,\bar{z}_1)\delta z^2 + C_{\bar{z}\bar{z}}(u,z_1,\bar{z}_1)\delta\bar{z}^2\Big). \end{aligned} \tag{3.76}$$

Thus, the relative displacement between retarded time u_i and u_f is given by

$$\Delta L = \frac{r_1}{2L_0^2}\Big(\Delta C_{zz}(z_1,\bar{z}_1)\delta z^2 + \Delta C_{\bar{z}\bar{z}}(z_1,\bar{z}_1)\delta\bar{z}^2\Big), \tag{3.77}$$

with ΔC_{zz} defined by (3.74). Summarising, the passing radiation induces a transition from one degenerate vacuum state to another, generating a shift in the metric, which results in a displacement of the detectors.
Since the displacement of the inertial detectors is incredibly subtle, it is difficult to detect, but [43–45] proposed methods of measuring it, based on sophisticated data analyses. If we one day succeed in measuring the memory effect, the world has obtained a direct measurable consequence of supertranslation symmetry, which is quite the exciting prospect.

3.7 Superrotations

In 2011, Glenn Barnich and Cédric Troessaert published papers which stated that "supertranslations call for superrotations" [46, 47], and argued to lift the restrictions (3.12) on ζ. As a result, one obtaines the infinite dimensional superrotation subgroup. In this section, we will analyse the superrotations.
For rotations and boosts, the diffeomorphism has the following form [1],

$$\begin{aligned}\zeta_Y =&\frac{u}{2}\Big(D_zY^z + D_{\bar z}Y^{\bar z}\Big)\partial_u - \frac{u+r}{2}\Big(D_zY^z + D_{\bar z}Y^{\bar z}\Big)\partial_r \\ &+ \Big[Y^z + \frac{u}{2r}(Y^z - D^zD_{\bar z}Y^{\bar z})\Big]\partial_z + \Big[Y^{\bar z} + \frac{u}{2r}(Y^{\bar z} - D^{\bar z}D_zY^z)\Big]\partial_{\bar z},\end{aligned} \tag{3.78}$$

where $(Y^z, Y^{\bar z})$ is a two-dimensional vector field on $\mathcal{C}S^2$, and is independent of u and r. Now take

$$Y^z = a + bz + cz^2. \tag{3.79}$$

For this specific choice of Y^z, it is shown in Appendix D.6.1 that the diffeomorphism (3.78) generates generic Lorentz transformations. However, instead of restricting ourselves to (3.79), we will now instead let Y^z be a general vector for the rest of this thesis. One discovers a restriction on Y^z by noting that

$$\begin{aligned}\mathcal{L}_\zeta g_{\bar z\bar z} =&\zeta^\rho\partial_\rho g_{\bar z\bar z} + 2g_{\bar z\rho}\partial_{\bar z}\zeta^\rho \\ =&2g_{\bar z z}\partial_{\bar z}\zeta^z + \mathcal{O}(r) \\ =&2r^2\gamma_{z\bar z}\partial_{\bar z}Y^z + \mathcal{O}(r).\end{aligned} \tag{3.80}$$

satisfies the falloff condition for $g_{\bar z\bar z}$ (3.4) if the $\mathcal{O}(r^{-2})$ term vanishes, i.e. if

$$\partial_{\bar z}Y^z = 0. \tag{3.81}$$

Our earlier choice for Y^z (3.79) obviously satisfies this condition. The action of superrotations on C_{zz} is given by $\mathcal{L}_\zeta C_{zz} = \frac{1}{r}\cdot\mathcal{L}_\zeta g_{zz}|_{\mathcal{O}(r)}$, with

$$\mathcal{L}_\zeta g_{zz} = r\left[\frac{u}{2}D\cdot YN_{zz} - \frac{1}{2}D\cdot YC_{zz} + Y\cdot DC_{zz} + 2C_{zz}D_zY^z - uD_z^3Y^z\right] + \mathcal{O}(1). \tag{3.82}$$

The derivation of (3.82) is placed in Appendix D.6.1. Thus,

$$\mathcal{L}_\zeta C_{zz} = \frac{u}{2} D \cdot Y N_{zz} - \frac{1}{2} D \cdot Y C_{zz} + Y \cdot D C_{zz} + 2 C_{zz} D_z Y^z - u D_z^3 Y^z. \tag{3.83}$$

Differentiating (3.83) with respect to u yields

$$\mathcal{L}_\zeta N_{zz} = \frac{u}{2} D \cdot Y \partial_u N_{zz} + Y \cdot D N_{zz} + 2 N_{zz} D_z Y^z - D_z^3 Y^z. \tag{3.84}$$

Having found the asymptotic symmetries (3.83) and (3.84), we will now construct the associated conserved charges. Subsequently, we will check whether these conserved charges generate the correct transformations.The superrotation charges are given by [1]

$$Q_Y^+ = \frac{1}{8\pi G} \int_{\mathcal{I}_-^+} \mathrm{d}^2 z [Y_{\bar{z}} N_z + Y_z N_{\bar{z}}], \tag{3.85}$$

$$Q_Y^- = \frac{1}{8\pi G} \int_{\mathcal{I}_+^-} \mathrm{d}^2 z [Y_{\bar{z}} N_z + Y_z N_{\bar{z}}]. \tag{3.86}$$

From the matching condition for N_z (3.11), it follows that we have conservation of the superrotation charges,

$$Q_Y^+ = Q_Y^-. \tag{3.87}$$

Once again, we will rewrite the charges until we have expressed them as the sum of a soft charge and a hard charge. Integrating the future charge (3.85) by parts with respect to u yields

$$Q_Y^+ = -\frac{1}{8\pi G} \int_{\mathcal{I}^+} \mathrm{d}^2 z \, \mathrm{d}u \, [Y_{\bar{z}} \partial_u N_z + Y_z \partial_u N_{\bar{z}}], \tag{3.88}$$

where Y_z does not depend on u. Computing the leading order uz-component of Einstein's equations yields a constraint on the angular momentum aspect N_z (see Appendix D.3), which is given by

$$\partial_u N_z = \frac{1}{4} \partial_z (D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u \partial_u \partial_z m_B - T_{uz}, \tag{3.89}$$

with

$$T_{uz} \equiv 8\pi G \lim_{r \to \infty} \left[r^2 T_{uz}^M \right] - \frac{1}{4} \partial_z (C_{zz} N^{zz}) - \frac{1}{2} C_{zz} D_z N^{zz}. \tag{3.90}$$

Plugging (3.89) into future charge (3.88) yields

$$\begin{aligned} Q_Y^+ = &- \frac{1}{8\pi G} \int_{\mathcal{I}^+} \mathrm{d}^2 z \, \mathrm{d}u \, Y_{\bar{z}} \left[\frac{1}{4} \partial_z (D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u \partial_u \partial_z m_B - T_{uz} \right] \\ &- \frac{1}{8\pi G} \int_{\mathcal{I}^+} \mathrm{d}^2 z \, \mathrm{d}u \, Y_z \left[\frac{1}{4} \partial_{\bar{z}} (D_{\bar{z}}^2 C^{\bar{z}\bar{z}} - D_z^2 C^{zz}) - u \partial_u \partial_{\bar{z}} m_B - T_{u\bar{z}} \right]. \end{aligned} \tag{3.91}$$

Subsequently, one uses the constraint on m_B (3.27), which yields

$$\begin{aligned}
Q_Y^+ =& Q_H^+ + Q_S^+,\\
Q_H^+ =& \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,\left(Y_{\bar z}T_{uz} + Y_z T_{u\bar z}\right) - \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,\left(uY_{\bar z}\partial_z T_{uu} + uY_z\partial_{\bar z}T_{uu}\right),\\
Q_S^+ =& -\frac{1}{32\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,Y_{\bar z}\Big(\partial_z\big[D_z^2C^{zz} - D_{\bar z}^2C^{\bar z\bar z}\big] - u\partial_z\big[D_z^2N^{zz} + D_{\bar z}^2N^{\bar z\bar z}\big]\Big)\\
&-\frac{1}{32\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,Y_z\Big(\partial_{\bar z}\Big[D_{\bar z}^2C^{\bar z\bar z} - D_z^2C^{zz}\Big] - u\partial_{\bar z}\Big[D_z^2N^{zz} + D_{\bar z}^2N^{\bar z\bar z}\Big]\Big).
\end{aligned}$$

The hard charge Q_H^+ contains all terms quadratic in C_{zz}, while the soft charge Q_S^+ is linear in C_{zz}. By integrating Q_H^+ by parts, we can re-express the hard charge as

$$Q_H^+ = \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,\left(Y_{\bar z}T_{uz} + Y_z T_{u\bar z} + u(\partial_z Y_{\bar z})T_{uu} + u(\partial_{\bar z}Y_z)T_{uu}\right). \tag{3.92}$$

Now note that it follows from constraint (3.7) that

$$\int_{\mathcal{I}^+} \mathrm{d}u\,\left(D_z^2C^{zz} - D_{\bar z}^2C^{\bar z\bar z}\right) = (\gamma^{z\bar z})^2\Big[\cancel{u(D_z^2C_{\bar z\bar z} - D_{\bar z}^2C_{zz})}\Big]_{\mathcal{I}_-^+}^{\mathcal{I}_+^+} - \int_{\mathcal{I}^+} \mathrm{d}u\,u\Big(D_z^2N^{zz} - D_{\bar z}^2N^{\bar z\bar z}\Big). \tag{3.93}$$

Using (3.93), the soft charge can be rewritten as

$$\begin{aligned}
Q_S^+ =& \frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,Y_{\bar z}u\partial_z\big[D_z^2N^{zz}\big] + \frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,Y_z u\partial_{\bar z}\big[D_{\bar z}^2N^{\bar z\bar z}\big]\\
=& -\frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,u\Big((D_z^3Y_{\bar z})N^{zz} + (D_{\bar z}^3Y_z)N^{\bar z\bar z}\Big).
\end{aligned} \tag{3.94}$$

In order to obtain (3.94), we integrated by parts three times, where the boundary terms vanish. We will now show that the bracket action of Q_Y^+ on C_{zz},

$$\big\{Q_Y^+, C_{zz}(u,z,\bar z)\big\} = \big\{Q_S^+, C_{zz}(u,z,\bar z)\big\} + \big\{Q_H^+, C_{zz}(u,z,\bar z)\big\}, \tag{3.95}$$

is equal to $\mathcal{L}_\zeta C_{zz}$ (3.83). The action of the soft charge Q_S^+ on C_{zz} is given by

$$\begin{aligned}
\big\{Q_S^+, C_{zz}(u,z,\bar z)\big\} =& -\frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\,u'(D_w^3Y^w)\gamma^{w\bar w}\\
&\qquad\qquad\cdot\big\{N_{\bar w\bar w}(u',w,\bar w), C_{zz}(u,z,\bar z)\big\}\\
&-\frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\,u'(D_{\bar w}^3Y^{\bar w})\gamma^{w\bar w}\\
&\qquad\qquad\cdot\cancel{\big\{N_{ww}(u',w,\bar w), C_{zz}(u,z,\bar z)\big\}}\\
=& -\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\,u'(D_w^3Y^w)\delta(u'-u)\delta^2(w-z)\\
=& -uD_z^3Y^z,
\end{aligned} \tag{3.96}$$

where one first manipulated the indices of (3.94) with γ's and then used the bracket relation (3.36). Furthermore, $\{N_{ww}, C_{zz}\} = 0$, since N_{ww} is not coupled to C_{zz} in the symplectic form (3.35). The next step is to derive $\{Q_H^+, C_{zz}(u, z, \bar{z})\}$, but since this process is quite tedious to read, it is placed in Appendix D.6.2. There, one finds that

$$\{Q_H^+, C_{zz}(u, z, \bar{z})\} = \frac{u}{2} D \cdot Y N_{zz} - \frac{1}{2} D \cdot Y C_{zz} + Y \cdot D C_{zz} + 2 C_{zz} D_z Y^z. \tag{3.97}$$

Combining (3.96) and (3.97) yields

$$\{Q_Y^+, C_{zz}(u, z, \bar{z})\} = \frac{u}{2} D \cdot Y N_{zz} - \frac{1}{2} D \cdot Y C_{zz} + Y \cdot D C_{zz} + 2 C_{zz} D_z Y^z - u D_z^3 Y^z, \tag{3.98}$$

which is exactly equal to $\mathcal{L}_\zeta C_{zz}$ (3.83). However, something is still a bit off when it comes to superrotations. If we do the same calculation with N_{zz} instead of C_{zz}, we end up with

$$\{Q_Y^+, N_{zz}\} = \mathcal{L}_Y N_{zz} - D_z^3 Y^z - D_z^2 D_{\bar{z}} Y^{\bar{z}} \neq \mathcal{L}_Y N_{zz}. \tag{3.99}$$

This is highly likely the result of a phase space that is not defined to be large enough, and remains a topic for future investigations. It is shown in [48] that the Ward identity constructed from the conserved superrotation charges (3.85) is related to an associated subleading soft graviton theorem. Furthermore, given the infrared triangle, it follows that there should also be an associated memory effect. This indeed turns out to be the case, and it is given by the gravitational spin memory effect, which is described in [49].

Chapter 4

Black Holes

From a non-subjective viewpoint - it's more like a lot of wibbly wobbly, time-y wimey... stuff.

The Doctor (2007).

4.1 Introduction

In this chapter, we apply the findings of Chapter 3 on Schwarzschild geometry. The content below is based on the marvellous work presented in [40].
The structure of this chapter is given as follows. In Section 4.2, we describe a black hole with linearised supertranslation 'hair'. Then, in Section 4.3 we show that by sending in a linearised shockwave, it is possible to physically construct such an object. Finally, in Section 4.4, it is shown that horizon supertranslation charges generate asymptotic symmetries on $r = 2m_B$.

4.2 Hairy Black Holes

In this section, we supertranslate Schwarzschild geometry and obtain a black hole with linearised supertranslation hair. Subsequently, we show that one can add superrotation charge to a black hole by supertranslating it.

Consider a Schwarzschild black hole that is formed by an incoming null spherical shockwave of infalling matter with total energy M, sent in at advanced time $v = 0$. The formation of such a black hole is depicted in Figure 4.1.

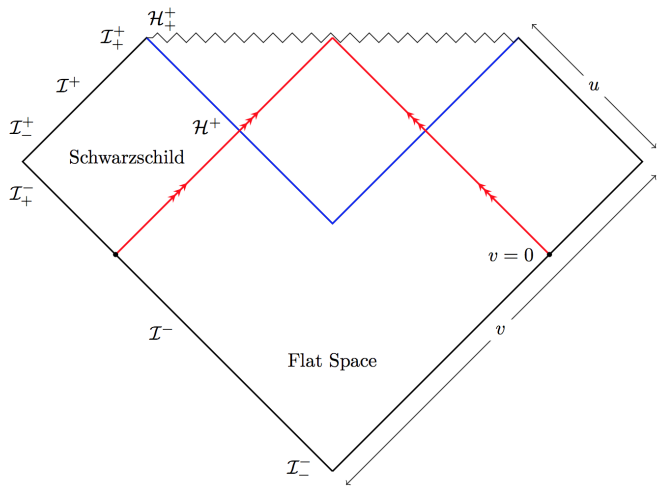


Figure 4.1: Penrose diagram of black hole formation. When the shockwave of collapsing infalling matter (red) passes the Schwarzschild radius $2GM$, the event horizon $\mathcal{H}^+$ forms (blue). Sourced from [1].

Before $v = 0$, spacetime is Minkowski, while it is Schwarzschild after $v = 0$. In advanced Bondi coordinates $(v, r, z, \bar{z})$, this implies a line element of the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2m_B\theta(v)}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}, \tag{4.1}$$

where $m_B = GM$, $c = 1$, and the step function is given by

$$\theta(v) = \begin{cases} 0, & \text{if } v < 0, \\ 1, & \text{if } v > 0. \end{cases} \tag{4.2}$$

After $v = 0$, the geometry is described by the Schwarzschild line element

$$\mathrm{d}s^2 = -V\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}\Theta^A\,\mathrm{d}\Theta^B, \qquad V \equiv 1 - \frac{2m_B}{r}, \tag{4.3}$$

with $\Theta^A = (z, \bar{z})$, and where the event horizon $\mathcal{H}^+$ is located at $r = 2GM$. The Schwarzschild Christoffel symbols can be found in Appendix E.1. Furthermore, the form of the diffeomorphism that generates supertranslations is derived in Appendix E.2, and is given by

$$\zeta_f = f\partial_v + \frac{1}{r}D^A f\partial_A - \frac{1}{2}D^2 f\partial_r, \qquad f = f(z, \bar{z}), \tag{4.4}$$

which holds up for all orders of r. Note that the fact that ζ has the following large r falloffs

$$\zeta^v, \zeta^r \sim \mathcal{O}(1), \qquad \zeta^z, \zeta^{\bar{z}} \sim \mathcal{O}(r^{-1}), \tag{4.5}$$

implies that it generates supertranslations, and not boosts or rotations. A supertranslation acts on coordinate r as

$$r \to r + \frac{1}{2}D^2 f. \tag{4.6}$$

Thus, the infinitesimally supertranslated event horizon is located at $r = 2MG + \frac{1}{2}D^2 f$. Supertranslating the metric components yields[1]

$$\begin{aligned} \mathcal{L}_\zeta g_{vv} &= \frac{m_B D^2 f}{r^2}, & \mathcal{L}_\zeta g_{vA} &= -D_A(Vf + \frac{1}{2}D^2 f), \\ \mathcal{L}_\zeta g_{AB} &= 2rD_A D_B f - r\gamma_{AB} D^2 f, & \mathcal{L}_\zeta g_{rr} &= \mathcal{L}_\zeta g_{rA} = 0. \end{aligned} \tag{4.7}$$

By adding (4.7) to (4.3), one obtains the infinitesimally supertranslated Schwarzschild metric:

$$\begin{aligned} \mathrm{d}s^2 + \mathcal{L}_\zeta \,\mathrm{d}s^2 = &-\left(V - \frac{m_B D^2 f}{r^2}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r - \mathrm{d}v\,\mathrm{d}\Theta^A D_A(2Vf + D^2 f) \\ &+ (r^2\gamma_{AB} + 2rD_A D_B f - r\gamma_{AB} D^2 f)\,\mathrm{d}\Theta^A\,\mathrm{d}\Theta^B. \end{aligned} \tag{4.8}$$

Since there exist infinitely many supertranslations, this describes the geometry for a black hole with a lush head of infinite supertranslation hair. As was mentioned in Section 3.3, the conserved past supertranslation charge is given by

$$Q_f^- = \frac{1}{4\pi G}\int_{\mathcal{I}_+^-} \mathrm{d}^2 z \gamma_{z\bar{z}} f m_B. \tag{4.9}$$

Supertranslating a black hole does not result in added supertranslation charges, which can be observed by noting that supertranslating the Bondi mass yields

$$\hat{m}_B = \mathcal{L}_\zeta m_B = \frac{r}{2}\cdot \mathcal{L}_\zeta g_{vv}\Big|_{\mathcal{O}(r^{-1})} = 0, \tag{4.10}$$

which implies that $\hat{Q}_f^- = 0$. However, by supertranslating a black hole, one does add superrotation charges to the black hole. The conserved past superrotation charge is given by

$$Q_Y^- = \frac{1}{8\pi G}\int_{\mathcal{I}_+^-} \mathrm{d}^2 z \left[Y_{\bar{z}} N_z + Y_z N_{\bar{z}}\right]. \tag{4.11}$$

Ignoring terms quadratic in C_{zz}, it follows from (4.7) and advanced line element (3.6) that

$$\mathcal{L}_\zeta N_A = -\frac{3r}{2}\cdot \mathcal{L}_\zeta g_{vA}\Big|_{\mathcal{O}(r^{-1})} = -\frac{3r}{2}\cdot D_A\left(\frac{2m_B}{r}f\right) = -3MG\partial_A f. \tag{4.12}$$

[1] The derivation of the supertranslated metric components can be found in Appendix E.2.

