Family symmetry Grand Unified Theories

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Abstract

The mixing amongst the fermions of the Standard Model is a feature that has up to yet not been understood. It has been suggested that an A_4 symmetry can be responsible for the specific mixing within the lepton sector. This thesis investigates the embedding of this A_4 symmetry in a continuous family symmetry, $SU(3)_F$. This continuous embedding allows for the construction of GUTs which include a family symmetry. Constructing and outlining these GUTs is the ultimate objective of this thesis.

Author:
Jelle Thole (s2183110)
University of Groningen
j.j.e.thole@student.rug.nl

Supervisor:
Prof. Dr. D. Boer

2nd examinator:
Prof. Dr. E. Pallante

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1 Introduction

The Standard Model (SM) of particle physics is a remarkably complete theory. It contains all fundamental particles (currently discovered) and, excluding gravity, all the fundamental interactions amongst these particles. At the core of the SM lie the 3 local gauge symmetries, corresponding to the 3 fundamental forces of the SM, these are

$$SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{1.1}$$

Using these symmetries to construct an invariant Lagrangian and subsequently an invariant action one can very accurately describe the interactions in the world around us.

The SM is however unable to explain some things as well, it furthermore includes features that are up to now not understood. Some of the things that are left unexplained by the SM are e.g. the amount of CP violation (CPV) in our universe and other cosmological phenomena like dark matter and energy. Next to the things that are left unexplained there are features within the SM, which are not understood. Examples of these phenomena are the hierarchy of the particles masses, CPV in the strong interactions and the mixing amongst the fermions of different generations [1].

It has been suggested [2] that the specific form of mixing amongst leptons of the different generations can be explained by adding a discrete family symmetry, D_F, to the SM. After having outlined this discrete symmetry, this thesis will then investigate whether such a D_F including SM can be embedded into a grand unified theory (GUT). A GUT extends the SM by embedding its symmetry group into a larger symmetry group. This ensures that the 3 gauge couplings of the fundamental SM interactions will unify into 1 single gauge coupling corresponding to a single grand unified force. In short, this thesis will try to explain the mixing amongst the 3 generations of the SM fermions by devising a GUT that includes a discrete family symmetry.

In order to achieve this, this thesis will be structured as follows. First the SM will be outlined in chapter 2 starting at the structure of the SM itself. Important to this structure is the role spontaneous symmetry breaking (SSB) plays, a consequence of this SSB is the Yukawa mechanism which gives all SM fermions their mass. It is this Yukawa mechanism that furthermore leads to the mixing amongst generations as well. Having outlined the SM and especially the mixing amongst the generations, it will then be shown in chapter 3 how a D_F is able to explain this mixing.

Before then trying to embed this D_F including SM into a GUT first some group-theoretical tools will be outlined in chapter 4. These tools will help explain how GUTs can be devised in a systematic way, to help show this some examples of GUTs will be outlined in chapter 5. This thesis greatly benefits from the work of Slansky [3] in systematically outlining the symmetry groups that underly GUTs. In chapter 6 then 2 family symmetry GUTs will be constructed, from the specific forms of these 2 GUTs some conclusions on family symmetry GUTs can be drawn.

2 The Standard Model of particle physics

As mentioned above this thesis tries to explain one of the features of the SM that has up to now not been understood; the specific way in which the the fermions of the different generations within the SM mix with each other. This thesis tries to do so by adding a family symmetry to the SM, however before implementing this symmetry first the SM and therefore this mixing between generations will be outlined below, many of the derived results on the SM that are used in this section are taken from [4].

In order to outline the SM this section will start by outlining the group-theoretical structure of the SM. After outlining this group-theoretical structure the Higgs mechanism, which is responsible for the breaking of the electroweak symmetry will be described. Next to breaking this symmetry the Higgs mechanism, via the Yukawa mechanism, gives fermions their masses as well. It is in giving fermions their masses this Yukawa mechanism actually creates terms which mix particles amongst fermion generations.

This mixing occurs due to a difference between the flavour and mass eigenstates of the fermions, which will first be shown for the quarks. For the quarks the information on how quarks mix is contained in a matrix which is called the CKM matrix. The chapter will then conclude by showing how leptons get their mass via the Yukawa mechanism and how leptons mix. This is somewhat different from the masses and mixing of quarks, leading to a significantly different mixing matrix as well. The matrix containing the information for mixing amongst the leptons is called the PMNS matrix. It is the specific form of the PMNS matrix that this thesis tries to explain by implementing a family symmetry. Now in order to explain this first the structure of the SM will be described.

2.1 Structure of the Standard model

At the core of the SM lie the 3 gauge symmetries corresponding to the fundamental interactions of the SM itself. These 3 symmetries are the following

$$SU(3)_C \times SU(2)_L \times U(1)_Y, \tag{2.1}$$

where $SU(3)_C$ is the symmetry of the strong (coloured) interactions and $SU(2)_L \times U(1)_Y$ is the symmetry of the electroweak interactions. This electroweak symmetry is broken by spontaneous symmetry breaking (SSB) leaving the SM with the following symmetry

$$SU(3)_{\mathrm{C}} \times U(1)_{\mathrm{Q}}, \tag{2.2}$$

where $U(1)_{\mathrm{Q}}$ is the symmetry of quantum electrodynamics (QED). All fundamental particles transform according to a irreducible representation of these 3 symmetries. The way in which these particles transform then actually determines how they interact with each other. These interactions are mediated by the different gauge bosons of the SM, these gauge bosons correspond to the different generators of the symmetries above. The gauge bosons for the different symmetries are listed in table 2.1 on the next page.

Fundamental interaction	Symmetry	Number of generators	Gauge bosons
Strong	$SU(3)_\mathrm{C}$	8	8 gluons
Weak	$SU(2)_\mathrm{W}$	3	$W^\pm, Z^0$
Electromagnetic	$U(1)_\mathrm{Q}$	1	photon

Table 2.1: A summary of all the gauge bosons of the (unbroken) SM, note that every generator of the SM gauge symmetries corresponds to exactly one SM gauge boson.

As mentioned above whether and how a particle interacts via these gauge bosons depends on the (irreducible) representation of the 3 symmetries a particle transforms according to. For instance the left-handed quarks of the first generation are in the same representation

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \begin{pmatrix} u_L^r & u_L^g & u_L^b \\ d_L^r & d_L^g & d_L^b \end{pmatrix} . \tag{2.3}$$

This multiplet of particles corresponds to the following representation of the unbroken gauge symmetries of the SM $(\mathbf{3}, \mathbf{2}, \frac{1}{3})$. This notation denotes the following; the left-handed quarks transform as a triplet, $\mathbf{3}$, for the strong interaction, a doublet, $\mathbf{2}$, for the left-handed $SU(2)_\mathrm{L}$ symmetry of the electroweak interactions. Finally, the particles in this multiplet all have hypercharge, $Y = \frac{1}{3}$. Using this notation, the particles of the first generation of the SM can be summarized in table 2.2 below.

Particles	Notation	Representation
Left-handed quarks	$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3}, \mathbf{2}, \frac{1}{3})$
Left-handed leptons	$L = \begin{pmatrix} e_L \\ v_L \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, -1)$
Right-handed up quark	u_R	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{4}{3})$
Right-handed down quark	d_R	$(\overline{\mathbf{3}}, \mathbf{1}, \frac{2}{3})$
Right-handed electron	e_R	$(\mathbf{1}, \mathbf{1}, 2)$

Table 2.2: The 15 fermions that together form the first generation of the SM fermions.

The fermion content of the SM is a threefold copy of the particles listed above. The only way in which these copies or families are different from each other are their masses. The differences in masses is explained by the Yukawa mechanism, a concept closely related to the Higgs mechanism, which both will be explained below.

An interesting thing to note from table 2.2 above is the difference between right and left-handed fermions, this is due to the $SU(2)_L$ symmetry of the electroweak interactions. Each right-handed fermion corresponds to a $SU(2)_L$ singlet, $\mathbf{1}$, and each left-handed fermion corresponds to a $SU(2)_L$ doublet, **2**. For a general fermion (ψ) its left and right-handed (ψ_L and ψ_R) parts are defined as

$$\psi_L = \frac{1-\gamma^5}{2}\psi \text{ and } \psi_R = \frac{1+\gamma^5}{2}\psi, \tag{2.4}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Now that the different bosons and fermions have been outlined, the final ingredient needed to understand the interactions of the SM is the Lagrangian. The Lagrangian of the SM carries the information on how the different bosons and fermions actually interact with each other. Using the Lagrangian the SM action can be constructed, which determines which interactions actually are possible. Futhermore the SM action allows for the calculation of the likelihood of a given interaction.

The Lagrangian of the SM is, in a very condensed fashion, given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi + \psi_i y_{ij}\psi_j\phi + h.c. + |D_\mu\phi|^2 - V(\phi). \tag{2.5}$$

The terms of this Lagrangian are gauge invariant under the different symmetries mentioned above. This gauge invariance of the different terms of the Lagrangian shapes the Lagrangian itself and the interactions it convenes. So when mentioning the symmetries of the SM, $SU(3)_C \times SU(2)_L \times U(1)_Y$, it is actually meant that the terms of the Lagrangian are locally gauge invariant under those symmetries. For instance the SM electromagnetic interaction, Quantum Electrodynamics, is locally gauge invariant under

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \tag{2.6}$$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x). \tag{2.7}$$

This corresponds to the Lagrangian being invariant under the $U(1)_Q$ symmetry associated with QED. Next to this gauge invariance having a large impact on the SM, breaking these symmetries has large implications as well. As previously mentioned the symmetry of the SM is broken by SSB, a mechanism devised by Higgs et al. [5] [6] [7]. This next section will show how the Higgs mechanism does so and which consequences arise from this SSB.

2.2 Standard Model Higgs

The electroweak symmetry of the SM is broken by SSB, also known as the Higgs mechanism. For a formal treatment of the Higgs [1] mechanism the reader is referred to [5] [6] [7]. Here the mechanism will be explained in words, after that using examples [4] it will be shown how this mechanism breaks the symmetry of the SM. Furthermore, the Yukawa coupling which gives particles their masses and is a consequence of the Higgs mechanism will be described as well.

The SM symmetry breaking is accommodated by the Higgs mechanism. In short, the Higgs mechanism minimizes a scalar potential, $V(\phi)$ that is added to the Lagrangian. In minimizing this potential, the complex scalar field on which this potential depends

[1] The Higgs mechanism gets its name from the paper of Higgs on SSB [5], the effort of both Englert & Brout [6] and Guralnik, Hagen & Kibble [7] on SSB should however be noted as well.

obtains a vacuum expectation value (VEV). It is this VEV that gives the scalar field a preferred value. If a scalar field has a preferred value, it is no longer invariant under a transformation for some symmetry, effectively breaking the symmetry. In the process of breaking this symmetry the gauge bosons corresponding to the generators of the broken symmetry acquire a mass. Furthermore, one of the fields used in the Higgs mechanism acquires a mass as well. For the SM SSB this massive particle is the well-known Higgs boson.

Now 2 examples will be used to show what this concretely means for a given theory, before showing the SM symmetry breaking. The first example that will be shown is the SSB of the Abelian $U(1)$ symmetry.

2.2.1 Breaking U(1)

Consider the electromagnetic Lagrangian plus a complex scalar field that couples to itself and to the electromagnetic field as well. This Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D_\mu\phi) - V(\phi), \tag{2.8}$$

with the following covariant derivative $D_\mu = \partial_\mu\phi + ieA_\mu$. Furthermore, A_μ is the gauge boson, $\phi(x)$ is the complex scalar field and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic tensor. This Lagrangian is invariant under the following local $U(1)$ transformation

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \tag{2.9}$$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x). \tag{2.10}$$

For this Lagrangian $V(\phi)$ is the following scalar potential

$$V(\phi) = -\mu^2\phi^2 + \frac{\lambda}{2}\phi^4 \tag{2.11}$$

with $\phi = |\phi| = \sqrt{\phi^*\phi}$. It is this scalar potential that does the symmetry breaking described before. If $\mu < 0$ the field ϕ will acquire a VEV, which breaks the $U(1)$ symmetry. The minimum of $V(\phi)$ occurs at the following VEV

$$\langle\phi\rangle = \phi_0 = \sqrt{\frac{\mu^2}{\lambda}} \tag{2.12}$$

or a rotation of this value in the complex plane corresponding to the $U(1)$ rotations mentioned before. Since ϕ now has a preferred direction in the complex plane the $U(1)$ symmetry is broken. The VEV mentioned above allows for rewriting the complex scalar field $\phi(x)$ as an expansion around this minimum

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \tag{2.13}$$

where $\phi_1(x)$ and $\phi_2(x)$ are real scalar fields. $V(\phi)$ then becomes

$$V(\phi) = -\frac{1}{2\lambda}\mu^4 + \frac{1}{2}\cdot 2\mu^2\phi_1^2 + \mathcal{O}(\phi_i^3). \tag{2.14}$$

The second term of the potential then actually is the mass term of the field ϕ_1, which acquires a mass of $m = \sqrt{2}\mu$, therefore corresponding to the massive Higgs boson. The kinetic term of the Lagrangian becomes

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}e\phi_0\cdot A_\mu\partial^\mu\phi_2 + e^2\phi_0^2A_\mu A^\mu + \ldots, \tag{2.15}$$

where cubic and quartic terms are omitted. The last term corresponds to a mass term of the form

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m_A^2 A_\mu A^\mu, \tag{2.16}$$

with $m_A^2 = 2e^2\phi_0^2$ being the photon mass. This shows how the Higgs mechanism can break a gauge symmetry, including the occurence of the massive Higgs boson and the photon acquiring a mass. $U(1)$, however is an Abelian symmetry, for the actual (electroweak) symmetry breaking of the SM, an understanding of the breaking of $SU(2) \times U(1)$ is needed. This is somewhat more involved as $SU(2)$ is a non-Abelian symmetry and the symmetry consists of the product of 2 separate symmetry groups. First it will now be shown how $SU(2)$ can be spontaneously broken, before turning to the $SU(2) \times U(1)$ of the SM.

2.2.2 Breaking SU(2)

In the previous section it was shown how a VEV can break $U(1)$. In order to break $SU(2)$ again a scalar potential $V(\phi)$ is used, the $SU(2)$ gauge field couples to the scalar field ϕ, which is a $SU(2)$ doublet. The covariant derivative therefore is

$$D_\mu \phi = (\partial_\mu - igA_\mu^a \tau^a)\phi, \tag{2.17}$$

with $\tau^a = \frac{\sigma^a}{2}$. The VEV that ϕ will acquire can be defined, using $SU(2)$ rotations, as

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.18}$$

Using this VEV the kinetic term of the Lagrangian becomes

$$|D_\mu \phi|^2 = \frac{1}{2} g^2 \begin{pmatrix} 0 & v \end{pmatrix} \tau^a \tau^b \begin{pmatrix} 0 \\ v \end{pmatrix} A_\mu^a A^{b\mu} + \dots. \tag{2.19}$$

Using $\{\tau^a, \tau^b\} = \delta^{ab}$ to symmetrize the matrix product under the interchange of a and b, the following mass term can be found

$$\mathcal{L}_{\text{mass}} = \frac{g^2 v^2}{8} A_\mu^a A^{a\mu}. \tag{2.20}$$

leading to a broken symmetry with all 3 gauge bosons of $SU(2)$ acquiring the mass $m = \frac{gv}{2}$. Having shown how $SU(2)$ can spontaneously broken, this principle will now be used to break the electroweak symmetry $SU(2)_L \times U(1)_Y$ of the SM.

2.2.3 Electroweak symmetry breaking

To break the electroweak symmetry, $SU(2)_L \times U(1)_Y$, of the SM again a scalar field ϕ in the spinor representation of $SU(2)$ is used, ϕ then has a hypercharge $Y = +\frac{1}{2}$ for the $U(1)_Y$ symmetry. Therefore, the gauge transformation of ϕ for this $SU(2)_L \times U(1)_Y$ symmetry is

$$\phi \to e^{i\alpha^a \tau^a} e^{i\beta Y} \phi = e^{i\alpha^a \tau^a} e^{i\frac{\beta}{2}} \phi. \tag{2.21}$$

Now again ϕ acquires the following VEV

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{2.22}$$

Again a similar procedure as above can be used, the covariant derivative however is somewhat different from the $SU(2)$ example mentioned above. It is given by

$$D_\mu \phi = (\partial_\mu - igA^a_\mu \tau^a - \frac{i}{2} g' B_\mu)\phi. \tag{2.23}$$

Using this covariant derivative and $\langle \phi \rangle$, the following kinetic term arises

$$|D_\mu \phi|^2 = \frac{1}{2} \begin{pmatrix} 0 & v \end{pmatrix} \left(gA^a_\mu \tau^a + \frac{1}{2} g' B_\mu \right) \left(gA^b_\mu \tau^b + \frac{1}{2} g' B_\mu \right) \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{2.24}$$

Using the anti-commutation relations and the definition of τ^a as used above the following mass term in the Lagrangian is found

$$\mathcal{L}_{\text{mass}} = \frac{v^2}{8} \left((gA^1_\mu)^2 + (gA^2_\mu)^2 + (-gA^3_\mu + g' B_\mu)^2 \right). \tag{2.25}$$

This mass term includes 4 different gauge bosons, however the SM only includes 3 massive gauge bosons. These 3 massive gauge bosons and the massless gauge boson are actually all included in the mass term above. This can be shown by writing the mass term in the following way

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} V_{\mu,i} (M^2)_{ij} V^\mu_j, \tag{2.26}$$

where $V_\mu = (A^1_\mu, A^2_\mu, A^3_\mu, B_\mu)$ and M^2 is the mass matrix squared, which is given by

$$M^2 = \frac{v}{4} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix} \tag{2.27}$$

The off-diagonal terms of this mass matrix are unpractical, since it are only the eigenvalues of the mass matrix that can get measured in experiments. This means that A^3_μ and B_μ have to mix to find the measured mass eigenstates, in order to find these eigenstates M has to be diagonalized. Actually it suffices to diagonalize only one part of the matrix, the one that has the off-diagonal terms

$$M^2_{\text{OD}} = \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}. \tag{2.28}$$

This matrix can be diagonalized by the following matrix [8]

$$O = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix}, \tag{2.29}$$

where θ_W is the so-called Weinberg angle, which is defined in the following way

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}. \tag{2.30}$$

Then M^2_{OD} can be diagonalized in the following way

$$\begin{aligned} O^{-1} M^2_{\text{OD}} O &= \frac{1}{g^2 + g'^2} \begin{pmatrix} g & -g' \\ g' & g \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \\ &= \begin{pmatrix} g^3 + gg'^2 & -g^2 g' - g'^3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \\ &= \begin{pmatrix} (g^2 + g'^2)^2 & 0 \\ 0 & 0 \end{pmatrix} = M^2_{\text{D}}. \end{aligned} \tag{2.31}$$

This leads to a mass matrix that has 3 massive gauge bosons and 1 massless gauge boson, like the SM has as well. The matrix that diagonalizes the lower part of the mass matrix mixes the gauge bosons as well. The mass eigen states are

$$\begin{pmatrix} Z^0_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}. \tag{2.32}$$

This gives the following massive gauge boson Z^0_μ

$$Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gA^3_\mu - g'B_\mu) \text{ with mass } M_Z = \sqrt{g^2 + g'^2}\frac{v}{2} \tag{2.33}$$

and the massless gauge boson A_μ

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'A^3_\mu + gB_\mu), \tag{2.34}$$

where $m_A = 0$ obviously. The 2 remaining massive gauge bosons are then A^1_μ and A^2_μ, however these gauge bosons are not charge eigenstates. Therefore to diagonalize the charge matrix, the physical eigenstates again have to be mixed. The following bosons are then arrived at

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1\mu \mp iA^2_\mu) \text{ with mass } M_W = \frac{gv}{2} \tag{2.35}$$

These 4 bosons correspond to the 3 massive gauge bosons of the weak interaction and the massless photon of QED. Now that it is shown how gauge bosons acquire their mass through the Higgs mechanism, the mechanism that gives fermions their mass, the Yukawa mechanism, will be outlined.

2.2.4 The Yukawa mechanism

As mentioned above left-handed and right-handed fermions transform differently under $SU(2)_\text{L}$ and have different $U(1)_\text{Y}$ charge as well. Ordinary mass terms of the form $-m_e e_L e_R$ therefore violate the gauge invariance of the SM. The term above for instance corresponds to a non-zero hypercharge Y and transforms as a $\mathbf{2} \otimes \mathbf{1} \sim \mathbf{2}$ $SU(2)$ doublet, therefore violating gauge invariance. These mass terms are therefore not allowed to be included in the SM Lagrangian. The Yukawa mechanism is able to circumvent this problem, which will be shown below.

