# The swampland conjectures and inflationary model-building

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# Contents

# Abstract

I review many of the swampland conjectures and the constraints they place on several models of large-field inflation. I start with recapitulating elements of inflationary cosmology, effective field theory, supersymmetry, string theory and specific large-field models with emphasis on $\alpha$-attractors. From the several swampland conjectures I elaborate on the weak gravity conjecture and the swampland distance conjecture. I discuss both bottom-up and top-down arguments supporting ideas surrounding the swampland. I also briefly discuss recent ideas of emergence. In chapter 8 I proceed with a discussion of the Kaloper-Sorbo mechanism of axion monodromy and how this mechanism might be implemented in $\alpha$-attractors. This requires extending the Kaloper-Sorbo gauge theory to a non-abelian three-form gauge theory. I also comment on the implications of the weak gravity conjecture for the abelian monodromy scenario. This requires extending the weak gravity conjecture to massive abelian three-form discrete gauge theories. In the final chapter, I discuss positivity constraints on the space of effective field theories and how this has been used in the context of the weak gravity conjecture.

# Acknowledgements

I want to thank my supervisor Diederik Roest for allowing me to keep the thesis topic very broad so that I was able to study a lot of new topics not covered in the standard curriculum of the Quantum Universe program. I also want to thank David Stefanyszyn for helping me regarding the project as well as during my search for a PhD position.

# Chapter 1

# Introduction

One of the most powerful physical concepts developed in the second half of the previous century are the ideas of effective field theory (EFT). To describe physics at macroscopic scales we do not need to know the details of a theory of quantum gravity. Instead, EFT provides the simplest framework that captures the *essential* physics in a manner that can be corrected to arbitrary precision by including additional subleading corrections.

Constructing quantum EFT's is a systematic procedure. One needs to identify the quantum fields, the expansion parameter(s) and the symmetries and this constrains to some extent the structure of the EFT. But how constraining is this really? A quick comparison with our best theory of quantum gravity, which is string theory, shows that low energy EFT's are not so constrained at all (see figure 1.1). The self-consistency of string theory is so enormously constraining that the theory is uniquely fixed. However, the low energy EFT's derived from it, the so-called landscape string vacua, constitute an immense set of consistent theories and this reflects the enormous freedom one has in writing down a consistent EFT.

But not anything goes. That is to say, not every EFT does arise from a theory of quantum gravity. In fact, the set of theories that does not arise as an EFT from quantum gravity is much larger than the landscape of string vacua. This set has been called the swampland [1]. Now, such a distinction between EFT's is of course useless if there is no prescription of *how* to make this distinction. This "prescription" that defines the boundary between the landscape and the swampland consists of a set conjectures, none of which have been proven.

Only in the last 5-6 years the idea of the swampland started to receive considerable attention from physicists, in particular from those working in early universe cosmology. The reason being that some of the swampland conjectures seemed to provide an explanation of *why* theories of cosmic inflation resisted an embedding in string theory. Equivalently, this suggests that theories of cosmic inflation are at odds with quantum gravity consistency requirements. The fate of popular models of cosmic inflation in light of the swampland conjectures is the main subject in this thesis.

Recent work on the swampland focuses on trying to prove and derive, or at least motivate, the various swampland conjectures from established microphysical principles or from constructing very general examples to check the validity of the conjectures. This constitutes the second main topic.

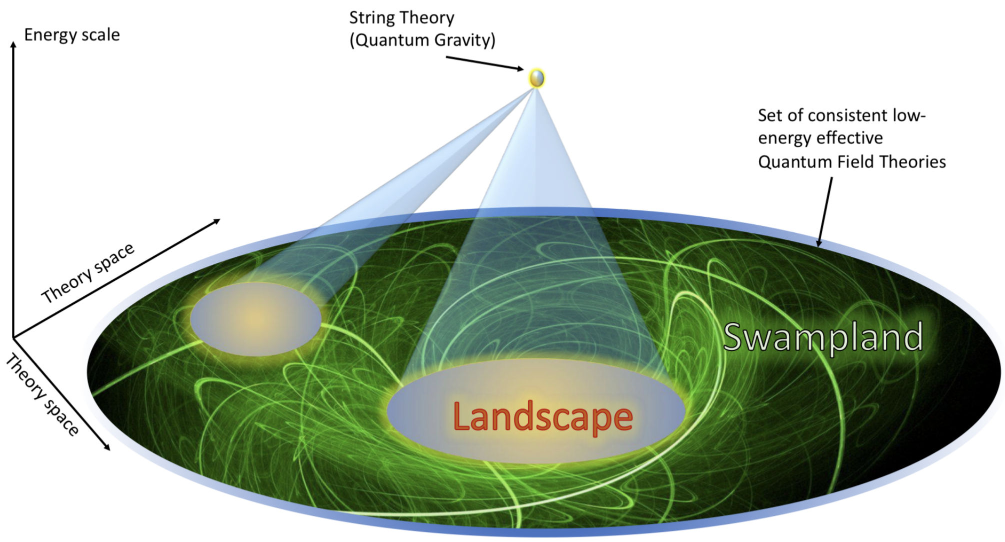



Figure 1.1: A plot of theory space versus energy scales. The dot at high energies indicates the hypothesis that the theory of quantum gravity is unique, which is true in the case of string theory. This picture is taken from [2].

## 1.1 Outline and research questions

This Master thesis is almost entirely a review of existing literature on the swampland and therefore adds nothing new. Below I enumerate -and describe the content of the various chapters and the research questions answered in them but let me list here the two main topics (or research questions) already mentioned above:

1. *What are the consequences of the swampland conjectures for various large-field models of inflation?*

2. *Can we prove or derive the swampland conjectures from a set of well-established physical ideas/principles?*

Before answering such questions one needs to obtain a fair amount of preliminary knowledge so it is not until the end of **Chapter 5** that we start answering question 1. In **Chapter 7** we begin with answering question 2. By far the largest part of this thesis is an attempt to answer question 1.

- **Chapter 2** provides an overview of the essentials of inflationary cosmology and single-field slow-roll inflation. We also briefly discuss the problem of initial conditions.

- **Chapter 3** is about concepts in Wilsonian quantum effective field theory. It describes the naturalness problem of inflation and the formulation of inflation as an EFT. In particular, *we want to understand what it means for inflation* *to be UV sensitive.* The final section collects some elements from supersymmetric field theory that we need in order to describe the $\alpha$-attractor models in **Chapter 4**.

- In **Chapter 4** we describe those models of large-field inflation whose fate we want to study in the context of the swampland conjectures. At the end we return to the problem of initial conditions, now specificially in the context of $\alpha$-attractors. At the very end of **Chapter 5** we will return to this again.

- In **Chapter 5** we have our first encounter with the swampland. For a proper discussion of the swampland, however, we first need to introduce a minimal set of tools from string theory. In this chapter we have two aims that are intertwined. Eventually, we would like to answer the question *what the so-called swampland distance conjecture implies for some of the theories of large-field inflation* discussed in **Chapter 4**. To answer this question we first summarise the results of the very interesting work of [3]. This also provides an answer to our second research question, namely, *how strong is the evidence for the validity of the distance conjecture in explicit string theory constructions*? Along the way we also mention the connection between string dualities and some of the conjectures. We further discuss the consequences of the more recently proposed (and more controversial) de Sitter conjecture for the general idea of single-field slow-roll inflation.

- **Chapter 6** deals with the weak gravity conjecture (WGC). This is the most well-established swampland conjecture. We view the WGC as a statement that makes the idea that quantum gravity abhors continuous global symmetries more quantitative. Therefore, we first want to understand *why quantum gravity forbids continuous global symmetries.* Eventually, we want to again consider the question *what the fate is of inflationary models but now in light of the WGC.* This question is impossible to answer if we do not first *obtain a thorough understanding of the various formulations of the WGC.* This will make up the largest part of the chapter. Having established this, studying the consequences for inflation become relatively simple.

- In **Chapter 7** we encounter for the first time our second thesis subject, namely, the microphysical motivation for some of the swampland conjectures. In this chapter we discuss a possible physical understanding of the swampland distance conjecture that goes by the name of the "emergence proposal"[1].

- **Chapter 8** introduces the Kaloper-Sorbo-Lawrence (KSL) mechanism. We want to study *how it works and why it is useful.* We also want to find out whether *the WGC places any constraints on this mechanism.* Finally, *we make an attempt to implement the KSL mechanism into $\alpha$-attractors.*

- **Chapter 9** proceeds with the same philosophy as in **Chapter 7**, namely, the physical understanding of the swampland. Here we want to study *how one derives positivity bounds, what the connection is between positivity and the WGC and how it can be used to prove a particular formulation of the WGC.*

[1] The emergence proposal actually refers to a more general proposal, namely, that all dynamics at low energies emerges from dynamics in the ultraviolet by integrating-out these dynamical ultraviolet fields.

- In **Chapter 10** we draw our conclusions and suggest further lines of research.
- I have not yet included an **Appendix** with various computations (in particular some regularization calculations).

# Chapter 2

# Inflationary theory

## 2.1 The big-bang puzzles and their resolution

Inflation provides an excellent solution to the Big-Bang puzzles [4, 5, 6]. These problems go by the name of the flatness problem and the horizon problem[1]. These problems are all related to the extremely fine-tuned initial conditions of the universe. A philosophical point worth remarking is the following. One could ask whether a theory of physics should predict its own initial conditions. In all theories of physics we just specify the initial conditions for the computation of dynamics. On the other hand, it would be extremely satisfying if a theory is capable of explaining its initial conditions, in particular when they need to take very specific values. Inflation is such an explanation for the initial conditions of big-bang cosmology. In the next two subsections we want to discuss these cosmological problems and their resolution in some detail[2].

### 2.1.1 The horizon problem

The horizon problem is the problem of initial homogeneity. We know from observations of WMAP, Planck and others [9, 10] that the CMB temperature is extremely homogeneous across the universe with only tiny fluctuations of $\mathcal{O}(10^{-5}K)$. These temperature inhomogeneities grow as a function of time as a consequence of gravitational instability. This implies that the inhomogeneities were much smaller in the early universe. Hence, the early universe was extremely homogeneous. Now, this in itself is not necessarily a problem. However, according to the classical big-bang model of cosmology, the early universe contains a particle horizon of finite size. This limits the causal connections between different spacetime regions. We will now discuss this. Consider a coordinate transformation of the time-coordinate $t \to \tau = \int_0^t \frac{dt'}{a(t')}$ where $a(t')$ denotes the scale-factor. $\tau$ is known as conformal time because this transformation transforms the Euclidean (flat) Friedmann-Robertson-Walker (FRW) metric to a metric that is conformally equivalent to the Minkowski metric:

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 \to ds^2 = a^2(\tau)\Big(d\tau^2 - d\vec{x}^2\Big) \tag{2.1}$$

[1] There are more problems, for example the monopole problem, but we won't discuss those here

[2] This section is heavily inspired by the excellent lecture notes [7, 8]

We can rewrite $\tau$ as follows:

$$\tau = \int_0^{a'} \frac{da'}{Ha^2} = \int_0^{a'} d\ln a' \Big(\frac{1}{a'H}\Big) \tag{2.2}$$

We define the fundamental quantity $\frac{1}{aH}$ as the *comoving Hubble radius* or *comoving horizon*[3]. The particle horizon is defined using this expression for conformal time but with different integration limits in general:

$$d_{PH} = \int_{\ln a_i}^{\ln a} d\ln a' \frac{1}{a'H} \tag{2.3}$$

So the particle horizon is just the difference in conformal time. To proceed we need a few standard results from big-bang cosmology. One can solve Einstein's equation for a perfect fluid energy-momentum tensor $T_{\mu\nu}$ and the FRW-metric ansatz, i.e. the assumption of spatial homogeneity -and isotropy. A perfect-fluid $T_{\mu\nu}$ is the only compatible energy-momentum tensor with these assumed symmetries. This gives the famous Friedmann equations:

$$H^2 \equiv \Big(\frac{\dot{a}}{a}\Big)^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} \tag{2.4}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\Big(\rho + 3P\Big) \tag{2.5}$$

In these expressions the cosmological constant $\Lambda \equiv 0$. $k$ is the curvature parameter and $\rho, P$ are the mass-density respectively pressure of the perfect fluid. These two equations can be combined into a continuity equation:

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{2.6}$$

Upon division this equation by $\rho$ and a trivial rewriting we get:

$$\frac{d\ln\rho}{d\ln a} = -3(1+\omega) \tag{2.7}$$

where we have defined the equation of state of the perfect fluid $\omega = \frac{P}{\rho}$. This differential equation is easily solved giving the energy-density as a function of the scale-factor: $\rho(a) \sim a^{-3(1+\omega)}$. This shows that the Hubble-radius scales as: $\frac{1}{aH} \sim a^{\frac{1}{2}(1+3\omega)}$. For all familiar matter sources we have that $1 + 3\omega > 0$ which is a special case of the strong-energy condition (SEC). If the SEC is satisfied, the Hubble-radius is an increasing function of time. But then the integral defining the particle horizon is dominated by the upper intergration limit. More explicitly:

$$d_{PH} \sim \int_{\ln a_i}^{\ln a} d\ln a' a'^{\frac{1}{2}(1+3\omega)} \sim \frac{1}{1+3\omega}\Big(a^{\frac{1}{2}(1+3\omega)} - a_i^{\frac{1}{2}(1+3\omega)}\Big) \tag{2.8}$$

Obviously, if we take the early-time limit $t \to 0$, so that $a_i(t) \to 0$, and we assume the SEC, then the second term between brackets vanishes and we find that the particle horizon is proportional to the Hubble-radius: $d_{PH} \sim a^{\frac{1}{2}(1+3\omega)} \sim \frac{1}{aH}$. This also explains the confusing terminology of calling the Hubble-radius a "horizon",

[3]This is confusing terminology but nevertheless common in all textbooks and lecture notes.

even though they are two completely different concepts. Now, a finite particle horizon limits communication (with the past), so two spacetime regions separated by a comoving distance larger than the particle horizon can have *never* influenced each other by sending signals. In particular, slightly separated points of the CMB could not have been in causal contact in the past. This defines the *horizon problem*: Why is the CMB nearly uniform (in temperature) today if in the past many points on the CMB were not in causal contact? At the time of recombination one can estimate that the amount of causally disconnected patches is about $10^4$ since the particle horizon is estimated to be about 100Mpc.

The solution to the horizon problem is not hard to guess if the difference between the particle horizon and the Hubble-radius is appreciated. Two points separated by a comoving distance larger than the Hubble-radius cannot communicate *now* while two points separated by a comoving distance larger than the particle horizon can have never communicated. To solve the horizon problem, we need to generate causal contact between the different points on the CMB separated by more than the particle horizon at that time. Thus, we need a mechanism that yields $d_{PH} \gg \frac{1}{aH}$, so that particles that could not communicate when the CMB was created were in causal contact *before* that time. This statement is equivalent to the statement that the Hubble-radius should decrease as a function of time and we will take this as the fundamental definition of a period of inflation:

$$\boxed{\textbf{Inflation}: \quad \frac{d}{dt}(aH)^{-1} < 0} \tag{2.9}$$

Note that this definition is equivalent to the more familiar notion of inflation as a phase of accelerated expansion of space, $\ddot{a} > 0$. For a decreasing Hubble-radius, the contribution to the particle horizon is dominated by the lower integration limit (see 2.8). In fact, the particle horizon becomes infinite, so that there is an infinite amount of *conformal time* before the time of last scattering to have causal contact. Note that a decreasing Hubble-radius requires the existence of matter that violates the SEC, $\omega < -\frac{1}{3}$.

### 2.1.2 The flatness problem

The flatness problem relates to the initial velocities of the particles composing the fluid. These need to be extremely fine-tuned for the universe to remain homogeneous at late times. If the initial velocities are too small the universe would re-collapse and if they are too large the universe would expand too quickly and we end up with an empty universe. Why is this called the "flatness" problem? In big-bang cosmology we can define density parameters by dividing the energy-density by a critical energy-density $\rho_c$. The critical density is defined to be $\rho_c \equiv \frac{3H^2}{8\pi G_N}$ so that $k = 0$ and the universe is flat[4]. Generally, a density parameter for matter type $i$ is defined as $\Omega_i^o \equiv \frac{\rho_i^o}{\rho_c^o}$ where the superscript $o$ indicates that this is the density parameter *today*. Using the first Friedmann equation and the fact that the energy-densities evolve in

[4]This can be seen from the first Friedmann equation 2.4.

time as $\rho_i(a(t)) = \rho_i^o a^{-3(1+\omega^i)}$ we can derive the relation:

$$\left(\frac{H}{H^o}\right)^2 = \sum_i \frac{\rho_i}{3(H^o)^2} - \frac{k}{(a^o H^o)^2} a^{-2} \tag{2.10}$$

We now change to Planck units. The (reduced) Planck scale is defined as $M_{pl}^2 \equiv \frac{1}{8\pi G_N}$ and Planck units are defined by $M_{pl} = 1$. In Planck units the critical density today equals $3(H^o)^2$. Hence, we can rewrite the above equation as:

$$\left(\frac{H}{H^o}\right)^2 = \sum_i \Omega_i a^{-3(1+\omega_i)} - \Omega_k a^{-2} \tag{2.11}$$

where we have defined the curvature-density parameter $\Omega_k \equiv \frac{k}{(a^o H^o)^2}$ which is the relevant parameter for the flatness problem. This expression can be evaluated at the time $t^o$ *today*. The FRW metric has the following rescaling symmetry:

$$a(t) \to \lambda a(t) \tag{2.12}$$

$$r \to \frac{r}{\lambda} \tag{2.13}$$

$$k \to \lambda^2 k \tag{2.14}$$

This allows us to set the scale-factor $a^o \equiv a(t^o) \equiv 1$. Therefore, we arrive at:

$$1 = \sum_i \Omega_i + \Omega_k \quad \text{when } t = t^o \tag{2.15}$$

The flatness problem follows from this relation. If we put back in the time-dependence of the scale-factor we have the relation: $1 - \sum_i \Omega_i(a(t)) = -\frac{k}{(aH)^2}$. In standard big-bang cosmology, the Hubble-radius is an increasing function of time. Thus, the absolute value $|1 - \sum_i \Omega_i(a(t))| \to \infty$ as the universe expands. However, according to observations $\Omega_k^o \sim 0$ today, or equivalently $\sum_i \Omega_i^o \sim 1$ today. But at early times the Hubble-radius is a relatively small quantity so that the quantity $\sum_i \Omega_i^o$ was even closer to unity. This implies that at the Planck-time $t_{pl} \sim \frac{1}{M_{pl}}$, $|\sum_i \Omega_i - 1| \sim \mathcal{O}(10^{-61})$, explaining why this problem is called the flatness problem.

Again, note that the Hubble-radius plays a fundamental role in the flatness problem. Its resolution in inflation is almost trivial. The decreasing Hubble-radius implies $|1 - \sum_i \Omega_i(a(t))| \to 0$, so $\sum_i \Omega_i \to 1$ and $\Omega_k \to 0$, hence the universe is driven towards flatness through inflation.

We have not yet discussed "how much inflation" we need to solve the Horizon and -flatness problem. Under the assumptions that the universe is radiation-dominated since the end of inflation and the temperature was of the order of the GUT-scale at the end of inflation (which is roughly $10^{16}$GeV), the amount of inflation to solve the big-bang puzzles corresponds to 64 *e-folds*: $\frac{a_E}{a_I} > e^{64}$. Note that this does not provide an upper-bound on the number of e-folds. In fact, it is not even clear whether the number of e-folds has an upper-bound. It has, though, a lower-bound. This is provided by the temperature of reheating.

To end this discussion of the big-bang puzzles we again note that these puzzles are not inconsistencies within the theory. We could just assume that initially $\Omega \sim 1$ and that the universe was extremely homogeneous with the right amount of anisotropies to develop structure formation via gravitational instability. The inflationary mechanism of a decreasing Hubble-radius is so attractive because it explains these initial conditions.

## 2.2 Single-field slow-roll inflation

From the fundamental definition of inflation we can define the fundamental inflationary parameters. Taking the time-derivative of the Hubble-radius gives:

$$\frac{d}{dt}(aH)^{-1} = -\frac{1}{a}(1-\epsilon) < 0 \quad \text{where } \epsilon \equiv -\frac{\dot{H}}{H^2} \tag{2.16}$$

Demanding that inflation *happens* means that the Hubble parameter varies slowly in time, or:

$$\boxed{\epsilon < 1} \tag{2.17}$$

The limit $\epsilon \to 0$ is the perfect de Sitter limit since in this limit the spacetime metric becomes:

$$ds^2 = dt^2 - e^{Ht} d\vec{x}^2 \tag{2.18}$$

which is a representation of a de Sitter metric[5] obtained via flat slicing of the spacetime. Note that we could similarly define $\epsilon$ as parametrizing the deviation from the de Sitter equation of state $\omega = -1$ as:

$$\epsilon \equiv \frac{3}{2}(1+\omega) \tag{2.19}$$

Hence, we speak of inflation as a *quasi* de Sitter period[6]. The end of inflation is defined as $\epsilon = 1$. From the discussion above we know that the scale-factor must grow roughly by the exponential factor $\sim e^{60}$ between the end and the beginning of inflation so in addition to the condition $\epsilon < 1$ we also need to formulate a parametric condition of the duration of inflation. This implies that $\epsilon$ needs to remain small for a sufficiently long period. We define the number of e-folds infinitesimally as $dN \equiv d\ln a = Hdt$[7]. Then, we require $\epsilon$ to change slowly per Hubble-time:

$$\eta \equiv \frac{d\ln\epsilon}{dN} = \frac{\dot{\epsilon}}{\epsilon H} \tag{2.20}$$

The condition that inflation endures long enough to solve the big-bang puzzles then becomes:

$$\boxed{|\eta| < 1} \tag{2.21}$$

Note that we can also rewrite $\epsilon$ as:

$$\epsilon = \frac{d\ln H}{dN} \tag{2.22}$$

[5]The form of the de Sitter metric follows from the Friedmann equation 2.4. For $\omega = -1$ we find that $H^2 = const.$ and therefore that $a(t) \sim e^t$.

