University of Groningen

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Master Thesis

Exploring the Unitarity of Renormalizable Quantum Gravity Theories

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January 9, 2019

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Abstract

We investigate higher derivative theories of quantum gravity. In particular we study a new prescription to make gravitational theories quadratic in the curvature, shown by Stelle to be renormalizable, into unitary theories. The prescription, designed by Anselmi et al.[1], turns the ghosts that are present in higher derivative gravity, into a new type of particles, namely fakeons. This not directly observable particle appears in loops and thus contributes to the renormalization of the theory, but does not appear in the asymptotic states, so it does not contribute to the S-matrix of the theory.

Acknowledgements

I would like to thank Elisabetta Pallante for supervising me during this project, for continuous support during the research for the thesis and the writing of it. Next to her I would like to thank Anupam Mazumdar for being the second supervisor and useful discussions. I would also like to thank Marco Boers, Casper Dijkstra and Ivo Ertman for useful discussions during the research of my thesis. I am of course also grateful for my friends and family for their support and lifting me up when the project of writing a thesis got me a bit down. Thank you.

Contents

1 Introduction to Quantum Gravity

Quantizing gravity, as we would normally do, leads to a unitary theory, but also a theory that is not renormalizable. Proposals have been made to formulate gravity in a renormalizable way, but this led to the loss of unitarity. The task for formulating a proper theory of quantum gravity lies thus in combining renormalizability with unitarity. In this thesis we investigate a new attempt to do this, recently proposed by Anselmi et al [1]. This new attempt utilizes a new quantization prescription based on Lee-Wick models. In this introduction we first discuss the problems of quantizing gravity as we would naively do. At the end of this introduction there is an outline of the thesis.

1.1 Problems of quantum gravity from a field theoretic perspective

An important problem of quantum field theories is that we can only consider them as being consistent up to arbitrary high energies if they are renormalizable[1], that is, the UV divergencies and the counterterms associated with them we encounter when doing perturbation theory only renormalize the term already present in the Lagrangian. This means that there are no new operators generated after renormalization. Another constraint that we put on (quantum) field theories is that we require them to be unitary. We discuss the renormalizability of the Einstein-Hilbert action and address the importance of unitarity.

Renormalizability
We consider the Einstein-Hilbert action in d dimensions:

$$S_{EH} = \frac{1}{2\kappa^2} \int d^d x \sqrt{-g} \left(R - 2\Lambda_C \right), \tag{1.1}$$

or, more conveniently at this moment for our investigation, the same action without the cosmological constant Λ_C

$$S_{EH} = \frac{1}{2\kappa^2} \int d^d x \sqrt{-g} R. \tag{1.2}$$

In this action κ is defined as $\sqrt{8\pi G}$, in which G is the Gravitational constant, g is the determinant of the metric tensor $g_{\mu\nu}$ and R is the Ricci scalar(i.e. the Ricci tensor contracted with the metric $R = g^{ab} R_{ab}$ and the Ricci tensor is the trace of the Riemann tensor $R_{ab} = R^c_{abc}$). To see whether (1.2) is renormalizable, we can use a power counting argument[2]. We derive the contributions of the Ricci scalar R to the superficial degree of divergence D. First we note that for each loop integral there is a momentum integration

[1] We can view theories that are not renormalizable in full, such as the Standard Model(which itself is renormalizable) plus extensions to explain phenomena not explained by the Standard Model(which can be non-renormalizable theories), as a renormalizable part and a part that acts as an effective field theory. We then use a cutoff for the non-renormalizable part up to which the theory is valid. We would like, however, to have a theory that is completely renormalizable.

over d^dk, thus giving a contribution of dL to D for L loops. Second we note that the Ricci scalar contains two derivatives of the metric, so that the graviton propagator goes like $\frac{1}{k^2}$ and the vertex function like k^2 so they contribute $-2I$ and $2V$, respectively, where I denotes the number of internal lines and V denotes the number of vertices. This leads to the superficial degree of divergence of a generic Feynman diagram of L loops i d spacetime dimensions

$$D = dL + 2V - 2I. \tag{1.3}$$

We can make use of the topological relation for the number of loops

$$L = 1 + I - V \tag{1.4}$$

to eliminate V and I in (1.3) to write it as

$$D = 2 + (d - 2)L, \tag{1.5}$$

which for $d > 2$ gives a superficial degree of divergence that keeps growing with each order in perturbation theory, i.e. with the number of loops L. In $d = 4$ it gives $D = 2L + 2$ and this positive value of the superficial degree of divergence leads us to conclude that the theory with the action (1.2) is non-renormalizable. 't Hooft and Veltman showed in [3] that the divergencies occurring in one-loop calculations of pure gravity (1.2) in four dimensions do in fact generate counterterms proportional to R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, i.e. terms that are quadratic in the curvature. Using the fact that the Gauss-Bonnet[2] term is a total derivative in four dimensions, we can write all surviving counterterms in four dimensions in terms of only R^2 and $R_{\mu\nu}R^{\mu\nu}$. In the case of pure gravity at one loop, these counterterms can be absorbed in field redefinitions, so that no physically relevant divergencies remain and the theory is finite(at one loop)[3]. 't Hooft and Veltman also showed, however, that when the pure gravity theory is coupled to a scalar field, the finiteness that is present in pure gravity at one loop is destroyed. The explicit proof of the non-renormalizability of (1.2) was given by Goroff and Sagnotti in [4]. In this work it was shown that at two loops, the counterterm proportional to $R^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}R_{\gamma\delta}R^{\gamma\delta}R_{\mu\nu}$ could not be absorbed in the field redefinitions, hence the theory could not be made finite at two loops. The occurrence of the higher derivative terms in the counterterms, that is, the terms at least quadratic in the curvature, led to research of quantum theories of gravity that include the higher derivative terms directly in the action, as we will see in chapter 2.

Unitarity

In the search for a renormalizable quantum theory of gravity we have to ensure that the theory is unitary as well for it to be an acceptable candidate. The importance of unitarity, in particular that of the *scattering* or *S-matrix* relating initial and final states in scattering processes, has been put nicely into words by Veltman:

> "[Unitarity of the S-matrix] is truly important, because unitarity implies conservation of probability, and probability is the link between the formalism and

[2]Gauss-Bonnet $= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

observed data. If we have no unitarity, then we have nothing that can be interpreted as probability and the link to observed processes disappears." - M. Veltman [5]

Several attempts of quantizing gravity, especially those involving higher derivative terms, turn out to lead to a non-unitary theory due to the presence of ghosts, as we will see in the next chapter. As we have just learned from Veltman, it makes no sense to have a non-unitary theory, so we must make sure that a new quantum theory of gravity is unitary.

1.2 Constraints from a gravitational perspective

In addition to solving the problems from a field theoretic perspective we have to keep in mind that a new theory still has to satisfy the constraints of the classical version of General Relativity and the experimental observations scientists have made so far. We now address the ones most relevant for the theories discussed in this thesis.

Newtonian Potential
When gravitational theories with terms quadratic in curvature were first considered, experimental tests of General Relativity all followed form the Schwarzschild line element. It was thus believed that having the Schwarzschild metric as a solution of a gravitational action satisfies the same constraints as General Relativity. This is not the complete story, as having the Schwarzschild metric as a solution is only valid in empty space and we also need solutions for a non-vanishing energy-momentum tensor [6]. The most trivial check of a gravitational theory, however, is that it satisfies the Newtonian limit. That is to say, in the weak field limit it reproduces the potential

$$V(r) = -\frac{GMm}{r} \tag{1.6}$$

for masses M and m separated by a distance r, G being the gravitational constant. So a theory can deviate form this limit only in ways that we have not been able to measure yet.

Cosmological Constant
Another important aspect of a quantum theory of gravity is that it satisfies the constraint of having a cosmological constant to be in accordance with cosmological observations, most recently with Planck data [7]. Another option is that a new theory provides a new mechanism to explain the observations that we currently ascribe to the cosmological constant.

1.3 Outline of the thesis

The aim of this thesis is to review higher derivative theories of gravity and in particular to review a recent proposal by Anselmi[1] to make the renormalizable but non-unitary higher derivative theory by Stelle [8] into a both renormalizable *and* unitary theory by means of a new prescription. To do this we proceed in the following way.

In chapter 2 we first discuss the original proposal by Stelle in section 2.1 to learn why this theory is interesting, but also to see why it cannot be a physical theory. In section 2.2 we briefly address a special case of higher derivative gravity that is *Conformal Gravity*. There we will see how the classical constraints prevent us from pursuing any direction we like. We will also turn our attention to a theory called *Agravity* by Salvio and Strumia [9] in section 2.3 to explore the possibilities of designing a theory of quantum gravity that also solves other problems in particle physics and cosmology.

In chapter 3 we arrive at the main section of this thesis, namely the one concerning the new prescription by Anselmi [1] to turn the theory by Stelle [8] into a unitary theory. The new prescription originates from the Lee-Wick theory [10] designed to formulate a finite theory of Electrodynamics in the 1960's, so we first review that proposal and the problems it encountered[11] in section 3.1. We then turn our attention to the new formulation of that theory by Anselmi [12] in section 3.2 and study the unitarity of the bubble diagram in a toy model in section 3.2.3. Finally we see how we can use this new prescription to make the Stelle theory unitary in section 3.3.2 and we dissect the different components of the original action to see what possibilities the new prescription has to offer in section 3.3.4. We end this chapter with the first phenomenological aspects of the new prescription.

We conclude in chapter four with a discussion of the general aspects of higher derivative theories of gravity that have been discussed in the previous chapters. We also present a brief outlook on the new prescription by Anselmi and the obstacles it has yet to overcome to be considered a viable candidate of quantum gravity.

2 Higher derivative gravity

Knowing that the quantization of the Einstein-Hilbert action (1.2) generates counterterms that are quadratic in the curvature, thus containing derivatives of the metric tensor higher than the ones in the Einstein-Hilbert action (1.2), inspired physicists to consider theories of gravity that already contain such terms in the classical action in addition to the Einstein-Hilbert term(hence the name *Higher derivative gravity*). These new terms should be useful to regulate the ultraviolet(high-energy) divergencies, hopefully leading to a renormalizable theory, while having little effect on the infrared(low-energy) behavior of the theory.

2.1 Stelle Theory

Such a theory was indeed shown to be renormalizable by Stelle in 1977 and the proof thereof can be found in [8]. Gravity quadratic in the curvature in general has this form:

$$S_{quadratic} = -\int d^4x\sqrt{-g}\left[\alpha' R^2 + \beta' R_{\mu\nu}R^{\mu\nu} + \delta R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \kappa^{-2}\gamma R\right] \tag{2.1}$$

with R^2 the Ricci scalar squared, $R_{\mu\nu}R^{\mu\nu}$ the Ricci tensor squared and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ the Riemann tensor squared, i.e. all terms quadratic in curvature. The couplings α', β', δ γ are all dimensionless numbers and κ is defined as in (1.2). The term $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ in four dimensions can be related to the Gauss-Bonnet term

$$Gauss - Bonnet = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}_{\rho\sigma}, \tag{2.2}$$

in which $\epsilon_{\alpha\beta\gamma\delta}$ is the antisymmetric Levi-Civita tensor. In four dimensions, the Guass-Bonnet term is a total derivative and does not contribute to the field equations. It is usually ignored in the action, assuming that the fields vanish sufficiently fast so that the integral of the total divergence is zero. So we can make the replacement

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4R_{\mu\nu}R^{\mu\nu} - R^2 \tag{2.3}$$

in (2.1) to obtain the gravitational action shown to be renormalizable by Stelle:

$$S_{Stelle} = -\int d^4x\sqrt{-g}\left[\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \kappa^{-2}\gamma R\right] \tag{2.4}$$

in which the couplings α, β and γ are dimensionless numbers.

This gravitational action is associated with the following gravitational potential:

$$V(r) = -\frac{GMm}{r}\left(1 + \frac{1}{3}e^{-m_0 r} - \frac{4}{3}e^{-m_2 r}\right) \tag{2.5}$$

in which the constants m_0 and m_2 are related to the couplings of the theory as follows:

$$m_0 = \left([3\beta - \alpha]\kappa^2\right)^{-1/2} \quad \text{and} \quad m_2 = \left(\frac{1}{2}\alpha\kappa^2\right)^{-1/2}. \tag{2.6}$$

So we see that the gravitational potential associated with a point source in this theory has one contribution that is Newtonian, the first term, and two contributions that are Yukawa-like potentials, the latter two terms in (2.5). One requirement is that the masses m_0 and m_2 are real, otherwise there would be an oscillating potential and the particles associated with these masses would be tachyons[3] The absence of tachyons thus implies $\alpha > 0$ and $\beta > \frac{1}{3}\alpha$. The complete potential has an appropriate Newtonian limit at large distances, since the effects due to the Yukawa terms would only be visible at length scales of order $\frac{1}{m_0}$ and $\frac{1}{m_2}$.

We might be tempted to consider a theory with only terms quadratic in the curvature and omit the Einstein-Hilbert term. While this is interesting to investigate, and we will do so in the next two sections of this chapter, it does not lead to a gravitational potential like the one in (2.5). The potential would actually be linearly proportional to the distance between two point masses, instead of being inversely proportional, as Stelle points out[13].

There are eight degrees of freedom present in the action (2.1), two of which are propagating via the massless spin-2 graviton that we know from quantizing just the Einstein-Hilbert action. Next to the graviton there is one degree of freedom propagating in the form of a massive scalar with which we associate the mass m_0 in (2.6). The other five degrees of freedom are in a massive spin-2 particle with which we associate the mass m_2 in (2.6). This contribution turns out to be problematic for the theory as it has a negative kinetic energy, a property of higher derivative theories in general(see Appendix A). Such a particle is known as a ghost. In quantum field theory we can transform this negative energy state into a negative norm state, resulting in a negative contribution to the total probability. The negative probabilities are responsible for the loss of unitarity, which we know to be a very important feature of a proper quantum field theory. The non-unitary nature of the higher derivative theories of gravity was known by Stelle when showing the renormalizability of such theories and *"no sensible physical interpretation of these higher derivative models"* is possible before the unitarity problem is resolved[8]. The main part of this thesis, chapter 3 in particular, is such an attempt to resolve the unitarity problem in higher derivative models by quantizing the ghosts in a new way. Before turning to that attempt we will first for a moment turn our attention to other directions within the higher derivative models of gravity, that do not specifically address the ghost problem but are useful for the general understanding of the following chapters, namely conformal gravity and "agravity".

2.2 Conformal Gravity

Conformal gravity, or conformal Weyl gravity, is a theory of gravity that is invariant under conformal transformations and thus possesses no scales. The theory is based on the Weyl tensor(see equation 2.7), named after Hermann Weyl, who already in 1918 showed that this tensor has remarkable properties[14]. But, as already mentioned, such a theory by itself does not contain any scales and this was considered unacceptable at the time of its

[3]Tachyons are particles that, if they were to exist, move faster than the speed of light at all times.

construction. Decades later the process of spontaneous symmetry breaking was used to implement scales in a scaleless theory, which revived interest in conformal gravity - see Mannheim[15] and references therein for a review. We will in this section briefly discuss some properties of conformal gravity and in the next section we discuss whether conformal gravity could be the high energy limit of a new theory called "agravity".

The Weyl tensor $C_{\mu\nu\rho\sigma}$ in d dimensions is given by

$$\begin{aligned} C_{\mu\nu\rho\sigma} =& R_{\mu\nu\rho\sigma} - \frac{1}{(d-2)}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma}\right) \\ &+ \frac{1}{(d-2)(d-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right)R \end{aligned} \tag{2.7}$$

in which $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. The action for conformal Weyl gravity in d=4 constructed using the Weyl tensor with coupling $\zeta_{cw}$is

$$\begin{aligned} S_{Weyl} =& -\zeta_{cw}\int d^dx\sqrt{-g}\left[C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right] \\ =& -\zeta_{cw}\int d^dx\sqrt{-g}\left[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right] \end{aligned} \tag{2.8}$$

and by now we know that in four dimensions we can get rid of the Riemann tensor squared by using the fact that the Gauss-Bonnet is a total derivative(see equation 2.3) so we insert (2.3) into (2.8) to obtain

$$\begin{aligned} S_{Weyl} =& -\zeta_{cw}\int d^4x\sqrt{-g}\left[2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2\right] \\ =& -2\zeta_{cw}\int d^4x\sqrt{-g}\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right]. \end{aligned} \tag{2.9}$$

For later use we define

$$\frac{1}{2}W^2 = \left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right). \tag{2.10}$$

Although this theory is renormalizable due to specific structure of the higher derivatives contained in it, we have learned that unitarity is lost precisely due to the presence of those terms. Recently the case was made that unitarity in this theory can be recovered due to Parity-Time symmetry[15]. Even if this is the case, the problem that remains with such a theory lies in the fact that *only* terms with four derivatives are present, i.e. the Einstein-Hilbert term is left out. This leads to a gravitational potential linearly proportional to instead of inversely proportional to the distance between masses, as noted by Stelle in [8]. Adding the Einstein-Hilbert term, however, would destroy the conformal symmetry of the theory and would return the Stelle theory with a particular choice of the coefficients α and β in (2.4).

2.3 Agravity

Before considering a possible solution for the loss of unitarity due to the presence of ghosts in the higher derivative theory with action (2.1) in the next chapter, we first consider a theory called *Agravity* to explore the possibilities of designing a theory that solves other problems in particle physics and cosmology.