Plugging (4.12) into (4.11) directly yields the added superrotation charge,

$$\hat{Q}_Y^- = -\frac{3}{8\pi}\int_{\mathcal{I}_+^-} \mathrm{d}^2 z[Y_{\bar{z}} M \partial_z f + Y_z M \partial_{\bar{z}} f]. \tag{4.13}$$

Since there are infinitely many supertranslations, there is also an infinite number of superrotation charges that can be added to the black hole. From the added superrotation charges, black holes that are supertranslated with a different function f can be distinguished.

4.3 Hair Implants

In the previous section, we derived the metric for a supertranslated Schwarzschild black hole (4.8). In this section, we will show how it is possible to physically implant supertranslation hair on a bald Schwarzschild black hole. Subsequently, we show how one is able to implant hair on a black hole that is formed from the vacuum by a shockwave at $v = 0$.

Consider a linearised shockwave with total mass μ, that is sent in at advanced time v_s in Schwarzschild geometry, for a black hole with mass M. Near $\mathcal{I}^-$, this shockwave has energy-momentum density

$$\hat{T}_{vv} = \frac{\mu + \hat{T}(z,\bar{z})}{4\pi r^2}\delta(v - v_s). \tag{4.14}$$

For simplicity, we assume that $\hat{T}$ only depends on $(z,\bar{z})$. It was presented in [40] that, due to stress-energy conservation $\nabla_a \hat{T}^{ab} = 0$, one requires subleading-in-$\frac{1}{r}$ corrections

$$\hat{T}_{vv} = \left(\frac{\mu + \hat{T}(z,\bar{z})}{4\pi r^2} + \frac{\hat{T}^{(1)}(z,\bar{z})}{4\pi r^3}\right)\delta(v - v_s), \quad \hat{T}_{vA} = \frac{\hat{T}_A(z,\bar{z})}{4\pi r^2}\delta(v - v_s), \tag{4.15}$$

with $A = (z,\bar{z})$,

$$(D^2 + 2)\hat{T}^{(1)} = -6MG\hat{T}, \quad \text{and} \quad D^A \hat{T}_A = \hat{T}^{(1)}. \tag{4.16}$$

In Section 3.6, we introduced the Green's function

$$G_r(z,\bar{z};w,\bar{w}) = \frac{1}{\pi} S \log S, \qquad S \equiv \sin^2\frac{\Delta\Theta}{2} = \frac{(z-w)(\bar{z}-\bar{w})}{(1+z\bar{z})(1+w\bar{w})}, \tag{4.17}$$

with property

$$D_z^2 D_{\bar{z}}^2 G_r(z,\bar{z};w,\bar{w}) = \gamma_{z\bar{z}}\delta^2(z-w) + \ldots, \tag{4.18}$$

where a subscript r is added to avoid confusion with the gravitational constant G. It is shown in Appendix E.3 that $D_z^2 D_{\bar{z}}^2 G_r = \frac{(\gamma_{z\bar{z}})^2}{4} D^2(D^2+2)G_r$, which implies

$$\frac{\gamma_{z\bar{z}}}{4} D^2(D^2+2)G_r(z,\bar{z};w,\bar{w}) = \delta^2(z-w) + \dots \tag{4.19}$$

Now define

$$\hat{C}(z,\bar{z}) \equiv G \int \mathrm{d}^2 w \, \gamma_{w\bar{w}} \, G_r(z,\bar{z};w,\bar{w}) \hat{T}(w,\bar{w}). \tag{4.20}$$

Using (4.19), one finds

$$\begin{aligned} \frac{\gamma_{z\bar{z}}}{4} D^2(D^2+2)\hat{C} &= G \int \mathrm{d}^2 w \, \gamma_{w\bar{w}} \, \delta^2(z-w) \hat{T}(w,\bar{w}) \\ &= G \gamma_{z\bar{z}} \hat{T}(z,\bar{z}), \end{aligned} \tag{4.21}$$

or

$$\hat{T}(z,\bar{z}) = \frac{1}{4G} D^2(D^2+2)\hat{C}. \tag{4.22}$$

It follows from (4.16) and (4.22) that

$$\begin{aligned} (D^2+2)\hat{T}^{(1)} = -6MG\hat{T} = -\frac{3M}{2} D^2(D^2+2)\hat{C}, \qquad &\Rightarrow \quad \hat{T}^{(1)} = -\frac{3M}{2} D^2 \hat{C}, \\ D^A \hat{T}_A = -\frac{3M}{2} D^A D_A \hat{C}, \qquad &\Rightarrow \quad \hat{T}_A = -\frac{3M}{2} D_A \hat{C}. \end{aligned} \tag{4.23}$$

Plugging the above into (4.15) yields

$$\begin{aligned} \hat{T}_{vv} &= \frac{1}{4\pi r^2} \left[\mu + \frac{1}{4G} D^2(D^2+2)\hat{C} - \frac{3M}{2r} D^2 \hat{C} \right] \delta(v - v_s), \\ \hat{T}_{vA} &= -\frac{3M}{8\pi r^2} D_A \hat{C} \delta(v - v_s). \end{aligned} \tag{4.24}$$

On $\mathcal{I}^-$, one finds a leading order constraint on m_B by computing the vv-component of the Einstein's equations[2], which results in

$$\partial_v m_B = \frac{1}{4} D^A D^B N_{AB} + T_{vv}, \quad \text{with} \quad T_{vv} \equiv \frac{1}{4} N_{zz} N^{zz} + 4\pi G \lim_{r\to\infty} \left[r^2 T^M_{vv} \right]. \tag{4.25}$$

By using (4.14), and ignoring terms quadratic in C_{zz}, one finds up to leading order

$$\begin{aligned} \partial_v m_B &= \frac{1}{4} D^A D^B N_{AB} + 4\pi G \lim_{r\to\infty} \left[r^2 \hat{T}_{vv} \right] \\ &= \frac{1}{4} D^A D^B N_{AB} + G \left(\mu + \hat{T}(z,\bar{z}) \right) \delta(v - v_s). \end{aligned} \tag{4.26}$$

[2]See Appendix D.3 for the derivation of the m_B-constraint on $\mathcal{I}^+$.

Integrating (4.26) over the sphere gives our shifted Bondi mass

$$m_B = MG + \frac{1}{4}D^A D^B C_{AB} + G\left(\mu + \hat{T}(z,\bar{z})\right)\theta(v - v_s), \tag{4.27}$$

where θ is the step function (4.2), with $\partial_v \theta(v - v_s) = \delta(v - v_s)$. The Bondi mass aspect m_B is only allowed to shift by a constant. Thus, $\partial_A m_B = 0$, which implies that

$$D^A D^B C_{AB} = -4G\hat{T}(z,\bar{z})\theta(v - v_s), \tag{4.28}$$

and subsequently,

$$m_B = MG + \mu G\theta(v - v_s). \tag{4.29}$$

Before the passing of the shockwave, at $v < v_s$, the Bondi mass equals MG, whereas one has $m_B = (M + \mu)G$ at $v > v_s$. The shockwave causes a small perturbation $h_{\mu\nu}$ of the eternal Schwarzschild metric, i.e.

$$\mathrm{d}s^2 = -V\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}\Theta^A\,\mathrm{d}\Theta^B + h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \quad V \equiv 1 - \frac{2MG}{r}. \tag{4.30}$$

We restrict $h_{\mu\nu}$ by requiring that it preserves the structure of the metric components $g_{\mu\nu}$, i.e.

$$h_{rr} = h_{rA} = \gamma^{AB}h_{AB} = 0. \tag{4.31}$$

The linearised Einstein equations are given by [1,41]

$$\nabla_\rho\nabla_\mu h_\nu{}^\rho + \nabla_\rho\nabla_\nu h_\mu{}^\rho - \nabla^2 h_{\mu\nu} - \nabla_\mu\nabla_\nu h - g_{\mu\nu}\nabla_\rho\nabla_\sigma h^{\rho\sigma} + g_{\mu\nu}\nabla^2 h = 0, \tag{4.32}$$

which, for (4.15), is solved by metric perturbations of the form[3]

$$\begin{aligned} h_{vv} &= \theta(v - v_s)\left(\frac{2G\mu}{r} - \frac{MGD^2\hat{C}}{r^2}\right), \\ h_{vA} &= \theta(v - v_s)\partial_A\left(V + \frac{1}{2}D^2\right)\hat{C}, \\ h_{AB} &= -2r\theta(v - v_s)\left(D_A D_B\hat{C} - \frac{1}{2}\gamma_{AB}D^2\hat{C}\right). \end{aligned} \tag{4.33}$$

By comparing this solution with the supertranslated Schwarzschild metric components (4.7), one can see that it can be expressed as

$$h_{\mu\nu} = \theta(v - v_0)\left(\mathcal{L}_{f=-\hat{C}}g_{\mu\nu} + \frac{2G\mu}{r}\delta^v_\mu\delta^v_\nu\right). \tag{4.34}$$

[3]See page 134-136 of [1] for the derivation of the linearised metric perturbations.

This implies that, at $v > v_s$, one obtains a black hole with linearised supertranslation hair, with mass parameter $m_B = (M + \mu)G$. Note that by replacing (4.29) with

$$m_B = MG\theta(v) + \mu G\theta(v - v_s), \quad \text{where} \quad v_s > 0, \tag{4.35}$$

one physically implants hair on a black hole that is formed from the vacuum by a shock-wave sent in at $v = 0$.

4.4 Asymptotic Symmetries on the Horizon $\mathcal{H}^+$

In this section, we show that the horizon supertranslation charge $\hat{Q}_f^{\mathcal{H}^+}$ generates asymptotic symmetries. Since the metric perturbation $h_{\mu\nu}$ has to preserve Bondi gauge, its Lie derivative has the following form [41]

$$\mathcal{L}_\zeta h_{AB} = \nabla_A \zeta_B + \nabla_B \zeta_A. \tag{4.36}$$

In Appendix E.4, it is derived that on the event horizon $r = 2m_B$, the supertranslation action on h_{AB} is given by

$$\mathcal{L}_\zeta h_{AB} = 2m_B(2D_A D_B f - \gamma_{AB} D^2 f). \tag{4.37}$$

A Cauchy surface is a spacelike hypersurface that is intersected exactly once by every non-spacelike curve. For Schwarzschild geometry, $\mathcal{I}^+$ is not a Cauchy surface. Nevertheless, for massless fields, $\mathcal{I}^+ \cup \mathcal{H}^+$ is a Cauchy surface, which leads us to suspect that the supertranslation charge can be written as

$$\hat{Q}_f^+ = \hat{Q}_f^{\mathcal{I}^+} + \hat{Q}_f^{\mathcal{H}^+}. \tag{4.38}$$

After an incredibly lengthy derivation, it is shown in [40] that $\hat{Q}_f^{\mathcal{H}^+}$ is given by

$$\hat{Q}_f^{\mathcal{H}^+} = \frac{1}{32\pi G m_B} \int_{\mathcal{H}^+} \mathrm{d}v\, \mathrm{d}^2 z \gamma_{z\bar{z}} D^A D^B f \partial_v h_{AB}. \tag{4.39}$$

In the same paper, the symplectic form on $\mathcal{H}^+$ is derived, and subsequently, also the following Dirac bracket relation:

$$\left\{\partial_v h_{CD}, h_{AB}\right\} = 64\pi G m_B^2 (\gamma_{CA}\gamma_{DB} + \gamma_{CB}\gamma_{DA} - \gamma_{CD}\gamma_{AB})\delta(v - v')\delta^2(z - z'). \tag{4.40}$$

Using (4.40), one finds

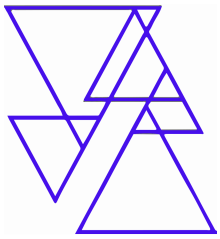
$$\begin{aligned}
\left\{\hat{Q}_f^{\mathcal{H}^+}, h_{AB}\right\} &= \frac{1}{32\pi G m_B} \int_{\mathcal{H}^+} \mathrm{d}v\, \mathrm{d}^2 z D^C D^D f \left\{\partial_v h_{CD}, h_{AB}\right\} \\
&= 2m_B \int_{\mathcal{H}^+} \mathrm{d}v\, \mathrm{d}^2 z D^C D^D f (\gamma_{CA}\gamma_{DB} + \gamma_{CB}\gamma_{DA} - \gamma_{CD}\gamma_{AB})\delta(v - v')\delta^2(z - z') \\
&= 2m_B D^C D^D f (\gamma_{CA}\gamma_{DB} + \gamma_{CB}\gamma_{DA} - \gamma_{CD}\gamma_{AB}) \\
&= 2m_B (2D_A D_B f - \gamma_{AB} D^2 f),
\end{aligned} \tag{4.41}$$

which is exactly equal to (4.37). This implies that $\hat{Q}_f^{\mathcal{H}^+}$ generates asymptotic symmetries on the event horizon.

Chapter 5

Discussion and Concluding Remarks

In any field, find the strangest thing
and then explore it.



John Archibald Wheeler (1911-2008)

In this thesis, we have shown the equivalence relation between the asymptotic symmetry group and the soft theorems, in the context of massless QED and gravity. This feat is quite remarkable, because it implies that the soft theorem can be understood as a consequence of the asymptotic symmetry group. A generalisation to massive QED is given in [3], whereas [50] includes magnetic charges in the analysis.
We have seen that we directly obtain the conserved charges that generate the asymptotic symmetries, simply by constructing Lorentz invariant boundary conditions near spatial infinity i^0. This highlights the importance of Lorentz invariance for charge conservation. Many subtleties arise while constructing the phase space, from which one obtains the symplectic form, and subsequently the Dirac brackets. It is important to identify proper asymptotic boundary constraints, otherwise the Dirac brackets will fail to generate the asymptotic symmetries. For superrotation symmetries, the construction of a properly defined phase space is still an outstanding issue.
In the case of gravity, we also discussed the third corner of the triangle, the memory effect. Although hard to detect, the memory effect offers a physical way to measure the asymptotic symmetry group and their associated conserved charges.

It was shown for both QED and gravity, that the asymptotic symmetry group is spontaneously broken by the vacuum. This implies that **the vacuum is not unique**, but

infinitely degenerate. The vacua differ from each other by the creation and annihilation of soft bosons. In the case of gravity, it was observed that the different vacua are related by supertranslations for which $C \to -2f$, where C is the soft Goldstone graviton with zero mass.
In Chapter 4, it was shown that black holes can carry soft supertranslation hair, which has implications for the infamous black hole information paradox: in 1976, Stephen Hawking argued that information is destroyed during the process of black hole formation and evaporation [51]. However, we have seen in chapter 4 that the soft supertranslation hairdo is rearranged by throwing something into the black hole, which implies that information about the black hole formation is partially stored [40, 52]. Currently, it is still unknown what the exact nature of this information is, but the realisation that information can be stored on the event horizon in the form of soft photons and gravitons, appears to be a promising start to resolving the information paradox [53].

With regards to future outlook, it is not difficult to imagine that there exists a triangular equivalence relation for every type of massless particle. Many of these infrared triangles still have to be constructed. Furthermore, one can also try to derive the infrared triangle for dimension $D \neq 4$, or generalise the soft theorem to two or more external soft particles. In gravity, there is an equivalence relation between superrotation symmetries and the subleading graviton theorem, which is defined in [54]. It is also possible to construct a sub-subleading infrared triangle, and triangles of any subleading order. In short, the infrared triangle is still in its infancy, and there are loads of infrared triangular relations left to be explored.

In conclusion, the infrared triangle is a dazzling step towards a more unified theory of (infrared) physics. The fact that three seemingly disparate topics actually turn out to describe the same underlying physical system, can be seen as a reflection of the intricate, enigmatic beauty of nature. May the exploration continue!