Remember that the scalar field ϕ, which does the original symmetry breaking is a $SU(2)$ doublet and has hypercharge $Y = \frac{1}{2}$. This allows for the construction of the following gauge invariant term within the Lagrangian

$$\mathcal{L}_\text{Yukawa} = \lambda_e \overline{L_L}\phi\, e_R. \tag{2.36}$$

Now plugging in the VEV that ϕ acquires, amongst other things the following term is arrived at

$$\mathcal{L}_\text{electron mass} = -\frac{1}{\sqrt{2}}\lambda_e v \overline{L_L} e_R, \tag{2.37}$$

which corresponds to an electron mass term with mass $m_e = \frac{1}{\sqrt{2}}\lambda_e v$. It is interesting to note that since there exists no right-handed neutrino ν_r, there exists no Yukawa term that gives neutrino their mass. There is another mechanism that gives neutrinos their small mass, which will be explained later. Now first the Yukawa mass terms for quarks will be explained, in deriving these terms a discrepancy between the flavour and mass eigenstates of the quarks will be encountered. This discrepancy is described by the CKM matrix, which has a specific form that is unexplained by the SM.

2.3 Quark masses

Quarks get their mass via the Yukawa mechanism as used above for the electron mass. This leads to a similar mass term in the Lagrangian for the first generation of quarks

$$\mathcal{L}_Q = \frac{1}{\sqrt{2}}\lambda_d \overline{d_L}\phi d_R - \frac{1}{\sqrt{2}}\lambda_u \overline{u_L}\phi u_R + h.c.\ , \tag{2.38}$$

then plugging in the VEV ϕ acquires lead to the up and the down quark having the masses $m_d = \frac{1}{\sqrt{2}}\lambda_d v$ and $m_u = \frac{1}{\sqrt{2}}\lambda_u v$. However, when the second and third generations of quarks are added the mass term becomes more complicated

$$\mathcal{L}_Q = -\overline{Q_{i,L}}(m_d)_{ij}d_{j,R} - \overline{Q_{i,L}}(m_u)_{ij}u_{j,R}, \tag{2.39}$$

where i and j denote the generations, such that $u_i = (u, c, t)$ and $d_i = (d, s, b)$. M_d and M_u are non-diagonal matrices, which means that these interactions mix different flavours with each other. However, as already mentioned above experiments can only measure mass eigenstates. It is therefore that diagonalizing M_u and M_d is more convenient, in order to achieve this a new basis for the quarks is needed. The new basis of the quarks is

$$\begin{aligned} u'_{i,L} &= (u'_L, c'_L, t'_L) \\ u'_{i,R} &= (u'_R, c'_R, t'_R) \\ d'_{i,L} &= (d'_L, s'_L, b'_L) \\ d'_{i,R} &= (d'_R, s'_R, b'_R). \end{aligned} \tag{2.40}$$

These are the mass eigenstates, which are related to the flavour eigenstates by

$$\begin{aligned} u^i_L &= U^{ij}_u u'^j_L \\ u^i_R &= V^{ij}_u u'^j_R \\ u^i_L &= U^{ij}_d u'^j_L \\ d^i_R &= V^{ij}_d d'^j_R, \end{aligned} \tag{2.41}$$

where U^{ij}_u, U^{ij}_d, V^{ij}_u and V^{ij}_d are unitary matrices. The mass matrices can then be diagonalized in the following way

$$\begin{aligned} M_u &= U^\dagger_u m_u V_u = \mathrm{diag}(m_u, m_c, m_t) \\ M_d &= U^\dagger_d m_d V_d = \mathrm{diag}(m_d, m_s, m_b). \end{aligned} \tag{2.42}$$

2.3.1 The CKM Matrix

Now the off-diagonal terms of the matrices in the flavour basis allow for the mixing of quarks between the different generations, via the weak interaction. The new basis actually does not change the pure kinetic terms of the Lagrangian between quarks, since

$$\overline{u^i_L}\gamma^\mu u^i_L \rightarrow \overline{u^i_L}(U^\dagger_u)_{ij}\gamma^\mu (U_u)_{jk}u^k_L = \overline{u^i_L}\gamma^\mu u^i_L \tag{2.43}$$

However, the charged interactions get changed by the new basis, they become

$$\mathcal{L}_{W^\pm} = \frac{-g}{\sqrt{2}}\overline{u^i_L}\gamma^\mu \frac{1-\gamma^5}{2}(V_{CKM})_{ij}d^j W^+_\mu + h.c. \,, \tag{2.44}$$

where the Cabbibo-Kobayashi-Maskawa (CKM) matrix is defined as

$$(V_{CKM})_{ij} = (U_u^\dagger U_d)_{ij}. \tag{2.45}$$

The CKM matrix relates the weak and mass eigenstates of the down quarks in the following way $d'_i = (V_{\text{CKM}})_{ij}d_j$.

The CKM matrix is a 3×3 unitary matrix, which means it has 9 parameters. These paramaters are the 3 rotation angles between the different generations and 6 phases. 5 of the phases of the CKM matrix can actually be rotated away. This is due to the fact that the Lagrangian is invariant under the following transformations [9]

$$\begin{aligned} u^i_L &\to e^{i\phi(i)}u^i_L \\ d^i_L &\to e^{i\phi(i)}d^i_L. \end{aligned} \tag{2.46}$$

The CKM matrix can therefore be rewritten as

$$V'_{\text{CKM}} = \begin{pmatrix} e^{-i\phi(u)} & 0 & 0 \\ 0 & e^{-i\phi(c)} & 0 \\ 0 & 0 & e^{-i\phi(t)} \end{pmatrix} V_{CKM} \begin{pmatrix} e^{-i\phi(d)} & 0 & 0 \\ 0 & e^{-i\phi(s)} & 0 \\ 0 & 0 & e^{-i\phi(b)} \end{pmatrix} \tag{2.47}$$

This rewriting allows for the rotating of the 5 phases. The CKM matrix is then left with 3 angles and one phase, it can therefore be written in the following way

$$V'_{\text{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\text{CP}}} \\ 0 & 1 & 0 \\ s_{13}e^{-i\delta_{\text{CP}}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.48}$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$ and θ_{ij} is the rotation angle between the ith and the jth generation. δ_{CP} is the phase that is left in the CKM matrix this phase is actually CP violating, since it makes the CKM matrix complex. For more on the CKM matrix and CP violation (CPV) see for instance [10].

The interest of this thesis lies not on the CPV of the CKM matrix, it lies on the specific form of the CKM matrix. This form is one of the things that the SM fails to explain. It might however be explained by a discrete family symmetry, which relates the different families in the SM amongst each other. The form of the CKM matrix is the following [11]

$$V_{\text{CKM}} = \begin{pmatrix} 0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 & 0.00365 \pm 0.00012 \\ 0.22438 \pm 0.00044 & 0.9759^{+0.00010}_{-0.00011} & 0.04214 \pm 0.00076 \\ 0.00896^{+0.00024}_{-0.00023} & 0.04133 \pm 0.00074 & 0.999105 \pm 0.000032 \end{pmatrix}, \tag{2.49}$$

which is remarkably close to the $\mathbb{1}_{3\times 3}$. Its form actually is approximately [2]

$$|V_{\text{CKM}}| \approx \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \text{ where } \lambda \ll 1. \tag{2.50}$$

The aim of the thesis is finding a family symmetry GUT, which then explains the form of the mixing amongst the generations. However, the CKM matrix does (except for some off-diagonal corrections) not point in the direction of any special type of mixing. Since the CKM matrix can be furthermore be approximated using these corrections, the focus of this thesis will be on explaining the PMNS matrix, which points in the direction of a discrete family symmetry. The lepton masses and the PMNS matrix will therefore now first be outlined.

2.4 Lepton masses

As mentioned above the CKM matrix is not the only matrix that mixes mass and flavour eigenstates within the SM. Analogous to the CKM matrix for quark masses is the PMNS matrix for lepton masses. However, the mechanism that gives leptons their masses is somewhat different from that for the quark masses. This is due to the fact that there as of yet is no evidence for a right-handed neutrino, ν_R to exist, therefore there is no possibility for a Yukawa terms that let neutrinos acquire a mass.

The mechanism that gives the charged leptons their mass is the same as that for quarks. The Yukawa terms therefore look like

$$\mathcal{L}_Y = -(M_l)_{ij}\overline{L_{L,i}}e_{R,j} \ , \tag{2.51}$$

where $e_j = (e, \mu, \tau)$. Now since M_l is a non-diagonal matrix a new mass basis has to be introduced for the charged leptons as well. In order to diagonalize the mass matrix the following new basis is introduced

$$e'_{L,i} = (U_L)_{ij}e_{L,j}e'_{R,i} = (V_R)_{ij}e_{R,j}. \tag{2.52}$$

This is all very similar to quark masses, however the absence of the right-handed neutrinos makes it impossible to have a similar Yukawa term for the neutrinos. The fact that neutrinos are experimentally proven to oscillate implies however that they actually must have a non-zero mass. Therefore there has to be a mechanism beyond the SM that gives neutrinos their mass.

A first option to give neutrinos their mass is postulating a right-handed neutrino, ν_R. As mentioned above there has up to now not been found any evidence for ν_R. This might be due to it being absent from the SM, another option however is that this is due to the nature ν_R. If ν_R exits it corresponds to a $(\mathbf{1}, \mathbf{1}, 0)$ SM irrep, which makes it invariant under every force and therefore very hard to measure experimentally.

If the right-handed neutrino does exist, it allows for a Yukawa mass term of the form

$$\mathcal{L}_Y = -\overline{L_{L,i}}(M_\nu)_{ij}\nu_{R,j} + h.c. \ . \tag{2.53}$$

Such a Yukawa mass term, that up to now has been used to give all particles their mass is a Dirac mass which in general is of the form

$$\mathcal{L}_D = -\overline{\psi_{L,i}}(M)_{ij}\psi_{R,j} + h.c. \ , \tag{2.54}$$

where ψ is a fermion field and i and j are indices denoting the generation. A Dirac mass term always couples a left-handed particle to a right-handed particle. The neutrino is then called a Dirac neutrino.

Another type of mass terms that can be added to the Lagrangian is a Majorana term. A Majorana term couples a (right) left-handed particle with the charge conjugate of that particle in the following way

$$\mathcal{L}_M = -\frac{1}{2}\overline{\psi^c_{L(R),i}}(m_{L(R)})_{ij}\psi_{L(R),j} - \frac{1}{2}\overline{\psi_{L(R),i}}(m_{L(R)})_{ij}\psi^c_{L(R),j} + h.c.\,, \tag{2.55}$$

which is the left(right)-handed Majorana mass term. In order for a Majorana mass term to give a particle a mass, this particle has to be its own particle. This is the second option that can be used to achieve neutrino masses by extending the SM, assuming the neutrino is its own antiparticle a so-called Majorana neutrino.

There is a lot of experimental effort being done on determining whether a neutrino is Dirac or Majorana, these efforts have up to now not been conclusive. For the remainder of this thesis both options will therefore be considered. Whether a neutrino is either Dirac or Majorana determines the form of the neutrino mass term in the Lagrangian. The Dirac mass term is already mentioned above the Majorana mass term will now be outlined below.

The lowest dimensional Majorana mass term that can be achieved using SM particles is the Weinberg operator [12], which for neutrinos is

$$\mathcal{L}_M = -\frac{v^2}{2M}g_{ij}\overline{\nu^c_{i,L}}\nu_{j,L} + h.c.\,. \tag{2.56}$$

Here g_{ij} is a matrix with dimensionless coupling constants, $v^2 = \langle\phi\rangle^2$ and M is a large mass scale, which occurs due to the fact that this operator is non-renormalizable.

The interesting thing about this Weinberg operator is that $M_\nu = \frac{v^2 g_{ij}}{2M}$ is a symmetric matrix, due to the nature of a Majorana mass term. Therefore diagonalizing the mass matrix is somewhat different then for a Yukawa mass term. Above it was already mentioned that any complex matrix can be diagonalized by 2 unitary matrices U and V in the following way

$$m_\nu = U^\dagger M_\nu V, \tag{2.57}$$

where m_ν is the diagonal mass matrix. However since M_ν is symmetric the following 2 expressions are equal to each other

$$\begin{aligned} M_\nu(M_\nu)^\dagger &= U m_\nu V^\dagger V m_\nu U^\dagger = U m_\nu^2 U^\dagger \\ M_\nu^T(M_\nu^T)^\dagger &= V^* m_\nu U^T U^* m_\nu V^T = V^* m_\nu^2 V^T. \end{aligned} \tag{2.58}$$

Therefore the following can be derived

$$\begin{aligned} U m_\nu^2 U^\dagger &= V^* m_\nu^2 V^T \\ \Rightarrow\ V^T U m_\nu^2 U^\dagger U &= V^T V^* m_\nu^2 V^T U \\ \Rightarrow\ V^T U m_\nu^2 &= m_\nu^2 V^T U. \end{aligned} \tag{2.59}$$

Now since $V^T U$ is a unitary matrix the final expression above implies that $V^T U$ actually has to be a diagonal matrix of phases. This in turn implies that another diagonal matrix of phases D can be defined such that if $V_\nu = VD$ and $U_\nu = UD$ then

$$V_\nu^T U_\nu = \mathbb{1}, \tag{2.60}$$

therefore $V_\nu^T = U_\nu^\dagger$. This result can be used to diagonalize the Majorana mass matrix

$$\begin{aligned} &DU_\nu^\dagger M_\nu V_\nu D^* = m \\ &\to V_\nu^T M_\nu V_\nu = m. \end{aligned} \tag{2.61}$$

The Majorana mass matrix can therefore be diagonalized using only one unitary matrix V_ν [2] as opposed to the Dirac mass matrix, which needs 2 unitary matrices to be diagonalized.

2.4.1 PMNS matrix

The PMNS matrix can be defined as the product of 2 matrices from (2.52), like the CKM matrix, in the following way

$$U_{\text{PMNS}} = U_e^\dagger V_\nu. \tag{2.62}$$

The PMNS matrix then connects the flavour eigenstates of the neutrinos to the mass eigenstates of the neutrinos in the following way

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_L = U_{\text{PMNS}} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L , \tag{2.63}$$

A thing to note here is the peculiar form of the PMNS matrix itself. Like for the CKM matrix the parameters fully determining the PMNS matrix can be reduced to 4 parameters which are the 3 angles and 1 phase. The PMNS matrix can therefore be written in the following form as well

$$U_{\text{PMNS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\text{CP}}} \\ 0 & 1 & 0 \\ s_{13}e^{-i\delta_{\text{CP}}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} P. \tag{2.64}$$

Here the only thing that differs from the CKM matrix is the addition of the matrix P, which is either $\mathbb{1}_{3\times3}$ in the case of the Dirac neutrino, or in case of the Majorana neutrino matrix is a diagonal matrix. This diagonal matrix contains the 2 phases that are additionally needed when a neutrino is its own antiparticle [13].

The current experimental constraints (3σ bound) on the PMNS matrix are the following [13]

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.800 - 0.844 & 0.515 - 0.581 & 0.139 - 0.155 \\ 0.229 - 0.516 & 0.438 - 0.699 & 0.614 - 0.790 \\ 0.249 - 0.528 & 0.462 - 0.715 & 0.595 - 0.776 \end{pmatrix} . \tag{2.65}$$

This PMNS matrix has an interesting form which, opposed to the CKM matrix, actually differs from the unit matrix significantly. This specific form actually points in the direction of a certain discrete family symmetry. The next chapter will show how such a family symmetry can then be used to explain the form of the PMNS matrix.

3 Family Symmetries

As mentioned above one of the features that is not understood about the SM is the form of both the CKM and the PMNS matrix. The form of both these matrices might be explained by adding a family symmetry to the SM, however due to the CKM matrix being so close to the unit matrix this thesis focuses on the PMNS matrix. The specific form of the PMNS matrix points towards a discrete family. This chapter will therefore show how such a discrete family symmetry can explain the experimentally observed mixing of the leptons amongst the different generations.

Adding a discrete family symmetry to the SM would make the symmetry group of the SM

$$SU(3)_C \times SU(2)_L \times U(1)_Y \times D_F, \tag{3.1}$$

where D_F denotes a discrete family symmetry. Before diving into specific discrete symmetries, some important observations can be made that can help in finding a suitable discrete symmetry.

The SM has 3 generations, therefore D_F has to have a 3-dimensional irreducible representation. This is due to the fact that a 3D irrep is needed for governing the mixing amongst the 3 different generations. Furthermore, the PMNS matrix has quite a specific form.

It was for a while thought that U_{PMNS} was of the following form

$$|U_{\text{PMNS}}| \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \tag{3.2}$$

This form is called the Tri-Bimaximal form, due to the fact that ν_2 is the maximal mix of the three flavour eigenstates and ν_3 is a maximal mix of two of these eigenstates

$$\begin{aligned} \nu_2 &= \sqrt{\frac{1}{3}}\nu_e + \sqrt{\frac{1}{3}}\nu_\mu + \sqrt{\frac{1}{3}}\nu_\tau \\ \nu_3 &= \sqrt{\frac{1}{2}}\nu_\mu - \sqrt{\frac{1}{2}}\nu_\tau. \end{aligned} \tag{3.3}$$

Tri-Bimaximal mixing (TBM) has however recently been excluded up to at least 3σ since experiments determined that $(U_{\text{PMNS}})_{13} \neq 0$ [13]. Before that TBM was excluded up to this level, it was already shown that an A_4 symmetry can explain this form of the PMNS matrix.

The current experimental data point towards another type of mixing in the PMNS matrix, that of Cobimaximal mixing (CBM), which corresponds to $\theta_{13} \neq 0$, $\theta_{23} = \frac{\pi}{4}$ and $\delta_{CP} = \pm\frac{\pi}{2}$ [14]. This type of mixing can actually be explained by adding a A_4 symmetry to the SM as well. Furthermore, A_4 is the smallest discrete group that has a 3-dimensional irrep.

Since A_4 is the smallest group with a 3D irrep and is able to explain both TBM and CBM, A_4 is a very interesting canditate for the discrete family symmetry that explains the PMNS matrix. The remainder of this chapter will therefore focus on the A_4 family symmetry. First, the A_4 symmetry and some of this characteristics will be outlined. After that it will be shown how an A_4 symmetry can be used to explain the specific form of the PMNS matrix.

3.1 An A_4 family symmetry

An excellent review of an A_4 family symmetry is given in [2], here the most important features of that review are given. As mentioned above A_4 is the smallest discrete symmetry that has an 3-dimensional irrep. A_4 is the alternating group with $n = 4$, which consists of the even permutations of 4 elements. A_4 is called the tetrahedral group as well, since it can be used to represent the 12 possible symmetry rotations that leave a tetrahedron invariant. Since A_4 consists of 12 different group elements it is of order 12. These 12 elements can be generated by 2 elements

$$S = (14)(23) \text{ and } T = (123), \tag{3.4}$$

where (12) denotes the permutation of the elements 1 and 2. These generators satisfy the following $S^2 = (ST)^3 = T^3 = 1$. It can then be shown that the elements of A_4 fall into 4 conjugacy classes, therefore the following relation has to be satisfied

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 = 12, \tag{3.5}$$

where d_i is the dimension of the ith irrep. This equation has only one (integer) solution $d_1 = d_2 = d_3 = 1$ and $d_4 = 3$, the irreps of A_4 are therefore the following

$$\mathbf{1}, \mathbf{1'}, \mathbf{1''}, \mathbf{3}. \tag{3.6}$$

These one-dimensional irreps can be represented in the following way

$$\begin{aligned} \mathbf{1} &: S = 1 \;\; T = 1 \\ \mathbf{1'} &: S = 1 \;\; T = \omega \\ \mathbf{1''} &: S = 1 \;\; T = \omega^2, \end{aligned} \tag{3.7}$$

where $\omega = e^{\frac{2\pi i}{3}}$. There are 2 convenient basises which can be used to represent the **3** representation. These are the Ma-Rajasekaran (MR) basis for which

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

This basis is the most commonly used, there is however an another basis that is convenient for explaining the PMNS matrix as well. This is the Altarelli-Feruglio (AF) basis which is

$$S' = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = V^\dagger S V \quad T' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} = V^\dagger T V, \tag{3.9}$$

which as shown can be related to the MR basis using the unitary transformation matrix where

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \tag{3.10}$$

Now that the A_4 has been outlined it is time to show how this symmetry can be used to explain the specific form of the PMNS matrix. In order to do so first it has to be shown how A_4 invariant terms can be constructed.