2.3.1 Motivation

Higher derivative quantum field theories offer the possibility of having a renormalizable quantum theory of gravity. Within the branch of higher derivative field theories, however, we still need other arguments to choose specific theories to investigate. This allows us to look for theories that have beneficial properties other than being renormalizable. Salvio and Strumia [9] proposed a theory based on the general principle that nature does not possess any scale. The principle is based on two arguments which in itself are supported by fairly recent experimental results. The first argument concerns naturalness and the second argument concerns inflation.

Naturalness

The criterion of naturalness is something that is present in many fields of science, but in no other field is it as influential as in particle physics [16]. Our current theories describing the Standard Model are considered to be effective field theories. This means they are low-energy approximations of some hypothesized more fundamental theory that we do not know (yet). These low-energy descriptions are valid up to some ultraviolet energy scale Λ, above which the details of the more fundamental theory play a non-negligible role. Many variations of the naturalness criterion exist, but a rather clean and quantifiable version, dubbed *Technical Naturalness* was devised by 't Hooft[17]. He utilizes the notion of effective field theory in the following way: All the parameters of a theory, made dimensionless by measuring them in units of Λ(and using "natural units" $\hbar$=c=1), should be of order unity unless setting the parameter to zero increases the symmetry of the theory. If this condition is not satisfied, the theory is said to violate the principle of naturalness or be an unnatural theory. Whether this is a good principle to follow is debatable, but the fact is that many physicists prefer a natural theory over an unnatural theory. But how does naturalness, or rather the violation thereof, manifest itself in particle physics? One of the biggest achievements of naturalness is the prediction of the existence of the charm quark before its discovery[18]. The mass of the charm quark was even predicted by invoking naturalness, so it can be a very useful asset in theory choice.

In effective field theories physical scalar quantities are expressed as the sum of a bare quantity(q_{bare}) and contributions from quantum fluctuations(Δq)[19]

$$q_{physical} = q_{bare} + \Delta q \tag{2.11}$$

These quantum fluctuations depend on the energy scales so in effective field theories they will contribute up to the cutoff energy Λ. We can regard the bare quantity as a black box that includes the effects associated with the physics that happens above the cutoff energy Λ. What naturalness would suggest is that the value of $q_{physical}$ should not be highly dependent on a delicate cancellation of q_{bare} and Δq, because this would mean the low energy physics is highly dependent on the high energy physics. To see how this type of violation of naturalness occurs in the Standard Model, we turn to the mass of a scalar (spin-0) particle. The Standard Model has one fundamental electrically neutral scalar: the Higgs boson. The dependence of the squared mass of the Higgs boson on the UV cutoff is an example of(2.11):

$$M^2_{physical} = M^2_{bare} + \Delta M^2$$

The quantum corrections ΔM^2 to the squared mass depend quadratically on the cutoff energy Λ in any regularization that preserves power-like divergencies. The Higgs mechanism gives mass to all the fermions in the Standard Model, and since the top quark is the heaviest it has the largest Yukawa coupling. The dominant contribution to the quantum fluctuations thus comes from the top quark, such that we have:

$$M^2_{physical} = M^2_{bare} - \frac{\kappa_t^2}{16\pi^2}\Lambda^2 + ...$$

where the dots denote contributions from other, lighter, particles and κ_t determines the strength of the Yukawa interaction between the top quark and the Higgs boson. The fraction $\kappa_t^2/16\pi^2$ is known to be of the order 10^{-2}. The physical mass of the Higgs boson is known to be approximately $M_{physical} \simeq$125 GeV. When looking for an appropriate value for the cutoff energy of the Standard Model we have to take into account its successful features such as conservation of baryon number and lepton number, which leads to a cutoff that is larger than 10^{16}GeV. Quantum gravitational effects are expected to play a role above the Planck scale $M_{Planck} \simeq 10^{19}$GeV such that we can regard gravity as a non-renormalizable interaction suppressed by a cutoff $\Lambda \sim 10^{19}$GeV. If we take the actual cutoff of the Standard model to be the Planck scale there would need to be a cancellation to 34 orders of magnitude between the squared bare mass and the quantum fluctuations to obtain a physical mass of 125 GeV. This is far from natural and the separation between these two scales (that of the Higgs mass and the Planck scale) is called the hierarchy problem. In order to have a more natural physical Higgs mass the Standard Model would have to be modified and one of the most promising candidates has for a long time been supersymmetry[20]. The options for such a modification have decreased with each run of the LHC, as no new physics has been observed. Salvio and Strumia, for this exercise, take these observations as the final verdict and study the implications thereof. In doing so they explore the possibility of a theory that, on a fundamental level, does not possess any scales, i.e. its Lagrangian does not contain super-renormalizable nor non-renormalizable terms. This leaves us only with the renormalizable interactions, i.e. dimensionless couplings.

Inflation

The second argument to explore a quantum theory of gravity that has no fundamental scales is based on cosmological observations. From those observations we have learned that we need an inflation model with small anisotropies to describe the period shortly after the Big Bang [21]. To have such small anisotropies from quantum fluctuations would require a model with flat potentials and field values that exceed the Planck scale. To obtain such potentials, one can turn to Starobinsky-like inflation models [22], which are inflation models that can be described by a single scalar field. As we only know one fundamental scalar in the Standard Model, it is interesting to explore whether the Higgs could be the inflaton field [23]. The latter must be a scalar field S with a potential V(S) with specific properties and a coupling to gravity $-\frac{1}{2}f(S)R$ [9] in the Jordan frame. The potential in the Einstein frame is then $V_{Einstein} = \bar{M}^4_{Planck}V(S)/f(S)^2$.[4] This potential $V_{Einstein}$ becomes flat and gives predictions compatible with results from Planck data[7] for V(S) $\propto f^2(S)$ at S $\gg M_{Planck}$. But the flatness of this potential is, as Salvio and Strumia claim [9], the result of a fine-tuning[5], because in the presence of Planck suppressed operators, V(S) and f(S), and thus $V_{Einstein}$, are themselves functions of S/M_{Planck}. Having a flat potential $V_{Einstein}$ for S $\gg M_{Planck}$ is considered by Salvio and Strumia as a clue dropped by nature that only adimensional couplings exist.

2.3.2 Adimensional couplings

Since naturalness is a problem of the Standard Model formulated as an effective field theory, let us look at the Higgs potential terms of the Standard Model Lagrangian:

$$\mathscr{L} \sim \Lambda^4 + \Lambda^2|H|^2 + \lambda|H|^4 + \frac{|H|^6}{\Lambda^2} + ... \tag{2.12}$$

As $\Lambda \gg M_{Higgs}$ it makes sense that we only see those terms that are not suppressed by powers of Λ. The idea of having only adimensional couplings in the Lagrangian leaves only the $\lambda|H|^4$ term of the Higgs potential. The complete theory is build upon the simple principle: "*The fundamental theory of nature does not possess any mass or length scale and thereby only contains 'renormalizable' interactions - i.e. interactions with dimensionless couplings*"[9]. Now the problems that led to this idea, naturalness and inflation, are solved by construction. The only term in (2.12) satisfying this constraint is $\lambda|H|^4$, hence the power divergencies of the Higgs mass are not present. For the inflation potential we also only have a quartic term $V(S) = \lambda_S|S|^4$ and its coupling to gravity is, by dimensional reasoning, determined to be $-\xi_S|S|^2R$ which leads to $V_E = \bar{M}^4_{Planck}(\lambda_S|S|^4)/(\lambda_S|S|^2)^2 = \bar{M}^4_{Planck}\lambda_S/\xi_S^2$. This potential is flat at tree-level and the couplings run at the quantum level, such that their beta-functions can take the role of the slow-roll parameters. The scale invariance that we have in such a theory at the classical level is an accidental symmetry, as there are no masses present, and is broken at the quantum level by quantum corrections, just like lepton and baryon number conservation is broken in the Standard Model. The different scales that we encounter in the Standard Model arise through what is called

[4] $\bar{M}_{Planck}$ is the reduced Planck mass: $\bar{M}_{Planck} = M_{Placnk}/\sqrt{8\pi} = 2.4 * 10^{18}GeV$

[5] Naturalness, as described in the previous subsection, can be seen as a special case of the more general notion of fine-tuning.

dimensional transmutation [24]. This process comes down to trading a degree of freedom for a mass scale via the Coleman-Weinberg mechanism[25].

2.3.3 Agravity

The theory that we will consider consists of those gravitational terms that are renormalizable and of dimension D, all couplings being dimensionless.

$$S_{agravity} = \int d^4x\sqrt{-g}\left[\mathscr{L}_{agravity} + \mathscr{L}_{matter}\right] \tag{2.13}$$

The gravitational part consists of quadratic terms, divided by their dimensionless couplings:

$$\mathscr{L}_{agravtiy} = \frac{R^2}{6f_0^2} + \frac{\frac{1}{3}R^2 - R_{\mu\nu}^2}{f_2^2} \tag{2.14}$$

which is the most generic form that is relativistically invariant. The quadratic curvature R^2 term is divided by the gravitational coupling f_0^2. The second term, $(\frac{1}{3}R^2 - R_{\mu\nu}^2)$, is the Weyl tensor squared from (2.10). In the case of agravity this term is divided by f_2^2. As discussed, it does not contain the Einstein-Hilbert term $-M_{Planck}^2 R/16\pi$ in the gravitation sector nor does it contain the Higgs mass term $\frac{1}{2}M_H^2|H|^2$ in the matter sector, as these would introduce a dimensionful parameters.

The matter Lagrangian in (2.13) contains only the adimensional part of the Standard Model (SM)

$$\mathscr{L}_{SM}^{adimensional} = -\frac{F_{\mu\nu}^2}{4g^2} + \bar{\psi}i\not{D}\psi + |D_\mu H|^2 - (yH\bar{\psi}\psi + h.c.) - \lambda_H|H|^4 - \xi_H|H|^2R \tag{2.15}$$

in which $-\frac{F_{\mu\nu}^2}{4g^2}$ and $\bar{\psi}i\not{D}\psi$ denote the interactions of the gauge group $SU(3)\otimes SU(2)\otimes U(1)$ and those interactions extended to all quarks and gluons, respectively. One can also add terms beyond the Standard Model(BSM) to the action (2.13). Salvio and Strumia, for example, add a scalar singlet S to the action:

$$\mathscr{L}_{BSM}^{adimensional} = |D_\mu S|^2 - \lambda_S|S|^4 + \lambda_{HS}|S|^2|H|^2 - \xi_S|S|^2R. \tag{2.16}$$

Such a scalar can be responsible for generating the Planck scale via dimensional transmutation, as mentioned in section 2.3.2, by obtaining a vacuum expectation value such that $\xi_S\langle S\rangle^2 = M_{Planck}^2/16\pi$. The total action is then:

$$S_{agravity} = \int d^4x\sqrt{-g}\left[\frac{R^2}{6f_0^2} + \frac{\frac{1}{3}R^2 - R_{\mu\nu}^2}{f_2^2} + \mathscr{L}_{SM}^{adimensional} + \mathscr{L}_{BSM}^{adimensional}\right] \tag{2.17}$$

As the gravitational kinetic terms contain four derivatives, the corresponding graviton propagator is proportional to $1/p^4$ in the case of a vanishing Planck mass M_{Planck}, which is what we desire from a renormalization perspective. The propagator related to the Weyl term suppressed by f_2^2 in (2.17), in the presence of a non-zero Planck mass M_{Planck}, is proportional to:

$$\frac{1}{M_2^2 p^2 - p^4} = \frac{1}{M_2^2}\left[\frac{1}{p^2} - \frac{1}{p^2 - M_2^2}\right] \tag{2.18}$$

resulting in a massless spin-2 graviton and where $M_2^2 = \frac{1}{2} f_2^2 \bar{M}_{Planck}^2$ is the mass of a spin-2 state with negative norm. There is also a spin-0 graviton related to the R^2 term suppressed by f_0^2 with a mass $M_0^2 = \frac{1}{2} f_0^2 \bar{M}_{Planck}^2 + ...$(The dots denote possible contributions from mixing with other scalars) and positive norm. The negative norm of the spin-2 state, a bad ghost(in contrast with Fadeev Popov ghosts), is a consequence of formulating gravity using terms quadratic in the curvature or quartic derivative terms, as mentioned in section 2.1. It is problematic as it is responsible for breaking unitarity, but consequences of formulating a theory in this way will be evaluated nonetheless. A possible way to treat ghosts in higher derivative theories will be explored in detail in chapter 3, but for now, as Salvio and Strumia put it "Sometimes in physics we have the right equations before having their right interpretation".

2.3.4 Renormalization group equations of Adimensional Gravity

Although we have formulated a renormalizable quantum theory of gravity, we still need to establish the UV-behaviour of the theory and see whether it is well-behaved or not. To do this, we will look at the gravitational couplings of the theory, f_0^2 and f_2^2. A first thing to note is the rather unusual way to denote couplings as squared parameters in the denominator of a term. In the previous section we saw the masses of the graviton states involving f_0^2 and f_2^2 and to avoid tachyonic instabilities for these states we require $f_0^2 > 0$ and $f_2^2 > 0$. These couplings are related to the ones in the Stelle theory considered in (2.4) by

$$f_0^2 = \frac{1}{2\alpha - 6\beta}, \qquad f_2^2 = -\frac{1}{\alpha} \tag{2.19}$$

The renormalization group equations (RGE's) to one loop can be determined independent of the renormalization scheme. The first more or less successful attempt to determine the RGE's of the gravitational couplings in higher derivative gravity was done by Fradkin and Tseytlin[26][27], but still contained a sign error in the RGE for f_0^2. The first completely correct calculation for pure gravity was done by Avramidi and Barvinsky[28]. We report here the beta-functions to one loop for the agravity theory[9]

$$\beta_{f_2^2} = -\frac{1}{(4\pi)^2} f_2^4 \left(\frac{133}{10}\right) \tag{2.20}$$

$$\beta_{f_0^2} = +\frac{1}{(4\pi)^2}\left(\frac{5}{3} f_2^4 + 5 f_2^2 f_0^2 + \frac{5}{6} f_0^4\right) \tag{2.21}$$

When matter is included and scalars are minimally coupled to gravity with a dimensionless coupling ξ_{ab}via

$$-\frac{1}{2}\xi_{ab}\phi_a\phi_b R$$

the beta-functions become[29]

$$\beta_{f_2^2} = -\frac{1}{(4\pi)^2} f_2^4 \left(\frac{133}{10} + \frac{N_V}{5} + \frac{N_f}{20} + \frac{N_s}{60} \right) \tag{2.22}$$

$$\beta_{f_0^2} = +\frac{1}{(4\pi)^2} \left(\frac{5}{3} f_2^4 + 5 f_2^2 f_0^2 + \frac{5}{6} f_0^4 + \frac{f_0^4}{12} (\delta_{ab} + 6\xi_{ab})(\delta_{ab} + 6\xi_{ab}) \right) \tag{2.23}$$

where N_V, N_f and N_s are the number of gauge vector fields, Weyl fermions and real scalars, respectively. The idea for a consistent high energy limit of the *agravity* theory is to let the coupling f_0^2 grow as a function of energy, to have the R^2 term disappear from the action (2.17). In this way the high energy limit of agravity would be just conformal gravity as we saw in section 2.2 and this option for the behavior of f_0^2 is thus favored by Salvio and Strumia, the authors of [9] and [29]. The beta-function at one loop of f_0^2 would allow this possibility, but it is in no way guaranteed that at higher orders in perturbation theory this possibility remains.

In this particular version of higher derivative quantum gravity the problem of the ghosts, i.e. the loss of unitarity, that we saw in the general version of higher derivative quantum gravity remains. Moreover, the purely quadratic curvature nature of this theory does not recover, analogous to conformal gravity, a Newtonian like potential at large distances. The ultraviolet limit of quantum gravity is treated in[29], but the constraints concerning the problematic infrared limit are not discussed by these authors.

After this intermezzo on conformal gravity and agravity, we continue in the next chapter with the attempt by Anselmi to quantize Stelle theory in a new way to obtain a theory of quantum gravity both renormalizable and unitary.

3 Anselmi Theory

In section 2.1 we saw that quantizing an action of the form

$$S_{quadratic} = -\frac{1}{2\kappa^2}\int\sqrt{-g}\left[2\Lambda_C + \zeta R + \alpha\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) - \frac{\xi}{6}R^2\right] + S_{matter}(g,\Psi) \quad (3.1)$$

in the usual way, by means of the Feynman prescription for all of the fields, leads to a renormalizable but non-unitary theory, due to ghosts generated by the higher derivative terms with couplings α and ξ in the action (3.1). Recently it was proposed by Anselmi [1] to quantize higher derivative theories in a new way, based on an idea originally proposed by Lee an Wick [10, 30]. The idea of Lee and Wick ought to complete Quantum Electrodynamics in the UV(that is, get rid of the Landau pole in the high energy limit) while keeping the theory both renormalizable and unitary, but the original formulation was incomplete and led to violation of Lorentz invariance [11], among other things. Attempts to cure the theory by additional prescriptions [31] were never truly convincing. Until recently, when a new formulation was proposed [12], that could also be used to quantize gravity in a new way. Before starting to discuss this new formulation, let us first review the original idea by Lee and Wick and see what problems are encountered by that theory.