Acknowledgments

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Appendix A

Differential Forms

A differential p-form is a $(0,p)$ tensor which is completely antisymmetric. Suppose that we have a n-dimensional manifold M. The **Hodge star operator** $\star$ is defined on M as a map from p-forms to $(n-p)$-forms,

$$(\star A)_{\mu_1\ldots\mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1\ldots\nu_p}{}_{\mu_1\ldots\mu_{n-p}}A_{\nu_1\ldots\nu_p}, \tag{A.1}$$

where the Levi-Civita tensor is given by

$$\epsilon_{\mu_1\mu_1\ldots\mu_n} = \sqrt{-g}\tilde{\epsilon}_{\mu_1\mu_1\ldots\mu_n}, \tag{A.2}$$

with

$$\tilde{\epsilon}_{\mu_1\mu_1\ldots\mu_n} = \begin{cases} +1, & \text{if } \mu_1\mu_1\ldots\mu_n \text{ is an even permutation of } 01\ldots(n-1), \\ -1, & \text{if } \mu_1\mu_1\ldots\mu_n \text{ is an odd permutation of } 01\ldots(n-1), \\ 0, & \text{otherwise.} \end{cases}$$

The **exterior derivative d** differentiates p-form tensors as follows,

$$(dA)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}. \tag{A.3}$$

Now suppose we have a p-form A and a q-form B. We are able to construct a $(p+q)$-form known as the **wedge product** $A\wedge B$ by taking the antisymmetrised tensor product,

$$(A\wedge B)_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}A_{[\mu_1\ldots\mu_p}B_{\mu_{p+1}\ldots\mu_{p+q}]}. \tag{A.4}$$

Note that

$$A\wedge B = (-1)^{pq}B\wedge A. \tag{A.5}$$

We conclude this section by introducing **Stokes' theorem**. Suppose that we have a n-dimensional manifold M with boundary ∂M and a $(n-1)$-form ω on M. Stokes' theorem says that the integral of n-form ω over the boundary ∂M is equal to the integral of its exterior derivative $d\omega$ over the whole of M, i.e.

$$\int_M d\omega = \int_{\partial M}\omega. \tag{A.6}$$

Appendix B

Hamiltonian Formalism

The section below is largely based on [55].
Suppose one has a Lagrangian $\mathcal{L}(q, \dot{q})$, where $\{q_k\}$ are called the generalised coordinates. The action is then given by

$$\mathcal{S} = \int \mathcal{L}(q, \dot{q})\, \mathrm{d}t. \tag{B.1}$$

Now suppose there are N degrees of freedom. By using the principle of least action, one can derive the Euler-Lagrange equation from equation (B.1),

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = 0 \qquad (1 \leqslant k \leqslant N). \tag{B.2}$$

The associate conjugate momentum to q_k is defined as

$$p_k = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k}. \tag{B.3}$$

Furthermore, the space with coordinates (q_k, p_k) is called phase space Γ. Using (B.3), one can rewrite the Euler-Lagrange equation as

$$\frac{\mathrm{d}p_k}{\mathrm{d}t} = \frac{\partial \mathcal{L}}{\partial q_k}. \tag{B.4}$$

The Hamiltonian is defined as a Legendre transformation of variables,

$$H(q, p) \equiv \sum_k p_k \dot{q}_k - \mathcal{L}(q, \dot{q}), \tag{B.5}$$

By considering a variation of the Hamiltonian, one obtains

$$\begin{aligned} \delta H &= \sum_k \left(\delta p_k \dot{q}_k + \cancel{p_k \delta \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k}\delta q_k - \cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\delta \dot{q}_k} \right) \\ &= \sum_k \left(\delta p_k \dot{q}_k - \frac{\partial \mathcal{L}}{\partial q_k}\delta q_k \right), \end{aligned} \tag{B.6}$$

where the second and the fourth terms in the brackets cancel, since $p_k = \partial\mathcal{L}/\partial\dot{q}_k$. By comparing equation (B.6) with

$$\delta H = \sum_k \left(\frac{\partial H}{\partial p_k}\delta p_k + \frac{\partial H}{\partial q_k}\delta q_k \right), \tag{B.7}$$

one acquires Hamilton's equations of motion,

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}. \tag{B.8}$$

Given two functions $A(q,p)$ and $B(q,p)$ defined on the phase space, the **Poisson bracket** is defined as

$$\{A, B\} = \sum_k \left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k} \right). \tag{B.9}$$

Now note that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \sum_k \left(\frac{\partial A}{\partial q_k}\frac{\mathrm{d}q_k}{\mathrm{d}t} + \frac{\partial A}{\partial p_k}\frac{\mathrm{d}p_k}{\mathrm{d}t} \right) \tag{B.10}$$

$$= \sum_k \left(\frac{\partial A}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial q_k} \right) \tag{B.11}$$

$$= \{A, H\}. \tag{B.12}$$

Theorem B.0.1. (**Noether's theorem**) *Consider an infinitesimal coordinate transformation $q_k \to q'_k = q_k + \varepsilon f_k(q)$. Let $H(q_k, p_k)$ be a Hamiltonian which is invariant under this transformation, i.e. $H(q_k, p_k) = H(q'_k, p'_k)$. Then*

$$Q = \sum_k p_k f_k(q) \tag{B.13}$$

is a conserved quantity.

Proof. We have

$$\Lambda_{ij} = \frac{\partial q'_i}{\partial q_j} \simeq \delta_{ij} + \varepsilon\frac{\partial f_i(q)}{\partial q_i}, \tag{B.14}$$

where the Jacobian is given by

$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial q'_i}{\partial q_j} & \frac{\partial q'_i}{\partial p_j} \\ \frac{\partial p'_i}{\partial q_j} & \frac{\partial p'_i}{\partial p_j} \end{pmatrix} = \begin{pmatrix} \Lambda_{ij} & 0 \\ \frac{\partial p'_i}{\partial q_j} & \frac{\partial p'_i}{\partial p_j} \end{pmatrix}. \tag{B.15}$$

In order to have invariant Hamilton's equations, the momenta has to transform under this coordinate change as [56]

$$p_i \to \sum_j p_j \Lambda^{-1}_{ji} \simeq p_i - \varepsilon \sum_j p_j \frac{\partial f_j}{\partial q_i}. \tag{B.16}$$

Subsequently,

$$\begin{aligned}
0 &= H\left(q'_k, p'_k\right) - H\left(q_k, p_k\right) \\
&= \sum_k \left(\frac{\partial H}{\partial q_k} \delta q_k + \frac{\partial H}{\partial p_k} \delta p_k \right) \\
&= \sum_k \left(\frac{\partial H}{\partial q_k} \varepsilon f_k(q) - \frac{\partial H}{\partial p_k} \varepsilon p_j \frac{\partial f_j}{\partial q_k} \right) \\
&= \varepsilon \sum_k \left(\frac{\partial H}{\partial q_k} f_k(q) - \frac{\partial H}{\partial p_k} p_j \frac{\partial f_j}{\partial q_k} \right) \\
&= \varepsilon \sum_k \left(\frac{\partial H}{\partial q_k} \frac{\partial Q}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial Q}{\partial q_k} \right) \\
&= \varepsilon \{H, Q\} = -\varepsilon \frac{\mathrm{d}Q}{\mathrm{d}t}.
\end{aligned}$$

Thus, Q is conserved. □

This implies that a transformation that leaves the Hamiltonian invariant leads to a conserved quantity, and vice versa. The associated conserved charge Q is called the symmetry of the Hamiltonian. Note that

$$\{q_i, Q\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial Q}{\partial p_k} - \cancel{\frac{\partial q_i}{\partial p_k}} \frac{\partial Q}{\partial q_k} \right) = \sum_k \delta_{ik} f_k(q) = f_i(q). \tag{B.17}$$

Hence, the conserved quantity Q acts as the generator of the infinitesimal symmetry transformation, $\delta q_i = \varepsilon f_i(q) = \varepsilon \{q_i, Q\}$. By multiplying the Poisson bracket with $i\hbar$, we can promote it to a quantum commutator

$$i\hbar \{\ ,\ \}_{\text{classical}} \leftrightarrow [\ ,\ ]_{\text{quantum}}, \tag{B.18}$$

where the commutator is given by

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{B.19}$$

We employ natural units in which $\hbar = 1$. Throughout this thesis, the Dirac bracket is used, which is a generalisation of the Poisson bracket. [57]

Appendix C

QED: Additional Details

C.1 Symplectic Form $\Omega_{\mathcal{I}^+}$

By integrating the symplectic current form ω over a Cauchy surface Σ, one obtains the symplectic form Ω_Σ,

$$\Omega_\Sigma = \int_\Sigma \omega = -\frac{1}{e^2}\int_\Sigma \delta(\star F) \wedge \delta A. \tag{C.1}$$

As our Cauchy surface, we choose $\Sigma = \mathcal{I}^+$. In order to find ω, one follows a procedure described in [58]. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{4e^2}\left(\partial_\mu A_\nu - \partial_\mu A_\nu\right)\left(\partial^\mu A^\nu - \partial^\mu A^\nu\right).\end{aligned} \tag{C.2}$$

A variation of the Lagrangian yields

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu)}\right)\delta A_\mu + \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu)}\delta A_\mu\right). \tag{C.3}$$

By comparing (C.3) to

$$\delta\mathcal{L} = E^a\delta A_\mu + \nabla_\alpha\Theta^\alpha, \tag{C.4}$$

one can see that the pre-symplectic potential current density Θ^α is given by

$$\Theta^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu)}\delta A_\mu = -\frac{1}{e^2}F^{\alpha\mu}\delta A_\mu. \tag{C.5}$$

Now consider two linearised perturbations of the gauge field, $\delta_1 A_\mu$ and $\delta_2 A_\mu$. The symplectic current form ω is given by an antisymmetrised variation of Θ^α,

$$\begin{aligned}\omega^\alpha &= \delta_1\Theta^\alpha(A_\mu, \delta_2 A_\mu) - \delta_2\Theta^\alpha(A_\mu, \delta_1 A_\mu) \\ &= -\frac{1}{e^2}\left(\delta_1 F^{\alpha\mu}\delta_2 A_\mu - \delta_2 F^{\alpha\mu}\delta_1 A_\mu\right) \\ &= -\frac{1}{e^2}\left(\delta F^{\alpha\mu} \wedge \delta A_\mu\right) \\ &= -\frac{1}{e^2}\delta(\star F) \wedge \delta A.\end{aligned} \tag{C.6}$$

In order to find the presymplectic form $\Omega_{\mathcal{I}^+}$, one needs to find the $(uz\bar{z})$- component of ω,

$$\omega_{uz\bar{z}}\big|_{\mathcal{I}^+} = -\frac{1}{e^2}\Big(\delta(\star F)_{uz} \wedge \delta A_{\bar{z}} + \delta(\star F)_{z\bar{z}} \wedge \delta A_u + \delta(\star F)_{\bar{z}u} \wedge \delta A_z\Big). \tag{C.7}$$

The Hodge dual of F is given by,

$$\begin{aligned}\star F =& iF^z{}_z\, \mathrm{d}v \wedge \mathrm{d}r - ir^2\gamma_{z\bar{z}}F_{ru}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &+ \Big[i\left(F_{uz} - F_{rz}\right)\mathrm{d}u \wedge \mathrm{d}z + iF_{rz}\, \mathrm{d}r \wedge \mathrm{d}z + \text{c.c.}\Big],\end{aligned} \tag{C.8}$$

where 'c.c.' stands for 'complex conjugates'. The components of $\star F$ were derived using $(\star F)_{ab} = \frac{1}{2}\epsilon^{cd}{}_{ab}F_{cd}$ (See Appendix A). As an example, we will derive the $z\bar{z}$- and uz-component of $\star F$,

$$\begin{aligned}(\star F)_{z\bar{z}} =& \frac{1}{2}\epsilon^{cd}{}_{z\bar{z}}F_{cd} & (\star F)_{uz} =& \frac{1}{2}\epsilon^{cd}{}_{uz}F_{cd} \\ =& \frac{1}{2}g^{ca}g^{db}\epsilon_{abz\bar{z}}F_{cd} & =& \frac{1}{2}g^{ca}g^{db}\epsilon_{abuz}F_{cd} \\ =& \frac{1}{2}g^{ru}g^{ur}\epsilon_{urz\bar{z}}F_{ru} & =& \frac{1}{2}\left(g^{cr}g^{d\bar{z}}\epsilon_{r\bar{z}uz}F_{cd} + g^{c\bar{z}}g^{dr}\epsilon_{\bar{z}ruz}F_{cd}\right) \\ &+ \frac{1}{2}g^{ur}g^{ru}\epsilon_{ruz\bar{z}}F_{ur} & =& -g^{ur}g^{z\bar{z}}\epsilon_{urz\bar{z}}F_{uz} - g^{rr}g^{z\bar{z}}\epsilon_{urz\bar{z}}F_{rz} \\ =& \epsilon_{urz\bar{z}}F_{ru} & =& \frac{\gamma^{z\bar{z}}}{r^2}\epsilon_{urz\bar{z}}\left(F_{uz} - F_{rz}\right) \\ =& \sqrt{-g}\tilde{\epsilon}_{urz\bar{z}}F_{ru} & =& \frac{\gamma^{z\bar{z}}}{r^2}\sqrt{-g}\tilde{\epsilon}_{urz\bar{z}}\left(F_{uz} - F_{rz}\right) \\ =& ir^2\gamma_{z\bar{z}}F_{ru} & =& i\left(F_{uz} - F_{rz}\right) \\ \cong& i\gamma_{z\bar{z}}F^{(2)}_{ru}, & \cong& iF^{(0)}_{uz},\end{aligned} \tag{C.9}$$

where $F_{\mu\nu} = -F_{\nu\mu}$. The g in above derivations denotes the determinant of $g_{\mu\nu}$, which is given by $r^4(\gamma_{z\bar{z}})^2$. Furthermore, $F_{ru} \sim \mathcal{O}(r^{-2})$, $F_{uz} \sim \mathcal{O}(1)$ and $F_{rz} \sim \mathcal{O}(r^{-2})$. The other components are derived in a similar fashion. Note that

$$(\star F)_{\bar{z}u} = -(\star F)_{u\bar{z}} \cong iF^{(0)}_{u\bar{z}}. \tag{C.10}$$

By using the components of $\star F$, and the fact that $A_z \sim \mathcal{O}(1)$ and $A_u \sim \mathcal{O}(r^{-1})$, one finds up to leading order

$$\omega_{uz\bar{z}}\big|_{\mathcal{I}^+} = -\frac{i}{e^2}\left(\delta F^{(0)}_{uz} \wedge \delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}} \wedge \delta A^{(0)}_{z}\right). \tag{C.11}$$

Plugging this result into equation (C.1) gives

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2}\int \mathrm{d}u\, \mathrm{d}^2z\Big(\delta F^{(0)}_{uz} \wedge \delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}} \wedge \delta A^{(0)}_{z}\Big). \tag{C.12}$$

Subsequently, split up $A_z^{(0)}$ as follows,

$$A_z^{(0)}(u,z,\bar{z}) = \hat{A}_z^{(0)}(u,z,\bar{z}) + \partial_z\phi(z,\bar{z}), \tag{C.13}$$

with $\hat{A}_z\big|_{\mathcal{I}_\pm^+} = 0$ and $\partial_z\phi \equiv \frac{1}{2}\big[A_z^{(0)}|_{\mathcal{I}_+^+} + A_z^{(0)}|_{\mathcal{I}_-^+}\big]$.
Substituting equation (C.13) into equation (C.12) yields,

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2}\int \mathrm{d}u\,\mathrm{d}^2z\Big[\delta F_{uz}^{(0)} \wedge \delta\left(\hat{A}_{\bar{z}}^{(0)} + \partial_{\bar{z}}\phi\right) + \delta F_{u\bar{z}}^{(0)} \wedge \delta\left(\hat{A}_z^{(0)} + \partial_z\phi\right)\Big]. \tag{C.14}$$

In order to re-express this, one first takes a closer look at the sum of all terms containing $\hat{A}_z^{(0)}$ or $\hat{A}_{\bar{z}}^{(0)}$. Note that $F_{uz}^{(0)} = \partial_u A_z^{(0)} = \partial_u \hat{A}_z^{(0)}$, so

$$\begin{aligned}
&\frac{1}{e^2}\int \mathrm{d}u\,\mathrm{d}^2z\Big(\delta F_{uz}^{(0)} \wedge \delta\hat{A}_{\bar{z}}^{(0)} + \delta F_{u\bar{z}}^{(0)} \wedge \delta\hat{A}_z^{(0)}\Big)\\
&\quad= \frac{1}{e^2}\int \mathrm{d}u\,\mathrm{d}^2z\Big(\delta\partial_u\hat{A}_z^{(0)} \wedge \delta\hat{A}_{\bar{z}}^{(0)} + \delta\partial_u\hat{A}_{\bar{z}}^{(0)} \wedge \delta\hat{A}_z^{(0)}\Big)\\
&\quad= \frac{1}{e^2}\int \mathrm{d}u\,\mathrm{d}^2z\Big(\delta\partial_u\hat{A}_z^{(0)} \wedge \delta\hat{A}_{\bar{z}}^{(0)} - \delta\hat{A}_{\bar{z}}^{(0)} \wedge \delta\partial_u\hat{A}_z^{(0)}\Big) + \cancel{\frac{1}{e^2}\Big[\int \mathrm{d}^2z\delta\hat{A}_{\bar{z}}^{(0)} \wedge \delta\hat{A}_z^{(0)}\Big]_{u=-\infty}^{u=\infty}}\\
&\quad= \frac{2}{e^2}\int \mathrm{d}u\,\mathrm{d}^2z\Big(\delta\partial_u\hat{A}_z^{(0)} \wedge \delta\hat{A}_{\bar{z}}^{(0)}\Big).
\end{aligned} \tag{C.15}$$

In the third line, we integrated the second term by parts over u, and used the fact that $\hat{A}_z\big|_{\mathcal{I}_\pm^+} = 0$. Furthermore, the last line was obtained by using the antisymmetric properties of the wedge product.
Now analyse the sum of all terms containing $\delta_z\phi$ or $\delta_{\bar{z}}\phi$ in equation (C.14),

$$\begin{aligned}
&\frac{1}{e^2}\int \mathrm{d}u\,\mathrm{d}^2z\left(\delta F_{uz}^{(0)} \wedge \delta\partial_{\bar{z}}\phi + \delta F_{u\bar{z}}^{(0)} \wedge \delta\partial_z\phi\right)\\
&\qquad= \int \mathrm{d}^2z\left(\delta\partial_z N^+ \wedge \delta\partial_{\bar{z}}\phi + \delta\partial_{\bar{z}}N^+ \wedge \delta\partial_z\phi\right)\\
&\qquad= \int \mathrm{d}^2z\left(-\delta N^+ \wedge \delta\partial_{\bar{z}}\partial_z\phi + \delta\partial_{\bar{z}}N^+ \wedge \delta\partial_z\phi\right)\\
&\qquad= 2\int \mathrm{d}^2z\left(\delta\partial_{\bar{z}}N^+ \wedge \delta\partial_z\phi\right)\\
&\qquad= -2\int \mathrm{d}^2z\left(\delta\partial_z\phi \wedge \delta\partial_{\bar{z}}N^+\right).
\end{aligned} \tag{C.16}$$