3.2 An A_4 invariant Lagrangian

In order to explain the PMNS matrix the A_4 symmetry should be able to explain the specific form of the terms in the Lagrangian that give neutrinos their mass. However, adding a family symmetry means that the terms in the Lagrangian should be invariant under this new symmetry as well. It is therefore important to know how different irreps of A_4 multiply. It can be shown using the character table of A_4 that the Clebsch-Gordan decomposition of direct products are the following [12]

$$\begin{aligned} \mathbf{1} \otimes \mathbf{1} &= \mathbf{1} \\ \mathbf{1'} \otimes \mathbf{1"} &= \mathbf{1} \\ \mathbf{1'} \otimes \mathbf{1'} &= \mathbf{1"} \\ \mathbf{1(')\ (")} \otimes \mathbf{3} &= \mathbf{3} \\ \mathbf{3} \otimes \mathbf{3} &= \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1"} \oplus \mathbf{3} \oplus \mathbf{3}. \end{aligned} \tag{3.11}$$

Now every SM particle should transform according to an A_4 irrep, like every SM particle transforms according to a specific irrep of the SM gauge symmetries as well. There actually is some freedom in choosing these irreps, an important thing to note however is the irrep the Higgs field ϕ corresponds to. [2] shows that if it is demanded that one mechanism explains all lepton masses ϕ can not transform as a singlet of A_4. Since this thesis looks for a family symmetry that explains all lepton masses, this only leaves one option: a Higgs A_4 triplet ϕ_i, which consists of 3 $SU(2)$ doublets ϕ_1, ϕ_2 and ϕ_3.

The SSB of the SM is then somewhat changed since the covariant derivative is changed to $D_\mu \phi_i$, which leaves the following kinetic term

$$\mathcal{L} = \sum_{i=1}^{3} (D_\mu \phi_i)^\dagger D_\mu \phi_i, \tag{3.12}$$

with i denoting the triplet index. All the ϕ_i then acquire a VEV

$$\langle \phi_i \rangle = \begin{pmatrix} 0 \\ v_i \end{pmatrix}, \tag{3.13}$$

which leads to the following mass term for the Lagrangian

$$\mathcal{L} = \frac{gv^2}{8} \left((A_\mu^1)^2 + (A_\mu^2)^2 \right) + \frac{v^2}{8} (g A_\mu^3 - g' B_\mu)^2, \tag{3.14}$$

which is similar to the SM SSB, except here $v^2 = |v_1|^2 + |v_2|^2 + |v_3|^2$. The gauge bosons are then the same combinations of A_μ^i and B_μ as before. Their masses however are somewhat changed due to v being redefined as above. [2] then shows that there are 3 viable options for the 3 VEVS (v_1, v_2, v_3). These 3 options are $\frac{1}{\sqrt{3}}(v, v, v)$, $(v, 0, 0)$ and $(\frac{1}{\sqrt{2}} v e^{-i\alpha/2}, \frac{1}{\sqrt{2}} v, 0)$, which will be plugged in the A_4 invariant terms of the Lagrangian explaining fermion masses below.

Now using this Higgs triplet, ϕ_i, Yukawa mass terms for the charged leptons and the quarks of the following form

$$\mathcal{L}_m^Y = -\frac{1}{2} \lambda \overline{L_L} \phi_i e_R \tag{3.15}$$

can be constructed. In order for this part of the Lagrangian to be A_4 invariant, either L_L or e_R or both L_L and e_R have to be A_4 triplets. [2] shows that Yukawa terms that consist of 3 triplets actually are not able to give correct mass terms. Therefore either L_L or e_R is an A_4 triplet, most models choose the former. It is therefore that the SM particles are assigned to the following A_4 irreps

$$\begin{aligned}
(L_L)_i &= \begin{pmatrix} l_L \\ \nu_{l,L} \end{pmatrix}_i \sim \mathbf{3} \\
(Q_L)_i &= \begin{pmatrix} u_L \\ d_L \end{pmatrix}_i \sim \mathbf{3} \\
u_{R1}, d_{R1}, l_{R1} &\sim \mathbf{1} \\
u_{R2}, d_{R2}, l_{R2} &\sim \mathbf{1'} \\
u_{R3}, d_{R3}, l_{R3} &\sim \mathbf{1''} \\
\phi_i &= \begin{pmatrix} 0 \\ \phi^0 \end{pmatrix}_i \sim \mathbf{3}\ ,
\end{aligned} \tag{3.16}$$

where it should be remembered that $u_i = (u, c, t)$, $d_i = (d, s, b)$, $l_i = (e, \mu, \tau)$ and $\nu_i = (\nu_e, \nu_\mu, \nu_\tau)$. Having outlined these representations now it can be shown how the invariant terms of the Lagrangian can be constructed. If there are 2 triplets $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ the resulting product is denoted as $(a \times b)_{\mathbf{1}}$, where the subscript denotes to which irrep the product corresponds. This product should then comply to the transformation laws which were listed above, for instance remembering the MR basis

$$(a \times b)_{\mathbf{1}} \to ((S_{\mathbf{3}}a) \times (S_{\mathbf{3}}b))_{\mathbf{1}} = S_{\mathbf{1}}(a \times b)_{\mathbf{1}}. \tag{3.17}$$

Using these rules and the Clebsch-Gordan decomposition above the singlet terms of the Lagrangian will look like

$$\begin{aligned}
(a \times b)_{\mathbf{1}} &= a_1b_1 + a_2b_2 + a_3b_3 \\
(a \times b)_{\mathbf{1'}} &= a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3 \\
(a \times b)_{\mathbf{1''}} &= a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3
\end{aligned} \tag{3.18}$$

in the MR basis. For the AF basis the singlets will be

$$\begin{aligned}
(a \times b)_{\mathbf{1}} &= a_1b_1 + a_2b_3 + a_3b_2 \\
(a \times b)_{\mathbf{1'}} &= a_1b_2 + a_2b_1 + a_3b_3 \\
(a \times b)_{\mathbf{1''}} &= a_1b_3 + a_2b_2 + a_3b_1.
\end{aligned} \tag{3.19}$$

Now that these product representations have been outlined, A_4 invariant Yukawa terms can be conceived that give both quarks and charged leptons their masses.

3.3 Explaining neutrino mixing

As previously mentioned in order to find a neutrino mass using a Dirac term, the SM has to be extended with a right-handed neutrino ν_R. This right-handed neutrino corresponds to the following SM irrep $(\mathbf{1}, \mathbf{1}, 0)$. The 3 different generations then all correspond to a different 1D irrep of A_4, $\mathbf{1}$, $\mathbf{1'}$ and $\mathbf{1''}$. The following Yukawa mass term can then be constructed using the right-handed neutrino

$$-\mathcal{L}^Y = y_1(\overline{\nu_L}\phi)_{\mathbf{1}}\nu_{1,R} + y_2(\overline{\nu_L}\phi)_{\mathbf{1''}}\nu_{2,R} + y_3(\overline{\nu_L}\phi)_{\mathbf{1'}}\nu_{3,R}. \tag{3.20}$$

This gives the following mass matrix

$$M_\nu = \begin{pmatrix} y_1 v_1 & y_2 v_1 & y_3 v_1 \\ y_1 v_2 & \omega y_2 v_2 & \omega^2 y_3 v_2 \\ y_1 v_3 & \omega^2 y_2 v_3 & \omega y_3 v_3 \end{pmatrix}, \tag{3.21}$$

where the MR basis and (3.18) are used to obtain this form of M. Plugging in the VEV (v, v, v) allows this matrix to be diagonalized by the following matrix

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} = U_{\mathrm{PMNS}}, \tag{3.22}$$

then $U_{\mathrm{PMNS}} M_\nu = \sqrt{3}\,\mathrm{diag}(y_1 v, y_2 v, y_3 v)$. Therefore the neutrinos obtain the following masses at $m^2_{\nu_e} = 3y_1^2$, $m^2_{\nu_\mu} = 3y_2^2$ and $m^2_{\nu_\tau} = 3y_3^2$. The reason that $V = U_{\mathrm{PMNS}}$ is due to the fact that for the charged leptons exactly the same derivation can be done, where then $U_{\mathrm{PMNS}} M_e = \sqrt{3}\,\mathrm{diag}(y_4, y_5, y_6)$. However this form of the U_{PMNS} means that this model is unable to explain the actual (experimental) form of the PMNS matrix itself.

Ma [14] puts forward another model that combines a Dirac term for the charged leptons and a Majorana term for the neutrino masses. First of all, the Yukawa mass terms for the charged leptons will look like

$$-\mathcal{L}_Y = y_4(\overline{l_L}\phi)_{\mathbf{1}} l_{1,R} + y_5(\overline{l_L}\phi)_{\mathbf{1}'} l_{2,R} + y_6(\overline{l_L}\phi)_{\mathbf{1}''} l_{3,R}, \tag{3.23}$$

this term was actually already mentioned above and is diagonalized by (3.22) to yield $\sqrt{3}\,\mathrm{diag}(y_4, y_5, y_6)$ for the masses. The matrix (3.22) then links M_e and M_ν as well and allows for writing the neutrino mass matrix in the following way

$$M_\nu = U_\omega M_A U_\omega^T, \tag{3.24}$$

where it should be noted that M_ν is in the flavour basis and that as mentioned before due to the mass term being Majorana M_A has to be symmetric i.e.

$$M_A = \begin{pmatrix} a & c & e \\ c & d & b \\ e & b & f \end{pmatrix}. \tag{3.25}$$

If the neutrinos mix tri-bimaximally then instead of U_ω diagonalizing M_ν, M_ν would be diagonalized in the following way

$$M_\nu = U_B M_B U_B^T, \tag{3.26}$$

where it should be remembered that

$$U_B = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}, \tag{3.27}$$

and where M_B is diagonal for TBM. In general M_B is the following symmetric Majorana mass matrix given by

$$\begin{pmatrix} m_1 & m_6 & m_4 \\ m_6 & m_2 & m_5 \\ m_4 & m_5 & m_3 \end{pmatrix}. \tag{3.28}$$

Non-zero m_4, m_5 and m_6 therefore correspond to deviations from TBM.

The A_4 mass matrix M_A is then related to the TBM mass matrix M_B by

$$M_B = U_A^\dagger M_A U_A^*, \tag{3.29}$$

where

$$U_A = U_\omega^\dagger U_B = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{\frac{1}{2}} & 0 & i\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 & -i\sqrt{\frac{1}{2}} \end{pmatrix}. \tag{3.30}$$

It can then be shown [14] that the paramaters of this specific A_4 Majorana mixing matrix M_A and the parameters of the TBM matrix M_B are related by

$$\begin{aligned} m_1 = b + \frac{d+f}{2},\ m_2 = a,\ m_3 = b - \frac{d+f}{2} \\ m_4 = \frac{i(f-d)}{2},\ m_5 = \frac{i(e-c)}{\sqrt{2}},\ m_6 = \frac{e+c}{\sqrt{2}}. \end{aligned} \tag{3.31}$$

So in order to achieve TBM with an A_4 Majorana mass term the following conditions have to be satisfied $c = e = 0$ and $d = f$ [14].

However, TBM is not supported by current experimental evidence anymore, [14] notes that current experimental data actually supports CBM. To obtain this kind of mixing $c = e = 0$, $d \neq f$ and a, b, d and f all have to be real [14]. This scheme, however fails to fully predict U_{PMNS} since it leaves parameters that are as of yet undetermined.

There are more elaborate models that are able to fully explain neutrino mixing patterns. These models add multiple (supersymmetric) particles and are for instance models that use the Seesaw mechanisms I, II or III, a combination of the different Seesaw mechanisms or the Altarelli and Feruglio model [2]. These more complicated models are however beyond the scope of this thesis. Some of these models are able to explain the form of the CKM matrix as well, however most of the models treat the CKM matrix as a unit matrix, which has no mixing. Corrections of the order λ, λ^2 and λ^3 then explain the off-diagonal terms. An exception to this is for instance the supersymmetry A_4 model put forward by Babu, Ma and Valle [15] which explains both the CKM and the PMNS matrices.

The aim of this thesis however is not to explain these elaborate models, its aim is implementing a discrete family symmetry into a GUT. Before doing this, now first some group theoretical tools will be outlined. These tools will allow for constructing the symmetries of GUTs. Having constructed these GUTs, this thesis will then conclude with implementing a discrete family symmetry into 2 particular GUTs.

4 Group theoretical tools

A grand unified theory (GUT) is the collective term for a theory that extends the SM by unifying the 3 fundamental forces of the SM into one "grand" force. As mentioned before, all of the 3 fundamental forces correspond to a fundamental gauge symmetry. Furthermore, it was shown that the weak and the electromagnetic interaction unify into the electroweak interaction. It is therefore expected that at an even higher energy[2] the 3 forces of the SM unify into one grand unified force.

Such a force then corresponds to an even larger symmetry group, the nature of this group has implications for the SM itself. Furthermore, the way in which a GUT breaks down to the SM influences the SM as well, as there might be different paths that can be taken from the symmetry of the GUT to the symmetry of the SM. In order to understand these implications a thorough understanding of symmetries and how symmetries break is needed. This section will therefore outline some group-theoretical tools that are useful for understanding the implications of different GUTs. In order to do so this section will now start by outlining the concept of Lie groups and especially their associated Lie algebras.

4.1 Lie groups and algebras

The symmetries of the fundamental forces mentioned above are all Lie groups. A Lie group is a special kind of group since its elements continuously depend on some parameter(s). If g is an element of the Lie group G it depends on a set of parameters $\alpha = (\alpha_1, \dots, \alpha_n)$ with n being the dimension of G. For a complete and a more formal treatment of Lie groups the reader is referred to [17].

Underlying every Lie group there is a Lie algebra, these Lie algebras are convenient to use for analyzing Lie groups as a Lie algebra locally completely characterizes its corresponding Lie group. Lie algebras therefore are convenient for characterizing the symmetry groups of GUTs as well.

A Lie algebra, L, is a vector space with a specific algebraic structure. This vector space contains the generators of the Lie group. The reason the elements within a Lie algebra are called generators is the fact that locally they can be used to generate elements of their associated Lie group. If $g(\alpha)$ is an element of a n-dimensional Lie group G, then the d-dimensional matrix representation $D(\alpha) = D(g(\alpha))$ can be generated in the following way

$$D(d\alpha) = \mathbb{1} + i d\alpha_a X^a + \dots, \tag{4.1}$$

where $a = 1, \dots, n$ and is summed over.

The algebraic structure of this vector space is governed by the Lie product $[a, b]$, where a, b are 2 generators in L. The Lie product has the following properties

1. $[a.b] \in L \; \forall a, b \in L$
2. $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c] \; \forall a, b, c \in L \; \forall \alpha, \beta \in \mathbb{R}$
3. $[a, b] = -[b, a]$

[2] $E_{\text{GUT}} \approx 10^{15} - 10^{16}\; GeV$ [16]

4. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \; \forall a, b, c \in L$.

Property 1 guarantees that every Lie product is in L as well, therefore a basis $\{a_1, \ldots, a_n\}$ that spans L can be chosen. For every basis every Lie product can be written as

$$[a_p, a_q] = \sum_{i=1}^{n} f^r_{pq} a_r, \tag{4.2}$$

where f^r_{pq} are the so-called structure constants. The structure of a Lie group around the identity element is completely determined by the structure constants of the associated Lie algebra. This is the reason that Lie algebras lie at the heart of studying Lie groups.

Similar to Lie groups, Lie algebras can be represented by matrices as well like in (4.1), where $X^i = D(a^i)$. The structure constants actually themselves form such a representation

$$ad(a^i)_{jk} = f^j_{ik} \tag{4.3}$$

This representation is called the adjoint representation. The adjoint representation is key in classifying Lie algebras, as the structure constants actually determine a Lie algebra completely. In classifying Lie algebras a more convenient for the generators can be chosen. The next section will outline this basis and how roots, associated with this basis, can be used in analyzing Lie algebras.

4.2 Roots

As mentioned above the generators of L can be written in a more convenient basis; the Cartan-Weyl basis. This basis is convenient, since this basis has the maximal set of commuting generators within it. This means that for L the maximum amount of structure constants equal to zero. Furthermore, the rest of the structure constants which are non-zero are non-degenerate. These non-zero structure constants are called roots. It will be now be shown how they are defined.

To write an algebra in the Cartan-Weyl basis first the Cartan subalgebra must be defined. The Cartan subalgebra contains l hermitian generators, H_i, that satisfy the following commutation relations

$$[H_i, H_j] = 0 \text{ for } i, j = 1, \ldots, l. \tag{4.4}$$

The generators within the Cartan subalgebra can be diagonalized simultaneously. l is then called the rank of an algebra and therefore equals the number of simultaneously diagonalizable generators. The remaining generators of G, E_α, can then be chosen to satisfy the following commutation relation

$$[H_i, E_\alpha] = \alpha_i E_\alpha \text{ for } i = 1, \ldots, l. \tag{4.5}$$

These numbers α_i are the non-zero structure constants. For each operator E_α there are l such numbers, which together make up a vector in an l-dimensional space. This vector is called the root vector of E_α. As mentioned before these root vectors then completely specify the Lie algebra they correspond to.

To retrieve a Lie algebra, L, from its roots, some additional commutation relations are needed. Firstly by taking the hermitian conjugate of (4.5) the following can be derived

$$[H_i, E_\alpha]^\dagger = (H_i E_\alpha)^\dagger - (E_\alpha H_i)^\dagger = E_\alpha^\dagger H_i - H_i E_\alpha^\dagger = -[H_i, E_\alpha^\dagger] \tag{4.6}$$

$$[H_i, E_\alpha^\dagger] = -[H_i, E_\alpha]^\dagger = -\alpha_i E_\alpha^\dagger \tag{4.7}$$

therefore $E_\alpha^\dagger = E_{-\alpha}$. This makes E_α very similar to the raising and lowering operator J^+ and J^- of $SU(2)$. Actually every pair of roots $\pm\alpha$ corresponds to an $SU(2)$ subalgebra of the group, with $E_{\pm\alpha}$ acting as the raising and lowering operators [18]. Now using property 4 of a Lie algebra, the so-called Jacobi identity, the following can be derived

$$\begin{aligned}
[H_i, [E_\alpha, E_{-\alpha}]] &= -[E_\alpha, [E_{-\alpha}, H_i]] - [E_{-\alpha}, [H_i, E_\alpha]] \\
&= [E_\alpha, -\alpha_i E_{-\alpha}] - [E_{-\alpha}, \alpha_i E_\alpha] \\
&= -\alpha_i [E_\alpha, E_{-\alpha}] - \alpha_i [E_{-\alpha}, E_\alpha] \\
&= -\alpha_i [E_\alpha, E_{-\alpha}] + \alpha_i [E_\alpha, E_{-\alpha}] = 0.
\end{aligned} \tag{4.8}$$

This actually means that if $[E_\alpha, E_{-\alpha}] \neq 0$ it is in the Cartan subalgebra. Taking $\{H_1, \ldots, H_l\}$ as the basis that spans the Cartan subalgebra, $[E_\alpha, E_{-\alpha}]$ can be chosen in a specific way such that

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^{l} \alpha_i H_i = \alpha \cdot H. \tag{4.9}$$

Furthermore by taking the Jacobi identity with E_β where $\beta \neq -\alpha$, the following can be derived

$$\begin{aligned}
[H_i, [E_\alpha, E_\beta]] &= -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] \\
&= (\alpha_i + \beta_i)[E_\alpha, E_\beta] \\
&= N_{\alpha\beta} E_{\alpha+\beta},
\end{aligned} \tag{4.10}$$

where for the final equation it is assumed that $\alpha + \beta \neq 0$ and $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root. To summarize the generators can be put in the Cartan-Weyl basis, for which the structure constants are called roots, which completely determine the Lie algebra. The Cartan-Weyl basis is defined by the following commutation relations

$$[H_i, H_j] = 0 \tag{4.11}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{4.12}$$

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H \tag{4.13}$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}. \tag{4.14}$$

The Lie algebra can be completely retrieved from the roots using the 4 equations above. These properties of roots make them very useful for analyzing a specific Lie algebra or group. Roots can be used to classify Lie algebras as well. The tool that is very convenient to do so is the Dynkin diagram, which summarizes the roots of a Lie algebra in a graphical way. The next section will show how a Dynkin diagram does this and how they can be used to analyze Lie algebras.

4.3 Dynkin diagrams

As mentioned above Dynkin diagrams are a graphical way of summarizing the roots of a Lie algebra. They do not actually summarize all the roots of a Lie algebra, they only summarize the simple roots. The simple roots are a subset of the roots that can be obtained in the following way [3]:

1. Define all the roots in a Cartesian basis
2. Take the roots that have a positive number as the first entry of their root vector, these are the positive roots.
3. Find the l positive roots that can not be written as a linear combination of the other positive roots.

These l linearly independent roots are the simple roots, where l again is the rank of the Lie algebra, L. It is these simple roots that are summarized in a Dynkin diagram.