3.1 Introduction to Lee-Wick models

3.1.1 Original proposal by Lee and Wick

The original proposal by T. D. Lee and G. C. Wick [30] for the so-called *Lee-Wick models* was an attempt to construct a divergence-free theory of quantum electrodynamics. This was done by re-evaluating the Pauli-Villars regularization [32]. As Pauli and Villars introduce a fictitious mass scale as a regulator in the propagator, Lee and Wick promoted the regulator to carrying a physical degree of freedom. The usual massless photon field A_μ is replaced in the Lagrangian by the complex field $\phi_\mu = A_\mu + iB_\mu$ were B_μ is a massive photon field. Having the regulator as a part of the theory leads to a higher derivative theory of QED with Lagrangian of the free theory

$$\mathscr{L}_{LWQED} = -\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu} + G_{\mu\nu}G^{\mu\nu}\right) - \frac{1}{2}(MB_\mu)^2 \quad (3.2)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \quad (3.3)$$

The propagator will, in such a theory, have multiple poles. To first order in perturbation theory the propagator will be of the Pauli-Villars form[33]

$$\begin{aligned} D &= \frac{1}{k^2} - \frac{1}{k^2 - M^2} \\ &= \frac{-M^2}{k^2(k^2 - M^2)} \end{aligned} \quad (3.4)$$

So including the regulator term in the propagator leads to the propagator having multiple poles, one belonging to the massless photon($k^2 = 0$) and the others belonging to the massive photon($k^2 = M^2$) which we will call its Lee-Wick partner. The mass of the Lee-Wick partner is thus the real valued M. This particle, though it is not in the physical spectrum, is responsible for removing the divergencies of the theory. Since the B_μ field is a negative metric field, unitarity requires that its poles are off the real axis [10], one on each side of the real axis actually. The negative metric of the Lee-Wick partner, sometimes called a Lee-Wick ghost, can be seen explicitly when going from a higher derivative formulation to an equivalent auxiliary field formulation. To see this we can view both the higher derivative formulation and the auxiliary formulation of a scalar theory with cubic self-interactions[34]. The full derivation can be found in Appendix A. The higher derivative Lagrangian is given by:

$$\mathscr{L}_{HD} = \frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2M^2}(\partial^2 A)^2 - \frac{1}{2}m^2 A^2 - \frac{\lambda}{3!}A^3 \tag{3.5}$$

We introduce an auxiliary field B and define $C = A + B$ after which we diagonalize the Lagrangian with the fields B and C to obtain the final Lagrangian:

$$\mathscr{L}_{Auxiliary} = \frac{1}{2}\partial_\mu C' \partial^\mu C' - \frac{1}{2}m'^2 C'^2 - \frac{1}{2}\partial_\mu B' \partial^\mu B' + \frac{1}{2}M'^2 B'^2 - \frac{\lambda'}{3!}(C' - B')^3 \tag{3.6}$$

in which we have rewritten the Lagrangian in terms of the mass eigenstates B' and C', with m' and M' being the masses of the fields C' and B', respectively. The important thing to note here is the opposite sign of the kinetic term of field B', which indicates the negative norm/negative metric of the Lee-Wick states of the theory. The diagonalization is only possible if $M > 2m$ in the original Lagrangian, which makes sense as the Lee-Wick state of mass M cannot be in the physical spectrum but has to decay into two states of mass m.

3.1.2 Problems in the original formulation

Combining renormalizability and unitarity in this special higher derivative theory is not without cost. The price to pay is twofold. The most severe problem with the original Lee-Wick theory is that at a certain point in doing calculations(when calculating the self energy at second order in perturbation theory to be exact), Lorentz invariance is violated, as pointed out by Nakanishi [11]. This problem, however, can be overcome by formulating the theory in a different way [31]. The authors of [31], Cutkosky, Landshoff, Olive and Polkinghorne(CLOP), also encountered another problem, namely that the integration path that we would like to use to perform a contour integration in the complex energy plane can turn out to be non-analytic due to pinching of it by the Lee-Wick poles, a phenomenon that we will call Lee-Wick pinching (which we will come back to when discussing the new formulation of the theory).

Their solution to this problem, known as the *CLOP prescription*, involves assigning different values to the imaginary part of the mass of the Lee-Wick particles. This allows one to deform the contour such that the integral can be calculated analytically. After that, the limit should be taken in which the masses of the Lee-Wick particles are the same again. Although this prescription still leads to the desired Lorentz-invariant and unitary theory it is not without problems itself [11]. The prescription may look fine in the simplest situations, but does lead to ambiguities in higher order diagrams. Another problem is that the prescription is ad hoc, since it has not been derived from a Lagrangian field theory, and possibly never will be, as Nakanishi points out.

The aforementioned issues stimulate one to keep the ideas behind the original Lee-Wick theory, but incorporate them in a new formalism. But when one tries to come up with a new formulation of Lee-Wick theories, the options turn out to be very limited. The easiest option would seem to be to define the theory directly in Minkowski spacetime, which means the energy integrals are performed with an integration path along the real axis. This version of a Lee-Wick theory (and higher derivative theories in general), as was shown in [35], turns out to be inconsistent and violate unitarity, as the counterterms generated by the theory are non-local and are not Hermitian. This leaves only the option to define the theory in Euclidean spacetime and defer part of the problem and its solution to the Wick rotation.

3.2 New formulation

In this section we describe the basics of the new formulation of the Lee-Wick models that can be used to quantize the Stelle Lagrangian(3.1) in a new way, leading to the reconciliation of renormalizability and unitarity in quantum gravity. In doing so, we follow closely [12, 36, 37] by D. Anselmi and in part M. Piva. The new formulation of the theory as described here will, for simplicity, focus mainly on the bubble diagram (Fig.2.a) to show the peculiarities of the theory. For proofs and higher order effects we refer the interested reader to the aforementioned articles. The new formulation will ultimately come down to the fact that we can use a simple arithmetic average of two analytic continuations to do calculations, the average continuation, but we first turn to the problematic parts of the old formulation.

When we consider a massive scalar theory with quartic self-interactions we have a Lagrangian

$$\mathscr{L}_{Scalar} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{3.7}$$

and we define a physical particle using the Feynman prescription[6]. The poles of the propagator

$$iD(p^2, m^2, \epsilon) = \frac{i}{p^2 - m^2 \pm i\epsilon} \tag{3.8}$$

are displaced by a distance ϵ off the real axis on the complex p^0 plane to define a physical particle. When calculating a loop diagram (Fig.2a), we perform a contour integration on the real axis, as indicated in Fig.1, and we close the contour in either the upper or the lower half plane. This integration path leaves the left pole above the path and the right pole

[6] See for example [38]

below, when we close the contour in the upper half plane. When we calculate the bubble diagram of Fig.2a, we get a loop integral that consists of two propagators of momentum k and $p+k$, which is an analytic function everywhere in the complex energy plane, except for two branch cuts slightly above and below the real axis (or on the real axis for $\epsilon \to 0$) as indicated in Fig.2b. Although the branch cuts are a violation of analyticity, they have a physical meaning, since they determine when the loop momenta are on mass shell. When the loop momenta are on shell, the loop can be cut into two tree diagrams, because the process inside the loop diagram is a *physical* process. This is all valid in the usual Feynman prescription.

3.2.1 Lee-Wick pinching

We can turn the scalar theory of (3.7) into a Lee-Wick theory by adding higher derivative terms that we divide by a mass scale M, which we call the Lee-Wick scale.[36]

$$\mathscr{L}_{ScalarLee-Wick} = \frac{1}{2}(\partial_\mu\phi)(1+\frac{\Box}{M^4})(\partial^\mu\phi) - \frac{1}{2}m^2\phi(1+\frac{\Box}{M^4})\phi - \frac{\lambda}{4!}\phi^4 \tag{3.9}$$

in which $\Box = \partial_\mu\partial^\mu$ is the d'Alembert operator. In the limit $M \to \infty$ (3.9) returns to (3.7), but for a finite Lee-Wick scale, the propagator

$$iD(k^2, m^2, M^2, \epsilon) = \frac{iM^4}{(k^2 - m^2 + i\epsilon)((k^2)^2 + M^4)} \tag{3.10}$$

admits extra Lee-Wick poles, so that all the poles are given by

$$k^0 = \pm\omega_m(\vec{k}) \mp i\epsilon \quad k^0 = \pm\Omega_M(\vec{k}) \quad k^0 = \pm\Omega^*_M(\vec{k}) \tag{3.11}$$

where

$$\omega_m(\vec{k}) = \sqrt{\vec{k}^2 + m^2} \quad and \quad \Omega_M(\vec{k}) = \sqrt{\vec{k}^2 + iM^2}$$

The poles are indicated in Fig.3 in which the crosses denote the normal poles as in the Feynman prescription and the circles denote the Lee-Wick poles. The Lee-Wick poles come in complex conjugate pairs, which makes them special in two ways. One reason is

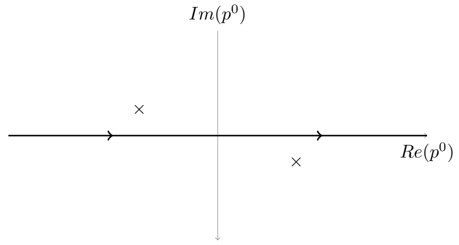
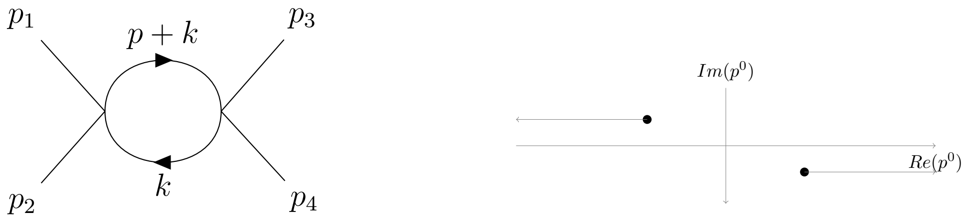


Figure 1: Integration path for the Feynman propagator, to be closed in either the upper or lower half plane. Crosses denote the poles of the propagator.



(a) The bubble diagram with $p = p_1 + p_2 = p_3 + p_4$

(b) Branch cuts of the bubble diagram

Figure 2: Bubble diagram and its branch cuts

that their pair-wise structure will allow them to cancel and keep the theory unitary. The other reason is that having pairs on opposite sides of the real axis violates analyticity in a non-trivial way, as we will see. The integration path that is the most obvious is going along the real axis, the Minkowski framework, as is done in the Feynman prescription. It turns out, as was shown in [35], that this leads to an inconsistent theory due to the poles in the first and third quadrant in the complex plane. This leaves only the option to start from the Euclidean framework, that is, integrating on the imaginary axis, and Wick-rotating to Minkowski via a modified Wick rotation. The integration path for the Lee-Wick propagator(in Fig.3) after the Wick rotation can be seen in Fig. 4. This is still a nice contour leading to an analytic function, but problems originate when loop diagrams are considered. Let's go back to the bubble diagram of Fig.2 but let the propagators now be Lee-Wick propagators. This leads to an integral proportional to

$$\mathscr{I}(p) = \int \frac{d^D k}{(2\pi)^D} D(k^2, m_1^2, \epsilon_1) D((p+k)^2, m_2^2, \epsilon_2) \tag{3.12}$$

and consequently an amplitude proportional to

$$\mathcal{M}(p) = \frac{-i\lambda^2}{2} \int \frac{d^D k}{(2\pi)^D} D(k^2, m_1^2, \epsilon_1) D((p+k)^2, m_2^2, \epsilon_2) \tag{3.13}$$

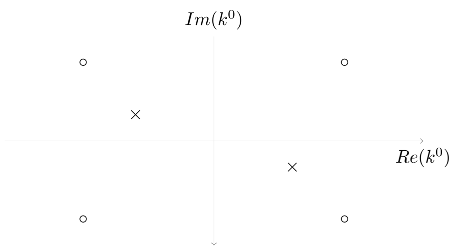


Figure 3: Poles of the Lee-Wick propagator (3.10). The crosses denote the normal poles as in the Feynman prescription and the circles denote the Lee-Wick poles.

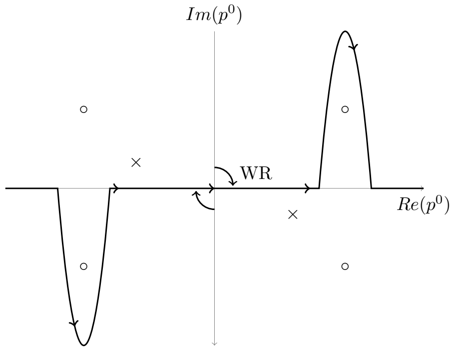


Figure 4: Wick-rotated integration path for the Lee-Wick propagator

with the coupling λ and Lee-Wick propagators:

$$iD(k^2, m_1^2, \epsilon_1) = \frac{iM^4}{(k^2 - m_1^2 + i\epsilon_1)((k^2)^2 + M^4)} \tag{3.14}$$

$$iD((p+k)^2, m_2^2, \epsilon_2) = \frac{iM^4}{((p+k)^2 - m_2^2 + i\epsilon_2)(((p+k)^2)^2 + M^4)} \tag{3.15}$$

which give poles due to the k^2 propagator in the complex k^0-plane

$$k^0 = \pm\omega_{m_1}(\vec{k}) \mp i\epsilon_1 \quad k^0 = \pm\Omega_M(\vec{k}) \quad k^0 = \pm\Omega_M^*(\vec{k}) \tag{3.16}$$

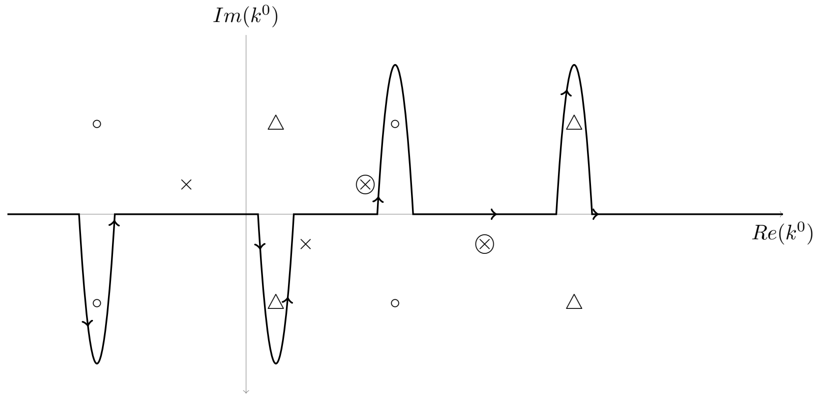
$$\omega_{m_1}(\vec{k}) = \sqrt{\vec{k}^2 + m_1{}^2} \quad \Omega_M(\vec{k}) = \sqrt{\vec{k}^2 + iM^2} \tag{3.17}$$

and poles due to the $(k+p)^2$ propagator in

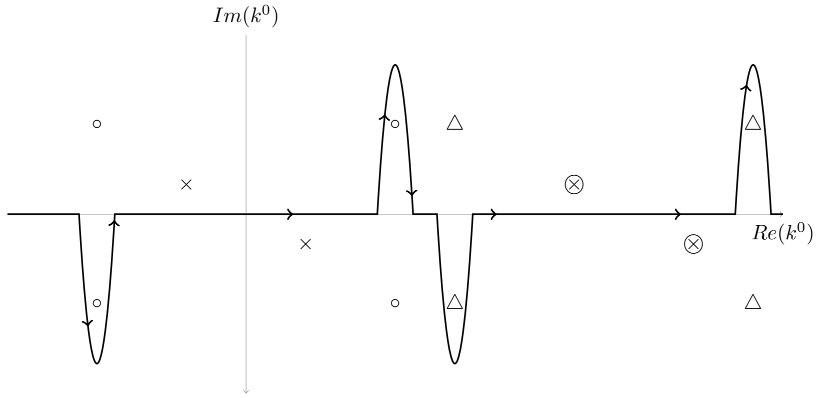
$$k^0 = \pm\omega_{m_2}(\vec{p}+\vec{k}) - p^0 \mp i\epsilon_2 \quad k^0 = \pm\Omega_M(\vec{p}+\vec{k}) - p^0 \quad k^0 = \pm\Omega_M^*(\vec{p}+\vec{k}) - p^0 \tag{3.18}$$

$$\omega_{m_2}(\vec{p}+\vec{k}) = \sqrt{(\vec{p}+\vec{k})^2 + m_2^2} \quad \Omega_M(\vec{p}+\vec{k}) = \sqrt{(\vec{p}+\vec{k})^2 + iM^2} \tag{3.19}$$

Having the poles of the k^2 propagator (3.14) fixed, the poles of the $(p+k)^2$ propagator (3.15) move on the k^0-plane by varying the external momentum p. Since we start by drawing the integration contour on the imaginary axis, the poles of propagator (3.15) can cross the integration path and we have to either deform the path or, equivalently, exclude the crossing poles from the contour to keep the function analytic. After we perform the Wick rotation, we end up with a situation like Fig.5.a or Fig.5.b, depending on the external momentum p. The poles for propagator (3.14) are the same as in Fig.4 and encircled crosses and triangles represent the normal poles and Lee-Wick poles of propagator (3.15), respectively. The general rule for the contour is similar to the Feynman prescription, meaning that the left half of the poles of each propagator are above the integration path and the right half of the poles are below. We say half of the poles in the Lee-Wick case, because it concerns both a normal pole and the Lee-Wick pair consisting of a pole on both



(a) Integration path corresponding to the $\mathscr{I}_1$ contribution to (3.28)



(b) Integration path corresponding to the $\mathscr{I}_2$ contribution to (3.28)

Figure 5: Integration paths for Lee-Wick bubble diagram $\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2$

sides of the real axis. Going from the situation in Fig.5.a to the one in Fig.5.b can be done by varying the external momentum p, but in doing so there is a point where the right Lee-Wick poles of (3.14) coincide with the left Lee-Wick poles of (3.15) and at this point the integration path gets pinched. We call this pinching *Lee-Wick pinching*, as mentioned in the previous section. Going back to the p^0 plane, now, we find two disjoint regions, each one belonging to an opposite side of the pinching. The amplitude (3.13) has to be computed in both regions separately, meaning we get two contributions to the integral (3.12), $\mathscr{I}_1(p)$ and $\mathscr{I}_2(p)$ but they are not analytically related. The violation of analyticity in this case means that Lorentz invariance is violated as well. What the previously mentioned CLOP prescription[31] proposed, was starting with different Lee-Wick scales $M'$and M in the different propagators, calculate the (bubble)diagram and then take the limit of $M' \to M$. This works for the bubble diagram with $m_1 = m_2$, but the procedure is *ad hoc* in the sense that it is not clear how this should be incorporated into the Feynman rules and moreover it is non-ambiguous for the bubble diagram with $m_1 = m_2$ only. That is, different values are obtained for some physical quantities depending on the sign of $M' - M$, as indicated in Fig.9.

