University of Groningen

Master thesis

A holographic derivation of the linearized equations of gravity

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December 19, 2018

Abstract

Using the Ryu-Takyanagi conjecture it will be argued that spacetime emerges from the entanglement entropy in the AdS/CFT correspondence. We will use the Iyer-Wald formalism to derive the linearized equations of gravity in the AdS space. To do this we will consider an exact quantum generalization of the thermodynamic first law on the CFT side and translate this to a holographic restraint in the form of a bulk first law. The linearized equations hold for any diffeomorphic invariant theory of gravity and in particular we shall explicitly derive the linearized Einstein equations. Finally extensions to this derivation and similar ways of deriving the Einstein equations will be discussed.

Contents

Introduction

In recent years the AdS/CFT correspondence and holographic principle have shed new light on a possible explanation of gravity. The holographic principle states that theories of gravity can be equivalently described by a quantum theory in a lower dimensional space. In the holographic context there is evidence [**?**] that seems to suggest that spacetime and restrictions thereof, i.e. classical gravity, are emergent. With emergent we mean that spacetime and gravity are in a way a macroscopic quantity derived from the lower dimensional quantum theory. The emergent theory is then posed to be described by the holographic dual of this lower dimensional theory.

Exactly how this emergence of spacetime is encoded in the microscopic degrees of freedom is still an open question. In this thesis we will investigate recent proposals [**?**] that this is encoded by the structure of quantum entanglement of the underlying theory. More specifically the entanglement entropy will be studied which quantifies the degree of entanglement between certain regions of the quantum theory. We will assume that the dual quantity of this is given by the Ryu-Takyanagi formula[14] in the context of AdS/CFT and shall derive that this implies that entanglement is necessary for spacetime.

To find the restrictions on the emergent spacetime it has been proposed [**?**] to study a specific restriction of the entanglement structure. Concretely, this restriction is a quantum generalization of the thermodynamical first law on the CFT. It states that the change in entropy of a region is equal to its change in energy. By considering the holographic equivalent of this first law the dual spacetime is found to be restricted in such a way that up to linear order the dual spacetime must satisfy the Einstein equations. In fact the Ryu-Takyanagi proposal only holds when the bulk theory is GR but we will show for arbitrary diffeomorphic invariant theories of gravity that this first law implies bulk gravity to linear order.

We will first introduce Anti de-Sitter (AdS) space and then briefly introduce the holographic principle and show how spacetime emerges in this setting. In section 4 the linear Einstein equations in an AdS background will be derived. Then the Iyer-Wald formalism will be introduced which shall be crucial in our derivation of the linearised equations of gravity. In section 6 we will introduce the first law restriction on the CFT side and then translate this to a bulk constraint as a holographic first law. In section 7 everything will be put together and the linearized equations will be derived. Explicitly we will also derive the Einstein equations to linear order and show that this matches the result of section 4. Finally we will investigate some recent proposals to find the gravitational equations beyond linear order.

Chapter 1

Anti-de Sitter spacetime and causal structure

In this section the Anti de-Sitter spacetime or AdS will be reviewed. It describes a maximally symmetric spacetime with negative cosmological constant. We shall derive the linearized Einstein equations explicitly in the framework of the AdS/CFT correspondence, to be introduced in chapter 3 and therefore different representations of AdS will now be reviewed.

Embedding of AdS

AdS_d is a maximally symmetric spacetime, meaning that it has $d(d+1)/2$ Killing vectors, and has a negative cosmological constant. It is embedded in $\mathcal{R}^{2,d-1}$

$$ds^2 = -dT_0^2 + \sum_{i=1}^{d} dX_i^2 - dT_{d+1}^2 \tag{1.1}$$

constrained by

$$-T_1^2 + \sum_{i=1}^{d-1} dX_i^2 - T_{d+1}^2 = -L_{AdS}^2 \tag{1.2}$$

It is a space with Lorentzian signature and negative constant curvature. L_{AdS} denotes AdS radius. The space in which it is embedded has $SO(2,d)$ symmetry by construction so AdS_d is constrained under the same group. In order to make an explicit coordinate realization of AdS_d we will consider two coordinate systems: the Poincaré and the global coordinates.

Poincaré patch

Let us start with the Poincaré patch which is defined in terms of the coordinates t, $\vec{x}$ and z. We parametrize AdS as follows:

$$\begin{aligned} T^1 + X^d &= \frac{L^2}{z} \\ T^0 + X^i &= \frac{L}{z} x^i \end{aligned} \tag{1.3}$$

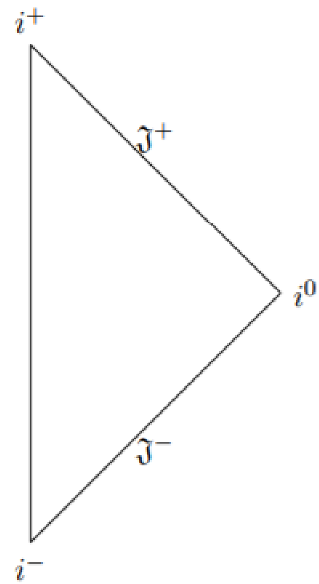

Figure 1.1: The Penrose diagram of Minkowski and the Poincaré patch. $i^\pm$ denote the future and past temporal infinities. The $\mathcal{J}^\pm$ lightlike boundaries denote the origin of incoming and outgoing lightrays. Finally i^0 denotes spatial infinity.

with $x^i \in (-\infty, \infty)$ and $z \in (0, \infty)$. The z coordinate is like a radial coordinate and since $z > 0$ or $z < 0$ it implies that only a patch of full AdS is covered.

Substituting this into the hyperbolic constraint gives:

$$-\frac{L^2}{z}(T^1 - X^d) + \frac{L^2}{z^2}\eta_{\mu\nu}x^\mu x^\nu = -L^2 \implies (T^1 - X^d) = z + \frac{1}{z}\eta_{\mu\nu}x^\mu x^\nu$$

This allow us to calculate the induced metric using $g_{\mu\nu} = \partial_\mu x^\alpha \partial_\nu x^\beta g_{\alpha\beta}$ giving the metric in Poincaré coordinates

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \tag{1.4}$$

The Poincaré patch has the structure of Minkowski space warped by a total prefactor $\frac{L^2}{z^2}$. It can be viewed as a foliation of Minkowski sheets. To investigate the boundary structure we perform a Weyl transformation. It relates two conformally equivalent metrics $g'_{\mu\nu} = \Omega(\vec{x})g_{\mu\nu}$, meaning that angles between four-vectors are preserved which also implies that the causal structure is preserved. After this transformation it is seen that its boundary structure is given by Minkowski space. The penrose diagram is plotted in figure 1. It seems strange that even though AdS and Minkowski space are different spacetimes they have the same metric. This contradiction is resolved by noting that the Poincaré coordinates only cover a part of AdS.

The Poincaré patch is very useful since it allows us to define a boundary theory in the AdS/CFT correspondence where the CFT lives on Minkowski space. From now on, when talking about AdS, the Poincaré patch is understood which we define as having $z > 0$.

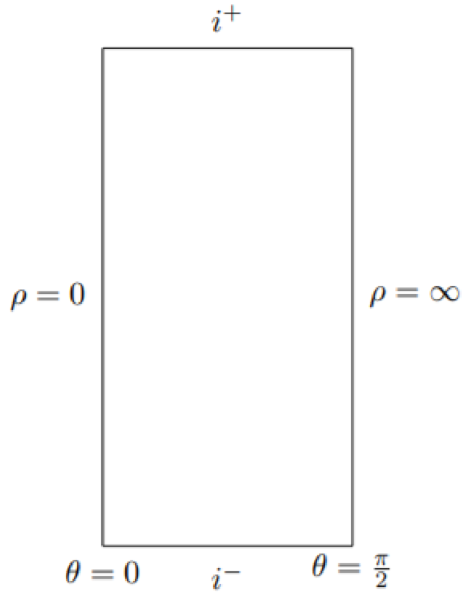

Figure 1.2: This is the Penrose diagram of global AdS space. $i^{\pm}$ denote temporal future and past infinities. $\rho = 0$ is the center of global AdS and $\rho = \infty$ denotes the boundary at spatial infinity.

Global AdS

The other explicit realization is the global covering by global coordinates. In this system the embedding coordinates of AdS_d are

$$\begin{aligned}
T^1 &= L\cosh(\rho)\sin(\tau) \\
T^2 &= L\cosh(\rho)\cos(\tau) \\
X^i &= \rho\sinh(\rho)\sin(\phi_1)...\sin(\phi_{d-3})\cos(\phi_{d-2}) \quad i = 1..d-1
\end{aligned} \tag{1.5}$$

where $\rho \in [0, \infty)$, $\tau \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$. It is of course strange that $\tau \in [0, 2\pi)$ but usually this is considered an artifact of the embedding and therefore gets extended to be $\tau \in (-\infty, \infty)$. The induced metric is now

$$ds^2 = L^2(-\cosh^2 \rho d\eta^2 + d\rho^2 + \sinh \rho d\Omega_i^2) \tag{1.6}$$

where Ω_i parametrizes the d-1 dimensional sphere. Letting $\tan(\theta) = \sinh(\rho)$ with $\theta \in [0, 1/2\pi]$ gives us

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\eta^2 + d\theta^2 + \sin^2\theta d\Omega_i^2) \tag{1.7}$$

The terms in the brackets define a cylinder and the boundary structure at $\theta = \pi/2$ so after a Weyl transformation we note that the boundary structure is that of a cylinder.

Chapter 2

Density matrices and entanglement entropy

2.1 Density matrix

Consider a quantum mechanical system. If it is a pure system then it can be expressed by a single wavefunction $|\psi\rangle$ in a N-dimensional Hilbert space $\mathcal{H}$ otherwise otherwise it can be described as an ensemble of states $|\phi_i\rangle$ each with a classical probability p_i. This can be neatly expressed in the density matrix formalism.

A density matrix ρ is in general described by

$$\rho \equiv \sum_i p_i |\phi_i\rangle \langle\phi_i| \tag{2.1}$$

such that it reduces for a pure state $|\psi\rangle$ to

$$\rho \equiv |\psi\rangle \langle\phi| \tag{2.2}$$

The density matrix is hermitian by construction and thus diagonalizable in which form it has the properties that

- $tr\rho = 1$ (normalized probability)
- $\rho^\dagger = \rho$ (hermicity)
- $\rho_{ii} \geq 0$ (non-negative eigenvalues)

The classical diagonal probabilities p_i can have two origins. They can originate from our ignorance of the state meaning that the exact state is not known but with a probability p_i it is the state $|\phi_i\rangle$. If we consider entangled states however then these classical probabilities are not due to our ignorance of the state but because there is no underlying structure, i.e. the wavefunctions can not be decomposed into a waveproduct.

Consider a quantum system with a subsystem A and a complementary subsystem $\bar{A}$ such that the Hilbert space of this quantum system can be written as a product state of its subsystems namely

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \tag{2.3}$$

Let $|\Psi\rangle \in \mathcal{H}$ then $\Psi = \Psi(\alpha, \beta)$ where α, β are the set of commuting observables of A and $\bar{A}$ respectively. In general this wave is not decomposable into the form of an uncorrelated wave product

$$\Psi(\alpha, \beta) \neq \phi_A(\alpha)\phi_{\bar{A}}(\beta) \tag{2.4}$$

We can decompose the full system state in a product state of two basises for A and $\bar{A}$. Let $|\psi_i\rangle$ and $|\psi_j\rangle$ be a basis for the respective subsystems. Then we can express the full state as:

$$|\Psi\rangle = \sum_{i,j} c_{i,j} |\psi_i\rangle \otimes |\psi_j\rangle \tag{2.5}$$

So $\rho = |\Psi\rangle \langle\Psi|$ is the density operator for the full system. We can now calculate the expectation value of the operator acting only on subsystem A: $\mathcal{O}^A \otimes \mathbb{1}$

$$\begin{aligned} tr((\mathcal{O}_A \otimes \mathbb{1})\rho) &= tr(\sum_{i,j}\sum_{a,b} c^*_{a,b} c_{i,j} \mathcal{O}_A |\psi_i\rangle \langle\psi_a| \otimes |\psi_j\rangle \langle\psi_b|) \\ &= tr(\sum_{i,j}\sum_{a,b} c^*_{a,b} c_{i,j} \mathcal{O}_A |\psi_i\rangle \langle\psi_j)|) \\ &\equiv tr(\mathcal{O}_A \rho_A) \end{aligned}$$

Thus by tracing over the degrees of freedom of the complementary subsystem we define a new density matrix ρ_A which we call the reduced density matrix.

2.2 Von Neumann and entanglement entropy

A quantitative measure of how close a certain ensemble is to being a pure state is given by the Von Neumann entropy. It is defined by

$$S = -tr[\rho \log \rho] = -\sum_i \rho_i \log \rho_i \tag{2.6}$$

First of all it is a non-negative quantity $S \geq 0$. Secondly, it is zero if and only if the density matrix is a pure state, in other words when there exists only one non-zero eigenvalue and maximal if the probabilities are spread evenly.

Now consider again a system, either pure or mixed, that is separated into two subsystems A and $\bar{A}$ with their corresponding density matrices ρ_A and $\rho_{\bar{A}}$. Consider the von Neumann entropy of the reduced density matrix

$$S_A = -tr\rho^A \log \rho^A \tag{2.7}$$

For the reduced density matrix the von Neumann entropy again is zero for a pure state. An increase in entanglement entropy is now entirely due to the entanglement between the two subsystems. The reduced density matrix gives the result of measurements on subsystem A. If the two subsystems are in a product state then obviously no additional information can be gained. The probabilities p_i can be seen as the extend to which the system is entangled. The Von Neumann entropy of a reduced density matrix is therefore also called the entanglement entropy.

When the total system ρ is a pure state then the reduced density matrices have identical eigenvalues and thus $S(\rho_A) = S(\rho_{\bar{A}})$. Since the regions A and $\bar{A}$ are arbitrary this shows that the entanglement entropy an intrinsic value as opposed to the thermodynamical entropy. When the total system is however a mixed state then the reduced density matrix will contain both thermodynamical and entanglement contributions.

Let us now introduce some new definitions and constraints for the entanglement entropy.

Relative entropy

Relative entropy is a quantity that can be used as a measure of the differences between two given density matrices ρ and σ. It is defined as follows [?]

$$S(\rho|\sigma) = tr(\rho \log \rho) - tr(\rho \log \sigma) \tag{2.8}$$

. measuring the difference of ρ with respect to the reference state σ. When the density matrices are identical than the relative entropy is obviously zero. Otherwise it is a non-negative, increasing function [?] and thus a good measure of differences.

Subadditivity

The subadditivity constraint states that the entanglement entropy of a total system is less or equal than the sum of its parts. This can be quickly shown starting from the relative entropy and setting $\rho = \rho^{A\bar{A}}$ and $\sigma = \rho^{A} \otimes \rho^{\bar{A}}$. Now:

$$\begin{aligned} 0 &\leq tr[\rho^{A\bar{A}} \log(\rho^{A} \otimes \rho^{\bar{A}})] \\ S(\rho^{A\bar{A}}) &\leq tr[\rho^{A\bar{A}}(\log(\rho^{A}) - \log(\rho^{\bar{A}}))] \end{aligned}$$

Working this out gives the subadditivity constraint

$$S(\rho^{A\bar{A}}) \leq S(\rho^{A}) + S(\rho^{\bar{A}}) \tag{2.9}$$

The equality only holds when the total state is a product state.