Now remember that the root vectors live in a l-dimensional euclidean space. Such a space can be spanned by l l-dimensional vectors. It can be proven [19] that the simple roots actually span the space where roots live in (the root space). Therefore any root, α, can be written as a linear combination of the simple roots

$$\alpha = \sum_{i}^{l} n_i \alpha_i. \tag{4.15}$$

Furthermore it can be proven [3] that every algebra has a highest root from which all roots can be derived by subtracting the simple roots. Another very important statement on roots that is needed for constructing the Dynkin diagram is the following theorem

Theorem 1. *If α and β are two arbitrary roots, then the ratio $\frac{2\alpha\cdot\beta}{\alpha^2}$ is an integer.*

This implies that

$$2\,\alpha \cdot \beta = k\alpha^2 = l\beta^2 \quad k, l \in \mathbb{Z}. \tag{4.16}$$

Now remembering that $a \cdot b = |a||b| \cos\theta_{ab}$. This leads to

$$\cos^2\theta_{\alpha\beta} = \frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \frac{kl}{4}. \tag{4.17}$$

This equation is restricted by the Schwarz inequality [19]

$$\left(\frac{2\alpha\cdot\beta}{\alpha^2}\right)\left(\frac{2\beta\cdot\alpha}{\beta^2}\right) = kl \leq 4. \tag{4.18}$$

Georgi [18] furthermore shows that for two simple roots α and β, $\frac{\pi}{2} \leq \theta_{\alpha\beta} \leq \pi$. Now assuming that $k \geq l$ only leaves 4 options for the relative lengths and angles between 2 simple roots α and β, these are

1.	$n = m = 0$	$\theta_{\alpha\beta} = \frac{\pi}{2}$	$\frac{\beta^2}{\alpha^2}$ arbitrary
2.	$n = 1, m = 1$	$\theta_{\alpha\beta} = \frac{2\pi}{3}$	$\beta^2 = \alpha^2$
3.	$n = 1, m = 2$	$\theta_{\alpha\beta} = \frac{3\pi}{4}$	$\beta^2 = 2\alpha^2$
4.	$n = 1, m = 3$	$\theta_{\alpha\beta} = \frac{5\pi}{6}$	$\beta^2 = 3\alpha^2$.

The angle, $\theta_{\alpha\beta}$, between 2 simple roots thus can have 4 different values. These options can actually be restricted further, Slansky [3] shows that the simple roots of a single Lie algebra, L, can have at most 2 different lengths.

The results on the simple roots, the angles between them and their relative lengths derived above can then be summarized using a Dynkin diagram. In a Dynkin diagram one dot represents one simple root. If the simple roots have different lengths then the circles corresponding to the shorter root are filled [16]. These dots are then connected by lines which represent the angles between the simple roots. The rules for constructing these Dynkin diagrams are summarized in table 4.1 below.

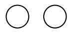
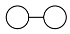
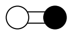
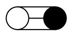
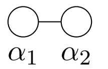
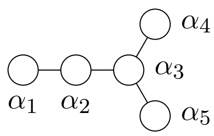
$\theta_{\alpha\beta}$	$\frac{\lvert\beta\rvert}{\lvert\alpha\rvert}$	Diagram
$\frac{\pi}{2}$	arbitrary	
$\frac{2\pi}{3}$	1	
$\frac{3\pi}{4}$	2	
$\frac{5\pi}{6}$	3	

Table 4.1: Rules for constructing the Dynkin diagram of a give Lie algebra L, by using these rules every simple Lie algebra can be represented by a single Dynkin diagram. Note that the filled root corresponds to the shorter root.

Using these rules all the Lie algebras can be represented by a Dynkin diagram. For instance, $SU(3)$ has the following Dynkin diagram



This diagram represents the 2 simple roots of $SU(3)$, α_1 and α_2, and the fact these simple roots lie at an angle of 120°. A more complicated diagram is the one representing $SO(10)$



It can actually be shown that, apart from 5 exceptions every Lie algebra can be divided into 4 categories. The proof of this classification lies beyond the scope of this thesis, the interested reader is refered to [18]. The 4 categories and 5 exceptions of Lie algebras are listed in table 4.2 on the next page.

As mentioned before, Dynkin diagrams are not only useful for summarizing Lie algebras. Dynkin diagrams can be used to visualize symmetry breaking as well. Using Dynkin diagrams it can be shown to which symmetries a given symmetry, say $SO(10)$, can break. This allows for the construction of 'paths' from a given symmetry towards the symmetry group of the SM, $SU(3) \times SU(2) \times U(1)$. Exploring which symmetries can actually break to the SM, and which symmetries lie between the GUT and the SM, tells a lot about the SM itself. How Dynkin diagrams can be used to construct these paths will be shown in the next section.

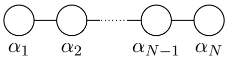
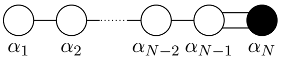
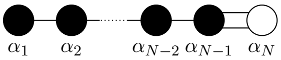
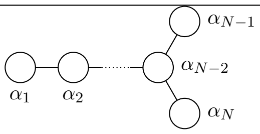
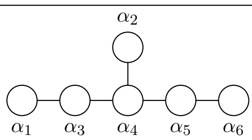
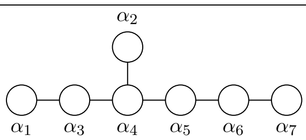
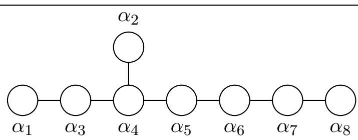
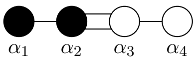
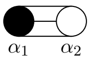
Dynkin diagram	Lie group	Dimension	Rank = number of simple roots
	$SU(N+1)$	N^2-1	N
	$SO(2N+1)$	$\frac{N(N+1)}{2}$	N
	$Sp(2N)$	$N(2N+1)$	N
	$SO(2N)$	$\frac{N(N+1)}{2}$	N
	E_6	78	6
	E_7	133	7
	E_8	248	8
	F_4	52	4
	G_2	14	2

Table 4.2: Dynkin diagrams for all the possible simple Lie algebras and the dimension and rank of these Lie algebras. These Dynkin diagrams fall into 2 categories, the top 4 Dynkin diagrams are the infinite series of the 4 different algebras. The bottom 5 Dynkin diagrams then correspond to the 5 exceptions on these 4 infinite series.

4.4 Symmetry breaking using Dynkin diagrams

As shown above Dynkin diagrams summarize Lie algebras by depicting the simple roots and the respective angles between them graphically. They can however be used to show which subalgebras a certain Lie algebra has as well. A Lie subalgebra L' is defined as a set satisfying the following [17]

1. $L' \subset L$
2. L' has the same Lie product as L
3. If $a', b' \in L'$, then $[a', b'] \in L'$.

If and only if these properties are satisfied then L' is a subalgebra of L. L' is called a maximal subalgebra of L, if there exists no subalgebra L^* such that

$$L' \subset L^* \subset L. \tag{4.19}$$

There are two type of subalgebras for a simple Lie algebra [3]

1. Regular subalgebras, R, which can be obtained by looking directly at the Dynkin diagram.
2. Special subalgebras, S, which can be discovered by comparing the irreps of an algebra L with the irreps of a canditate subalgebra.

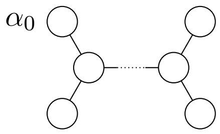
Since finding special subalgebras is done by comparing specific irreps it is hard to come up with a systematic way of finding these irreps, whereas (as will be shown below) Dynkin diagrams can be found for finding all regular subalgebras of a Lie algebra. Furthermore, the SM below EWSB is a regular subgroup of the SM above EWSB [16]. Special subalgebras are therefore not of interest for this thesis, the interested reader is referred to [3] [16]for more on these special subalgebras. The focus of this paragraph will therefore from here on be on regular subalgebras.

The roots of a regular subalgebra, R, are a subset of the roots of the algebra itself, L. Since the simple roots of R are a subset of the simple roots of L, R can be obtained by leaving one or multiple circles out from the Dynkin diagram of L. This results in a new diagram, which then corresponds to a subset of the original simple roots of the Dynkin diagram. Therefore leaving out one or multiple circles from the Dynkin diagram of L leaves us with the Dynkin diagram of a regular subalgebra R of L.

However by removing a circle from the Dynkin diagram not every subalgebra of a Lie algebra L can be obtained, since removing one simpe root from a subalgebra decreases the rank of the subalgebra to $l - 1$. This means that using this procedure the maximal subalgebras of a Lie algebra can not be found, since a maximal subalgebras R of L has rank l, which equals that of the algebra L itself. In order to obtain these maximal subalgebras from the Dynkin diagram of L, the Dynkin diagram first has to be extended, where extending the diagram means adding a circle to the diagram. Adding a circle to the Dynkin diagram is the graphical representation of adding an additional root to the set of simple roots.

The root that is added can not be any root, it actually has to be the lowest root,

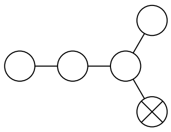
$\alpha_0 = -\alpha_H$, where α_H is the highest root. Adding this root to the set of simple roots, creates a linear dependence between the simple roots. Except for that linear dependence amongst this set of roots, every rule for a Dynkin diagram is still satisfied. The fact that only the lowest root, α_0, can be added means that corresponding to every Dynkin diagram there is a unique extended Dynkin diagram [18]. This means that all the diagrams in the table above have a unique way of being extended. For instance the extended Dynkin diagram for a $SO(2N)$ algebra is



,

where for convenience the added root α_0 is labeled. Now removing different circles from such an extended Dynkin diagram then yields the different maximal regular subalgebras of L.

Having outlined how Dynkin diagrams and their extended versions can be used to show symmetry breaking, now an example will be used to further show the convenience of Dynkin diagrams. The example that will be shown here is that of $SO(10)$ breaking down to the SM, which is quite a well known GUT. Here $SO(10)$ breaks to $SU(5)$ first before breaking to the SM. First remember the diagram for $SO(10)$ already outlined above



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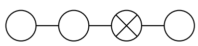
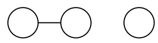
where the simple root that has to be removed is marked by a cross, $\times$. This leaves the Dynkin diagram of $SU(5)$



,

where again the root that will be removed in the next symmetry breaking step is marked. This ultimately leaves the symmetry group of the SM $SU(3) \times SU(2) \times U(1)$, which corresponds to the following Dynkin diagram



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Now in removing a circle from a Dynkin diagram a Cartan generator gets removed from the algebra, this actually generates an additional $U(1)$ algebra [3]. This is explicitly shown in the final step $SU(5) \to SU(3) \times SU(2) \times U(1)$. The first step thus actually is $SO(10) \to SU(5) \times U(1)$. However in going down from a high rank algebra to the SM a lot of these additional $U(1)$ are created. If they are of no physical significance for the GUT they are therefore, for convenience, omitted, as is the case above.

Now that it has been shown how Dynkin diagrams can be used to show the different subalgebras of a Lie group, another tool that is needed will be explained. Knowing which subalgebra and therefore which subgroup lies within a Lie group, is important knowledge, however for building a GUT knowledge of irreps is needed as well. The method of highest weights, which is closely related to the roots of an algebra, allows us to build and break irreps of different algebras. It will be therefore be explained below.

4.5 Method of highest weights

As shown in chapter 2, particles transform according to different irreps of the symmetries of the SM. The way in which they transform determines how these different particles interact with the forces within the SM. When breaking symmetries towards the SM the irreps of these symmetries get broken as well. The SM particles therefore correspond to one irrep for every symmetry in the 'path' from the SM to the GUT.

As in general these irreps get larger as the symmetries grow, this means that particles which previously were not in the same irrep might be in the same irrep corresponding to a larger symmetry. Therefore, particles that previously could not interact with each other will interact with each other at higher energies in moving on to the GUT. Which makes sense, as it is expected that for the GUT, all particles will interact with each other according to one unified force.

In order to see how these larger irreps work, which particles can be in them and how they can break to the SM irreps the method of highest weights is needed. To explain this method, first weights in general have to be outlined. As mentioned above weights are closely related to roots, roots actually are the weights of the adjoint representation, which will be shown below.

Now first remember the generators inside the Cartan subalgebra H_i and the fact that they can be diagonalized simultaneously. Therefore, the Hilbert space vectors $|\lambda\rangle$ of an irrep of L can be characterized by l labels using H_i in the following way

$$H_i\,|\lambda\rangle = \lambda_i\,|\lambda\rangle \;\text{ for } = 1\ldots l \tag{4.20}$$

these l coefficients λ_i are called weights, which together compose the weight vector λ [3]. Now remember that in the adjoint representation the generators are the representations themselves. Consider the hermitian generators $X_i = -ia_i$, which have the associated states $|X_i\rangle$. Then the following can be derived

$$\begin{aligned} X_a\,|X_b\rangle &= \sum_c |X_c\rangle\,\langle X_c|\,X_a\,|X_b\rangle = \sum_c |X_c\rangle\,[ad(X_a)]_{cb} \\ &= \sum_c if_{abc}\,|X_c\rangle = |\sum_c if_{abc}X_c\rangle = |[X_a, X_b]\rangle\,. \end{aligned} \tag{4.21}$$

After changing to the Cartan-Weyl basis, this leads to

$$H_i\,|H_j\rangle = 0 \tag{4.22}$$

$$H_i\,|E_\alpha\rangle = \alpha_i\,|E_\alpha\rangle\,. \tag{4.23}$$

Therefore, the weights of the adjoint representation are the roots [17]. Now remember that all the roots can be retrieved by subtracting the simple roots from the highest root. A similar procedure can be used to construct an irrep from a weight, this is the method of the highest weight. Before outlining this method, first the highest weights has to be defined. In order to define the highest weight the following theorem from [17] is needed

Theorem 2. *Every weight λ can be written in terms of the simple roots*

$$\lambda = \sum_{i=1}^{l} \mu_i \alpha_i \tag{4.24}$$

with the coefficients μ_i being real and rational.

A weight λ is then is defined as positive if its first non-vanishing component μ_i is positive. This allows for the definition of the most positive and therefore highest weight Λ, for which $\Lambda > \lambda$ for every other weight. In order to then use the method of highest weights it is convenient to switch to the Dynkin basis. The Dynkin basis is defined in the following manner, if λ is a weight vector than its components in the Dynkin basis a_i are given by

$$a_i = \frac{2\lambda \cdot \alpha_i}{\alpha_i^2}, \tag{4.25}$$

where α_i is the ith root. Now the reason for switching to Dynkin basis is the fact that for all roots and weights their associated Dynkin labels a_i are integers, making the computation with weights boil down to integer arithmetic.

Now in this convenient basis the method of highest weights will be introduced. It was Dynkin that proved the following theorem [3]

Theorem 3. *The highest weight of an irrep can be selected in a way such that the Dynkin labels are all non-negative integers. Furthermore each and every irrep is uniquely identified by a set of integers* $(a_1, \ldots, a_l)$ *and each such set is a highest weight of one and only one irrep.*

Using this theorem and the fact that the simple roots act as ladder operators an irrep can be completely be built out of the highest weight by subtracting the simple roots from it. The different simple roots can be subtracted from the highest weight, until the lowest weight for a given simple root α_i is reached. This is analogous to raising and lowering operators, $J^{\pm}$, of the $SU(2)$ spin symmetry of classical quantum mechanics. Every simple root actually acts as such a lowering operator for the $SU(2)$ subalgebra it corresponds to.

The $SU(2)$ analogy then is the following, from the highest weight Λ for a spin state with spin l the J^- can be subtracted $2l + 1$ times, before reaching the lowest weight $-\Lambda$. The method of the highest weights is the multi-dimensional version of this, for more on this analogy the interested reader is referred to [18]. Since there is a unique way of moving down from this highest weight by using the simple roots, a highest weight completely and uniquely specifies a given irrep.

In order to show how this procedure works, now some irreps will be built. Starting simple with 2 irreps from $SU(3)$, remember that $SU(3)$ has 2 roots α_1 and α_2, which in the Dynkin basis are

$$\begin{aligned} \alpha_1 &= (2, -1) \\ \alpha_2 &= (-1, 2). \end{aligned} \tag{4.26}$$

Now using these 2 simple roots irreducible representations can be built, starting at the fundamental representation. The fundamental representation is 3 dimensional and corresponds to the highest weight $\Lambda = (1\ 0)$, from which the simple roots α_1 and α_2 can be subtracted from in the following way

$$\begin{tikzcd} \boxed{1\ 0} \arrow[d, "\alpha_1"'] \\ \boxed{-1\ 1} \arrow[d, "\alpha_2"'] \\ \boxed{0\ -1} \end{tikzcd},$$

which shows that the fundamental representation is indeed 3-dimensional. Note that is the only way of building the fundamental, **3**, irrep from the highest weight Λ as was already explained above. Again this is due to the multi-dimensional ladder operators being restricted the weights corresponding to the specific irrep.

Another example that is a little more involved is the construction of the 6-dimensional representation of $SU(3)$, that corresponds to $\Lambda = (2\ 0)$

$$\begin{tikzcd}
& \boxed{2\ 0} \arrow[d, "\alpha_1"'] & \\
& \boxed{0\ 1} \arrow[dl, "\alpha_2"'] \arrow[dr, "\alpha_1"] & \\
\boxed{1\ \text{-}1} \arrow[dr, "\alpha_1"'] & & \boxed{\text{-}2\ 2} \arrow[dl, "\alpha_2"] \\
& \boxed{\text{-}1\ 0} \arrow[d, "\alpha_2"'] & \\
& \boxed{0\ \text{-}2} &
\end{tikzcd}.$$

Now these 2 examples show how irreps can be built using the method of highest weights, as mentioned before this method can be used to show the breaking of irreps as well. Now in order to show how this irrep breaks down use the fact that when breaking a symmetry a simple root is removed from the Lie algebra. This means that this root is also removed from the highest weight diagram. Therefore, parts of the irrep become disconnected as they can no longer be connected by the simple root which is removed. This means that an irrep gets broken in a way which is specific to the symmetry breaking being done. In order to show this a more complicated example will be used.

Remember from the previous section the path from $SO(10)$ to the SM. For this GUT it is believed that the 15 fermions of a generation are in the same 16-dimensional irrep of $SO(10)$ effectively coupling these particles together. The final particle that completes this representation is then believed to be the right-handed neutrino, ν_R. In order to show how this **16** irrep can be broken down to the SM particle irreps, remember that in the symmetry breaking $SO(10) \to SU(5)$ the simple root α_5 is removed from the Dynkin diagram. This means α_5 has to be removed from the highest weight diagram of **16**, corresponding to $\Lambda = (0\ 0\ 0\ 0\ 1)$, this is shown in figure 4.1 on the next page.

In the weight diagram the α_5 'transitions' are dashed since this simple root is removed from the Dynkin diagram and therefore does not belong to the subalgebra of $SU(5)$. From the diagram it can therefore be deduced how the **16** irrep of $SO(10)$ splits into 3 different $SU(5)$ irreps

$$\mathbf{16} \to \mathbf{10} \oplus \overline{\mathbf{5}} \oplus \mathbf{1}. \tag{4.27}$$

It can then be shown using the weight diagrams of these 3 irreps, how they break down to the SM irreps. [3] gives an excellent review of all the possible breaking patterns, which can be deduced using the method above. The $SU(5)$ irreps can be shown to break in the following way

$$\begin{aligned}
\mathbf{10} &\to (\mathbf{1}, \mathbf{1}, 1) \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}) \oplus (\mathbf{3}, \mathbf{2}, \tfrac{1}{6}) = \overline{e_R} + \overline{u_R} + Q_L \\
\overline{\mathbf{5}} &\to (\mathbf{1}, \mathbf{2}, -\tfrac{1}{2}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \tfrac{1}{3}) = d_R + \overline{L_L},
\end{aligned} \tag{4.28}$$

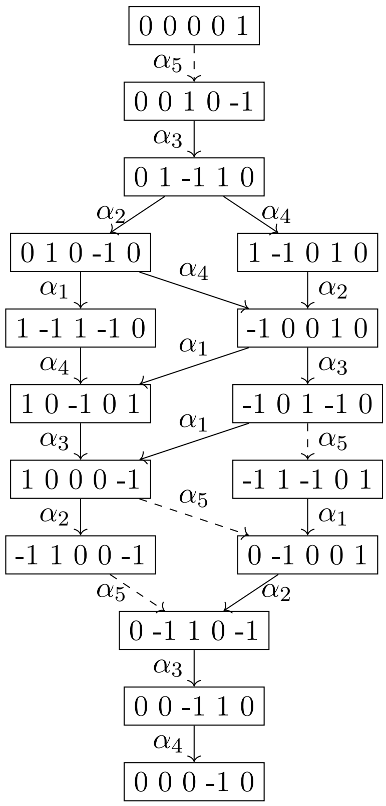


Figure 4.1: Method highest weight for the **16** irrep of $SO(10)$. When breaking the $SO(10)$ symmetry to $SU(5)$ the α_5 root is removed from the Dynkin diagram, this root has to be removed from the weight diagram as well. Removing this root, denoted by the dashed line, breaks the irrep into 3 pieces corresponding to the **10**, **5** and **1** irreps of $SU(5)$ the **16** irrep of $SO(10)$ breaks to.

therefore retrieving all the SM particles.