3.2.2 How to treat the Lee-Wick pinching

We will now discuss the treatment of Lee-Wick pinching. The result will be that we need to perform a highly nontrivial domain deformation, but this procedure happens to coincide with a much easier one, namely that of averaging two analytic continuations. This procedure will be the one that we can use to do calculations. In the following we consider the pure Lee-Wick pinching case(there is also a possibility of pinching of one Lee-Wick pole and one regular pole in higher order loops) at non-zero external space momentum $\vec{p}$, as the amplitude is ill-defined in some parts for the $\vec{p} = 0$ case, but this problem can be overcome by solving the problems in the $\vec{p} \neq 0$ case which will lead to the consistent theory eventually. For simplicity we only look at the specific case where both the poles above and below the real axis can pinch the integration path, as is the case in fig.5, albeit the pinching of the top pole of one propagator and the bottom pole of the other propagator is also possible. The poles that are relevant for this pinching are

$$\frac{1}{k^0 + p^0 + \Omega_M^*(\vec{k})} \frac{1}{k^0 - \Omega_M(\vec{k})} \tag{3.20}$$

for the pinching above the real axis and the complex conjugate of 3.20 for the pinching below the real axis. The pinching occurs when the poles overlap, giving an equation for the pinching:

$$p^0 = -\sqrt{\vec{k}^2 + iM^2} - \sqrt{\vec{k}^2 - iM^2} \tag{3.21}$$

which is solved for

$$\vec{k}^2 = \frac{(p^0)^4 - 4M^4}{4(p^0)^2} \tag{3.22}$$

Naively, we expect that the $\vec{k}$ integration path is on the positive real axis, which solves the $\vec{k}^2$ formula in (3.22) for ${p^0}^2 > 2M^2$, resulting in a branch cut as displayed in Fig.6.a. The cut starts at the Lee-Wick threshold $p^2 > 2M^2$ and continues along the positive real

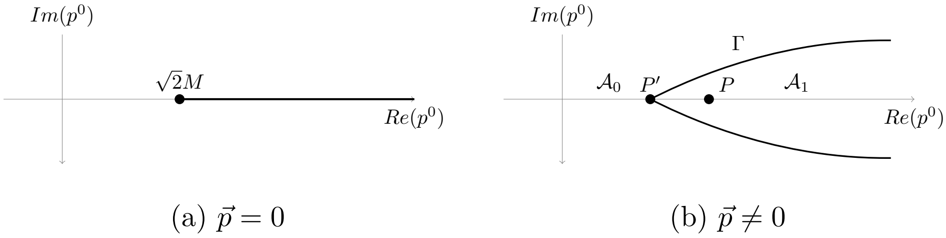


Figure 6: Branch cut and region for Lee-Wick pinching, the shape of the region is merely qualitative.

axis. As is shown in [12], this is not the actual situation, as the integration path for $\vec{k}$ has to be deformed to define the integral (3.12) properly, but we go straight to the case where $\vec{p} \neq 0$. There, the equation for the pinching (3.22) turns into

$$p^0 = -\sqrt{\vec{k}^2 + iM^2} + \sqrt{(\vec{k} - \vec{p})^2 - iM^2} \tag{3.23}$$

The solutions for some fixed value of $\vec{p}$ do not give a branch cut as in Fig.6.a, but give an entire region $\mathcal{A}_1$ as shown in Fig.6.b. This results in two analytically disconnected regions(In this oversimplified example. There are also regions within the first and second quadrant, while the mirrored situation is understood for the negative real half of the complex plane). The region containing the imaginary axis is called the *main region*, denoted by $\mathcal{A}_0$ and the region surrounded by the bounding curve Γ is the one denoted by $\mathcal{A}_1$. The region $\mathcal{A}_1$ contains the point P at

$$p^0 = \sqrt{M^2 + \vec{p}^2} \tag{3.24}$$

that can be seen as the $\vec{p} \neq 0$ equivalent of the Lee-Wick threshold $\sqrt{2}M$. The point where the boundary Γ crosses the real axis we denote by P' and is defined by

$$p^0 = \sqrt{\frac{\vec{p}^2}{2} + \sqrt{\frac{(\vec{p}^2)^2}{4} + 4M^4}} \tag{3.25}$$

so that we have $P > P' > \sqrt{2}M$. As Nakanishi showed in [11], the point P' and the entire *"non-analytic barrier"*, as Nakanishi called the boundary Γ, are violating Lorentz invariance. To retrieve a Lorentz invariant version of the integral (3.12), we would have to deform the integration contour for integrating on $\vec{k}$ to include complex values. The deformation must be so that the boundary Γ is squeezed onto the real axis(or other Lorentz invariant conditions for the regions left out in this overview), effectively turning the situation in Fig.6.b into one resembling the situation in Fig.6.a.

To do this, we merely have to focus on the contributions to (3.12) of the Lee-Wick poles, so we separate (3.10) into a physical part and a Lee-Wick part. We start by writing (3.10) as:

$$iD(p^2, m^2, M, \epsilon) = \left(\frac{M^4 + (p^2)^2 - (p^2 + m^2 - i\epsilon)(p^2 - m^2 + i\epsilon)}{M^4 + (m^2 + i\epsilon)(m^2 - i\epsilon)}\right)\left(\frac{iM^4}{(p^2 - m^2 + i\epsilon)((p^2)^2 + M^4)}\right)$$

and for convenience we rewrite it as

$$iD(p^2, m^2, M, \epsilon) = \frac{iM^4}{M^4 + (m^2 + i\epsilon)(m^2 - i\epsilon)}\left(\frac{M^4 + (p^2)^2 - (p^2 + m^2 - i\epsilon)(p^2 - m^2 + i\epsilon)}{(p^2 - m^2 + i\epsilon)((p^2)^2 + M^4)}\right),$$

where we can now split the second fraction to obtain

$$iD(p^2, m^2, M, \epsilon) = \frac{iM^4}{M^4 + (m^2 + i\epsilon)(m^2 - i\epsilon)}\left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{p^2 + m^2 - i\epsilon}{(p^2)^2 + M^4}\right)$$

and we can drop the $i\epsilon$ prescription everywhere but the physical part thus obtaining

$$iD(p^2, m^2, \epsilon) = \frac{iM^4}{M^4 + m^4}\left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{p^2 + m^2}{(p^2)^2 + M^4}\right) \tag{3.26}$$

so that

$$iD(p^2, m^2, \epsilon) = iD_0(p^2, m^2, \epsilon) + iD_{LW}(p^2, m^2) \tag{3.27}$$

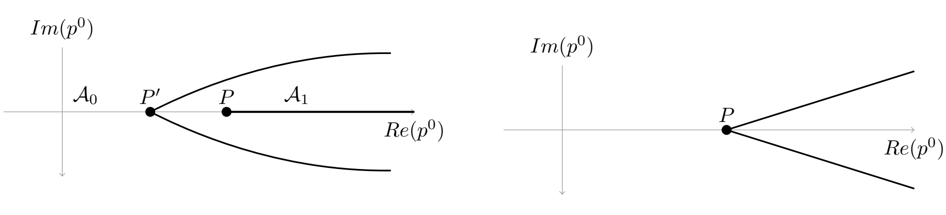
with

$$D_0(p^2, m^2, \epsilon) = \frac{M^4}{M^4 + m^4}\frac{1}{p^2 - m^2 + i\epsilon},$$
$$D_{LW}(p^2, m^2) = -\frac{M^4}{M^4 + m^4}\frac{p^2 + m^2}{(p^2)^2 + M^4}.$$

The non-analytic contribution to the integral (3.12) is then just the Lee-Wick part:

$$\mathscr{I}_{LW}(p) = \int \frac{d^D k}{(2\pi)^D} D_{LW}(k^2, m_1^2) D_{LW}((k+p)^2, m_2^2) \tag{3.28}$$

By integrating on k^0 we end up with an integral on $\vec{k}$ which is, as in Fig.7.a, divided into the analytic region $\mathcal{A}_0$ containing the imaginary axis and the non-analytic and Lorentz invariance-violating region $\mathcal{A}_1$ starting from point P' and containing the Lee-Wick threshold P. We denote the integral (3.28) in the region $\mathcal{A}_0$ by $\mathscr{I}^0_{LW}(p)$ and in the region $\mathcal{A}_1$ by $\mathscr{I}^1_{LW}(p)$. The region $\mathcal{A}_0$ can be maximally extended up to the Lee-Wick threshold P. To overcome this threshold, we go inside the region $\mathcal{A}_1$ to evaluate the deformed integral $\mathscr{I}^{deformed}_{LW}(p)$ there. The integration domain for $\vec{k}$ should now be deformed in such a way that the region $\mathcal{A}_1$ is squeezed onto the half line on the real axis starting from point P as in 3.24. The result of the deformed integral $\mathscr{I}^{deformed}_{LW}(p)$ that was originally in $\mathcal{A}_1$ is now both analytic and Lorentz invariant. Let us briefly compare this new procedure with the Feynman procedure. When we evaluate a bubble diagram using the Feynman prescription, we integrate on the real axis, meaning that we are always either above or below the branch cuts as in Fig.2.b. With the new prescription, we evaluate inside the enlarged branch region and then shrink the region onto a line, leaving the integral effectively *inside* the now created branch cut.



(a) Lee-Wick branch region and deformation

(b) Lee-Wick analytic regions

Figure 7: Lee-Wick deformation and analytic regions

As is shown in [12] and more extensively in [37], the domain deformation makes the procedure tremendously complicated, especially with more higher order diagrams. Luckily, it turns out that the result of this procedure coincides with a much easier one based on averaging two analytic procedures to circumvent the branch point. To use this procedure, we merely have to evaluate the integral (3.28) in the original Euclidean region $\mathcal{A}_0$, using the analytic function $\mathscr{J}^0_{LW}(p)$, and analytically continue it from both the imaginary positive and negative half plane to the half-line starting from P. We can define the analytic continuation of $\mathscr{J}^0_{LW}(p)$ in the positive imaginary half plane to be $\mathscr{J}^{0+}_{LW}(p)$ and the one in the negative imaginary half plane to be $\mathscr{J}^{0-}_{LW}(p)$. The arithmetic average of these analytically continued functions turns out to coincide with the integral $\mathscr{J}^{deformed}_{LW}(p)$ in the region $\mathcal{A}_1$ that is shrunk onto the half line by the procedure described before, so that we denote it by

$$\mathscr{J}^{averaged}_{LW}(p) = \frac{1}{2}\big[\mathscr{J}^{0+}_{LW}(p) + \mathscr{J}^{0-}_{LW}(p)\big]. \tag{3.29}$$

Note that, although $\mathscr{J}^{0+}_{LW}(p)$ and $\mathscr{J}^{0-}_{LW}(p)$ are themselves analytically continued functions, and the final answer $\mathscr{J}^{averaged}_{LW}(p)$ is analytic as well, the procedure to obtain it is not. The average (3.29) can be uniquely determined from (3.28) in $\mathcal{A}_0$, but not the other way around. It can nonetheless be used to define $\mathscr{J}_{LW}(p)$ beyond the Lee-Wick threshold P and can also be used to analytically continue the integral in the neighbourhood of the half-line from P. This way we can create regions of analyticity that are disjoint. Anselmi therefore calls the result *regionwise analytic*. The final result also allows us to have a well-defined function above the Lee-Wick threshold in the $\vec{p} = 0$ case, something that was briefly mentioned at the start of this section.

In the next section we will discuss the unitarity of the new formulation, but let us first quickly review the renormalizability of a non-analytically Wick-rotated theory. As is shown in [1] and [37], the renormalization of (3.1) when formulating it as a non-analytically Wick-rotated higher derivative theory of gravity, coincides with its Euclidean version. One easy argument to see this is looking at the average continuation (3.29). It consists of the average of two analytic continuations, for which it holds that the convergence of the original functions remains after the analytic continuation. As the divergencies of the theory are removed in the main, Euclidean, region to ensure convergence of the amplitudes, the convergence remains after the non-analytic Wick rotation as it coincides with the average of two analytic functions.

3.2.3 Unitarity

To show that the new formulation of the Lee-Wick models is unitary, we will look at the discontinuity of the bubble diagram in the usual case and extend it to the Lee-Wick bubble diagram. The perturbative unitarity to all orders can be found in section 7 of [37]. In this section we make use of the Optical Theorem and the cutting rules, which can be found in Appendix B.1 and B.3, respectively.

We start with the amplitude for the bubble diagram (3.13) with regular propagators, and we write

$$i\mathcal{M}(p) = \frac{\lambda^2}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2 - m_1^2 + i\epsilon_1}\frac{1}{(k-p)^2 - m_2^2 + i\epsilon_2}, \tag{3.30}$$

which we can rewrite as

$$\begin{aligned} i\mathcal{M}(p) = \frac{\lambda^2}{2}\int \frac{dk^0 d^{D-1}\vec{k}}{(2\pi)^D} & \frac{1}{(e_1 - \omega_1 + i\epsilon_1)}\frac{1}{(e_1 + \omega_1 - i\epsilon_1)} \\ & \times \frac{1}{(e_2 - \omega_2 + i\epsilon_2)}\frac{1}{(e_2 + \omega_2 - i\epsilon_2)} \end{aligned} \tag{3.31}$$

with $e_1 = k^0, e_2 = k^0 - p^0, \omega_1 = \sqrt{\vec{k}^2 + m_1^2}$, $\omega_2 = \sqrt{(\vec{k} - \vec{p})^2 + m_2^2}$ and $\epsilon_i > 0$. We now integrate on k^0 by closing the contour in the lower half plane, which gives poles at

$$\begin{aligned} k^0 &= z_1 & k^0 &= z_2 \\ z_1 &= \omega_1 - i\epsilon_1 & z_2 &= \omega_2 - i\epsilon_2. \end{aligned} \tag{3.32}$$

Applying the residue theorem we obtain

$$i\mathcal{M}(p) = -\frac{i\lambda^2}{2}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\Big[\mathrm{Res}(z_1) + \mathrm{Res}(z_2)\Big]. \tag{3.33}$$

We can express the individual residues in a way that is useful in the Lee-Wick case later on:

$$\begin{aligned} \mathrm{Res}(z_1) &= \frac{1}{2z_1(z_1 - z_2)(z_1 + z_2 - 2p^0)} = \frac{1}{2(\omega_1 - i\epsilon_1)\Delta(\omega_1 + \omega_2 - p^0 - i\epsilon_+)} \equiv r_1, \\ \mathrm{Res}(z_2) &= -\frac{1}{2(z_2 - p^0)(z_1 - z_2)(z_1 + z_2)} = -\frac{1}{2(\omega_2 - i\epsilon_2)\Delta(\omega_1 + \omega_2 + p^0 - i\epsilon_+)} \equiv r_2, \end{aligned} \tag{3.34}$$

with $\Delta = (z_1 - z_2) = (\omega_1 - \omega_2 - p^0 - i\epsilon_-)$ and $\epsilon_\pm = \epsilon_1 \pm \epsilon_2$. The sum of the residues, as they appear in (3.33), is

$$\mathrm{Res}(z_1) + \mathrm{Res}(z_2) = -\frac{1}{4\omega_1\omega_2}\Big(\frac{1}{\omega_1 + \omega_2 - p^0 - i\epsilon_+} + \frac{1}{\omega_1 + \omega_2 + p^0 - i\epsilon_+}\Big) \tag{3.35}$$

in which the unimportant ϵ's have been left out so we can write in terms of ω_i.

Applying the Sokhotski–Plemelj theorem we can evaluate a complex fraction as

$$\frac{1}{a \pm i\epsilon} = \mathcal{P}\left(\frac{1}{a}\right) \mp i\pi\delta(a) \tag{3.36}$$

in which $\mathcal{P}$ denotes the Cauchy principal value.