[6]In a de Sitter space, the expansion of the universe never stops. Therefore, inflation cannot have taken place in an exact de Sitter space.

[7]The opposite convention $dN = -Hdt$ can also be found in the literature.

Thus, we are interested in models that give rise to the conditions $\{|\eta|, \epsilon\} < 1$. The most influential model of inflation consists of a single dynamical scalar field $\phi(x^\mu)$ called the *inflaton* with a canonical kinetic term $\frac{1}{2}(\partial\phi)^2$ and a potential energy density $V(\phi)$, minimally coupled to Einstein gravity. The action for this model is:

$$S[\phi, g] = \int d^4x\sqrt{-g}\Big(\frac{M_{pl}^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big) \tag{2.23}$$

The inflaton is a purely hypothetical field whose origin is yet to be explained. It is useful to think of $\phi$ as a clock measuring the duration of inflation. What conditions do we need to impose on the inflaton field to satisfy to inflationary conditions $\{|\eta|, \epsilon\} < 1$? To find the answer we compute the equation of motion with respect to the inverse metric and the inflaton. Varying the action with respect to the inverse metric is the definition of the energy-momentum tensor associated to the inflaton. Together with the equation of motion for $\phi$ we have:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\Big) \tag{2.24}$$

$$\frac{\delta S}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + \frac{dV}{d\phi} = 0 \tag{2.25}$$

If we assume that the gradient of the inflaton vanishes so that the field is homogeneous in space (i.e. $\phi(x^\mu) = \phi(t)$) and we assume that the background spacetime is spatially homogeneous -and isotropic the energy-momentum tensor reduces to that of a perfect fluid and the equation of motion of the inflaton simplifies drastically. The dynamics of the inflaton field is governed by the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{2.26}$$

We see that the expansion of the universe acts as a friction term for the inflaton while its potential acts as a force. The inflaton might contain initial inhomogeneities, i.e. a non-vanishing gradient. Numerically, it can be shown in specific examples that the inflaton field needs to be homogeneous over a few times the horizon-size at that time in order for inflation to start. This is sometimes called the patch-problem. The other equation of motion imposes conditions on the mass-density and energy-density of the inflaton field and hence on the equation of state $\omega$:

$$\rho(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.27}$$

$$P(\phi) = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.28}$$

$$\omega(\phi) = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)} \tag{2.29}$$

Since inflation requires $\omega < -\frac{1}{3}$ we need to require that the kinetic energy density of the inflaton is subdominant compared to the potential energy density: $\frac{1}{2}\dot{\phi}^2 < V(\phi)$. This can also be seen from the Friedmann equations with $k = 0$. Expressing the Friedmann equations in terms of the inflaton field using the expressions for the mass -and energy density we can derive the relation:

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{pl}^2} \tag{2.30}$$

Using the definition of the $\epsilon$-parameter (2.17) and the condition for inflation, we find:

$$\epsilon = \frac{\dot{\phi}^2}{2M_{pl}^2 H^2} < 1 \tag{2.31}$$

Hence, $\frac{1}{2}\dot{\phi}^2 < M_{pl}^2 H^2 \sim \rho(\phi)$, so inflation happens when the kinetic energy of the inflaton field contributes less than the potential energy to the total energy. This condition defines *slow-roll* inflation. Note that, just like with the initial homogeneity of the field, this initial condition is also quite specific. It could very well be that in a region of the potential where inflation is supposed to occur, the kinetic energy has a reasonable value so that the field will "overshoot" that region to source inflation. This is the overshoot-problem. Together with the patch-problem this is related to the more general problem of defining probability measures in the eternal inflation scenario. We will not further discuss eternal inflation but we will get back to the problem of initial conditions of inflation at the end of this chapter.

Until now, everything has been exact. To simplify the equations of motion, one makes the *slow-roll approximation*. The benefit of this is that the question of whether inflation occurs can be answered by looking at the potential $V(\phi)$ only. The first approximation is that the kinetic energy is totally negligible compared to the potential energy: $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$, which is equivalent to $\epsilon \ll 1$, so that the first Friedmann equation becomes:

$$H^2 \approx \frac{V(\phi)}{3M_{pl}^2} \Longleftrightarrow \epsilon \ll 1 \tag{2.32}$$

The Klein-Gordon equation is simplified by the approximation that the acceleration of the inflaton is small so that its kinetic energy remains subdominant compared to the potential energy for a sufficiently long time to maintain inflation:

$$3H\dot{\phi} \approx -\frac{dV}{d\phi} \Longleftrightarrow |\eta| \ll 1 \tag{2.33}$$

The requirement that the acceleration is small can be formalized by defining a parameter $\lambda \equiv -\frac{\ddot{\phi}}{\dot{\phi}H}$. From the first approximation 2.32 we find that the $\epsilon$-parameter is related to the potential via:

$$\epsilon = \frac{\dot{\phi}^2}{2M_{pl}^2 H^2} \approx \frac{M_{pl}^2}{2}\Big(\frac{V'}{V}\Big) \equiv \epsilon_V \tag{2.34}$$

The approximation to the Klein-Gordon equation relates the $\eta$-parameter to the potential. Taking a time-derivative of the approximated KG equation and dividing by the quantity $\frac{1}{3H^2\dot{\phi}}$ we get:

$$\frac{\dot{H}}{H^2} + \frac{\ddot{\phi}}{\dot{\phi}H} = -\frac{V''}{3H^2} \Rightarrow \epsilon + \lambda \approx M_{pl}^2 \frac{|V''|}{V} \equiv |\eta_V| \tag{2.35}$$

The collection of parameters $\{|\eta_V|, \epsilon_V\}$ are called the *slow-roll parameters*. In the slow-roll approximation, succesful inflation corresponds to $\{|\eta_V|, \epsilon_V\} \ll 1$. Using these parameters a useful expression for the number of e-folds can be obtained. Since $dN \equiv d\ln a = Hdt$ we have that:

$$N = \int_{t_i}^{t_f} Hdt = \int_{\phi_i}^{\phi_f} d\phi \frac{H}{\dot{\phi}} = \frac{1}{M_{pl}} \int_{\phi_i}^{\phi_f} d\phi \frac{1}{\sqrt{2\epsilon}} \approx \frac{1}{M_{pl}} \int_{\phi_i}^{\phi_f} d\phi \frac{1}{\sqrt{2\epsilon_V}} \tag{2.36}$$

where in the last step we used the slow-roll approximation. Here $\phi_i, \phi_f$ indicate the initial respectively final value of the field during inflation. In terms of a differential equation we get:

$$\frac{d\phi}{dN} = \sqrt{2\epsilon} \tag{2.37}$$

Note that this equation is exact and can be interpreted as a redefinition of the field $\phi \rightarrow N$. This leads to the *N-formalism* of inflation. To end this section, we note that the inflationary parameters $\{\epsilon, |\eta|\}$ are related to their slow-roll partners $\{\epsilon_V, |\eta_V|\}$ as:

$$\boxed{\epsilon = \epsilon_V \quad \text{and} \quad \eta = -4\epsilon_V + 2\eta_V}$$

## 2.3 Quantum fluctuations

The above analysis assumed exact spatial homogeneity -and isotropy. However, we know that the CMB is not exactly isotropic which indicates that the early universe was not exactly homogeneous but contained small inhomogeneities. The beauty is that the CMB temperature anisotropies can be explained via quantum mechanical fluctuations in the inflaton field during inflation. Roughly speaking, quantum fluctuations $\delta\phi(\vec{x}, t)$ alter the time at which inflation would have ended because the particular spacetime region may remain potential-dominated for a longer or shorter time depending on the sign of the fluctuation. Because of these local time-delays, inflaton fluctuations induce perturbations in the primordial density $\delta\rho$. These density perturbations source the temperature fluctuations in the CMB temperature.

Because the deviation from spatial homogeneity is small, the inhomogeneities can be treated in linear relativistic perturbation theory by perturbing around a homogeneous FRW background. The idea is to decompose all relevant quantities as $\delta A(\vec{x}, t) = A(\vec{x}, t) - \bar{A}(t)$. A fundamental feature of *linear* perturbation theory is that the metric fluctuations can be decomposed into scalar, vector -and tensor perturbations that can be treated independently. This is the well-known SVT-decomposition theorem. The form of the perturbed metric depends on whether one wants to calculate the power-spectrum of scalar -or tensor perturbations. A subtlety of relativistic perturbation theory is the coordinate invariance of general relativity, but this plays only a role in the computation of scalar perturbations as tensor perturbations are automatically gauge invariant. One can create perturbations that are gauge artifacts. One thus needs to define gauge invariant perturbations, known as the *Bardeen variables*. Among other gauge invariant constructions in terms of the Bardeen variables, an extremely important quantity can be defined called the *comoving curvature perturbation* $\mathcal{R}$. Here "comoving" refers to a particular gauge choice. It provides the connection between the fluctuations in the inflaton field and the late-time fluctuations in the universe. It is defined as:

$$\mathcal{R} \equiv \psi - \frac{H}{\bar{\rho} + P} \delta q \tag{2.38}$$

Here $\psi$ is a particular Bardeen variable and $\delta q$ is a perburbation of the momentum density which arises by considering a SVT-decomposition of the perturbed energy-momentum tensor. The comoving curvature perturbation can be found by computing the scalar curvature of the induced metric on 3-surfaces of constant time and then fix the gauge to be comoving gauge. For adiabatic perturbations this quantity is constant on superhorizon scales, i.e. length scales $L > (aH)^{-1}$, and thus larger than the Hubble-radius, we have $\dot{\mathcal{R}}_{|\vec{k}|} \approx 0$. This property of $\mathcal{R}_{|\vec{k}|}$ allows to link cosmological observables to the quantum fluctuations in the inflaton field that exit the horizon roughly 60 e-foldings before the end of inflation. When we have in addition to the inflaton other dynamically relevant scalar fields during inflation, the comoving curvature perturbation is not constant on superhorizon scales but receives contributions from *entropic* -or *isocurvature* perturbations. Thus, the comoving curvature perturbation of each Fourier mode is constant on superhorizon scales. The goal is to compute quantum fluctuations in the quantity $\mathcal{R}$ which corresponds to the quantity:

$$\langle \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{K}'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \mathcal{P}_{\mathcal{R}}(k) \quad \text{at} \quad k = aH \quad \text{(horizon-crossing)} \tag{2.39}$$

where $\mathcal{P}_{\mathcal{R}}(k)$ is the power-spectrum of scalar fluctuations. This is the two-point function and contains all the statistical properties of the perturbations if the perturbations $\mathcal{R}_{\vec{k}}$ are exactly Gaussian. Whether this is true is a question of observational cosmology but the most recent Planck data [] places very strong constraints on *non-gaussianity*. Deviations from gaussianity are contained in higher-order correlation functions of $\mathcal{R}$ and characterised by their amplitude $F_{NL}$ in Fourier space. Non-gaussianity can be large in models with non-canonical kinetic terms or multi-field models, compared to single-field slow-roll models, where the non-gaussianity is of the order of the slow-roll parameters.

The calculation of the two-point function of scalar perturbations is one of the most famous calculations in theoretical cosmology and can be found in any textbook. Hence, we will only highlight the main steps. There are multiple ways to do this calculation but arguably the cleanest method is to expand the inflaton action:

$$S[\phi, g] = \int d^4x \sqrt{-g} \Big( \frac{M_{pl}^2}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \Big) \tag{2.40}$$

to second-order in the curvature perturbation which is contained in the Ricci scalar. This is the method of Maldacena [11]. This gives the action:

$$S^{(2)} = \int d^4x a^3 \frac{\dot{\bar{\phi}}^2}{H^2} \Big( \dot{\mathcal{R}}^2 - a^{-2} (\partial_i \mathcal{R})^2 \Big) \tag{2.41}$$

The computation of the equation of motion yields, after a field redefinition $v \equiv z\mathcal{R}$ with $z^2 \equiv a^2 \frac{\dot{\bar{\phi}}^2}{H^2}$ and a change to conformal time, a harmonic-oscillator type of equation known as the *Mukhanov-Sasaki* equation:

$$v_k'' + (k^2 - \frac{z''}{z}) v_k \equiv v_k'' + \omega^2(\tau) v_k = 0 \tag{2.42}$$

The primes denote differentiation with respect to conformal time $\tau$. In canonical quantization we interpret the classical Mukhanov fields $v(z)_{\vec{k}}$ as quantum operators obeying canonical commutation relations. To solve equation 2.42 is quite hard because the variable $z$ depends on the background dynamics. An exact solution can be found in the pure de Sitter limit and slow-roll approximation, in which the Hubble-parameter is constant and the acceleration of the inflaton is negligible. These approximations imply that:

$$\frac{z''}{z} \approx \frac{a''}{a} \approx \frac{2}{\tau^2} \tag{2.43}$$

The solutions to the Mukhanov-Sasaki equation are then:

$$v_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{(2k)}}\Big(1 - \frac{i}{k\tau}\Big) + \beta \frac{e^{ik\tau}}{\sqrt{(2k)}}\Big(1 - \frac{i}{k\tau}\Big) \tag{2.44}$$

To specify the constants $\alpha, \beta$ we need to impose two boundary conditions. The first one is derived from the canonical commutation relations obeyed by the field $v_k$. Its Wronskian is demanded to be unity so that the commutation relations of creation -and annihilation operators equals a delta-function. For details see []. The second condition leads to the definition of the inflationary vacuum state. In the limit $\tau \to -\infty$, the frequency $\omega^2(\tau) \approx k^2$, so that the Mukhanov-Sasaki equation becomes a Klein-Gordon equation of a free-field in Minkowski space whose solution is $v_k \sim e^{-ik\tau}$ [8]. Note that the limit of infinite negative conformal time corresponds to the deep-horizon limit, i.e. $k \gg aH$. So the second boundary condition is:

$$\lim_{\tau \to -\infty} v_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \tag{2.45}$$

The set of functions $v_k(\tau)$ satisfying these two conditions define a *unique* vacuum state known as the *Bunch-Davies* vacuum. This vacuum state is typically assumed in inflationary models. It is possible to have models of inflation that have a different vacuum state. With these initial conditions the constants $\alpha = 1, \beta = 0$.

We can now compute the power-spectrum of scalar fluctuations. Typically, one defines a dimensionless power-spectrum:

$$\Delta_s^2 \equiv \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2}\mathcal{P}_{\mathcal{R}}(k) \tag{2.46}$$

Using the result[9]:

$$\langle |\mathcal{R}|^2 \rangle = \frac{|v|^2}{a^2}\Big(\frac{H}{\dot{\phi}}\Big)^2 \tag{2.47}$$

and since we know the form of the functions $v_k(\tau)$ we find the famous result:

$$\Delta_{\mathcal{R}}^2(k) = \Big(\frac{H}{2\pi}\Big)^2\Big(\frac{H}{\dot{\phi}}\Big)^2 \quad \text{evaluated at } k = aH \tag{2.48}$$

In terms of the inflationary parameter $\epsilon$ the result is:

$$\Delta_{\mathcal{R}}^2(k) = \frac{1}{8\pi^2}\frac{1}{\epsilon}\frac{H^2}{M_{pl}^2} \quad \text{at } k = aH \tag{2.49}$$

Scalar perturbations source the anisotropies in the CMB temperature. The prediction of the power spectrum of these perturbations has been a remarkable success for the theory of inflation. The inflationary prediction matches the power spectrum of temperature anisotropies (figure 2.1) up to a very high accuracy. The peaks corre-

[8]Here we take the minus-sign solution because this gives the positive-frequency mode which corresponds to the ground state.

[9]Here $\dot{\phi}$ denotes the unperturbed inflaton field.

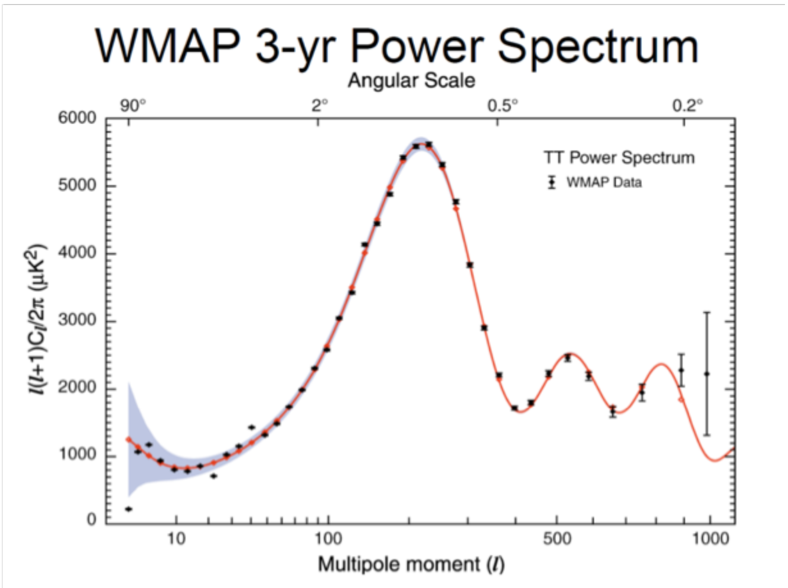



Figure 2.1: Power spectrum of the fluctuations in the temperature of the CMB as measured by WMAP in 2003 [12]. The red line is the prediction of single-field slow-roll inflation. The top horizontal axis labels the angular scales in degrees. The bottom horizontal axis labels the angular scale in multipole moments $l$. Large angular scales in degrees correspond to small $l$. The vertical axis labels the power in units of temperature fluctuations.

spond to baryonic acoustic oscillations (BAO) in the baryon-photon fluid. Note that for small angular scales the oscillations are damped and for scales $< 0.01$ deg, which cannot be seen on this figure but more recent data probes smaller angular scales, the power spectrum practically vanishes. This is the reason why observations of the CMB cannot probe the entire inflationary trajectory of the inflaton and hence they cannot constrain the form of the scalar potential completely. The whole analysis for tensor perturbations is very similar. The resulting dimensionless power-spectrum is:

$$\Delta_t^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{pl}^2} \quad \text{at } k = aH \tag{2.50}$$

The important difference between the power-spectra of scalar -and tensor perturbations is that the one for tensors only depends on $H$ and not on its time-derivative (i.e. not on $\epsilon$). The prediction of primordial gravitational waves is a key prediction of inflation. This prediction can be verified by observations of the polarization of the CMB. A detection of so-called *B-modes* will provide evidence for this prediction[10]. B-modes are parity-odd polarization modes which can, roughly speaking, be thought of as the curl of a vector field. Scalar perturbations cannot generate B-modes. Instead, their polarization signature corresponds to *E-modes*.

Now that we have derived the power-spectra, we can define inflationary observables. First we define the *scale-dependence* of the power-spectra. Consider a reference momentum scale $k_\star$. Near this momentum-scale the spectra take the form of a

[10]This is not completely true. B-modes can also be produced by vector perturbations, however such perturbations quickly decay due to the quasi-exponential expansion.

power-law:

$$\Delta^2_{\mathcal{R}}(k) \equiv A_s\Big(\frac{k}{k_\star}\Big)^{n_s-1} \quad \text{and} \quad \Delta^2_t \equiv A_t\Big(\frac{k}{k_\star}\Big)^{n_t} \tag{2.51}$$

The amplitude $A_s$ of the scalar-spectrum has been measured at a reference scale $k_\star \approx 0.05Mpc^{-1}$ and is approximately $10^{-9}$. The momentum dependence of the power-spectra introduces the notion of *tilt*:

$$n_s - 1 \equiv \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} \quad \text{and} \quad n_t \equiv \frac{d\ln\Delta^2_t}{d\ln k} \tag{2.52}$$

where the $-1$ in the definition of the scalar-tilt is conventional. A *scale-invariant* power-spectrum corresponds to $n_s = 1, n_t = 0$ so that the spectra are momentum-independent. Such a power spectrum is called a *Harrison-Zel'dovich* spectrum. $n_s, n_t$ are referred to as *spectral indices*. Note that in the parametrizations of the power-spectra above we have only included first order coefficients of scale-dependence. The spectral indices may also be scale-dependent, a phenomenon called *running*. The spectral indices can be related to the slow-roll -and inflationary parameters and hence measurements of $n_s, n_t$ provide information about the dynamics of inflation:

$$n_s - 1 = 2\eta_V - 6\epsilon_V \quad \text{and} \quad n_t = -2\epsilon_V \tag{2.53}$$

$$n_s - 1 = \eta - 2\epsilon \tag{2.54}$$

It is often stated that one of the predictions of single-field inflation is a nearly scale-invariant spectrum but not everyone agrees upon that [13, 14]. We can rewrite the above as:

$$1 - n_s = 2\epsilon - \frac{d\ln\epsilon}{dN} \tag{2.55}$$

To have a nearly scale-invariant spectrum we need to tune $\epsilon$ since inflation only demands that $\epsilon < 1$. This leaves room for large deviations of scale-invariance. To obtain near scale-invariance and consistency with observations demands $\epsilon < 0.03$. Hence, it seems that near scale-invariance is not a natural prediction of inflation. This is an example of the universal fine-tuning problem of parameters in inflationary models. One explanation is that $\epsilon$ scales as an inverse power-law in the number of e-folds $\epsilon \sim \frac{1}{N^p}$ where the number $p$ is model-dependent. We will see a little more on this in section 2.3.1). Alternatively, we can understand the deviation from $n_s = 1$ as an $\frac{1}{N}$ effect. A second inflationary observable is defined by normalizing the amplitude of tensor perturbations to scalar perturbations:

$$r \equiv \frac{\Delta^2_t}{\Delta^2_{\mathcal{R}}} \tag{2.56}$$

$r$ is called the *tensor-to-scalar* ratio. The most recent observations by Planck of $n_s, r$ are indicated in figure 2.2. Since non-gaussianities are very small ($F_{NL} \sim 0$), this plot suffices to distinguish between the various single-field models. Generically, a point $n_s, r$ does not fix the inflationary field range $\Delta\phi$ uniquely. This is not too surprising as $\Delta\phi$ is a quantity that depends on the entire inflationary trajectory. Currently, $n_s$ is determined to be 0.9663 and $r$ is upper-bounded by 0.0065. Because $n_s < 1$ we say it is *red-shifted*. The value of $r$ quoted above rules out the most simple potentials such as $m^2\phi^2$, $\lambda\phi^4$ because such potentials yield $r \sim 0.2$.