The above examples show the strength of using the method of highest weight for building and breaking irreps. Now that it is outlined how Dynkin diagrams and the method of highest weight can be used to break Lie algebras and irreps corresponding to these algebras the final thing missing for building GUTs is the mechanism that does the actual symmetry breaking. This mechanism will now be outlined below

4.6 Symmetry breaking in grand unified theories

As shown in the chapter 2 symmetries can be broken by SSB. However, SSB was only shown for the SM case, in which a complex scalar field ϕ, that is a $SU(2)$ doublet, does the actual symmetry breaking. For larger symmetries SSB becomes more involved. Li [20] shows how the principle of SSB can be extended to include larger symmetries.

The field that then does the symmetry breaking actually is something more complicated then the compex scalar field used before. Li uses multiple fields to construct a singlet and then show how these different singlets can do actual symmetry breaking. This would mean that the Higgs particles used for this kind of symmetry breaking

Representation	Dimension	Transformation law	Covariant derivative
vector	N	$\psi_i \to \psi_i + i\epsilon_i^j \psi_j$	$\partial_\mu \psi_i - igW_{\mu i}^j \psi_j$
2nd-rank symmetric tensor	$\frac{N(N+1)}{2}$	$\psi_{ij} \to \psi_{ij} + i\epsilon_i^k \psi_{kj} + i\epsilon_j^k \psi_{ik}$	$\partial_\mu \psi_{ij} - igW_{\mu i}^l \psi_{lj} - igW_{\mu j}^l \psi_{il}$
adjoint representation	$N^2 - 1$	$\psi_i^j \to \psi_i^j + i\epsilon_i^k \psi_k^j + i\epsilon_k^j \psi_i^k$	$\partial_\mu \psi_i^j - igW_{\mu i}^l \psi_l^j - igW_{\mu l}^j \psi_i^l$

Table 4.3: Summary of the different representations that can be responsible for the breaking of $SU(N)$ and their corresponding transformation laws and covariant derivatives.

would be consisting of multiple particles, therefore being a compound Higgs. Up to now however no evidence has been found for the SM Higgs particle consisting of other particles.

This section will now show how SSB can be extended to include all $SU(N)$ symmetries. For the complete derivation and the derivation of the SSB of other symmetries like for instance $O(N)$ the interested reader is referred to [20]. The results for $SU(N)$ will be shown below.

The $SU(N)$ Lie groups have $N^2 - 1$ hermitian generators U_i^j $(i, j = 1, \dots N)$ [3] which satisfy the following commutation relation

$$[U_i^j, U_k^l] = \delta_j^k U_i^l - \delta_i^l U_k^j. \tag{4.29}$$

There are then $N^2 - 1$ vector gauge bosons $W_{\mu j}^i$, for which $W_{\mu j}^i = (W_{\mu i}^j)^*$ and $W_{\mu i}^i = 0$. These vector gauges bosons transform according to

$$W_{\mu i}^j \to W_{\mu i}^j + i\epsilon_i^l W_{\mu l}^j - i\epsilon_k^j W_{\mu l}^k, \tag{4.30}$$

where $\epsilon_i^j = (\epsilon_j^i)^*$. The Yang-Mills Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu i}^j F_j^{\mu\nu i}, \tag{4.31}$$

where

$$F_{\mu\nu i}^j = \partial_\mu W_{\nu i}^j - \partial_\nu W_{\mu i}^j + ig(W_{\mu i}^k W_{\nu k}^j - W_{\nu i}^k W_{\mu k}^j). \tag{4.32}$$

Now breaking the $SU(N)$ of this Lagrangian via SSB can be done using different irreducible representations of $SU(N)$. The irreps of interest for this thesis are the vector irrep, the 2nd-rank symmetric tensor and the adjoint representation. These irreps are summarized in table 4.3 on the next page.

Now that these irreps and their transformation laws are outlined, the way in which these irreps can break a $SU(N)$ symmetry group will be outlined. Since for breaking

[3] Note that matrix indices are shown here as opposed to the rest of this thesis. These indices are shown since they are needed in some parts of the derivation of this section.

the $SU(N)$ symmetries SSB is used, the first thing to do is to construct an invariant under the given symmetry, using this invariant then an invariant potential can be constructed.

The vector irrep is the first irrep that will be looked at, using this irrep the following invariant can be constructed

$$\phi = \psi^i \psi_i. \tag{4.33}$$

By using this irrep the following invariant potential can be constructed

$$V(\psi) = -\frac{\mu^2}{2}\psi_i\psi^i + \frac{\lambda}{4}(\psi_i\psi^i)^2 \tag{4.34}$$

where μ and λ are real. Taking the first derivative to find the minimum gives the following N conditions

$$\frac{\partial V}{\partial \psi_i} = (-\mu^2 + \lambda\psi_j\psi^j)\psi_i = 0. \tag{4.35}$$

This is satisfied if $\psi_i\psi^i = \frac{\mu^2}{\lambda}$, which is very similar to the SM result. Actually the $N = 2$ case is the SM case, only now with the scalar Higgs of the SM would consisting of two fermions, again this has up to now not been experimentally witnessed.

For $N > 2$ the vector ψ_i can be chosen to have the following form $\psi_i = (0, 0, \dots, \frac{\mu^2}{\lambda})$, which satisfies the condition above. The other solutions to this condition can be found using a $SU(N)$ transformation. This solution then is invariant under transformation, which leave the Nth axis invariant. Therefore SSB of $SU(N)$ using one vector irrep leaves an $SU(N-1)$ symmetry.

The SSB using one vector can actually be extended to multiple vectors. If 2 vectors ψ_1 and ψ_2 are used, then a potential V can be constructed using ψ_1^2, ψ_2^2 and $\psi_1 \cdot \psi_2$. The vectors that then minimize this V can be written as $\psi_1 = (0, \dots, \alpha, \beta)$ and $\psi_2 = (0, \dots, \gamma, \delta)$, where α, β, γ and δ are coefficients such that V is minimized. Now these vectors can again be rotated into other solutions using $SU(N)$ transformations. Now rotations around the Nth and the $N-1$th axis leave these solutions invariant, therefore leaving a $SU(N-2)$ symmetry. This can be further extended to the general result that SSB with k vectors break $SU(N)$ to a $SU(N-k)$ symmetry.

Now that the results of using vectors for SSB have been obtained the symmetry breaking with a 2nd-rank asymmetric tensor will be looked at. The invariant potential then becomes

$$V(\psi) = -\frac{\mu^2}{2}\psi_{ij}\psi^{ij} + \frac{\lambda_1}{4}(\psi_{ij}\psi^{ij})^2 + \frac{\lambda_2}{4}(\psi_{ij}\psi^{jk}\psi_{kl}\psi^{il}), \tag{4.36}$$

where $\psi_{ij} = \psi_{ji} = (\psi^{ij})^*$ and the invariant scalar now is $\phi = \psi_{ij}\psi^{ij}$. Now in order to minimize $V(\psi)$ the following conditions need to be met

$$\frac{\partial V}{\partial \psi_{ij}} = -\frac{\mu^2}{2}\psi^{ij} + \frac{\lambda_1}{2}(\psi_{lm}\psi^{lm})\psi^{ij} + \frac{\lambda_2}{2}(\psi^{jk}\psi_{kl}\psi^{li}) = 0. \tag{4.37}$$

These equations can be simplified by introducing the following Hermitian matrix

$$X_l^k \equiv \psi_{lm}\psi^{mk}. \tag{4.38}$$

The conditions to be met then become

$$\mu^2\psi^{ij} + \lambda_1(X_l^l)\psi^{ij} + \lambda_2 X_l^j \psi^{li} = 0. \tag{4.39}$$

Now since X is hermitian it can be diagonalized using a basis transformation, this leads to the following equation

$$\left(\mu^2 + \lambda_1 \sum_{k=1}^{l} X_k + \lambda_2 X_i\right) \psi^{ij} = 0. \tag{4.40}$$

So assuming that $\psi_{ij} \neq 0$, the following condition has to be met to minimize $V(\psi)$

$$\mu^2 + \lambda_1 \sum_{k=1}^{l} X_k + \lambda_2 X_i = 0. \tag{4.41}$$

Now the solutions that are of interest are the solutions for which not all $X_i = 0$. Now suppose that for $i = 1, \ldots, k$ $X_i \neq 0$, then (4.41) holds for $i = 1, \ldots, k$, therefore

$$X_i = \frac{\mu^2}{2\lambda_1 k + \lambda_2} \text{ for } i = 1, \ldots, k. \tag{4.42}$$

Therefore the minimium of $V(\psi)$ lies at

$$V(\psi) = \frac{k\mu^4}{2k\lambda_1 + \lambda_2}. \tag{4.43}$$

The potential $V(\psi)$ as a function of the number of nonzero X_i is monotonically increasing if $\lambda_2 > 0$, and monotonically decreasing if $\lambda_2 < 0$ and $\lambda_1 k + \lambda_2 > 0$. For the former case it can be shown[4] that the minimum lies at $X = c^2\mathbb{1}$ this leads to $\psi_{ij} = c\mathbb{1}$ at the minimum. This form of ψ actually has an $O(N)$ symmetry, since $\psi \to U^T \psi U = U^T U \psi = \psi$. For this case the $SU(N)$ is broken to a $O(N)$ symmetry, this case is however not of interest for this thesis.

For the latter case it can be shown that the minimum is at

$$X = d^2 \begin{pmatrix} 1 & \ldots & & 0 \\ & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}, \tag{4.44}$$

where $d^2 = \frac{\mu^2}{\lambda_1 + \lambda_2}$ and $\lambda_1 + \lambda_2 > 0$. Therefore

$$\psi = d \begin{pmatrix} 1 & \ldots & & 0 \\ & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} \tag{4.45}$$

this is very similar to the 1 vector case derived above. Therefore, the 2nd-rank symmetric tensor in this case breaks $SU(N)$ to $SU(N-1)$.

Using a similar derivation it can be shown that an antisymmetric 2nd-rank tensor can be used to break down $SU(N)$ to either $O(l)$, where $l = \frac{N}{2}$ or $SU(N-2)$. Now that it has been shown how vectors and tensors can break $SU(N)$, now the adjoint

[4] Appendix C of [20] shows the proof for this.

representation will be looked at.

Imposing the additional symmetry $\psi \to -\psi$ on the adjoint representation the invariant potential becomes

$$V(\psi) = \frac{\mu^2}{2}\psi_i^j\psi_j^i + \frac{\lambda_1}{4}(\psi_i^j\psi_j^i)^2 + \frac{\lambda_2}{4}(\psi_i^j\psi_j^k\psi_k^l\psi_l^i) \tag{4.46}$$

for which $\psi_i^i = 0$. Since the adjoint rep is hermitian as well, ψ can again be diagonalized. Therefore ψ can be written in the following way $\psi_i^j = \delta_i^j\phi_j$. This leaves the following potential

$$V(\psi) = \frac{\mu^2}{2}\sum_{i=1}^{N}\phi_i^2 + \frac{\lambda_1}{4}\left(\sum_{i=1}^{N}\phi_i^2\right)^2 + \frac{\lambda_2}{4}\sum_{i=1}^{N}\phi_i^4 - g\sum_{i=1}^{N}\phi_i. \tag{4.47}$$

Here g is the Langangian multiplier which allows us to include the constraint of ψ being traceless into the potential. Minimizing the potential using the first derivatives then leads to the following constraints

$$\frac{\partial V}{\partial \phi} = -\mu^2\phi_i + \lambda_1\left(\sum_{j=1}^{N}\phi_j^2\right)\phi_i + \lambda_2\phi_i^3 - g = 0. \tag{4.48}$$

Since all the ϕ_i's have to satisfy the same cubic equation it can actually be shown that there can at most be two different solutions ϕ_1 and ϕ_2. The set of solutions ϕ then can be written as

$$\phi = \begin{pmatrix} \phi_1 & & \cdots & & & 0 \\ & \ddots & & & & \\ \vdots & & \phi_1 & & & \\ & & & \phi_2 & & \\ & & & & \ddots & \\ 0 & & & & & \phi_2 \end{pmatrix}. \tag{4.49}$$

For $\lambda_2 > 0$ this ϕ breaks $SU(N)$ to $SU(N-l) \times SU(l)$, where $l = N/2$ for even N and $l = \frac{N+1}{2}$ for odd N. This can be viewed as the ϕ_1's acting as l vectors and ϕ_2's acting as $N - l$ vectors. The split l is not arbitrary and can be determined using a computation, which determines it as mentioned above, this computation can be found in [20]. Again using this computation for $\lambda_2 < 0$ it can be shown that then $SU(N)$ breaks to $SU(N-1)$.

The different ways of SSB $SU(N)$ using the different irreps as derived above are summarized in table 4.4 on the next page.
These different ways of breaking $SU(N)$ symmetries can for instance be used in breaking down the $SU(5)$ to the SM, which is part of a GUT that will be elaborated on further in the next chapter. For now note that $SU(5)$ can be broken in different way, the 5-dimensional fundamental vector representation **5** for instance leads to the following symmetry breaking

$$SU(5) \to SU(4) \times U(1). \tag{4.50}$$

The symmetry breaking needed to arrive at the SM is the one achieved by using the adjoint representation which leads to

$$SU(5) \to SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{4.51}$$

Representation	Broken symmetry
Vector	$SU(N-1)$
k vectors	$SU(N-k)$
2nd-rank asymmetric tensor	$O(N)$ or $SU(N-1)$
Adjoint representation	$SU(l) \times SU(N-l)$, where $l = \frac{N}{2}$ if l is even $l = \frac{N+1}{2}$ if l is odd or $SU(N-1)$

Table 4.4: Different irreps of $SU(N)$ and the symmetry groups to which $SU(N)$ breaks by SSB using invariants constituting of these different irreps.

This is the symmetry of the SM before Electroweak Symmetry Breaking by the SM Higgs that is responsible for the final symmetry breaking step

$$SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_Q. \tag{4.52}$$

Now that it has been shown how different irreps can be responsible for different ways of breaking their respective group, an important observation can be made on these symmetry breaking irreps. This observation can be made by looking at the the $SU(5)$ SSB irreps **24** and **5** and their respective branching ratios. For $SU(5) \to SU(3) \times SU(2) \times U(1)$ the branching ratio of the symmetry breaking irrep **24** is [3]

$$\mathbf{24} \to (\mathbf{8}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{3}, \mathbf{2}, -5) \oplus (\overline{\mathbf{3}}, \mathbf{2}, 5). \tag{4.53}$$

Subsequently the branching rule of $SU(5) \to SU(4) \times U(1)$ breaking irrep **5** is

$$\mathbf{5} \to (\mathbf{1}, 4) \oplus (\mathbf{4}, -1). \tag{4.54}$$

Both these decompositions contain a singlet of their respective (broken) groups, which are $(\mathbf{1}, \mathbf{1}, 0)$ and $(\mathbf{1}, 4)$. This is actually a characteristic that all symmetry breaking irreps share [3]. Therefore in searching for an irrep that can be responsible for the SSB of its corresponding group to a certain subgroup, the branching rule of this irrep to the broken irreps should include a singlet of the subgroup.

Now that the different ways in which larger symmetries can be broken have been summarized, all tools that are needed have been mentioned. Using these tools different GUTs will now be outlined, therefore the consequences of these GUTs can be determined as well.

5 Grand unified theories

Now that these group-theoretical tools have been outlined above, GUTs based on 2 different symmetry groups will be described, showing how a specific symmetry group impacts the GUT. The idea for a GUT was first proposed by Georgi and Glashow [21] in their paper on $SU(5)$ unification. Although now proven to be experimentally excluded, the paper at that time was novel since for the first time it proposed the idea of extending the SM by embedding it in a larger symmetry group. This chapter will actually outline a $SO(10)$ GUT that via $SU(5)$ breaks down to the SM. Next to this model other GUTs based on the $SO(10)$ group will be outlined. The other symmetry group this sector will outline is the $(SU(3))^3$ Trinification model.

However before outlining models based on these 2 symmetry groups, a scheme is given, which outlines how different groups of rank $l \leq 6$ are interconnected. This scheme, given on the next page, is taken from [16], it shows how diverse the paths are that can be chosen for a GUT that ultimately breaks down to the SM. In choosing such a path and the irreps corresponding to the different symmetries involved a GUT is already quite heavily restricted. This was already shown above in the $SO(10) \to SU(5) \to SU(3)_C \times SU(2)_L \times U(1)_Y$ example, which shows how the fermions of one generation correspond to one representation, the **16** of $SO(10)$. This representation is then broken down to the SM representations corresponding to the particles witnessed in nature. This section will now show for the example above and the trinification model how these models are built up completely and what consequences can be derived from that, starting with the $SO(10)$ model.

5.1 SO(10) grand unification

As already mentioned above the fermions of one generation nicely fit in the **16** of $SO(10)$, this is one of the reasons which makes $SO(10)$ an interesting symmetry group for a GUT. The 16th fermion in this representation then actually corresponds to the right-handed neutrino, which would imply the neutrino being a Dirac particle. As can be seen from the figure on the next page $SO(10)$ can break down to the SM symmetry group via 3 different paths, these will now be outlined.

5.1.1 The Georgi-Glashow path

This path was already mentioned above and gets its name from Georgi and Glashow being the first one put forward $SU(5)$ unification [21]. The path therefore is the following

$$SO(10) \to SU(5) \to SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{5.1}$$

The SM fermions then are retrieved in the following 2 steps

$$\mathbf{16} \to \mathbf{10} + \overline{\mathbf{5}} + \mathbf{1} \tag{5.2}$$

and

$$\begin{aligned}
\mathbf{10} &\to (\mathbf{1},\mathbf{1},1) \oplus (\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3}) \oplus (\mathbf{3},\mathbf{2},\tfrac{1}{6}) = \overline{e_R} + \overline{u_R} + Q_L \\
\overline{\mathbf{5}} &\to (\mathbf{1},\mathbf{2},-\tfrac{1}{2}) \oplus (\overline{\mathbf{3}},\mathbf{1},\tfrac{1}{3}) = d_R + \overline{L_L} \\
\mathbf{1} &\to (\mathbf{1},\mathbf{1},0) = \nu_R.
\end{aligned} \tag{5.3}$$

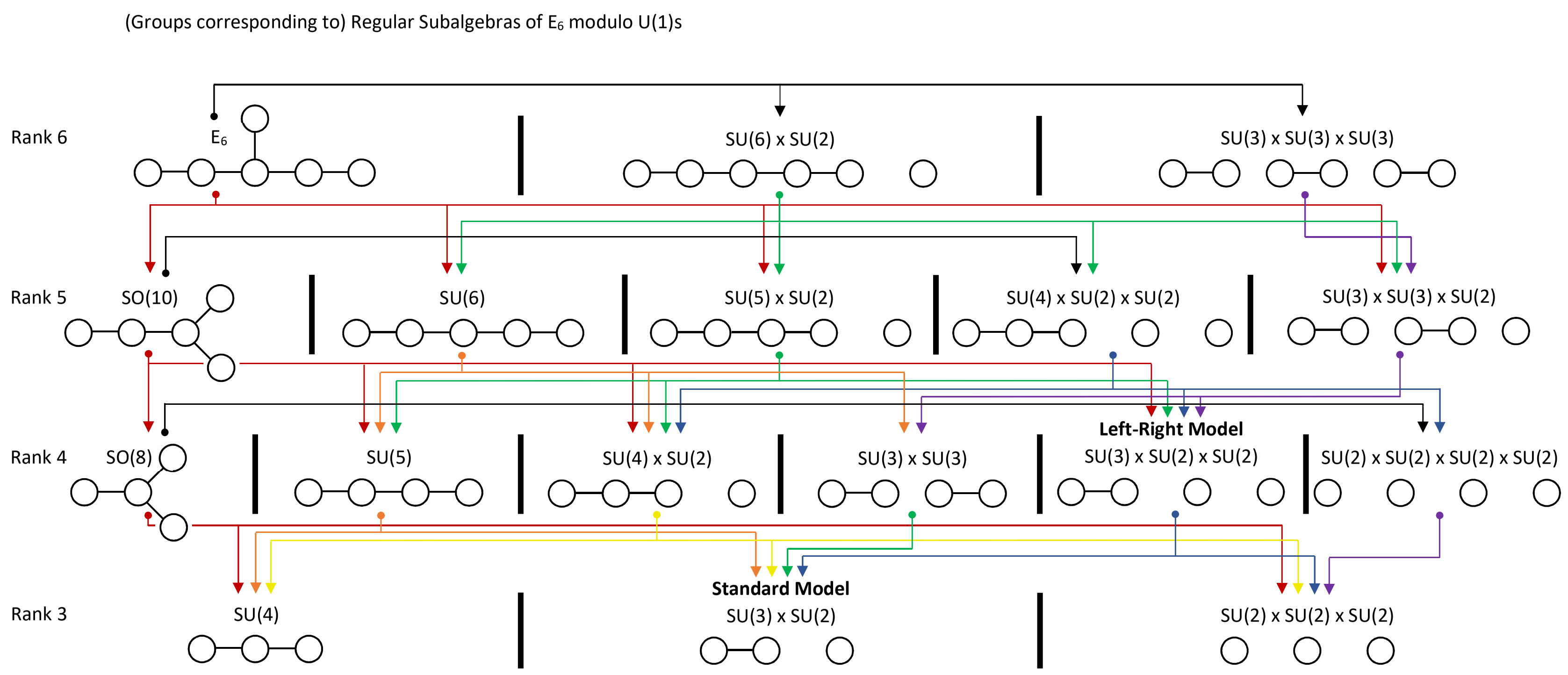


Figure 5.1: Figure showing all regular subalgebras of E_6 taken from [16], the arrows denote which subalgebras one can reach from a certain algebra via the use of the (extended) Dynkin diagram of that algebra as was outlined in chapter 4. The $U(1)$ groups which correspond to the root that is removed from the Dynkin diagram in going down a rank are left out for the sake of clarity.