In order to obtain the second line, equations (2.61) and (2.64) were used. In the third and fourth line, the first term was integrated by parts, where the boundary terms vanish.
Finally, by using (C.16) and (C.15), the symplectic form (C.14) is re-expressed as

$$\Omega_{\mathcal{I}^+} = \frac{2}{e^2}\int \mathrm{d}u\,\mathrm{d}^2z\left(\delta\partial_u\hat{A}_z^{(0)} \wedge \delta\hat{A}_{\bar{z}}^{(0)}\right) - 2\int \mathrm{d}^2z\,\left(\delta\partial_z\phi \wedge \delta\partial_{\bar{z}}N^+\right). \tag{C.17}$$

C.2 Dirac brackets

Given a symplectic two-form

$$\Omega = \frac{1}{2}\Omega_{IJ} dx^I \wedge dx^J, \tag{C.18}$$

the Dirac bracket can be constructed as

$$\{A, B\} = \Omega^{IJ}\partial_I A \partial_J B. \tag{C.19}$$

So in order to determine the Dirac brackets on $\mathcal{I}^+$, one needs to invert the symplectic form (C.17). Since the first term of equation (C.17) only involves $\hat{A}_z$ and $\hat{A}_{\bar{z}}$, and the second term only involves boundary fields ϕ and N^+, the two pieces of the symplectic form can be inverted separately. Focusing on the first term, one has

$$\begin{aligned}
&\left\{\partial_u \hat{A}_z^{(0)}(u,z,\bar{z})\ ,\ \hat{A}_{\bar{w}}^{(0)}(u',w,\bar{w})\right\} = \\
&\qquad = \frac{e^2}{2}\int \mathrm{d}u''\,\mathrm{d}^2z' \left(\cancel{\frac{\delta(\partial_u \hat{A}_z^{(0)})}{\delta \hat{A}_{\bar{z}'}^{(0)}}\frac{\delta \hat{A}_{\bar{w}}^{(0)}}{\delta(\partial_{u''}\hat{A}_{z'}^{(0)})}} - \frac{\delta(\partial_u \hat{A}_z^{(0)})}{\delta(\partial_{u''}\hat{A}_{z'}^{(0)})}\frac{\delta \hat{A}_{\bar{w}}^{(0)}}{\delta \hat{A}_{\bar{z}'}^{(0)}}\right) \\
&\qquad = -\frac{e^2}{2}\int \mathrm{d}u''\,\mathrm{d}^2z' \delta(u-u'')\delta^2(z-z')\delta(u'-u'')\delta^2(w-z') \\
&\qquad = -\frac{e^2}{2}\delta(u-u')\delta^2(z-w).
\end{aligned} \tag{C.20}$$

The integral representation of the delta function is given by

$$\delta(u) = \int \frac{\mathrm{d}w}{2\pi} e^{iwu}. \tag{C.21}$$

Integrating equation (C.20) with respect to u yields

$$\left\{\hat{A}_z^{(0)}(u,z,\bar{z})\ ,\ \hat{A}_{\bar{w}}^{(0)}(u',w,\bar{w})\right\} = -\frac{e^2}{4}\Theta(u-u')\delta^2(z-w), \tag{C.22}$$

where the integration constant equals zero, due to the antisymmetric properties of the bracket, and Θ is the sign function

$$\Theta(u) = \frac{1}{\pi i}\int \frac{d\omega}{\omega} e^{i\omega u}. \tag{C.23}$$

with $\Theta(u<0) = -1$ and $\Theta(u>0) = 1$. Note that $\partial_u \Theta(u-u') = 2\delta(u-u')$, whereas $\partial_{u'}\Theta(u-u') = -2\delta(u-u')$. Differentiating equation (C.22) with respect to u' yields

$$\left\{\hat{A}_z^{(0)}(u,z,\bar{z})\ ,\ \hat{A}_{\bar{w}}^{(0)}(u',w,\bar{w})\right\} = \frac{e^2}{2}\delta(u-u')\delta^2(z-w). \tag{C.24}$$

On the other hand, for the second term of (C.17), we have

$$\begin{aligned}
&\left\{\partial_z\phi(z,\bar{z})\ ,\ \partial_{\bar{w}}N^+(w,\bar{w})\right\} = \\
&\qquad = -\frac{1}{2}\int \mathrm{d}^2z' \left(\cancel{\frac{\delta(\partial_z\phi)}{\delta(\partial_{\bar{z}'}N^+)}\frac{\delta(\partial_{\bar{w}}N^+)}{\delta(\partial_{z'}\phi)}} - \frac{\delta(\partial_z\phi)}{\delta(\partial_{z'}\phi)}\frac{\delta(\partial_{\bar{w}}N^+)}{\delta(\partial_{\bar{z}'}N^+)}\right) \\
&\qquad = \frac{1}{2}\int \mathrm{d}^2z'\delta^2(z-z')\delta^2(w-z') \\
&\qquad = \frac{1}{2}\delta^2(z-w).
\end{aligned} \tag{C.25}$$

C.3 Plane Wave Expansion of $\partial_z N^+$

Near $\mathcal{I}^+$, the plane wave expansion of the outgoing gauge field is identified as

$$A_\mu(x) = e\sum_{\alpha=\pm}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega}\left[\varepsilon_\mu^{*\alpha}(\vec{q})a_\alpha^{\text{out}}(\vec{q})e^{iq\cdot x} + \varepsilon_\mu^{\alpha}(\vec{q})a_\alpha^{\text{out}\dagger}(\vec{q})e^{-iq\cdot x}\right]. \tag{C.26}$$

Observe that

$$\begin{aligned}
iq\cdot x &= -iq^0t + i\vec{q}\cdot\vec{x} \\
&= -i\omega(u+r) + ir\omega\hat{q}\cdot\hat{x} \\
&= -i\omega u - i\omega r(1-\hat{q}\cdot\hat{x}).
\end{aligned} \tag{C.27}$$

Thus, plane wave expansion (C.26) can be re-expressed as

$$\begin{aligned}
A_\mu(x) = e\sum_{\alpha=\pm}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega}\Bigg[&\varepsilon_\mu^{\alpha *}(\omega\hat{x})a_\alpha^{\text{out}}(\omega\hat{x})e^{-i\omega u - i\omega r(1-\hat{q}\cdot\hat{x})} \\
&+ \varepsilon_\mu^{\alpha}(\omega\hat{x})a_\alpha^{\text{out}\dagger}(\omega\hat{x})e^{i\omega u + i\omega r(1-\hat{q}\cdot\hat{x})}\Bigg].
\end{aligned} \tag{C.28}$$

By substituting $\mathrm{d}^3q = \omega^2\sin\theta\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}\phi$, and by using that $\hat{q}\cdot\hat{x} = \cos\theta$, equation (C.28) can be rewritten as

$$\begin{aligned}
A_\mu(x) = \frac{e}{8\pi^2}\sum_{\alpha=\pm}\int_0^\infty \mathrm{d}\omega\,\omega\int_0^\pi \mathrm{d}\theta\sin\theta\Bigg[&\varepsilon_\mu^{\alpha *}(\omega\hat{x})a_\alpha^{\text{out}}(\omega\hat{x})e^{-i\omega u - i\omega r(1-\cos\theta)} \\
&+ \varepsilon_\mu^{\alpha}(\omega\hat{x})a_\alpha^{\text{out}\dagger}(\omega\hat{x})e^{i\omega u + i\omega r(1-\cos\theta)}\Bigg].
\end{aligned} \tag{C.29}$$

Next, the stationary phase approximation will be used. The exponent is stationary at $\theta = 0$ and $\theta = \pi$. However, the Riemann-Lebesgue Lemma [59] states that for a function f that satisfies

$$\int_{\mathbb{R}} \mathrm{d}x |f(x)| < \infty, \tag{C.30}$$

one has

$$\int_{\mathbb{R}} \mathrm{d}x f(x) e^{-izx} \to 0, \text{ as } |z| \to \infty, \text{ for } z \in \mathbb{C}. \tag{C.31}$$

Thus, the saddle point at $\theta = \pi$ does not contribute. Expanding around $\theta = 0$ yields

$$\begin{aligned} A_\mu(x) =& \frac{e}{8\pi^2} \sum_{\alpha=\pm} \int_0^\infty \mathrm{d}\omega\, \omega \varepsilon_\mu^{\alpha *}(\omega\hat{x}) a_\alpha^{\mathrm{out}}(\omega\hat{x}) e^{-i\omega u} \int_0^\pi \mathrm{d}\theta\, \theta\, e^{-i\omega r\theta^2/2} \\ &+ \frac{e}{8\pi^2} \sum_{\alpha=\pm} \int_0^\infty \mathrm{d}\omega\, \omega \varepsilon_\mu^{\alpha}(\omega\hat{x}) a_\alpha^{\mathrm{out}\dagger}(\omega\hat{x}) e^{i\omega u} \int_0^\pi \mathrm{d}\theta\, \theta e^{i\omega r\theta^2/2} + \mathcal{O}(r^{-2}) \\ =& -\frac{ie}{8\pi^2 r} \sum_{\alpha=\pm} \int_0^\infty \mathrm{d}\omega \Big[\varepsilon_\mu^{\alpha *}(\omega\hat{x}) a_\alpha^{\mathrm{out}}(\omega\hat{x}) e^{-i\omega u} - \varepsilon_\mu^{\alpha}(\omega\hat{x}) a_\alpha^{\mathrm{out}\dagger}(\omega\hat{x}) e^{i\omega u} \Big] + \mathcal{O}(r^{-2}). \end{aligned}$$

The z-component of the gauge field is given by $A_z = (\partial_z x^\mu) A_\mu$. Thus, in order to find A_z one needs to determine $(\partial_z x^\mu)\varepsilon_\mu^\pm$. The polarisation tensors $\varepsilon_\mu^\pm$ are given by (2.90). Moreover, on $\mathcal{I}^+$, we have

$$x^\mu = \left(u + r,\ r\frac{z+\bar{z}}{1+z\bar{z}},\ -r\frac{i(z-\bar{z})}{1+z\bar{z}},\ r\frac{1-z\bar{z}}{1+z\bar{z}} \right). \tag{C.32}$$

Now note

$$(\partial_z x^\mu)\varepsilon_\mu^+ = (\partial_z x^\mu)\varepsilon_\mu^{-*} = 0, \quad \text{and} \quad (\partial_z x^\mu)\varepsilon_\mu^- = (\partial_z x^\mu)\varepsilon_\mu^{+*} = \frac{\sqrt{2}r}{1+z\bar{z}}, \tag{C.33}$$

from which it follows that

$$A_z(x) = -\frac{i}{8\pi^2} \frac{\sqrt{2}e}{1+z\bar{z}} \int_0^\infty \mathrm{d}\omega \Big[a_+^{\mathrm{out}}(\omega\hat{x}) e^{-i\omega u} - a_-^{\mathrm{out}\dagger}(\omega\hat{x}) e^{i\omega u} \Big] + \mathcal{O}(r^{-1}). \tag{C.34}$$

Since $\lim_{r\to\infty} A_z(u,r,z,\bar{z}) = A_z^{(0)}(u,z,\bar{z})$, one obtains

$$A_z^{(0)}(u,z,\bar{z}) = -\frac{i}{8\pi^2} \frac{\sqrt{2}e}{1+z\bar{z}} \int_0^\infty \mathrm{d}\omega \left[a_+^{\mathrm{out}}(\omega\hat{x}) e^{-i\omega u} - a_-^{\mathrm{out}\dagger}(\omega\hat{x}) e^{i\omega u} \right]. \tag{C.35}$$

Finally,

$$
\begin{aligned}
\partial_z N^+ &= \frac{1}{2e^2} \lim_{\omega\to 0^+} \int_{-\infty}^{\infty} \mathrm{d}u (e^{i\omega u} + e^{-i\omega u}) \partial_u A_z^{(0)} \\
&= -\frac{1}{2e} \frac{1}{8\pi^2} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega\to 0^+} \int_{-\infty}^{\infty} \mathrm{d}u \left(e^{i\omega u} + e^{-i\omega u}\right) \\
&\qquad\qquad \cdot \int_0^{\infty} \mathrm{d}\omega_q\, \omega_q \left[a_+^{\mathrm{out}}(\omega_q \hat{x})\, e^{-i\omega_q u} + a_-^{\mathrm{out}\dagger}(\omega_q \hat{x})\, e^{i\omega_q u}\right] \\
&= -\frac{1}{2e} \frac{1}{8\pi^2} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega\to 0^+} \int_{-\infty}^{\infty} \mathrm{d}u \int_0^{\infty} \mathrm{d}\omega_q\, \omega_q \\
&\qquad \cdot \left[a_+^{\mathrm{out}}\left(e^{i(\omega-\omega_q)u} + e^{-i(\omega+\omega_q)u}\right) + a_-^{\mathrm{out}\dagger}\left(e^{i(\omega+\omega_q)u} + e^{-i(\omega-\omega_q)u}\right)\right] \\
&= -\frac{1}{8\pi e} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega\to 0^+} \int_0^{\infty} \mathrm{d}\omega_q\, \omega_q \\
&\qquad \cdot \left[a_+^{\mathrm{out}}\left(\delta(\omega-\omega_q) + \cancel{\delta(\omega+\omega_q)}\right) + a_-^{\mathrm{out}\dagger}\left(\cancel{\delta(\omega+\omega_q)} + \delta(\omega-\omega_q)\right)\right] \\
&= -\frac{1}{8\pi e} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega\to 0^+} \left[\omega a_+^{\mathrm{out}}(\omega\hat{x}) + \omega a_-^{\mathrm{out}\dagger}(\omega\hat{x})\right],
\end{aligned}
\tag{C.36}
$$

where we used

$$
\int_{-\infty}^{\infty} \mathrm{d}u\, e^{i(\omega-\omega_q)u} = 2\pi\delta(\omega-\omega_q). \tag{C.37}
$$

Appendix D

Gravity: Additional Details

D.1 Metric

In retarded Bondi coordinates (u, r, Θ^A), with $\Theta^A = (z, \bar{z})$ and $u = t - r + \ldots$, any four-dimensional Lorentzian metric can be written as [60]

$$\mathrm{d}s^2 = -Ue^{2\beta}\,\mathrm{d}u^2 - 2e^{2\beta}\,\mathrm{d}u\,\mathrm{d}r + g_{AB}\left(\mathrm{d}\Theta^A - \mathcal{U}^A\,\mathrm{d}u\right)\left(\mathrm{d}\Theta^B - \mathcal{U}^B\,\mathrm{d}u\right), \tag{D.1}$$

where $A, B = (z, \bar{z})$, U, β, and $\mathcal{U}^A$ are functions of $(u, r, z, \bar{z})$, and where $g_{rr} = 0$, $g_{rA} = 0$ and $\partial_r \det(g_{AB}/r^2) = 0$. The expansions of U, g_{AB} and $\mathcal{U}^A$ are given by [60,61]

$$\begin{aligned}
U &= 1 - \frac{2m_B}{r} + \mathcal{O}(r^{-2}),\\
\beta &= -\frac{C_{AB}C^{AB}}{32r^2} + \mathcal{O}(r^{-3}),\\
g_{AB} &= r^2\gamma_{AB} + rC_{AB} + h^{(0)}_{AB} + \mathcal{O}(r^{-2}),\\
\mathcal{U}^A &= \frac{D_B C^{AB}}{2r^2} + \frac{1}{r^3}\Big[-\frac{2}{3}N^A + \frac{1}{16}D^A(C_{BC}C^{BC}) + \frac{1}{2}C^{AB}D^C C_{BC}\Big] + \mathcal{O}(r^{-4}),
\end{aligned} \tag{D.2}$$

The expansions (D.2) yield a retarded line element of the following form

$$\begin{aligned}
\mathrm{d}s^2 = &-\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}\\
&+ \frac{2m_B}{r}\,\mathrm{d}u^2 + rC_{zz}\,\mathrm{d}z^2 + rC_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z}^2 + D^z C_{zz}\,\mathrm{d}u\,\mathrm{d}z + D^{\bar{z}}C_{\bar{z}\bar{z}}\,\mathrm{d}u\,\mathrm{d}\bar{z}\\
&+ \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)\mathrm{d}u\,\mathrm{d}z + \text{c.c.} + \ldots,
\end{aligned} \tag{D.3}$$

From expansions (D.2), the large r falloffs for the metric components can be determined,

$$\begin{aligned}
g_{uu} &= -1 + \frac{2m_B}{r} + \mathcal{O}(r^{-2}),\\
g_{ur} &= -1 + \frac{1}{8r^2}C^{zz}C_{zz} + \mathcal{O}(r^{-3}),\\
g_{uz} &= \frac{1}{2}D^zC_{zz} + \frac{1}{2r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right) + \mathcal{O}(r^{-2}),\\
g_{zz} &= rC_{zz} + h^{(0)}_{zz} + \mathcal{O}(r^{-1}),\\
g_{z\bar{z}} &= r^2\gamma_{z\bar{z}} + h^{(0)}_{z\bar{z}} + \mathcal{O}(r^{-1}),\\
g_{rz} &= g_{rr} = 0.
\end{aligned} \tag{D.4}$$

Note that gauge condition (3.3) implies that ${C_A}^A = 0$,

$$\begin{aligned}
0 =& \partial_r \det\left(\frac{g_{AB}}{r^2}\right)\\
=& \partial_r \det\left(\gamma_{AB} + \frac{1}{r}C_{AB} + \mathcal{O}(r^{-2})\right)\\
=& \partial_r \det(\gamma_{AB})\left(1 + \frac{1}{r}{C_A}^A + \mathcal{O}(r^{-2})\right)\\
=& -\frac{\det(\gamma_{AB})}{r^2}{C_A}^A + \mathcal{O}(r^{-3}), \qquad \Rightarrow \qquad {C_A}^A = 0.
\end{aligned} \tag{D.5}$$

This, in turn, implies that $C_{z\bar{z}} = 0$, since $\gamma^{z\bar{z}}C_{z\bar{z}} = {C_z}^z = 0$.
The $\mathcal{O}(1)$ component of g_{AB} is defined as follows [**?**]

$$h^{(0)}_{AB} = \frac{1}{4}\gamma_{AB}C^C_DC^D_C + D_{AB}, \tag{D.6}$$

with ${D_A}^A = 0$, $D_{z\bar{z}} = 0$, and $\partial_u D_{AB} = 0$. Equation (D.6) means, for example, that

$$h^{(0)A}_A = \frac{1}{2}C_{zz}C^{zz}, \quad \text{and} \quad h_{zz} = D_{zz}. \tag{D.7}$$