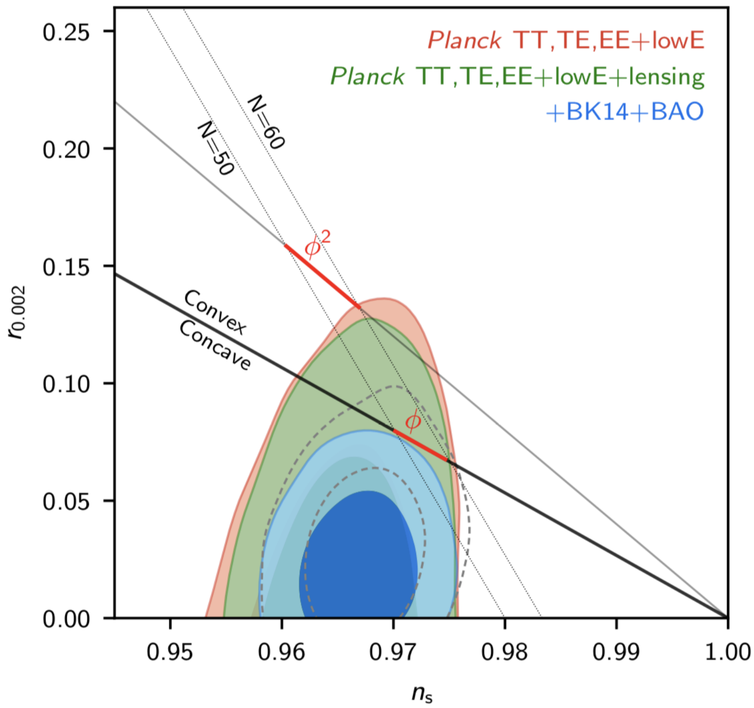



Figure 2.2: Scalar tilt vs. the tensor-to-scalar ratio as measured by Planck and published in 2015 [10]. The blue region is the favoured region.

### 2.3.1 Energy scale, Lyth bound and universality

The tensor-to-scalar ratio defines the energy-scale of inflation. In the slow-roll approximation we have $V \approx 3M_{pl}^2 H^2$ so that the inflationary energy scale is defined as:

$$E_{inf} \equiv V^{\frac{1}{4}} = (3M_{pl}^2 H^2)^{\frac{1}{4}} \tag{2.57}$$

From the tensor-to-scalar ratio we derive that:

$$\Delta_t = \frac{\sqrt{2}}{\pi}\frac{H}{M_{pl}} = \sqrt{r}\Delta_{\mathcal{R}} \Rightarrow H \approx 3 \times 10^{-5} M_{pl}\sqrt{\frac{r}{2}} \tag{2.58}$$

where we used the measured value of the amplitude of the scalar-spectrum and made some numerical approximations. Substituting this in the definition of the inflationary energy scale we get:

$$E_{inf} \approx 8 \times 10^{-3}\sqrt{\frac{r}{0.1}}M_{pl} \sim \left(\frac{r}{0.01}\right)^{\frac{1}{4}} 10^{16} GeV \tag{2.59}$$

So we see that the energy-scale of inflation depends on the magnitude of the gravitational waves signal.

The Lyth bound relates the distance traversed in scalar field space, measured in Planck units, to the tensor-to-scalar ratio. In fact, it shows that detectable gravitational waves correspond to super-planckian field displacements $\Delta\phi > M_{pl}$. To see this, note that the tensor-to-scalar ratio is linearly related to $\epsilon$:

$$r \equiv \frac{\Delta_t^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon(N) \tag{2.60}$$



22

This simply follows from the definitions of the power-spectra. Using expression 2.31 and the fact that $dN = Hdt$ we get:

$$r = 8\Big(\frac{\dot{\phi}}{M_{pl}H}\Big)^2 = \frac{8}{M_{pl}^2}\Big(\frac{d\phi}{dN}\Big)^2 \tag{2.61}$$

We can rewrite this as the differential equation:

$$\frac{1}{M_{pl}}\int_{\phi_i}^{\phi_{end}} d\phi = \int_{N_{end}}^{N_{CMB}} \sqrt{\frac{r(N)}{8}}dN \tag{2.62}$$

which after not too much work is:

$$\frac{\Delta\phi}{M_{pl}} \sim \mathcal{O}(1) \times \sqrt{\frac{r}{0.01}} \tag{2.63}$$

This is the famous Lyth bound. It is the simplest estimate one can make for the field range $\Delta\phi$ since it is assumed that $\epsilon(N)$ is constant during the whole period of inflation. Detectable gravitational waves ($r > 0.01$) are related to super-planckian field displacements. Note that this fixes the energy-scale of inflation to be above the GUT-scale of $10^{16}$GeV. The size of the order one coefficient is determined by the amount of e-folds taken into account. Observationally, one can only see $N_{eff} \approx 7$ e-folds. Then the coefficient is approximately 0.25. To arrive at the Lyth bound we have made a few assumptions. For example, we have assumed that the inflationary vacuum state in which we compute the power-spectra is the Bunch-Davies vacuum and that the kinetic term of the inflaton is canonically normalized. Altering these assumptions, one might evade the Lyth bound. The Lyth bound can also be defined away from the slow-roll regime.

### Universality

We already mentioned a couple of times that observations of the CMB do not allow to get insight into the complete inflationary trajectory. Here we want to elaborate slightly on this issue[11]. We denote the number of e-folds before the end of inflation when the modes that are observable in the CMB left the horizon as $N_\star$. This point is located around 60 e-folds to account for the uniformity of the universe. The period of inflation that can be probed via CMB observations corresponds to $\Delta N \approx 7$. The location of this *CMB-window* $N$ is determined by the difference $N_\star$ and $N_e$, the latter being the point where inflation ends: $N = N_\star - N_e$. This leads to the fact that models with different potentials can yield similar cosmological predictions, as long as the potentials agree on the CMB-window. This is called *universality*. We noted above that there might be no upper-bound on the number of e-folds and hence it is expected that $N_\star > 60$. This makes its inverse a natural expansion parameter. Furthermore, since $n_s$ is only very slightly red-shifted, we may parametrize this as $n_s = 1 - \frac{2}{N}$, which is accurate to the percentage level. These considerations lead to a natural power-law parametrization of $\epsilon$ in the large-$N$ limit:

$$\epsilon(N) = \frac{\beta}{N^p} \quad \text{with} \quad p, \beta \in \mathbb{R} \tag{2.64}$$

[11] For an extensive discussion on universality we refer to the PhD thesis of Marco Scalisi [15] and [16].

a parametrization we already mentioned above to explain the smallness of $\epsilon$. Here we suppress higher-order corrections in $\frac{1}{N}$ as these are irrelevant for cosmological predictions. Many well-known inflationary models allow for such a parametrization for $\epsilon$. Using the expression $n_s = 1 + \eta - 2\epsilon$ we derive that:

$$n_s = \begin{cases} 1 - \frac{2\beta+1}{N} & \text{if } p = 1 \\ 1 - \frac{p}{N} & \text{if } p > 1 \end{cases}$$

Together with $r = \frac{16\beta}{N^p}$ this defines the *perturbative* universality class. The case $p < 1$ can be neglected as it is in tension with observations. During slow-roll, the Friedmann equation simplifies to $H^2 \approx \frac{V}{3}$ in Planck units. Hence, we can express the potential in terms of $N$ by solving:

$$\frac{d}{dN} \ln\left(\frac{V}{3}\right) = \frac{2\beta}{N^p} \tag{2.65}$$

whose solution is:

$$V(N) = \begin{cases} V_0 N^{2\beta} & \text{if } p = 1 \\ V_0\Big(1 - \frac{2\beta}{(p-1)N^{p-1}}\Big) & \text{if } p > 1 \end{cases}$$

This can be expressed in terms of $\phi$ using $\frac{d\phi}{dN} = \sqrt{\frac{2\beta}{N^p}}$, showing very explicitly the difference in functional form of the potential in both formalisms:

$$V(\phi) = \begin{cases} V_0\phi^n & \text{if } p = 1 \text{ (chaotic monomial)} \\ V_0\Big(1 - e^{-\frac{\phi}{\mu}}\Big) & \text{if } p = 2 \text{ (Starobinsky, plateau models)} \\ V_0\Big(1 - \left(\frac{\phi}{\mu}\right)^n\Big) & \text{if } p > 1, p \neq 2 \text{ (hilltop)} \end{cases}$$

The parameters $n, \mu$ are related to $\beta, p$. The case $p = 2$ is an important case in this thesis. These are models whose potential approaches exponentially fast in the field value a plateau region. Interestingly, these models are favoured by the Planck data, yet in tension with some of the swampland conjectures (see section 5.7). We can derive an interesting relation for the field range $\Delta\phi$ in terms of $N$ that we need in section 5.7. One finds that [17]:

$$\Delta\varphi = \sqrt{\frac{3\alpha}{2}} \log N - \Delta\varphi_e \tag{2.66}$$

### 2.3.2 The problem of initial conditions

If inflation is allowed to start, one has a perfect well-defined theory with predictions that can be tested against observations of the CMB. In the case of high-energy scale inflation, i.e. models where inflation starts at the Planck density $\rho \sim 1$ in Planck units, the initial conditions for inflation seem quite natural [18, 19]. The argument is as follows. We consider a closed universe which initially has a planckian size $L \sim 1$ and is in a state with $\rho \sim 1$ so that a description in terms of classical space-time starts becoming reliable. Initially, we have $\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \sim 1$. We could ask what initial values of the field are natural. If the potential is constant, the scalar field enjoys a continuous shift-symmetry and all initial field values are equally probable. If the potential grows as a function of $\phi$ and becomes greater than the Planck density at field values $\phi > \phi_{pl}$, we can constrain the initial field value. If $V(\phi_{pl}) = 1$ then $\phi < \phi_{pl}$ initially. However, there seems to be no reason to impose that $\phi \ll \phi_{pl}$ and therefore a natural choice seems to be $\phi \sim \phi_{pl}$. Then, roughly speaking, the argument is that since all these energy densities span the same range and the total energy density is 1, they are probably all of the same order: $\frac{1}{2}\dot{\phi}^2 \sim \frac{1}{2}(\nabla\phi)^2 \sim V(\phi) \sim \mathcal{O}(1)$. For inflation to start, we need this planckian patch to expand to a patch of size $\frac{1}{H}$ while remaining homogeneous. This is perfectly fine, since evolving from the "all equal energy-density condition", due to the expansion the kinetic energy density and gradient inhomogeneities quickly dilute and the universe becomes very quickly potential-dominated and inflation starts.

Similarly, one can address the initial conditions problem for low-energy scale inflation, i.e. models where inflation starts for $V(\phi) \ll 1$. This means that initially, kinetic -and gradient energy of the inflaton field make up the dominant part of the energy density: $\dot{\phi}^2 \sim (\nabla\phi)^2 \sim 1$ and there exist only patches where $V(\phi)$ is extremely subdominant. $\alpha$-attractors are such low-energy scale models, where the inflationary regime starts at roughly $V \sim 10^{-10}$, ten orders of magnitude below the Planck scale. The initial condition problem is an open problem for such scenarios and hence it is fair to discuss two sides of the story. We start with arguments against the inflationary scenario.

If we evolve from such initial conditions, while the kinetic energy density quickly dilutes, gradient inhomogeneities seem to prevent inflation from starting. The argument is that, for inflation to start, the initial patch has to be extremely homogeneous. This makes the initial conditions problem for inflation more involved. Consider a *compact* flat or open universe which has the topology of a torus. This is very common in numerical investigations of the initial conditions of inflation, where a periodicity is imposed on the boundary conditions. We assume again that the size of the torus is planckian at the Planck time. Since the horizon $\frac{1}{H} \sim t$ grows faster than the scale factor, the latter determining the size of the torus $a \sim t^{\frac{1}{2}}$, the friction term $\sim H$ decreases, quickly becoming smaller than the momenta of the inhomogeneities $\nabla\phi$ in the inflaton field. The equation of state of the inflaton inhomogeneities approaches the one for relativistic matter. In such a radiation-dominated universe density perturbations do not grow. So it is expected that the universe remains relatively homogeneous prior to inflation.

In addition, since the mean free path $\frac{1}{H} \sim t$ increases faster with time than the scale factor, the ultra-relativistic particles have traversed the torus after $t \gg 1$ many times, leading to so-called *chaotic mixing*, quickly realizing homogeneity to a high degree. Hence, even though the inflaton had huge kinetic energy, inflation still seems to start. Note that if inflation takes place sufficiently long (60 e-folds), so that the universe becomes extremely huge, the effects of its compact topology on observations is negligible. To conclude, inflation is allowed to start if there exists a relatively homogeneous domain of planckian size.

Besides the initial conditions problem, there are two other subtleties that remain unexplained. One is the fine-tuning of parameters required for the correct normalization of the amplitude of scalar fluctuations. While supersymmetry or a weakly-broken shift-symmetry might explain why the (self)-coupling is radiatively stable, it does not explain why the amplitude of scalar fluctuations is small. While it is true that inflation solves the fine-tuning of the initial conditions of classical big-bang cosmology, fine-tuning seems to re-enter via the back door. The other problem is the measure problem implied by eternal inflation [20]. If inflation starts, it will end only locally but not globally. Quantum fluctuations on top of the classical trajectory of the inflaton down the potential hill towards the minimum push the field upwards in some spacetime regions so that they undergo additional inflation. These regions will dominate most part of the volume of the universe. This goes on forever, so that we end up with many different local universes in each of which the inflaton has followed a different trajectory leading to different cosmological predictions for $n_s, r$.

# Chapter 3

# Wilsonian effective field theory and supersymmetry

This chapter is devoted to the explanation of the statement that inflation is *UV-sensitive*. The chapter collects some essential concepts from effective field theory and supersymmetry for the rest of the thesis. I have tried to keep things as concise as possible while not rushing over key concepts. A good reference for effective field theory (with applications to particle physics) is [21][1].

## 3.1 The wilsonian effective action

At the heart of effective field theory lies the identification of an ultraviolet cut-off, denoted by $\Lambda$. With respect to this cut off one identifies the heavy -and light degrees of freedom. Note that there is no principle or law of nature that determines where to place the UV cut off. This is a subtle point and we come back to it in the next subsection. The light degrees of freedom are part of the effective theory. What happens to the heavy degrees of freedom depends on whether one has complete knowledge of the UV theory and whether it is computable. If the UV theory is strongly-coupled, perturbation theory is invalid and hence not computable. If one has a weakly-coupled UV-complete theory, one can *integrate out* the heavy degrees of freedom by performing a functional integral. This is sometimes called the top-down approach to effective field theory. We will use this approach to define the effective action [2]. Denote the light fields of the theory by $\phi_L$ and the heavy fields by $\phi_H$, then the Wilsonian path-integral is defined by:

$$e^{iS_{eff}[\phi_L]} = \int_{\mathcal{C}} \mathcal{D}[\phi_H] e^{iS[\phi_L,\phi_H]} \tag{3.1}$$

where we integrate only over the field-configuration space of the heavy fields. Here the effective action is defined as the integral of the effective Lagrangian density:

$$S_{eff}[\phi_L] = \int d^4x \mathcal{L}_{eff}[\phi_L] \tag{3.2}$$

[1]MIT open-courseware has uploaded on youtube an entire series on EFT back in 2013. Especially the first 4-5 lectures contain material that is discussed in the first sections of this chapter.

[2]This is the effective action $\Gamma$: the Legendre transform of the generating functional of connected diagrams $W$.

When building effective field theories such functional integrals are only computed perturbatively via a procedure known as *matching*. The idea of matching is very simple. Compute a physical observable in the UV theory at some order in perturbation theory, e.g. a scattering amplitude at tree-level and expand the result in the ratio $\frac{E}{\Lambda}$. Then compute the same process in the low-energy theory by writing down terms in the Lagrangian that give rise to the same tree-level process. Now the actual matching takes place. Demand that the result for the scattering amplitude in the low-energy theory is the same as in the UV theory. This fixes the undetermined coefficients in the low-energy theory in terms of the coupling between the heavy -and light field, at tree level. Generally, this leads to a series-expansion of the effective theory parameters in terms of the UV-theory parameters. Let us now look at the general structure of the effective Lagrangian. The effective Lagrangian is a functional only of the light degrees of freedom. When going through the matching procedure, it turns out that the heavy fields are parametrized by non-renormalizable interactions among the light fields. Here "non-renormalizabilty" refers to the mass dimensionality of the interaction being greater than the mass dimensionality of the Lagrangian density (which is 4 in $d = 4$)[3]. Generally, when we have integrated-out all the heavy fields except the lightest degree of freedom, the effective Lagrangian looks like:

$$\mathcal{L}_{eff}[\phi] = \mathcal{L}_{ren}[\phi, \partial_\mu \phi] + \sum_i c_i \frac{\mathcal{O}_i[\phi, \partial_\mu \phi]}{\Lambda^{\delta_i - 4}} \tag{3.3}$$

The renormalizable part of the Lagrangian contains terms of mass dimension up to four. These terms are called *relevant*[4] -and *marginal* operators[5]. The terms in the (in principle) infinite sum are non-renormalizable and are called *irrelevant* operators. They are irrelevant since they are suppressed by powers of the cut off scale $\Lambda$. The operators $\mathcal{O}$ are constructed from the light field $\phi$ and its derivatives. $\delta_i$ indicate their mass-dimensions. These operators are local, meaning that they depend only on the fields evaluated at a single point. This should be contrasted with for example Wilson lines or 't Hooft lines, which are operators whose values depend on the values of a gauge field along a curve.

From a top-down point of view one can calculate the exact structure of the operators. For example, consider an abelian Higgs model whose effective theory is a non-linear sigma model. We can integrate-out the Higgs boson at tree-level and this will yield a four-derivative operator. However, when one has no complete knowledge of the UV theory or it is not computable, one cannot integrate out degrees of freedom. Then one has to adopt the bottom-up approach to effective field theory which means parametrizing ignorance via assumptions about the UV theory. All operators that are consistent with the symmetries of the UV theory, which we do not know but assume remain intact in the UV, are allowed. For example, in the

[3]More generally, we consider the behaviour of fields under uniform scale transformations. For specific scaling dimensions, the (free) kinetic action is scale invariant. These scaling dimensions coincide with the mass dimensions of operators and parameters in the action. Thus, the adjectives relevant, irrelevant and marginal are defined with respect to the (canonical) kinetic term.

[4]Relevant operators can be dangerous at extreme long distances where they dominate over the kinetic term. Therefore, we need to restrict/fine-tune the uniform scale transformations $S$ to those that satisfy $m^2 S^2 \sim 1$ so that $(\partial\phi)^2 \sim m^2\phi^2$ at long distances.

[5]At tree-level marginal operators are always marginal. However, quantum loop corrections might change this into marginal-relevant or marginal-irrelevant.

case of general relativity, we would write down all diffeomorphism invariant terms under the assumption that this symmetry remains intact in the UV.

The coefficients $c_i$ are the *Wilson coefficients* and they are dimensionless. One can calculate the Wilson coefficients in terms of an expansion of the coupling constants between the heavy -and the light fields. Typically, they are assigned $\mathcal{O}(1)$ values but a priori this cannot be determined. Even if we do have a UV completion, the coefficients are not necessarily $\mathcal{O}(1)$. Symmetries in the UV might highly suppress their values and consistency requirements in the infra-red (such as causality) fixes some coefficients to be positive. Note also that, since the Wilson coefficients can be expressed in terms of the coupling constants $g_i$ among the heavy -and light fields, a weakly-coupled theory also gives suppressed coefficients.

The bottom-line is that, in the absence of a UV completion, we are rather ignorant about the values of the $c_i$'s. Only an explicit, model-dependent, calculation will tell us the answer. This is problematic since often the UV completion is absent. In an attempt to get a better grip on the bottom-up approach to effective field theory people have often found help using the concept of naturalness. The idea of naturalness also fixes to some extend where we should expect "new physics" and thus where we should place the UV cut off.