An important thing that has up to now been neglected is how the SM gauge bosons can be recovered in a GUT. One of the strenghts of $SU(5)$ unification actually is that all the SM bosons fit in the adjoint representation of $SU(5)$ in the following way

$$\mathbf{24} \to (\mathbf{8}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{3}, \mathbf{2}, -5) \oplus (\overline{\mathbf{3}}, \mathbf{2}, 5), \tag{5.4}$$

where the first 3 terms correspond subsequently to the 8 gluons and the A^i_μ and B_μ bosons. The 2 final terms both transform non-trivially under both $SU(3)_C$ and $SU(2)_L$. These bosons are therefore able to connect leptons and quarks with each other, they are therefore coined leptoquarks. These leptoquarks allow for proton decay which is shown in figure 5.2 below.

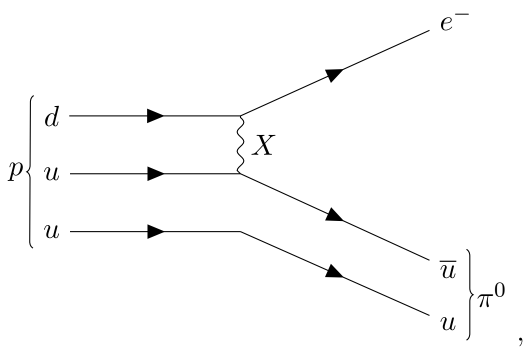


Figure 5.2: Feyman diagram which shows the $p \to \pi^0 + e^-$ type proton decay, this decay is mediated by a leptoquark X that due to its specific SM irrep is able to couple leptons to quarks.

This kind of proton decay is actually what excludes $SU(5)$ unification, due to it predicting a proton lifetime that has been experimentally excluded. However by embedding $SU(5)$ GUT into a $SO(10)$ GUT this proton lifetime can actually be shifted in a way that coincides with current experimental constraints. Since all additional particles occuring in a GUT are often postulated to be very heavy, measuring them directly is as of yet technically impossible. Measurements of the proton lifetime therefore give an interesting way of indirectly determining which GUTs are viable and which are not. For more on proton decay the reader is referred to [16].

The $SU(5)$ $\mathbf{24}$ adjoint representation which includes the SM bosons can then be embedded into $SO(10)$ in the following way

$$\mathbf{45} \to \mathbf{24} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}, \tag{5.5}$$

which is the adjoint representation of $SO(10)$. This embedding leaves this GUT with an additional 21 extra bosons on top of the 12 leptoquarks already included in the theory due to the specific $SU(5)$ symmetry breaking, that as mentioned above are all presumed very heavy.

A final thing that is interesting to examine is what the Higgs sector of this model is, which is responsible for the SSB of this theory. As was previously shown $SU(5)$ can be broken to the SM by its adjoint representation, $\mathbf{24}_H$, this $\mathbf{24}_H$ can then be embedded in a $SO(10)$ $\mathbf{45}_H$. It has however up to now not been outlined how a SSB of the type $SO(10) \to SU(5)$ actually works. It turns out that the Higgs sector is one of the most arbitrary things about a GUT, since there are a lot of choices that can be made. However the lowest dimensional Higgs capable of breaking $SO(10) \to SU(5)$ is the $\mathbf{16}_H$ [22]. Therefore the minimal Higgs sector of this specific $SO(10)$ GUT is $\mathbf{45}_H$ and $\mathbf{16}_H$. Having outlined the Georgi-Glashow path now the Left-Right path will now be looked at.

5.1.2 The Left-Right path

As can be seen from figure 5.2 there are 2 other ways of arriving at the SM from $SO(10)$, both of these paths go via the Left-Right (LR) model [23]. However one of these paths goes directly via the LR model, this path is the following

$$SO(10) \to SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \to SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{5.6}$$

The Left-Right model extends the SM in a very symmetric way, it imposes an additional $SU(2)_R$ symmetry, therefore removing the asymmetry of the SM between how left and right-handed particles interact. The particles of the first generation then correspond to the following multiplets

$$\begin{aligned} Q_L &= \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim (\mathbf{2}, \mathbf{1}, \tfrac{1}{3}) \quad & Q_R &= \begin{pmatrix} u_R \\ d_R \end{pmatrix} \sim (\mathbf{1}, \mathbf{2}, \tfrac{1}{3}) \\ L_L &= \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} \sim (\mathbf{2}, \mathbf{1}, -1) \quad & L_R &= \begin{pmatrix} e_R \\ \nu_R \end{pmatrix} \sim (\mathbf{1}, \mathbf{2}, -1), \end{aligned} \tag{5.7}$$

where this notation denotes the irreducible representation of subsequently $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. Note that the $U(1)_Y$ of the SM is replaced by the $U(1)_{B-L}$ symmetry, which corresponds to the Baryon and Lepton number symmetry, this global symmetry of the SM therefore has become a local gauge symmetry in this theory. The fermions of this model can be embedded in $SO(10)$ in the following way

$$\mathbf{16} \to (\overline{\mathbf{3}}, \mathbf{2}, \mathbf{1}, -\tfrac{1}{3}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, 1) \oplus (\mathbf{3}, \mathbf{1}, \mathbf{2}, \tfrac{1}{3}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, -1) = \overline{Q_L} + \overline{L_L} + Q_R + L_R. \tag{5.8}$$

The gauge bosons can again be included in the $\mathbf{45}$ adjoint representation of $SO(10)$, this now gives the following decomposition

$$\begin{aligned} \mathbf{45} \to\ & (\mathbf{1}, \mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1}, 0) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{8}, \mathbf{1}, \mathbf{1}, 0) \\ & \oplus (\mathbf{3}, \mathbf{1}, \mathbf{1}, \tfrac{4}{3}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, -\tfrac{4}{3}) \oplus (\mathbf{3}, \mathbf{2}, \mathbf{2}, -\tfrac{2}{3}) \oplus (\overline{\mathbf{3}}, \mathbf{2}, \mathbf{2}, \tfrac{2}{3}). \end{aligned} \tag{5.9}$$

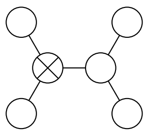
In this decomposition the first 4 terms correspond to the SM bosons plus the right-handed bosons $A^j_{\mu,R} \sim (\mathbf{1}, \mathbf{1}, \mathbf{3}, 0)$, which is the symmetric partner to the left-handed bosons. There are therefore corresponding to the $W^\pm_{\mu,L}$ and $Z^0_{\mu,L}$, the right-handed versions of these bosons $W^\pm_{\mu,R}$ and $Z^0_{\mu,R}$ [23]. The 2 final terms of this decomposition are again leptoquarks, therefore allowing for proton decay. Having outlined this Left-Right path, now the Pati-Salam path will be outlined.

5.1.3 The Pati-Salam path

The final possible path from $SO(10)$ to the SM is the one that goes via the $SU(4) \times SU(2) \times SU(2)$ symmetry group. The model based on this symmetry group was first put forward by Pati and Salam [24], it is therefore called the Pati-Salam path. This path is the following

$$\begin{aligned} SO(10) &\to SU(4) \times SU(2)_L \times SU(2)_R \to SU(3)_C \times SU(2)_L \times SU(2)_R \\ &\to SU(3)_C \times SU(2)_L \times U(1)_Y. \end{aligned} \tag{5.10}$$

Since the Pati-Salam symmetry group is of the same rank as the $SO(10)$ this symmetry group the breaking goes via the extended diagram, which is



.

The root that is crossed is the one that is removed from the Dynkin diagram to leave $SU(4) \times SU(2) \times SU(2)$. The **16** containing the fermions of one generation breaks down in the following way for this path

$$\mathbf{16} \to (\overline{\mathbf{4}}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2}). \tag{5.11}$$

These 2 multiplets correspond to the left and right-handed fermions, with the quarks and leptons being in the same multiplet. Lepton number therefore becomes the fourth colour, corresponding to these 4 colours is the $SU(4)$ symmetry of the PS model. The fact that leptons and quarks are actually connected to each other via this $SU(4)$ symmetry means that again proton decay is possible within this model. The adjoint representation, **45**, then again contains all the gauge bosons needed, for this path it breaks in the following way

$$\mathbf{45} \to (\mathbf{1}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{15}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{2}, \mathbf{2}). \tag{5.12}$$

To arrive at the SM the $SU(4)$ symmetry is broken in the following way

$$SU(4) \to SU(3)_C \times U(1)_{B-L}, \tag{5.13}$$

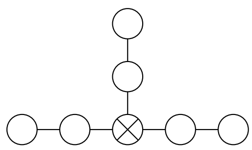
therefore leaving the LR-model. Arriving at the LR model via the PS model actually gives the exact same representations already listed above for the LR model. There is another model which also arrives at the SM via the LR model this is the so called Trinification model which corresponds to the following symmetry group $SU(3) \times SU(3) \times SU(3)$ or $(SU(3))^3$, this model will now be outlined below.

5.2 The Trinification model

The Trinification model [25] has the following symmetry group

$$SU(3)_C \times SU(3)_L \times SU(3)_R, \tag{5.14}$$

which is similar to the LR model. The left and right-handed symmetry groups however now are not $SU(2)$, they are $SU(3)$. This symmetry group in itself actually does not guarantee one single gauge coupling, which is one of the most important aspects of a GUT. This single gauge coupling can however be achieved by either implementing an additional discrete Z_3 symmetry, which allows for the rotation of the 3 $SU(3)$ symmetries in to each other. Another way of guaranteeing this single gauge coupling is embedding the trinification model in an E_6 symmetry. E_6 is the same rank as $(SU(3))^3$ and therefore the extended Dynkin diagram has to be looked at. The extended Dynkin diagram of E_6 is the following

 ,

where again the simple root that is crossed is removed from the Dynkin diagram leaving the 3 $SU(3)$ Dynkin diagrams. The $(SU(3))^3$ breaks down to the SM in the following way

$$\begin{aligned}(SU(3))^3 &\to SU(3)_C \times SU(3)_L \times SU(2)_R \to SU(3)_C \times SU(2)_L \times SU(2)_R \\ &\to SU(3)_C \times SU(2)_L \times U(1)_Y. \end{aligned} \tag{5.15}$$

All the SM fermions of one generation can then be contained in the following $SU(3)^3$ multiplet

$$(\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}) \equiv \psi_L \oplus \psi_{Q^c} \oplus \psi_Q. \tag{5.16}$$

This multiplet can then exactly be embedded into the fundamental representation of E_6, the **27**. They break down to the SM symmetry group in the following way

$$\begin{aligned}\psi_L &\to (\mathbf{1}, \mathbf{2}, \tfrac{1}{2}) \oplus 2(\mathbf{1}, \mathbf{2}, -\tfrac{1}{2}) \oplus (\mathbf{1}, \mathbf{1}, 1) \oplus 2(\mathbf{1}, \mathbf{1}, 0) \\ \psi_{Q^c} &\to (\overline{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}) \oplus 2(\overline{\mathbf{3}}, 1, \tfrac{1}{3}) \\ \psi_Q &\to (\mathbf{3}, \mathbf{2}, \tfrac{1}{6}) \oplus (\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}). \end{aligned} \tag{5.17}$$

[25] then shows how from these representations together make up to the SM fermions of one generation. The SM bosons can then be put in the following multiplet

$$(\mathbf{8}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{8}). \tag{5.18}$$

The first terms corresponds to the 8 gluons of the SM and the second term then contains the other 4 gauge bosons by breaking down to the SM bosons in the following way

$$(\mathbf{1}, \mathbf{8}, \mathbf{1}) \to (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{2}, 0) \oplus (\mathbf{1}, \mathbf{2}, 0) \tag{5.19}$$

An interesting thing to note is that the multiplet containing the SM bosons does not include leptoquarks since there are no representations that transform non-trivial under both $SU(3)_C$ and $SU(2)_L$ or $SU(2)_R$. However the Higgs sector of this theory does have representations that do transform non-trivially for both these symmetries.

Furthermore, if $SU(3)^3$ is embedded in E_6 as was mentioned before, then these 3 boson multiplets correspond to the adjoint representation, **78**, of E_6 in the following way

$$\mathbf{78} \to (\mathbf{8}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{8}) \oplus (\mathbf{3}, \mathbf{3}, \overline{\mathbf{3}}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{3}}, \mathbf{3}). \tag{5.20}$$

This theory does then actually include leptoquarks, therefore allowing for direct proton decay.

As for the Higgs sector, remember that a **3** vector is capable of breaking $SU(3) \to SU(2) \times U(1)$. Therefore $(\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}})$ is capable of breaking $SU(3)_L \times SU(3)_R \to SU(2)_L \times SU(2)_R \times U(1)$. Then another breaking using $(\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}})$ breaks down $SU(2)_L \times SU(2)_R \times U(1) \to SU(2)_L \times U(1)_Y$, therefore arriving at the SM. The Higgs sector therefore consists of 2 $(\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}})_H$ multiplets, which can be embedded in 2 E_6 $\mathbf{27}_H$ multiplets. These 2 multiplets are then able to mediate proton decay, acting as scalar leptoquarks, which are different from the $SU(5)$ leptoquarks.

From outlining these theories it can be seen that in picking a symmetry group for a GUT and then working down to the SM a lot of things actually can be implied by pure group-theoretical analysis. In trying to achieve a minimal GUT by picking one of the symmetry groups listed above a lot of consequences from the GUT can already be derived. The final chapter of this thesis will now try to extend this framework of GUTs by adding a discrete family symmetry to them.

6 Family symmetry GUTs

This chapter will show how a GUT can be built that has a discrete family symmetry embedded into it. To do so this chapter will first outline how a discrete family can actually be embedded in a continuous symmetry. Here A_4 will be embedded into the continuous $SU(3)_F$ symmetry, to do so this thesis will follow the procedure outlined in [26]. After having outlined this embedding the 2 GUTs mentioned in the previous chapter, $SO(10)$ and $SU(3)^3$, will be extended by this continuously embedded family symmetry. Before doing this now first it will be shown how a discrete family can be embedded in a continuous symmetry.

6.1 Embedding D_F in a continuous symmetry

In the previous chapter a discrete symmetry actually already was added to a GUT in the trinification model. The original model imposes a Z_3 symmetry to ensure there is one single gauge coupling for the $SU(3)^3$ model. This would be an option to implement D_F to a GUT as well, simply adding it to the desired symmetry group for the GUT e.g. $SO(10) \times A_4$. This approach however is quite unsatisfactory, since it would not explain how D_F got there in the first place. This problem can be circumvented by embedding D_F in a continuous symmetry group.

This sector will show by using A_4 as an example, how D_F can be embedded in a continuous group. Remember from chapter 3 that A_4 is a suitable D_F since it has a 3-dimensional irreducible representation. Therefore when embedding A_4 in a continuous symmetry, this continuous symmetry should have a 3-dimensional irreducible representation as well. Two suitable canditates are therefore $SU(3)$ and $SO(3)$, where $SO(3)$ can easily be embedded in the GUT structure listed above due to its (local) isomorphism with $SU(2)$. Which makes the embedding of A_4 via either $SU(3)$ or $SU(2)$ similar, the only real difference is the rank of $SU(3)$ being 2 instead of the rank 1 of $SU(2)$.

Using $SU(2)$ to embed the A_4 family symmetry into a GUT will therefore result in a smaller symmetry group then $SU(3)$. However using $SU(3)$ leads to the fundamental representation of $SU(3)$ the **3** being responsible for the mixing amongst the 3 generations. This is similar to the symmetry of $SU(3)_C$ being responsible for the mixing amongst 3 generations. It is therefore chosen to use $SU(3)$ as the symmetry group being responsible for the A_4 embedding, since this is esthetically more pleasing. It will therefore now be shown below how the discrete family symmetry A_4 can be embedded in the continuous $SU(3)_F$ flavour symmetry.

6.1.1 $A_4 \subset SU(3)$

This section will determine how the continuous family symmetry $SU(3)_F$ will break down to the discrete family symmetry A_4. The mechanism that will do the actual symmetry breaking will again be that of SSB. Now in order to achieve this SSB a $SU(3)$ potential has to be found that after symmetry breaking is A_4 invariant. Therefore the branching rules for $SU(3)$ irreps breaking to A_4 are needed.

These branching rules can be determined by comparing the tensor products of $SU(3)$ to the Kronecker products of A_4. These Kronecker products were already outlined in (3.11), for convenience the $\mathbf{3} \otimes \mathbf{3}$ is repeated here

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1''} \oplus \mathbf{3} \oplus \mathbf{3}. \tag{6.1}$$

This Kronecker product can then be compared to the analogous tensor product of $SU(3)$

$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6} \tag{6.2}$$

Since A_4 is real $\mathbf{3} = \overline{\mathbf{3}}$, therefore from comparing the $\mathbf{3} \otimes \mathbf{3}$ Kronecker and tensor products, the following can be deduced

$$\mathbf{6} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1''} \oplus \mathbf{3}. \tag{6.3}$$

Now another tensor product of $SU(3)$ is

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \tag{6.4}$$

From which it can be derived that

$$\mathbf{8} = \mathbf{1'} \oplus \mathbf{1''} \oplus 2 \cdot \mathbf{3} \tag{6.5}$$

This procedure can be used with even larger $SU(3)$ tensor products to arrive at the following list of $SU(3) \supset A_4$ branching rules listed in table 6.1 below.

$A_4 \subset SU(3)$
$\mathbf{3} = \mathbf{3}$
$\mathbf{6} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1''} \oplus \mathbf{3}$
$\mathbf{8} = \mathbf{1'} \oplus \mathbf{1''} \oplus 2 \cdot \mathbf{3}$
$\mathbf{10} = \mathbf{1} \oplus 3 \cdot \mathbf{3}$
$\mathbf{15} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1''} \oplus 4 \cdot \mathbf{3}$

Table 6.1: Decomposition of the $SU(3)$ irreps up to **15** into the different A_4 irreps by using the method of comparing Kronecker and tensor products which was mentioned above. Note that the $SU(3)$ irreps are on the left-hand side of the equations.