Mutual information

From subadditivity definition we can now define a new non-negative measure of entropy called the mutual information

$$I(A : \bar{A}) \equiv S(A) + S(\bar{A}) - S(A, \bar{A}) \geq 0 \tag{2.10}$$

If the total state is now a mixed thermal system then this gets subtracted from the entanglement entropies of the subsystems and thus provides us with a pure measure of entanglement.

2.3 Path integral representation of density matrices in QFT

Path integrals have many purposes in quantum field theory. It turns that they can also be used to represent density matrices. To show this we start out by considering the transition amplitude between two states.

$$\langle \phi_1(t_1)| e^{-iHt} |\phi_0(t_0)\rangle = \mathcal{N} \int_{\phi(t_0)=\phi_0}^{\phi(t_1)=\phi_1} d\phi e^{iS(\phi(t))} \tag{2.11}$$

Since we will be interested in the density matrix in the vacuum state a natural choice is to perform a Wick rotation $\tau = it$ into Euclidean space where time is now periodic with $\tau \in [0, \beta]$. The transition element in Euclidean space now reads:

$$\langle \phi_1| e^{-\beta H} |\phi_0\rangle = \mathcal{N} \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_1} d\phi e^{-S_{Euc}(\phi(\tau))} \tag{2.12}$$

A thermal density matrix is given by $e^{-\beta H}/Z(\beta)$ where $Z(\beta)$ is simply the partition function $Z(\beta) = tr(e^{-\beta H})$. The trace is found by integrating over ϕ with periodic conditions $\phi(0) = \phi(\beta)$. The trace is thus:

$$Z(\beta) = \mathcal{N} \int_{\phi(0)=\phi(\beta)} d\phi e^{-S_{Euc}(\phi(\tau))} \tag{2.13}$$

This also gives us a straightforward way to calculate the reduced density matrix of A by taking the partial trace with respect to its complement $\bar{A}$. We get

$$\langle \phi_1^A | \, \rho_A^\beta \, | \phi_0^A \rangle = \frac{1}{Z} \int_{\phi^A(0)=\phi_0^A, \phi_{\bar{A}}(0)=\phi^{\bar{A}}}^{\phi^A(\beta)=\phi_1^A} (\beta) d\phi e^{-S_{Euc}(\phi(\tau))} \tag{2.14}$$

We can now also find an expression for the vacuum state by letting the temperature go to zero. In this limit we find

$$\lim_{\beta \to \infty} e^{-\beta H} \, |\Phi\rangle = \mathcal{N} \, |\Omega\rangle \tag{2.15}$$

where $|\Omega\rangle$ is the vacuum state. This follows from the fact that we can write any state as a linear combination of its energy eigenstates which all have vanishing coefficients in this limit. To find the vacuum density matrix we want to calculate $\langle \phi_0 | \, |\Omega\rangle \, \langle \Omega | \, |\phi_1\rangle$. To calculate this we first evaluate

$$\langle \phi_0 | \, |\Omega\rangle \propto \int_{\phi(0)=\phi_0} d\phi e^{-S_{Euc}(\phi(\tau))} \tag{2.16}$$

which covers $\tau > 0$. Now by taking its conjugate we find the expression for the other braket which now covers $\tau < 0$. Thus we can combine the two vacuum path integrals:

$$\langle \phi_1^A | \, \rho_A^\beta \, | \phi_0^A \rangle = \langle \phi_1^A | \Omega \rangle \, \langle vac | \phi_0^A \rangle = \frac{1}{Z} \int_{\phi^A(0^+)=\phi_0^A, \phi_{\bar{A}}(0)=\phi^{\bar{A}}}^{\phi^A(0^-)=\phi_1^A} d\phi e^{-S_{Euc}(\phi(\tau))} \tag{2.17}$$

here we take the field ϕ to be continuous across $\tau = 0$. As will be seen later on in this thesis this formulation will allow us to find an explicit expression for the density matrix in the half space of Minkowski.

Chapter 3

AdS/CFT correspondence and emergence of spacetime

The AdS/CFT duality will form the cornerstone of the derivation in this thesis. This will be the way to relate entanglement entropy to geometrical properties. The holographic principle will be considered as a direct means of relating both sides of the duality. Next the Ryu-Takyanagi formula will be reviewed. It provides us with a direct link between the bulk geometry and boundary entanglement entropy. This will be one of the cornerstones in deriving the linearized Einstein equations. Finally, in the last section it will be shown how spacetime seems to be intimately related to entanglement.

3.1 The holographic principle

The holographic principle [8, 9] states that a d-dimensional quantum theory on a fixed spacetime background, thus lacking any gravitational interactions, is in fact equivalent to (d+1)-dimensional quantum theories of gravity. By equivalent we mean that they both describe the same physics. Although the interpretation on both sides of the duality can be different a state on one side must have a corresponding state in the dual theory and the same holds for observables.

The idea was inspired by the Bekenstein-Hawking entropy [10]

$$S = \frac{A_{BH}}{4G} \tag{3.1}$$

which states that the entropy of black holes is proportional to the area of their corresponding event horizon. The main idea for the holographic principle follows from the following thought experiment.

Assume that there is some sphere of volume V with boundary A in a space described by a theory of gravity. We take the volume to have energy E less than the mass M needed for a black hole with horizon area A to form. If we let a spherical shell with energy $M - E$ fall radially onto this object then a black hole will form with mass M and horizon area A. If we consider the total system and we do not want the entropy of it to decrease then we require that:

$$S \leq \frac{A}{4G} \tag{3.2}$$

for an arbitrary spherical area A. This shows that in fact the entropy of any volume enclosed by the surface A is bounded by the above form. This is called the Bekenstein bound [11].

Now in a statistical interpretation entropy is a measure of the degrees of freedom for a given system. If the system obeys Einstein gravity in (d+1)-dimensions we have that for a spatial d-dimensional slice $\Sigma_{M_{d+1}}$ of the manifold M_{d+1}:

$$S \propto Area(\Sigma_{d+1}) \propto Vol(\Sigma_{\partial M_{d+1}}) \tag{3.3}$$

Thus the degrees of freedom of a given volume are seen to scale with the boundary area, this is called the holographic principle. This bound on entropy is called the holographic entropy bound.

For a d-dimensional QFT the entropy of a d-1 dimensional spatial slice scales with volume. Assuming the duality is true one can imagine a QFT state living on the boundary dual to a spacetime M_{d+1} with a boundary ∂M_{d+1}. This then implies that the entropy scales with the boundary of the dual spacetime

$$S_{QFT} \propto Vol(\Sigma_{\partial M_{d+1}}) \tag{3.4}$$

The entropy is thus the same for the quantum field theory side as it is for a gravitational system thereby satisfying the holographic principle.

Maldacena proposed the first concrete realization [12] of this duality. He proposed a duality between $\mathcal{N} = 4$ Super Yang-Mills theory living on the boundary and type IIB string theory in $AdS_5 \times \mathcal{S}^4$ living on the corresponding dual bulk space. In this thesis we will however not require the full machinery of this correspondence for which more information can be found in [13]. We are instead mostly interested in the Ryu-Takyanagi formula entry of the dictionary to relate boundary entanglement and bulk geometry.

3.2 AdS/CFT

We will now investigate the AdS_{d+1}/CFT_d duality by first assuming that the CFT theory, living on Minkowski space, has a classical holographic dual description. Different states on the CFT side now correspond to different spacetime geometries even though asymptotically all spacetime geometries are equivalent, namely the background spacetime of the CFT. To study the CFT side a family of states is introduced where excitations of the CFT vacuum are parametrized by λ such that we have $|\phi(\lambda)\rangle$ where $|\phi(0)\rangle$ is the ground state as $z \to 0$. The state $|\phi(\lambda)\rangle$ lives on a fixed d-dimensional flat spacetime background, $\mathcal{R}^{d-1,1}$. The ground state of the CFT is now dual to a Poincaré patch of pure AdS:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu) \tag{3.5}$$

which we know is conformally equivalent to Minkowski space on the boundary where $z \to 0$. Spacetimes dual to excited states, i.e. $\lambda \neq 0$, are described by

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \Gamma_{\mu\nu}(x,z)dx^\mu dx^\nu) \tag{3.6}$$

where $\Gamma_{\mu\nu}$ dictates how much the spacetime differs from pure AdS. This reduces to pure AdS for $\lambda = 0$ since $\lim_{\lambda\to 0}\Gamma_{\mu\nu} = \eta_{\mu\nu}$. Since we will be interested in small perturbations in λ notice that for small z $\Gamma_{\mu\nu} = \eta_{\mu\nu} + z^d h_{\mu\nu}(x,z)$ giving us the metric in Fefferman-Graham coordinates

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu + z^d h_{\mu\nu} dx^\mu dx^\nu) \tag{3.7}$$

Thus we obtained a way to express perturbations to the CFT vacuum and relate them to perturbations near the boundary $z = 0$ of pure AdS on the dual spacetime side.

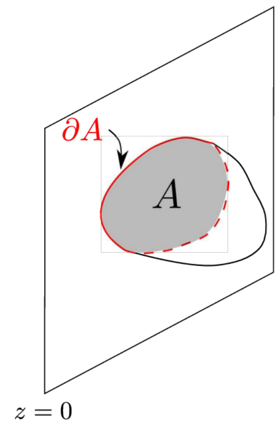

Figure 3.1: Here a subregion A of the CFT living on z=0 with boundary ∂_A is depicted. The surface going in z direction is the extremized surface in AdS.

3.3 The Ryu-Takyanagi conjecture

In the previous section we used the Bekenstein Hawking entropy to derive the holographic principle. It provided us with a relation between the entropy of a black hole and its event horizon area

$$s_{BH} = \frac{A_{BH}}{4G_N} \tag{3.8}$$

It was the motivation behind the holographic principle. If the AdS/CFT duality is assumed to hold it implies that there exists some surface in the bulk that is dual to a subsystem of the CFT side. Motivated by this seeming relation between surfaces in the bulk and the CFT side Ryu and Takyanagi [14] proposed a generalization of the holographic interpretion of the Bekenstein-Hawking entropy and conjectured that the entanglement entropy of any spatial region on the CFT side has a bulk interpretation in terms of a surface in the spacetime dual which minimizes the area functional of AdS spacetime.

We will use the notation and conventions of [15]. The conjecture states the following. Consider a CFT on a spacetime geometry $\mathcal{B}$ which has a holographic dual description. Now we let the CFT be in state $|\Phi\rangle$ which has a classical holographic dual description M_Φ. We now pick an arbitrary spatial slice $\Sigma_\mathcal{B}$ of $\mathcal{B}$ of which we choose an arbitrary subset $A \subset \Sigma_\mathcal{B}$. Now note that $\mathcal{B}$ and M_Φ share the same boundary which we denote by ∂M_Φ. This allows us to define regions on the boundary A and its complement $\bar{A}$. Now consider the entropy S_A of the subsystem A with the rest of the system $\bar{A}$. The conjecture is that this subsystem entropy is equal to a codimension-2 surface $\tilde{A}$ in M_Φ such that:

$$S(A) = \frac{1}{4G_N} Area(\tilde{A}) \tag{3.9}$$

where the surface $\tilde{A}$ is defined by the following properties:

- The surface $\tilde{A}$ and A share the same boundary
- $\tilde{A}$ is homologous to A.
- The surface $\tilde{A}$ extremizes the area functional of AdS.

$$A = \int d^d\sigma \sqrt{det(g_{\mu\nu})} \tag{3.10}$$

where $g_{\mu\nu}$ is the metric induced on $\tilde{A}$. When multiple solutions exist the one with minimal area is chosen.

Since the Ryu-Takyanagi formula was conceived as a generalization of the Bekenstein-Hawking formula a first check will be to see if we can recover the Bekenstein-Hawking formula. Interpret the entropy holographically as entanglement entropy over the boundary of some surface on the CFT side ∂S_A where A is the entangling surface. This implies that the horizon area must be a co-dimension two surface in the bulk satisfying the properties above for it to be consistent with the Ryu-Takyanagi formula.

The horizon area has no boundary since it encloses a singularity thus $\partial\tilde{A} = \emptyset$. This implies that the boundary of the entangling surface is also the empty set which means that A must cover the boundary spacetime entirely. Now the surface that would be homologous to A and minimizing the area functional would obviously be $A = \emptyset$. This contradiction is however resolved by noting that a spacetime with a black hole has a singular point in the metric and is thus not a simply connected region. Instead we look for a surface enclosing the singularity that extremizes the area functional, which is the black hole horizon. This shows that the Ryu-Takyanagi formula is at least consistent with the holographic Bekenstein-Hawking formula.

3.4 General spacetime from entanglement

In the previous section we saw the intimate relation between the entanglement entropy of the CFT and the geometry of the AdS bulk. The Ryu-Takyanagi formula allows us to investigate the effect of changes in entanglement to the bulk geometry. It can in fact be used to study the effect of changes in entanglement of any holographic CFT on the corresponding extremal bulk surfaces. It appears that entanglement controls the 'connectivity' [15, 17] of classical spacetime.

Consider a CFT state $|\phi\rangle$ on S^d living on the boundary ∂M_ϕ dual to an asymptotic AdS spacetime M_ϕ. Now arbitrarily divide the boundary ∂M_ϕ into two complementary spatial regions call them B and $\bar{B}$. The degrees of freedom of the subsystems of $|\phi\rangle$ on the complementary regions are now entangled with each other. This can be expressed by taking the tensor product of their respective Hilbert spaces

$$\mathcal{H}_{\partial M_\phi} = \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}} \tag{3.11}$$

Such that the state ϕ can be expressed in terms of a sum over the tensor product of states in B and $\bar{B}$:

$$|\Psi\rangle = \sum_{i,j} p_{i,j} \, |\psi_i^B\rangle \otimes |\psi_j^{\bar{B}}\rangle \tag{3.12}$$

which is in general an entangled state. Consider disentangling both states. By the Ryu-Takyanagi formula the extremal surface on the dual AdS space homologous to both A and $\bar{A}$ will shrink as the entanglement entropy decreases. Finally when the state is completely disentangled we end up with a product state

$$|\phi\rangle = \sum_i c_i \, |\psi_i^B\rangle \otimes \sum_j d_j \, |\psi_i^{\bar{B}}\rangle \tag{3.13}$$

corresponding to disconnected spacetimes since the extremal area joining them is now zero. This is depicted in figure 3.2.

An additional argument [18] making use of mutual information (2.10) shows that the geodesic distance between the points in the complementary regions also grows as entanglement decreases.