For large r, the inverse metric is determined to be

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0\\ -1 & 1 - \frac{2m_B}{r} + \frac{(D^zC_{zz})(D_zC^{zz})}{2r^2} & \frac{D_zC^{zz}}{2r^2} & \frac{D_{\bar{z}}C^{\bar{z}\bar{z}}}{2r^2}\\ 0 & \frac{D_zC^{zz}}{2r^2} & -\frac{C^{zz}}{r^3} & \frac{\gamma^{z\bar{z}}}{r^2}\\ 0 & \frac{D_{\bar{z}}C^{\bar{z}\bar{z}}}{2r^2} & \frac{\gamma^{z\bar{z}}}{r^2} & -\frac{C^{\bar{z}\bar{z}}}{r^3} \end{pmatrix}. \tag{D.8}$$

D.2 Christoffel Symbols

In order to determine the Christoffel symbols, one has to become accustomed with the way γ can be used to raise and lower indices. For example,

$$C_{zz}\partial_u C^{zz} = C^{\bar{z}\bar{z}}(\gamma_{z\bar{z}})^2\partial_u C^{zz} = C^{\bar{z}\bar{z}}\partial_u C_{\bar{z}\bar{z}}. \tag{D.9}$$

Note that γ does not depend on u, so it can be moved around freely around ∂_u.
The nonzero Christoffel symbols are given by

$$\begin{aligned}
\Gamma^u_{zz} &= \frac{1}{2}C_{zz} + \mathcal{O}(r^{-2}) & \Gamma^u_{uu} &= -\frac{1}{8r^2}(\partial_u(C_{zz}C^{zz}) + 8m_B) + \mathcal{O}(r^{-3}) \\
\Gamma^u_{z\bar{z}} &= r\gamma_{z\bar{z}} + \mathcal{O}(r^{-2}) & \Gamma^u_{uz} &= -\frac{1}{3r^2}\left(N_z + u\partial_z m_B\right) + \mathcal{O}(r^{-3}) \\
\Gamma^r_{uu} &= -\frac{\partial_u m_B}{r} + \mathcal{O}(r^{-2}) & \Gamma^r_{uz} &= \frac{1}{4r}\Big[(D_zC^{zz})(\partial_u C_{zz}) - 4\partial_z m_B\Big] + \mathcal{O}(r^{-2}) \\
\Gamma^r_{ur} &= \frac{m_B}{r^2} + \mathcal{O}(r^{-3}) & \Gamma^r_{zz} &= \frac{1}{2}r\partial_u C_{zz} - \frac{1}{2}\Big(D_zD^zC_{zz} + C_{zz}\Big) + \mathcal{O}(1) \\
\Gamma^r_{rr} &= \mathcal{O}(r^{-3}) & \Gamma^z_{uu} &= \frac{\partial_u D_z C^{zz}}{2r^2} + \frac{\gamma^{z\bar{z}}}{3r^3}\Big(2\partial_u N_{\bar{z}} + 2u\partial_u\partial_{\bar{z}}m_B - \partial_{\bar{z}}m_B\Big) \\
\Gamma^r_{rz} &= \frac{D^zC_{zz}}{2r} + \mathcal{O}(r^{-2}) & &\quad - \frac{\gamma^{z\bar{z}}}{8r^3}\Big(\partial_u\partial_{\bar{z}}(C_{zz}C^{zz}) + 4C^{zz}\partial_u\partial_{\bar{z}}C_{zz}\Big) + \mathcal{O}(r^{-4}) \\
\Gamma^r_{z\bar{z}} &= -r\gamma_{z\bar{z}} + \mathcal{O}(r^{-1}) & \Gamma^z_{uz} &= \frac{C_{zz}\partial_u C^{zz} - C^{zz}\partial_u C_{zz} + D_z^2C^{zz} - D_{\bar{z}}^2C^{\bar{z}\bar{z}}}{4r^2} \\
\Gamma^z_{ur} &= \mathcal{O}(r^{-4}) & &\quad + \mathcal{O}(r^{-3}) \\
\Gamma^z_{r\bar{z}} &= -\frac{C^{zz}\gamma_{z\bar{z}}}{2r^2} + \mathcal{O}(r^{-4}) & \Gamma^z_{u\bar{z}} &= \frac{\gamma^{z\bar{z}}\partial_u C_{\bar{z}\bar{z}}}{2r} + \mathcal{O}(r^{-3}) \\
\Gamma^z_{z\bar{z}} &= \mathcal{O}(r^{-2}) & \Gamma^z_{rz} &= \frac{1}{r} - \frac{C^{zz}C_{zz}}{2r^3} + \mathcal{O}(r^{-4}) \\
\Gamma^z_{\bar{z}\bar{z}} &= \frac{D^zC_{\bar{z}\bar{z}}}{2r} + \mathcal{O}(r^{-2}) & \Gamma^z_{zz} &= \gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}} - \frac{\gamma^{z\bar{z}}\partial_{\bar{z}}C_{zz}}{2r} + \mathcal{O}(r^{-2}),
\end{aligned} \tag{D.10}$$

plus their complex conjugates. We often used the definition of the covariant derivative with respect to $\gamma_{z\bar{z}}$ to simplify Christoffel symbols. For example, for $\Gamma^z_{z\bar{z}}$ and $\Gamma^z_{\bar{z}\bar{z}}$ we used, respectively

$$\begin{aligned}
D_zC_{\bar{z}\bar{z}} &= \partial_z C_{\bar{z}\bar{z}} - \cancel{\tilde{\Gamma}^a_{z\bar{z}}}C_{a\bar{z}} - \cancel{\tilde{\Gamma}^a_{z\bar{z}}}C_{\bar{z}a} = \partial_z C_{\bar{z}\bar{z}}, \\
D_{\bar{z}}C_{\bar{z}\bar{z}} &= \partial_{\bar{z}} C_{\bar{z}\bar{z}} - \tilde{\Gamma}^a_{\bar{z}\bar{z}}C_{a\bar{z}} - \tilde{\Gamma}^a_{\bar{z}\bar{z}}C_{\bar{z}a} = \partial_{\bar{z}} C_{\bar{z}\bar{z}} - 2\gamma^{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}C_{\bar{z}\bar{z}},
\end{aligned} \tag{D.11}$$

where the Christoffel symbols with respect to $\gamma_{z\bar{z}}$ are given by (2.15). Furthermore, note that you can freely move the γ's around covariant derivatives with respect to γ. For

example,

$$\begin{aligned}(\gamma_{z\bar{z}})^2 D_{\bar{z}} C^{\bar{z}\bar{z}} &= (\gamma_{z\bar{z}})^2 \partial_{\bar{z}} C^{\bar{z}\bar{z}} + 2(\gamma_{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}}) C^{\bar{z}\bar{z}} \\ &= \partial_{\bar{z}} \Big((\gamma_{z\bar{z}})^2 C^{\bar{z}\bar{z}} \Big) \\ &= \partial_{\bar{z}} C_{zz} = D_{\bar{z}} C_{zz}. \end{aligned} \tag{D.12}$$

In order to obtain $\Gamma^z_{u\bar{z}}$, we used the fact that $\partial_u h^{(0)}_{\bar{z}\bar{z}} = \partial_u D_{\bar{z}\bar{z}} = 0$. To reach the final form of Γ^z_{uz}, we had to use this particular piece of algebra,

$$\begin{aligned}\gamma^{z\bar{z}}[\partial_z D^{\bar{z}} C_{\bar{z}\bar{z}} - \partial_{\bar{z}} D^z C_{zz}] =& \gamma^{z\bar{z}}[\partial_z (D_z C^{zz} \gamma_{z\bar{z}}) - \partial_{\bar{z}} (D_{\bar{z}} C^{\bar{z}\bar{z}} \gamma_{z\bar{z}})] \\ =& \partial_z D_z C^{zz} + (\gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}}) D_z C^{zz} - \partial_{\bar{z}} D_{\bar{z}} C^{\bar{z}\bar{z}} - (\gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}}) D_{\bar{z}} C^{\bar{z}\bar{z}} \\ =& D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}. \end{aligned} \tag{D.13}$$

D.3 Einstein's Equations: a Constraint on m_B and N_z

Einstein's equations are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T^M_{\mu\nu}, \tag{D.14}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the Ricci scalar or the scalar curvature, G is Newton's gravitational constant, and $T^M_{\mu\nu}$ is a matter stress-energy tensor corresponding to massless modes.

Constraint on m_B

By computing the leading order uu-component of the Einstein's equation, we will obtain a constraint on m_B. In order to do so, we first need to compute the Ricci tensor R_{uu} and Ricci scalar R. The Ricci tensor is given by

$$R_{ab} = R^c_{acb}, \tag{D.15}$$

where R^c_{acb} denotes the Riemann curvature tensor

$$R^a_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{c\rho} \Gamma^\rho_{db} - \Gamma^a_{d\rho} \Gamma^\rho_{cb}. \tag{D.16}$$

Whilst computing the Riemann curvature tensors, note that $R^a_{bcd} = -R^a_{bdc}$. This implies that $R^a_{bcc} = 0$, and thus also that terms like R^u_{uuu} are zero. We have

$$\begin{aligned} R^r_{uru} =& \mathcal{O}(r^{-3}), \\ R^z_{uzu} =& \frac{\partial_z \partial_u D_z C^{zz} + (\gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}}) \partial_u D_z C^{zz}}{2r^2} - \partial_u \left(\frac{C_{zz} \partial_u C^{zz} - C^{zz} \partial_u C_{zz} + D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}}{4r^2} \right) \\ & - \frac{\partial_u m_B}{r^2} - \frac{(\partial_u C^{zz})(\partial_u C_{zz})}{4r^2} + \mathcal{O}(r^{-3}). \end{aligned}$$

Hence, R_{uu} is given by

$$\begin{aligned} R_{uu} =& R^r_{uru} + R^z_{uzu} + R^{\bar{z}}_{u\bar{z}u} \\ =& \frac{1}{2r^2}\left(\partial_z\partial_u D_z C^{zz} + (\gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}})\partial_u D_z C^{zz}\right) + \frac{1}{2r^2}\left(\partial_{\bar{z}}\partial_u D_{\bar{z}} C^{\bar{z}\bar{z}} + (\gamma^{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}})\partial_u D_{\bar{z}} C^{\bar{z}\bar{z}}\right) \\ & - \frac{1}{2r^2}\left(4\partial_u m_B + (\partial_u C^{zz})(\partial_u C_{zz})\right) + \mathcal{O}(r^{-3}). \end{aligned} \tag{D.17}$$

The Ricci scalar is specified by

$$\begin{aligned} R =& g^{ij}R_{ij} \\ =& 2g^{ur}R_{ur} + g^{rr}R_{rr} + 2g^{rz}R_{rz} + 2g^{r\bar{z}}R_{r\bar{z}} + g^{zz}R_{zz} + 2g^{z\bar{z}}R_{z\bar{z}} + g^{\bar{z}\bar{z}}R_{\bar{z}\bar{z}}, \end{aligned} \tag{D.18}$$

where

$$R_{ur} = \mathcal{O}(r^{-3}), \quad R_{rr} = \mathcal{O}(r^{-4}), \quad R_{rz} = \mathcal{O}(r^{-3}), \quad R_{zz} = \mathcal{O}(r^{-1}), \quad R_{z\bar{z}} = \mathcal{O}(r^{-1}).$$

Thus, it follows that $R = \mathcal{O}(r^{-3})$. Also note that [60]

$$T^M_{uu} = \sum_{n=2}^{\infty} \frac{T^{M(n)}_{uu}}{r^n} \quad \text{implies that} \quad T^{M(2)}_{uu} = \lim_{r\to\infty}\left[r^2 T^M_{uu}\right]. \tag{D.19}$$

After plugging everything into the uu-component of Einstein's equation (D.14) and multiplying both sides of the equation with $2r^2$, we end up with

$$\begin{aligned} 4\partial_u m_B =& \partial_z D_z \partial_u C^{zz} + (\gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}}) D_z\partial_u C^{zz} \\ &+ \partial_{\bar{z}} D_{\bar{z}}\partial_u C^{\bar{z}\bar{z}} + (\gamma^{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}) D_{\bar{z}}\partial_u C^{\bar{z}\bar{z}} \\ &- (\partial_u C^{zz})(\partial_u C_{zz}) - 16\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right] + \mathcal{O}(r^{-1}). \end{aligned} \tag{D.20}$$

Now note that

$$D^2_z N^{zz} = \partial_z D_z N^{zz} + (\gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}}) D_z N^{zz}. \tag{D.21}$$

Thus, we finally obtain

$$\partial_u m_B = \frac{1}{4}\left[D^2_z(N^{zz}) + D^2_{\bar{z}}(N^{\bar{z}\bar{z}})\right] - T_{uu}, \tag{D.22}$$

with

$$T_{uu} \equiv \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right]. \tag{D.23}$$

Similarly, on $\mathcal{I}^-$, one finds

$$\partial_v m_B = \frac{1}{4}\left[D^2_z(N^{zz}) + D^2_{\bar{z}}(N^{\bar{z}\bar{z}})\right] + T_{vv}, \tag{D.24}$$

with

$$T_{vv} \equiv \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{vv}\right]. \tag{D.25}$$

Constraint on N_z

Computing the leading order uz-component of Einstein's equations (D.14) yields a constraint on the angular momentum aspect N_z. We have,

$$\begin{aligned}
R^u_{uuz} =& -\frac{\gamma^{z\bar{z}}\partial_u\partial_{\bar{z}}C_{zz}}{2r} - \frac{1}{r^2}\Big(\partial_u N_z + u\partial_u\partial_z m_B - \partial_z m_B\Big) \\
&+ \frac{C_{zz}\partial_u D_z C^{zz}}{4r^2} + \frac{\partial_u\partial_z(C_{zz}C^{zz})}{4r^2} + \mathcal{O}(r^{-3}), \\
R^r_{urz} =& \mathcal{O}(r^{-3}), \\
R^{\bar{z}}_{u\bar{z}z} =& \frac{\gamma^{z\bar{z}}\partial_{\bar{z}}\partial_u C_{zz}}{2r} - \frac{\partial_z m_B}{r^2} - \frac{(D_z C_{zz})(\partial_u C^{zz})}{4r^2} \\
&- \partial_z\Big[\frac{-C_{zz}\partial_u C^{zz} + C^{zz}\partial_u C_{zz} - D_z^2 C^{zz} + D_{\bar{z}}^2 C^{\bar{z}\bar{z}}}{4r^2}\Big] + \mathcal{O}(r^{-3}).
\end{aligned} \tag{D.26}$$

Thus,

$$\begin{aligned}
R_{uz} =& -\frac{1}{r^2}\Big(\partial_u N_z + u\partial_u\partial_z m_B\Big) + \frac{1}{4r^2}\partial_z\Big[D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\Big] \\
&+ \frac{1}{4r^2}\partial_z\Big(C_{zz}\partial_u C^{zz}\Big) + \frac{1}{2r^2}C_{zz}D_z\partial_u C^{zz} + \mathcal{O}(r^{-3}).
\end{aligned} \tag{D.27}$$

The uz-component of the stress tensor is $\mathcal{O}(r^{-2})$, which implies [60]

$$T^M_{uz} = \sum_{n=2}^{\infty}\frac{T^{M(n)}_{uz}}{r^n}. \tag{D.28}$$

Combining everything, one obtains

$$\partial_u N_z = \frac{1}{4}\partial_z(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u\partial_u\partial_z m_B - T_{uz}, \tag{D.29}$$

with

$$T_{uz} \equiv 8\pi G \lim_{r\to\infty}\Big[r^2 T^M_{uz}\Big] - \frac{1}{4}\partial_z(C_{zz}N^{zz}) - \frac{1}{2}C_{zz}D_z N^{zz}. \tag{D.30}$$

D.4 Dirac Brackets

The symplectic form is given by [4]

$$\begin{aligned}
\Omega_{\mathcal{I}^+} &= \frac{1}{16\pi G}\int \mathrm{d}u\,\mathrm{d}^2z\gamma^{z\bar{z}}\left(\delta C_{zz}\wedge\delta N_{\bar{z}\bar{z}} - \delta N_{zz}\wedge\delta C_{\bar{z}\bar{z}}\right) \\
&= -\frac{1}{16\pi G}\int \mathrm{d}u\,\mathrm{d}^2z\gamma^{z\bar{z}}\left(\delta N_{\bar{z}\bar{z}}\wedge\delta C_{zz} + \delta N_{zz}\wedge\delta C_{\bar{z}\bar{z}}\right).
\end{aligned} \tag{D.31}$$

After inverting the symplectic form, one finds

$$\begin{aligned}
&\left\{N_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} \\
&\qquad = -16\pi G \int \mathrm{d}u''\,\mathrm{d}^2z'\gamma_{z\bar{z}} \left(\cancel{\frac{\delta N_{\bar{z}\bar{z}}}{\delta C_{z'z'}}\frac{\delta C_{ww}}{\delta N_{\bar{z}'\bar{z}'}}} - \frac{\delta N_{\bar{z}\bar{z}}}{\delta N_{\bar{z}'\bar{z}'}}\frac{\delta C_{ww}}{\delta C_{z'z'}} \right) \\
&\qquad = 16\pi G \int \mathrm{d}u''\,\mathrm{d}^2z'\gamma_{z\bar{z}}\delta(u-u'')\delta^2(z-z')\delta(u'-u'')\delta^2(w-z') \\
&\qquad = 16\pi G\gamma_{z\bar{z}}\delta(u-u')\delta^2(z-w).
\end{aligned} \tag{D.32}$$

Differentiating (D.32) with respect to u' yields

$$\left\{N_{\bar{z}\bar{z}}(u,z,\bar{z}), N_{ww}(u',w,\bar{w})\right\} = -16\pi G\gamma_{z\bar{z}}\partial_u\delta(u-u')\delta^2(z-w), \tag{D.33}$$

where one should note that

$$\partial_{u'}\delta(u-u') = -iw\int\frac{\mathrm{d}w}{2\pi}e^{iw(u-u')} = -\partial_u\delta(u-u'). \tag{D.34}$$