### 3.1.1 The principle of naturalness

We want to know where to put the UV cut off $\Lambda$ and what the values of the Wilson coefficients $c_i$ are. The principle of naturalness provides us with an answer. However, this principle is not sacred and can be violated. The most well-known violation of the principle is in the case of the cosmological constant. The principle is formulated in many different ways. One distinguishes between technical naturalness, 't Hooft naturalness, Dirac naturalness, top-down naturalness and bottom-up naturalness. For the time being, let us focus only on bottom-up naturalness, as this is the concept that is usually referred to. Bottom-up naturalness states that we should place the UV cut off $\Lambda$ at the point where quantum corrections to a parameter in the Lagrangian become larger than the experimentally measured value of that parameter. Equivalently, bottom-up naturalness states that new degrees of freedom appear when the quantum corrections become sufficiently large. We start by looking at an example where the principle does not work [21]. This happens when we have to integrate-out a heavy Dirac fermion $\psi$ at one-loop. Consider the following Yukawa Lagrangian:

$$\mathcal{L}[\phi,\psi] = \frac{1}{2}(\partial\phi)^2 + i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m^2\phi^2 - M\bar{\psi}\psi - g\phi\bar{\psi}\psi \tag{3.4}$$

where we assume the mass hierarchy $m \ll M$ and where $g$ denotes the Yukawa coupling. Integrating-out the fermion induces quantum corrections to the $\phi$-propagator (two-point function), that renormalizes the mass and the field strength (the kinetic term) of the scalar. At one-loop order we need to regularize the integral:

$$i\mathcal{A} = (-1)(-ig)^2\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p\!\!\!/ - k\!\!\!/) - M + i\epsilon}\frac{i}{k\!\!\!/ - M + i\epsilon} \tag{3.5}$$

This is a quadratically divergent Feynman integral. In this thesis, all integrals will be regularized using dimensional regularization. In the appendix all divergent integrals in this work are computed. The result is:

$$\mathcal{A} = -\frac{4ig^2}{(4\pi)^2}\Big[\Big(\frac{3}{\epsilon}+1+3\log\big(\frac{\mu^2}{M^2}\big)\Big)\Big(M^2-\frac{p^2}{6}\Big)+\frac{p^2}{6}-\frac{p^4}{20M^2}+\dots\Big] \tag{3.6}$$

where $p$ denotes the external momentum of the scalar which in position space translates into a derivative. The choice of the renormalization scale $\mu$ is arbitrary and we choose $\mu = M$. Choosing to appropriate counterterm $\delta$ to cancel the $\frac{1}{\epsilon}$ pole encoding the divergence, we find our effective scalar field theory:

$$\mathcal{L}_{eff} \supset \big(1-\frac{4g^2}{3(4\pi)^2}\big)\frac{(\partial_\mu\phi)^2}{2} - \big(m^2+\frac{4g^2M^2}{(4\pi)^2}\big)\frac{\phi^2}{2}+\dots \tag{3.7}$$

where the dots indicate higher-derivative corrections (higher powers in external momentum) and higher polynomial interactions of $\phi$ originating from corrections to other S-matrix elements (n-point function with $n > 2$). More interesting is the following observation. It is hard to explain why the one-loop renormalized mass $m_R^2 = m^2 + \frac{4g^2M^2}{(4\pi)^2}$ should be small. We might expect the Yukawa theory to be valid up to the Planck scale but certainly not higher. In this case, the scalar mass receives a radiative correction of $\Lambda^2 \sim M_{pl}^2$ which is enormous. Thus, naturalness tells us that the scalar has a mass of $\mathcal{O}(M_{pl})$. A small scalar mass requires a very precise cancellation between the radiative correction and the bare scalar mass. This would require a symmetry relating tree-level corrections to loop corrections but the Yukawa Lagrangian does not have such a symmetry. We say that the sensitivity of the scalar mass $m$ to the heavy fermion mass $M$ makes light scalars unnatural. Its small value could be explained by symmetries such as a weakly-broken shift-symmetry in the Yukawa Lagrangian, or supersymmetry. Note that the weakly-broken shift-symmetry requires that $g \ll 1$.

Translating the discussion to inflation, a light inflaton field $m \ll H$ is unnatural as long as it is not protected by a symmetry. This form of naturalness, in which symmetries cause all renormalized parameters to be smaller than the radiative corrections, is called technical naturalness. As technical naturalness is the weakest formulation of naturalness, all other formulations of naturalness are also violated in this case. Thus, a priori, the inflaton is not technically natural and certainly not natural. Now, lets do an example where some form of naturalness does work. This example will come back when discussing the magnetic weak gravity conjecture in chapter 6. The mass of the electron is about 0.5MeV. In Maxwell theory, electrons are point charges. The energy carried by the electric field of the electron diverges if we truly regard the electron as a point so we have to introduce a cut off:

$$\vec{E} \sim \frac{q}{e^2}\hat{r} \longrightarrow E^2 = \int_{r_e}^{\infty} dr\frac{e^2}{r^2} = \frac{e^2}{r_e} \tag{3.8}$$

where $r_e$ is the "radius" of the electron. We can determine this radius by equating the self-energy stored in the electric field to the rest mass of the electron: $r_e = \frac{e^2}{m_e c^2}$ where we have restored SI-units for the moment. This self-energy introduces a linear divergence in the electron mass:

$$\Delta m_e = \frac{e^2}{c^2 r_e} \equiv \frac{e^2}{c^2}\Lambda \tag{3.9}$$

Naturalness would tell us that new physics appears before we reach distances shorter than $r_e$. We can slightly rewrite the above divergence as $\Delta m = \alpha\Lambda$ with $\alpha \approx \frac{1}{137}$ the fine-structure constant. By demanding naturalness of the electron mass we need to require $\alpha\Lambda < 0.5$ MeV. We find that the cut off is about 70 MeV. Thus, we expect new physics below or around 70 MeV. Indeed, quantum field theory predicts the existence of positrons. It can be shown that virtual positrons around the electron exactly cancel the linear divergence in the electron mass. Still, the electron mass diverges, but now it scales only logarithmically with the cut off.

To end this section let us stress the differences between different formulations of naturalness. Technical naturalness states that the radiative corrections induced by the integrating-out procedure should always be smaller than the renormalized parameter. 'T Hooft naturalness states that in the limit of some renormalized parameter $g_R \to 0$ a symmetry is completely restored. Equivalently, it states that the "amount" of symmetry-breaking is controlled by the renormalized parameter $g_R$ so it is this parameter that controls the size of radiative corrections. Technical naturalness and 't Hooft naturalness are both concepts introduced by 't Hooft and hence are sometimes mixed up with each other in the literature. In the rest of this work we won't bother too much about this difference. Typically, in the literature, models of inflation are regarded as technically natural which in our language corresponds to 't Hooft natural. Top-down naturnalness is the statement that the procedure of integrating-out always yields Wilson coefficients of order unity.

## 3.2 Gravity as a field theory

Theories of inflation involve gravity so we better incorporate it. General relativity is a non-renormalizable quantum field theory and hence should be interpreted as an effective field theory. This point of view was pioneered in [22]. To see that it is non-renormalizable is actually easy because gravity involves a dimensionful coupling $[G] = -2$. Dimensionful interactions will violate perturbative unitarity at some critical center of mass energy, e.g. graviton-graviton scattering at the Planck scale. Therefore, GR cannot be the end of the story. However, let us nevertheless discuss why GR follows the pattern of an effective field theory in some more detail. This will also give us some insight into EFT's in general. The action of pure gravity is the Einstein-Hilbert action:

$$S_{EH}[g] = \frac{M_{pl}^2}{2}\int_{M_4} d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu} \equiv \frac{M_{pl}^2}{2}\int_{M_4} d^4x\sqrt{-g}R \tag{3.10}$$

$R$ and $R_{\mu\nu}$ are built from the trace parts of the Riemann tensor $R^{\alpha}_{\mu\nu\beta}$:

$$[D_\mu, D_\nu]V^\alpha = R^{\alpha}_{\mu\nu\beta}V^\beta \tag{3.11}$$

Where the D's are covariant derivatives constructed from the connection symbols $\Gamma$: $D_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\alpha}V^\alpha$. If the connection is symmetric in its lower indices (it is torsion free) and it is metric compatible ($\nabla g \equiv 0$), the connection is completely determined by the metric. This specific connection is known as the Christoffel connection or Levi-Civita connection. If we think of gravity as a gauge theory, the Riemann tensor can be thought of as the field strength tensor making the metric a dynamical symmetric tensor field. Working out the commutator gives the Riemann tensor in terms of the connection symbols:

$$R^{\alpha}_{\mu\nu\beta} = \partial_\nu \Gamma^{\alpha}_{\mu\beta} - \partial_\beta \Gamma^{\alpha}_{\nu\mu} + \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\beta\mu} - \Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\nu\mu} \tag{3.12}$$

We use the definition of the Ricci tensor as the contraction of the first -and third indices of the Riemann tensor. This yields:

$$R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu} \tag{3.13}$$

The mass dimensions of the curvature invariants are +2. Since the Christoffel symbols involve first derivatives of the metric $\Gamma \sim \partial g$, the Ricci tensor -and scalar involve second derivatives of the metric $R_{\mu\nu}, R \sim \partial^2 g, \partial g \partial g$.

We can consider weak gravitational fields and treat gravity in linearized perturbation theory. For the moment we set $\Lambda = T_{\mu\nu} \equiv 0$. We perturb the spacetime metric around flat Minkowski space: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The symmetric tensor field $h_{\mu\nu}(x)$ of spin-2 is interpreted as the graviton field. It is actually a gauge field whose redundancy is due to (linearized) diffeomorphisms:

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{3.14}$$

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{3.15}$$

The perturbation around the metric is exact, while the perturbation around the inverse metric is an infinite series with respect to the metric perturbation $h_{\mu\nu}$, so $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \dots$. We can now expand the curvature invariants. In particular, the Ricci scalar is:

$$R = (\partial_\mu \partial_\nu h^{\mu\nu} - \Box h^{\mu}_{\mu}) + \dots \tag{3.16}$$

While the expansion of the determinant of the metric $\sqrt{-g} \equiv \sqrt{-det g_{\mu\nu}}$ is:

$$\sqrt{-g} = \sqrt{-\eta}\Big(1 + \frac{1}{2}h^{\mu}_{\mu} + \dots\Big) \tag{3.17}$$

Suppressing Lorentz indices, an expansion of the pure gravity action in the field $h_{\mu\nu}(x)$ takes the form:

$$S_{EH}[h] = \frac{M_{pl}^2}{2} \int d^4x (\partial h \partial h + h \partial h \partial h + h^2 \partial h \partial h + \dots) \tag{3.18}$$

For this expansion to make sense we of course assume $h \ll M_{pl}$. This expansion is often compared with the pure Yang-Mills action:

$$S[A] = \int d^4x \frac{1}{g^2} Tr\Big[F_{\mu\nu}F^{\mu\nu}\Big] \supset \int d^4x \frac{1}{g^2}\Big((\partial A)^2 + (\partial A)A^2 + A^4\Big)$$

where the trace is over the Lie group indices. The similarity is that both gravity and Yang-Mills theory are self-interacting quantum field theories. The difference is that the Yang-Mills action terminates at quadratic order in the gauge field $A$ but the Einstein-Hilbert action is an infinite series in the (gauge) graviton field $h_{\mu\nu}$ due to the expansion of the determinant of the metric and its inverse. This can be recognized as an effective field theory in which the non-renormalizable graviton self-interactions parametrize quantum gravity degrees of freedom.

To see why, we consider the LHS of classical Einstein equation which involves the energy-momentum tensor $T_{\mu\nu}$. We can then consider loop corrections due to matter to the graviton propagator. The resulting divergence cannot be absorbed in the parameters present in the Einstein-Hilbert Lagrangian [23]. Hence we should add an additional parameter that achieves this, which is a four-derivative operator in the case of a scalar loop correction. This is a signature of a (non-renormalizable) effective field theory: the theory is renormalizable order by order in perturbation theory and so needs in principle an infinite number of parameters to absorb all UV divergences. We do not have to consider matter loops but we can also consider graviton loop corrections to the graviton propagator. Consider for example the Einstein-Hilbert action plus second-order curvature invariants like $R^2$. The one-loop divergence can be absorbed into a redefinition of the coefficients of these quadratic curvature terms [23].

Pure gravity, in four dimensions[6], is actually a finite theory at one-loop. One can always use the classical equations of motion to simplify higher-dimensional operators[7]. In this case we can use the vacuum tree-level Einstein equations $R_{\mu\nu} = 0$ and $R = 0$. This eliminates the four-derivative interactions

To write down an effective action for general relativity amounts to parametrizing our ignorance. The most obvious attempt is to write down all higher-dimensional operators consistent with the gauge symmetry of general relativity, diffeomorphism invariance. The light degree of freedom is the spacetime metric $g_{\mu\nu}$. The effective action has the form:

$$S_{eff}[g] = \int d^4x\sqrt{-g}\Big(M_{pl}^2 R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} + \\ + \frac{1}{M_{pl}^2}\left(d_1 R^3 + \dots\right) + \dots\Big)$$

The first dots indicate higher powers of curvature invariants of dimension six and the second dots indicate all other higher-dimensional operators. A technical detail is that the coefficient $c_3$ of the term quadratic in the Riemann tensor can be set to zero in four dimensions. This is a consequence of the Gauss-Bonnet theorem in four dimensions which implies that the term $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ is a total derivative and hence topological.[8]

## 3.3 Inflation in effective field theory

This section is largely based on [8]. We consider single-field slow-roll models of inflation in effective field theory. The effective action is a combination of the effective

[6]This is only true in four dimensions because of the Gauss-Bonnet theorem (see below).

[7]This is known as the representation independence theorem. It is not so obvious that this holds at loop-level and when propagators are involved. In tree-level processes without propagators this is obvious because the classical equations of motion just put the external legs on-shell.

[8]This combination of curvature invariants is equal to the Euler number $\chi(M)$ of the spacetime.

actions of gravity and a scalar field plus a coupling between these two light degrees of freedom:

$$S_{eff}[g,\phi] = S_{eff}[g] + S_{eff}[\phi] + S_{g,\phi}[g,\phi] \tag{3.19}$$

This "effective" interaction term consists of operators constructed from the metric, the scalar field and their derivatives:

$$S_{g,\phi}[g,\phi] = \int d^4x\sqrt{-g}\Big(\sum_i c_i \frac{\mathcal{O}_i[g,\phi]}{\Lambda^{\delta_i-4}}\Big) \tag{3.20}$$

If we consider a regime of a spacetime with weak curvature, we can neglect all higher curvature operators in $S[g]$ so that the only interaction term $S_{g,\phi}$ is of the form:

$$S_{g,\phi}[g,\phi] \approx \int d^4x\sqrt{-g}\xi\phi^2 R \tag{3.21}$$

Here "weak curvature" is defined with respect to the cut off scale $\Lambda$, i.e. the curvature is small in units of the cut off. The coefficient $\xi$ is dimensionless. When $\xi \neq 0$ we speak of *non-minimal* coupling to Einstein gravity. It is always possible to perform a transformation of the metric so that $\xi = 0$. The transformation is of the form $g_{\mu\nu} \to e^{2\omega(\phi)}g_{\mu\nu}$ so that by a clever choice of the function $\omega(\phi)$ we arrive at $\xi = 0$. This rescaling of the metric is known as a Weyl transformation which is a special case of a conformal transformation. When we are in a frame where $\xi = 0$, we speak of *minimal* coupling to Einstein gravity. This frame is known as the *Einstein frame*. Note that when the metric is rescaled by a Weyl-transformation, all terms in the action that are not invariant under this transformation change. In the regime of weak curvature the full action of single-field slow-roll inflation with canonical kinetic term in Einstein frame takes the form:

$$S_{eff}[\phi,g] = \int_{M_4} d^4x\sqrt{-g}\Big(\frac{M_{pl}^2}{2}R + \mathcal{L}_{Ren}[\phi] + \sum_i c_i\frac{\mathcal{O}_i[\phi]}{\Lambda^{\delta_i-4}}\Big) \tag{3.22}$$

From our discussion of the effective action for gravity we know that the cut off scale must be below the Planck-scale. The lower-bound on the cut off is fixed by consistency considerations of the effective theory on superhorizon scales $k = aH$. Thus, the cut off is at least above the Hubble scale, the characteristic energy scale of inflation. Hence, we have the hierarchy $H \lesssim \Lambda \lesssim M_{pl}$. This means that all fields with mass $m \lesssim H$ are relevant degrees of freedom and hence part of the effective field theory.

However, for inflation the situation is different compared to particle physics. There, heavy degrees of freedom decouple from the low-energy physics. In inflation, irrelevant operators arising upon integrating-out the heavy fields, suppressed by powers of $\Lambda$, do not decouple and therefore affect the dynamics. Even when $\Lambda = M_{pl}$, so that the irrelevant operators are Planck-suppressed, the dynamics is affected! This is the notorious UV-sensitivity of inflation and it is most clearly demonstrated via the so-called *eta-problem*.

For prolonged slow-roll inflation to cure the big-bang puzzles we need $\eta \equiv M_{pl}^2\frac{|V''|}{V} \ll 1$. This condition is radiatively unstable, or unnatural, and sensitive to higher-dimensional operators. The radiative instability is easily seen. We have seen that scalar masses are sensitive to radiative corrections induced by heavy particles $\Delta m^2 \sim \Lambda^2$. Since $\Lambda \gtrsim H$, we must have that $\Delta m^2 \gtrsim H^2$ which implies that $\Delta V'' \gtrsim H^2$. Using the Friedmann equation we see that $\Delta\eta \sim \frac{\Lambda^2}{H^2} \gtrsim 1$.

We can also consider a dimension six operator, e.g. $O_6[\phi] = cV(\phi)\frac{\phi^2}{\Lambda^2} \equiv \Delta V$, where generically $c \sim \mathcal{O}(1)$, so that we get:

$$\Delta\eta = 2c\Big(\frac{M_{pl}}{\Lambda}\Big)^2 \to 2c \quad \text{as } \Lambda \to M_{pl} \tag{3.23}$$

Note that even when $\Delta V \ll V$, and $\langle\phi\rangle < \Lambda$ so that the quantum correction to the potential is small, the situation remains the same. In fact, the situation is more general since higher-dimensional operators also induce corrections to $\eta$. A general operator of dimension $\delta$ takes the form $\mathcal{O}_\delta[\phi] = c\langle V\rangle\left(\frac{\phi}{\Lambda}\right)^{\delta-4}$. It is easy to see that we now get:

$$\Delta\eta = c(\delta-4)(\delta-5)\Big(\frac{M_{pl}}{\Lambda}\Big)^2\Big(\frac{\phi}{\Lambda}\Big)^{\delta-6} \tag{3.24}$$

Now, there is a very important distinction between the eta problem in small-field -and large-field models of inflation. Consider small-field inflation $\Delta\phi < M_{pl}$ and let $\Lambda = M_{pl}$. Suppose we are in a region of spacetime where the field value of the inflation is far below the Planck-scale, $\phi \ll M_{pl}$. Then:

$$\Delta\eta = c(\delta-4)(\delta-5)\Big(\frac{\phi}{M_{pl}}\Big)^{\delta-6} \tag{3.25}$$

For $\phi \ll M_{pl}$ and $\delta \gg 6$ we have that the factor $\left(\frac{\phi}{M_{pl}}\right)^{\delta-6} \ll 1$ so that $\Delta\eta \ll 1$. This does not solve the eta problem because we still need to explain the presence of the dimension six operator. Note also that this is the best case scenario since we assumed $\Lambda = M_{pl}$. In general, the ratios $\frac{M_{pl}}{\Lambda}$ and $\frac{\phi}{M_{pl}}$ determine whether higher-dimensional (beyond $\delta = 6$) are relevant or not. The problem with large-field inflation in effective field theory is worse. To see why we need to clearly identify what is so problematic in this class of models.

Suppose that for the moment we do not care about the UV completion of gravity and consider an effective field theory of inflation containing only the inflaton and the graviton. The first thing one might worry about is that for super-planckian field displacements the inflaton energy-density becomes super-planckian $m > M_{pl}$. However, this is ruled out by measurements of the amplitude of scalar perturbations which is roughly $10^{-9}$, evaluated at some reference momentum scale $k_\star$. To satisfy this normalization the inflaton mass must be subplanckian, of the order of $10^{-5}$ in Planck units. Another issue might be quantum corrections to the potential induced by inflaton loops and graviton loops. This problem is resolved if we assume an approximate shift-symmetry in the infra-red which renders the inflaton mass technically natural, as we discussed previously. The graviton loop corrections take the form:

$$\frac{\Delta V}{V} = c_1\frac{1}{M_{pl}^2}\frac{d^2V}{d\phi^2} + c_2\frac{V}{M_{pl}^4} \tag{3.26}$$

where $\{c_1, c_2\}$ are of order unity. Since for inflation $m \sim \sqrt{V''}$ and $V \ll M_{pl}^4$ this is guaranteed to be small. Thus, an important conclusion is that there is nothing wrong with large-field inflation in effective field theory from a *bottom-up* perspective. The problems arise in a *top-down* perspective, where we have to integrate-out heavy degrees of freedom, such as Kaluza-Klein modes, moduli-fields, string states and quantum gravity states. The couplings between the inflaton and these excitations, or equivalently the Wilson coefficients of the effective theory, are not necessarily small as we have seen: this is determined by the structure of the UV theory. Upon integrating-out these degrees of freedom the effective Lagrangian for the inflaton takes the form:

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \sum_{i=1}^{\infty}\left(\frac{c_i}{M_{pl}^{2i}}\phi^{4+2i} + \frac{d_i}{M_{pl}^{2i}}(\partial\phi)^2\phi^{2i} + \dots\right) \tag{3.27}$$

where we assumed a $\mathbb{Z}_2$-symmetry for ease of writing but this is of course not necessarily true. The dots indicate higher-derivative operators. We note that in the absence of fine-tuning of the Wilson coefficients, for super-planckian displacements an infinite series of higher-dimensional operators are relevant! This has to be contrasted with the class of small-field models, where we had to fine-tune only the dimension-six operator. Fine-tuning an infinite number of operators is referred to as functional fine-tuning. The terms with coefficients $c_i$ are corrections to the potential and the terms that scale with $d_i$ are corrections to the two-derivative kinetic term. For a single field the field space is just $\mathbb{R}$. In the case of multi-field models, the field space geometry is more complicated and it is defined by a Riemannian metric $G_{ab}$. The corrections to the two-derivative kinetic term are corrections to the metric on field space or moduli space. Now, we do not know the precise sizes of the Wilson coefficients but if we point to the principle of (top-down) naturalness, these are of order unity, which is problematic.