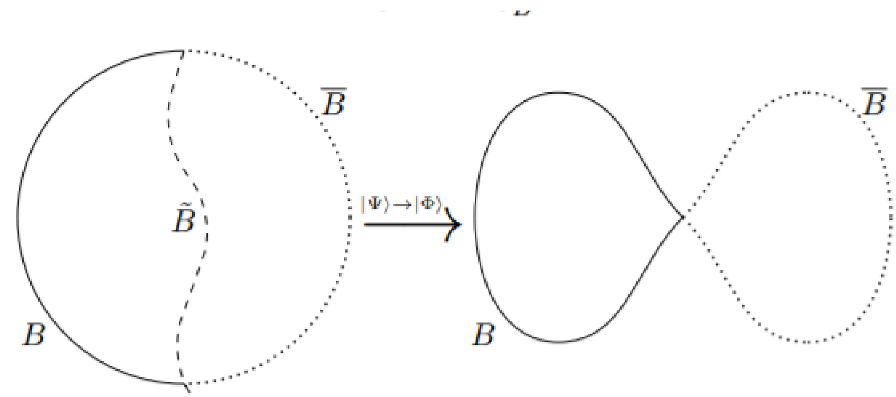

Figure 3.2: As the entanglement between B and $\bar{B}$ is decreased the shared extremal surface $\tilde{B}$ decrease to zero. Points in B move away from $\bar{B}$ so that both spacetimes seem to move apart giving the picture of the dual spacetime separating into two disconnected spacetimes.

Entanglement provides an upper bound on correlation functions between local operators in subsystems of regions B and $\bar{B}$. If we let $C \subset B$ and $D \subset \bar{B}$ then the bound on the correlation functions is given by:

$$I(C,D) \geq \frac{(\langle \mathcal{O}_C \mathcal{O}_D \rangle - \langle \mathcal{O}_C \rangle \langle \mathcal{O}_D \rangle)^2}{2|\mathcal{O}_C||\mathcal{O}_D|} \tag{3.14}$$

Now let us evolve the state such that the mutual information goes to zero, then the correlator functions must vanish as well.

Furthermore, in the AdS/CFT framework two-point correlation functions of local operators inserted on the boundary can be expressed in terms of their geodesic distance through the bulk [17]. More precisely the two point functions can be expressed as

$$\langle \mathcal{O}_C(x_C) \mathcal{O}_D(x_D) \rangle \sim e^{-mL} \tag{3.15}$$

where m is the mass of the dual particle in the bulk and L the length of the extremal geodesic connecting both x_C and x_D. Thus as the correlation function vanishes the distance L goes to infinity. The picture that we end up with is one where a decrease in entanglement entropy causes the spacetimes to move apart while the surface connecting them shrinks.

Chapter 4

Linearised Einstein equations in AdS

In this thesis we shall be primarily interested into linear order perturbations to the CFT vacuum state and thus also to pure AdS. Therefore we would like to first find the Einstein equations to linear order in an AdS background. First linearized gravity for a general background metric will be considered and then specialized to the case where the perturbed background metric is the Poincaré patch.

4.1 Linearized EFE in a general background

Since the Einstein tensor only depends on the metric $E(g_{\mu\nu}) = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}$ a natural way to linearize gravity is to consider a power series of the Einstein tensor in terms of derivatives of the metric. This is done by introducing a new parameter λ such that we can express the metric as:

$$g_{\mu\nu}(\lambda, x) = g^0_{\mu\nu}(0, x) + \lambda \frac{\partial g_{\mu\nu}(\lambda, x)}{\partial \lambda}|_{\lambda=0} + \frac{1}{2}\lambda^2 \frac{\partial^2 g_{\mu\nu}(\lambda, x)}{\partial \lambda^2}|_{\lambda=0} + ... \tag{4.1}$$

We will now however only restrict ourselves to perturbations linear in λ such that the metric perturbation $\delta g_{\mu\nu}$ is given by:

$$\frac{\partial g_{\mu\nu}(\lambda)}{\partial_\lambda}|_{\lambda=0} = \delta g_{\mu\nu} \tag{4.2}$$

such that in the linearized approximation scheme the full metric is given by

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu} \tag{4.3}$$

To find the linear Einstein equations we now insert the perturbed metric into the Einstein equations:

$$\delta G_{\mu\nu} \equiv \frac{\partial G_{ab}(g_{\mu\nu}(\lambda, x))}{\partial \lambda}|_{\lambda=0} = 8\pi G_N \frac{\partial T_{ab}(g_{\mu\nu}(\lambda, x))}{\partial \lambda}|_{\lambda=0} \tag{4.4}$$

The restraints that this equality imposes on $\delta g_{\mu\nu}$ will now be calculated by considering the change in the Einstein tensor $\delta G_{\mu\nu}$ under this perturbation

$$\begin{aligned} \delta G_{\mu\nu} &= \frac{\partial}{\partial \lambda}|_{\lambda=0}[(\delta^a_\mu \delta^b_\nu - \frac{1}{2}g^{ab}g_{\mu\nu})R_{ab} + \Lambda g_{\mu\nu}] \\ &= (\delta^a_\mu \delta^b_\nu - \frac{1}{2}g^{ab}g_{\mu\nu})\delta R_{ab} - \frac{1}{2}(g^{ab}\delta g_{\mu\nu} - \delta g^{ab}g_{\mu\nu})R_{ab} + \Lambda \delta g_{\mu\nu} \end{aligned} \tag{4.5}$$

where we raised and lowered indices using the background metric $g^0_{\mu\nu}$ since in linear perturbation theory using the full metric gives second order corrections in $\delta g_{\mu\nu}$ so these vanish anyway.

Now what remains is to find the linear change of the Ricci tensor in terms of the perturbation. This is just a matter of writing everything out and has been done explicitly in [20]. The result for the linear change of the Ricci tensor is:

$$\begin{aligned}\delta R_{\mu\nu} = -\frac{1}{2}(\nabla_c\nabla^c\delta g_{\mu\nu} - \nabla_\mu\nabla^c\delta g_{c\nu} - \nabla_\nu\nabla^c\delta g_{c\mu} \\ \nabla_\mu\nabla_\mu\delta g^c_c - R^d_\mu\delta g_\nu d - R^d_\nu\delta g_\mu d + 2R^{cd}_{\mu\nu}\delta g_{dc})\end{aligned} \tag{4.6}$$

So we now have all the ingredients of the linearized Einstein equations in terms of the linear order change $\delta g_{\mu\nu}$ since the .

4.2 Perturbing AdS in the Poincaré patch

Since AdS is a maximally symmetric spacetime the Riemann tensor takes on a particularly simple form [16]

$$R_{abcd} = \frac{-1}{L^2}(g_{ab}g_{bd} - g_{ad}g_{bc}) \tag{4.7}$$

and the Ricci tensor is:

$$R_{ab} = \frac{-d}{L^2}g_{ab} \tag{4.8}$$

From (1.4) we know that the metric is diagonal such that only the g_{ii} elements are non-zero and thus also only the R_{ii} components are non-zero.

Now we contract both tensors with the perturbation such that

$$R^{cd}_{\mu\nu}\delta g_{cd} = (g^{cd}g_{\mu\nu} - \delta^c_\mu\delta^d_\nu)\frac{-1}{L^2}\delta g_{cd} \quad R^c_\mu\delta g_{\nu c} = \frac{-d}{L^2}\delta g_{\mu\nu} \tag{4.9}$$

In the previous chapter we showed that for small perturbations, i.e. near the boundary, the AdS dual spacetime is described in terms of the Fefferman-Graham coordinates by the metric

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_i dx^j) + z^{d-2}h_{ij}(\vec{x}, z)dx^\mu dx^\nu \tag{4.10}$$

so we have an explicit description of the metric perturbation.

If we naïvely insert this into the expression for $\delta G_{\mu\nu}$ we obtain $\frac{d(d+1)}{2}$ PDE's with $\frac{d(d+1)}{2}$ independent variables $h_{\mu\nu}$. However there are an abundant number of degrees of freedom so to solve the Einstein equations we need to impose so called gauge conditions [19]. One restrains the metric tensor by fixing the coordinate system. So in this case gauge fix (d+1)-components, i.e. the number of degrees of freedom. We will choose the so called radial gauge such that all z-components vanish, $h_{zj} = 0$. So

$$\frac{L^2}{z^2}(dz^2 + dx_i dx^j) + z^{d-2}h_{ij}(\vec{x}, z)dx^\mu dx^\nu \equiv g^{AdS}_{\mu\nu} + z^{d-2}h_{\mu\nu} \tag{4.11}$$

where $h_{\mu\nu}$ is equal to h_{ij} with the additional gauge conditions.

Now with the gauge conditions in place we can finally derive the expression for $\delta G_{\mu\nu}$. However for simplicity we will concentrate us on the δG_{tt} components, all other components follow from a similar calculation. Using the gauge condition we substitute all the expressions of (4.9) back into (4.5) to find explicitly

$$\delta G_{tt} = \frac{d}{2L^2}(\delta^{ij}z^{d-2}h_{ij} + z^{d-2}h_{tt}) + \frac{1}{2}\delta^{ab}\delta R_{ab} \tag{4.12}$$

where the contraction with the Ricci tensor can now be explicitly calculated using (4.6):

$$\delta^{ab}\delta R_{ab} = \frac{z^2}{L^2}[-\frac{d}{z^2}z^{d-2}h_{tt} - (\partial_z^2 + \partial_i\partial^i - \frac{(d-5)}{z}\partial_z + \frac{d-4}{z^2})\delta^{kl}\delta g_{kl} + \partial^\mu\partial^\nu z^{d-2}h_{\mu\nu}] \quad (4.13)$$

where $\partial_i \equiv \delta_{ij}\partial^j$.

Combing the results we finally obtain the expression for the linearized (tt)-component of the Einstein equations

$$\delta G_{tt} = \frac{z^d}{2L^2}[-(\partial_z^2 + \frac{d+1}{z}\partial_z + \partial_i\partial^i)\delta^{ij}h_{ij} + \partial^i\partial^j h_{ij}] \quad (4.14)$$

The other components of $\delta G_{\mu\nu}$ are omitted but can be derived similarly.

Chapter 5

The Iyer-Wald formalism

Looking ahead to chapter 7 we will find a differential form χ satisfying the properties which are used to derive the linearised gravity equations. To derive this form we introduce the Iyer-Wald formalism [2]. The Iyer-Wald formalism will allow us to derive the linearized gravity equations for more general diffeomorphism invariant theories of gravity. Moreover we will derive expressions for the generalized Wald entropy and canonical energy, that shall be extensively used next chapter, in terms of the gravitational equations of motions which. The Iyer-Wald formalism uses the symplectic formalism for which a good introduction is given in [1]. In this thesis differential forms will be written in bold-faced notation.

5.1 Lagrangian and hamiltonian in classical field theory.

Consider the following Lagrangian

$$\mathbf{L} = \mathcal{L}\epsilon \tag{5.1}$$

where ϵ is the volume form given by

$$\epsilon = \frac{1}{(d+1)!}\sqrt{-det(g_{\mu\nu})}\epsilon_{a_1 a_2 \dots a_{d+1}} dx^{a_1} \wedge dx^{a_2} \wedge \dots \wedge dx^{a_{d+1}} \tag{5.2}$$

which is a (d+1)-form on a (d+1) dimensional spacetime. The Lagrangian density $\mathcal{L}$ is made up of the fields and a finite number of their derivatives. The fields ϕ consist of the metric tensor g_{ab} and arbitrary matter fields. We will be interested in theories of gravity which are invariant under a diffeomorphic mapping $f : M \to M$ such that $\mathbf{L}[f * (\phi) = f * \mathbf{L}[\phi]$

Analogous to the derivation of Euler-Lagrange in particle mechanics a first order variation of the fields will be considered. The variation of the Lagrangian was derived in [3] to be

$$\delta\mathbf{L} = \mathbf{E}^{\phi}\delta\phi + d\Theta(\phi, \delta\phi) \tag{5.3}$$

the equations of motions are defined as $\mathbf{E}^{\phi} \equiv 0$. The summation over the fields ϕ is implicit. The d-form Θ is referred to as the symplectic potential current form and is the exterior derivative of the boundary term which results from integration by parts to get the EOM. Normally the boundary term vanishes when minimizing the action but we keep it since it is used to define the Noether current form.

In order to define the hamiltonian we will be interested in finding a so-called symplectic form Ω which shall now be defined. We start by considering another variation δ_2 of the Lagrangian such that we have

$$\delta_1\delta_2\mathbf{L} = (\delta_1\mathbf{E}^{\phi})(\delta_2\phi) + \mathbf{E}^{\phi}(\delta_1\delta_2\phi) + d\delta_1\theta(\phi, \delta_2\phi) \tag{5.4}$$

next we switch the order of the variation and subtract the varied Lagrangians

$$\begin{aligned}\delta_1\phi\delta_2\phi L - \delta_2\phi\delta_1\phi L &= \delta_2\mathbf{E}_\phi\delta_1\phi - \delta_1\mathbf{E}_\phi\delta_2\phi + d\delta_2\mathbf{\Theta}\omega(\phi,\delta_1\phi) - d\delta_1\mathbf{\Theta}\omega(\phi,\delta_2\phi)\\ &= \delta_2\mathbf{E}_\phi\delta_1\phi - \delta_1\mathbf{E}_\phi\delta_2\phi + d\omega(\phi,\delta_1\phi,\delta_2\phi)\end{aligned}$$

where we defined the define symplectic current d-form to be

$$\omega(\phi,\delta_1\phi,\delta_2\phi) \equiv \delta_1[\Theta(\phi,\delta_2\phi)] - \delta_2[\Theta(\phi,\delta_1\phi)] \tag{5.5}$$

Now we can define the pre-symplectic form Ω as the integral of the above current over a Cauchy surface S

$$\Omega(\phi,\delta_1\phi,\delta_2\phi) \equiv \int_S \omega(\phi,\delta_1\phi,\delta_2\phi) \tag{5.6}$$

This holds independently of the cauchy surface when the perturbations of the field satisfy the linearized Einstein equations and S is compact or the fields have the right asymptotic behaviour [21]. The Hamiltonian associated to the vector field ξ is in this formalism defined as

$$\delta H_\xi = \Omega(\phi,\delta\phi,\mathcal{L}_\xi\phi) \tag{5.7}$$

where $\mathcal{L}_\xi\phi$ is the Lie derivative of ϕ with respect to ξ.

5.2 The Noether current and charge

As it turns out the entropy and canonical energy can be described as Noether charges associated to local symmetries of the Lagrangian field theory. We consider an arbitrary vector field ξ^a on the spacetime manifold M. The variation of the fields can then be described using the Lie derivative which describes the variation of fields along integral curves of ξ^a such that $\delta_\xi\phi \equiv \mathcal{L}_\xi\phi$. Under this variation of the field the Lagrangian transforms as

$$\begin{aligned}\delta_\xi L &\equiv \mathcal{L}_\xi\mathbf{L}\\ &= \xi\cdot d\mathbf{L} + d(\xi\cdot\mathbf{L})\\ &= d(\xi\cdot\mathbf{L})\end{aligned}$$

where in the second line Cartan's magic formula is used which holds for the Lie derivative of an arbitrary vector field on any differential form. To get to the last line we note that the Lagrangian is a (d+1)-form on a (d+1)-dimensional space and is thus closed since it is a top form. Finally, it is observed that under the variation of the fields the Lagrangian only varies by a total derivative. We thus conclude that the field transformation is a local symmetry.