Observe that N_{ww} is not coupled with N_{zz} in the symplectic form (D.31), which implies that $\{N_{zz}, N_{ww}\} = 0$. By using (3.8) and (3.9), one has the following relations

$$D_z^2\left\{N^+(z,\bar{z}), C_{\bar{w}\bar{w}}(u,w,\bar{w})\right\} = \int_{-\infty}^{\infty}\mathrm{d}u'\left\{N_{zz}(u',z,\bar{z}), C_{\bar{w}\bar{w}}(u,w,\bar{w})\right\}, \tag{D.35}$$

$$D_{\bar{w}}^2\left\{N^+(z,\bar{z}), C(w,\bar{w})\right\} = \lim_{u\to-\infty}\left\{N^+(z,\bar{z}), C_{\bar{w}\bar{w}}(u,w,\bar{w})\right\}. \tag{D.36}$$

Integrating (D.32) with respect to u yields

$$\left\{C_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = 8\pi G\gamma_{z\bar{z}}\Theta(u-u')\delta^2(z-w), \tag{D.37}$$

with sign function Θ (C.23), and $\partial_u\Theta(u-u') = 2\delta(u-u')$.
The angle between $(z,\bar{z})$ and $(w,\bar{w})$ on the sphere is denoted by $\Delta\Theta$. Using this angle, let us introduce

$$S \equiv \sin^2\frac{\Delta\Theta}{2} = \frac{(z-w)(\bar{z}-\bar{w})}{(1+z\bar{z})(1+w\bar{w})}, \tag{D.38}$$

with properties

$$\begin{aligned}
D_w^2\left(S\ln|z-w|^2\right) &= \partial_w^2\left(S\ln|z-w|^2\right) - \left(\gamma^{w\bar{w}}\partial_w\gamma_{w\bar{w}}\right)\left(S\ln|z-w|^2\right) \\
&= \frac{S}{(z-w)^2}, \\
D_{\bar{z}}^2D_w^2\left(S\ln|z-w|^2\right) &= \partial_{\bar{z}}^2\frac{S}{(z-w)^2} - (\gamma^{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}})\partial_{\bar{z}}\frac{S}{(z-w)^2} \\
&= \pi\gamma_{z\bar{z}}\delta^2(z-w),
\end{aligned} \tag{D.39}$$

where we used the fact that $\partial_{\bar{z}}(1/(z-w)) = 2\pi\delta^2(z-w)$. Furthermore, note that $D^2_{\bar{z}}D^2_w\left(S\ln|z-w|^2\right) = D^2_{\bar{w}}D^2_w\left(S\ln|z-w|^2\right)$, so we equally likely could have written $D^2_{\bar{z}}D^2_w\left(S\ln|z-w|^2\right) = \pi\gamma_{w\bar{w}}\delta^2(z-w)$. Equations (D.32) and (D.37) can now be rewritten as

$$\left\{N_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = 16G\delta(u-u')D^2_{\bar{z}}D^2_w\left(S\ln|z-w|^2\right). \tag{D.40}$$

$$\left\{C_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = 8G\Theta(u-u')D^2_{\bar{z}}D^2_w\left(S\ln|z-w|^2\right) \tag{D.41}$$

Now assume that u' is not on the boundary and that $u \to -\infty$. Using relation (3.8), we can rewrite (D.41) as

$$D^2_{\bar{z}}\left\{C(z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = -8GD^2_{\bar{z}}D^2_w\left(S\ln|z-w|^2\right), \tag{D.42}$$

or

$$\left\{C(z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = -8GD^2_w\left(S\ln|z-w|^2\right), \tag{D.43}$$

where one should note that if u' is not on the boundary, then $\Theta(u-u') = -1$ for $u \to -\infty$. Differentiating (D.43) with respect to u' yields

$$\left\{C(z,\bar{z}), N_{ww}(u',w,\bar{w})\right\} = \partial_{u'}\left\{C(z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = 0, \tag{D.44}$$

Using relation (3.9), one can integrate (D.40) with respect to u and obtain

$$D^2_{\bar{z}}\left\{N^+(z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = 16GD^2_{\bar{z}}D^2_w\left(S\ln|z-w|^2\right), \tag{D.45}$$

or

$$\left\{N^+(z,\bar{z}), C_{ww}(u',w,\bar{w})\right\} = 16GD^2_w\left(S\ln|z-w|^2\right). \tag{D.46}$$

Note that

$$\{N^+(z,\bar{z}), N_{ww}(u',w,\bar{w})\} = \partial_{u'}\{N^+(z,\bar{z}), C_{ww}(u',w,\bar{w})\} = 0. \tag{D.47}$$

Finally, let $u' \to -\infty$ for (D.46), which results in

$$D^2_w\left\{N^+(z,\bar{z}), C(w,\bar{w})\right\} = 16GD^2_w\left(S\ln|z-w|^2\right), \tag{D.48}$$

or

$$\left\{N^+(z,\bar{z}), C(w,\bar{w})\right\} = 16G\left(S\ln|z-w|^2\right). \tag{D.49}$$

D.5 Supertranslations

D.5.1 Diffeomorphism

In this section, we will determine the structure of the supertranslation diffeomorphism ζ that satisfies the falloff conditions for the metric components (D.4), where

$$\zeta^u, \zeta^r \sim \mathcal{O}(1), \qquad \zeta^z, \zeta^{\bar{z}} \sim \mathcal{O}(r^{-1}), \tag{D.50}$$

in order to eliminate boosts and rotations which grow with r at infinity.
The variation of the metric generated by ζ is given by Lie derivative [41]

$$\mathcal{L}_\zeta g_{\mu\nu} = \zeta^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \zeta^\rho + g_{\nu\rho}\partial_\mu \zeta^\rho. \tag{D.51}$$

While calculating the Lie derivative of the metric components at large r, we only keep track of the leading components. Since $g_{rr} = 0$, we have

$$\begin{aligned}
\mathcal{L}_\zeta g_{rr} &= \zeta^\rho \partial_\rho \cancel{g_{rr}} + 2 g_{r\rho}\partial_r \zeta^\rho = 0, \\
&\Rightarrow g_{ru}\partial_r \zeta^u = 0, \\
&\Rightarrow -\partial_r \zeta^u + \mathcal{O}(r^{-2}) = 0, \\
&\Rightarrow \zeta^u = \zeta^u(u, z, \bar{z}).
\end{aligned} \tag{D.52}$$

Since the Lie derivative of a scalar function ϕ is given by [41]

$$\mathcal{L}_\zeta \phi = \zeta^\rho \partial_\rho \phi, \tag{D.53}$$

it follows that $\mathcal{L}_\zeta g_{ur}|_{\mathcal{O}(1)} = -\mathcal{L}_\zeta 1 = 0$. Thus,

$$\begin{aligned}
\mathcal{L}_\zeta g_{ur} &= \zeta^\rho \partial_\rho g_{ur} + g_{u\rho}\partial_r \zeta^\rho + g_{r\rho}\partial_u \zeta^\rho \\
&= \underbrace{\zeta^\rho \partial_\rho g_{ur}}_{\mathcal{O}(r^{-2})} + g_{uu}\cancel{\partial_r \zeta^u} + \underbrace{g_{ur}\partial_r \zeta^r}_{\mathcal{O}(r^{-2})} + \underbrace{g_{uz}\partial_r \zeta^z}_{\mathcal{O}(r^{-2})} + \underbrace{g_{u\bar{z}}\partial_r \zeta^{\bar{z}}}_{\mathcal{O}(r^{-2})} + \underbrace{g_{ru}\partial_u \zeta^u}_{\mathcal{O}(1)} \\
&= -\partial_u \zeta^u + \mathcal{O}(r^{-2}), \\
&\Rightarrow \partial_u \zeta^u = 0, \\
&\Rightarrow \zeta^u = f(z, \bar{z}).
\end{aligned} \tag{D.54}$$

At this point, we will asymptotically expand ζ as follows,

$$\zeta = f\partial_u + \sum_{n=0}^{\infty} \frac{\zeta^{r(n)}}{r^n}\partial_r + \sum_{n=1}^{\infty} \frac{\zeta^{z(n)}}{r^n}\partial_z + + \sum_{n=1}^{\infty} \frac{\zeta^{\bar{z}(n)}}{r^n}\partial_{\bar{z}}. \tag{D.55}$$

Since $g_{rz} = 0$, we obtain

$$\begin{aligned}
\mathcal{L}_\zeta g_{rz} &= \zeta^\rho \partial_\rho \cancel{g_{rz}} + g_{r\rho}\partial_z \zeta^\rho + g_{z\rho}\partial_r \zeta^\rho \\
&= g_{ru}\partial_z f + g_{zu}\cancel{\partial_r \zeta^u} + g_{zz}\partial_r \zeta^z + g_{z\bar{z}}\partial_r \zeta^{\bar{z}} \\
&= -\partial_z f - \frac{1}{r} C_{zz}\zeta^{z(1)} - \gamma_{z\bar{z}}\zeta^{\bar{z}(1)} - \frac{2}{r}\gamma_{z\bar{z}}\zeta^{\bar{z}(2)} + \mathcal{O}(r^{-2}) \\
&= -\left(D_z f + \gamma_{z\bar{z}}\zeta^{\bar{z}(1)}\right) - \frac{1}{r}\left(C_{zz}\zeta^{z(1)} + 2\gamma_{z\bar{z}}\zeta^{\bar{z}(2)}\right) + \mathcal{O}(r^{-2}) = 0,
\end{aligned} \tag{D.56}$$

where $D_z f = \partial_z f$. Equation (D.56) implies that $\zeta^{\bar{z}(1)} = -D^{\bar{z}} f$. Thus,

$$\zeta^{\bar{z}} = -\frac{1}{r} D^{\bar{z}} f + \mathcal{O}(r^{-2}), \quad \text{and} \quad \zeta^z = -\frac{1}{r} D^z f + \mathcal{O}(r^{-2}). \tag{D.57}$$

Furthermore,

$$\begin{aligned}
\mathcal{L}_\zeta g_{z\bar{z}} =& \zeta^\rho \partial_\rho g_{z\bar{z}} + g_{z\rho}\partial_{\bar{z}}\zeta^\rho + g_{\bar{z}\rho}\partial_z \zeta^\rho \\
=& \zeta^r \partial_r g_{z\bar{z}} + \zeta^z \partial_z g_{z\bar{z}} + \zeta^{\bar{z}}\partial_{\bar{z}} g_{z\bar{z}} + g_{z\bar{z}}\partial_{\bar{z}}\zeta^{\bar{z}} + g_{\bar{z}z}\partial_z \zeta^z + \mathcal{O}(1) \\
=& 2r\gamma_{z\bar{z}}\zeta^r + r^2\zeta^z\partial_z\gamma_{z\bar{z}} + r^2\zeta^{\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}} + r^2\gamma_{z\bar{z}}\partial_{\bar{z}}\zeta^{\bar{z}} + r^2\gamma_{z\bar{z}}\partial_z\zeta^z + \mathcal{O}(1) \\
=& r\gamma_{z\bar{z}}\left[2\zeta^r + r(\partial_z\zeta^z + (\gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}})\zeta^z) + r(\partial_{\bar{z}}\zeta^{\bar{z}} + (\gamma^{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}})\zeta^{\bar{z}})\right] + \mathcal{O}(1) \\
=& r\gamma_{z\bar{z}}\left(2\zeta^r + rD_z\zeta^z + rD_{\bar{z}}\zeta^{\bar{z}}\right) + \mathcal{O}(1) \\
=& r\gamma_{z\bar{z}}\left(2\zeta^{r(0)} - D_z D^z f - D_{\bar{z}} D^{\bar{z}} f\right) + \mathcal{O}(1),
\end{aligned} \tag{D.58}$$

where we used (D.57) and $D_z\zeta^z = \partial_z\zeta^z + (\gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}})\zeta^z$. Since $g_{z\bar{z}}|_{\mathcal{O}(r)} = 0$, Lie derivative (D.58) implies that

$$\zeta^{r(0)} = \frac{1}{2}\left(D_z D^z f + D_{\bar{z}} D^{\bar{z}} f\right). \tag{D.59}$$

Note that

$$D_z D^z f + D_{\bar{z}} D^{\bar{z}} f = \gamma^{z\bar{z}}(\partial_z\partial_{\bar{z}} + \partial_{\bar{z}}\partial_z) f = 2\gamma^{z\bar{z}}\partial_{\bar{z}}\partial_z f = 2D^z D_z f. \tag{D.60}$$

Thus,

$$\zeta^r = D^z D_z f + \mathcal{O}(r^{-1}). \tag{D.61}$$

Combining (D.54), (D.57) and (D.61), one finds that the diffeomorphism has the following form up to leading order:

$$\zeta = f\partial_u + D^z D_z f \partial_r - \frac{1}{r}(D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + \dots. \tag{D.62}$$

Using above diffeomorphism, one can determine the action of supertranslations on m_B. Note that $\mathcal{L}_\zeta m_B = \frac{r}{2} \cdot \mathcal{L}_\zeta g_{uu}\Big|_{\mathcal{O}(r^{-1})}$, with

$$\begin{aligned}\mathcal{L}_\zeta g_{uu} =& \zeta^\rho \partial_\rho g_{uu} + 2 g_{u\rho} \partial_u \zeta^\rho = f \partial_u g_{uu} + 2 g_{ur} \partial_u \zeta^r + \mathcal{O}(r^{-2}) \\ =& -2\cancel{\partial_u \zeta^{r(0)}} + \frac{1}{r}\Big(2 f \partial_u m_B - 2\partial_u \zeta^{r(1)} + 4 m_B \cancel{\partial_u \zeta^{r(0)}}\Big) + \mathcal{O}(r^{-2}).\end{aligned} \tag{D.63}$$

In order to determine the $\mathcal{O}(r^{-1})$ of $\mathcal{L}_\zeta g_{uu}$, we need $\zeta^{r(1)}$, which is given by [1]

$$\zeta^{r(1)} = -\frac{1}{4} C^{AB} D_A D_B f - \frac{1}{2}(D_A f) D_B C^{AB}, \text{ with } A, B = z, \bar{z}. \tag{D.64}$$

By inserting (D.64) into equation (D.63), we obtain

$$\mathcal{L}_\zeta g_{uu}\Big|_{\mathcal{O}(r^{-1})} = \frac{1}{r}\Big(2 f \partial_u m_B + \frac{1}{2}(\partial_u C^{AB}) D_A D_B f + (D_A f) \partial_u D_B C^{AB}\Big). \tag{D.65}$$

Since we have

$$\partial_u (D_z C^{zz}) = \partial_u [\partial_z C^{zz} + 2(\gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}}) C^{zz}] = \partial_z \partial_u C^{zz} + 2(\gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}}) \partial_u C^{zz} = D_z (\partial_u C^{zz}),$$

it is allowed to switch the order of ∂_u and D_z in equation (D.65). Also, note that $C_{z\bar{z}} = 0$. Thus, we discover that

$$\mathcal{L}_\zeta m_B = f \partial_u m_B + \frac{1}{4}\left(N^{zz} D_z^2 f + 2 D_z N^{zz} D_z f + \text{c.c.}\right). \tag{D.66}$$

D.5.2 Plane Wave Expansion of $D_z^2 N^+$

From

$$D_z^2 N^+ = \frac{1}{2} \lim_{\omega \to 0^+} \int_{-\infty}^{\infty} \mathrm{d}u (e^{i\omega u} + e^{-i\omega u}) \partial_u C_{zz}, \tag{D.67}$$

and

$$C_{zz} = -\frac{i\sqrt{2G}}{\sqrt{\pi^3}(1 + z\bar{z})^2} \int_0^\infty \mathrm{d}\omega \Big[a_+^{\text{out}}(\omega \hat{x}) e^{-i\omega u} - a_-^{\text{out}\dagger}(\omega \hat{x}) e^{i\omega u}\Big], \tag{D.68}$$

it follows that

$$
\begin{aligned}
D_z^2 N^+ &= \frac{1}{2}\lim_{\omega\to 0^+}\int_{-\infty}^{\infty} \mathrm{d}u (e^{i\omega u} + e^{-i\omega u})\partial_u C_{zz} \\
&= -\frac{\sqrt{G}}{\sqrt{2\pi^3}(1+z\bar{z})^2}\lim_{\omega\to 0^+}\int_{-\infty}^{\infty} \mathrm{d}u \left(e^{i\omega u} + e^{-i\omega u}\right) \\
&\qquad\qquad \cdot \int_0^{\infty} \mathrm{d}\omega_q\, \omega_q \left[a_+^{\text{out}}(\omega_q \hat{x})\, e^{-i\omega_q u} + a_-^{\text{out}\dagger}(\omega_q \hat{x})\, e^{i\omega_q u}\right] \\
&= -\frac{\sqrt{G}}{\sqrt{2\pi^3}(1+z\bar{z})^2}\lim_{\omega\to 0^+}\int_{-\infty}^{\infty} \mathrm{d}u \int_0^{\infty} \mathrm{d}\omega_q\, \omega_q \\
&\qquad \cdot \left[a_+^{\text{out}}\left(e^{i(\omega-\omega_q)u} + e^{-i(\omega+\omega_q)u}\right) + a_-^{\text{out}\dagger}\left(e^{i(\omega+\omega_q)u} + e^{-i(\omega-\omega_q)u}\right)\right] \\
&= -\frac{\sqrt{2G}}{\sqrt{\pi}(1+z\bar{z})^2}\lim_{\omega\to 0^+}\int_0^{\infty} \mathrm{d}\omega_q\, \omega_q \\
&\qquad \cdot \left[a_+^{\text{out}}\left(\delta(\omega-\omega_q) + \cancel{\delta(\omega+\omega_q)}\right) + a_-^{\text{out}\dagger}\left(\cancel{\delta(\omega+\omega_q)} + \delta(\omega-\omega_q)\right)\right] \\
&= -\gamma_{z\bar{z}}\sqrt{\frac{G}{2\pi}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{out}}(\omega_q\hat{x}) + \omega a_-^{\text{out}\dagger}(\omega_q\hat{x})\right].
\end{aligned}
\tag{D.69}
$$

D.5.3 The Bracket Action of Q_f^+ on N_{zz}, N^+ and C

Using bracket relations derived in Appendix D.4, we will compute the action of Q_f^+ on N_{zz}, N^+ and C. One has

$$
\begin{aligned}
\{Q_f^+, N_{zz}(u,z,\bar{z})\} =& \frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\, \mathrm{d}^2w\, \gamma^{w\bar{w}} f N_{ww}\{N_{\bar{w}\bar{w}}(u',w,\bar{w}), N_{zz}(u,z,\bar{z})\} \\
&- \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2w\, \gamma^{w\bar{w}} f D_w^2 D_{\bar{w}}^2 \cancel{\{N^+(w,\bar{w}), N_{zz}(u,z,\bar{z})\}} \\
=& -\int_{\mathcal{I}^+} \mathrm{d}u'\, \mathrm{d}^2w\, \gamma^{w\bar{w}}\gamma_{z\bar{z}} f N_{ww}\partial_{u'}\delta(u-u')\delta^2(z-w) \\
=& \int_{\mathcal{I}^+} \mathrm{d}u'\, \mathrm{d}^2w\, \gamma^{w\bar{w}}\gamma_{z\bar{z}} f \partial_{u'} N_{ww}\delta(u-u')\delta^2(z-w) \\
=& f\partial_u N_{zz},
\end{aligned}
\tag{D.70}
$$

where N_{ww} was pulled out of the bracket in the first line, because $\{N_{ww}, N_{zz}\} = 0$. After plugging in the Dirac bracket relations (D.33) and (D.47), one integrates by parts with respect to u', and subsequently integrates over the delta functions.