The way out is to impose a weakly-broken shift-symmetry so that all corrections to the potential are forbidden, but we can still have corrections to the two-derivative kinetic term and higher-derivative operators. This seems to render a consistent large-field inflation scenario. However, such a symmetry must allow a UV-completion because we want it to provide protection against Planck-suppressed operators. Therefore, we should embed large-field inflation in a UV-complete theory like string theory to obtain information about the (approximate) symmetries at the Planck-scale. It is not clear whether this is possible, i.e. whether an effective field theory with some assumed structures in the IR can be embedded in a UV-complete theory. Vice versa, not every effective field theory can arise as a low-energy limit of a consistent quantum gravity theory. Essentially, this is because the UV-completion of gravity imposes strong consistency constraints on effective field theories and it might be that our effective theory is far from consistent.

This is the fundamental problem of the *swampland program*: which effective quantum field theories do arise as low-energy limits of (consistent) quantum gravity theories?

## 3.4 Supersymmetry

Supersymmetry (SUSY) is a conjectured spacetime symmetry organizing particles of different spin into the same multiplet. These multiplets are called *supermultiplets*.

Thus, SUSY relates fermions and bosons. There are several reasons why SUSY is important. Regarding string theory, the importance lies in the fact that when SUSY is implemented on the string world-sheet (RNS-formalism), the Hilbert space of the theory contains spacetime fermions. SUSY can be understood as an extension of Poincaré invariance. However, to extend the Poincaré algebra one has to evade the famous Coleman-Mandula theorem which states that the most general symmetry group of the S-matrix is the direct product of the Poincaré group and an internal symmetry. To evade the no-go theorem one has to consider graded Lie algebras, i.e. algebras containing commutation as well as anti-commutation relations. In other words, one adds spinorial generators to the Poincaré algebra, obtaining the super-poincaré algebra. Now, one can add several amounts of spinorial generators. When one set of spinorial generators are added, we speak of *simple* SUSY and in any other case of *extended* SUSY. Without further motivation we state the extended SUSY algebra.

$$\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\delta^A_B \tag{3.28}$$

$$\left\{Q^A_\alpha, Q^B_\beta\right\} = \epsilon_{\alpha\beta}Z^{AB} \tag{3.29}$$

$$\left[Q^A_\alpha, T_i\right] = S_i{}^A{}_B Q^B_\alpha \tag{3.30}$$

We suppressed the Poincaré part. This is the algebra in four spacetime dimensions. Here $A, B = 2, .., N$ and $\alpha, \dot{\beta} = 1, 2$. The dotted indices indicate left-handed Weyl spinors[9]. $Z^{AB}$ is known as the *central charge* of the algebra. It is an anti-symmetric matrix commuting with all the generators of the SUSY-algebra and hence defining an abelian invariant subalgebra. Note that the anti-symmetry implies that the central charge is only present for extended SUSY theories. $T_i$ denote the generators of an internal symmetry. This internal symmetry is called an *R-symmetry*. If the central charge vanishes (e.g. in simple SUSY) the R-symmetry is U(N) but with non-vanishing central charge the R-symmetry is a subgroup of U(N). When we consider massless representations of the algebra, the central charge vanishes. However, for massive representations, the central charge does not have to vanish. One can show that in the case of $Z^{AB} \neq 0$, the masses of the representations have a lower bound, known as the *BPS-bound*. The bound is the following:

$$M \geq \frac{1}{2N}Tr(\sqrt{Z^\dagger Z}) \tag{3.31}$$

States for which the mass *equals* the RHS are called *BPS states* and they are said to *saturate* the bound. These states form a short supermultiplet because they contain less states than the states that do not saturate the bound. These states form a long supermultiplet. BPS states are stable since they are the lightest charged particles. We will encounter several types of BPS states in chapter 6: supersymmetric extremal black holes and Euclidean Dp-branes. As a final remark we mention that BPS states have a remarkable property. They are invariant under the action of several supercharges and hence preserve some of the original supersymmetry.

[9]The dotted-index notation is convenient to distinguish between $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ irreducible representations of the Lorentz group which are the two-component Weyl spinors whose tensor product is the Dirac spinor. Recall that irreducible representations of the Lorentz group are characterized by two numbers since $SO(3, 1)$ is locally isomorphic to two copies of $SU(2)$.

The massless representations are derived from the algebra. We consider a representative momentum vector $p_\mu = (E, 0, 0, E)$. The algebra gives:

$$\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} = 2E(\sigma^0 + \sigma^3)\delta^A_B = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^A_B \tag{3.32}$$

implying that $Q^A_2 = 0$ and hence the central charge $Z^{AB} \equiv 0$ by the algebra. We then define creation -and annihilation operators from the generator $Q^A_1$ to built representations by acting on a vacuum state $|0\rangle$:

$$a^A \equiv \frac{Q^A_1}{2\sqrt{E}} \quad \text{and} \quad a^{\dagger A} \equiv \frac{Q^A_{\dot{1}}}{2\sqrt{E}} \tag{3.33}$$

This normalization gives the usual anti-commutation -and commutation relations among the $a, a^\dagger$. Massless representations, following Wigners classification of one-particle states as finite-dimensional unitary irreducible representations of the Poincaré group, are labelled by their helicity $\lambda$. The vacuum state is thus defined as the state of minimum helicity $\lambda_0$, it is unique and it is annihilated by $a^A$. Action of the creation-operator $a^\dagger$ yields $N$ states with helicity $\lambda = \lambda_0 + \frac{1}{2}$. Acting again with the creation-operator yields $\frac{1}{2!}(N-1)N$ states of helicity $\lambda_0 + 1$. It is readily noted that the massless state of maximum helicity is unique and has $\lambda_{max} = \lambda_0 + \frac{N}{2}$. It is also not hard to see that the total number of massless representations is equal to $2^N$. To introduce some terminology, if we consider $N = 1$ and set $\lambda_0 = 0, \frac{1}{2}, 1$ we have the following states:

| | | |
|---|---|---|
| Chiral multiplet | $\lambda_0 = 0$ | $\lambda = \frac{1}{2}$ |
| vector -or gauge multiplet | $\lambda_0 = \frac{1}{2}$ | $\lambda = 1$ |
| gauge multiplet | $\lambda_0 = 1$ | $\lambda = \frac{3}{2}$ |
| gravity multiplet | $\lambda_0 = \frac{3}{2}$ | $= 2$ |

### Superspace and superfields

Next, we briefly want to describe the formalism of superfields and superspace[10]. We assume we have $N = 1, D = 4$ SUSY from now on. Superspace is an extension of spacetime $x^\mu$ by adding anticommuting grasmannian coordinates $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ where $\alpha, \dot{\alpha} = 1, 2$. Scalar superfields are functions defined on superspace $S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$, just as scalar fields are functions defined on spacetime $\phi(x^\mu)$. The algebra of grassmannian variables is extremely simple. Due to their anti-commutativity, the Taylor expansion of any function $f(\theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ terminates at $\mathcal{O}(\theta\theta\bar{\theta}\bar{\theta})$. This allows us to write down the most general expression for a scalar superfield $S$:

$$S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) + \\ (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\bar{\theta}\bar{\theta})(\theta\theta)D(x)$$

[10]This section is mostly based on the part III lecture notes [24].

where we have suppressed Lorentz -and spinor indices for simplicity. We can figure out how a superfield transforms under translations of all the coordinates by comparison of its transformation as an operator and as a function of some Hilbert space. With the transformation rule of superfields at hand, we can derive how the component fields transform under a supersymmetry transformation. The important transformation rule is the one of the D-component as it is a total derivative.

It turns out that acting with $\partial_\mu$ on $S$ is still a superfield but the action $\partial_\alpha$ on a superfield is no longer a superfield. Hence, a covariant derivative (with respect to spinor indices) on superspace needs to be defined so that superfields remain superfields after taking derivatives. The proper definition turns out to be:

$$D_\alpha \equiv \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu$$
$$\bar{D}_{\dot{\alpha}} \equiv -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu$$

A superfield that obeys $\bar{D}_{\dot{\alpha}}S = 0$ is called a *chiral* superfield $\Phi$. Chiral superfields, as opposed to vector superfields, are complex. Their general form is:

$$\Phi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}(\bar{\theta}\bar{\theta})(\theta\theta)\Box\phi(x)$$

Upon a SUSY transformation, the F-term is invariant up to a total derivative. To built supersymmetric interacting quantum field theories we want to include terms such that upon a SUSY transformation $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$ with $\delta\mathcal{L}$ a total derivative. Hence, this term and the D-term can be used to determine the couplings among chiral superfields and so we can construct a chiral Lagrangian. The most general form of a chiral Lagrangian is:

$$\mathcal{L}[\Phi] = K(\Phi, \Phi^\dagger)\mid_D + \big(W(\Phi)\mid_F + h.c.\big) \tag{3.34}$$

This expression deserves some comments. The function $K$ is a real function known as the *Kähler potential*. The $D$ denotes that it arises from the D-term in the expansion of the scalar superfield. The Kähler potential governs the structure of the kinetic term. $W$ is a holomorphic function, and therefore only dependent on $\Phi$, known as the *superpotential*. Because $\Phi$ is chiral, $W(\Phi)$ is chiral too. It descends from the F-term in the expansion of the scalar superfield. It determines the form of the interactions. Based on simple dimensional analysis we can derive the structure of these functions. For example, in the case of constructing renormalizable Lagrangians, the Kähler potential can take the form $K = \Phi\bar{\Phi}$. This is referred to as a canonical Kähler potential.

There exist powerful non-renormalization theorems in the theory of supersymmetry. It turns out that the superpotential is not renormalized in perturbation theory but only receives quantum corrections non-perturbatively. In particular, in the context of string theory, $W$ receives no corrections in the perturbative expansions in $\alpha'$ and $g_s$. The Kähler potential, however, receives perturbative corrections. To find the Kähler potential and the superpotential in terms of the component functions we Taylor expand them around $\Phi^i = \phi^i$. In particular, for the Kähler potential this gives:

$$\frac{\partial^2 K}{\partial\phi^i\partial\phi^{\bar{j}*}}\partial_\mu\phi^i\partial^\mu\phi^{\bar{j}*} \equiv K_{i\bar{j}}\partial_\mu\phi^i\partial^\mu\phi^{\bar{j}*} \tag{3.35}$$

where we have defined the *Kähler metric*. The expansion of the superpotential leads to the definition of a scalar potential. This is particularly interesting in the case of *local* supersymmetry which defines a supersymmetric theory of gravity called a *supergravity*. The potential descends from the F-term and hence is called the F-term scalar potential:

$$V_F(\phi^i, \phi^{\bar{i}}) = e^{\frac{K}{M_{pl}^2}}\left[K^{i\bar{j}}D_iW\bar{D}_{\bar{j}}\bar{W} - \frac{3}{M_{pl}^2}|W|^2\right] \tag{3.36}$$

where $D_i, \bar{D}_{\bar{j}}$ denote the covariant derivatives defined on the *Kähler* manifold spanned by the $\phi^i$:

$$D_iW \equiv \partial_i W + K_i W \tag{3.37}$$

This Kähler manifold is sometimes also called the *scalar manifold* as scalar fields make up the coordinates of this space. The covariant derivative of the superpotential is referred to as a F-term. For interesting cosmological scenarios we need the scalar potential to be a positive definite function. Hence, we necessarily have $D_iW \neq 0$, i.e. SUSY is broken during inflation. Spontaneously breaking SUSY introduces Goldstone modes. Since we are breaking a supersymmetry, the situation is a little different than e.g. the Higgs mechanism in the standard model. Breaking SUSY introduces fermionic goldstone modes with their scalar superpartners. The scalar superpartners are called *sgoldstinos*. The direction(s) of the scalar manifold in which SUSY is broken are called *sgoldstino directions*. Interestingly, the dynamics of inflation is constrained by the directions in which SUSY is broken. The sgoldstino directions are *unstable* directions and hence need to be stabilized around their zero values by some mechanism. The case of interest in the next chapter is when the sgoldstino directions are orthogonal to the inflaton. We consider the situation where the field $\Phi$ acts as the inflaton and we have a complex superfield $S$. Typically, the superpotential depends linearly on this field [15]:

$$W = Sf(\Phi) \tag{3.38}$$

where $f(\Phi)$ depends holomorphically on $\Phi$. We now have four real degrees of freedom $\Phi, S$. We would like to truncate this to a single-field model. If we let the Kähler potential depend on the combination $S\bar{S}$, so that $S \to -S$ is a symmetry of the scalar potential, we can consistently truncate along the direction $S = 0$. The F-term scalar potential takes the form:

$$V_F = e^K K^{S\bar{S}}|f(\Phi)|^2 \tag{3.39}$$

which is positive definite since the negative definite term is killed along $S = 0$. Also, along $S = 0$, the covariant derivatives are:

$$D_\Phi W = 0 \tag{3.40}$$

$$D_S W = f \neq 0 \tag{3.41}$$

showing that the field $S$ belongs to the goldstino supermultiplet, containing a sgoldstino scalar and goldstino fermion. With $S = 0$ the scalar potential depends on two real degrees of freedom. Further truncation to a single field and the associated stability issues depend on the detailed structure of the Kähler potential. For the potential to be a viable inflationary potential we need to control the prefactor $e^K$. We impose a continuous shift-symmetry in the real part of $\Phi$ so we let $K$ take the form: $K = -\frac{(\Phi-\bar{\Phi})^2}{2}$. We note that $\Phi \to \bar{\Phi}$ is also a symmetry of the scalar potential. We can then consistently truncate along the extrema $\Phi - \bar{\Phi} = S = 0$ to a single field model. We arrive at the positive definite potential:

$$V(\mathrm{Re}\,\Phi) = |f(\mathrm{Re}\,\Phi)|^2 \tag{3.42}$$

Based on the assumed symmetries of the Kähler potential and the linear dependence of the superpotential on $S$ we have generated a general positive scalar potential. However, the question remains whether the masses squared of $S$ and $\mathrm{Im}\,\Phi$ are positive otherwise the extrema are saddle points or maxima instead of stable minima. Stabilization of all other directions besides the inflaton is a delicate issue and makes embedding inflation in supergravity non-trivial.

# Chapter 4

# Axionic models and pole inflation

In this chapter we want to discuss several models of large-field inflation that are radiatively stable. We focus on two classes of models: models in which the inflaton is identified with a compact pseudoscalar, the axion, and $\alpha$-attractors which form a subset of pole-inflation models. We discuss both single-axion -and multi-axion inflation. The single-field models we discuss are natural inflation and axion monodromy[1] and the multi-field model is N-flation. Eventually, we want to see what the fate of these models is in light of the swampland conjectures. Therefore, the purpose of this chapter is to introduce these models to the extend that we can understand how they are constrained. We pay a little more attention to pole-inflation because it has a geometric interpretation that deserves some extra clarification. We also discuss the initial conditions problem in the case of $\alpha$-attractors because it is interesting from the point of view of the swampland distance conjecture.

## 4.1 Single axion inflation

### Goldstone bosons

Axions are familiar from quantum chromodynamics, arising as goldstone bosons from the spontaneous breaking of a global U(1) symmetry. Spontaneous symmetry breaking (SSB) means that the vacuum of the field theory is no longer invariant under the action of the symmetry while the Lagrangian is. There are infinitely many physically equivalent vacua connected via the action of the spontaneously broken symmetry. These transformations between equivalent vacua are conveniently parametrized by particular values of a scalar field, a goldstone boson $\phi(x)$, about which more below. In general, given a group $G$, it can be spontaneously broken to a proper subgroup $H$. The vacuum state remains invariant under $H$ but it transforms under the action of the coset space $G/H$[2]. Goldstone bosons parametrize the action of the coset in the low-energy theory.

Recall that goldstone bosons $\phi$ are defined in a relativistic field theory as massless (gapless) particles that decouple from the other fields at low-energies. In the case of a broken global internal U(1), the goldstone boson realizes this broken symmetry in the low-energy Lagrangian via a continuous shift-symmetry, i.e. the U(1)

[1] The literature on axion monodromy is enormous so we will be extremely brief.

[2] The coset space is also a group if and only if the subgroup $H$ is an invariant subgroup of $G$.

acts on $\phi$ as $\phi \to \phi + c$. It is common to take this shift-symmetry as the defining property of a goldstone boson. Indeed, the shift demands the Lagrangian to depend on $\partial\phi$ and higher-derivatives only so that it is massless. This also makes the decoupling property manifest since at low spatial momentum $\vec{p} \to 0$ the derivatives $\partial\phi$ vanish and hence all the couplings to other fields. When the shift-symmetry is weakly-broken, e.g. due to a polynomial dependence in the potential, we speak of a pseudo goldstone boson. In the case of a broken continuous internal symmetry, the number of goldstone bosons equals the number of broken symmetry generators. This is equal to the dimension of the coset space: $\dim(G/H) = \#$goldstones. This is no longer true for spontaneously broken spacetime symmetries.

As an aside, note that by breaking more complicated internal symmetries than a global abelian U(1), for example non-abelian ones, the symmetries realized in the effective goldstone Lagrangian are more involved than a shift. These symmetries, which for broken non-abelian symmetries are non-linearly realized on the goldstone fields, are sometimes referred to as generalized shift-symmetries. For example, consider the symmetry-breaking pattern:

$$SO(3) \to SO(2) \cong U(1) \tag{4.1}$$

The number of broken generators is equal to two and so we have two goldstone bosons in the low energy theory. Roughly speaking, this means that we have two shift-symmetries. The coset space is the two-sphere $SO(3)/SO(2) \cong S^2$. This symmetry is non-linearly realized in the effective theory by the two goldstone modes and in particular the kinetic-sector of the Lagrangian is invariant under the action of this coset space. If we want to built a realistic inflationary scenario, we need to add a scalar potential that weakly breaks this non-linearly realized symmetry. Just as with the shift-symmetry, this symmetry can be used to protect the model against dangerous perturbative quantum corrections, as long as the symmetry is weakly-broken, rendering the model technically natural. When we discuss $\alpha$-attractors below, we will be interested in the pattern $SO(2,1) \to SO(2)$ where the two goldstones parametrize the coset space $SO(2,1)/SO(2) \cong \mathbb{H}^2$.

### Natural inflation

String theory contains a lot of axions, the exact number depending on the topology of the compactification manifold. Natural inflation typically involves string theory axions. These axions have an exact continuous shift-symmetry that is preserved to all orders in perturbation theory. This symmetry is too much as the model lacks a potential at this level. To generate a potential, we need some non-perturbative mechanism that breaks the continuous shift-symmetry. Instantons are such non-perturbative effects. Instantons are not ordinary physical particles because they are localized in time. They are stationary points of the Euclidean action. Switching between spacetimes with Minkowski -and Euclidean signature amounts to performing a Wick rotation. Instantons come in various flavours of which the field theory Yang-Mills instanton is the most familiar. In this stringy context, there are multiple types of instantons that break the shift-symmetry: worldsheet instantons and euclidean D3-branes.

Now, instantons break the exact continuous shift-symmetry to an exact *discrete* shift-symmetry $\phi \to \phi + 2\pi f$. The potential of single axion inflation is thus periodic and the Lagrangian typically takes the form [25]:

$$\mathcal{L} = \frac{f^2}{2}(\partial c)^2 - \Lambda^4(1 - \cos c) + \dots \tag{4.2}$$

where $\Lambda = M_{UV}e^{-Sinst}$ is a dynamical scale since it depends on the instanton action, $c$ is a non-canonical axion field and the dots indicate higher-order instanton corrections which we neglect here. Note that this amounts to assuming $S_{inst} \gg 1$. We will later see (in section 6.7.2) that this requirement on the action is not always necessary to suppress higher instanton harmonics. $f$ denotes a parameter of mass dimension one and is known as the axion decay constant. It sets the symmetry-breaking scale if we regard $\phi$ as arising from spontaneous symmetry breaking. Its value also determines whether we are dealing with small -or large-field inflation. The canonically normalized field is $\phi = fc$ and the action becomes:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \Lambda^4(1 - \cos\left(\frac{\phi}{f}\right)) + \dots \tag{4.3}$$

Note that for $f \gg \phi$ and $S_{inst} \gg 1$ we get $\Lambda\frac{\phi^2}{f^2}$ chaotic inflation. This is an example of a large-field model with $\Delta\phi \sim 10$ in Planck units. We can calculate the slow-roll parameter easily from the above potential and we see that demanding an inflationary scenario requires:

$$\epsilon_V \sim \left(\frac{V'}{V}\right)^2 \sim \left(\frac{M_{pl}}{f}\right)^2 \ll 1 \tag{4.4}$$

Hence, we need a super-planckian axion decay constant $f \gg M_{pl}$. Already a while ago it was noticed that constructing large-field models of inflation with a *single* axion in string theory might not be possible. [26] constructed several examples from the different string theories, finding $f$ parametrically larger than $M_{pl}$ only in cases when higher instanton contributions spoiled the flatness of the potential. This may give a hint for an underlying reason that quantum gravity somehow forbids such decay constants and indeed, it is stated that this is related to the weak gravity conjecture[3].