According to Noether's theorem we can always associate a conserved current to a local symmetry when the equations of motion are satisfied. Taking the variation in (5.3) to be a Lie derivative allows us to combine it with the equation above

$$\mathcal{L}_\xi\mathbf{L} = \mathbf{E}^\phi\mathcal{L}_\xi\phi + d\Theta = d(\xi\cdot\mathbf{L}) \tag{5.8}$$

Now if the equations of motions are satisfied $\mathbf{E}^\phi = 0$ then

$$d\Theta = d(\xi\cdot\mathbf{L}) \to d\Theta - d(\xi\cdot\mathbf{L}) = d(\Theta - \xi\cdot\mathbf{L}) = 0 \tag{5.9}$$

it is now obvious how to define the closed, on-shell Noether current d-form

$$\mathbf{J}[\xi] = \Theta - (\xi\cdot\mathbf{L}) \tag{5.10}$$

such that $\mathbf{J}[\xi] = 0$. Because the current form is closed there must exist a (d-1)-form of which the current is the exterior derivative. Let us denote such a form by $\mathbf{Q}[\xi]$ called the Noether charge (d-1)-form

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] \quad (onshell) \tag{5.11}$$

It is important to distinguish this from the Noether charge $Q[\xi]$ which is simply a scalar quantity defined to be the integral of the Noether current form over some spacelike surface Σ such that

$$Q[\xi] = \int_{\Sigma} \mathbf{J}[\xi] = \int_{\partial\Sigma} d\mathbf{Q}[\xi] \tag{5.12}$$

where the second equality holds due to Stokes' theorem. So the Noether charge is just an integral of the charge form over the boundary of Σ. The form of the Noether charge form was explicitly derived by Iyer and Wald[4] to be

$$\mathbf{Q}[\xi] = \mathbf{W}_i\xi^i + \mathbf{X}^{ij}\nabla_{[c\xi_d]} \quad \mathbf{X}^{ij} - E_R^{abij}\epsilon_{ab} \tag{5.13}$$

where E_R^{abij} is the variational derivative of the Lagrangian with respect to the Riemann tensor when this is considered as an independent field. The explicit form of $\mathbf{Q}[\xi]$ was generalized to the off-shell form in [5] such that

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] + \xi^i\mathbf{C}_i \tag{5.14}$$

where C_i denotes a set of constraint equations when evaluated at fixed time.

Wald entropy

In the next chapter we will find that the dual space of a reduced density matrix contains an AdS black hole with corresponding Killing horizon. Thus looking ahead we now restrict ourselves to spacetimes with a bifurcate Killing horizon and corresponding bifurcation surface $\mathcal{H}$. We let ξ be the Killing vector that generates the horizon. Here the Wald entropy is defined [6] to be

$$S_{Wald} = 2\pi \int_{\mathcal{H}} \mathbf{X}^{ij} n_{ij} \tag{5.15}$$

where n_{ij} is the binormal to $\mathcal{H}$. On the Killing horizon we have by definition that $\nabla_{[i\xi_j]} = \kappa n_{ij}$ with κ the surface gravity. Now we are ready to show that the Wald entropy is in fact the Noether charge of this Killing symmetry over $\mathcal{H}$

$$\begin{aligned} \frac{\kappa}{2\pi} S_{Wald} = Q[\xi] &= \int_{\mathcal{H}} \mathbf{Q}[\xi] \\ &= \int_{\mathcal{H}} \mathbf{W}_i\xi^i + X^{ij}\nabla_{i\xi_j} \\ &= \int_{\mathcal{H}} X^{ij}\nabla_{i\xi_j} \end{aligned} \tag{5.16}$$

where in going to the last line we used that the Killing vector vanishes on the bifurcation surface over which we are integrating.

5.3 Canonical energy

In this section we will derive the general expression for the variation of the bulk energy and will remark on finding a proper definition of this energy. Since the variation in energy is related to a variation in the Hamiltonian we will first derive the form of the variation of the Hamiltonian. We know that this is related to the symplectic current ω by (5.7). To express the symplectic current form consider the variation of the Noether current form under a variation of the fields ϕ satisfying the equations of motions

$$\begin{aligned}\delta\mathbf{J}[\xi] &= \delta\mathbf{\Theta}(\phi,\mathcal{L}_\xi\phi) - \xi\cdot\delta\mathbf{L}\\ &= \delta\mathbf{\Theta}(\phi,\mathcal{L}_\xi\phi) - \xi\cdot d\mathbf{\Theta} - \xi\cdot\mathbf{E}_\phi\delta\phi\\ &= \delta\mathbf{\Theta}(\phi,\mathcal{L}_\xi\phi) - \mathcal{L}_\xi\mathbf{\Theta}(\phi,\delta\phi) + d(\xi\cdot\mathbf{\Theta}(\phi,\delta\phi)) - \xi\cdot\mathbf{E}_\phi\delta\phi\end{aligned}$$

Where first (5.3) and then Cartan's magic formula were used. The first terms on the right hand side are recognized as the presymplectic current form (5.5) with $\delta_2\phi = \mathcal{L}_\xi\phi$. Hence

$$\omega(\phi,\delta\phi,\mathcal{L}_\xi\phi) = \delta\mathbf{J}[\xi] - d(\xi\cdot\mathbf{\Theta}) \tag{5.17}$$

Now integrating over some Cauchy surface $\mathcal{C}$ gives us the variation of the Hamiltonian

$$\delta H[\xi] = \int_{\mathcal{C}}(\delta\mathbf{J}[\xi] - d(\xi\cdot\mathbf{\Theta})) \tag{5.18}$$

Now inserting the general form of the off-shell Noether current form (5.14) gives

$$\delta H[\xi] = \delta\int_{\mathcal{C}}\xi^i\mathcal{C}_i + \int_{\partial\mathcal{C}}(\delta\mathbf{Q}[\xi] - \xi\cdot\mathbf{\Theta}) \tag{5.19}$$

after using Stokes' to get the second term. If the constraint equations $\mathbf{C}^i$ satisfy the linearized equations of motions we are left with only the boundary term since then $\delta\mathbf{C}^i = 0$. It is exactly this boundary term that is defined as the canonical energy as argued in [3]. The variation in energy is now the variation in conserved Noether charge associated to asymptotic time translations $\xi^i = \partial_t^i$. Replacing the Cauchy surface by the asymptotic boundary of the space at infinity we find for the boundary term

$$\delta E \equiv \int_\infty(\delta\mathbf{Q}[\partial_t] - \partial_t\cdot\mathbf{\Theta}) \tag{5.20}$$

Chapter 6

Bulk spacetime physics from entanglement entropy constraints

It has been showed that the geometrical structure of the spacetime is intimately connected to the entanglement structure of the CFT states on the boundary. It turns out that by considering perturbations to the vacuum CFT state the Einstein field equations to linearized order around the vacuum AdS bulk can be retrieved. The derivation in this and the following chapter will follow [6, 15] closely.

Consider the setup in 6.2. We have a CFT living in Minkowski space with a spherical entangling region B entangled with $\bar{B}$ over the boundary ∂B. Furthermore we will consider the causal development $\mathcal{D}$ of the region B which is defined as the union of past and future domains of dependence. All points in $\mathcal{D}$ can be described by evolving the state in B. The CFT will be in the vacuum state and we will consider the t=0 slice. The classical dual geometry is thus AdS space and we will consider the perturbations to the CFT side to correspond to perturbations in the AdS via the Fefferman-Graham metric.

The constraint on the CFT side is known as the first law of entanglement entropy. It will be explained in more detail but it is basically a quantum generalization of the thermodynamic first law. The first law of entropy states that variation of the entanglement entropy of a CFT for some region $A \subset B$ is equal to the variation of energy associated to that region.

$$\delta S_A = \delta E_A \tag{6.1}$$

Next we would like to translate this boundary first law to a contraint on the bulk spacetime. In this chapter we will interpret the two boundary quantities in terms of bulk quantities and show that they obey a holographic version of the first law. The entanglement entropy will be given by the Ryu-Takyanagi formula for GR and by the Wald entropy for more general diffeomorphic theories of gravity. The right hand side will be interpreted as the energy associated to the boundary metric.

$$\delta S_A^{grav} = \delta E_A^{grav} \tag{6.2}$$

This will result in both sides of (6.2) being expressed as an integral over the metric perturbation. This metric perturbation will then be calculated to satisfy the equations of gravity up to linear order.

6.1 First law of entanglement entropy

Consider a family of states $|\Psi(\lambda)\rangle$ parametrized by λ describing a general QFT system with some subsystem A. Using the definition of entanglement entropy we can directly calculate the variation of S under a variation of λ:

$$\frac{d}{d\lambda}S_A = -tr(\log \rho_A \frac{d}{d\lambda}\rho_A) - tr(\frac{d}{d\lambda}\rho_A) = -tr(\log \rho_A \frac{d}{d\lambda}\rho_A) \tag{6.3}$$

where the second term vanishes since the trace is one and commutes with the derivative. Since the density matrix can be expressed as e^H we define so called modular hamiltonian in terms of the unperturbed density matrix, i.e. $\lambda = 0$, as

$$H_A \equiv -\log \rho_A(\lambda = 0) \tag{6.4}$$

This allows us to relate the change in entropy to the change in expectation value of the modular hamiltonian since we have $\langle \mathcal{O} \rangle = tr[\mathcal{O}\rho]$ and thus since the modular hamiltonian is independent of λ:

$$\frac{d}{d\lambda}S_A = tr(H_A \frac{d}{d\lambda}\rho_A) = \frac{d}{d\lambda}tr[H_A\rho_A] = \frac{d}{d\lambda}\langle H_A \rangle \tag{6.5}$$

Note that we never assumed that ρ_A describes an equilibrium state and the first law thus holds for arbitrary states.

Lets consider the special case where the density matrix is in a thermal state $\rho_A = \frac{e^{-H_A/T}}{Z}$. Plugging this into (6.5) gives

$$\frac{d}{d\lambda}S_A = \frac{1}{T}\frac{d}{d\lambda}\langle H_A \rangle \tag{6.6}$$

which is recognized as the first law of thermodynamics for a closed, fixed-volume system by identifying the energy with the expectation value of the hamiltonian, i.e. $dE = TdS$. For this reason the quantum generalization to arbitrary states is known as the first law of entanglement.

Thus for an arbitrary region on the boundary spacetime $B \subset \Sigma_{\partial M_\phi}$ we have that

$$\delta S_A = \delta \langle H_A \rangle \tag{6.7}$$

where $\delta\mathcal{O} = \frac{d}{d\lambda}\mathcal{O}|_{\lambda=0}$ The first law of entanglement entropy is most useful when we have an explicit expression for the modular hamiltonian since it is in general not a local operator [6]. There are however cases for which an explicit local modular hamiltonian. In the next section we will consider a spherical entanglement region and find an explicit expression for the modular hamiltonian.

6.2 The first law of entanglement for a CFT

To continue our derivation it will be necessary to find an expression for the modular hamiltonian. As said it can, in general, not be written in terms of local operators. Fortunately if we consider a spherical entanglement surface of a CFT in Minkowski space then it turns out that the modular hamiltonian can be written as a local object. This was shown by [7] and the following section will be based on their work.

It will be shown that the domain of dependence, i.e. the set of points p for which all lightlike geodesics pass through a spatial slice Σ, of the spherical entanglement surface in the vacuum state of a CFT is equivalent to a thermal state in the Rindler wedge.

The Rindler wedge describes a part of Minkowski spacetime in which accelerated observers are able to send and receive signals to one another. In this section observers accelerated in the

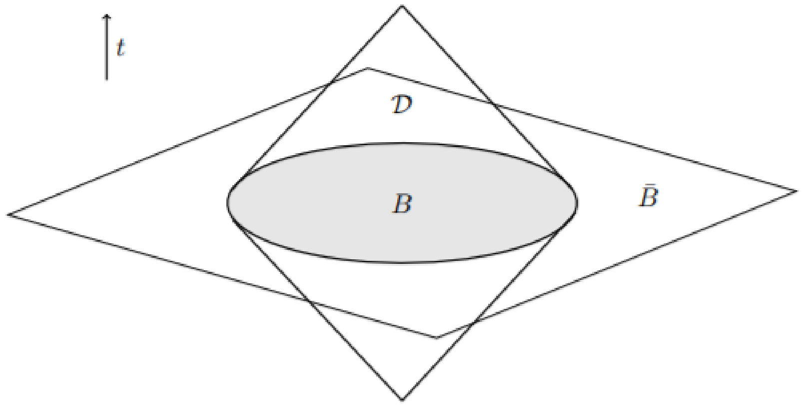

Figure 6.1: B describes a spherical region with $t = 0$ of flat spacetime on which a CFT lives in the vacuum state. $\mathcal{D}$ denotes the causal development of the region B. $\tilde{B}$ is the complementary region with which B is entangled.

X^1 Minkowski coordinate direction will be considered. Solving the equations of motions for a proper acceleration in the X^1 direction tells us that accelerated observers in Minkowski space are described by the following coordinate transformations

$$X^1 = r\cosh(\eta/R) \quad X^0 = r\sinh(\eta/R) \tag{6.8}$$

where r is the distance X^1 when η, the proper time along the trajectory, is zero. The metric in Rindler coordinates is thus

$$ds^2 = -\frac{r^2}{R^2}d\eta^2 + dr^2 + dX^i dX_i \tag{6.9}$$

The equivalence between the CFT on the domain of dependence $\mathcal{D}$ and the CFT on a Rindler wedge follows since they are related by a conformal transformation and CFT's are invariant under these transformations by construction. We will show that in the Rindler wedge the modular hamiltonian of the density matrix can be written as the boost generator in the X_1 direction or H_η in the Rindler wedge. Inverting the conformal transformation allows us to translate the modular hamiltonian for the CFT in the half space to the modular hamiltonian in $\mathcal{D}$

Consider the set up as in figure The modular Hamiltonian of the CFT will be denoted H_B.

6.2.1 Modular Hamiltonian of Rindler space

The description of a density matrix for subsystems in a quantum field theory is in general a difficult problem, even when simplified to the vacuum state. Consider Minkowski space with coordinates X^μ. The half space of Minkowski space $X_1 > 0$ turns out to have an explicit expression of its density matrix in terms of a local modular Hamiltonian.