We can now use this to determine the discontinuity of the amplitude (3.33) by the relation in (B.20):

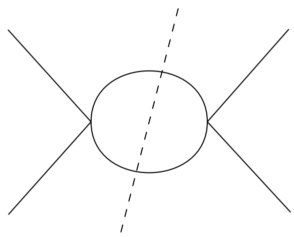


Figure 8: The cut bubble diagram

$$\text{Disc}\,(i\mathcal{M}(p)) = -2\text{Im}\mathcal{M}(p), \tag{3.37}$$

to obtain

$$\text{Disc}\,(i\mathcal{M}(p)) = \frac{i\lambda^2}{2}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\frac{2\pi}{(2\omega_1)(2\omega_2)}\big[\delta(p^0-\omega_1-\omega_2)+\delta(p^0+\omega_1+\omega_2)\big]. \tag{3.38}$$

We can relabel the momenta k and $p-k$ to be q_1 and q_2, respectively, to rewrite (3.38) as integrals over $q_1$andq_2. We then also need delta functions to ensure $q_i^0 = \pm\Omega_i$ with $\Omega_i = \sqrt{\vec{q_i}^2 + m_i^2}$ and $\vec{p} = \vec{q_1} + \vec{q_2}$ so that the delta functions in (3.38) can be viewed as conservation of the total energy $\delta(p^0 - q_1^0 - q_2^0)$. The discontinuity can then be written as

$$\begin{aligned}\text{Disc}\,(i\mathcal{M}(p)) =& \frac{i\lambda^2}{2}\int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}(2\pi)^D\delta^{(D)}(p-q_1-q_2)(2\pi)^D\delta(q_1^2-m_1^2)(2\pi)\delta(q_2^2-m_2^2)\\ &\times[\theta(q_1^0)\theta(q_2^0)+\theta(-q_1^0)\theta(-q_2^0)].\end{aligned} \tag{3.39}$$

The θ-terms in brackets show the direction of flow of the energy as the particles in the loop are put on shell by the δ-functions. Multiplied by $-i$, the discontinuity corresponds to the sum of two cut diagrams $\mathcal{F}_i$ as in Fig. 8[7]. We can also compute the cut diagrams by applying the cutting rules[39], as explained in Appendix B.3, which amounts to making the replacement

$$\frac{1}{k^2-m^2+i\epsilon} \to -i2\pi\theta(\pm k^0)\delta(k^2-m^2) \tag{3.40}$$

in the diagrams $\mathcal{F}_i$ to obtain

$$i\mathcal{M}^* - i\mathcal{M} = \mathcal{F}_1 + \mathcal{F}_2, \tag{3.41}$$

which is the exact bubble diagram version of (B.4), hence the unitarity equation is satisfied.

To show unitarity in the Lee-Wick case, we want to derive the cutting equation for the bubble diagram and show that it does not propagate anything other than physical degrees of freedom. This property coincides with the Lee-Wick poles not showing up in

[7]The diagrams $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to the external momentum propagating through the cut either from left to right and applying a minus sign to the right vertex or vice versa.[5]

the imaginary part of the amplitude, hence they are not in the discontinuity of the diagram. In the Lee-Wick case we start, of course, once more with the same amplitude (3.13), but now we use the Lee-Wick propagators (3.10). We can keep the definitions as in (3.31) and add the Lee-Wick definitions $\nu_1 = \sqrt{\vec{k}^2 + iM^2}$ and $\nu_2 = \sqrt{(\vec{k} - \vec{p})^2 + iM^2}$ to write (3.13) as

$$\begin{aligned} i\mathcal{M}(p) =& M^8 \frac{\lambda^2}{2} \int \frac{dk^0 d^{D-1}\vec{p}}{(2\pi)^D} \\ &\times \frac{1}{(e_1 - \nu_1)(e_1 + \nu_1)(e_1 - \nu_1^*)(e_1 + \nu_1^*)(e_1 - \omega_1 + i\epsilon_1)(e_1 + \omega_1 - i\epsilon_1)} \\ &\times \frac{1}{(e_2 - \nu_2)(e_2 + \nu_2)(e_2 - \nu_2^*)(e_2 + \nu_2^*)(e_2 - \omega_2 + i\epsilon_2)(e_2 + \omega_2 - i\epsilon_2)}. \end{aligned}$$

As in the regular case we integrate on k^0 by closing in the lower half plane, which gives us the six poles indicated in Fig.5:

$$\begin{aligned} z_1 &= \omega_1 - i\epsilon_1 \quad l_1 = \nu_1 \\ z_2 &= \omega_2 - i\epsilon_2 \quad l_2 = p^0 + \nu_2 \end{aligned} \tag{3.42}$$

and the complex conjugates of the l_i poles. This results in a amplitude

$$i\mathcal{M}(p) = -i\frac{\lambda^2}{2} \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} [\text{Res}(z_1) + \text{Res}(z_2) + \text{Res}(l_1) + \text{Res}(l_2) + \text{Res}(l_1^*) + \text{Res}(l_2^*)], \tag{3.43}$$

which consists of the residues due to normal poles and two pairs of Lee-Wick poles, the pairs consisting of l_i and l_i^*. For the residues due to the regular poles z_1 and z_2 we cannot take the limit $\epsilon_i \to 0$, but for the residues due to the Lee-Wick poles we can, because the are still off the real axis without that prescription. If we perform the integration deformation per region $\mathcal{D}_i$ in a complex conjugate way for the complex conjugate residues, it turns out that in that limit $\epsilon_i \to 0$

$$[\text{Res}(l_i^*)]^* = \text{Res}(l_i), \tag{3.44}$$

in the different analytic regions, corresponds to

$$\left[\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \text{Res}(l_i^*) \right]^* = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \text{Res}(l_i), \tag{3.45}$$

where the deformation of the left integration contour is the complex conjugate of that of the right, but the complex conjugation does not affect $\vec{k}$. So if we take this limit $\epsilon_i \to 0$ for the Lee-Wick poles and keep $\epsilon_i \neq 0$ for the normal poles the contributions to the amplitude are:

$$
\begin{aligned}
i\mathcal{M}(p) =& -i\frac{\lambda^2}{2}\int\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}[\mathrm{Res}(z_1)+\mathrm{Res}(z_2)]\\
&-i\frac{\lambda^2}{2}\mathrm{Re}\left[\int\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\mathrm{Res}(l_1)+\mathrm{Res}(l_2)+\mathrm{Res}(l_1^*)+\mathrm{Res}(l_2^*)\right]\\
=& -i\frac{\lambda^2}{2}\int\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}[\mathrm{Res}(z_1)+\mathrm{Res}(z_2)]-i\lambda^2\mathrm{Re}\left[\int\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\mathrm{Res}(l_1)+\mathrm{Res}(l_2)\right].
\end{aligned}
\tag{3.46}
$$

As we are working with i times the amplitude, the Lee-Wick contributions to it are purely real, so they do not contribute to its discontinuity:

$$
\mathrm{Disc}\mathcal{M} = 2i\mathrm{Im}\mathcal{M} = -i\lambda^2\int\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\mathrm{Im}[\mathrm{Res}(z_1)+\mathrm{Res}(z_2)]. \tag{3.47}
$$

This compensation mechanism of having the residues due to the Lee-Wick poles in (3.46) giving compensating imaginary contributions and thus giving no contribution to the imaginary part of the diagram is the key result for turning ghost-plagued theories into unitary theories.

To show that the cutting equations are also satisfied in the Lee-Wick case we extract the imaginary part from the residues, which in this case we define using r_1 and r_2 from (3.34):

$$
\mathrm{Res}(z_1)+\mathrm{Res}(z_2) = h(z_1)r_1+h(z_2)r_2 \tag{3.48}
$$

with

$$
h(k^0) = \frac{M^4}{((k^2)^2+M^4)}\frac{M^4}{(((k-p)^2)^2+M^4)} \tag{3.49}
$$

Inserting these residues into (3.47) the expression ultimately reduces to

$$
\mathrm{Disc}\mathcal{M} = \frac{i\lambda^2}{2}\frac{M^8}{(M^4+m_1^4)(M^4+m_2^4)}\int\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\frac{2\pi}{(2\omega_1)(2\omega_2)}\left[\delta(p^0-\omega_1-\omega_2)+\delta(p^0+\omega_!+\omega_2)\right]. \tag{3.50}
$$

To this equation we can apply the same procedure as we did with (3.38) to conclude that the discontinuity of $\mathcal{M}$ is equal to

$$
\begin{aligned}
\mathrm{Disc}\mathcal{M} =& \frac{i\lambda^2}{2}\int\frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}(2\pi)^D\delta^{(D)}(p-q_1-q_2)\frac{(2\pi)M^4}{M^4+m_1^4}\delta(q_1^2-m_1^2)\frac{(2\pi)M^4}{M^4+m_2^4}\delta(q_2^2-m_2^2)\\
&\times[\theta(q_1^0)\theta(q_2^0)+\theta(-q_1^0)\theta(-q_2^0)],
\end{aligned}
\tag{3.51}
$$

which is equal to making the replacement of the cut propagators

$$
\mathcal{C}(p^2,m^2)\to 2\pi\frac{M^4}{M^4+m^4}\delta(p^2-m^2) \tag{3.52}
$$

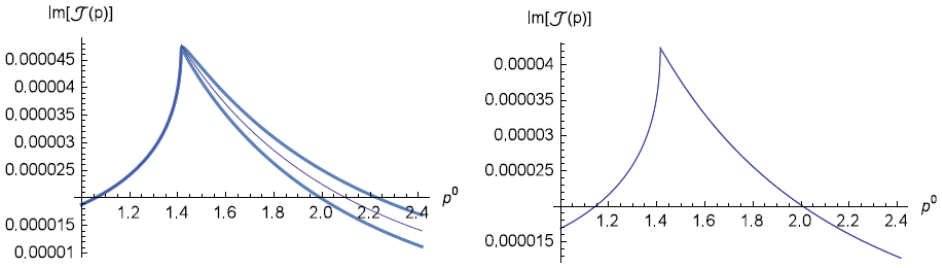


Figure 9: New formulation vs. CLOP prescription for $m_1 \neq m_2$(left) and $m_1 = m_2$(right) *picture taken from section 5 of* [12]

in the direct computation of the discontinuity through the cutting equations

$$\begin{aligned}\text{Disc}\mathcal{M} =& \frac{i\lambda^2}{2} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} (2\pi)^D \delta^{(D)}(p - q_1 - q_2) \mathcal{C}(q_1^2, m_1^2) \mathcal{C}(q_2^2, m_2^2) \\ & \times [\theta(q_1^0)\theta(q_2^0) + \theta(-q_1^0)\theta(-q_2^0)],\end{aligned} \tag{3.53}$$

hence the cutting equations are satisfied as we conclude that the cut propagators do not propagate Lee-Wick degrees of freedom, but only the physical ones.

This section has merely been an attempt to show that the bubble diagram satisfies unitarity. A proof of perturbative unitarity to all orders needs a more generalized notion of the cutting rules, called algebraic cutting equations[40] and can be found in [37].

3.2.4 Comparison with the CLOP prescription and Minkowski versions

As a conclusion of this section on the new formulation we will now briefly come back to the older formulation. The Lee-Wick models have been revisited recently(recent with respect to their original formulation) in [34], among other papers. In that paper, the CLOP prescription[31] was used to come to a more or less consistent theory. The result of the calculation of a simple bubble diagram in that paper coincides with the new formulation. This is the case only because the propagators that have been used have equal physical masses ($m_1 = m_2$ case). Taking different masses would already result in an ambiguity. An explicit check has been done by Anselmi in [12] of which the results can be seen in Fig.9. The right picture shows that the new formulation and the CLOP prescription coincide for the equal mass case, while the left picture shows the ambiguous result for the CLOP prescription. In the left picture, the new formulation and the CLOP prescription coincide up to the Lee-Wick threshold, but after that the CLOP prescription gives the upper and lower curve while the middle curve belongs to the new formulation. The average of the ambiguous result of the CLOP prescription happens to coincide with the new formulation. One could argue that the CLOP prescription can be kept if we average the ambiguous results afterwards, but this would only solve the ambiguity problem, while the new formulation both solves that *and* can be implemented at the Lagrangian level.

We also come back to the possibility of defining the theory directly in Minkowski spacetime as this would be troublesome according to the statements made at the beginning of this section [35]. If such a formulation led to a consistent and unitary theory, its cutting equations would also coincide with those of the unitary Euclidean version, because the degrees of freedom propagating through the cuts are the same. Looking at Fig.5 we see that integrating directly on the real axis would pick up poles z_1, z_2, l_1^* and l_2^*(defined in (3.42)), but instead of picking up the poles l_1 and l_2 we would now be picking up $l_1' = -\nu_1$ and $l_2' = p^0 - \nu_2$ that are on the other side of the contour in Fig.5. The total discontinuity of the difference between the Minkowski version of the amplitude and the Euclidean version of the amplitude should be vanishing. This is not the case, as is shown in [36], because the difference between the residues in l_1' and l_2', due to the Minkowski version and l_1^* and l_2^*, due to the Euclidean version, has an imaginary part. We can therefore conclude that, indeed, the Minkowski version is not perturbatively unitary.

3.3 Anselmi Quantum Gravity

The compensation mechanism in (3.46) and thus the absence of the discontinuity in the Lee-Wick behavior of the newly formulated theory plays the key role in the formulation of quantum gravity based on the newly formulated Lee-Wick models. This absence amounts to not having physical degrees of freedom propagating through the cuts of a diagram, due to the Lee-Wick contributions, and thus no physical degrees of freedom propagating in the loop of, say, the bubble diagram, as was discussed before. Quantizing a gravitational action with terms quadratic in the curvature in the usual way renders the theory renormalizable, but is plagued with ghosts as we saw in section 2.1. The propagators of this theory, quantized in the usual way, do not admit the structure that would render the theory unitary via the Lee-Wick mechanism just described, but can we come up with a solution to this problem? The new proposal for a quantum theory of gravity by Anselmi [1] starts with the renormalizable Stelle type action. The theory is then turned into a Lee-Wick type theory by doubling the ghost poles and adding a fictitious Lee-Wick scale Υ as we will discuss in the context of a ghost containing scalar model in the next section, 3.3.1. We will subsequently look at how to apply this new prescription to the graviton propagator of the Stelle action. The theory then becomes a Lee-Wick type theory that is renormalizable due to the original action and that is unitary due to the cancellation of the Lee-Wick behavior of the doubled ghost poles. We will study the contributions to the absorptive part of the graviton self-energy and then discuss the different possibilities of applying the prescription to the Stelle action by going to an auxiliary field formulation. In this formulation we discuss the first phenomenological results of the new prescription. We end this chapter by stating the one-loop beta-functions of the theory.

3.3.1 Turning ghosts into fakes

The doubling of the ghost poles and quantizing them as Lee-Wick particles is what Anselmi dubs *turning the ghost degrees of freedom into fake degrees of freedom*. The resulting particles are therefore also called "*fake particles*", or fakeons, a new type of particle the properties of which we will investigate in more depth later on in this section. As the Stelle action (3.1) is in principle a higher derivative scalar theory, we can start to discuss the new proposal for quantum gravity by looking how the process of turning ghosts into fake particles(fakeons) works in a self interacting scalar theory. Consider the action

$$S = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)\left(\alpha - \beta\frac{\Box}{m^2}\right)(\partial^\mu \phi) - \frac{\lambda}{4!}\phi^4\right] \tag{3.54}$$

in which the couplings α positive, β negative, m is a mass parameter and $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator. In Euclidean spacetime the propagator is:

$$D = \frac{1}{\alpha p^2} + \frac{\beta}{\alpha(\alpha m^2 - \beta p^2)} \tag{3.55}$$

which, after Wick rotating to Minkowski, would propagate a massless scalar due to the first fraction and a massive ghost due to the second fraction. Instead of directly performing the Wick rotation, we first double the ghost poles in the second propagator using $\frac{1}{p^2} = \frac{p^2}{(p^2)^2}$ and adding an infinitesimal Lee-Wick scale parameter Υ, which we will remove at the end, to make the propagator resemble the structure of the Lee-Wick propagator (3.10):

$$\begin{aligned} D =& \frac{1}{\alpha p^2} + \frac{\beta(\alpha m^2 - \beta p^2)}{\alpha[(\alpha m^2 - \beta p^2)^2 + \Upsilon^4]} \\ =& \frac{1}{\alpha p^2} + \frac{1}{2}\left[\frac{\beta}{\alpha(\alpha m^2 - \beta p^2) - i\Upsilon^2} + \frac{\beta}{\alpha(\alpha m^2 - \beta p^2) + i\Upsilon^2}\right] \end{aligned} \tag{3.56}$$

and when we naively Wick rotate we get a propagator in the form of

$$\begin{aligned} D =& \frac{i}{\alpha(p^2 + i\epsilon)} - \frac{i\beta(\alpha m^2 + \beta p^2)}{\alpha[(\alpha m^2 + \beta p^2)^2 + \Upsilon^4]} \\ =& \frac{i}{\alpha(p^2 + i\epsilon)} - \frac{i}{2}\left[\frac{\beta}{\alpha(\alpha m^2 + \beta p^2) - i\Upsilon^2} + \frac{\beta}{\alpha(\alpha m^2 + \beta p^2) + i\Upsilon^2}\right] \end{aligned} \tag{3.57}$$

where we left out the iϵ prescription in the terms in brackets as the poles there are already off the real axis. The propagator in (3.57) is propagating a massless scalar due to the $p^2 = 0$ pole, but the ghost is not present anymore. The ghost has been turned into a fakeon, as the poles in $p^2 = \frac{-1}{\beta}(\alpha m^2 + i\Upsilon^2)$ and $p^2 = \frac{-1}{\beta}(\alpha m^2 - i\Upsilon^2)$ disappear in the cutting equation, due to the compensation mechanism we just saw in section 3.2.3. We can now do calculations using this propagator and after the calculation we send the Lee-Wick parameter Υ to zero, just like, but after, we send ϵ to zero. In the limit $\Upsilon \to 0$ we are still left with a unitary theory, as the compensation mechanism remains intact during the calculation. The renormalizability also remains valid, since the divergent parts of the amplitudes to be calculated are only in the main region $\mathcal{A}_0$, that contains the imaginary axis as in Fig.6, and this region is analytic both in the case of $\Upsilon \neq 0$ and $\Upsilon \to 0$.