### 4.1.1 Axion monodromy and tunneling

The basic idea of axion monodromy inflation is very simple, yet explicit constructions require advanced string theory techniques and constructing consistent models is a difficult task [28, 29]. All monodromy models have to cope with a general problem which is related to quantum mechanical tunneling. The whole idea of axion monodromy is that the axion winds multiple times over its fundamental domain, while its potential returns to its initial value up to a monodromy transformation. In this way the potential increases and so does the field excursion. One might think that the field excursion can be arbitrarily large, while staying in the *same branch* of the potential. This is impossible, due to tunneling of the axion field to the next branch of the potential. Tunneling always happens in axion monodromy models due to the presence of membranes[4]. Their nucleation mediates the tunneling process.

[3] We will extensively review this in section 6.7 based on the work in [27]

[4] We will return to this problem in section 8.3.

From the point of view of effective field theory, in which monodromy potentials take the form:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \Lambda^4_{dyn}\cos\left(\frac{\phi}{f}\right) \tag{4.5}$$

we can calculate the tunneling rate using various approximations.

## 4.2 Multi axion inflation

### 4.2.1 N-flation

The fact that obtaining $f > M_{pl}$ seemed impossible in string theory, motivated people to investigate whether there were other methods to realize superplanckian field displacements. N-flation is such an attempt [30][5]. In N-flation we have several axions that together form a collective field. Each individual axion field moves over a subplanckian distance while the collective field moves over a superplanckian distance. We mention its most important characteristics here. N-flation is a natural generalization of natural inflation. The potential of this model has a special structure:

$$V(\Phi) = \sum_{i=1}^{N} V_i(\phi_i) \tag{4.6}$$

$\Phi(\phi_i)$ denotes the collective field. Each $V_i$ depends only on a single field $\phi_i$. Each $V_i(\phi_i)$ has the potential of natural inflation:

$$V_i(\phi_i) = \Lambda_i^4 \cos\left(\frac{2\pi\phi_i}{f_i}\right) + (\Lambda_i^{(2)})^4 \cos\left(\frac{4\pi\phi_i}{f_i}\right) + \ldots \tag{4.7}$$

where we now have explicitly written the higher instanton contributions. The second term is proportional to the scale $(\Lambda_i^{(2)})^4$. The size of this scale relative to the first one defines a UV-scale:

$$\Lambda_i^{(2)} = \frac{\Lambda_i^2}{M} \tag{4.8}$$

We demand that we are in a regime where $\Lambda \ll M$ so that the higher harmonics are negligible. We further demand that each $f_i < M_{pl}$. In the regime we are considering, we can also neglect cross-couplings which are induced by multi-instanton effects:

$$V_{ij}^{(2)} = \frac{\Lambda_i^4\Lambda_j^4}{M^4}\cos\left(\frac{2\pi\phi_i}{f_i}\right)\cos\left(\frac{2\pi\phi_j}{f_j}\right) \tag{4.9}$$

Hence, the dynamics is specified by the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{N}\left(\frac{1}{2}(\partial\phi_i)^2 - \Lambda_n^4\big(1-\cos\big(\frac{2\pi\phi_i}{f_i}\big)\big)\right) \tag{4.10}$$

Next, we Taylor-expand the potential for every action around $\phi_i = 0$. This gives:

$$\begin{aligned} V_i(\phi_i) &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nV(\phi_i)}{d\phi_i^n}\,|_{\phi_i=0}\cdot\phi_i^n = -\frac{1}{2!}\Big(\frac{2\pi\Lambda_i^2}{f_i}\Big)^2\phi_i^2 - \frac{1}{4!}\Big(\frac{2\pi\Lambda_i}{f_i}\Big)^4\phi_i^4 + \mathcal{O}(\phi_i^6) \\ &= \frac{1}{2}m_i^2\phi_i^2 - \frac{\lambda_i}{4!}\phi_i^4 + \mathcal{O}(\phi_i^6) \quad \text{where} \quad m_i \equiv \frac{2\pi\Lambda_i^2}{f_i} \quad \text{and} \quad \lambda_i \equiv \Big(\frac{2\pi\Lambda_i}{f_i}\Big)^4 \end{aligned}$$

[5] There are more attempts, for example axion alignment.

We take each $\phi_i \ll M_{pl}$ and each $m_i = m$. Note that the equation of motion of each field is a Klein-Gordon equation:

$$\ddot{\phi}_i + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi_i} \quad i = 1, \ldots, N \tag{4.11}$$

In the slow-roll approximation we have $3M_{pl}^2 H^2 \approx \sum_{i=1}^{N} V_i(\phi_i)$. Thus, each field feels the Hubble-friction of *all* potentials. This is the physical mechanism for achieving super-planckian displacements even though each component-field moves over a sub-planckian distance. Consider the simplest initial configuration, where each field is displaced an equal distance from its minimum, bounded from above by:

$$\Delta\phi_i^2 \leq f_i^2 \tag{4.12}$$

It is for sure sub-planckian since $f_i < M_{pl}$. Now suppose that $\Delta\phi_i \ll f_i$ so that we can neglect the quartic self-coupling $\lambda_i$. In this case the potential is simply:

$$V(\Phi) = \frac{1}{2}m^2 \sum_{i=1}^{N} \phi_i^2 \equiv \frac{1}{2}m^2\Phi^2 \tag{4.13}$$

where we have defined the collective field $\Phi$ as $\Phi^2 \equiv \sum_{i=1}^{N} \phi_i^2$. In this case the collective field displacement is:

$$(\Delta\Phi)^2 = \sum_{i=1}^{N}(\Delta\phi_i)^2 = N(\Delta\phi)^2 \Rightarrow \Delta\Phi = \sqrt{N}\Delta\phi \tag{4.14}$$

which can be super-planckian for sufficiently large N[6]. Even though N-flation is an elegant idea, it also does not seem to meet the consistency requirements of a theory of quantum gravity. Again, this is due to the weak gravity conjecture. The $\sqrt{N}$ factor is precisely canceled as we will see in section 6.8.

## 4.3 Pole inflation

Next we describe the essentials of pole-inflation and a special subset of these models known as $\alpha$-attractors[7]. The initial motivation for constructing these models came from the release of the Planck-data from 2013. Two models, the Starobinsky model and Higgs inflation, despite being very different, made similar predictions for the inflationary observables $(n_s, r)$. The Starobinsky model is a special case of a so-called *conformal attractor* which is a subset of $\alpha$-attractors. Higgs inflation is a special case of universal attractors which are a subset of $\xi$-attractors. Both $\xi$ -and $\alpha$-attractors are subsets of pole-inflation. We will not further discuss $\xi$-attractors.

[6]This is not completely true. If $N$ gets very large, the Planck mass will get renormalized due to the many particle species running in loops in graviton-graviton scattering. This is a manifestation of Dvali's species bound (see sec 7.1).

[7]This is no longer true when we embed $\alpha$-attractors in string theory because the inversion symmetry is lost.

### Conformal attractors - Clever gauge choices

We start by discussing conformal attractors [31, 32]. Consider Einstein gravity with a cosmological constant minimally coupled to a canonical scalar field $\phi$:

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_{pl}^2}{2}R + \frac{1}{2}(\partial\phi)^2 - \Lambda\Big] \tag{4.15}$$

The kinetic term of the scalar is canonically normalized. As the name suggest, these models posses a conformal symmetry. This is not obvious from the above action. In fact, the conformal symmetry is hidden due to gauge-fixing. To make the conformal symmetry manifest, we add a scalar field called the conformon $\chi$ and demand that it forms a doublet with $\phi$ under global SO(1,1) transformations. The conformon is not a physical field as we will see. The form of the action is now restricted to be of the form:

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{12}(\chi^2-\rho^2)R + \frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\chi)^2 - \frac{\lambda}{4}(\rho^2-\chi^2)^2\Big] \tag{4.16}$$

Note that we have replaced $\phi$ by $\rho$ for reasons that become clear below. Also note that $\chi$ is a ghost-field since its kinetic term has the "wrong" sign. This action is invariant under the transformations:

$$\begin{aligned} g_{\mu\nu} &\to e^{2\sigma(x)}g_{\mu\nu} \\ \rho &\to e^{-\sigma(x)}\rho \\ \chi &\to e^{-\sigma(x)}\chi \end{aligned}$$

Thus, the dynamics is invariant under rescalings of the metric and hence the theory is conformal. Note that the symmetry parameter is a function on spacetime so we are dealing with gauge symmetries. The conformal symmetry of this action is related to the conformal symmetry of the original action. We can use the function $\sigma(x)$ to gauge-fix the scalar fields $\rho$ and $\chi$. The clever choice is to choose $\sigma(x)$ such that $\chi^2 - \rho^2 = 6M_{pl}^2$. Note that this also respects the global SO(1,1) symmetry as it should. This particular gauge choice screems for hyperbolic functions:

$$\begin{aligned} \rho(\phi) &= \sqrt{6}M_{pl}\sinh\big(\frac{\phi}{\sqrt{6}M_{pl}}\big) \\ \chi(\phi) &= \sqrt{6}M_{pl}\cosh\big(\frac{\phi}{\sqrt{6}M_{pl}}\big) \end{aligned}$$

Thus, $\rho$ is related to the canonical field $\phi$ via this relation. With this gauge choice, we reproduce the original action we started with by substituting these expressions into 4.16. This is called gauge-fixing to Einstein frame and the gauge choice is termed "rapidity-gauge" because the relation between canonical -and non-canonical field is the same as the rapidity-velocity relation in special relativity. The cosmological constant turns out to be $\Lambda = 9\lambda M_{pl}^4$. Depending on the sign of the coupling $\lambda$ we have either a de Sitter -or anti de Sitter spacetime. Thus, the conclusion is that Einstein gravity with cosmological constant, minimally-coupled to a scalar-field, has conformal symmetry, but it is hidden due to the gauge symmetry.

Next, the potential is deformed by multiplying it with a function $F(\frac{\rho}{\chi})$:

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{12}(\chi^2-\rho^2)R + \frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\chi)^2 - \frac{\lambda}{4}F\big(\frac{\rho}{\chi}\big)(\rho^2-\chi^2)^2\Big] \tag{4.17}$$

Note that this explicitly breaks the global SO(1,1) symmetry unless $F(\frac{\rho}{\chi}) = const.$ but spontaneously breaks the local conformal symmetry. We consider symmetry breaking because this will generate an inflationary potential. If we again make the same gauge choice $\chi^2 - \rho^2 = 6M_{pl}^2$ so that we gauge fix to Einstein frame, we get:

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_{pl}^2}{2}R + \frac{1}{2}(\partial\phi)^2 - 9\lambda M_{pl}^4 F\Big(\tanh\big(\frac{\phi}{\sqrt{6}M_{pl}}\big)\Big)\Big] \tag{4.18}$$

The function $F(\frac{\rho}{\chi})$ is arbitrary. One can think of it as parametrizing deformations from an inflationary model with a pure cosmological constant potential. The arbitrariness of this function can be employed to generate arbitrary inflationary potentials. For example, if we consider a monomial structure $F = x^{2n}$, then we get the potential $V(\phi) = \tanh^{2n}\big(\frac{\phi}{\sqrt{6}M_{pl}}\big)$. Models with this potential are called *T-models* because of the shape of the potential. Note that when $\phi \to \infty$, in this case (but also in many other choices of $F$) $F \to const.$ and the global SO(1,1) symmetry is restored[8].

### Cosmological predictions

The potential becomes nearly constant at super-planckian values of the canonical field. According to the slow-roll conditions, inflation is expected to happen at such large field values. We can trivially rewrite the T-model potential as:

$$V(\phi) = \Big(\frac{1-e^{-\sqrt{\frac{2}{3}}\phi}}{1+e^{-\sqrt{\frac{2}{3}}\phi}}\Big)^{2n} \tag{4.19}$$

where $M_{pl} = 1$ for ease of notation. At large field values of the canonical field, we can Taylor expand the potential and approximate it as:

$$V(\phi) \sim 1 - 4ne^{-\sqrt{\frac{2}{3}}\phi} + \dots \tag{4.20}$$

We see that the potential approaches a plateau at large field values. From the potential we can compute the slow-roll parameters and the inflationary observables $(n_s, r)$. This is easily done by using the relation between the number of e-folds and the canonical inflaton field:

$$N(\phi) = \int \frac{d\phi}{\sqrt{2\epsilon_V}} \implies \frac{d\phi}{dN} = \sqrt{2\epsilon_V} = \frac{V'}{V} \tag{4.21}$$

which implies the differential equation:

$$\frac{d\phi}{dN} = 4n\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\phi} \tag{4.22}$$

[8]Broken global symmetries that become exact again at infinite field values is a very generic property of moduli/field spaces and will re-appear a few times later on.

whose solution is:

$$e^{-\frac{2}{3}\phi(N)} = \frac{3}{8n}\Big(\frac{1}{N}\Big) \tag{4.23}$$

in the limit of large $N$, in which we can neglect the contribution of the lower-integration limit. In this limit, the Taylor expansion of the potential is an expansion in $\frac{1}{N}$. Now we can find $\epsilon_V$ and $|\eta_V|$ as functions of $N$ and hence the observables $(n_s, r)$ in terms of $N$. The result is:

$$r = \frac{12}{N^2} \quad \text{and} \quad n_s = 1 - \frac{2}{N} + \mathcal{O}\Big(\frac{1}{N^2}\Big) \tag{4.24}$$

It turns out that for many different choices of the potential, this result for $n_s$ and $r$ is found (we saw that it is independent of the integer $n \in \mathbb{Z}$). Therefore, these models are called (conformal) attractors, since they are "attracted" towards the above inflationary predictions, despite having (very) different potentials.

### The boundary of moduli space

There is geometric understanding why conformal inflation leads to these universal predictions [31, 33, 34]. We noted that the slow-roll conditions are readily satisfied for large values of the canonical inflaton field. This flattening of the potential always starts close to the boundary of the moduli space[9]. To see this, we start by writing the action in *Jordan frame* and choose the gauge $\chi = \sqrt{6}$ known as conformal gauge [31]:

$$S_J = \int d^4x\sqrt{-g_J}\Big[\frac{R(g_J)}{2}\big(1-\frac{\rho^2}{6}\big) - \frac{1}{2}(\partial\rho)^2 - \frac{\lambda}{4}(\rho^2-6)^2\Big] \tag{4.25}$$

Note that now the non-canonical field $\rho$ has a canonical kinetic term but it is non-minimally coupled to the metric. To go to Einstein frame we need to conformally rescale the metric and simultaneously redefine the canonical field $\rho$:

$$g^E_{\mu\nu} = \Big(1-\frac{\rho^2}{6}\Big)^{-1} g^J_{\mu\nu} \tag{4.26}$$

$$\frac{d\phi}{d\rho} = \frac{1}{1-\frac{\rho^2}{6}} \tag{4.27}$$

This yields the Einstein frame action but with different functional form of the potential because it is obviously not conformally invariant. The function $F$ becomes in this gauge $F(\frac{\rho}{\sqrt{6}})$. The above relation shows that $\phi$ and $\rho$ are related as:

$$\phi(\rho) = \tanh^{-1}\frac{\rho}{\sqrt{6}} \Rightarrow \frac{\rho(\phi)}{\sqrt{6}} = \tanh\frac{\phi}{\sqrt{6}} \tag{4.28}$$

The boundary of the moduli space is located at $\rho = \pm\sqrt{6}$ since there the flattening of the potential kicks in. We can show that, if we Taylor expand the potential near the boundary of the moduli space, we arrive at the same inflationary predictions. We define a variable $x \equiv 1 - \frac{\rho}{\sqrt{6}} = 1 - \tanh\frac{\phi}{\sqrt{6}}$, measuring the deviation from

[9]This terminology will be explained in greater detail in the following chapter but let us mention here that "moduli space" is the same thing as field space. It is the space spanned by, strictly speaking, massless scalar fields. However, typically also the space spanned by massive scalar fields is called "moduli space".

the boundary. Using the Taylor expansion of the hyperbolic tangent that we used above, for large field values of $\phi$ this can be approximated as $x \approx 2e^{-\sqrt{\frac{2}{3}}\phi}$. Taylor expanding the potential near the boundary of the moduli space means:

$$V(\phi) = V_0\Big[1 - \sum_n c_n x^n\Big] \quad \text{with} \quad x(\rho) \equiv 2e^{-\sqrt{\frac{2}{3}}\phi} \tag{4.29}$$

where $V = V_0$ coincides with the canonical field being at infinity. To first order in the deviation $x$ we get:

$$V(\phi) \approx V_0\Big[1 - 2c_1 e^{-\sqrt{\frac{2}{3}}\phi} + \mathcal{O}(x^2)\Big] \tag{4.30}$$

Note that this is exactly the same expansion as 4.20. In the large $N$ limit, we now have, by replacing $4n \to 2c_1$,:

$$2c_1 e^{-\frac{2}{3}\phi(N)} = \frac{3}{2N} \tag{4.31}$$

We observe the same result: the Taylor expansion in the deviation from the moduli space boundary $x$ is equivalent to the expansion in the inverse number of e-folds $\frac{1}{N}$ in the large $N$-limit. Hence, we obtain the same results for the inflationary observables. This also shows that the closer we are to the pole in non-canonical field space, or the boundary in canonical field space, the larger the number of e-foldings becomes. This means inflation happens in $\alpha$-attractor models as the canonical field moves away from the boundary.

Note that the inflationary plateau in these models is infinitely long. We will see in the next chapter that quantum gravity requirements demand the traversed distance of $\phi$ to be (at least) finite otherwise the effective description will breakdown due to the appearance of infinitely many light states. This breakdown cannot be cured by switching to a different description. This implies that the inflationary plateau cannot be infinitely long. This has also implications for the problem of initial conditions of inflation in these models. In fact, one can show under particular circumstances how far away from infinity the canonical field has to be in order to obey quantum gravity restrictions.

To recap, this analysis works when the Taylor expansion of the potential exists near the point $\rho = \sqrt{6}$ in non-canonical field space. Thus, the scalar potential is required to be a regular function near the boundary of the canonical field space. Secondly, we assumed a large number of e-foldings to rewrite the Taylor expansion in terms of the number of e-folds. In this limit the inflationary predictions are given by 4.24. Besides the conformal universality class there also exists a *superconformal* universality class and the T-model discussed above can arise as a supergravity version of a particular superconformal action [32]. Again, via spontaneous symmetry, but now of the superconformal symmetry, we can arrive at the same predictions for the inflationary observables by clever choices of gauge. In the literature, $\alpha$-attractors were first discussed as generalizations of superconformal attractors. In that framework the parameter $\alpha$ was also related to the scalar curvature of the underlying Kähler manifold. We will first take a slightly different route to the notion of $\alpha$-attractors that is more closely connected to the previous discussion and then go back to the superconformal framework.

### Non-canonical kinetic terms

Everything discussed above, in particular the T-model, can be equivalently formulated as simple as an inflationary action with a non-canonical kinetic term [35]. The universal predictions of $(n_s, r)$ will be generalized accordingly. Consider the following action in Einstein frame:

$$S_E = \int d^4\sqrt{-g}\Big[\frac{M_{pl}^2}{2}R - \frac{1}{2}\frac{(\partial\rho)^2}{\left(1-\frac{\rho^2}{6\alpha}\right)^2} - \frac{1}{2}m^2\rho^2\Big] \tag{4.32}$$

where we introduced the parameter $\alpha \in \mathbb{R}$. Note that for $\alpha \to \infty$ we get $m^2\rho^2$-inflation which has $r = \frac{8}{N}$. What happens for $\alpha \to 0$ is discussed below. The metric on moduli/field space is in this case non-trivial: in fact, it is singular since it has a pole of order two at $\rho = \pm\sqrt{6\alpha}$. Note that these field values correspond to the location of the boundary of the moduli space. This action turns out to be precisely the T-model action, as can be seen by switching to the canonical field $\phi$. We get that $\rho = \sqrt{6\alpha}\tanh\frac{\phi}{\sqrt{6\alpha}}$ and hence the potential becomes $V(\phi) = 3\alpha m^2\tanh^2\frac{\phi}{\sqrt{6\alpha}}$, which is the T-model potential. The inflationary predictions are generalized to:

$$\boxed{n_s = 1 - \frac{2}{N} \quad \text{and} \quad r = \frac{12\alpha}{N^2}} \tag{4.33}$$

When $\alpha \sim \mathcal{O}(1)$, modifications of the potential do not alter the inflationary predictions. This value is also in agreement with observations of Planck. So we see that $r$ interpolates between $\frac{12}{N^2}$ and $\frac{8}{N}$. We already noted the pole of order two in the non-canonical kinetic term above. The presence of a pole in the kinetic term is the characteristic of pole-inflation. It turns out that the properties of the leading pole in the Laurent expansion of the kinetic term in Einstein frame are responsible for the robustness of the predictions 4.33. More precisely [35]:

**Origin of attractor nature**
*Consider an inflationary action of the form:*

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}R - \frac{1}{2}K_E(\rho)(\partial\rho)^2 - V_E(\rho)\Big] \tag{4.34}$$

*If $K_E$ is given by a Laurent series, then the inflationary predictions for the above model are determined by the order -and residue of the leading pole of the series, provided the scalar potential is regular near the pole.*

In [] it is shown that in the large $N$ limit, for a pole order two with residue $a_2$, the inflationary observables are:

$$n_s = 1 - \frac{2}{N} \quad \text{and} \quad r = \frac{8a_2}{N^2} \tag{4.35}$$

For the T-model action we have:

$$K_E \equiv \frac{\alpha}{\left(1-\frac{\rho^2}{6}\right)^2} \tag{4.36}$$

This function has poles of order two at $\psi = \pm\sqrt{6}$. For the pole at $\rho = \sqrt{6}$ one can show that the Laurent expansion is:

$$K_E = \frac{3\alpha}{2}\frac{1}{\left(1-\frac{\rho^2}{6}\right)^2} - \dots \tag{4.37}$$

The residue is $a_2 = \frac{3\alpha}{2}$, the coefficient of this term. Therefore, we see that we indeed recover the predictions of $\alpha$-attractors for $(n_s, r)$.