Furthermore,

$$\begin{aligned}\{Q_f^+, N^+(z,\bar{z})\} &= \frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\,\gamma^{w\bar{w}} f\{\cancel{N_{ww}N_{\bar{w}\bar{w}}(u',w,\bar{w}), N^+(z,\bar{z})}\} \\ &\quad - \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\gamma^{w\bar{w}} f D_w^2 D_{\bar{w}}^2 \cancel{\{N^+(w,\bar{w}), N^+(z,\bar{z})\}} = 0,\end{aligned} \tag{D.71}$$

where both bracket relations equal zero. Lastly, one finds

$$\begin{aligned}\{Q_f^+, C(z,\bar{z})\} &= \frac{1}{16\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\,\gamma^{w\bar{w}} f\{\cancel{N_{ww}N_{\bar{w}\bar{w}}(u',w,\bar{w}), C(z,\bar{z})}\} \\ &\quad - \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\gamma^{w\bar{w}} f D_w^2 D_{\bar{w}}^2 \{N^+(w,\bar{w}), C(z,\bar{z})\}. \\ &= -\frac{2}{\pi}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\gamma^{w\bar{w}} f D_w^2 D_{\bar{w}}^2 \left(S \ln|z-w|^2\right) \\ &= -2\int_{\mathcal{I}^+} \mathrm{d}^2w\,\gamma^{w\bar{w}} f \gamma_{z\bar{z}}\delta^2(z-w) \\ &= -2f,\end{aligned} \tag{D.72}$$

where bracket relations (D.44), (D.49) and property of S (D.39) were used.

D.6 Superrotations

D.6.1 Diffeomorphism

For rotations and boosts, the diffeomorphism has the following form [1],

$$\begin{aligned}\zeta_Y = \frac{u}{2}\Big(D_zY^z + D_{\bar{z}}Y^{\bar{z}}\Big)\partial_u - \frac{u+r}{2}\Big(D_zY^z + D_{\bar{z}}Y^{\bar{z}}\Big)\partial_r \\ + \Big[Y^z + \frac{u}{2r}(Y^z - D^zD_{\bar{z}}Y^{\bar{z}})\Big]\partial_z \\ + \Big[Y^{\bar{z}} + \frac{u}{2r}(Y^{\bar{z}} - D^{\bar{z}}D_zY^z)\Big]\partial_{\bar{z}},\end{aligned} \tag{D.73}$$

where $(Y^z, Y^{\bar{z}})$ is a two-dimensional vector field on $\mathcal{C}S^2$, and is independent of u and r. Now take

$$Y^z = a + bz + cz^2. \tag{D.74}$$

We will now show that, for this specific choice of Y^z, the diffeomorphism given by (D.73) generates generic Lorentz transformations. Note that

$$\begin{aligned}D_zY^z + D_{\bar{z}}Y^{\bar{z}} &= \partial_zY^z - \frac{2\bar{z}}{1+z\bar{z}}Y_z + \partial_{\bar{z}}Y^{\bar{z}} - \frac{2z}{1+z\bar{z}}Y_{\bar{z}} \\ &= \frac{2z(c-a^*) + 2\bar{z}(c^*-a) + (b+b^*)(1-z\bar{z})}{1+z\bar{z}},\end{aligned} \tag{D.75}$$

and

$$\begin{aligned} D^z(D_zY^z + D_{\bar{z}}Y^{\bar{z}}) &= \gamma^{z\bar{z}}D_{\bar{z}}(D_zY^z + D_{\bar{z}}Y^{\bar{z}}) \\ &= \frac{(1+z\bar{z})^2}{2}\partial_{\bar{z}}(D_zY^z + D_{\bar{z}}Y^{\bar{z}}) \\ &= -((a-c^*) + (b+b^*)z + (c-a^*)z^2) \end{aligned} \tag{D.76}$$

Furthermore, note that $D^zD_zY^z = -Y^z$. Thus, with this choice for Y^z, one is able to rewrite ζ_Y as a vector field with components

$$\begin{aligned} \zeta^u &= \frac{u}{2}\frac{2z(c-a^*) + 2\bar{z}(c^*-a) + (b+b^*)(1-z\bar{z})}{1+z\bar{z}}, \\ \zeta^r &= -\frac{u+r}{2}\frac{2z(c-a^*) + 2\bar{z}(c^*-a) + (b+b^*)(1-z\bar{z})}{1+z\bar{z}}, \\ \zeta^z &= a + bz + cz^2 + \frac{u}{2r}((a-c^*) + (b+b^*)z + (c-a^*)z^2). \end{aligned} \tag{D.77}$$

In Cartesian coordinates, rotations are given in their more familiar form

$$J_{ij} = x^i\partial_j - x^j\partial_i, \tag{D.78}$$

whereas boosts are given by

$$K_i = x^0\partial_i + x^i\partial_0. \tag{D.79}$$

In retarded Bondi coordinates $(u, r, z, \bar{z})$, rotations and boosts can be re-expressed as

$$\begin{aligned} J_{12} &= iz\partial_z - i\bar{z}\partial_{\bar{z}}, \\ J_{23} &= \frac{1}{2}i(z^2-1)\partial_z - \frac{1}{2}i(\bar{z}^2-1)\partial_{\bar{z}}, \\ J_{31} &= \frac{1}{2}(z^2+1)\partial_z + \frac{1}{2}(\bar{z}^2+1)\partial_{\bar{z}}, \\ K_1 &= -\frac{u(z+\bar{z})}{1+z\bar{z}}\partial_u + \frac{(r+u)(z+\bar{z})}{1+z\bar{z}}\partial_r \\ &\qquad - \frac{(z^2-1)(r+u)}{2r}\partial_z - \frac{(\bar{z}^2-1)(r+u)}{2r}\partial_{\bar{z}}, \\ K_2 &= \frac{iu(z-\bar{z})}{1+z\bar{z}}\partial_u - \frac{i(r+u)(z-\bar{z})}{1+z\bar{z}}\partial_r \\ &\qquad + \frac{i(z^2+1)(r+u)}{2r}\partial_z - \frac{i(\bar{z}^2+1)(r+u)}{2r}\partial_{\bar{z}}, \\ K_3 &= -\frac{u(1-z\bar{z})}{1+z\bar{z}}\partial_u + \frac{(r+u)(1-z\bar{z})}{1+z\bar{z}}\partial_r - \frac{z(r+u)}{r}\partial_z - \frac{\bar{z}(r+u)}{r}\partial_{\bar{z}}. \end{aligned} \tag{D.80}$$

It is easy to see that one is able to obtain the individual boosts and rotations (D.80) from (D.77) by making particular choices for a, b, c,

$$
\begin{aligned}
J_{12} &= \zeta|_{\substack{a=0,\\ b=i\\ c=0}}, & J_{23} &= \zeta|_{\substack{a=-i/2,\\ b=0\\ c=i/2}}, & J_{31} &= \zeta|_{\substack{a=1/2,\\ b=0\\ c=1/2}}, \\
K_1 &= \zeta|_{\substack{a=1/2\,,\\ b=0\\ c=-1/2}}, & K_2 &= \zeta|_{\substack{a=i/2,\\ b=0\\ c=i/2}}, & K_3 &= \zeta|_{\substack{a=0\,.\\ b=-1\\ c=0}}
\end{aligned}
\tag{D.81}
$$

Thus, we have shown that if Y^z is given by (D.74), then the diffeomorphism (D.73) generates rotations and boosts. The action of superrotations on the zz-component of the metric is given by

$$
\begin{aligned}
\mathcal{L}_\zeta g_{zz} =& \zeta^\rho \partial_\rho g_{zz} + 2 g_{z\rho} \partial_z \zeta^\rho \\
=& \zeta^u \partial_u g_{zz} + \zeta^r \partial_r g_{zz} + \zeta^z \partial_z g_{zz} + \zeta^{\bar{z}} \partial_{\bar{z}} g_{zz} + 2 g_{zz} \partial_z \zeta^z + 2 g_{z\bar{z}} \partial_z \zeta^{\bar{z}} + \mathcal{O}(1) \\
=& r\Big[\frac{u}{2}\Big(D_z Y^z + D_{\bar{z}} Y^{\bar{z}}\Big)\partial_u C_{zz} - \frac{1}{2}\Big(D_z Y^z + D_{\bar{z}} Y^{\bar{z}}\Big) C_{zz}\Big] \\
&+ r\Big[Y^z \partial_z C_{zz} + Y^{\bar{z}} \partial_{\bar{z}} C_{zz} + 2 C_{zz} \partial_z Y^z\Big] \\
&+ 2r^2 \gamma_{z\bar{z}} \Big[\cancel{\partial_z Y^{\bar{z}}} + \frac{u}{2r}(\cancel{\partial_z Y^{\bar{z}}} - \partial_z D^{\bar{z}} D_z Y^z)\Big] + \mathcal{O}(1) \\
=& r\Big[\frac{u}{2} D \cdot Y N_{zz} - \frac{1}{2} D \cdot Y C_{zz} + Y \cdot D C_{zz} + 2 C_{zz} D_z Y^z - u D_z^3 Y^z\Big] + \mathcal{O}(1).
\end{aligned}
\tag{D.82}
$$

Here, we used the condition on Y^z (3.81), which implies that $\partial_z Y^{\bar{z}} = 0$. The last line of (D.82) was obtained by using the relations

$$
\gamma_{z\bar{z}} \partial_z D^{\bar{z}} D_z Y^z = \gamma_{z\bar{z}} D_z D^{\bar{z}} D_z Y^z = D_z^3 Y^z, \tag{D.83}
$$

and

$$
\begin{aligned}
Y^z \partial_z C_{zz} + Y^{\bar{z}} \partial_{\bar{z}} C_{zz} + 2 C_{zz} \partial_z Y^z =& Y^z [D_z C_{zz} + 2(\gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}}) C_{zz}] + Y^{\bar{z}} D_{\bar{z}} C_{zz} + 2 C_{zz} \partial_z Y^z \\
=& Y \cdot D C_{zz} + 2 C_{zz} D_z Y^z.
\end{aligned}
$$

From $\partial_z Y^{\bar{z}} = 0$, one is able to derive that

$$
\begin{aligned}
D^{\bar{z}} D_{\bar{z}} Y^{\bar{z}} &= \gamma^{z\bar{z}} D_z D_{\bar{z}} Y^{\bar{z}} = \gamma^{z\bar{z}} \partial_z \Big(\partial_{\bar{z}} Y^{\bar{z}} + (\gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}}) Y^{\bar{z}}\Big) \\
&= \gamma^{z\bar{z}} \Big(\cancel{\partial_{\bar{z}} \partial_z Y^{\bar{z}}} + \underbrace{\partial_z (\gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}})}_{=-\gamma_{z\bar{z}}} Y^{\bar{z}} + (\gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}}) \cancel{\partial_z Y^{\bar{z}}}\Big) \\
&= -Y^{\bar{z}}.
\end{aligned}
\tag{D.84}
$$

The Lie derivative of g_{zr} is given by

$$\begin{aligned}\mathcal{L}_\zeta g_{zr} =&\zeta^\rho \partial_\rho \cancel{g_{zr}} + g_{z\rho}\partial_r \zeta^\rho + g_{r\rho}\partial_z \zeta^\rho \\ =&g_{z\bar{z}}\partial_r \zeta^{\bar{z}} + g_{ru}\partial_z \zeta^u + \mathcal{O}(r^{-1}) \\ =&-\frac{u}{2}\Big(Y_z - D_z D_z Y^z + \partial_z D_z Y^z + \partial_z D_{\bar{z}} Y^{\bar{z}}\Big) + \mathcal{O}(r^{-1}) \\ =&\mathcal{O}(r^{-1}),\end{aligned} \tag{D.85}$$

where we used that $D_z(D_z Y^z) = \partial_z(D_z Y^z)$ and

$$\partial_z D_{\bar{z}} Y^{\bar{z}} = D_z D_{\bar{z}} Y^{\bar{z}} = \gamma_{z\bar{z}} D^{\bar{z}} D_{\bar{z}} Y^{\bar{z}} = -\gamma_{z\bar{z}} Y^{\bar{z}} = -Y_z. \tag{D.86}$$

For the other metric components, we have

$$\begin{aligned}\mathcal{L}_Y g_{uu} =&\zeta^\rho \partial_\rho g_{uu} + 2g_{u\rho}\partial_u \zeta^\rho \\ =&2g_{uu}\partial_u \zeta^u + 2g_{ur}\partial_u \zeta^r + \mathcal{O}(r^{-1}) \\ =&-\left(D_z Y^z + D_{\bar{z}} Y^{\bar{z}}\right) + \left(D_z Y^z + D_{\bar{z}} Y^{\bar{z}}\right) + \mathcal{O}(r^{-1}) \\ =&\mathcal{O}(r^{-1}),\end{aligned}$$

$$\begin{aligned}\mathcal{L}_Y g_{ur} =&\zeta^\rho \partial_\rho g_{ur} + g_{u\rho}\partial_r \zeta^\rho + g_{r\rho}\partial_u \zeta^\rho \\ =&g_{ur}\partial_r \zeta^r + g_{ru}\partial_u \zeta^u + \mathcal{O}(r^{-2}) \\ =&\frac{1}{2}D_z Y^z - \frac{1}{2}D_z Y^z + \mathcal{O}(r^{-2}) \\ =&\mathcal{O}(r^{-2}),\end{aligned} \tag{D.87}$$

$$\begin{aligned}\mathcal{L}_Y g_{z\bar{z}} =&\zeta^\rho \partial_\rho g_{z\bar{z}} + g_{z\rho}\partial_{\bar{z}} \zeta^\rho + g_{\bar{z}\rho}\partial_z \zeta^\rho \\ =&\zeta^r \partial_r g_{z\bar{z}} + \zeta^z \partial_z g_{z\bar{z}} + \zeta^{\bar{z}} \partial_{\bar{z}} g_{z\bar{z}} + g_{z\bar{z}}\partial_{\bar{z}} \zeta^{\bar{z}} + g_{\bar{z}z}\partial_z \zeta^z + \mathcal{O}(r) \\ =&r^2 \gamma_{z\bar{z}}\Big[-D_z Y^z - D_{\bar{z}} Y^{\bar{z}} \\ &\qquad + (\gamma^{z\bar{z}}\partial_z \gamma_{z\bar{z}})Y^z + (\gamma^{z\bar{z}}\partial_{\bar{z}} \gamma_{z\bar{z}})Y^{\bar{z}} + \partial_{\bar{z}} Y^{\bar{z}} + \partial_z Y^z\Big] + \mathcal{O}(r) \\ =&\mathcal{O}(r).\end{aligned}$$

D.6.2 The Bracket Action of Q_H^+ on C_{zz}

For the hard charge Q_H^+,

$$Q_H^+ = \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}u\,\mathrm{d}^2z\,\Big(Y_{\bar z}T_{uz} + Y_z T_{u\bar z} + u(\partial_z Y_{\bar z})T_{uu} + u(\partial_{\bar z}Y_z)T_{uu}\Big), \tag{D.88}$$

one has

$$\begin{aligned}\{Q_H^+, C_{zz}(u,z,\bar z)\} &= \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\; Y_{\bar w}\{T_{u'w}, C_{zz}(u,z,\bar z)\}\\ &\quad + \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\; Y_w\{T_{u'\bar w}, C_{zz}(u,z,\bar z)\}\\ &\quad + \frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}u'\,\mathrm{d}^2w\; u'\big[(\partial_w Y_{\bar w}) + (\partial_{\bar w}Y_w)\big]\{T_{u'u'}, C_{zz}(u,z,\bar z)\}.\end{aligned} \tag{D.89}$$

We assume massless fields, so it follows from (D.23) and (D.30) that

$$T_{uz} = -\frac{1}{4}\partial_z(C_{zz}N^{zz}) - \frac{1}{2}C_{zz}D_zN^{zz} \qquad \text{and} \qquad T_{uu} = \frac{1}{4}N_{zz}N^{zz}.$$

In order to determine the first term of (D.89), one needs

$$\begin{aligned}\{T_{u'w}, C_{zz}(u,z,\bar z)\} &= -\frac{1}{4}\{\partial_w[(\gamma^{w\bar w})^2 C_{ww}N_{\bar w\bar w}], C_{zz}(u,z,\bar z)\}\\ &\quad -\frac{1}{2}\{(\gamma^{w\bar w})^2 C_{ww}D_wN_{\bar w\bar w}, C_{zz}(u,z,\bar z)\}\\ &= -\frac{1}{4}\partial_w\Big[(\gamma^{w\bar w})^2 C_{ww}\{N_{\bar w\bar w}, C_{zz}(u,z,\bar z)\}\Big]\\ &\quad -\frac{1}{2}(\gamma^{w\bar w})^2 C_{ww}D_w\{N_{\bar w\bar w}, C_{zz}(u,z,\bar z)\},\end{aligned} \tag{D.90}$$

where $\{C_{ww}, C_{zz}\} = 0$, so it was taken out of the Dirac bracket. The first term of (D.89) now becomes

$$\begin{aligned}&\frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\mathrm{d}u'\; Y_{\bar w}\{T_{u'w}, C_{zz}(u,z,\bar z)\}\\ &\quad= -\frac{1}{2}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\mathrm{d}u'\; Y_{\bar w}\delta(u'-u)\partial_w\Big[\gamma^{w\bar w}C_{ww}\delta^2(w-z)\Big]\\ &\qquad -\int_{\mathcal{I}^+} \mathrm{d}^2w\,\mathrm{d}u'\; Y^wC_{ww}\delta(u'-u)D_w\Big[\delta^2(w-z)\Big]\\ &\quad= -\frac{1}{2}\int_{\mathcal{I}^+} \mathrm{d}^2w\; Y_{\bar w}\partial_w\Big[\gamma^{w\bar w}C_{ww}\delta^2(w-z)\Big] - \int_{\mathcal{I}^+} \mathrm{d}^2w\; Y^wC_{ww}D_w\delta^2(w-z)\\ &\quad=\frac{1}{2}\int_{\mathcal{I}^+} \mathrm{d}^2w\; (D_wY_{\bar w})\gamma^{w\bar w}C_{ww}\delta^2(w-z) + \int_{\mathcal{I}^+} \mathrm{d}^2w\; D_w(Y^wC_{ww})\delta^2(w-z)\\ &\quad=\frac{1}{2}(D_zY^z)C_{zz} + D_z(Y^zC_{zz})\\ &\quad=\frac{3}{2}(D_zY^z)C_{zz} + Y^zD_zC_{zz}.\end{aligned} \tag{D.91}$$