3.3.2 The graviton propagator with the new prescription

We will now apply the prescription of section 3.3.1 to the propagator of the metric fluctuation to quantize the ghosts of the Stelle theory, discussed in section 2.1, as fakeons. We write the action for the Stelle theory a bit different from hereon out to follow the notation of Anselmi et al.[1]:

$$S_{quadratic} = -\frac{\mu^{-\epsilon}}{2\kappa^2}\int d^Dx\sqrt{-g}\left[2\Lambda_C + \zeta R + \alpha\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) - \frac{\xi}{6}R^2\right], \tag{3.58}$$

Λ_C, ζ, α, ξ and κ are real constants, with $\zeta > 0$, $\alpha > 0$ and $\xi > 0$, while μ is the dynamical scale and $\epsilon = 4 - D$, D being the continued spacetime dimension introduced by the dynamical regularization. In this section and the next three sections we closely follow Anselmi and Piva in [41] and [42]. We start with the free propagator of the metric fluctuation $h_{\mu\nu}$ of (3.58) at vanishing cosmological constant:

$$\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_0 = \frac{i\mathcal{I}_{\mu\nu\rho\sigma}}{2p^2(\zeta - \alpha p^2)} + \frac{i(\alpha-\xi)\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{6(p^2)^2(\zeta-\alpha p^2)(\zeta - \xi p^2)} \tag{3.59}$$

with

$$\mathcal{I}_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma} \tag{3.60}$$

and

$$\bar{\omega}_{\mu\nu} = p^2\eta_{\mu\nu} + 2p_\mu p_\nu \tag{3.61}$$

We can now directly apply the procedure from 3.3.1 to this propagator, but it is more convenient to separate the physical contributions in the propagator from the ghost, or soon to be the fake contributions.

$$\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_0 = \langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_{0grav} + \langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_{0fake} \tag{3.62}$$

The physical pole in (3.59) is the one corresponding to $p^2 = 0$ to which we apply the ordinary Feynman prescription. The poles at $p^2 = \zeta/\alpha$ and $p^2 = \zeta/\xi$ are the ones to which we apply the fakeon prescription. To separate these contributions we can make use of the fraction identities:

$$\frac{1}{x(1-ax)} = \frac{1}{x} + \frac{a}{1-ax} \tag{3.63}$$

and

$$\frac{1}{x^2(1-ax)(1-bx)} = \frac{1}{x^2} + \frac{a+b}{x} + \frac{1}{a-b}\left(\frac{a^3}{1-ax} - \frac{b^3}{1-bx}\right) \tag{3.64}$$

To obtain the $(1-ax)$ and $(1-bx)$ terms in the denominators we first divide the terms inside the brackets in (3.59) by ζ to get:

$$\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_0 = \frac{i\mathcal{I}_{\mu\nu\rho\sigma}}{2\zeta p^2(1-\frac{\alpha}{\zeta}p^2)} + \frac{i(\alpha-\xi)\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{6\zeta^2(p^2)^2(1-\frac{\alpha}{\zeta}p^2)(1-\frac{\xi}{\zeta}p^2)} \tag{3.65}$$

Applying the identity (3.63) to the first fraction in (3.65) we obtain:

$$\frac{i\mathcal{I}_{\mu\nu\rho\sigma}}{2\zeta p^2(1-\frac{\alpha}{\zeta}p^2)} = \frac{i\mathcal{I}_{\mu\nu\rho\sigma}}{2\zeta}\left[\frac{1}{p^2} + \frac{\frac{\alpha}{\zeta}}{1-\frac{\alpha}{\zeta}p^2}\right] \tag{3.66}$$

and applying the identity (3.64) to the second fraction in (3.65) we obtain:

$$\frac{i(\alpha-\xi)\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{6\zeta^2(p^2)^2(1-\frac{\alpha}{\zeta}p^2)(1-\frac{\xi}{\zeta}p^2)} = \frac{i(\alpha-\xi)\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{6\zeta^2}\left[\frac{1}{(p^2)^2}+\frac{\frac{\alpha}{\zeta}+\frac{\xi}{\zeta}}{p^2}+\frac{1}{\frac{\alpha}{\zeta}-\frac{\xi}{\zeta}}\left(\frac{(\frac{\alpha}{\zeta})^3}{1-\frac{\alpha}{\zeta}p^2}-\frac{(\frac{\xi}{\zeta})^3}{1-\frac{\xi}{\zeta}p^2}\right)\right] \tag{3.67}$$

After rewriting the fractions containing ζ in the denominators by appropriately multiplying them by ζ we can collect the $p^2 = 0$ pole-terms in one term, i.e. the graviton term, and collect the other terms in the fakeon term, resulting in:

$$\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_{0grav} = \frac{i}{2\zeta p^2}\left[\mathcal{I}_{\mu\nu\rho\sigma} + \frac{(\alpha-\xi)\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{3\zeta^2}\left(\frac{\zeta}{p^2}+\alpha+\xi\right)\right] \tag{3.68}$$

and

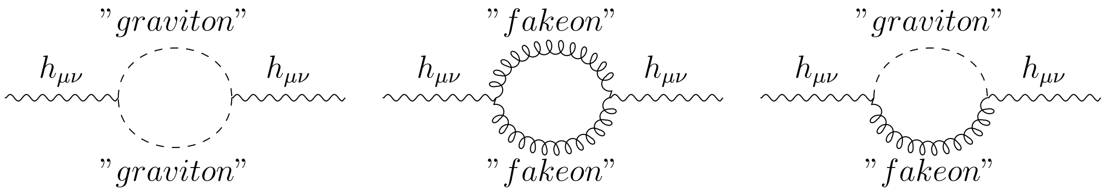
$$\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_{0fake}\frac{i\alpha\mathcal{I}_{\mu\nu\rho\sigma}}{2\zeta(\zeta-\alpha p^2)} + \frac{i\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{6\zeta^3}\left(\frac{\alpha^3}{\zeta-\alpha p^2}-\frac{\xi^3}{\zeta-\xi p^2}\right). \tag{3.69}$$

The complete fakeon procedure now consists of three steps:

1. The usual Feynman replacement of $p^2 \to p^2 + i\epsilon$ in the denominators
2. Then the fakeon replacement $\frac{1}{\zeta-up^2} \to \frac{\zeta-up^2}{(\zeta-u(p^2+i\epsilon))^2+\Upsilon^4}$ with u equal to α or ξ.
3. Now we can calculate the diagrams and let ϵ tend to zero before letting Υ tend to zero.

After step 1. and 2. we end up with the propagator contributions

$$\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_0 = \underbrace{\frac{i}{2\zeta(p^2+i\epsilon)}\left[\mathcal{I}_{\mu\nu\rho\sigma} + \frac{(\alpha-\xi)\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{3\zeta^2}\left(\frac{\zeta}{p^2+i\epsilon}+\alpha+\xi\right)\right]}_{graviton}$$
$$+\underbrace{\frac{i\alpha\mathcal{I}_{\mu\nu\rho\sigma}(\zeta-\alpha p^2)}{2\zeta[(\zeta-\alpha(p^2+i\epsilon))^2+\Upsilon^4]} + \frac{i\bar{\omega}_{\mu\nu}\bar{\omega}_{\rho\sigma}}{6\zeta^3}\left(\frac{\alpha^3(\zeta-\alpha p^2)}{(\zeta-\alpha(p^2+i\epsilon))^2+\Upsilon^4}-\frac{\xi^3(\zeta-\xi p^2)}{(\zeta-\xi(p^2+i\epsilon))^2+\Upsilon^4}\right)}_{fakeon} \tag{3.70}$$



(a) pure graviton diagram (b) pure fakeon diagram (c) mixed diagram

Figure 10: Metric fluctuation diagrams for the one loop graviton self energy decomposed into three contributions.

3.3.3 Contributions to the apsorbtive part of the graviton self energy at one loop

Now that we have obtained the propagator with the new presciption we discuss something that is crucially different from the old quantization of (3.58), namely the absorptive part of the graviton self energy at one loop. Anselmi and Piva[41] couple the gravity sector to N scalar fields with the action

$$S_\phi = \frac{1}{2}\sum_{i=1}^{N}\int \sqrt{-g}g^{\mu\nu}(\partial_\mu\phi^i)(\partial^\nu\phi^i) \tag{3.71}$$

and calculate the absorptive part of the graviton self energy. Their computation is done at $\Lambda_C = 0$, as more powerful computational techniques are required to perform the calculation at non-vanishing cosmological constant. Also, the gauge is fixed using the de Donder gauge as gauge-dependent results would result in too complicated expressions to report, but the gauge-dependent results can be obtained via [41]. We will not do the actual computation, but merely discuss the different contributions to the absorptive part of the graviton self energy.

The absorptive part of an amplitude is equal to its imaginary part. Since we are here investigating diagrams contributing to the amplitude, we will look for minus the real part of the diagram(i.e. a diagram $\sim i\mathcal{M}$). When looking at the self energy at $\Lambda_C = 0$, coupled to N_S scalar fields, the self energy consists of three diagrams. The loop can be made of three propagators, namely scalars, Faddeev-Popov ghosts or the metric. The scalars and Faddeev-Popov ghosts do not involve fakeons hence they give no contributions that are different from the ones in Einstein gravity, so we look at just the metric fluctuation bubble diagram.

Since we decomposed the metric fluctuation propagator into a graviton and a fakeon part, we can view the bubble diagram as the sum of three diagrams, namely the pure graviton diagram(Fig.10.a), the pure fakeon diagram(Fig.10.b) and the mixed diagram, i.e. one graviton propagator and one fakeon propagator to make the loop(Fig.10.c).

It turns out that the diagrams that include fakeon propagators (both pure and mixed) are purely imaginary, so the only contribution to the absorptive part of the amplitude is coming from the pure graviton loops that contain a real part. To see this we can look at what would be the contribution by a fakeon-containing diagram. By following the theory outlined in the previous sections, the diagrams must be calculated in Euclidean space before being non-analytically Wick-rotated to Minkowski. Before letting $\Upsilon \to 0$ we are dealing with a Lee-Wick theory so we can use the results of section 3.2.

- Mixed diagram(fig.10.c): There is no Lee-Wick threshold located on the real p^0-axis, because the Lee-Wick pinching occurs at a distance Υ away from the real axis. For real p then, the Wick-rotation is analytic. The final result is purely imaginary, because an i is picked up by the Cauchy-theorem for the integral of the k^0 loop and below the threshold the $i\epsilon$ is redundant so we can drop it ($\epsilon \to 0$) which leaves the remaining integrand real and hence the total contribution purely imaginary.

- Pure fakeon diagram(fig.10.b): we can ignore the $i\epsilon$ prescription from the beginning because there are only fakeons in the loop. Lee-Wick thresholds can lie on the real p^0-axis in this case. Below the thresholds the result is purely imaginary the same way as in the above case. The region above the thresholds can be reached by the average continuation using (3.29). The analytic continuation of a function that is purely imaginary on some part of the real axis is itself purely imaginary on the entire real axis and so is the average of the continuation from above and below the real axis.

So the only diagram contributing to the absorptive part of the graviton self energy is the pure graviton loop of Fig.10. This also means that we can use as a propagator just the graviton part $\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_{0grav}$ in (3.62), that is the graviton underlined part in (3.70).

As is shown in [41], the absorptive part of the self energy can also be derived from the divergent part of the self energy in the low energy expansion of the action. We state here once more how the Feynman prescription differs from the fakeon prescription.

The loop integrals of the graviton self energy are related to the external momentum p via some polynomial of the couplings multiplied by $ln(-p^2)$. The poles using the Feynman prescription give contributions like $-\frac{1}{2}ln(-p^2-i\epsilon)$ from which we can extract the absorptive part, which is proportional to $i\frac{\pi}{2}\theta(p^2)$. When we use the fakeon prescription for the ghost poles(that we turned into Lee-Wick poles), by the average continuation, we study $f(z) = ln(z)$[37]. Its branch cut is on the negative real axis. Analytically continuing from above and below we get $f_{average}(z) = \frac{1}{2}[f_+(z)+f_-(z)] = \frac{1}{2}ln(z^2)$. So using the fakeon prescription the contribution is proportional to $-\frac{1}{4}ln(-p^2)^2$ which has no absorptive part. The quantum gravity theory involving the fakeon prescription thus gives predictions radically different from the usual quantization as used in the Stelle theory.

3.3.4 Auxiliary field formulation of fakeon quantum gravity

In the previous section the massive scalar of the graviton multiplet was implicitly turned into a fakeon as the fakeon prescription was used on both poles that were not the graviton. This does not have to be the case as we can also let this scalar propagate as a physical degree of freedom by quantizing it in the usual way. To separate the different contributions in the gravitational action (3.58), we can formulate the action using auxiliary fields similar to the toy model in Appendix A. In this section we follow closely Anselmi and Piva in [42].

We can couple (3.58) to the matter sector, which for simplicity we will make up of scalars that are at least quadratic in the action to obtain:

$$S_{quadratic} = -\frac{1}{2\kappa^2}\int\sqrt{-g}\left[2\Lambda_C + \zeta R + \alpha\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) - \frac{\xi}{6}R^2\right] + S_m(g,\varphi_m), \quad (3.72)$$

with the coefficients defined in (3.58) and φ_m denotes the generic matter content of the matter action S_m.

In terms of the Weyl tensor in (2.10) we have

$$S_{quadratic} = -\frac{1}{2\kappa^2}\int\sqrt{-g}\left[2\Lambda_C + \zeta R + \frac{\alpha}{2}W^2 - \frac{\xi}{6}R^2\right] + S_m(g,\varphi_m)n \quad (3.73)$$

We then rewrite the pure gravity part(i.e. without the matter sector) conveniently as follows:

$$-\frac{1}{2\kappa^2}\int\sqrt{-g}\left[2\Lambda_C + \zeta R + \frac{\alpha}{2}W^2 - \frac{\xi}{6}R^2 + \frac{8}{3}\frac{\xi}{\zeta^2}\Lambda_C^2 + \frac{4}{3}\frac{\xi}{\zeta}\Lambda_C R - \frac{8}{6}\xi\frac{\Lambda_C}{\zeta}R - \frac{16}{6}\xi\frac{\Lambda_C^2}{\zeta^2}\right] \quad (3.74)$$

and regroup as:

$$-\frac{1}{2\kappa^2}\int\sqrt{-g}\left[2\left(\Lambda_C + \frac{4}{3}\frac{\xi}{\zeta^2}\Lambda_C^2\right) + \left(\zeta + \frac{4}{3}\frac{\xi}{\zeta}\Lambda_C\right)R + \frac{\alpha}{2}W^2 - \frac{\xi}{6}\left(R + 4\frac{\Lambda_C}{\zeta}\right)^2\right], \quad (3.75)$$

so that we can make the field redefinitions

$$\hat{\Lambda}_C = \Lambda_C\left(1 + \frac{4\xi\Lambda_C}{3\zeta^2}\right), \quad \hat{\zeta} = \zeta\frac{\hat{\Lambda}_C}{\Lambda_C}, \quad \hat{R} = R + \frac{4\Lambda_C}{\zeta}, \quad (3.76)$$

to obtain the more convenient expression

$$S_{quadratic} = -\frac{1}{2\kappa^2}\int\sqrt{-g}\left[2\hat{\Lambda}_C + \hat{\zeta}R + \frac{\alpha}{2}W^2 - \frac{\xi}{6}\hat{R}^2\right] + S_m(g,\varphi_m) \quad (3.77)$$

We can also write (3.77) in terms of the Einstein-Hilbert and Weyl actions

$$S_{quadratic} = \hat{S}_{EH}(g) + S_W(g) + \frac{\xi}{12\kappa^2}\int\sqrt{-g}\hat{R}^2 + S_m(g,\varphi_m) \quad (3.78)$$

in which

$$\hat{S}_{EH} = -\frac{1}{2\kappa^2}\int\sqrt{-g}\left(2\hat{\Lambda}_C + \hat{\zeta}R\right) \quad \text{and} \quad S_W(g) = -\frac{\alpha}{4\kappa^2}\int\sqrt{-g}W^2 \quad (3.79)$$

are the Einstein-Hilbert action and the Weyl action, respectively.