### Suppression of field theory corrections

The description 4.34 of $\alpha$-attractors is of course an effective description below some scale $\Lambda$. In effective field theory, we suppressed corrections by a weakly-broken shift-symmetry. In $\alpha$-attractors, or more generally pole-inflation, we have the same situation. The non-canonical field space pole of the kinetic function is translated, upon transforming to canonical field space, into a potential with an infinitely long approximately shift-symmetric dS-plateau. The shift-symmetry is only broken by exponentially suppressed corrections and the shift-symmetry is restored completely at the boundary of the moduli space, i.e. when $\phi \to \infty$. We have seen that a shift-symmetry is equivalent to a hierarchy of the Wilson coefficients of higher-dimensional operators. It is then natural to ask how this approximate shift-symmetry of the canonical potential translates into properties of the kinetic function in non-canonical field space. It has been shown that suppression of higher-dimensional operators is translated into a hierarchy of the residues of the poles in the kinetic function [36]. Suppose for the moment that the pole is located at $\rho = 0$. Then terms higher-order in the pole, i.e. terms higher-order in $\frac{1}{\rho}$, become increasingly important as $\rho \to 0$. We write the kinetic function as a Laurent expansion:

$$K(\rho) = \frac{a_q}{\rho^q} + \frac{a_p}{\rho^p} + \dots \quad \text{with} \quad q > p \tag{4.38}$$

where the dots denote terms higher-order in $\rho$ and are hence less important upon approaching the pole. This higher-order correction would introduce a correction to the effective potential. For this correction not to dominate over the leading term we need to require the hierarchy:

$$\frac{a_q}{\rho^q} \ll \frac{a_p}{\rho^p} \quad \forall q > p \tag{4.39}$$

This relation is to be understood as a "dual" statement to the weakly-broken shift-symmetry of the scalar potential in canonical variables. Just as a shift-symmetry prevents field theory corrections from being relevant, this hierarchy suppresses higher-order pole corrections from being relevant. However, a priori there is no justification for a hierarchy like 4.39, just as there is no justification a priori for a weakly-broken shift-symmetry in the top-down approach to effective field theory for the suppression of higher-order field theory corrections. Whether this hierarchy of the residues of the kinetic function is natural is a question for the UV completion but the absence of global symmetries[10] should warn us that this is probably not the case.

[10]Exact continuous global symmetries are thought to be forbidden in a theory of quantum gravity [37]. We will discuss this in section 6.1.1.

With respect to the purposes of inflation, this is not necessarily problematic. Inflation was invented to solve the big-bang puzzles. Thus, we should care about whether we can achieve 60 e-foldings of exponential expansion and this seems to be allowed.

### 4.3.1 Interpretations of $\alpha$

**1. $\alpha$ parametrizes the scalar curvature**

In [32], superconformal attractors were studied. These models contained three complex superfields: the conformon $X^0$, the inflaton $X^1 \equiv \Phi$ and a sGoldstino $X^2 \equiv S$. The superconformal action is formulated in terms of a Kähler-embedding function $\mathcal{N}(X, \bar{X})$, with a SU(1,1) symmetry between the conformon and the inflation, and a superpotential $\mathcal{W}$ that preserves the subgroup SO(1,1) of SU(1,1):

$$\mathcal{N}(X, \bar{X}) = -\left|X^0\right|^2 + |\Phi|^2 + |S|^2 \tag{4.40}$$

$$\mathcal{W} = Sf\left(\frac{\Phi}{X^0}\right)\left[(X^0)^2 - (\Phi)^2\right] \tag{4.41}$$

This superconformal attractor corresponds to $\alpha = 1$. By deforming the function $f\left(\frac{\Phi}{X^0}\right)$ the SO(1,1) symmetry is explicitly broken. Upon substitution in the superconformal action one finds the standard inflationary action with the potential a function of the hyperbolic tangent, i.e. the T-models.

To incorporate $\alpha \neq 1$ and generalize superconformal attractors one considers the Kähler-embedding function -and superpotential:

$$\mathcal{N}(X, \bar{X}) = -\left|X^0\right|^2\left[1 - \frac{|\Phi|^2 + |S|^2}{|X^0|^2}\right]^{\alpha} \tag{4.42}$$

$$\mathcal{W} = S\left|X^0\right|^2 f\left(\frac{\Phi}{X^0}\right)\left[1 - \frac{\Phi^2}{(X^0)^2}\right]^{\frac{(3\alpha-1)}{2}} \tag{4.43}$$

$\alpha = 1$ and a constant function $f$ preserves the SO(1,1) symmetry but otherwise it is explicitly broken. Gauge-fixing the conformal symmetry by setting the complex conformon to a constant value $X^0 = \bar{X}^0 = \sqrt{3}$ a supergravity is obtained in terms of:

$$K = -3\alpha \log\left[1 - \frac{S\bar{S} + \Phi\bar{\Phi}}{3}\right] \tag{4.44}$$

$$W = Sf\left(\frac{\Phi}{\sqrt{3}}\right)\left(3 - \Phi^2\right)^{\frac{3\alpha-1}{2}} \tag{4.45}$$

Note that the sgoldstino is part of the logarithmic structure in the Kähler potential. Below we will consider a canonical Kähler potential for $S$. Truncating the model to a single-field model $S = \Phi - \bar{\Phi} = 0$ we obtain a supergravity model with Kähler potential:

$$K = -3\alpha \ln\left(1 - \frac{\Phi\bar{\Phi}}{3}\right) \tag{4.46}$$

Stability of this truncation is achieved by adding a term to the Kähler potential, i.e. by deforming the Kähler manifold []. We gauge-fixed a conformal $\mathbb{R}$-symmetry, meaning that we also have gauge-fixed a U(1) symmetry. The Kähler potential is therefore invariant under the coset group $SU(1,1)/U(1)$. This is a very interesting coset space: it is the double-cover of the hyperbolic plane. A Kähler potential defines a Kähler metric on this $SU(1,1)/U(1)$ manifold:

$$K_{\Phi\bar{\Phi}} = \partial_\Phi \partial_{\bar{\Phi}} K = \frac{\alpha}{\left(1 - \frac{\Phi\bar{\Phi}}{3}\right)^2} \tag{4.47}$$

Note that this potential has a pole of order 2 at $|\Phi| = \pm\sqrt{3}$. Since the Kähler metric also determines the form of the kinetic term of the inflaton, such Kähler potentials can be used for supergravity embeddings of pole-inflaton. To proceed, we use a standard formula for the scalar curvature of a Kähler manifold:

$$R = -K^{\Phi\bar{\Phi}} \partial_\Phi \partial_{\bar{\Phi}} \ln K_{\Phi\bar{\Phi}} \tag{4.48}$$

We find the interesting relation [32]:

$$R = -\frac{2}{3\alpha} \tag{4.49}$$

$\alpha$ is inversely proportional to the scalar-curvature of the manifold $SU(1,1)/U(1)$.

**2. $\alpha$ is a radius**

There is another interpretation of $\alpha$ related to this one [33, 34]. The Poincaré-disk representation of the hyperbolic plane $\mathbb{H}^2$ has negative curvature $-\frac{2}{R^2}$ for a Poincaré-disk of radius $R$. If we set $R = \sqrt{3\alpha}$, we obtain the curvature of the above symmetric space. Thus, $\alpha$ sets the size of a Poincaré-disk representation of $\mathbb{H}^2$. Furthermore, we can identify the Kähler geometry, defined by the Kähler metric, with the metric of the Poincaré disk-model of $\mathbb{H}^2$:

$$K_{\Phi\bar{\Phi}} = \frac{\alpha}{\left(1 - \frac{\Phi\bar{\Phi}}{3}\right)^2} = \frac{dx^2 + dy^2}{\left(1 - \frac{x^2+y^2}{3\alpha}\right)^2} \tag{4.50}$$

for $\Phi = \frac{x+iy}{\sqrt{3\alpha}}$. $(\Phi, \bar{\Phi})$ are disk-variables and obey the condition $\Phi\bar{\Phi} < 1$. This leads to the insight that the moduli space of $\alpha$-attractors is a hyperbolic space that can be represented by a Poincaré-disk model with radius $R = \sqrt{3\alpha}$. Since the tensor-to-scalar ratio depends on $\alpha$, the amplitude of gravitational waves is determined by the curvature of the moduli space.

It is a well-known geometric fact that a two dimensional hyperbolic geometry $\mathbb{H}^2$ can equivalently be represented by a Poincaré half-plane model. These two representations are related to each other via a conformal mapping known as a *Caley* transformation:

$$\Phi = \frac{T-1}{T+1} \quad \text{and} \quad T = \frac{1+\Phi}{1-\Phi} \tag{4.51}$$

where $T, \bar{T}$ denote the half-plane variables and they obey $T + \bar{T} > 0$. The Kähler potential -and metric in these variables are:

$$K = -3\alpha \ln\left(T + \bar{T}\right) \quad \textit{and} \quad K_{T\bar{T}} = \frac{3\alpha}{\left(T + \bar{T}\right)^2} \tag{4.52}$$

Half-plane variables reveal an interesting property of the line element of the moduli space of $\alpha$-attractors. Define a variable $\tau = iT$. Then the line element becomes

$$ds^2 = 3\alpha \frac{d\tau d\bar{\tau}}{(2\mathrm{Im}(\tau))^2} \tag{4.53}$$

This is invariant under transformations:

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \quad \text{where} \quad ac - bd \neq 0 \tag{4.54}$$

In other words, the line element is invariant under the group of real linear invertible transformations GL(2, $\mathbb{R}$).

### 4.3.2 A different Kähler frame

The symmetry group of the kinetic term of the inflaton is the Möbius group which, among other transformations, includes a shift-symmetry. More precisely, we write the complex half-plane variable $T$ as:

$$T = \exp\left\{\sqrt{\frac{2}{3\alpha}}\varphi\right\} + ia \tag{4.55}$$

where $\varphi$ is the inflaton (dilaton) and $a$ an axion. One can decompose an arbitrary Möbius transformation in terms of a product of compact, abelian and nilpotent transformations. The abelian subgroup of GL(2, $\mathbb{R}$) acts as a shift on the inflaton field and as a dilation on the axion. It is convenient to have a Kähler potential in which this abelian symmetry, containing the shift-symmetry of the inflaton, is manifest during inflation (when $\mathrm{Im}\, T = 0$), and which is only weakly-broken by a superpotential. This makes radiative stability of the superpotential manifest, demonstrating that it really is the underlying hyperbolic geometry that protects the model against perturbative quantum corrections. It turns out that right choice of Kähler potential is [34]:

$$K = -\frac{3\alpha}{2} \log\left(\frac{\left(T + \bar{T}\right)^2}{4(T\bar{T})}\right) \tag{4.56}$$

in half-plane variables and in disk variables:

$$K = -\frac{3\alpha}{2} \log\left(\frac{(1 - Z\bar{Z})^2}{(1 - Z^2)(1 - \bar{Z}^2)}\right) \tag{4.57}$$

and the two are related by a Caley transformation. It is important that there is no need for an additional Kähler transformation. This would alter the obvious symmetries of the Kähler potential. Stabilization of the inflaton partner is realized by adding terms to $K$ that preserve the shift-symmetry in the inflaton direction. However, stabilization requires the curvature to take rather specific values, i.e. stabilization cannot be achieved for arbitrary values of $\alpha$. When we add the second superfield $S$, along which SUSY is broken, with canonical Kähler potential:

$$K = -\frac{3\alpha}{2} \log\left(\frac{\left(T + \bar{T}\right)^2}{4(T\bar{T})}\right) + S\bar{S} \tag{4.58}$$

the model is stable for all $\alpha$ during inflation and *no* additional stabilization terms are required. If $S$ was part of the logarithmic structure, we had to add additional powers of $S\bar{S}$ to the potential. Hence, the structure 4.58 is preferred.

To summarize, the universal predictions for $(n_s, r)$ in $\alpha$-attractor models are a consequence of a pole of order two in the kinetic term and its residue sets the size of $r$. This shifts the complexity of the inflationary dynamics from the potential to the kinetic term. The existence of a pole has a geometric interpretation, in the context of supergravity, in terms of a boundary in the moduli space. The moduli space of $\alpha$-attractors, based on a single complex superfield, is a two-dimensional hyperbolic geometry. It therefore allows for a Poincaré-disk -or half-plane representation, making the Möbius symmetry of the kinetic term manifest. It is this symmetry that protects the model against perturbative quantum gravity corrections. Inflation happens as the canonical inflaton field moves away from the boundary.

### 4.3.3 On initial conditions

#### Inflation in a FRW spacetime

A particular clever choice of coordinates on the moduli space is the following [38]. We let $\Phi = \phi + i\theta$ and $Z = \tanh \frac{\Phi}{\sqrt{6\alpha}}$ so that $Z = \tanh \frac{\phi + i\theta}{\sqrt{6\alpha}}$. We take $\phi$ to be a Killing direction of the moduli space geometry, meaning that the moduli space metric does not depend on this coordinate. We can consider the T-model again. The superpotential is $W = \sqrt{\alpha} m S Z$ and the inflaton potential is $V = \alpha m^2 \tanh^2 \frac{\phi}{\sqrt{6\alpha}}$. In the variables $\phi, \theta$, the total (two-dimensional) potential has a minimum at $\theta = 0$ and at large values of the inflaton $\phi$ has a constant infinite de Sitter valley of $\phi$-independent width. The existence of this infinite dS vally is important for addressing the problem of initial conditions for inflation.

First, we want to consider the inflationary dynamics of $\phi$ in a homogeneous FRW spacetime along $\theta = 0$. Furthermore, we consider large field values $\phi \to \infty$ so that the inflationary potential is constant $V_{dS} = \alpha m^2$. Recall that the amplitude of scalar perturbations is measured to be $A_s \sim 10^{-9}$. This fixed the inflaton mass $m \sim 10^{-5}$. Therefore, we have for $\alpha \sim \mathcal{O}(1)$ $V_{dS} \sim 10^{-10}$. Now, we want to discover when $\phi$ enters the slow-roll regime. We take the initial state to be at the Planck density $\frac{1}{2}\dot{\phi}^2 = 1$ where the field has value $\phi = \phi_0$. The equation of motion is:

$$\ddot{\phi} + 3H\dot{\phi} = \ddot{\phi} + \sqrt{3}\dot{\phi}\sqrt{\frac{1}{2}\dot{\phi}^2 + V_{ds}} \approx \ddot{\phi} + \sqrt{\frac{3}{2}}\dot{\phi}^2 = 0 \tag{4.59}$$

where we used the Friedmann equation. This equation is easily solved:

$$\frac{\dot{\phi}^2}{2} = \frac{1}{3t^2}, \quad \phi(t) = \sqrt{\frac{2}{3}} \ln\left(\frac{t}{t_0}\right) + \phi_0(t_0) \tag{4.60}$$

The time $t$ is the time where the kinetic energy density equals the Planck density and at the time $t_0$ it equals $V_{ds}$. This interval defines the kinetic energy dominated regime. Hence, we find:

$$\Delta\phi \equiv \phi(t) - \phi_0(t_0) = \sqrt{\frac{2}{3}} \ln\left(10^{10}\right)^{\frac{1}{2}} \approx 9.4 \tag{4.61}$$

Next, we enter the potential-dominated phase and the dynamics is approximately governed by:

$$\ddot{\phi} + \sqrt{3V_{ds}}\dot{\phi} = 0 \tag{4.62}$$

$\phi$ is exponentially damped and quickly becomes constant, after roughly one Planck unit. Interestingly, after 10 Planck units in field space, all memory about the initial velocity is lost. The situation remains the same when we take into account the kinetic energy of the field $\theta$ and study its evolution at $\phi \to \infty$. Thus, the slow-roll regime starts after $\Delta\phi \sim 10$. In the case of the T-model, during slow-roll the field traverses a distance:

$$\Delta\phi_{SR} = \sqrt{\frac{3}{2}} \ln\Big(\frac{8N}{3}\Big) \tag{4.63}$$

where $N$ is the number of e-folds. For $N \approx 60$ we have $\Delta\phi_{SR} \approx 6$. Hence, the evolution of the inflaton from the initial Planck density state to the end of inflation does not depend on the initial velocities of $\phi, \theta$ if the initial field value $\phi_0$ is separated 16 Planck units from its value at the end of inflation.

#### Inflation in an inhomogeneous universe

We now want to address the initial conditions problem of inflation in an inhomogeneous universe for $\alpha$-attractors [38, 18]. The key to the solution is the realization that, for T-models[11], the potential of $\alpha$-attractors is equal to a cosmological constant $V_{ds} = \alpha m^2 \sim 10^{-10}$ for almost all values of the inflaton field, except for $|\phi| < \sqrt{\alpha}$. Hence, if the problem for an expanding de Sitter space is solved, it is to an exponentially accurate degree solved for $\alpha$-attractors[18]. Therefore, we start with a de Sitter model with $V_{dS} \ll 1$:

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{2} - \frac{1}{2}(\partial\phi)^2 - V_{ds}\Big) \tag{4.64}$$

The initial state is at the Planck density and dominated by the kinetic -and gradient energy of the inflaton: $\frac{1}{2}\dot{\phi}^2 + (\nabla\phi)^2 \sim 1$. If the universe has a compact flat or open topology, its evolution was already described at the end of the first chapter. It remains relatively homogeneous due to chaotic mixing and an ultra-relativistic equation of state that does not support the growth of inhomogeneities. When the inflaton energy density drops below the potential energy density, exponential expansion kicks in, in a perfect de Sitter space.

Instead of considering an initial universe that was very small and compact, we next consider the dynamics of a universe that was initially very large. Two things can happen for such a universe. First, it can collapse before inflation starts. This collapse has to occur rather quickly, since it can only happen before the energy density of the scalar field has dropped below $V_{ds} \sim 10^{-10}$. In fact, it has to occur after $t \sim \frac{1}{H} \sim \frac{1}{\sqrt{V}} \sim 10^5$ Planck seconds, or $10^{-28}$ seconds. The other option is that it not collapses but just keeps expanding. It might contain initial inhomogeneities that become large in some regions. As the matter density drops as $\rho \sim t^{-2}$, it will quickly drop below $V_{ds}$ and exponential expansion kicks in. All inhomogeneities will

[11] In fact, this argument holds for all models with an inflationary plateau at large values of the canonical field.

move away from each other exponentially fast and therefore are irrelevant. What is left is a large de Sitter space. In the context of the above model, the question is not whether inflation will happen in a universe containing large initial inhomogeneities but whether it is possible *not* to happen. In other words, there is no problem of initial conditions.

We can relate this pure de Sitter model to initial conditions for inflation in a more realistic model but not yet a $\alpha$-attractor. Consider the following model:

$$W = \sqrt{\alpha} m S \tag{4.65}$$

$$K = -\frac{3\alpha}{2}\log\left(\frac{(1-Z\bar{Z})^2}{(1-Z^2)(1-\bar{Z}^2)}\right) + S\bar{S} \tag{4.66}$$

The superpotential is $Z$-independent. When we consider the convenient moduli space coordinates $(\phi, \theta)$ from above, the potential has a dS minimum $V_{dS} = \alpha m^2$ at $\theta = 0$ and an infinite dS valley, coinciding with the dS valley in the large $\phi$ limit of the T-model potential:

$$V = \alpha m^2 \Big(\cos\sqrt{\frac{2}{3\alpha}}\theta\Big)^{-3\alpha} \tag{4.67}$$

Since the dS valley is $\phi$-independent, it is exactly shift-symmetric with respect to $\phi$. For $\theta = 0$ we enter the stage of exponential dS expansion because, according to the above argument, there is nothing preventing the field $\theta$ from settling in its minimum. Indeed, $\theta$ starts rolling first, which is clear when switching to canonical variables, after which inflation becomes driven by $\phi$. Due to the exact shift-symmetry of the dS valley with respect to $\phi$, all initial conditions for the inflaton have equal probability. When $\theta$ is at the minimum, different regions of the universe have different field values $\phi$, all of which have equal probability. Due to the exponential expansion all initial homogeneities in $\phi$ are exponentially killed.

However, this is not yet a satisfying inflationary scenerio as the inflaton does not enter a slow-roll regime, since the dS valley is infinite. We have to introduce a $Z$-dependence into the superpotential. In that case we have a $\alpha$-attractor. We already mentioned above that this potential is almost a cosmological constant except for field values $|\phi| < \sqrt{\alpha}$. There it rolls down to its minimum at $\phi = 0$. In the case of inflation in a homogeneous universe, we found that even if the initial kinetic energy of the inflaton is planckian, after a distance of 10 Planck units it reaches the slow-roll regime. The stage of slow-roll inflation occurs for all initial values of $\phi$ at the Planck density that satisfy $\sqrt{\alpha} + \mathcal{O}(10) < |\phi| < \infty$ with all values being equally probably due to the shift-symmetry. Therefore, to a very high degree of accuracy the discussion is the same as for the pure dS case, the inaccuracy due to the probability that the field $\phi$ might land in the region $|\phi| < \sqrt{\alpha} + \mathcal{O}(10)$.

We conclude that for all values $\phi$ in the infinite regime of initial conditions, the probability for slow-roll inflation in $\alpha$-attractors is equal to the probability that the universe ends up in an exponentially expanding dS space.