Consider the Wick rotated path integral representation of the vacuum density matrix (2.17). We denote the region $X_1 > 0$ by R. The representation now reads:

$$\langle\phi_1^R|\,\rho_R^\beta\,|\phi_0^R\rangle = \frac{1}{Z}\int_{\phi^R(0^+)=\phi_0^A}^{\phi^R(0^-)=\phi_1^R} d\phi e^{-S_{Euc}(\phi(\tau))} \tag{6.10}$$

Now we make a change of coordinates to describe the half space in terms of polar coordinates. With $X_1 = r\cos(\theta/R)$ and $\tau = r\sin(\theta/R)$ we get:

$$\langle\phi_1^R|\,\rho_R^\beta\,|\phi_0^R\rangle = \frac{1}{Z}\int_{\phi^R(\theta=0)=\phi_0^R}^{\phi^R(\theta=2\pi R)=\phi_1^R} d\phi e^{-S_{euc}(\phi(t))} \tag{6.11}$$

which is of the same form but with $\tau \to \theta$ and $\beta = 2R\pi$. (2.17) defined a density matrix $\rho \propto e^{-\beta H}$. Since H generated evolution in the variable $t = i\tau$ we have that in this half space the density matrix is defined by the Hamiltonian H_η which generates evolution in $\eta = i\theta$. Thus for a half space we find that the density matrix is given by $\rho_R \propto e^{-2R\pi H_\eta}$. The Lorentzian metric under this coordinate transformation becomes (where we surpressed the X^i coordinates)

$$dr^2 + r^2 d\theta^2 \to dr^2 - \frac{r^2}{R^2}d\eta^2 \tag{6.12}$$

where we now of course recognize this metric as the Rindler wedge of Minkowski space defined on $X_1 > 0$. So this shows that the vacuum state of any Lorentz invariant QFT is in fact a thermal density matrix in the half space. The density matrix for the Rindler observer is thus thermal with respect to H_η.

$$\rho_R = \frac{1}{Z}e^{-2R\pi H_\eta} \tag{6.13}$$

The modular Hamiltonian of Rindler space is thus $H_{\mathcal{R}} = 2\pi R H_\eta$. H_η generates translations in the η which we can recognize as the proper time along boost orbits. The boost orbits parametrized by η are generated by the vector field

$$\partial_\eta = X^1\partial_{X^0} + X^0\partial_{X^1} \tag{6.14}$$

Since H_η is the Noether charge associated to η-translations. It is thus described by

$$H_\eta = \int_S d\Sigma^\mu T_{\mu\nu}\partial_\eta^\nu \tag{6.15}$$

Let us consider the vector ∂_η in more detail. Since its vector field is just a change in η the lie derivative reduces to a simple differentiation of the metric with respect to η [22].

$$\mathcal{L}_{\partial_\eta} g_{\mu\nu}^{Rindler} = \partial_\eta g_{\mu\nu} = 0 \tag{6.16}$$

which is zero since the metric components are independent of η and thus the vector field is a Killing vector of the Rindler wedge. The norm of the Killing vector tells us that there are two Killing horizons since:

$$\langle\partial_\eta, \partial_\eta\rangle = (X^1 + X^0)(X^1 - X^0) \tag{6.17}$$

which is zero when $X^1 = \pm X^0$.

6.2.2 Map from Rindler space to causal development of the CFT

The density matrix of Rindler space holds for any arbitrary QFT. In this section we will specialize to the case in which the boundary theory is described by a CFT. Consider the modular flow of the null coordinates $X^\pm = X^1 \pm X^0$ of the Rindler wedge parametrized by s is given by a boost operation on the null coordinates

$$X^\pm(s) = X^\pm e^{\pm 2\pi s} \tag{6.18}$$

keeping other coordinates constant under the flow. The modular flow thus parametrizes the orbits of constant acceleration.

Now as claimed a map will be established between the causal development $\mathcal{D}$ of a spherical region of the CFT and Rindler space. The map will then be used to map the modular flow of the Rindler wedge to the CFT in Minkowski space from which we can find derive H_B. The final step is to relate the modular hamiltonian of the Minkowski region B H_B to $H_{\mathcal{R}}$.

We first parametrize the causal development $\mathcal{D}$ by defining null coordinates on the boundary

$$x^{\pm} = r \pm t$$
$$r = \sqrt{\sum_{i}^{(d-1)} (x_i)^2}$$

such that the causal development is given by $x^+ \leq R \cap x^- \leq R$. The map from the Rindler wedge to the $\mathcal{D}$ was derived in [7] to be

$$x^{\mu} = \frac{X^{\mu} - (X \cdot X)C^{\mu}}{1 - 2(X \cdot C) + (X \cdot X)(C \cdot C)} + 2R^2 C^{\mu} \tag{6.19}$$

with $C^{\mu} = (0, -1/2R, ..., 0)$. To find the expression for the modular Hamiltonian H_B we must map the modular flow $X^{\pm}(s) = X^{\pm}e^{\pm 2\pi s}$ to $\mathcal{D}$. First the mapping is inverted in order to express $X^{\pm}(s)$. Only the $X^{\pm}$ directions have to be considered since the modular flow leaves other coordinates unaffected. This gives $C^{\pm} = C^1 \pm C^0 = -1/2R$ and $(C \cdot C) = -1/4R^2$ such that

$$\begin{aligned} x^{\pm} &= \frac{X^{\pm} + \frac{X^+X^-}{2R}}{1 + \frac{X^+ + X^-}{2R} + \frac{X^+X^-}{4R^2}} \\ &= R\frac{(X^{\pm} - 2R)}{(X^{\pm} + 2R)} \end{aligned} \tag{6.20}$$

inverting (6.20) gives

$$X^{\pm} = -2R\frac{(x^{\pm} + R)}{(x^{\pm} - R)} \tag{6.21}$$

finally plugging this back in the right hand side of (6.20) gives, after some algebra, the modular flow in $\mathcal{D}$

$$X^{\pm}(s) = R\frac{(R + x^{\pm}) - e^{\mp 2\pi s}(R - x^{\pm})}{(R + x^{\pm}) - e^{\mp 2\pi s}(R - x^{\pm})} \tag{6.22}$$

which was shown in [7] to correspond to the modular flow in $\mathcal{D}$. This expression allows us to construct the modular hamiltonian H_B by investigating what happens to the null coordinates of the causal development $\mathcal{D}$ under evolution of the Rindler parameter s.

Consider for now the effect of a shift in δs on the surface where $x^0 = \frac{1}{2}(x^+(s) - x^-(s)) = 0$. First let us write $x^0(s)$ in terms of (6.22)

$$x^0(s) = \frac{1}{2}R\Big\{\frac{(R + r) - e^{-2\pi s}(R - r)}{(R + r) - e^{-2\pi s}(R - r)} - \frac{(R + r) - e^{+2\pi s}(R - r)}{(R + r) - e^{+2\pi s}(R - r)}\Big\} \tag{6.23}$$

the shift in s is then given by considering $\delta x^0(s) = \frac{d}{ds}x^0(s)|_{s=0}\delta s$ so

$$\partial_{x^0} = 2\pi\frac{R^2 - r^2}{2R}\partial_s \tag{6.24}$$

Doing the same calculation for $x^1(s)$ shows that $\delta r = 0$ and thus the H_B only shifts x^0. Now since the modular Hamiltonian is the generator of shifts in B we can write the Noether charge associated to this modular flow as the following operator in the CFT

$$H_B = 2\pi \int_B d^{d-1}x \frac{R^2 - r^2}{2R} T^{00} \tag{6.25}$$

More generally we can map the Killing vector $2\pi\partial_\eta$ of the Rindler wedge to the causal development $\mathcal{D}$ which turns out to be a conformal Killing vector on the causal development. To show this let us first invert the map (6.20), in a similar fashion to (6.22), for all coordinates X^μ

$$\begin{aligned} X^0 &= \frac{4R^2 x_0}{R^2 - x_0^2 + 2Rx_1 + \vec{x}^2} \\ X^1 &= -\frac{2(R^3 + Rx_0^2 - R\vec{x}^2}{R^2 - x_0^2 + 2Rx_1 + \vec{x}^2} \\ X^i &= \frac{4R^2 x_i^2}{R^2 - x_0^2 + 2Rx_1 + \vec{x}^2} \end{aligned}$$

By a coordinate transformation the Killing vector ∂_η of the Rindler wedge is now given in terms of the coordinates of the causal development

$$\zeta_{\mathcal{D}} = 2\pi[\frac{R^2 - t^2 - \vec{x}^2}{2R}\partial_0 - \frac{x^0 x^i}{R}\partial_i] \tag{6.26}$$

as claimed it will now be shown that this is in fact also a Killing vector on the causal development. Moreover the flow remains in $\mathcal{D}$ which allows us to construct the Hamiltonian H_B. To explicitly see that the flow is contained in $\mathcal{D}$ consider first the norm. It vanishes on the boundary of the causal development $x^\pm = R$

$$\langle \zeta_{\mathcal{D}}, \zeta_{\mathcal{D}} \rangle = 4\pi^2[-(\frac{R^2 - x_0^2 + \vec{x}^2}{2R})^2 + \frac{x_0^2\vec{x}^2}{R^2}]|_{x^\pm = R} = 0 \tag{6.27}$$

Secondly, $\zeta_{\mathcal{D}}$ vanishes on ∂B and on both ends of the causal development

$$\zeta_{\mathcal{D}}(x^0 = 0, r = R) = 0, \quad \zeta_{\mathcal{D}}(x^0 = \pm R, r = 0) = 0 \tag{6.28}$$

Finally, it is now possible to write the Hamiltonian associated to the conformal vector field covariantly as in (6.15)

$$H_B = \int_B d\Sigma^\mu \zeta_{\mathcal{D}}^\nu T_{\mu\nu} \tag{6.29}$$

The hyperbolic energy E_B is now simply the expectation value of the (6.25) modular hamiltonian.

$$E_B \equiv \langle H_B \rangle = \int_B d\Sigma^0 \zeta_{\mathcal{D}}^0 \langle T_{00} \rangle \tag{6.30}$$

We can now finally express the first order variation of the hyperbolic energy

$$\delta E_B = 2\pi \int_B d^{d-1} \frac{R^2 - r^2}{2R} \delta \langle T^{00} \rangle \tag{6.31}$$

6.3 Holographic interpretation of the first law

We have now established the first law for a ball shaped region B of a vacuum CFT. Since we assume that the CFT has a holographic interpretation. This means that the bulk dual can be described by a classical dual spacetime.

In general the holographic entanglement functional, such as RT-formula, is not known. However since we are considering spherical regions B on the CFT side we can show that this is equivalent to a black hole in the dual spacetime with some Killing vector generating the horizon. This allows us to calculate the black hole entropy using the Wald entropy δS^{wald}. This then implies that $\delta S^{wald} = \delta S^{grav}$ using the holographic dictionary.

First we show that the dual geometry to a reduced density matrix is a bulk black hole spacetime. The horizon is generated by a Killing vector which allows us to calculate the black hole entropy using the killing vector of the dual space.

6.3.1 AdS-Rindler space

In section 3.2 it was shown that the holographic dual of the total vacuum state of a CFT was pure AdS. The first law however holds for arbitrary spherical regions B of the boundary and the CFT is described on the causal development of that region. The state is thus described by a reduced density matrix instead $\rho_{\mathcal{D}}$.

A natural interpretation is that the reduced density matrix is dual to a subregion of pure AdS. We will now review [7] where they argued that this dual region is in fact described by AdS-Rindler space endowed with a Killing vector ξ_{AdS} which will be essential in deriving the linearised Einstein equations. We will first show that we can map the causal development to a hyperbolic geometry and subsequently derive the dual spacetime.

To show that we can map the causal development of the vacuum CFT to a hyperbolic geometry $\mathcal{H} = R \times H^{d-1}$ we start from the map to the Rindler wedge. The Rindler metric can be written as follows

$$ds^2_{\mathcal{R}} = (r/R)^2(-d\eta^2 + \frac{R^2}{r^2}(dr^2 + dX^i dX_i)) \tag{6.32}$$

since $(r/R)^2$ is an overall prefactor we perform a Weyl transformation to eliminate the conformal prefactor and obtain the geometric structure of $\mathcal{H}$

$$ds^2_{\mathcal{H}} = \frac{R^2}{r^2} ds^2_{\mathcal{R}} = -d\eta^2 + \frac{R^2}{r^2}(dr^2 + dX^i dX_i) \tag{6.33}$$

where the last term is now recognized as $\mathcal{H}$ completing the map, see figure 6.3.1. The Weyl prefactor is not dependent on η and therefore the modular hamiltonian is the same since it remains the generator of translations in η. The hyperbolic geometry is thus described by the same thermal density matrix as the Rindler wedge

$$\rho_{\mathcal{H}} = \rho_{\mathcal{R}} = \frac{e^{-}2\pi R H_\eta}{Z} \tag{6.34}$$

Moreover, in [7] they argued that also the Von Neumann entropy is invariant under these conformal transformations since there is now a unitary map from $\mathcal{D}$ to $\mathcal{H}$. So we mapped the entanglement entropy of the vacuum state to a thermodynamic entropy on $\mathcal{H}$. This allows us to calculate the holographic entropy which will be done in the next section.

Now lets perform one more coordinate transformation to get the metric in a form that will be useful when we consider the metric of the dual space in the next section. We basically

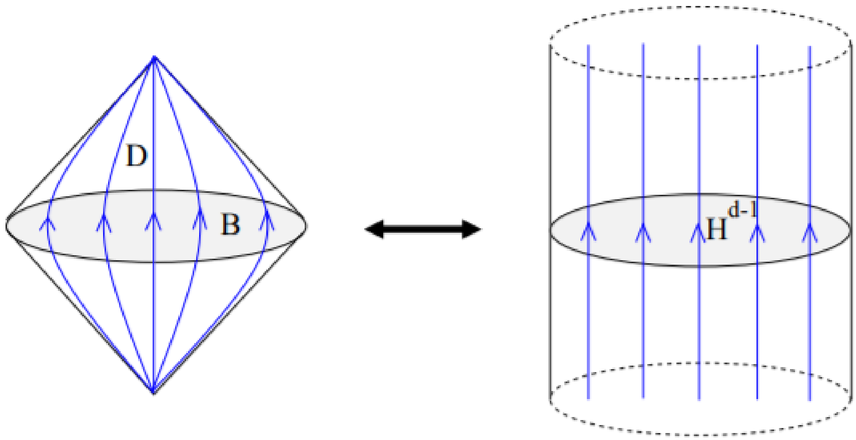

Figure 6.2: jo

restrict the coordinates of the hyperbolic spacetime to cover only the coordinates of the causal development $\mathcal{D}$

$$\frac{1}{r} = \sinh u cos\phi_1 + \cosh u$$
$$X^{d-1} = y \sinh u \sin \phi_1 ... \sin \phi_{d-1}$$

such that

$$ds^2_{\mathcal{H}} = -d\eta^2 + R^2(du^2 + \sinh^2 u d\Omega^2_{d-2}) \tag{6.35}$$

so as $\eta \to \infty$ then $(t, r) \to (\pm R, 0)$ and as $u \to \infty$ then $(t, r) \to (0, R)$ hence covering the causal development.

6.3.2 AdS-Rindler space and Killing vector

The AdS/CFT holographic dictionary states that a thermal CFT state can be described by a black hole in the dual space [**?**]. This so-called AdS-Rindler metric describes a black hole with a corresponding event horizon. It is given by an embedding in global coordinates[7]

$$\begin{aligned} T^1 &= \rho \cosh(u) \\ T^2 &= \tilde{\rho} \sinh(\eta/L) \\ X^d &= \tilde{\rho} \cosh(\eta/L) \\ X^i &= \rho \sinh(u) \sin(\phi_1) ... \sin(\phi_{d-3}) \cos(\phi_{d-2}) \quad i = 1..d-1 \end{aligned} \tag{6.36}$$

such that the AdS constraint gives

$$\tilde{\rho}^2 - \rho^2 = -L^2 \tag{6.37}$$

The induced AdS-Rindler metric is calculated to be

$$ds^2 = (\frac{\rho^2}{L^2} - 1)^{-1} d\rho^2 - (\frac{\rho^2}{L^2} - 1) d\tau^2 + \rho^2(du^2 + \sinh^2(u) d\Omega^2_{d-2}) \tag{6.38}$$

the last term between the brackets can be recognized as the hyperbolic plane (6.35) with unit curvature $R \times H^{d-1}$. The AdS-Rindler patch can be decomposed into a foliation of parallel lower dimensional submanifolds by taking constant ρ-slices. This decomposes the AdS space into hyperbolic planes $R \times H^{d-1}$.