We can now introduce an auxiliary field $\hat{\phi}$ in the $\hat{R}^2$ term to write (3.78) as

$$S_{quadratic} = \hat{S}_{EH}(g) + S_W(g) + \frac{\xi}{12\kappa^2}\int\sqrt{-g}(2\hat{R}+\hat{\phi})\hat{\phi} + S_m(g,\varphi_m) \tag{3.80}$$

We then perform the Weyl transformation $g_{\mu\nu} \to g_{\mu\nu}e^{\kappa\phi}$ with the conveniently chosen

$$\phi = -\frac{1}{\kappa}ln\left(1-\frac{\zeta\hat{\phi}}{3\hat{\zeta}}\right) \quad \text{or} \quad \hat{\phi} = \frac{3\hat{\zeta}}{\xi}(1-e^{-\kappa\phi}), \tag{3.81}$$

The Weyl term in (3.78) is evidently invariant under the Weyl transformation and the Ricci scalar transforms as

$$R \to e^{\kappa\phi}\left(R - 3\kappa\nabla^2\phi - \frac{3\kappa^2}{2}\nabla^\mu\phi\nabla_\mu\phi\right), \tag{3.82}$$

such that the $\hat{R}$ term in the auxiliary field action transforms as:

$$\begin{aligned}\frac{\xi}{12\kappa^2}\int\sqrt{-g}2R\hat{\phi} &\to \int\sqrt{-g}e^{2\kappa\phi}e^{-\kappa\phi}R(1-e^{-\kappa\phi})\frac{3\hat{\zeta}}{\xi}\frac{\xi}{6\kappa^2}\\ &= \int\sqrt{-g}\left[e^{\kappa\phi}R\frac{\hat{\zeta}}{2\kappa^2} - R\frac{\hat{\zeta}}{2\kappa^2}\right]\end{aligned} \tag{3.83}$$

and the term proportional to the Ricci scalar in the $\hat{S}_{EH}(g)$ term transforms as

$$-\frac{1}{2\kappa^2}\int\sqrt{-g}\hat{\zeta}R \to -\frac{\hat{\zeta}}{2\kappa^2}\int\sqrt{-g}e^{2\kappa\phi}e^{-\kappa\phi}R = -\frac{\hat{\zeta}}{2\kappa^2}\int\sqrt{-g}e^{\kappa\phi}R \tag{3.84}$$

and we can conclude that the sum of the transformed contributions makes the $e^{\kappa\phi}$ terms in (3.83) and (3.84) cancel, leaving $\hat{S}_{EH}(g)$ unchanged and the action of ϕ independent of R. The covariant derivatives on ϕ, due to the transformation (3.82), and the other couplings recombine into the following action for ϕ

$$S_\phi(g,\phi) = \frac{3\hat{\zeta}}{4}\int\sqrt{-g}\left[D_\mu\phi D^\mu\phi - \frac{1}{\kappa^2}\frac{\zeta}{\xi}(1-e^\kappa\phi)^2\right] \tag{3.85}$$

in which we can identify the mass of the scalar ϕ as $m^2 = \zeta/\xi$. The total action that we end up with is

$$S_{quadratic} = \hat{S}_{EH}(g) + S_W(g) + S_\phi(g,\phi) + S_m(ge^{\kappa\phi},\varphi_m). \tag{3.86}$$

Now that we have isolated the massive scalar of the graviton multiplet we turn to the isolation process of the fakeon. We first rewrite the action (3.86) using:

$$\tilde{\Lambda}_C = \Lambda_C\left(1+\frac{2(\alpha+2\xi)\Lambda_C}{3\zeta^2}\right), \quad \tilde{\zeta} = \zeta\frac{\tilde{\Lambda}_C}{\Lambda_C}, \quad \tilde{R}_{\mu\nu} = R_{\mu\nu} + \frac{\Lambda_C}{\zeta}g_{\mu\nu}, \tag{3.87}$$

so that we obtain the action

$$S_{quadratic} = \tilde{S}_{EH}(g) - \frac{\alpha}{2\kappa^2}\int \sqrt{-g}\left(\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} - \frac{1}{3}\tilde{R}^2\right) + S_\phi(g,\phi) + S_m(ge^{\kappa\phi},\varphi_m). \quad (3.88)$$

We now introduce another auxiliary field. The auxiliary field, that we will identify as the fakeon, is $\chi_{\mu\nu}$ and we write the action as

$$\begin{aligned} S_{quadratic} =& \tilde{S}_{EH}(g) - \frac{\tilde{\zeta}}{2\kappa^2}\int\sqrt{-g}\left[2\chi^{\mu\nu}\left(\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}\right) - \frac{\tilde{\zeta}}{\alpha}(\chi_{\mu\nu}\chi^{\mu\nu} - \chi^2)\right] \\ &+ S_\phi(g,\phi) + S_m(ge^{\kappa\phi},\varphi_m) \end{aligned} \quad (3.89)$$

The redefinition of the metric that we ought to apply now is

$$g_{\mu\nu} \to g_{\mu\nu} + 2\chi_{\mu\nu} + \chi_{\mu\nu}\chi - 2\chi_{\mu\rho}\chi^{\rho}_{\nu}$$

in which $\chi = \chi_{\mu\nu}g^{\mu\nu}$. Since we cannot perform calculations at non-vanishing Λ_C we can use the simpler redefinition:

$$g_{\mu\nu} \to g_{\mu\nu} + 2\chi_{\mu\nu} \quad (3.90)$$

to write (3.72) at $\Lambda_C = 0$ as the equivalent action

$$S_{quadratic} = S_H(g) + S_\chi(g,\chi) + S_\phi(g + 2\chi,\phi) + S_m(ge^{\kappa\phi} + 2\chi e^{\kappa\phi},\varphi_m) \quad (3.91)$$

in which S_H is just the Hilbert action

$$S_H = -\frac{1}{2\kappa^2}\int\sqrt{-g}\hat{\zeta}R. \quad (3.92)$$

The actual computation of S_χ is rather messy, so we will just report here that the mass of the fakeon field χ corresponds to $m_\chi^2 = \frac{\zeta}{\alpha}$ or $m_\chi^2 = \frac{\tilde{\zeta}}{\alpha}$ in the $\Lambda_C \neq 0$ case. We can see this by looking at the coefficient in front of the term $(\chi_{\mu\nu}\chi^{\mu\nu} - \chi^2)$ in (3.89), which gives the mass, and in the case of $\Lambda_C = 0$ the coefficient $\tilde{\zeta} = \zeta\left(1 + \frac{2(\alpha+2\xi)\Lambda_C}{3\zeta^2}\right)$ reduces to ζ. The action for S_χ can be found in [42] (in which S_χ corresponds to the $\Lambda_C \neq 0$ case and S'_χ to the $\Lambda_C = 0$ case).

As mentioned before, when we expand the metric $g_{\mu\nu}$ in the usual way around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$, the action (3.91) gives three particles for the gravity sector, namely the massless spin 2 graviton that is the fluctuation $h_{\mu\nu}$, the massive spin 0 scalar ϕ and the massive spin 2 ghost $\chi_{\mu\nu}$. We recognize $\chi_{\mu\nu}$ as a ghost by its wrong sign in the action as in Appendix A and as described in the previous section, we quantize the ghost as a fakeon to satisfy the optical theorem. But in the previous section we implicitly also quantized the scalar ϕ as a ghost. Now that we have separated the different actions of the gravity sector we are able to consider the quantization of the scalar particle separate from the ghost/fakeon particle.

The free propagator of the metric fluctuation is

$$\langle h_{\mu\nu}(p)h_{\rho\sigma}(-p)\rangle_0 = \frac{i(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma})}{2\tilde{\zeta}(p^2+\frac{2\Lambda_C}{\zeta}+i\epsilon)}, \tag{3.93}$$

which satisfies the optical theorem as long as Λ_C is nonpositive.

For the free ghost/fake particle we have the propagator

$$\langle \chi_{\mu\nu}(p)\chi_{\rho\sigma}(-p)\rangle_0 = -\frac{i\kappa^2}{\tilde{\zeta}}\frac{1}{p^2-m_\chi^2+i\epsilon}\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}(p,m_\chi^2) \tag{3.94}$$

with the mass of χ defined above as $m_\chi^2 = \frac{\tilde{\zeta}}{\alpha}$.[8] Note the minus sign in front of the propagator that characterizes it as a ghost. We turn the ghost into a fakeon by quantizing it as

$$\langle \chi_{\mu\nu(p)}\chi_{\rho\sigma}(-p)\rangle_0 = -\frac{i\kappa^2}{\tilde{\zeta}}\frac{(p^2-m_\chi^2+i\epsilon)}{(p^2-m_\chi^2+i\epsilon)^2+\Upsilon^4}\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}(p,m_\chi^2) \tag{3.95}$$

and perform calculations involving fakeons using the average continuation to overcome the thresholds to satisfy the optical theorem.

Now we turn to the massive spin 0 scalar ϕ which has the free propagator

$$\langle \phi(p)\phi(-p)\rangle_0 = \frac{2i}{3\tilde{\zeta}}\frac{1}{p^2-m_\phi^2+i\epsilon} \tag{3.96}$$

with the mass of ϕ defined above as $m_\phi^2 = \zeta/\xi$. This amounts to a physical scalar particle, and calculations involving this propagator already satisfy the optical theorem, but without experimental constraints we can choose to quantize it as a fakeon as well. Doing so we obtain

$$\langle \phi(p)\phi(-p)\rangle_0 = \frac{2i}{3\tilde{\zeta}}\frac{(p^2-m_\phi^2+i\epsilon)}{(p^2-m_\phi^2+i\epsilon)^2+\Upsilon^4} \tag{3.97}$$

and we treat calculations involving this "former scalar" fakeon propagator the same as the "former ghost" fakeon. This options thus leads to two unitary and renormalizable quantum theories of gravity, one consisting of a graviton, a fakeon and a scalar and the other consisting of a graviton and two fakeons.

So far we have discussed the fakeon as a particle that does not propagate a physical degree of freedom, but merely a *fake degree of freedom.* In discussing the absorptive part of the self energy we saw that a theory containing fakeons is inherently different from a theory containing ghosts. We did not yet get to physical quantities that would be measurable in experiments. We do know, however, that using the sum over the contributions to the self energy, we can write the dressed propagator in terms of the width of a particle, as is shown in Appendix B.2. Such a dressed propagator is valid at a certain peak in the energy,

[8] $\mathcal{P}^{(2)}$ is the spin 2 projection operator on shell defined by $\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}(p,m_\chi^2) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma}+\theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}$ with $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m_\chi^2}$

namely when the center of mass energy is close to the particle mass. This expression has been derived for the fakeon in [42] and reads

$$iD(p^2 \sim m_\chi^2) = -\frac{i\kappa^2}{\zeta}\frac{Z_\chi}{p^2 - \bar{m}_\chi^2 + i\bar{m}_\chi\Gamma_\chi}\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}(p, m_\chi^2). \tag{3.98}$$

Anselmi and Piva define a constant $\alpha_\chi = m_\chi^2/M_{Planck}^2$ as a *sort of "fakeon/gravtion structure constant"* to express the quantities appearing in (3.98) in terms of this constant. To first order in α_χ the quantity Z_χ and the corrected mass $\bar{m}_\chi$ are simply given by

$$Z_\chi = 1 + \mathcal{O}(\alpha_\chi), \quad \bar{m}_\chi^2 = m_\chi^2[1 + \mathcal{O}(\alpha_\chi)], \tag{3.99}$$

and the width of the fakeon Γ_χ is given by

$$\Gamma_\chi = -m_\chi\alpha_\chi C + \mathcal{O}(\alpha_\chi^2). \tag{3.100}$$

The term C appearing in (3.100) is related to the matter content of the total action and can contain only the matter added to the quadratic gravity action but it can also receive a contribution form the massive spin zero scalar (3.96) if it is not quantized as a fakeon. It is given by

$$C(m_\chi^2) = C_m + C_\phi(m_\chi), \quad C_m = \frac{N_s + 6N_f + 12N_v}{120}, \quad C_\phi(m_\chi^2) = \frac{1}{120}\theta(1 - r_\phi)(1 - r_\phi)^{5/2}, \tag{3.101}$$

with $r_\phi = 4m_\phi^2/m_\chi^2$. The quantity C_m is known in conformal field theory as the *central charge* with N_s the number of scalars, N_f the number of Dirac fermions plus half of the Weyl fermions and N_v the number of massless vectors, all in the matter sector. The value for C_ϕ is zero if the massive scalar (3.96) is quantized as a fakeon since it is not a real particle. Its contribution will also be zero if its mass is larger than $m_\chi/2$. The width of the fakeon, Γ_χ, remains a negative quantity which points to the violation of causality. Anselmi and Piva dub this a *violation of microcausality*, as the inverse of the absolute value of the width should not be interpreted as the lifetime of the fakeon but as the amount of time during which causality is meaningless. The violation of causality is a known feature of Lee-Wick models as discussed in [43] and remains in its new formulation.

For completion we also report the width of the scalar particle.

$$\Gamma_\phi = m_\phi\alpha_\phi\frac{\eta_s^2}{6}, \tag{3.102}$$

with the "scalar/graviton constant" $\alpha_\phi \equiv m_\phi^2/M_{Planck}^2$ and η_s the scalar coupling. For m_χ and m_ϕ the same order of magnitude, $|\Gamma_\chi|$ is expected to be much larger than Γ_ϕ, because with the Standard Model matter content $C \approx \frac{5}{2}(N_s = 4, N_f = 45/2, N_v = 12)$, which gives $|\Gamma_\chi|/\Gamma_\phi \approx 15/\eta_s^2(\eta_s^2$ is expected to be much smaller than 15).

3.3.5 Renormalization with new prescription

Since the one-loop renormalization of (3.58) in the non-analytically Wick-rotated theory coincides with its Euclidean version we will just report here the results in Euclidean spacetime obtained by Anselmi and Piva in [41], in which they made use of the Batalin-Vilkovisky formalism [44],[45]. This allowed them to both derive previously unknown results, that are not of our interest at this moment, and provide a check of the results derived by Salvio and Strumia in [29] and before that in [26], [27] and [28]. After perturbative renormalization the counterterms can be collected in a counterterm action:

$$S_{couterterm} = -\frac{\mu^{-\epsilon}}{(4\pi)^2\epsilon}\int\sqrt{-g}\Big[2\Delta\Lambda_C + \Delta\zeta R + \Delta\alpha\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big) - \frac{\Delta\xi}{6}R^2\Big] + (S,\mathcal{F}), \tag{3.103}$$

in which the term $(S,\mathcal{F})$ is a term called the *antiparentheses* of the action and some functional $\mathcal{F}$(which can be found in [41]). It is inherent to the Batalin-Vilkovisky formalism and contains, among other things, the newly derived results in [41]. We merely have to look at the usual counterterms. The beta functions and the corresponding counterterm coefficients are

$$\beta_\alpha = -\frac{2\kappa^2}{(4\pi)^2}\Delta\alpha \quad \text{with} \quad \Delta\alpha = -\frac{133}{10} \tag{3.104}$$

$$\beta_\xi = -\frac{2\kappa^2}{(4\pi)^2}\Delta\xi \quad \text{with} \quad \Delta\xi = \frac{5}{6} + \frac{5\xi}{\alpha} + \frac{5\xi^2}{3\alpha^2} \tag{3.105}$$

$$\beta_\zeta = -\frac{2\kappa^2}{(4\pi)^2}\Delta\zeta \quad \text{with} \quad \Delta\zeta = \zeta\Big(\frac{5}{6\xi} + \frac{5\xi}{3\alpha^2} + A\Big) \tag{3.106}$$

$$\beta_{\Lambda_C} = -\frac{2\kappa^2}{(4\pi)^2}\Delta\Lambda_C \quad \text{with} \quad \Delta\Lambda_C = \Lambda_C\Big(-\frac{5}{\alpha} + \frac{2}{\xi} + 2A\Big) - \frac{5\zeta^2}{4\alpha^2} - \frac{\zeta^2}{4\xi^2} \tag{3.107}$$

where A is a gauge dependent quantity. It is a parametrization of the gauge dependence of a quantity(t_0 in [41]) that can be seen as the renormalization of the flat-space background metric $\eta_{\mu\nu}$. This gauge dependence is eventually responsible for the surviving gauge dependence in the renormalization of our coefficients Λ_C and ζ, so that they can only occur in the combination Λ_C/ζ^2. Notice the corrected version of (3.107) with +2A, instead of -2A in [41].

The full beta functions at one loop are thus

$$\beta_\alpha = +\frac{2\kappa^2}{(4\pi)^2}\frac{133}{10} \tag{3.108}$$

$$\beta_\xi = -\frac{2\kappa^2}{(4\pi)^2}\Big(\frac{5}{6} + 5\frac{\xi}{\alpha} + \frac{5}{3}\frac{\xi^2}{\alpha^2}\Big) \tag{3.109}$$

$$\beta_\zeta = -\frac{2\kappa^2}{(4\pi)^2}\zeta\Big(\frac{5}{6\xi} + \frac{5\xi}{3\alpha^2} + A\Big) \tag{3.110}$$

$$\beta_{\Lambda_C} = -\frac{2\kappa^2}{(4\pi)^2}\left[\Lambda_C\left(-\frac{5}{\alpha}+\frac{2}{\xi}+2A\right)-\frac{5\zeta^2}{4\alpha^2}-\frac{\zeta^2}{4\xi^2}\right] \tag{3.111}$$

To look at the renormalization group equation of the gauge independent combination Λ_C/ζ^2 we have to differentiate:

$$\frac{\Lambda_C}{\zeta^2} \rightarrow \Delta\Lambda_C\frac{1}{\zeta^2} - 2\Lambda_C\frac{1}{\zeta^3}\Delta\zeta = \frac{1}{\zeta^3}\Big[\zeta\Delta\Lambda_C - 2\Lambda_C\Delta\zeta\Big], \tag{3.112}$$

where the part in brackets on the right hand side is the gauge independent variation we are looking for. So we can fill in the Δ-coefficients of (3.106) and (3.107) to obtain the β-function of the gauge independent combination:

$$\beta_{(\zeta\Delta\Lambda_C-2\Lambda_C\Delta\zeta)} = -\frac{2\kappa^2}{(4\pi^2)}\left[\zeta\Lambda_C\left(\frac{2}{6}\frac{1}{\xi}-5\frac{1}{\alpha}-\frac{10}{3}\frac{\xi}{\alpha^2}\right)-\zeta^3\left(\frac{5}{4}\frac{1}{\alpha^2}+\frac{1}{4}\frac{1}{\xi^2}\right)\right]. \tag{3.113}$$

This is indeed independent of A and hence gauge independent. Indeed the corrected version of (3.107) with +2A, instead of -2A in [41] does lead to the gauge independent combination in (3.113).