First, we plugged in Dirac bracket relation (D.32) and integrated over $\delta(u'-u)$. Subsequently, we integrated by parts with respect to w, where the boundary terms vanish. For the second term of (D.89), one has

$$\begin{aligned}\{T_{u'\bar{w}}, C_{zz}(u,z,\bar{z})\} =& -\frac{1}{4}\{\partial_{\bar{w}}[C_{\bar{w}\bar{w}}N^{\bar{w}\bar{w}}], C_{zz}(u,z,\bar{z})\}\\ &-\frac{1}{2}\{C_{\bar{w}\bar{w}}D_{\bar{w}}N^{\bar{w}\bar{w}}, C_{zz}(u,z,\bar{z})\}\\ =& -\frac{1}{4}\partial_{\bar{w}}\Big[N^{\bar{w}\bar{w}}\{C_{\bar{w}\bar{w}}(u',w,\bar{w}), C_{zz}(u,z,\bar{z})\}\Big]\\ &-\frac{1}{2}(D_{\bar{w}}N^{\bar{w}\bar{w}})\{C_{\bar{w}\bar{w}}(u',w,\bar{w}), C_{zz}(u,z,\bar{z})\}.\end{aligned} \tag{D.92}$$

Using (D.92), one obtains

$$\begin{aligned}&\frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\mathrm{d}u'\,Y_w\{T_{u'\bar{w}}, C_{zz}(u,z,\bar{z})\}\\ &=-\frac{1}{4}\int_{\mathcal{I}^+}\mathrm{d}^2w\,\mathrm{d}u'\,Y_w\partial_{\bar{w}}\Big[N^{\bar{w}\bar{w}}\gamma_{w\bar{w}}\Theta(u'-u)\delta^2(w-z)\Big]\\ &\quad-\frac{1}{2}\int_{\mathcal{I}^+}\mathrm{d}^2w\,\mathrm{d}u'\,Y^{\bar{w}}(D_{\bar{w}}N_{ww})\Theta(u'-u)\delta^2(w-z)\\ &=\frac{1}{4}\int_{\mathcal{I}^+}\mathrm{d}^2w\,\mathrm{d}u'\,(D_{\bar{w}}Y^{\bar{w}})(\partial_u C_{ww})\Theta(u'-u)\delta^2(w-z)\\ &\quad-\frac{1}{2}\int_{\mathcal{I}^+}\mathrm{d}u'\,Y^{\bar{z}}(D_{\bar{z}}\partial_{u'}C_{zz})\Theta(u'-u)\\ &=\frac{1}{4}\int_{\mathcal{I}^+}\mathrm{d}u'\,(D_{\bar{z}}Y^{\bar{z}})(\partial_{u'}C_{zz})\Theta(u'-u)-\frac{1}{2}\int_{\mathcal{I}^+}\mathrm{d}u'\,Y^{\bar{z}}(\partial_{u'}D_{\bar{z}}C_{zz})\Theta(u'-u)\\ &=-\frac{1}{4}\int_{\mathcal{I}^+}\mathrm{d}u'\,(D_{\bar{z}}Y^{\bar{z}})C_{zz}\partial_{u'}\Theta(u'-u)+\frac{1}{2}\int_{\mathcal{I}^+}\mathrm{d}u'\,Y^{\bar{z}}(D_{\bar{z}}C_{zz})\partial_{u'}\Theta(u'-u)\\ &=-\frac{1}{2}(D_{\bar{z}}Y^{\bar{z}})C_{zz}+Y^{\bar{z}}D_{\bar{z}}C_{zz},\end{aligned} \tag{D.93}$$

where Dirac bracket relation (D.37) was used. Furthermore, we integrated the first term by parts with respect to $\bar{w}$, while integrating the second term over $\delta^2(w-z)$. Near the end, both terms were integrated by parts with respect to u', which gives a delta function, since $\partial_{u'}\Theta(u'-u)=2\delta(u'-u)$.
Using Dirac bracket relation (D.32), one finds

$$\begin{aligned}\{T_{u'u'}, C_{zz}(u,z,\bar{z})\} =&\frac{1}{4}\{N_{ww}N^{ww}, C_{zz}(u,z,\bar{z})\}\\ =&\frac{1}{4}N_{ww}(\gamma^{w\bar{w}})^2\{N_{\bar{w}\bar{w}}, C_{zz}(u,z,\bar{z})\}\\ =&4\pi G N_{ww}\gamma^{w\bar{w}}\delta(u'-u)\delta^2(w-z).\end{aligned} \tag{D.94}$$

Thus, the third term of (D.89) becomes

$$\begin{aligned}
&\frac{1}{8\pi G}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\mathrm{d}u'\; u'\big[(\partial_w Y_{\bar{w}}) + (\partial_{\bar{w}} Y_w)\big]\big\{T_{u'u'}, C_{zz}(u,z,\bar{z})\big\} \\
&\quad = \frac{1}{2}\int_{\mathcal{I}^+} \mathrm{d}^2w\,\mathrm{d}u'\; u'\big[(D_w Y_{\bar{w}}) + (D_{\bar{w}} Y_w)\big] N_{ww}\gamma^{w\bar{w}}\delta(u'-u)\delta^2(w-z) \\
&\quad = \frac{u}{2}\big[(D_z Y^z) + (D_{\bar{z}} Y^{\bar{z}})\big] N_{zz} = \frac{u}{2} D\cdot Y N_{zz}.
\end{aligned} \tag{D.95}$$

Finally, using (D.91), (D.93) and (D.95) results in the action of the hard charge on C_{zz}:

$$\big\{Q_H^+, C_{zz}(u,z,\bar{z})\big\} = \frac{u}{2} D\cdot Y N_{zz} - \frac{1}{2} D\cdot Y C_{zz} + Y\cdot D C_{zz} + 2 C_{zz} D_z Y^z. \tag{D.96}$$

Appendix E

Black Holes: Additional Details

E.1 Christoffel Symbols

In advanced Bondi coordinates (v, r, Θ^A), with $\Theta^A = (z, \bar{z})$, the eternal Schwarzschild line element is given by

$$\mathrm{d}s^2 = -V\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}\Theta^A\,\mathrm{d}\Theta^B, \qquad V \equiv 1 - \frac{2m_B}{r}. \tag{E.1}$$

The inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - \dfrac{2m_B}{r} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\gamma^{z\bar{z}}}{r^2} \\ 0 & 0 & \dfrac{\gamma^{z\bar{z}}}{r^2} & 0 \end{pmatrix}, \tag{E.2}$$

and the non-zero Christoffel symbols are

$$\begin{aligned} &\Gamma^v_{AB} = -r\gamma_{AB}, \quad \Gamma^v_{vv} = \frac{m_B}{r^2}, \quad \Gamma^r_{vv} = \frac{m_B V}{r^2}, \quad \Gamma^r_{vr} = -\frac{m_B}{r^2}, \\ &\Gamma^r_{AB} = -rV\gamma_{AB}, \ \Gamma^A_{rB} = \frac{\delta^A{}_B}{r}, \quad \Gamma^A_{AA} = \gamma^{AB}\partial_A\gamma_{AB}. \end{aligned} \tag{E.3}$$

E.2 Diffeomorphism

In this section, we will determine the structure of the diffeomorphism ζ that generates supertranslations. The diffeomorphism has to preserve gauge conditions $g_{rr} = 0$ and $g_{rA} = 0$. Since $g_{rr} = 0$, it follows that

$$\begin{aligned}\mathcal{L}_\zeta g_{rr} &= \zeta^\rho \partial_\rho \cancel{g_{rr}} + 2g_{r\rho}\partial_r \zeta^\rho \\ &= 2\partial_r \zeta^v = 0, \quad \Rightarrow \quad \zeta^v = f(v, z, \bar{z}).\end{aligned} \tag{E.4}$$

On the other hand, $g_{rA} = 0$ implies that

$$\begin{aligned}\mathcal{L}_\zeta g_{rA} &= \zeta^\rho \partial_\rho \cancel{g_{rA}} + g_{r\rho}\partial_A \zeta^\rho + g_{A\rho}\partial_r \zeta^\rho \\ &= \partial_A f + r^2 \gamma_{AB}\partial_r \zeta^B = 0, \\ &\Rightarrow \zeta^B = \frac{1}{r}\gamma^{AB}\partial_A f = \frac{1}{r}\gamma^{AB} D_A f = \frac{1}{r} D^B f.\end{aligned} \tag{E.5}$$

Since $\mathcal{L}_\zeta g_{vr} = \mathcal{L}_\zeta 1 = 0$, one has

$$\begin{aligned}\mathcal{L}_\zeta g_{vr} &= \zeta^\rho \cancel{\partial_\rho g_{vr}} + g_{v\rho}\partial_r \zeta^\rho + g_{r\rho}\partial_v \zeta^\rho \\ &= g_{vv}\cancel{\partial_r f} + \partial_r \zeta^r + \partial_v f = 0, \quad \Rightarrow \quad \partial_r \zeta^r = -\partial_v f.\end{aligned} \tag{E.6}$$

Note that $g^{AB}\mathcal{L}_\zeta g_{AB} = 0$, so it follows that

$$\begin{aligned}g^{AB}\mathcal{L}_\zeta g_{AB} =& g^{AB}\left(\zeta^\rho \partial_\rho g_{AB} + g_{A\rho}\partial_B \zeta^\rho + g_{B\rho}\partial_A \zeta^\rho\right) \\ =& g^{AB}\left(\zeta^r \partial_r g_{AB} + \zeta^C \partial_C g_{AB} + g_{AC}\partial_B \zeta^C + g_{BC}\partial_A \zeta^C\right) \\ =& \frac{2}{r}\zeta^r + (\gamma^{AB}\partial_C \gamma_{AB})\zeta^C + \partial_B \zeta^B + \partial_A \zeta^A \\ =& \frac{2}{r}\zeta^r + D_C \zeta^C = 0, \\ \Rightarrow& \zeta^r = -\frac{r}{2} D_C \zeta^C = -\frac{1}{2} D_C f = -\frac{1}{2} D^2 f,\end{aligned} \tag{E.7}$$

where we used (E.5), and where $D^2 = \gamma^{AB} D_A D_B = D_z D^z + D_{\bar{z}} D^{\bar{z}}$. Thus,

$$\partial_r \zeta^r = -\frac{1}{2} D^2 \partial_r f = 0, \tag{E.8}$$

which, combined with (E.6), implies that $\partial_v f = 0$, so

$$\zeta^v = f(z, \bar{z}). \tag{E.9}$$

Thus, one finally finds that the diffeomorphism has the following form,

$$\zeta_f = f\partial_v + \frac{1}{r} D^A f \partial_A - \frac{1}{2} D^2 f \partial_r, \tag{E.10}$$

which holds up for all orders of r. The Lie derivatives of the other metric components are given by

$$\begin{aligned}
\mathcal{L}_\zeta g_{vv} &= -\cancel{\mathcal{L}_\zeta 1} + \mathcal{L}_\zeta \left(\frac{2m_B}{r}\right) \\
&= \zeta^\rho \partial_\rho \left(\frac{2m_B}{r}\right) = \zeta^r \partial_r \left(\frac{2m_B}{r}\right) \\
&= \frac{m_B D^2 f}{r^2},
\end{aligned} \tag{E.11}$$

$$\begin{aligned}
\mathcal{L}_\zeta g_{vA} &= \zeta^\rho \partial_\rho \cancel{g_{vA}} + g_{v\rho}\partial_A \zeta^\rho + \cancel{g_{A\rho}\partial_v \zeta^\rho} \\
&= g_{vv}\partial_A \zeta^v + g_{vr}\partial_A \zeta^r \\
&= -D_A(Vf + \frac{1}{2}D^2 f), \qquad \text{where } \partial_A D^2 f = D_A D^2 f,
\end{aligned} \tag{E.12}$$

$$\begin{aligned}
\mathcal{L}_\zeta g_{AB} &= \zeta^r \partial_r g_{AB} + \zeta^C \partial_C g_{AB} + g_{AC}\partial_B \zeta^C + g_{BC}\partial_A \zeta^C \\
&= -r\gamma_{AB} D^2 f + r(D^C f)\partial_C \gamma_{AB} + r\gamma_{AC}\partial_B D^C f + r\gamma_{BC}\partial_A D^C f \\
&= 2r D_A D_B f - r\gamma_{AB} D^2 f.
\end{aligned} \tag{E.13}$$

Note that

$$D_A D_B f = \partial_A \partial_B f - \tilde{\Gamma}^a_{AB} D_a f = \partial_B \partial_A f - \tilde{\Gamma}^a_{BA} D_a f = D_B D_A f, \tag{E.14}$$

which implies that

$$\mathcal{L}_\zeta g_{z\bar{z}} = 2r D_z D_{\bar{z}} f - r(D_z D_{\bar{z}} + D_{\bar{z}} D_z) f = 0. \tag{E.15}$$

E.3 Green's Function G

Given the Green's function

$$G(z,\bar{z};w,\bar{w}) = \frac{1}{\pi} S \ln S, \tag{E.16}$$

with property

$$D_z^2 D_{\bar{z}}^2 G(z,\bar{z};w,\bar{w}) = \gamma_{z\bar{z}} \delta^2(z-w) + \dots, \tag{E.17}$$

it will now be shown that property (E.17) can be written as

$$D_z^2 D_{\bar{z}}^2 G = \frac{(\gamma_{z\bar{z}})^2}{4} D^2(D^2+2)G. \tag{E.18}$$

Note that

$$\begin{aligned} D_z D_{\bar{z}} D_{\bar{z}} D_z G =& D_z \Big(\partial_{\bar{z}}^2 \partial_z G - (\gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}}) \partial_{\bar{z}} \partial_z G \Big) \\ =& D_z \Big(\partial_z \big[\partial_{\bar{z}}^2 G - (\gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}}) \partial_{\bar{z}} G \big] + \underbrace{\partial_z (\gamma^{z\bar{z}} \partial_{\bar{z}} \gamma_{z\bar{z}})}_{=-\gamma_{z\bar{z}}} \partial_{\bar{z}} G \Big) \\ =& D_z \left(\partial_z D_{\bar{z}}^2 - \gamma_{z\bar{z}} D_{\bar{z}} \right) G \\ =& D_z^2 D_{\bar{z}}^2 G - \gamma_{z\bar{z}} D_z D_{\bar{z}} G. \end{aligned} \tag{E.19}$$

Likewise,

$$D_z D_{\bar{z}} D_z D_{\bar{z}} G = D_z^2 D_{\bar{z}}^2 G - \gamma_{z\bar{z}} D_{\bar{z}} D_z G. \tag{E.20}$$

By using (E.19), (E.20), and their complex conjugates, one finds

$$\begin{aligned} &\frac{(\gamma_{z\bar{z}})^2}{4} D^2(D^2+2)G \\ &\quad = \frac{1}{4} (D_z D_{\bar{z}} D_z D_{\bar{z}} + D_z D_{\bar{z}} D_{\bar{z}} D_z + D_{\bar{z}} D_z D_z D_{\bar{z}} + D_{\bar{z}} D_z D_{\bar{z}} D_z) G \\ &\qquad + \frac{\gamma_{z\bar{z}}}{2} (D_z D_{\bar{z}} + D_{\bar{z}} D_z) G \\ &\quad = D_z^2 D_{\bar{z}}^2 G, \end{aligned} \tag{E.21}$$

which implies that

$$\frac{\gamma_{z\bar{z}}}{4} D^2(D^2+2)G = \delta^2(z-w) + \dots \tag{E.22}$$

E.4 Supertranslation Action on h_{AB}

On the horizon $\mathcal{H}^+$, with $r = 2m_B$, the supertranslation action on h_{AB} is given by

$$\begin{aligned}
\mathcal{L}_\zeta h_{AB} &= \nabla_A \zeta_B + \nabla_B \zeta_A \\
&= \partial_A(g_{BC}\zeta^C) + \partial_B(g_{AC}\zeta^C) - 2\Gamma^v_{AB}(g_{vr}\zeta^r) - 2\Gamma^r_{AB}(g_{rv}\zeta^v) - 2\Gamma^C_{AB}(g_{CE}\zeta^E) \\
&= r\Big((\partial_A\gamma_{BC})D^C f + \gamma_{BC}\partial_A D^C f + (\partial_B\gamma_{AC})D^C f + \gamma_{AC}\partial_B D^C f\Big) \\
&\quad + r\Big(-\gamma_{AB}D^2 f + 2\left(1 - \frac{2m_B}{r}\right) f - 2\delta^A{}_B(\partial_A\gamma_{AE})D^E f\Big) \\
&\overset{r=2m_B}{=} 2m_B\Big((\gamma^{BC}\partial_A\gamma_{BC})D_B f + \gamma_{BC}\partial_A(\gamma^{BC}D_B f) + (\gamma^{AC}\partial_B\gamma_{AC})D_A f\Big) \\
&\quad - 2m_B\Big(-\gamma_{AC}\partial_B(\gamma^{AC}D_A f) + \gamma_{AB}D^2 f + 2\delta^A{}_B(\partial_A\gamma_{AE})D^E f\Big) \\
&= 2m_B\Big(\partial_A D_B f + \partial_B D_A f - \gamma_{AB}D^2 f - 2\delta^A{}_B(\gamma^{AE}\partial_A\gamma_{AE})D_A f\Big) \\
&= 2m_B(2D_A D_B f - \gamma_{AB}D^2 f).
\end{aligned} \tag{E.23}$$

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