From this table it can be seen that the **6**, **10** and **15** irreps of $SU(3)$ contain an A_4 singlet in their decomposition. The next step in breaking $SU(3)$ to A_4 then is finding how the singlets within these representations actually look. This can be shown by looking at the fundamental triplet representation of $SU(3)$, a general triplet can be written as

$$|\phi\rangle = \sum_{i=1}^{3} \phi_i |i\rangle , \tag{6.6}$$

where the 3 orthonormal states are

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.7}$$

Such a triplet state then transforms according to the **3** representation of A_4 which remembering the MR basis given in (3.8) can be represented by the following 2 matrices

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.8}$$

Using these matrices it can be deduced how the **6** transforms under an A_4 transformation, in order to do so first this **6** has to be given in the triplet basis. In order to do this remember that the **6** corresponds to the symmetric product of 2 triplets. The most general sextet can be given in terms of the following 6 orthonormal states

$$\begin{aligned} &|1\} = |11\rangle \quad |2\} = |22\rangle \quad |3\} = |33\rangle \\ |4\} = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle) \quad &|5\} = \frac{1}{\sqrt{2}}(|23\rangle + |32\rangle) \quad |6\} = \frac{1}{\sqrt{2}}(|13\rangle + |31\rangle), \end{aligned} \tag{6.9}$$

where $|ij\rangle = |i\rangle \otimes |j\rangle$. The most general sextet, **6**, therefore is given by

$$|\chi\} = \sum_{i=1}^{6} \chi_i |i\}. \tag{6.10}$$

Now in order to find the A_4 singlet that is contained inside this most general sextet, it should be checked for which coefficients the following holds true: $D|\chi\} = |\chi\}$ and $A|\chi\} = |\chi\}$. In order to be able to check this the transformation laws for the orthonormal states should be used, which are

$$\begin{aligned} S|1\} &= S\,|11\rangle = S\,|1\rangle \otimes S\,|1\rangle = |11\rangle = |1\} \\ T|1\} &= T\,|11\rangle = T\,|1\rangle \otimes T\,|1\rangle = |33\rangle = |3\} \end{aligned} \tag{6.11}$$

and therefore similarly

$$\begin{aligned} S|2\} = (-1)^2\,|22\rangle = |2\} \quad & T|2\} = |11\rangle = |1\} \\ S|3\} = |3\} \quad & T|3\} = |22\rangle = |2\}. \end{aligned} \tag{6.12}$$

The orthonormal state $|4\}$ then transforms in the following way

$$\begin{aligned} S|4\} &= \frac{1}{\sqrt{2}}(S\,|1\rangle \otimes S\,|2\rangle + S\,|2\rangle \otimes S\,|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle \otimes -1 \cdot |2\rangle - 1 \cdot |2\rangle \otimes |1\rangle) = -|4\} \\ T|4\} &= \frac{1}{\sqrt{2}}(T\,|1\rangle \otimes T\,|2\rangle + T\,|2\rangle \otimes T\,|1\rangle) = \frac{1}{\sqrt{2}}(|3\rangle \otimes |1\rangle + |1\rangle \otimes |3\rangle) = |6\}. \end{aligned} \tag{6.13}$$

Therefore $|5\}$ and $|6\}$ transform in the following way

$$\begin{aligned} S|5\} = -|5\} \quad & T|5\} = |4\} \\ S|6\} = -|6\} \quad & T|6\} = |5\}. \end{aligned} \tag{6.14}$$

Now using these transformations laws it can be shown that the most general sextet transforms under D in the following way

$$\begin{aligned} S|\chi\} &= S\chi_1|1\} + S\chi_2|2\} + S\chi_3|3\} + S\chi_4|4\} + S\chi_5|5\} + S\chi_6|6\} \\ &= \chi_1|1\} + \chi_2|2\} + \chi_3|3\} - \chi_4|4\} - \chi_5|5\} - \chi_6|6\}, \end{aligned} \tag{6.15}$$

therefore in order for **6** to be invariant under S $\chi_4 = \chi_5 = \chi_6 = 0$. The sextet that is left transforms under T in the following way

$$\begin{aligned} T|\chi\} &= T\chi_1|1\} + T\chi_2|2\} + T\chi_3|3\} \\ &= \chi_1|3\} + \chi_2|1\} + \chi_3|2\}, \end{aligned} \tag{6.16}$$

therefore in order for this sextet to be invariant under T $\chi_1 = \chi_2 = \chi_3$. The A_4 invariant **6** therefore looks the following

$$|\chi\} \propto (1,1,1,0,0,0)^T. \tag{6.17}$$

By using this invariant then a Lagrangian can be built that breaks $SU(3)$, before outlining how this SSB the other A_4 invariant $SU(3)$ representations should be given. **10** corresponds to the symmetric product of 3 triplets, it can be shown using a similar procedure [26] that the A_4 invariant for this symmetric product is the following

$$\mathbf{10} \propto (0,0,0,0,0,0,0,0,0,1)^T. \tag{6.18}$$

For the **15** representation the A_4 invariant form is the following

$$\mathbf{15} \propto (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1)^T. \tag{6.19}$$

The derivation of both these results is given in Appendix A. As mentioned above it is in using such an A_4 singlet a potential can be built that breaks $SU(3)$. However, before building such a potential it is interesting to see whether such a potential actually does break $SU(3)$ down to A_4. This can be seen by evaluating which subgroup is left invariant when transforming the A_4 invariant using a general $SU(3)$ transformation. Such a general $SU(3)$ transformation can be parameterized in the following way

$$U = P_1 \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} P_2, \tag{6.20}$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$ and

$$P_1 = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{-i(\alpha_1+\alpha_2)} \end{pmatrix} \quad P_2 = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{-i(\beta_1+\beta_2)} \end{pmatrix}. \tag{6.21}$$

The triplet then transforms according to

$$|i\rangle \to \sum_{j=1}^{3} U_{ij}\,|j\rangle \tag{6.22}$$

The A_4 invariant sextet then transforms in the following way

$$\sum_{i=1}^{3} |ii\rangle \to \sum_{i,j,k=1}^{3} U_{ij}U_{ik}\,|jk\rangle \tag{6.23}$$

In order for this to be $SU(3)$ invariant the following condition has to hold

$$\sum_{i=1}^{3} U_{ij}U_{ik} = \sum_{i=1}^{3} U_{ji}^T U_{ik} = \delta_{jk}, \tag{6.24}$$

which actually is the definition of the $SO(3)$ symmetry group. The **6** therefore does not allow for the SSB of $SU(3)$ to A_4, it breaks down $SU(3)$ to $SO(3)$ instead. [26] shows that similar to the **6**, the **10** is incapable of breaking $SU(3)$ down to A_4 as well. A combination of both the **6** and the **10** however does break down $SU(3)$ to A_4. Furthermore it can be shown that a single **15** is by itself capable of breaking $SU(3)$ to A_4. Now it will be shown how using such an A_4 invariant $SU(3)$ irrep $SU(3)$ can break to A_4.

6.1.2 Breaking SU(3)

Having determined which irreps are capable of breaking down $SU(3)$ to A_4 now the actual symmetry breaking has to be done, again this symmetry breaking is done by SSB. As the VEVs capable of doing the symmetry breaking are already known the thing missing is the potential that is minimized by these VEVs. The most general **15** potential that involves up to quartic terms is

$$\begin{aligned} V(T) = &- m^2 T^k_{ij} T^{ij}_k + \lambda (T^k_{ij} T^{ij}_k)^2 + \kappa T^i_{jm} T^{jn}_i T^k_{ln} T^{lm}_k \\ &+ \rho T^i_{jm} T^{jn}_i T^m_{kl} T^{kl}_n + \tau T^m_{ij} T^{ij}_n T^n_{kl} T^{kl}_m + \nu T^i_{jm} T^j_{in} T^{km}_l T^{ln}_k . \end{aligned} \tag{6.25}$$

The first derivative of this potential is the following

$$\begin{aligned} \frac{\partial V}{\partial T^k_{ij}} = &- 2m^2 T^{ij}_k + 2\lambda T^{ij}_k (T^l_{mn} T^{mn}_l) + 2\kappa T^{jn}_i T^k_{ln} T^{lm}_k + 2\rho T^{jn}_i T^m_{kl} T^{kl}_n \\ &+ \tau T^{ij}_n T^n_{kl} T^{kl}_m + \nu T^j_{in} T^{km}_l T^{ln}_k . \end{aligned} \tag{6.26}$$

In order to minimize the potential $\frac{\partial V}{\partial T^k_{ij}} = 0$, then plugging in the symmetry breaking VEV for **15** which is

$$T^k_{ij} = a(0, 0, \ldots, 0, 1, 1, 1)^T \tag{6.27}$$

and noting that $T^l_{mn} T^{mn}_l = 3a^2$ leads to the following condition for $V(T)$ to be minimized

$$- m^2 + 6\lambda a^2 + 2\kappa a^2 + 2\rho a^2 + 2\tau a^2 + 2\nu a^2 = 0. \tag{6.28}$$

This condition is satisfied if

$$a^2 = \frac{m^2}{2(3\lambda + \kappa + \rho + \tau + \nu)}, \tag{6.29}$$

Then it has to be checked via the matrix of 2nd derivatives, the Hessian matrix, whether or not this is really a minimum. This leads to some additional conditions on the different coefficients, the interested reader is referred to [26] [27]. [26] shows as well how $SU(3)$ can be broken down to A_4 using a combination of the **6** and the **10**. The smallest representation that is able to break $SU(3)$ to A_4 single-handedly is the **15** which in order to do so has to acquire the following VEV

$$T^k_{ij} = \sqrt{\frac{m^2}{2(3\lambda + \kappa + \rho + \tau + \nu)}} (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)^T \tag{6.30}$$

in order to minimize the potential $V(T)$ given in (6.25). Minimizing this potential by using this VEV then breaks $SU(3)_F \to A_4$, now that is has been shown how the discrete family symmetry A_4 can be embedded in a continuous symmetry $SU(3)$, now the next section will show how a family symmetry GUT can be built.

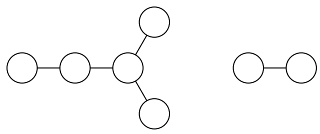
6.2 Family symmetry GUTs

As it was shown above A_4 can be embedded in $SU(3)$, therefore embedding A_4 into a GUT, means actually embedding $SU(3)_F$ in to a GUT. The 2 GUTs examined in the previous chapter therefore become $SO(10) \times SU(3)_F$ and $SU(3)^3 \times SU(3)_F$. The remainder of this chapter will determine how they will break down to the SM and what consequences can be drawn from that specific breaking. Furthermore, it is interesting to see whether these GUTs can still be embedded in the same large symmetry group as was mentioned before. This section will therefore investigate the following 2 paths

$$\begin{aligned}
&G_{\text{GUT}} \to SO(10) \times SU(3)_F \to \ldots \to SU(3)_C \times SU(2)_L \times U(1)_Y \\
&G_{\text{GUT}} \to SU(3)^3 \times SU(3)_F \to \ldots \to SU(3)_C \times SU(2)_L \times U(1)_Y.
\end{aligned} \tag{6.31}$$

6.2.1 A $SO(10) \times SU(3)_F$ GUT

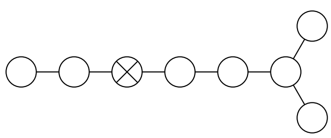
This section will determine some of the implications of adding the $SU(3)_F$ symmetry to the $SO(10)$ GUT. In order to derive these implications it is insightful to first start at the Dynkin diagrams of both $SO(10)$ and $SU(3)_F$, which are



.

It is then desired to embed these 2 symmetries in one larger symmetry group, otherwise embedding a Family Symmetry would be just adding a given symmetry to the GUT. Making it esthetically just the same as simply adding an A_4 symmetry to the symmetry group of the SM, which is a very ad hoc and unsatisfactory way of trying to explain the mixing amongst generations in the SM. Looking at tables 14 and 15 of [3] the symmetries which are capable of breaking to $SO(10) \times SU(3)_F$ can be found. Since $SO(10)$ is rank 5 and $SU(3)$ is rank 2, a symmetry capable of breaking to this specific group should be either rank 7 or rank 8, where the rank 7 breaking has to be done via the extended Dynkin diagram of that group.

When looking at these tables one way of embedding this symmetry is in the rank 8 group $SO(16)$, corresponding to the following Dynkin diagram



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By removing the root that is crossed in the Dynkin diagram above the following breaking is achieved

$$SO(16) \to SO(10) \times SU(3)_F \times U(1). \tag{6.32}$$

The fundamental representation of $SO(16)$ is the **16** which breaks to [28] $SO(10) \times SU(3)_F$ in the following way

$$\mathbf{16} \to (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \overline{\mathbf{3}}). \tag{6.33}$$

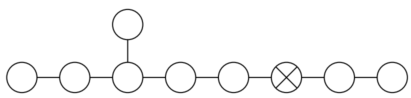
The **16** thus decomposes into 2 $SO(10)$ singlets that are an $SU(3)_F$ (anti-) triplet and 1 $SO(10)$ decuplet, **10**, which is a $SU(3)_F$ one singlet. Both these multiplets are not capable of holding either all the SM bosons or all the SM fermions. The following 2 decompositions however do contain the SM fermions and bosons

$$\begin{aligned}
&\mathbf{120} \rightarrow (\mathbf{45}, \mathbf{1}) \oplus (\mathbf{10}, \mathbf{3}) \oplus (\mathbf{10}, \overline{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \overline{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{1}) \\
&\mathbf{128} \rightarrow (\mathbf{16}, \mathbf{3}) \oplus (\mathbf{16}, \mathbf{1}) \oplus (\overline{\mathbf{16}}, \overline{\mathbf{3}}) \oplus (\overline{\mathbf{16}}, \mathbf{1}).
\end{aligned} \tag{6.34}$$

As was mentioned above the $(\mathbf{45}, \mathbf{1}$ multiplet can contain all the SM bosons, the SM bosons therefore fit into the $\mathbf{120}$ representation of $SO(16)$. The $\mathbf{128}$ than holds all the SM fermions since it decomposes in (amongst other things) a $(\mathbf{16}, \mathbf{3})$ multiplet, which corresponds to 3 generations the $15 + 1$ SM fermions contained in the $\mathbf{16}$ representation of $SO(10)$. The smallest representation that can be responsible for the SSB of $SO(16) \rightarrow SO(10) \times SU(3)_F$ is $\mathbf{135}_H$ since this is the smallest representation that (amongst other things) branches to a $(\mathbf{1}, \mathbf{1})$ representation.

An interesting thing to note here is that all the SM fermions correspond to a $\overline{\mathbf{3}}_{SU(3)} \sim \mathbf{3}_{A_4}$ triplet. This means that both left and right-handed fermions correspond to an A_4 triplet, this however contradicts with the Dirac mass terms as outlined in (3.20). These are of the form $\mathbf{3} \otimes \mathbf{3}_H \otimes \mathbf{1}$ in terms of A_4 irreps, however due to the right-handed fermions corresponding to a $\mathbf{3}$ as well these are then impossible. Therefore the lepton mass terms will not be of the simplest Dirac form as outlined in chapter 3. There are however more complicated mass terms possible still, as was already mentioned in chapter 3, these are however beyond the scope of this thesis.

Another way of achieving the embedding is by extending the Dynkin diagram of E_8 which has to be done in the following way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which leaves $E_6 \times SU(3)$, E_6 can then be broken to $SO(10)$ as was already shown in figure 5.2. These are the two possible ways in which a $SO(10)$ GUT combined with a $SU(3)_F$ family symmetry can be embedded in one large symmetry group. For $E_8 \rightarrow E_6 \times SU(3)_F$ the $\mathbf{248}$ fundamental representation branches in the following way [3]

$$\mathbf{248} \rightarrow (\mathbf{27}, \overline{\mathbf{3}}) \oplus (\mathbf{78}, \mathbf{1}) \oplus (\overline{\mathbf{27}}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{8}). \tag{6.35}$$

These representations then branch to the following $SO(10) \times SU(3)_F$ representations

$$\begin{aligned}
(\mathbf{27}, \overline{\mathbf{3}}) &\rightarrow (\mathbf{16}, \overline{\mathbf{3}}) \oplus (\mathbf{10}, \overline{\mathbf{3}}) \oplus (\mathbf{1}, \overline{\mathbf{3}}) \\
(\mathbf{78}, \mathbf{1}) &\rightarrow (\mathbf{45}, \mathbf{1}) \oplus (\overline{\mathbf{16}}, \mathbf{1}) \oplus (\mathbf{16}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \\
(\overline{\mathbf{27}}, \mathbf{3}) &\rightarrow (\overline{\mathbf{16}}, \mathbf{3}) \oplus (\mathbf{10}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}) \\
(\mathbf{1}, \mathbf{8}) &\rightarrow (\mathbf{1}, \mathbf{8}).
\end{aligned} \tag{6.36}$$

There however is one large problem with embedding $SO(10) \times SU(3)_F$ or any symmetry in E_8. Since all representations of E_8 are self-conjugate [3], there therefore exist no E_8 irreps that are different from their complex conjugate. Complex irreps are however needed for a theory to contain chiral fermions and C, P or CP violation. There might be a solution to this problem, which lies in creating the embedding of $E_6 \subset E_8$ via string theory [29], this embedding however is beyond the scope of this thesis. Having determined the consequences of embedding a $SU(3)_F$ to a $SO(10)$ GUT now the consequences for embedding a $SU(3)_F$ into a $(SU(3))^3$ GUT will be outlined.

6.2.2 A $SU(3)^3 \times SU(3)_F$ GUT

Having determined how $SU(3)_F$ can be embedded in a $SO(10)$ GUT and which conclusions can be drawn from this embedding, now the $(SU(3))^3 \times SU(3)_F$ or $SU(3)^4$ will be outlined. A logical starting point again is determining how the 4 rank 2 $SU(3)$ symmetries can be embedded in one large symmetry group. As was shown in the previous section $E_6 \times SU(3)$ can be embedded in E_8, since E_6 can then subsequently break to $(SU(3))^3$ this allows for the following embedding

$$E_8 \to E_6 \times SU(3)_F \to SU(3)^3 \times SU(3)_F, \tag{6.37}$$

where the $SU(3)^3$ is contained in the E_6 symmetry group instead of e.g. $SU(3)_L \times SU(3)_R \times SU(3)_F$, this ensures the single gauge coupling of $SU(3)^3$. Another way of embedding $SU(3)^4$ is by considering the Dynkin diagram of $SU(12)$ which is

$$\bigcirc - \bigcirc - \otimes - \bigcirc - \bigcirc - \otimes - \bigcirc - \bigcirc - \otimes - \bigcirc - \bigcirc \quad ,$$

where the breaking that is needed already is depicted. However, due to the rank 11 of $SU(12)$ being so much larger than other symmetries previously mentioned this $SU(12)$ model seems unnecessarily large. The same argument applies for the breaking of $SO(24) \to SU(4)^2 \times SU(3)^2 \to SU(3)^4$, where the family symmetry GUT originates from a rank 12 symmetry.

The only possible minimal embedding of $SU(3)^4$ therefore seems E_8, however as already mentioned above it is apart from string theory impossible to embed the symmetry groups of the above mentioned GUTs into E_8. There therefore is no possible way to embed $(SU(3))^3 \times SU(3)_F$ into one single larger symmetry group without resorting to either E_8 or unnecessarily large symmetry groups.

There is however one other option for a $(SU(3))^4$ GUT. Remember that for the trinification model gauge coupling unification was ensured, by either an E_6 embedding or by adding a Z_3 to the $(SU(3))^3$ symmetry group. A Z_4 symmetry can therefore ensure gauge couping unification for the $(SU(3))^4$ model. The SM fermions might then be contained in for instance the following Z_4 symmetric multiplet

$$(\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}) \oplus (\overline{\mathbf{3}}, \mathbf{3}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{3}, \mathbf{1}). \tag{6.38}$$

And the SM gauge bosons might then be contained in the following Z_4 symmetric multiplet

$$(\mathbf{8}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8}). \tag{6.39}$$

It however has to be determined whether this leads to a theory that can boil down to the SM and explain the specific form of the PMNS matrix as well, this is however beyond the scope of this thesis. Having built and outlined these family symmetry GUTs this thesis will now be concluded upon.

7 Conclusion

This thesis tried to find an explanation for a SM phenomenon that is up to now not understood; the specific form of mixing of quarks and leptons between generations. It tried explaining this specific type of mixing by adding a discrete family, D_F, symmetry to the SM. Since the CKM matrix, which carries the information on quark mixing, is up to corrections of order λ ($\lambda \ll 1$) the same as the unit matrix it was chosen to focus on the lepton mixing. The specific discrete family symmetry that was chosen to explain this mixing is A_4. It proved that relatively simple A_4 symmetric models which extend the SM minimally are capable of explaining the PMNS matrix which determines the specific form of the lepton mixing.

This thesis then tried to embed the A_4 family symmetry including SM into a GUT. Before doing so some group-theoretical tools were outlined, these tools were needed to be able to systematically build GUTs. After having outlined these tools some GUTs were built to provide an example of the strength of these tools. It was then tried to embed the A_4 family symmetry into the GUTs that were previously outlined. This embedding was indirect, since A_4 was embedded into the continuous symmetry group $SU(3)_F$ first. Therefore adding a discrete family symmetry into a GUT was reduced to seeing whether a $SU(3)_F$ symmetry could be embedded into a specific GUT.

Trying to find one symmetry group of a GUT that contains both the symmetry group of the SM and $SU(3)_F$ proved to be very restricting. Apart from the $SO(16) \to SO(10) \times SU(3)_F$ and the $(SU(3))^4 \times Z_4$ GUTs, there are no candidates for a minimal A_4 family symmetry GUT. This is due to the fact that E_8 is unable to serve as a GUT symmetry due to it having only self-conjugate irreps. Except for the $E_8 \times E_8$ string theory embedding E_8 therefore is not a viable family symmetry GUT candidate. Other candidates then become unnecessarily large, which in view of trying to extend the SM as minimal as possible is not viable either.

A solution to this problem might be in using the $A_4 \subset SU(2)_F$ embedding instead of $A_4 \subset SU(3)_F$. This might be esthetically somewhat less pleasing, $SU(2)$ does however have the advantage of being smaller and therefore leads to a family symmetry GUT with a smaller rank. The $(SU(3))^3 \times SU(2)_F$ GUT then however has to be embedded in a larger symmetry to ensure a unified gauge coupling. However except for diminishing the $SO(16)$ to a $SO(14)$ symmetry, still the other minimal embedding go via E_8, therefore not significantly changing the outcome above.