4 Conclusions and outlook

We will first briefly state our conclusions concerning the general formulation of gravitational theories containing terms quadratic in the curvature, i.e. higher derivative theories of gravity. We then turn to more specific conclusions about the proposal by Anselmi to utilize a new quantization prescription based on a new formulation of Lee-Wick models. We close with an outlook for future research in this direction and point out some ongoing research not yet discussed in this thesis.

4.1 Conclusions on higher derivative theories in general

Theories quadratic in the curvature can be used to make quantum gravity renormalizable at the cost of losing unitarity, so by itself the addition of higher derivative terms is not a solution to the problem of quantizing gravity in a consistent way.
Having a higher derivative theory of quantum gravity without the Einstein-Hilbert term R might be appealing for some reasons, but it is not a valid candidate for quantum gravity as it does not have the correct large distance limit, namely the Newtonian potential or some potential closely resembling that one within present experimental uncertainties. Conformal gravity and agravity, discussed in chapter 2, suffer from this infrared problem and they can thus not be regarded as a viable complete candidate for a quantum theory of gravity satisfactory both in the ultraviolet and the infrared.

4.2 Conclusions on the new quantization prescription for higher derivative quantum gravity

The newly formulated Lee-Wick models, described in section 3.2, are consistent. The ambiguity resulting from the CLOP prescription to recover Lorentz invariance, discussed in section 3.1.2 is gone in the new formulation (see Fig.9). The new prescription for quantizing theories that have negative kinetic energy, i.e. ghosts, when using the Feynman prescription, can be used to turn ghost particles into fake particles, thus turning ghost-plagued theories into unitary theories. This new prescription can be applied to the higher derivative theory of gravity, shown by Stelle to be renormalizable, to turn it into a unitary theory. The current status of the new prescription leaves us two options for the scalar particle ϕ in the graviton multiplet. We can either let it propagate as a physical scalar or we can turn it into a fake particle by applying the new quantization prescription, as both formulations are self-consistent. Some remarks on the new prescription resulting in the fakeon have to be made and some improvements are still needed though.

We know from the general higher derivative quantum gravity theory discussion that we need a non-vanishing ζ coupling in (3.58) (i.e. a non vanishing Einstein-Hilbert term) to arrive at a suitable classical limit, meaning a gravitational potential resembling a Newtonian potential as we saw in section 2.1. And since we also know from the renormalization discussion in section 3.3.5 that $\frac{\Lambda_C}{\zeta^2}$ is the only gauge independent combination of Λ_C and ζ, we need the tools to do calculations for a non-vanishing cosmological constant $\Lambda_C \neq 0$. Another indication that we are directed to a model with a non-vanishing cosmological constant comes from cosmological observation, or we would need another mechanism to account for the observations made so far. A non-vanishing cosmological constant also means that the theory has to be consistently formulated in de Sitter space, while the theory so far has been formulated in Euclidean and is then Wick-rotated to Minkowski. The proof of unitarity to all orders in perturbation theory as mentioned in section 3.2.3 and stated in [37] is, however, only valid without a cosmological constant, as Anselmi states in [1]: *"a non-vanishing cosmological constant prevents us from proving unitarity in a strict sense"*.

So the new quantization prescription can be used to unify renormalizability and unitarity in quantum gravity, but does come with big consequences.

A remnant of the fakeons is the violation of microcausality. First we can conclusde that for $E > m_\chi$ causality is violated, so the fakeons have to be massive. As the violation of microcausality only occurs for energies larger than the energies of the fakeons, a fakeon that is heavy with respect to energies currently available in experiments means we will not see the violation of microcausality in those experiments.

Another observation concerning the fakeons is that the inverse of the width $\frac{1}{|\Gamma_\chi|}$ should in this case not be regarded as the lifetime of the fakeon, but rather as the time during which causality is violated.

4.3 Outlook for future research

The conclusion on the non-vanishing cosmological constant and thus the requirement of a formulation in de Sitter space brings us to the outlook for future research, as this is a task that has to be undertaken to arrive at a proper theory of quantum gravity. The issue of whether a non-vanishing cosmological constant still leads to a unitary theory should also be addressed in future research.

Concerning the possibilities of measurements we can say that measuring the *sign of the width* of a particle, in this case Γ_χ, is not the way to go, as a negative width is not something we can measure in an experiment, but is merely a mathematical formulation of the violation of microcausality. And even though the *masses* of the fakeon m_χ and the scalar m_ϕ could be relatively light with respect to the Planck mass M_{Planck}, they might still be far too heavy to detect in present-day collision experiments.

A possibly more fruitful phenomenological approach is looking for deviations from General Relativity that occur due to the fakeons, since they would survive the classical limit. The violations of causality would require corrections to the field equations and it might be interesting to study them to probe the existence of the fakeons. The Einstein field equations classically have the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{4.1}$$

with $T_{\mu\nu}$ the stress-energy tensor. If the fakeons are a feature of quantum gravity the field equations would change into

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \langle T_{\mu\nu} \rangle \tag{4.2}$$

where $\langle T_{\mu\nu} \rangle$ is now an average due to the fakeons, and it is a little bit acausal. A start of investigating the effect of fakeons and the violation of microcausality in the classical limit of quantum gravity can be found in [46].

Appendix A Auxiliary field formulation of higher derivative theories

This appendix follows closely the analysis and conventions of [34]. To see the equivalence between a higher derivative theory and an auxiliary field formulation of that theory in which the extra degree of freedom is explicit, we start with a cubic self interacting scalar theory with the following Lagrangian:

$$\mathscr{L}_{HD} = \frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2M^2}(\partial^2 A)^2 - \frac{1}{2}m^2 A^2 - \frac{\lambda}{3!}A^3 \tag{A.1}$$

containing the scalar field A with mass m and the Lee-Wick scale M. The propagator for the field A has multiple poles, which means that more than one degree of freedom is described by the theory which we can make explicit. We can introduce an auxiliary field B such that we can write the theory as

$$\mathscr{L} = \frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2}m^2 A^2 - B\partial^2 A + \frac{1}{2}M^2 B^2 - \frac{\lambda}{3!}A^3 \tag{A.2}$$

which would return A.1 if we substitute B as solution of its equation of motion.

If we now introduce the field $C = A + B$ and integrate by parts, thus disregarding total derivative terms, we obtain the new Lagrangian in terms of the fields B and C

$$\mathscr{L} = \frac{1}{2}\partial_\mu C \partial^\mu C - \frac{1}{2}\partial_\mu B \partial^\mu B - \frac{1}{2}m^2 (C-B)^2 + \frac{1}{2}M^2 B^2 - \frac{\lambda}{3!}(C-B)^3 \tag{A.3}$$

We have a regular scalar field C and a Lee-Wick type scalar field B as it is associated with the Lee-Wick scale M. The regular scalar field and the Lee-Wick scalar field mix in the presence of the mass m. We can, however, diagonalize the Lagrangian while preserving the diagonal form of the derivative terms by performing a symplectic rotation

$$\begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} C' \\ B' \end{pmatrix}$$

which diagonalizes the Lagrangian for

$$\tanh 2\theta = \frac{-2m^2/M^2}{1 - 2m^2/M^2} \tag{A.4}$$

and with new coupling λ' to be

$$\lambda' = \lambda(\cosh\theta - \sinh\theta)^3$$

We end up with the diagonalized Lagrangian

$$\mathscr{L}_{Auxiliary} = \frac{1}{2}\partial_\mu C' \partial^\mu C' - \frac{1}{2}m'^2 C'^2 - \frac{1}{2}\partial_\mu B' \partial^\mu B' + \frac{1}{2}M'^2 B'^2 - \frac{\lambda'}{3!}(C' - B')^3 \tag{A.5}$$

with m' and M' being the masses of the fields C' and B', respectively. The kinetic term for the field B' has the opposite sign to the usual one, which makes it the Lee-Wick (ghost) field. The formula A.4 has a solution only if θ is $M > 2m$. This corresponds to the condition that this negative norm state has to be able to decay into two lighter positive norm states.

Appendix B Optical Theorem

B.1 Optical Theorem in general

Unitarity means conservation of probability[47]. Considering quantum states, conservation of probability over time can be expressed as:

$$\langle\psi;t|\psi;t\rangle = \langle\psi;0|\psi;0\rangle$$

We know the Hamiltonian to be a hermitian matrix due to $|\psi;t\rangle = e^{-iHt}|\psi;0\rangle$. We then look at the S-matrix which is defined by: $S = e^{-iHt}$, which allows us to express unitarity in terms of the S-matrix:

$$S^\dagger S = 1 \tag{B.1}$$

In a free theory, the S-matrix is equal to the identity matrix $\mathbb{1}$ and in an interacting theory we add deviations from the free theory by $S = \mathbb{1} + i\mathcal{T}$ where $\mathcal{T}$ is called the transfer matrix, the non-trivial part of the S-matrix, and it is defined by:

$$\mathcal{T} = (2\pi)^4\delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)\mathcal{M} \tag{B.2}$$

$\mathcal{M}$ is the non-trivial matrix element Sandwiching between final and initial states f and i, respectively, and using $i\mathcal{T} = S - \mathbb{1}$

$$\langle f|S - \mathbb{1}|i\rangle = i(2\pi)^4\delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)\langle f|\mathcal{M}|i\rangle$$

Which we will write as:

$$\langle f|\mathcal{T}|i\rangle = i(2\pi)^4\delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)\mathcal{M}(i \to f)$$

Inserting the S-matrix expressed in terms of the transfer matrix into the unitarity equation of the S-matrix, we obtain:

$$S^\dagger S = (\mathbb{1} - i\mathcal{T}^\dagger)(\mathbb{1} + i\mathcal{T}) = 1 \tag{B.3}$$

$$i(\mathcal{T}^\dagger - \mathcal{T}) = \mathcal{T}^\dagger\mathcal{T} \tag{B.4}$$

Sandwiching the lefthand-side of (B.4) between final and initial states we obtain:

$$\langle f|(\mathcal{T}^\dagger - \mathcal{T})|i\rangle = i\langle i|\mathcal{T}|f\rangle^* - i\langle f|\mathcal{T}|i\rangle \tag{B.5}$$

Evaluating this equation by inserting (B.2) into the equation we find:

$$\langle f|(\mathcal{T}^\dagger - \mathcal{T})|i\rangle = i(2\pi)^4\delta^4(p_i - p_f)(\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f) \tag{B.6}$$

To evaluate the righthand-side of (B.4) we sandwich between final and initial state as well, but we also make use of the particular property of Hilbert space, namely that it is complete:

$$\sum_X \int d\prod_X |X\rangle\langle X| = \mathbb{1}$$

with $d\prod_X \equiv \prod_{i\in X} \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}$ and the sum over states $|X\rangle$. So we get:

$$\langle f|\mathcal{T}^\dagger\mathcal{T}|i\rangle = \sum_X \int d\prod_X \langle f|\mathcal{T}^\dagger|X\rangle\langle X|\mathcal{T}|i\rangle$$

$$\langle f|\mathcal{T}^\dagger\mathcal{T}|i\rangle = \sum_X (2\pi)^4\delta^4(p_f - p_X)(2\pi)^4\delta^4(p_i - p_X)\int d\prod_X \mathcal{M}(i\to X)\mathcal{M}^*(f\to X) \quad \text{(B.7)}$$

Now we can insert (B.7) and (B.6) in (B.4) to obtain what can be called the *generalized optical theorem*:

$$\mathcal{M}(i\to f) - \mathcal{M}^*(f\to i) = i\sum_X \int d\prod_X (2\pi)^4\delta^4(p_i - p_X)\mathcal{M}(i\to X)\mathcal{M}^*(f\to X) \quad \text{(B.8)}$$

The requirement that we posited at the beginning of this derivation was that the theory, to which we apply this theorem, is unitary. The theorem relates the difference of two matrix elements on the left hand side to the product of two matrix elements on the right hand side.

B.2 Self energy

A special case of the optical theorem occurs when the initial and final state are the same state B.

$$2iIm[\mathcal{M}(B\to B)] = i\sum_X \int d\prod_X (2\pi)^4\delta^4(p_B - p_X)|\mathcal{M}(B\to X)|^2 \quad \text{(B.9)}$$

If this state B is a single particle state, it's decay rate is given by

$$\Gamma(B\to X) = \frac{1}{2m_B}\int d\prod_X (2\pi)^4\delta^4(p_B - p_X)|\mathcal{M}(B\to X)|^2$$

such that we can relate:

$$Im[\mathcal{M}(B\to B)] = m_B\sum_X \Gamma(B\to X) = m_B\Gamma_{total}, \quad \text{(B.10)}$$

where Γ_{total} is the total width and hence the inverse of the lifetime of state B. This general expression $Im[\mathcal{M}(B\to B)] = m\Gamma_{total}$ we can use to obtain the dressed propagator in terms of the width of a particle. When all the self-energy contributions are resummed, we can write the dressed propagator as

$$iD(p^2) = \frac{i}{p^2 - m_R^2 + \Sigma(p^2) + i\epsilon} \quad \text{(B.11)}$$

in which m_R is the renormalized mass in the Lagrangian and $i\Sigma(p^2)$ is equal to the sum of all 1PI self energy diagrams. The pole mass $m_p^2 = p^2$, or the mass in the undressed propagator, gives for the relation between $\Sigma(m_p^2)$ and m_R^2 the expression $m_p^2 - m_R^2 + \Sigma(m_p^2) = 0$. In the limit of $\Gamma_{total} \ll m_p$ the relation

$$Im[\mathcal{M}(B \to B)] = m\Gamma_{total} \tag{B.12}$$

reduces to

$$Im\Sigma(m_p^2) = m_p\Gamma_{total}. \tag{B.13}$$

For unstable particles this is nonzero, but we would like to keep the mass real. Therefor we can redefine the pole mass as

$$m_p^2 = m_R^2 - Re\Sigma(m_p^2) \tag{B.14}$$

Using now

$$\Sigma(m_p^2) = Re\Sigma(m_p^2) + iIm\Sigma(m_p^2) \tag{B.15}$$

to write (B.14) as

$$m_R^2 = m_p^2 - \Sigma(m_p^2) + iIm\Sigma(m_p^2) \tag{B.16}$$

We can insert this equation together with (B.13) into the dressed propagator to obtain the dressed propagator near the pole mass in terms of the width

$$iD(p^2) = \frac{i}{p^2 - m_p^2 + im_p\Gamma_{total}}. \tag{B.17}$$

B.3 Cutting rules

We evaluate the imaginary part of a Feynman propagator. We can write

$$\text{Im}\left[\frac{1}{k^2 - m^2 + i\epsilon}\right] = \frac{1}{2i}\left(\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right) = \frac{-\epsilon}{(k^2 - m^2)^2 + \epsilon^2} \tag{B.18}$$

for ϵ going to zero this vanishes, except for $p^2 = m^2$. If we integrate on momentum k^2 we get

$$\int_0^\infty dk^2 \frac{-\epsilon}{(k^2 - m^2)^2 + \epsilon^2} = -\pi \tag{B.19}$$

This means that the propagator has no imaginary part, i.e. is purely real, except for when the propagating particle goes on shell. So the imaginary part in loop amplitudes constitute the on-shell particles.

The discontinuity of an amplitude is due to the branch cuts corresponding to the thresholds of particles. Such a discontinuity separates parts of the positive imaginary half plane from the negative imaginary half plane. The discontinuity of an amplitude can thus be expressed as:

$$\text{Disc}(i\mathcal{M}(k^0)) = i\mathcal{M}(k^0 + i\epsilon) - i\mathcal{M}(k^0 - i\epsilon) = i\mathcal{M} - i\mathcal{M}^* = -2\text{Im}\mathcal{M}(k^0) \tag{B.20}$$

The computation of discontinuities in Feynman diagrams has been generalized by Cutkosky[39] and are now known as the *cutting rules* and are true for all amplitudes of unitary theories.

Applying the cutting rules amounts to cutting the diagram in all possible ways such that we can put the intermediate particles on shell and obeying momentum conservation. Putting the (former) intermediate particles on shell is equal to making the replacement of the propagator by

$$\frac{1}{q^2 - m^2 + i\epsilon} \rightarrow -2i\pi\delta(q^2 - m^2)\theta(q^0), \tag{B.21}$$

where θ is the step function. The discontinuity is then given by the summation over all possible cuts with the replaced propagators. In terms of the $\mathcal{T}$ matrix we can evaluate (B.4) between two possible states f and i:

$$\begin{aligned} i(\mathcal{T}^\dagger - \mathcal{T}) &= \mathcal{T}^\dagger \mathcal{T} \\ i(\langle f|\mathcal{T}^\dagger|i\rangle - \langle f|\mathcal{T}|i\rangle) &= \sum_X \langle f|\mathcal{T}^\dagger|X\rangle\langle X|\mathcal{T}|i\rangle \end{aligned} \tag{B.22}$$

and viewing this equation as diagrams we see that it is closely related to the cutting equations. The left hand side is the imaginary part of the cut diagram and the right hand side is, up to a minus sign, the sum over all the possible cuttings. [5]

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