Let us now consider the boundary structure of the AdS-Rindler patch by considering a constant ρ slice in the limit $\rho \to \infty$

$$\begin{aligned}\lim_{\rho\to\infty} ds^2_{AdS-\mathcal{R}} &= \lim_{\rho\to\infty} -(\frac{\rho^2}{L^2}-1)d\eta^2 + \rho^2(du^2+\sinh^2(u)d\Omega^2_{d-2}) \\ &= \lim_{\rho\to\infty} \frac{\rho^2}{L^2}(-d\eta^2 + \frac{L^2}{\rho^2}d\eta^2 + L^2(du^2+\sinh^2(u)d\Omega^2_{d-2})) \\ &= -d\eta^2 + L^2(du^2+\sinh^2(u)d\Omega^2_{d-2})\end{aligned}$$

where in the last equality first a Weyl transformation was done to get rid of the overall prefactor and then took the limit. For $L = R$ the boundary geometry is thus exactly (6.35).

$$ds^2_{AdS-\mathcal{R}} = -(\frac{\rho^2}{L^2}-1)d\eta^2 + (\frac{\rho^2}{L^2}-1)^{-1}d\rho^2 + \rho^2(du^2+\sinh^2 u d\Omega^2_{d-2}) \tag{6.39}$$

as in the Rindler wedge considered in the context of the CFT, η describes the worldline of an accelerating observer and is a ∂_η is a Killing vector of the metric.

In the following part it is shown that the black hole horizon in Poincaré coordinates is an extremal bulk surface homologous to the spherical CFT region B. This shows first of all that the reduced density matrix on the causal development of B is dual to the AdS-Rindler space. Secondly, it allows us to find a Killing vector associated to the black hole horizon which is essential in order to use the Iyer-Wald formalism.

To show that the boundary of the black hole is the region B we first must go from global to Poincaré coordinates. Just setting (1.3) equal to (6.3.2) allows us to express the boundary coordinates in terms of the global coordinates used to describe the black hole spacetime. The boundary coordinates are calculated to be

$$\begin{aligned} t &= L\frac{\tilde{\rho}\sinh(\tau/L)}{\rho\cosh(u)+\tilde{\rho}\cosh(\tau/L)} \\ r \equiv x_i^2 &= L\frac{\rho\sinh(u)}{\rho\cosh(u)+\tilde{\rho}\cosh(\tau/L)}\end{aligned} \tag{6.40}$$

In the boundary limit (6.37) reduces to

$$\lim_{\rho\to\infty}\tilde{\rho} = \lim_{\rho\to\infty}\rho\sqrt{1-\frac{L^2}{\rho^2}} = \rho \tag{6.41}$$

thus in (6.40) ρ and $\tilde{\rho}$ are divided out.

Next to proof our claim that the horizon is homologous to the spherical boundary region B note that at the black hole horizon, $\rho = L$, (6.37) gives $\tilde{\rho} = 0$. Moreover

$$T^1 + X^d = \frac{L^2}{z} = \rho\cosh(u) + \tilde{\rho}\cosh(\tau/L) \tag{6.42}$$

so $z \to 0$ corresponds to $u \to \infty$ for $\tilde{\rho} = 0$. For the boundary coordinates this limit implies that $\lim_{u\to\infty} t \to 0$ and $\lim_{u\to\infty} r \to L$. It thus intersects with the boundary in Poincaré coordinates

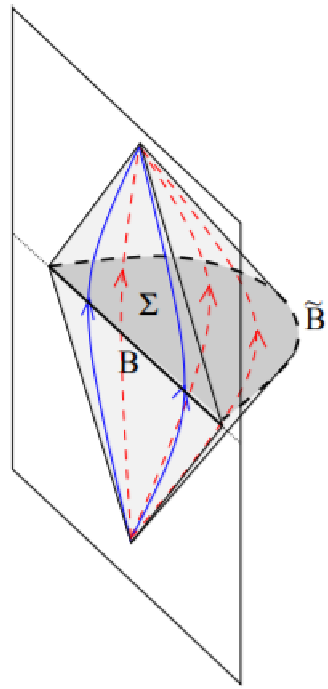

Figure 6.3: A depiction of the AdS-Rindler wedge dual to a spherical region B. The red dashed lines indicate the flow of ξ_{AdS-R} and the blue lines indicate the flow of $\zeta_{\mathcal{D}}$ where on the boundary the flow in the Rindler wedge asymptotes to the boundary flow. $\tilde{B}$ can be seen as the horizon of the dual black hole in Poincaré coordinates with Σ the exterior of the black hole.

as a circle with radius L at $t = 0$. Letting $L = R$ the boundary of the black hole spacetime thus exactly asymptotes to the boundary CFT region. By fixing $\rho = L$, the value of ρ for the horizon in AdS-Rindler, we find that the horizon surface is $\tilde{B} = (t = 0; \vec{x}^2 + z^2 = L)$. For $L = R$ this bulk surface $\tilde{B}$ is extremal and homologous to B at the boundary. This situation is depicted in figure 6.3.2

Since the event horizon of a black hole is also a Killing horizon we would like to find the Killing vector ξ_{AdS-R} that generates it. The Killing vector for the AdS-Rindler patch is given by ∂_η which is directly seen by noting that the AdS-Rindler metric is independent of η. To get an expression of the Killing vector in Poincaré coordinates, which we can compare to the conformal Killing vector $\zeta_{\mathcal{D}}$, consider first the embedding coordinate transformations. We first find the Killing vector in embedding space by the coordinate transformation (6.3.2). This gives us the Killing vector in embedding coordinates

$$\partial_\eta = \frac{1}{L}(X^d \partial_{T^1} + T^1 \partial_{X^d}) \tag{6.43}$$

To get the Killing vector in Poincaré coordinates simply use the Poincaré embedding (1.3)

$$\xi_{AdS-\mathcal{R}} = 2\pi L \partial_\eta = 2\pi[-\frac{1}{L}(tz\partial_z + tx^i\partial_i) + \frac{1}{2L}(L^2 - t^2 - \vec{x}^2 - z^2)\partial_t] \tag{6.44}$$

Next it is shown that the Rindler Horizon is in fact at the extremal bulk surface $\tilde{B} = (\vec{x}^2 + z^2 = L^2)$

$$\langle \xi_{AdS-\mathcal{R}}, \xi_{AdS-\mathcal{R}} \rangle \,|_{x^2+z^2=(L\pm t)^2} = 0 \tag{6.45}$$

Furthermore, $\tilde{B}$ is also the bifurcation surface of the Killing horizon

$$\xi_{AdS-\mathcal{R}}|_{\tilde{B}} = 0, \quad \xi_{AdS-\mathcal{R}}(t = \pm L, x^i = z = 0) = 0 \tag{6.46}$$

this verifies that the flow generated by the Killing vector remains in the AdS-Rindler space.

With the explicit expression of the AdS-Rindler Killing vector a non-trivial test of the duality between AdS-Rindler and the reduced boundary state $\rho_{\mathcal{D}}$ can be done by considering the asymptotic behaviour of $\xi_{AdS-\mathcal{R}}$ for $L = R$.

$$\lim_{z\to 0} \xi_{AdS-\mathcal{R}} = \zeta_{\mathcal{D}}, \tag{6.47}$$

This gives us the required asymptotic behaviour. We will set $L = R$ from now on.

6.3.3 Entanglement entropy and its holographic interpretation

We will take the holographic interpretation of the entanglement entropy to be provided by the Ryu-Takyanagi formula. In section 3.2 we assumed that a perturbation to the CFT vacuum state is dual to a perturbation of vacuum AdS given by a classical spacetime in the form of the Fefferman-Graham metric. Now we will make the assumption that the entanglement entropy of this perturbed state is given by the extremal surface in the perturbed dual spacetime.

Using the Ryu-Takyanagi formula we calculate the perturbation in black hole entropy

$$\delta S^{grav} = \frac{\delta Area(\tilde{B})}{4G_N} \tag{6.48}$$

This will give us the holographic interpretation of the left hand side of the holographic first law (6.2). Let us now explicitly calculate δS^{grav}. The perturbation on the CFT side will correspond to a variation away from pure AdS.

The area functional is given by

$$A(g, X) = \int d^{d-1}\sigma \sqrt{\det(\gamma_{\mu\nu})} \tag{6.49}$$

where $X^a(\sigma)$ is the function that parametrizes the surface and gamma is the induced metric on the surface. The extremal surface will be parametrizes by the functions $X^a_{ext}(\sigma)$.

Now the metric is perturbed such that

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu} \tag{6.50}$$

The change in the extremal surface is given by

$$\delta A = A[\tilde{B} + \delta\tilde{B}, g + \delta g] - A[\tilde{B}, g] \tag{6.51}$$

We can expand the first term on the right hand side to find that

$$A[\tilde{B} + \delta\tilde{B}, g + \delta g] = A[\tilde{B}, g + \delta g] + \mathcal{O}(\delta\tilde{B}^2) \tag{6.52}$$

where the linear order term vanishes due to the extremality of $\tilde{B}$. So the change in area only depends to linear order only the original extremal surface. The embedding X is thus fixed and the area depends only on the metric perturbation

$$\delta A = A[\tilde{B}, g + \delta g] - A[\tilde{B}, g] \tag{6.53}$$

Via (6.49) we then see that the variation in area is determined by the variation of $\gamma_{\mu\nu}$. The variation in the induced metric is now proportional to the variation of the metric because the embedding function is still extremal and thus vanishes at first order variation of $X^a(\sigma)$. The induced metric can thus, to first order in g, also be parametrized by λ

$$\frac{d}{d\lambda}\gamma(\lambda)|_{\lambda=0} = \delta\gamma \tag{6.54}$$

to find the variation of the area functional with respect to λ consider:

$$\begin{aligned}\frac{d}{d\lambda}\gamma|_{\lambda=0} = \sqrt{\det\gamma(\lambda)} &= \frac{1}{2\sqrt{\det\gamma(\lambda)}}\frac{d}{d\lambda}e^{tr(\ln\gamma(\lambda))}|_{\lambda=0} \\ &= \frac{1}{2\sqrt{\det\gamma(\lambda)}}\det(\gamma(0))\frac{d}{d\lambda}tr[\ln\gamma(\lambda)]_{\lambda=0} \\ &= \frac{1}{2\sqrt{\det\gamma(\lambda)}}\det(\gamma(0))tr[\gamma(0)\delta\gamma]\end{aligned}$$

So restoring tensor notation we finally obtain

$$\frac{d}{d\lambda}\gamma|_{\lambda=0} = \frac{1}{2}\sqrt{\det(\gamma^0_{\mu\nu}(\lambda))}tr(\gamma_0^{ab}\delta\gamma_{ab}) \tag{6.55}$$

The perturbed AdS metric is given by the Fefferman-Graham metric, recall:

$$g_{ab} = g_{ab}^{ads} + z^{d-2}h_{ab} \tag{6.56}$$

The original extremal surface $\tilde{B}$ of pure AdS on the t=0 slice is parametrized by $z(x^i) = \sqrt{R^2 - \vec{x}^2}$. This parametrization allows us to calculate the unperturbed induced metric γ_0. The Poincaré patch has only diagonal $\frac{L^2}{Z^2}$ entries so

$$\sum_{k=1}^{d-1}\frac{\partial X^k_{ext}}{\partial x^i}\frac{\partial X^k_{ext}}{\partial x^j} = \delta_{ij} \tag{6.57}$$

and

$$\frac{\partial Z}{\partial x^i}\frac{\partial Z}{\partial x^j} = \frac{x^i x^j}{z^2} \tag{6.58}$$

contracting the unperturbed induced metrics gives $\gamma_0^{ij}\gamma^0_{ij}$. The expression for the perturbed metric is now given by

$$\begin{aligned}\delta\gamma_{ij} &= z^{d-2}h_{ab}\frac{\partial X^a_{ext}}{\partial x^i}\frac{\partial X^b_{ext}}{\partial x^j} \\ &= z^{d-2}(h_{mn}\frac{\partial X^m_{ext}}{\partial x^i}\frac{\partial X^n_{ext}}{\partial x^j} + h_{z\mu}(..) + h_{zz}(..)) \\ &= z^{d-2}h_{mn}\delta_{im}\delta_{jn} = z^{d-2}h_{ij}\end{aligned}$$

where we use the radial gauge to get rid of the $h_{z\mu}$-components. Putting everything together allows us to finally calculate the variation of the extremal surface for a ball shaped region centered at $\vec{x}_0$

$$\begin{aligned}\delta A &= \int d^{d-1}\sigma\frac{1}{2}\sqrt{\det(\gamma^0_{\mu\nu}}\gamma_0^{ab}\delta\gamma_{ab} \\ &= \int d^{d-1}x\frac{1}{2}\frac{L^{d-1}R}{z^d}\frac{z^2}{L^2}(\delta^{ij} - \frac{x^i x^j}{R^2})z^{d-2}h_{ij} \\ &= \frac{L^{d-3}R}{2}\int_{|\vec{x}-\vec{x}_0|\leq R} d^{d-1}x(\delta^{ij} - \frac{1}{R^2}(x^i - x^i_0)(x^j - x^j_0))h_{ij}\end{aligned}$$

which is now combined with the Ryu-Takyanagi formula to give a final expression of the variation in the entropy

$$\delta S = \frac{\delta A}{4G_N} = \frac{L^{d-3}}{8G_N R}\int_{|\vec{x}-\vec{x}_0|\leq R} d^{d-1}x(R^2\delta^{ij} - (x^i - x_0^i)(x^j - x_0^j))h_{ij} \tag{6.59}$$

Wald entropy

In a more general diffeomorphic invariant theory of gravity a similar argument is used. First we note that the entanglement entropy of $\rho_{\mathcal{D}}$ or equivalently of the thermal state on the hyperbolic geometry is equal to the black hole entropy. The difference is that the black hole entropy is now described by the Noether charge (5.16) associated to a Killing vector field ξ. A perturbation to the state $\rho_{\mathcal{D}}$ should be equivalent to a perturbation of the associated Noether charge. Explicitly this gives

$$\delta S_B^{grav} \equiv \delta S_B^{Wald} = \int_{\tilde{B}} \delta \mathbf{Q}[\xi_{AdS-R}] \tag{6.60}$$

where the perturbation is evaluated on $\tilde{B}$ because of (6.53). Note the Wald entropy differs in extrinsic curvature terms for arbitrary entangling surfaces. However since we for spherical entangling regions $\tilde{B}$ is the horizon of a black hole these terms vanish.