Another option is turning to other discrete symmetries in explaining the specific form of the PMNS matrix. [12] gives an excellent review of these different discrete symmetries and their ability of explaining the PMNS matrix. [27] then shows how different discrete symmetries can be embedded in continuous symmetries. Advances in experiments furthermore might lead to evidence that points towards a specific discerete symmetry as well.

The only viable candidates that this thesis found is the one of $SO(16)$ and $(SU(3)) \times Z_4$, it might be interesting to research this option further. By computing aspect of these family symmetry GUTs, like proton decay and the running of the coupling constant one could further check how viable this candidate actually is.

To conclude this thesis tried to explain the specific form of the mixing of the leptons within the 3 SM generations by building a family symmetry GUT. To do so it used an discrete A_4 family symmetry to explain th specific form of lepton mixing. By embedding this discrete A_4 family symmetry into a $SU(3)_F$ continuous family symmetry this allowed for the creation of a family symmetry GUT. It was found that $SO(16)$ and $(SU(3))^4 \times Z_4$ are viable candidates for such a family symmetry GUT. These symmetry groups ultimately break down to $SU(3)_C \times SU(2)_L \times U(1)_Y \times A_4$ which corresponds to a SM including a discrete family symmetry. This is therefore a SM that explains the specific form of mixing between the leptons amongst generations.

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I would like to thank Ruud Peters and Sybrand Zeinstra for their help at the beginning of my thesis. During the meetings I had with them they improved my understanding of the SM and some of the shortcomings the SM has, this really helped during the start of my thesis.

Attending these meetings as well was Anno Touwen. After helping me with the start of my thesis as well, Anno and I together set out on trying to understand the group-theoretical structure of GUTs. During our weekly meetings we slowly unravelled the difficulties of this group theory. This not only was vital in writing this thesis, it proved to be a joy to do as well. Anno furthermore provided me with schemes that greatly helped my understanding and benefited this thesis as well. I would therefore like to thank Anno for his significant contribution to this thesis.

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Appendix A

This appendix will derive the form of the A_4 singlets within both the **10** and **15** of $SU(3)$, starting at the **10**.

Derivation of the 10

As mentioned before the **10** is the symmetric product of the triplets, a general decuplet can therefore be written using the following orthonormal states

$$
\begin{gathered}
|1\succ = |111\rangle \quad |2\succ = |222\rangle \quad |3\succ = |333\rangle \\
|4\succ = \frac{1}{\sqrt{3}}(|112\rangle + |121\rangle + |211\rangle) \quad |5\succ = \frac{1}{\sqrt{3}}(|113\rangle + |131\rangle + |311\rangle) \\
|6\succ = \frac{1}{\sqrt{3}}(|221\rangle + |212\rangle + |122\rangle) \quad |7\succ = \frac{1}{\sqrt{3}}(|223\rangle + |232\rangle + |322\rangle) \\
|8\succ = \frac{1}{\sqrt{3}}(|331\rangle + |313\rangle + |133\rangle) \quad |9\succ = \frac{1}{\sqrt{3}}(|332\rangle + |323\rangle + |233\rangle) \\
|10\succ = \frac{1}{\sqrt{6}}(|123\rangle + |231\rangle + |312\rangle + |321\rangle + |213\rangle + |132\rangle),
\end{gathered}
\tag{7.1}
$$

where now $|ijk\rangle = |i\rangle \otimes |j\rangle \otimes |k\rangle$. The most general decuplet then is given by

$$
|\phi\succ = \sum_{i=1}^{10} \phi_i |i\succ . \tag{7.2}
$$

In order to break $SU(3)$ the most general A_4 invariant decuplet has to be found, this again can be done by checking the transformation of all the orthonormal states under D and A. It is however more convenient to check how the most general state transforms under these transformations, starting at the D transformation $|\phi\succ$ transforms as

$$
\begin{aligned}
S|\phi\succ = {} & S\sum_{i=1}^{10} \phi_i |i\succ \\
& = S\phi_1 |111\rangle + S\phi_2 |222\rangle + S\phi_3 |333\rangle + S\frac{\phi_4}{\sqrt{3}}(|112\rangle + |121\rangle + |211\rangle) \\
& + S\frac{\phi_5}{\sqrt{3}}(|113\rangle + |131\rangle + |311\rangle) + S\frac{\phi_6}{\sqrt{3}}(|221\rangle + |212\rangle + |122\rangle) \\
& + S\frac{\phi_7}{\sqrt{3}}(|223\rangle + |232\rangle + |322\rangle) + S\frac{\phi_8}{\sqrt{3}}(|331\rangle + |313\rangle + |133\rangle) \\
& + S\frac{\phi_9}{\sqrt{3}}(|332\rangle + |323\rangle + |233\rangle) + S\frac{\phi_{10}}{\sqrt{6}}(|123\rangle + |231\rangle + |312\rangle + |321\rangle + |213\rangle + |132\rangle).
\end{aligned}
\tag{7.3}
$$

Now remembering that $S\left|ijk\right\rangle = S\left|i\right\rangle \otimes S\left|j\right\rangle \otimes S\left|k\right\rangle$ leads to

$$\begin{aligned}
S|\phi\succ &= \phi_1\left|111\right\rangle - \phi_2\left|222\right\rangle - \phi_3\left|333\right\rangle - \frac{\phi_4}{\sqrt{3}}(\left|112\right\rangle + \left|121\right\rangle + \left|211\right\rangle) \\
&\quad - \frac{\phi_5}{\sqrt{3}}(\left|113\right\rangle + \left|131\right\rangle + \left|311\right\rangle) + \frac{\phi_6}{\sqrt{3}}(\left|221\right\rangle + \left|212\right\rangle + \left|122\right\rangle) \\
&\quad - \frac{\phi_7}{\sqrt{3}}(\left|223\right\rangle + \left|232\right\rangle + \left|322\right\rangle) - \frac{\phi_8}{\sqrt{3}}(\left|331\right\rangle + \left|313\right\rangle + \left|133\right\rangle) \\
&\quad + \frac{\phi_9}{\sqrt{3}}(\left|332\right\rangle + \left|323\right\rangle + \left|233\right\rangle) + \frac{\phi_{10}}{\sqrt{6}}(\left|123\right\rangle + \left|231\right\rangle + \left|312\right\rangle + \left|321\right\rangle + \left|213\right\rangle + \left|132\right\rangle),
\end{aligned} \tag{7.4}$$

therefore in order for a **10** to be invariant under the S transformation $\phi_2 = \phi_3 = \phi_4 = \phi_5 = \phi_7 = \phi_8 = 0$. The remaining decuplet then transforms under T in the following way

$$\begin{aligned}
T|\phi\succ &= T\phi_1\left|111\right\rangle + T\frac{\phi_6}{\sqrt{3}}(\left|221\right\rangle + \left|212\right\rangle + \left|122\right\rangle) + T\frac{\phi_9}{\sqrt{3}}(\left|332\right\rangle + \left|323\right\rangle + \left|233\right\rangle) \\
&\quad + T\frac{\phi_{10}}{\sqrt{6}}(\left|123\right\rangle + \left|231\right\rangle + \left|312\right\rangle + \left|321\right\rangle + \left|213\right\rangle + \left|132\right\rangle) \\
&= \phi_1\left|222\right\rangle + \frac{\phi_6}{\sqrt{3}}(\left|113\right\rangle + \left|131\right\rangle + \left|311\right\rangle) + \frac{\phi_9}{\sqrt{3}}(\left|221\right\rangle + \left|212\right\rangle + \left|122\right\rangle) \\
&\quad + \frac{\phi_{10}}{\sqrt{6}}(\left|123\right\rangle + \left|231\right\rangle + \left|312\right\rangle + \left|321\right\rangle + \left|213\right\rangle + \left|132\right\rangle),
\end{aligned} \tag{7.5}$$

which means that for an A_4 invariant **10** $\phi_1 = \phi_6 = \phi_9 = 0$ as well. Therefore an A_4 invariant **10** can be written in the following way

$$\mathbf{10} \propto (0,0,0,0,0,0,0,0,0,1)^T. \tag{7.6}$$

Having determined the A_4 invariant **10** it is interesting to check which subgroup of $SU(3)$ this decuplet actually breaks $SU(3)$ to. In order to check this, it should be determined under which $SU(3)$ subgroup the A_4 invariant decuplet mentioned above is invariant. The decuplet mentioned above can in the triplet base be written as

$$\left|123\right\rangle + \left|231\right\rangle + \left|312\right\rangle + \left|321\right\rangle + \left|213\right\rangle + \left|132\right\rangle = \left|123\right\rangle + \text{permutations}, \tag{7.7}$$

where the $\left|123\right\rangle$ transforms as

$$\left|123\right\rangle \to \sum_{i,j,k=1}^{3} U_{1i}U_{2j}U_{3k}\left|ijk\right\rangle. \tag{7.8}$$

Due to the decuplet containing all the permutations of $\left|123\right\rangle$, $\left|123\right\rangle$ can be transformed in any of these, there are however terms that are forbidden such as $\left|333\right\rangle$. When looking at the specific transformation law for $\left|123\right\rangle$ this for instance means that

$$U_{13}U_{23}U_{33} = s_{13}c_{13}^2 s_{23}c_{23}e^{-i(3\beta_1+3\beta_2+\delta)} = 0 \tag{7.9}$$

From which it follows that either $\theta_{13} = 0, \frac{\pi}{2}$ and/or $\theta_{23} = 0, \frac{\pi}{2}$, by looking at the other conditions which are found by imposing that $\left|123\right\rangle$ should transform to itself or to a permutation of $\left|123\right\rangle$ it can be found that $SU(3)$ actually does not break down to A_4. Having found how the A_4 invariant **10** looks and to which subgroup of $SU(3)$, $SU(3)$ is broken by it now the A_4 invariant **15** will be determined.

Derivation of the 15

The **15** comes from the following product $\mathbf{3} \otimes \mathbf{3} \otimes \overline{\mathbf{3}}$, the part of this product that is symmetric in the 2 first indices i, j and from which the 3 traces is the **15** [27]. The most general 15-plet can be written as a sum of the following 15 orthonormal states

$$
\begin{aligned}
|1) &= \frac{1}{\sqrt{3}}(|11\overline{1}\rangle - |12\overline{2}\rangle - |21\overline{2}\rangle) \\
|2) &= \frac{1}{2\sqrt{6}}(2 \cdot |11\overline{1}\rangle + |12\overline{2}\rangle + |21\overline{2}\rangle - 3 \cdot |13\overline{3}\rangle - 3\,|31\overline{3}\rangle) \\
|3) &= \frac{1}{\sqrt{3}}(|22\overline{2}\rangle - |23\overline{3}\rangle - |32\overline{3}\rangle) \\
|4) &= \frac{1}{2\sqrt{6}}(2 \cdot |22\overline{2}\rangle + |23\overline{3}\rangle + |32\overline{3}\rangle - 3 \cdot |21\overline{1}\rangle - 3\,|12\overline{1}\rangle) \\
|5) &= \frac{1}{\sqrt{3}}(|33\overline{3}\rangle - |31\overline{1}\rangle - |13\overline{1}\rangle) \\
|6) &= \frac{1}{2\sqrt{6}}(2 \cdot |33\overline{3}\rangle + |31\overline{1}\rangle + |13\overline{1}\rangle - 3 \cdot |32\overline{2}\rangle - 3\,|23\overline{2}\rangle) \\
|7) &= |11\overline{2}\rangle \quad |8) = |11\overline{3}\rangle \quad |9) = |22\overline{3}\rangle \quad |10) = |22\overline{1}\rangle \quad |11) = |33\overline{1}\rangle \quad |12) = |33\overline{2}\rangle \\
|13) &= \frac{1}{\sqrt{2}}(|12\overline{3}\rangle + |21\overline{3}\rangle) \quad |14) = \frac{1}{\sqrt{2}}(|23\overline{1}\rangle + |32\overline{1}\rangle) \quad |15) = \frac{1}{\sqrt{2}}(|31\overline{2}\rangle + |13\overline{2}\rangle).
\end{aligned}
\tag{7.10}
$$

An interesting thing to not here is that some of the $|ij\overline{k}\rangle$ occur multiple times in the different orthonormal states. Using these orthonormal states the most general **15** can be written as

$$
|\phi) = \sum_{i=1}^{15} \phi_i |i). \tag{7.11}
$$

Under the S transformation this 15-plet then transforms as

$$
\begin{aligned}
S|\phi) &= S \sum_{i=1}^{15} \phi_i |i) \\
&= S\frac{\phi_1}{\sqrt{3}}(|11\overline{1}\rangle - |12\overline{2}\rangle - |21\overline{2}\rangle) + S\frac{\phi_2}{2\sqrt{6}}(2 \cdot |11\overline{1}\rangle + |12\overline{2}\rangle + |21\overline{2}\rangle - 3 \cdot |13\overline{3}\rangle - 3\,|31\overline{3}\rangle) \\
&+ S\frac{\phi_3}{\sqrt{3}}(|22\overline{2}\rangle - |23\overline{3}\rangle - |32\overline{3}\rangle) + S\frac{\phi_4}{2\sqrt{6}}(2 \cdot |22\overline{2}\rangle + |23\overline{3}\rangle + |32\overline{3}\rangle - 3 \cdot |21\overline{1}\rangle - 3\,|12\overline{1}\rangle) \\
&+ S\frac{\phi_5}{\sqrt{3}}(|33\overline{3}\rangle - |31\overline{1}\rangle - |13\overline{1}\rangle) + S\frac{\phi_6}{2\sqrt{6}}(2 \cdot |33\overline{3}\rangle + |31\overline{1}\rangle + |13\overline{1}\rangle - 3 \cdot |32\overline{2}\rangle - 3\,|23\overline{2}\rangle) \\
&+ S\phi_7\,|11\overline{2}\rangle + S\phi_8\,|11\overline{3}\rangle + S\phi_9\,|22\overline{3}\rangle + S\phi_{10}\,|22\overline{1}\rangle + S\phi_{11}\,|33\overline{1}\rangle + S\phi_{12}\,|33\overline{2}\rangle \\
&+ S\frac{\phi_{13}}{\sqrt{2}}(|12\overline{3}\rangle + |21\overline{3}\rangle) + S\frac{\phi_{14}}{\sqrt{2}}(|23\overline{1}\rangle + |32\overline{1}\rangle) + S\frac{\phi_{15}}{\sqrt{2}}(|31\overline{2}\rangle + |13\overline{2}\rangle).
\end{aligned}
\tag{7.12}
$$

Note that since A_4 is real that $S\,|\bar{1}\rangle = |\bar{1}\rangle$, $S\,|\bar{2}\rangle = -\,|\bar{2}\rangle$ and $S\,|\bar{3}\rangle = -\,|\bar{3}\rangle$, which is similar to the ordinary triplet transformation. Using this the following can then be derived

$$\begin{aligned}
S|\phi) = &\frac{\phi_1}{\sqrt{3}}(|11\bar{1}\rangle - |12\bar{2}\rangle - |21\bar{2}\rangle) + \frac{\phi_2}{2\sqrt{6}}(2\cdot|11\bar{1}\rangle + |12\bar{2}\rangle + |21\bar{2}\rangle - 3\cdot|13\bar{3}\rangle - 3\,|31\bar{3}\rangle) \\
&- \frac{\phi_3}{\sqrt{3}}(|22\bar{2}\rangle - |23\bar{3}\rangle - |32\bar{3}\rangle) - \frac{\phi_4}{2\sqrt{6}}(2\cdot|22\bar{2}\rangle + |23\bar{3}\rangle + |32\bar{3}\rangle - 3\cdot|21\bar{1}\rangle - 3\,|12\bar{1}\rangle) \\
&- \frac{\phi_5}{\sqrt{3}}(|33\bar{3}\rangle - |31\bar{1}\rangle - |13\bar{1}\rangle) - \frac{\phi_6}{2\sqrt{6}}(2\cdot|33\bar{3}\rangle + |31\bar{1}\rangle + |13\bar{1}\rangle - 3\cdot|32\bar{2}\rangle - 3\,|23\bar{2}\rangle) \\
&- \phi_7\,|11\bar{2}\rangle - \phi_8\,|11\bar{3}\rangle - \phi_9\,|22\bar{3}\rangle + \phi_{10}\,|22\bar{1}\rangle + \phi_{11}\,|33\bar{1}\rangle - \phi_{12}\,|33\bar{2}\rangle \\
&+ \frac{\phi_{13}}{\sqrt{2}}(|12\bar{3}\rangle + |21\bar{3}\rangle) + \frac{\phi_{14}}{\sqrt{2}}(|23\bar{1}\rangle + |32\bar{1}\rangle) + \frac{\phi_{15}}{\sqrt{2}}(|31\bar{2}\rangle + |13\bar{2}\rangle). \qquad (7.13)
\end{aligned}$$

Things are a little more involved since different states, $|ij\bar{k}\rangle$ appear in multiple parts of 15-plet above. In order for $|\phi)$ to be an A_4 invariant 15-plet it can be seen that $\phi_7 = \phi_8 = \phi_9 = \phi_{12} = 0$. Furthermore from the $|31\bar{3}\rangle$, $|12\bar{1}\rangle$ and $|23\bar{2}\rangle$ it can be seen that $\phi_2 = \phi_4 = \phi_6 = 0$. This then leads to $\phi_1 = \phi_3 = \phi_5 = 0$. This S invariant 15-multiplet then transforms under the T transformation in the following way

$$\begin{aligned}
T|\phi) &= T\phi_{10}\,|22\bar{1}\rangle + T\phi_{11}\,|33\bar{1}\rangle + T\frac{\phi_{13}}{\sqrt{2}}(|12\bar{3}\rangle + |21\bar{3}\rangle) \\
&\quad + T\frac{\phi_{14}}{\sqrt{2}}(|23\bar{1}\rangle + |32\bar{1}\rangle) + T\frac{\phi_{15}}{\sqrt{2}}(|31\bar{2}\rangle + |13\bar{2}\rangle) \\
&= \phi_{10}\,|11\bar{3}\rangle + \phi_{11}\,|22\bar{3}\rangle + \frac{\phi_{13}}{\sqrt{2}}(|31\bar{2}\rangle + |13\bar{2}\rangle) \\
&\quad + \frac{\phi_{14}}{\sqrt{2}}(|12\bar{3}\rangle + |21\bar{3}\rangle) + \frac{\phi_{15}}{\sqrt{2}}(|23\bar{1}\rangle + |32\bar{1}\rangle) \stackrel{?}{=} |\phi), \qquad (7.14)
\end{aligned}$$

where the final equation is satisfied when $\phi_{10} = \phi_{11} = 0$ and $\phi_{13} = \phi_{14} = \phi_{15}$. The A_4 invariant $SU(3)$ **15** therefore is

$$\mathbf{15} \propto (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1)^T. \qquad (7.15)$$

Now using this A_4 invariant a A_4 invariant potential can be build, however as mentioned before to actually see to which $SU(3)$ subgroup this potentials breaks $SU(3)$ it should be determined under which subgroup of $SU(3)$ the 15-plet is invariant. $|ij\bar{k}\rangle$ transforms in the following way for a general $SU(3)$ transformation

$$|ij\bar{k}\rangle \to \sum_{l,m,n=1}^{3} U_{il}U_{jm}U^*_{ln}\,|lm\bar{n}\rangle\,, \qquad (7.16)$$

where the complex conjugate of U is given by

$$U^* = P_1^* \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta} & s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta} \\ s_{12}c_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta} \\ s_{13}e^{i\delta} & s_{23}c_{13} & c_{23}c_{13} \end{pmatrix} P_2^*, \qquad (7.17)$$

where $P_1^* = \mathrm{diag}(e^{-i\alpha_1}, e^{-i\alpha_2}, e^{i(\alpha_1+\alpha_2)})$ and $P_2^* = \mathrm{diag}(e^{-i\beta_1}, e^{-i\beta_2}, e^{i(\beta_1+\beta_2)})$. The A_4 invariant 15-plet then transforms as

$$|12\bar{3}\rangle + \text{perm.} \to \sum_{i,j,k=1}^{3} U_{1i}U_{2j}U^*_{3k}\,|ij\bar{k}\rangle\,, \qquad (7.18)$$

which leads to the following condition

$$U_{13}U_{23}U_{33}^* = s_{13}c_{13}^2 s_{23}c_{23}e^{i(2\alpha_1+2\alpha_2-\beta_1-\beta_2-\delta)} = 0, \tag{7.19}$$

that is satisfied if $\theta_{13} = 0, \frac{\pi}{2}$ and/or $\theta_{23} = 0, \frac{\pi}{2}$. Plugging in these angles will lead into the most general $SU(3)$ transformation will reduce this transformation to A_4.

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