6.3.4 Holographic interpretation modular energy

The expectation value of the stress tensor is used to calculate the modular energy E_B on the CFT side. As it turns out this expectation value is calculated in the bulk by the so called holographic stress tensor [23] $T_{\mu\nu}^{grav}$. This bulk stress tensor is formed by the boundary metric and we can thus simply replace the expectation value by the holographic stress tensor in (6.15) since we integrate over the same boundary region. Thus we get

$$E_B^{grav} = 2\pi \int_{B(r,\vec{x}_0)} \frac{R^2 - |X|^2}{2R} T_{tt}^{grav}(t_0, \vec{x}) \tag{6.61}$$

Alternatively we can derive the holographic stress tensor using the first law combined with the Ryu-Takyangi formula. Let us first take the ball shaped region B around x_0 to be infinitesimal by taking $R \to 0$. This allows us to make the assumption that the expectation value of the conformal stress tensor is uniform throughout the ball which implies $\frac{d}{d\lambda}|_{\lambda=0} \langle T^{tt}(x)\rangle = \frac{d}{d\lambda}|_{\lambda=0} \langle T^{tt}(x_0)\rangle$. The variation as $R \to 0$ is

$$\begin{aligned}\delta E^{grav} &= \lim_{R\to 0}[2\pi \int_B d^{d-1}x \frac{R^2 - \vec{x}^2}{2R}\delta \langle T_{tt}(x)\rangle] \\ &= \frac{2\pi R^d S^{d-2}}{d^2-1}\delta \langle T_{tt}(x_0)\rangle\end{aligned}$$

where S^{d-2} is the volume of a (d-2)-sphere obtained by the integral over x. This must now, by the first law, be equal to $\lim_{R\to 0}\delta S$ such that we can now express the variation of the stress tensor as

$$\delta \langle T_{tt}(x_0)\rangle = \frac{d^2-1}{2\pi S^{d-2}} \lim_{R\to 0}(R^{-d}\delta S)$$

Now insert (6.59) into the variation of the entropy to get

$$\delta \langle T_{tt}\rangle = \frac{d^2-1}{2\pi S^{d-2}} \lim_{R\to 0}[R^{-d}\frac{L^{d-3}R}{8G_N}\int_{|\vec{x}|\leq R} d^{d-1}x(\delta^{ij} - \frac{1}{R^2}x^i x^j h_{ij})] \tag{6.62}$$

In the limit $R \to 0$ the metric perturbations in the region enclosed by the boundary spherical surface B and the extremal surface $\tilde{B}$ are now uniform such that $h_{ij}(x,z)|_{R\to 0} = h_{ij}(x_0, z = 0) \equiv h_{ij}$. Noting that the integral is symmetric ensures that all terms for which $i \neq j$ vanish

$$\delta \langle T_{tt} \rangle = \frac{d^2-1}{2\pi S^{d-2}} [\frac{L^{d-3}}{8G_N} h_{ii} \int_{|\vec{x}|\leq 1} d^{d-1}x(1 - x^i x^i) \tag{6.63}$$

where we replaced the coordinates by $x^i \equiv x^i/R$ such that they do not scale under $R \to 0$. Solving the last integral we find

$$\begin{aligned} \delta\langle T\rangle_{tt}(x_0) &= \frac{d^2-1}{2\pi S^{d-2}} \frac{L^{d-3}}{8G_N} \frac{S^{d-2}d}{d^2-1} h^i_i \\ &= \frac{L^{d-3}d}{16\pi G_N} h^i_i \end{aligned}$$

such that we can now insert the stress tensor back into (6.61) and get

$$\delta E^{grav}_B = \frac{dL^{d-3}}{16 G_N R} \int_B d^{d-1}x(R^2 - \vec{x}^2) h^i_i \tag{6.64}$$

Again for a more general theory this is described by the canonical energy given in (5.20).

The holographic first law is now thus

$$\delta S^{grav}_B = \delta E^{grav}_B \tag{6.65}$$

The first is an integrand of the metric perturbation evaluated on $\tilde{B}$ and the second on B. We therefore get a set of non-local integral equations of the metric perturbation. As it turns out this infinite set of non-local constraints will provide us with a local constraint, namely the linearised equations of gravity.

Chapter 7

Deriving linearized gravitational equations

In this chapter, which will be primarily based upon [6], we will proof that the holographic first law is sufficient to constrain the bulk metric perturbation to obey the linearized gravitational equations for any diffeomorphism invariant theory of gravity. More specifically, the linearized equations will be derived for pure AdS_d with a small metric perturbation. It will be shown that if the holographic first law holds for all AdS-Rindler patches then the perturbations have to satisfy linearized gravity. Finally for the case of Einstein gravity we will explicitly show that the metric perturbation satisfies the linearized gravitational equations (4.14) using the Ryu-Takyanagi formula.

To proof the linearised equations we introduce a differential form χ which can be used in the Iyer-Wald formalism and must satisfy three properties

1. $\int_B \chi = \delta E_B^{grav}, \quad \int_{\tilde{B}} \chi = \delta S_B^{grav}$
2. $d\chi = -2\xi^a_{AdS} \delta E^g_{ab} \epsilon^b$
3. $d\chi|_{\partial M} = 0$

notice that in the second property the linearised equations enter for the first time. We will first proof these three properties using the Iyer-Wald formalism. Subsequently we will proof that the linearised equations are satisfied and then we use the Ryu-Takyanagi formula for the entanglement entropy to find explicitly the linearised equations of general relativity (4.14).

7.1 Definition and properties of χ

In this section we will prove the three properties of the differential form χ necessary to derive the linearized Einstein equations. We will work in the AdS-Rindler patch where we choose the normalization of the Killing vector ξ_{AdS-R} to be 2π. The Killing vector of Ads-Rindler space is associated to asymptotic time translations $\xi_{AdS} = 2\pi R \partial_\tau$ and hence it defines an energy. Moreover since the spacetime is static so there is no angular momentum. The first law for general diffeomorphic theories of gravity is now

$$\delta S_B^{Wald} = \delta E_B^{grav}[\xi_{AdS-R}] \tag{7.1}$$

Now we define the following differential form

$$\chi = \delta \mathbf{Q}[\xi_{AdS}] - \xi_{AdS} \cdot \mathbf{\Theta}(\phi, \delta\phi) \tag{7.2}$$

By noting that ξ_{AdS-R} vanishes on the bifurcation surface $\tilde{B}$ we can express the Wald entropy in terms of χ

$$\int_{\tilde{B}} \chi = \int_{\tilde{B}} \delta \mathbf{Q}[\xi_{AdS}] - \xi_{ads} \cdot \Theta = \int_{\tilde{B}} \delta \mathbf{Q}[\xi_{AdS}] = \delta S_B^{Wald} \tag{7.3}$$

and for the variation of the energy the identification with χ is immediate

$$\delta E = \int_B (\delta \mathbf{Q}[\xi_{AdS}] - \xi_{AdS} \cdot \mathbf{\Theta}) = \int_B \chi \tag{7.4}$$

thus translating the first law into the first property

$$\int_B \chi = \int_{\tilde{B}} \chi \tag{7.5}$$

Next consider the exterior derivative of χ since ξ_{Ads} is a Killing vector describing a symmetry of the fields we have $\mathcal{L}_{\xi_{AdS}} \phi = 0$ and hence (5.17) is zero by anti-symmetry of ω. Hence

$$\begin{aligned} d\chi &= \delta(d\mathbf{Q}[\xi_{AdS}] - d(\xi_{AdS} \cdot \mathbf{\Theta})) \\ &= \delta(d\mathbf{Q}[\xi_{AdS}] - \omega(\phi, \delta\phi, \mathcal{L}_{\xi_{AdS}} \phi) - \mathbf{J}[\xi_{AdS}]) \\ &= \delta(d\mathbf{Q}[\xi_{AdS}] - \mathbf{J}[\xi_{AdS}]) \\ &= -\xi^i_{AdS} \delta \mathbf{C}_i \end{aligned}$$

where to get to the last line we used (5.14). As mentioned the form of C_i was derived in [5] to be

$$C_i = \sum_\phi [\sum_{i=1}^{r} (E^\phi)^{b_1 \ldots b_s}_{c_1 \ldots a \ldots c_r} \phi^{c_1 \ldots c_r}_{b_1 \ldots b_s} \epsilon_{c_i} - \sum_{i=1}^{s} (E^\phi)^{b_1 \ldots b_s}_{c_1 \ldots c_r} \phi^{c_1 \ldots c_r}_{b_1 \ldots a \ldots b_s} \epsilon_{b_i}] \tag{7.6}$$

Since we are considering pure AdS all matter fields vanish on the background and thus C_i will be simply the constraint equation for the metric. The constraint equations for the metric, a rank (0,2)-tensor, is thus simply

$$\mathbf{C}_i = 2E^g_{ij} \epsilon^j \tag{7.7}$$

hence we show the second property that the exterior derivative of χ is

$$d\chi = -2\xi^i_{AdS} \delta_{ij} \epsilon^j \tag{7.8}$$

To show the last property: $d\chi|_{\partial M} = 0$ consider first the definition of canonical energy in the Iyer-Wald formalism (5.20) and secondly the definition (6.61). These two definitions describe are equivalent [6] so on the boundary

$$\chi|_{\partial M} = d\Sigma^\mu T_{\mu\nu} \zeta_\nu \tag{7.9}$$

The CFT stress tensor is a conserved, traceless quantity and thus

$$d\chi|_{\partial M} = 0 \tag{7.10}$$

so we proved all properties of the differential form χ.

7.2 Linearized gravity equations of motion

Now we shall us the properties of χ to prove that the holographic first law implies that the gravitational equations of motion are satisfied in the bulk. Note that we are still using the same set-up as in which the first law was proven. Using these properties we have that

$$\delta S_B^{grav} - \delta E_B^{grav} = \int_{\tilde{B}} \chi - \int_B \chi = \int_{\partial\Sigma} \chi = \int_\Sigma d\chi = -2 \int_\Sigma \xi^t_{AdS-R} \delta E^g_{tt} \epsilon^t \tag{7.11}$$

in the second equality we used that B and $\tilde{B}$ enclose the spatial region Σ set at $t = 0$, as shown in figure 6.3.2 such that we can write both integrals as one integral over the boundary of Σ. In the third equality we invoked Stokes' theorem. The last equality follows from the using the second property of χ. On the $t = 0$ only the time-component of the Killing vector ξ_{AdS} is non-zero.

We now want to show that this implies that $\delta E_{tt}^{grav} = 0$ which is not clear from the above expression. To show this first multiply the last term by R and then take its with respect to R. The derivative of this multidimensional integral is performed by using the general Leibniz' integration rule:

$$\begin{aligned} 0 &= \frac{\partial}{\partial R} \int_\Sigma R d\chi \\ &= \int_\Sigma \langle \vec{v}, d(R d\chi \rangle + \int_{\partial\Sigma} \langle \vec{v}, R d\chi \rangle + \int_\Sigma \frac{\partial}{\partial R} R d\chi \\ &= \int_{\partial\Sigma} \langle \vec{v}, R d\chi \rangle + \int_\Sigma \frac{\partial}{\partial R} R d\chi \end{aligned} \tag{7.12}$$

where $\langle , \rangle$ denotes the interior product and $\vec{v}$ is a vector orthogonal to R. In the last equality we used that $d(d\omega) = 0$ and the second equality just uses Leibniz' rule. The integral over $\partial\Sigma$ also vanishes since

$$\begin{aligned} \int_{\partial\Sigma} \langle \vec{v}, R d\chi \rangle &= \int_{\tilde{B}} \langle \vec{v}, R d\chi \rangle - \int_B \langle \vec{v}, R d\chi \rangle \\ &= \int_{\tilde{B}} \langle \vec{v}, R d\chi \rangle \\ &= \int_{\tilde{B}} \xi^t_{AdS} \delta E^g_{tt} \epsilon^t = 0 \end{aligned}$$

where first we split the boundary into B and $\tilde{B}$ again. In the second equality we used the property of χ that on the boundary $d\chi|_{\partial M} = 0$ and thus the integral over B is zero. Finally we inserted the explicit expression for χ and observed that the Killing vector ξ vanishes on the bifurcation surface.

Now we are only left with the second term in (7.12). Inserting the explicit expression for the Killing vector ξ_{AdS} and taking the derivative now yields

$$\int_\Sigma \delta E^g_{tt} \epsilon^t = 0 \tag{7.13}$$

leaving us with an infinite set of equations, one for every spatial region Σ. This integral form implies that $\delta E_{tt}^{grav} = 0$ for the integral to be zero. It follows from a simple calculus argument as shown in for appendix A of [6]. To summarize, considering the holographic first law for spherical regions of a fixed timeslice we derived the (tt)-component of the linearized gravitational equations.

We have chosen a particular region B but the first law should hold in any arbitrary Lorentz frame. Lets consider what happens to the equations of motion by switching to an arbitrary Lorentz' frame defined by a d-velocity vector u^μ. The equations of motions in an arbitrary frame are $u^\mu u^\nu \delta E_{\mu\nu} = 0$. Which implies that $\delta E_{\mu\nu} = 0$ since this holds for arbitrary d-velocity vectors. Note that this does not yet mean that all the components δE_{ij} are zero since so far we only considered the boundary coordinates of the region B. To show that $\delta E_{zi} = 0$, where i runs over all bulk coordinates, it will first be shown that they are zero on the AdS boundary. Using the last two properties of χ one finds:

$$d\chi|_{\partial M} = -2\xi^a \delta E_{ab} \epsilon^b|_{z=0} \implies \delta E_{ab} = 0 \tag{7.14}$$

Then, as argued in [6] the initial value formulation of gravity is invoked explained in [16] such that if the equations of motions hold for all boundary coordinates and for $z = 0$ then they are required to hold for all z. Thus finally giving us the desired result that the linearized gravitational equations of motions are satisfied for all directions

$$\delta E^g_{\mu\nu} = 0 \tag{7.15}$$

So in this section we turned the non-local constraints of the holographic first law into a local set of equations of motions. This was possible due to the fact that for each spherical region B there is a non-local integral equation constraint. This infinite set of constraints allowed us to obtain a local constraint equations which is exactly the linearized equations of gravity.

7.3 Explicit realization of linearized Einstein equations

Now the linearized Einstein equations will be explicitly derived by using the Iyer-Wald formalism. The differential form χ will be derived from the Lagrangian of general relativity

$$\mathbf{L} = \frac{1}{16\pi G_N}\epsilon(R - 2\Lambda) \tag{7.16}$$

pure AdS is considered so there are no couplings to matter fields in the Lagrangian. The variation of the Lagrangian is given by (5.3). The following was explicitly calculated in [4]

$$\mathbf{\Theta} = \frac{1}{16\pi G_N}\epsilon_a(\nabla_c \delta g^{ac} - \nabla^a \delta g^c_c) \tag{7.17}$$

To find the Noether charge form required to define χ the symplectic current form is first calculated using (5.10)

$$\mathbf{J}[\xi] = \frac{1}{8\pi G_N}\epsilon_a \nabla_c(\nabla^{[c}\xi^{a]}) \tag{7.18}$$

now $\mathbf{J} = d\mathbf{Q}$ so one obtains

$$\mathbf{Q} = -\frac{1}{16\pi G_N}\epsilon_{ab}\nabla^a \xi^b \tag{7.19}$$

where ϵ_{ab} is the volume form for codimension-2 surfaces. Putting everything together the definition of $\chi = \delta\mathbf{Q} - \xi \cdot \theta$ gives us χ

$$\chi = -\frac{1}{16\pi G_N}[\delta(\nabla^a \xi^b \epsilon_{ab}) + \xi\epsilon_{ab}(\nabla_c \delta g^{ac} - \nabla^a \delta g^c_c)] \tag{7.20}$$

Now we will show that the translation of the first law to the holographic quantities δS_B^{grav} (6.59) and δE_B^{grav} (6.64) using the Ryu-Takyanagi prescription follow from this construction of χ. The three properties of χ are by construction satisfied but it will be explicitly shown to hold.

First the generator of diffeomorphisms ξ^a will be replaced by ξ^a_{AdS-R} in order to restrict the flow to the AdS-Rindler patch. All calculations will be done on a spatial slice Σ where $t=0$ for simplicity, the other components can be found in a similar fashion [20]. This implies that only the t-component of the AdS-Rindler Killing vector is non-zero and we find in radial gauge that

$$\begin{aligned}\chi|_\Sigma =& \frac{1}{16\pi G_N}[\epsilon[\delta(g_{tt}g^{zz}(\partial_z\xi^t+\Gamma^t_{zt}\xi_t))+g_{tt}\xi^t(\Gamma^z_{tt}\delta g^{tt}+\Gamma^z_{ij}\delta g^{ij}-g^{zz}\partial_z\delta g^c_c)]\\ &+\epsilon^t_i[\delta(g_{tt}g^{ij}\partial_j\xi^t)+g_{tt}\xi^t(\partial_j\delta g^{ij}-g^{ij}\partial_j\delta g^c_c)]]\end{aligned}$$

where it was used that most Christoffel symbols of the Poincaré patch are zero. Now plugging in the non-zero components and writing the metric and variation of the metric in terms of the linear perturbation we obtain

$$\chi|_\Sigma = \frac{z^d}{16\pi G_N}[\epsilon^t_z[(\frac{2\pi z}{R}+\frac{d}{z}\xi^t+\xi^t\partial_z)h^i_i]+\epsilon^t_i[(\frac{2\pi x^i}{R}+\xi^t\partial^i)h^i_i-(\frac{2\pi x^j}{R}+\xi^t\partial^j)h^i_j] \tag{7.21}$$

Finally to verify the first property and also E we restrict to the spherical boundary region B. On this restriction ϵ^t_i vanishes since the boundary is located at z=0 and thus dz=0

$$\chi|_B = \frac{z^d}{16\pi G_N}[d^{d-1}x\sqrt{\det g_{ind}}[(\frac{2\pi z}{R}+\frac{d}{z}\xi^t+\xi^t\partial_z)h^i_i]|_{z=0} \tag{7.22}$$

where g_{ind} denotes the induced metric on B. Now using $\sqrt{\det g_{ind}} = L^{d-1}/z^{d-1}$

$$\begin{aligned}\chi|_B &= \frac{z^d}{16\pi G_N}[d^{d-1}x\frac{L^{d-1}}{z^{d-1}}[(\frac{2\pi z}{R}+\frac{d}{z}\xi^t+\xi^t\partial_z)h^i_i]|_{z=0}\\ &= \frac{L^{d-1}}{16\pi G_N}[d^{d-1}x[(\frac{2\pi z^2}{R}+\frac{d^t}{\xi}+\xi z^t\partial_z)h^i_i]|_{z=0}\\ &= \frac{dL^{d-1}}{16\pi G_N}d^{d-1}x(R^2-\vec{x}^2)h^i_i\end{aligned}$$

which is the integrand of (6.64)

Restricting to the extremal bulk surface $\tilde{B}$ on the t=0 timeslice and using that the extremal pure AdS surface is parametrized by $\vec{x}+z^2=R^2$ gives us

$$\begin{aligned}\chi_{\tilde{B}} &= \frac{L^{d-1}}{16\pi G_N}d^{d-1}x[\frac{2\pi z^2}{R}h^i_i+x^i(\frac{2\pi x^i}{R}h^i_i-\frac{2\pi x^j}{R}h^i_j)]|_{\vec{x}^2+z^2=R^2}\\ &= \frac{z^d}{8RG_N}[d^{d-1}x(R^2h^i_i-x^ix^jh_{ij})\end{aligned}$$

we used that ξ_t vanishes on the extremal surface $\tilde{B}$ and that the other volume form is given by $\epsilon^t_i = \frac{L^{d-1}x^i}{z^d}d^{d-1}x$. This confirms that the restriction to $\tilde{B}$ gives the integrand of (6.59).

To verify the second property of χ we compute the exterior derivative restricted to Σ. Notice from (7.21) that $\chi|_\Sigma$ is of the form $\chi = a\epsilon_{zt}+\sum_i b_i\epsilon_{it}$. This allows us to take the exterior derivative and turn it into an expression that we can explicitly calculate. Namely

$$\begin{aligned}d\chi|_\Sigma &= d(a\epsilon_{zt}+b^i\epsilon_{it})\\ &= -d(a\sqrt{-\det g_{ind}}\bar{\epsilon}_{tz}+b^i\bar{\epsilon}_{ti})\\ &= -[\partial_z(a\sqrt{-\det g_{ind}})+\partial_i(b^i\sqrt{-\det g_{ind}})]\bar{\epsilon}_t\end{aligned} \tag{7.23}$$

where in going to the third line we defined $\bar{\epsilon}_{ab} = \epsilon_{ab}/\sqrt{\det g_{ind}}$ and we switched the indices of the volume form such that the t-component remains when taking the exterior derivative. The last equality just follows from the definition of the exterior derivative.

The two partial derivatives are given by

$$\begin{aligned}\partial_z(a\sqrt{-\det g_{ind}}) &= \partial_z \frac{L^{d-1}}{z^{d-1}} \frac{z^d}{16\pi G_N L^2}(\frac{2z\pi}{R} + \frac{d}{z}\xi^t + \xi^t\partial_z)h^i_i \\ &= \frac{L^{d-1}z}{16\pi G_N}(\frac{2\pi}{R}(2-d) + \frac{\xi^t}{z}(d+1)\partial_z + \xi^t\partial_z^2)h^i_i\end{aligned}$$

and

$$\begin{aligned}\partial_i(b^i\sqrt{-\det g_{ind}}) &= \partial_i \frac{L^{d-1}}{z^{d-1}} \frac{z}{16\pi G_N L^2}((\frac{2\pi x^i}{R} + \xi^t\partial^i)h^i_i - (\frac{2\pi x^j}{R} + \xi^t\partial^j)h^i_j) \\ &= \frac{L^{d-1}z}{16\pi G_N}((\frac{2\pi}{R}(d-2) + \xi^t\partial_i\partial^i)h^i_i - \xi^t\partial_i\partial^j h^i_j)\end{aligned}$$

Now we can combine them and plug them back into (7.23)

$$d\chi|_\Sigma = -\frac{L^{d-1}z}{16\pi G_N}[(\frac{\xi^t}{z}(d+1) + \xi^t\partial_z^2 + \xi^t\partial_i\partial^i)h^i_i - \xi^t\partial_i\partial^j h^i_j]\bar{\epsilon}_t \tag{7.24}$$

In order to identify the term δE^g_{tt} from the exterior derivative $d\chi|_\Sigma = -\frac{1}{8\pi G_N}\xi^t_{AdS}E^G_{tt}\epsilon^t$ we need to set

$$\bar{\epsilon}_t = g_{tt}\frac{z^{d-1}}{L^{d-1}}\epsilon_t = -\frac{z^{d+1}}{L^{d+1}}\epsilon^t \tag{7.25}$$

after which we now finally get the result

$$d\chi|_\Sigma = -\frac{z^d L^{2-d}}{16\pi G_N}\xi^t_{AdS}[-(\frac{d+1}{z}\partial_z + \partial_z^2 + \partial_i\partial^i)h^i_i + \partial_i\partial^j h^i_j]\epsilon^t \tag{7.26}$$

Now comparing this result with the exterior derivative we read of

$$\delta E^g_{tt} = -\frac{z^d L^{2-d}}{32\pi G_N}(\partial_z^2 h^i_i + \frac{d+1}{z}\partial_z h^i_i + \partial_j\partial^j h^i_i - \partial^i\partial^j h_{ij}) \tag{7.27}$$

which we now recognize as (4.14), the (tt)-component of the linearized Einstein equations!

Chapter 8

Beyond linearized gravity

Different attempts have been made to find the equations of gravity beyond linear order. In this section we will restrict ourselves to the case where the bulk gravity theory is described by general relativity. First we will heuristically consider extensions to the derivation above to non-linear order and secondly a different derivation by [26] yielding the full Einstein equations will be considered.

1/N corrections to the Ryu-Takyanagi formula

The first law on the CFT side is an exact result which holds to any order in the perturbation. If we however include order 1/N corrections to the CFT side we will have to include matter fields in the bulk [25]. The AdS side gets described by a classical spacetime with quantum matter fields $|\Phi\rangle$. The Ryu-Takyanagi formula takes on a new form [25] in this region due to entanglement of the bulk matter fields over the extremal area $\tilde{B}$ between the region Σ and the rest of the bulk

$$S_B|\Phi\rangle = \frac{A(\tilde{B})}{4G_N} + S_\Sigma(|\Phi\rangle) \tag{8.1}$$

This allows us to perform the translation of the CFT first law to the holographic first law analogous to the derivation done in section 6.3 and we will consider the derivation of [24]. The variation of the entanglement entropy thus corresponds to the original variation of $\tilde{B}$ and the variation of the bulk entanglement entropy over that surface. This entanglement entropy in turn depends on the variation of the extremal surface, which we already argued is zero, and the variation of the bulk fields. Thus we only need to focus on the variation of entanglement entropy due to the variation of the bulk fields.

The variation of entanglement entropy is now

$$\delta S_\Sigma(|\Phi\rangle) = -tr(\delta\rho_\Sigma^{bulk} \log \rho_\Sigma^{bulk}) \tag{8.2}$$

in [24] they then showed that for the bulk region the bulk modular hamiltonian can be written as a local expression

$$H_\Sigma = \int_\Sigma \xi^a T_{ab}^{bulk} \epsilon^b \tag{8.3}$$

giving us finally

$$\delta_\Sigma(|\Phi\rangle) = \int_\Sigma \xi^a \delta \left\langle T_{ab}^{bulk} \right\rangle \epsilon^b \tag{8.4}$$

Moreover they argued that the variation to the holographic energy $\delta_B E^{grav}$ is unchanged when considering $1/N$ corrections. The first law is now again derived from the first law by

$$\begin{aligned}\delta S^{grav} - \delta E^{grav} &= \int_\Sigma (-2\xi^a \delta E^g_{ab} \epsilon^b - \xi^a \delta \langle T^{bulk}_{ab} \rangle \epsilon^b) \\ &= \int_\Sigma -2\xi^a (\delta E^g_{ab} - \frac{1}{2} \delta \langle T^{grav}_{ab} \rangle) \epsilon^b = 0\end{aligned}$$

now by the identical argument as in section 7.2 considered before this has to hold for all spherical boundary regions B which means that the integrand has to vanish. This implies that

$$\delta E^g_{ab} = \frac{1}{2} \delta \langle T^{grav}_{ab} \rangle \tag{8.5}$$

so for a holographic dual description in terms of a classical spacetime with quantum bulk fields we find that the linearized Einstein equations includes a source term.

Equations of gravity to second order from first law

In a recent paper by Van Raamsdonk and Faulkner et al. [27] it was argued that the gravitational equations can be derived up to second order. Using the Holland-Wald gauge [29] they showed that up to any order in λ we have

$$\int_\Sigma d\chi(\phi, \frac{d}{d\lambda}\phi) = \frac{d}{d\lambda}(E^{grav}_B - S^{grav}_B) \tag{8.6}$$

for the AdS black hole spacetime that we also considered in this thesis. By considering the variation of the relative entropy up to second order they found that the gravitational equations are satisfied as long as the geometry of the spacetime $g_{\mu\nu}(\lambda)$ gives the entanglement entropy in the CFT.

Full Einstein equations by Jacobson

In 1995 Jacobson published a paper where he derived the full Einstein equations as an equation of state [26]. It served as an inspiration for the derivation of the linearized Einstein equations covered in this thesis and as we shall see the derivations have a lot of similarities. First of all entropy is assumed proportional to the area and secondly the heat, entropy and temperature are assumed to be related by:

$$\delta Q = T dS \tag{8.7}$$

Now at every spacetime point p there exists some local Rindler horizons. Note that these are simply causal horizons caused by the accelerated reference frame not by black holes. These will have the role of the diathermal wall in standard thermodynamics. That is, they form the barrier between the "system" outside the horizon and the local environment. δQ is the energy flux across the horizon and T the Unruh temperature seen by the local Rindler observer.

Since these causal horizons hide information of the system, mostly in correlations between the local and external system, they have a natural association with entropy. This led to the assumption that the entropy should be proportional to the horizon area of the Rindler observer.

We find, for the details of the calculation we refer to [26], for both sides of (8.7) that

$$\delta Q = -\kappa \int_{\mathcal{H}} T^{\mu\nu} k_\mu k_\nu \lambda d\lambda dA = T dS = \frac{h\kappa}{2\pi} \eta \delta A \tag{8.8}$$

now the change in area is found by the Raychaudhuri equation such that

$$\delta A = -\kappa \int_{\mathcal{H}} R^{\mu\nu} k_\mu k_\nu \lambda d\lambda dA \tag{8.9}$$

for the null vectors k_μ generating the horizon we thus have

$$\frac{2\pi}{\eta h} T^{\mu\nu} k_\mu k_\nu = R^{\mu\nu} k_\mu k_\nu \tag{8.10}$$

Now for general k_μ we can write this as

$$\frac{2\pi}{\eta h} T^{\mu\nu} = R^{\mu\nu} + f g_{\mu\nu} \tag{8.11}$$

since $g_{\mu\nu} k^\mu k^\nu = 0$ for null vectors. The scalar function f can be shown to be equal to $R/2 + \Lambda$ and with the proportionality constant $\eta = 1/4Gh$ we find the full Einstein equations when plugging this into the above equation.

What the derivation by Jacobson fails to explain however is the reason as to why the first law should hold for local causal horizons in the first place. The origin of the thermodynamic quantities used is not supported by a microscopic argument. On the contrary we proved the microscopic origin of the first law in terms of entanglement entropy and the expectation value of the modular hamiltonian. Since we are considering global AdS-Rindler horizons instead of local horizons it might is interesting to see whether our argument can be generalized to local AdS-Rindler horizons. Moreover since the first law on the CFT side is an exact result it could in principle be used to study quantum corrections to the classical theory in the context of AdS/CFT.

Chapter 9

Conclusion and outlook

In this thesis it was tried to make plausible that quantum entanglement is fundamentally important for the existence of space and gravity. Firstly we showed how the Ryu-Takyanagi conjecture defined a geometrical quantity holographically equivalent to the entanglement entropy. It thus suggests that entanglement in some sense binds spacetime since without entanglement all the microscopic degrees of freedom would be disconnected.

Secondly we have seen how the dynamical nature of spacetime was encoded in the CFT by considering an exact quantum first law. By translating this first law to a bulk constraint on the metric perturbation and by introducing the Iyer-Wald formalism it was shown that this holographic first law implies that the linearized equations of any diffeomorphic invariant theory of gravity are satisfied.

So far we have only been able to derive the linearized equations. We have seen that it is even possible to find the Einstein equations to second order but it seems that we need something more to find the full Einstein equations. The result by Jacobson seemed promising because it resulted in the full Einstein equations. The microscopic origin of the local Rindler horizons is however not well understood. An interesting avenue of research would be to see if we can give a microscopic explanation in the CFT for the first law of local Rindler horizons.

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