# Chapter 5

# The swampland conjectures

In 2005 Ooguri and Vafa proposed several criteria based on case-by-case examples in string theory and quantum field theory to distinguish consistent effective field theories that can be embedded in a theory of quantum gravity from those that cannot [39]. All these conjectures are statements about the properties of the *moduli space* of a theory of quantum gravity. In the literature, this set of conjectures is referred to as the *Ooguri-Vafa conjectures* or *Swampland conjectures*.

This chapter focusses on the so-called *swampland distance conjecture*. At the time of writing, its validity is extensively investigated. For a proper understanding of it, as well as for constructions that support it, an understanding of the concept of moduli space is required. From a mathematical point of view, a moduli space is an extremely general concept. It serves to classify geometric structures. Here we will start with an example of a moduli space that arises in compactifications of Type II string theories on Calabi-Yau manifolds.

Next, we zoom in on one component of the total moduli space of Calabi-Yau compactifications of Type IIB, the *complex-structure* moduli space. Many mathematical and physical properties of this moduli space have been extensively studied over the years by mathematicians and physicists. This makes it an excellent candidate to test the distance conjecture. We then proceed with the formulation of the conjecture and the discussion of a very general construction in this moduli space that supports it [3].

The swampland distance conjecture, if true, has rather dramatic phenomenological consequences, in particular its stronger (refined) formulation [40]. This will be discussed in the last section.

Before we can properly speak about the swampland conjectures, we first introduce the concept of the swampland with an "example conjecture". Then we briefly collect some string theory elements that play a role in motivating the conjectures via specific constructions. This will involve discussions on the construction of the Hilbert space of Type II theories and string dualities.

## 5.1 The idea of the swampland

Extensive studies of F-theory, M-theory and string theories have shown that the number of consistent compactifications from twelve, eleven respectively ten dimensions to realistic four-dimensionsional physics is astronomical [41]. Estimates are of the order of googols or even googolplexes, i.e. a number as large as $10^{100} - 10^{10^{100}}$. This space of theories is known as the *landscape of string theory*. It is the set of consistent solutions of string theory, i.e. the set of field configurations that solves the equations of motion. This led people to think that any effective field theory coupled to gravity in four dimensions can in some way consistently realized via a compactification of one of the above-mentioned theories. If this is true, why would one bother to study complicated geometric compactifications? One could just write down a seemingly consistent effective field theory and the job is finished.

For Cumrun Vafa this went too far. Starting in 2005, he argued that not all seemingly consistent-looking, low-energy effective theories *coupled to gravity* are actually consistent [1]. Instead of belonging to the string landscape, these theories belong to the *string swampland*, and they cannot arise from a compactification of string theory. Vice versa, these theories do not admit a UV completion in quantum gravity. When one decouples gravity, the conjectures no longer hold or are trivially satisfied. To distinguish the landscape from the swampland Vafa developed swampland criteria, or rather conjectures, as they lack rigorous mathematical proofs. The swampland conjectures are motivated by what are thought to be characteristics of quantum gravity, e.g. the Hawking evaporation of black holes, and by non-trivial constructions in sting theory.

One of the prime motivations of finding swampland criteria is that they give insight into characteristic properties of quantum theories of gravity. However, it turned out that some conjectures also have rather dramatic phenomenological implications for cosmology. In particular, *the weak gravity conjecture* and the *swampland distance conjecture* have a large impact on inflationary model building. It has been argued that models of large-field models of inflation reside in the swampland [42, 43, 27, 44]. It should be noted that these conjectures come in various forms, some of which put stronger constraints on parameters than others. In particular, the weak gravity conjecture has been studied extensively in various contexts, so that there exist multiple formulations and generalizations of the original statement. The next chapters are devoted to a careful study of several swampland conjectures, emphasising the aforementioned ones. We will also point towards connections between the various conjectures.

Besides the string swampland there also exists a swampland of effective field theories. This should come as no surprise as quantum field theories may contain *anomalies*. An anomaly arises when a symmetry of the classical Lagrangian is lost upon quantization. Anomalies manifest themselves in various ways, for instance by the non-conservation of the expectation value of the quantum current: $\partial_\mu \langle J^\mu \rangle \neq 0$. Now, anomalies associated to global symmetries are not problematic. However, anomalies associated to gauge symmetries, known as gauge anomalies, cause inconsistencies. Certain conditions have to be imposed to guarantee that the QFT is *anomaly-free*.

Such an anomaly cancellation is a crucial ingredient for the standard model to be a consistent quantum theory. Anomalies can be abelian or non-abelian. The standard example of an abelian global anomaly is the chiral anomaly in QED. There are also gravitational anomalies, supersymmetry anomalies and many more. Field theories with anomalies belong to the *EFT-swampland*.

There are also criteria beyond anomalies that a low-energy effective theory has to satisfy in order for it to admit a field theoretic UV completion. Recently, there is a lot of attention for the *positivity bounds*, i.e. the Wilson coefficients $c_i$ of certain irrelevant operators are strictly positive, demanded by analyticity of the S-matrix, or equivalently locality, in the UV completion [45]. For example, consider the effective field theory for a U(1) gauge theory. Its effective Lagrangian is:

$$\mathcal{L}[A] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda^4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{c_2}{\Lambda^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 + \ldots \tag{5.1}$$

where $\tilde{F}$ is the Hodge-dual tensor to $F$, $\tilde{F} = \star F$. At the level of effective field theory, the Wilson coefficients are in principle completely arbitrary dimensionless numbers. However, the claim is that $c_i > 0$ for it to be UV completed in a local quantum field theory. For instance, the above effective Lagrangian arises when integrating-out electrons from the QED Lagrangian at one-loop. This Lagrangian is known as the Euler-Heisenberg Lagrangian and it indeed satisfies $c_1, c_2 > 0$:

$$c_1 = \frac{\alpha^2}{90m_e^4} \quad \text{and} \quad c_2 = \frac{\alpha^2}{360m_e^4} \tag{5.2}$$

These positivity bounds have proven to be very important in many areas of research. For instance, it is used in theories of massive gravity, the EFT of quantum gravity and composite Higgs models. The very same ideas of positivity have been used in providing a field theory, or "S-matrix proof", of the mild weak gravity conjecture. Positivity is the topic of the very last chapter. Now that we have made the distinction between the EFT swampland and the string-swampland, let us formulate the concept of the string swampland precisely to avoid confusion about the terminology.

**Definition 1.** *The string swampland is the set of quantum effective field theories weakly coupled to Einstein gravity that cannot be embedded consistently in a UV complete theory. This UV complete theory can in principle be any quantum theory of gravity.*

From now on, the adjective "string" will be dropped and we will just refer to the swampland. As a first example of a swampland criterion we discuss the finiteness of the number of massless gauge fields. This discussion is based on [8, 46]. All string theory actions contain terms quadratic in the field strengths :

$$S \sim \int_{M_{10}} d^{10}x\sqrt{-G}F_{\mu_1+\cdots+\mu_{p+1}}F^{\mu_1+\cdots+\mu_{p+1}} \tag{5.3}$$

Upon a gauge choice analogous to the Lorentz gauge of Maxwell theory, the ten-dimensional $r$-form gauge field $A_r$ obeys a ten-dimensional Laplace equation:

$$\Delta_{10}A_r = 0 \tag{5.4}$$

Generally, a compactification of string theory means that the ten-dimensional spacetime decomposes into a product of a four-dimensional non-compact spacetime and a six-dimensional compact spacetime[1]:

$$M_{10} = M_4 \times M_6 \tag{5.5}$$

The ten-dimensional r-form and Laplacian operator decomposes analogously:

$$A_r = B_p \wedge C_{r-p} \qquad \Delta_{10} = \Delta_4 + \Delta_6 \tag{5.6}$$

Here $B_p$ is a $p$-form on the four-dimensional non-compact spacetime and $C_{r-p}$ a $(r-p)$-form on the internal manifold. Suppose that $C_{r-p}$ is an eigenfunction of the six-dimensional Laplacian:

$$\Delta_6 C_{r-p} = \lambda C_{r-p} \tag{5.7}$$

Substituting this into 5.4, the equation of motion of $A_r$, we get:

$$(\Delta_4 + \lambda) B_p = 0 \tag{5.8}$$

In four dimensions the Laplacian is the familiar box-operator, known as the D'Alembertian. We see that the eigenvalue of $C_{r-p}$ sets the mass for the four-dimensional p-form field. For $\lambda = 0$, $C_{r-p}$ is an *harmonic* form on the internal manifold and the $p$-form field is massless. We conclude that the number of massless $p$-form fields in the four-dimensional theory equals the number of harmonic $(r-p)$-forms on the internal manifold. To state the conjecture we need the following chain of isomorphisms provided by DeRham's theorem and Hodge's theorem.

DeRham's theorem establishes the vector space isomorphism between the r-th *homology group* and the *$r-th$ De Rham cohomology* group, $H_r(M) \cong H^r(M)$. Hodge's theorem then provides the isomorphism between the r-th cohomology group and the space of harmonic r-forms, $H^r(M) \cong \mathcal{H}^r(M)$. This implies that the dimension of the space of harmonic $r$-forms equals the dimension of the $r-th$ cohomology group. The dimension of the $r-th$ cohomology group is known as the *$r-th$ Betti number*.

We see that the number of massless fields equals the $r-th$ Betti number. Betti numbers are a sum of *Hodge numbers*, which are the dimensions of Dolbeault cohomology groups. For all known Calabi-Yau manifolds, the Hodge numbers are finite. There is no known example of a compactification manifold that has infinite-dimensional cohomology classes. Hence, we are led to the first finiteness conjecture [1, 46]:

**Conjecture 1.** *The number of massless gauge fields in a consistent theory of quantum gravity is finite. It is bounded from above by an integer N(d) that depends on the dimensionality of spacetime and bounded from below by the number of coupling constants in the theory.*

The lower bound is related to another conjecture that will be mentioned in the next chapter. The upper bound is deeply connected to *compactness* of the

[1] The compactification manifold does not have to be compact, it can also be non-compact. When one wants to study gravity one has to compactify on a compact manifold but in studies of gauge theories one usually compactifies on non-compact manifolds.

compactification manifold. As an example of the conjecture, consider Type IIA compactified on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$, i.e we consider the decomposition [46]:

$$M_{10} = M_6 \times \frac{\mathbb{C}^2}{\mathbb{Z}_N} \tag{5.9}$$

This compactification orbifold realizes SU(N) gauge symmetry in the six-dimensional non-compact spacetime. Now, there is no problem in taking N extremely large, meaning that we have $N^2 - 1$ many massless gauge bosons which is the dimension of the Lie group SU(N). It seems like we have violated the finiteness conjecture, but note that $\mathbb{C}^2$ is non-compact meaning that this is a field theory not coupled to gravity. For gravity we need a compact manifold. This can be seen as follows. Denote the compactification manifold by $X$. The gravitational part of the low-energy effective action derived from string theory is:

$$S \sim Vol(X) \int_{M_4} d^4x \sqrt{-g} R \tag{5.10}$$

whereas the Einstein-Hilbert action is:

$$S_{EH} = \frac{M_{pl}^2}{2} \int_{M_4} d^4x \sqrt{-g} R \tag{5.11}$$

Hence, $Vol(X) \to \infty$ (X is non-compact) corresponds to $M_{pl} \to \infty$. To couple the field theory to gravity we have to embed the $\mathbb{C}^2/\mathbb{Z}_\mathbb{N}$ into a compact manifold. It is known that if $X = K3$, which is a Calabi-Yau manifold in four real dimensions so that we study quantum gravity in six dimensions, then $N \leq 20$.

This finiteness criterium illustrates some typical aspects of swampland conjectures. First, it cannot be rigorously proven, only supported by explicit "case-by-case" constructions. It just seems to be a pattern. Second, decoupling gravity invalidates the conjecture as expected for a conjecture on the string swampland.

## 5.2 Concepts from string theory

A lot of the material can be found in the standard string theory textbooks [47, 48, 49] as well as the excellent lecture notes [50]. Historically, string theory was formulated as a candidate for a theory of the strong interaction. However, it got replaced by a succesful quantum field theory, quantum chromodymamics. String theory was put aside for a while, until people noticed that the closed-string Hilbert space contains a traceless-symmetric rank-2 tensor state, identified as the graviton. This made string theory a quantum theory of gravity and interest in the theory was completely revived. It turned out that there exist five perturbative superstring theories living in ten dimensions which are all related via *dualities*. Examples of these dualities are *T-duality* and *S-duality*. Besides the five perturbative theories there is also another theory known as *M-theory* which lives in eleven dimensions and is not a string theory. M-theory is obtained via two of the perturbative string theories. At strong string-coupling, Type IIA and heterotic $E_8 \times E_8$ obtain an additional dimension and this eleven-dimensional limit is called M-theory. M-theory is regarded as the most fundamental theory.

In addition to the aforementioned theories we have Type I, heterotic SO(32) and Type IIB superstring theory. All theories can be organized in the duality web illustrated in figure 5.1. In this work all string theories and M-theory will pass by every

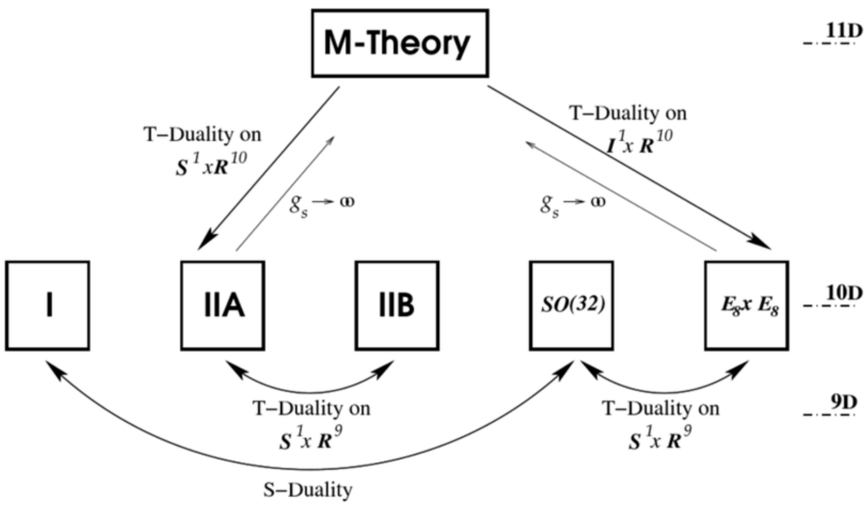



Figure 5.1: The famous duality web of string theory. All perturbative string theories are related via string dualities. This is a picture from [51].

now and then. Before getting into more technical aspects let us introduce the basic ingredients of string theory [47].

As the name suggests it is a theory of one-dimensional extended objects. Strings can be closed or open. These objects sweep out a surface in spacetime known as the world-sheet which is a Riemann surface. The world-sheet is parametrized by two coordinates $(\tau, \sigma)$. $\tau$ is a time-like coordinate for the string while $\sigma$ is a space-like coordinate along the string. The two-dimensional parameter space $(\tau, \sigma)$ is also called the world-sheet. The world-sheet is mapped into spacetime via embedding functions $X^\mu(\tau, \sigma)$ in the case of bosonic string theory and it is mapped into superspace (in the GS-formalism) in the case of superstring theory. Superspace extends spacetime through the addition of anti-commuting Grassmann coordinates $(\theta, \bar{\theta})$. We thus get additional embedding mappings $\Theta^{\dot{\alpha}}(\tau, \sigma), \bar{\Theta}^{\dot{\beta}}(\tau, \sigma)$. There are several equivalent actions for describing the bosonic string. The most convenient one from the perspective of path-integral quantization is the Polyakov action:

$$S_p = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} h^{ab} \partial_a X^M(\sigma) \partial_b X^N(\sigma) \eta_{MN} \tag{5.12}$$

The string propagates in 26-dimensional Minkowski spacetime. The integration is carried out over the world-sheet $\Sigma$. Here $\alpha'$ is known as the Regge-slope and it is inversely proportional to the string tension. It has dimensions of length squared. The Polyakov action contains a dynamical metric $h^{ab}$ on the world-sheet. It can be interpreted as scalar fields $X^M(\tau, \sigma)$ coupled to 2-dimensional gravity. It enjoys global Poincaré invariance and two gauge symmetries. The two gauge symmetries are diffeomorphism or reparametrization invariance and Weyl-symmetry. These gauge symmetries can be employed to simplify the action. A useful gauge is known as conformal gauge. The action becomes:

$$Sp = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \partial_a X_\mu \partial^a X^\mu \tag{5.13}$$

Upon quantization we get three massless bosonic states that appear in every string theory. Those are a symmetric 2-rank traceless tensor identified as the graviton $G_{MN}$, an anti-symmetric 2-rank tensor known as the Kalb-Ramond field $B_{MN}$ and a scalar called the dilaton $\Phi$. The exponential of the expectation value of the dilaton is related to the string coupling $g_s$. We observe something very important. The interaction strength is dynamical in string theory. This is in fact one of the swampland conjectures as we will see below.

To add fermions to the bosonic string action we add the Dirac action for free massless fermions:

$$S_F = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \bar{\psi}^M \rho^a \partial_a \psi_M \tag{5.14}$$

Here the $\rho^a$ form a two-dimensional Clifford algebra. The spinors in this action are Majorana spinors[2]. Recall that a Majorana spinor is obtained from a Dirac spinor via charge conjugation: $\psi^c = C\psi^\star$ with $C^\dagger C = 1, C^\dagger = -C$ a $4\times 4$ matrix which defines a charge-conjugated spinor. We then impose the Majorana reality condition on this Dirac spinor: $\psi^c = \psi$. This defines a Majorana spinor in four dimensions[3]. We can always choose a basis of the Clifford algebra such that the Majorana condition becomes $\psi = \psi^\star$ in two dimensions. It is more convenient to write down the fermionic action in light-cone coordinates on the world-sheet:

$$S_F = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+) \tag{5.15}$$

Light-cone coordinates are just linear combinations of $\tau$ and $\sigma$. Note that in this action it becomes apparent that we have separated the string motion into left -and right moving modes. This can be shown by calculating the equations of motion, which demonstrates that the $\psi$'s obey the Weyl equations[4]. In this expression the Lorentz index $M$ is suppressed. One can apply Hamilton's principle and demand the vanishing of the variation of the action. This gives us the equations of motion and some boundary terms. We will focus on the boundary terms as it gives rise to the notion of different sectors. It is conventional to set the endpoints of the string at $\sigma = 0, \pi$. Varying the action and writing down the boundary terms only one gets:

$$\delta S \sim \int d\tau(\psi_+\delta\psi_+ - \psi_-\delta\psi_-)\mid_{\sigma=\pi} -(\psi_+\delta\psi_+ - \psi_-\delta\psi_-)\mid_{\sigma=0} \tag{5.16}$$

So we note that the action only vanishes if we impose particular periodicity conditions on the fermions. Imposing periodicity conditions $\psi^M_\pm(\sigma+\pi) = +\psi^M_\pm(\sigma)$ is known as the *Ramond-sector* abbreviated as the R-sector. Imposing anti-periodicity conditions $\psi^M_\pm(\sigma+\pi) = -\psi^M_\pm(\sigma)$ gives the *Neveu-Schwartz sector* or NS-sector. For

[2]The index $M$ does not refer to Majorana but it is a Lorentz index.

[3]In four dimensions, the reality condition takes the form $\psi = \psi^\star$ only in the Majorana basis in which all gamma matrices are purely imaginary.

[4]Thus, the fermions are Majorana-Weyl fermions. Majorana -and Weyl conditions can be imposed simultaneously in two dimensions but not in four.

the open strings we have specified the dynamics entirely at this point. For closed strings however, we can impose these periodicity conditions on the left -and right moving modes separately. This gives us four sectors in total: the NS-NS sector, the NS-R sector, the R-NS sector and the R-R sector. It is worth remarking that we have these four sectors in the Type II theories but that we only have two sectors in the Type I theory since Type I is an open-string theory. For the heterotic theories the situation is a little different which has to do with the nature of the heterotic string. The heterotic string is, roughly speaking, a combination of the bosonic string and the superstring.

### 5.2.1 State space of Type II theories

Here we discuss the *massless* bosonic sectors of Type IIA and Type IIB as they feature in some arguments [47]. This is equivalent to the state space of Type IIA and Type IIB supergravity. We consider in this section the quantization of the closed-string in the GS-formalism. The most convenient quantization procedure in this formalism is known as quantization in the *light-cone gauge*. In this gauge the world-sheet actions for Type IIB and Type IIA are:

$$S_{IIB} = -\frac{1}{2\pi}\int d^2\sigma \partial_\alpha X_i \partial^\alpha X^i + \frac{i}{\pi}\int d^2\sigma \big(S_1^a \partial_+ S_1^a + S_2^a \partial_- S_2^a\big) \tag{5.17}$$

$$S_{IIA} = -\frac{1}{2\pi}\int d^2\sigma \big(\partial_\alpha X_i \partial^\alpha X^i + \bar{S}^a \rho^\alpha \partial_\alpha S^a\big) \tag{5.18}$$

These give rise to the same field equations as in the RNS-formalism for the bosonic fields $X^i$ and fermionic fields $S^{1a}, S^{2a}$. In light-cone gauge the ten-dimensional Lorentz symmetry $SO(9,1)$ is far from obvious but a $SO(8)$ rotational invariance is manifest. Recall that in the GS-formalism the world-sheet is embedded in superspace so in addition to spacetime coordinates we also have anticommuting grassmannian coordinates $\Theta(\tau, \sigma)$. In the light-cone gauge, the eight components of $\Theta$ that survive the gauge-fixing form an eight-dimensional representation of the double-cover of $SO(8)$ known as the spin-group *Spin(8)*. There are two inequivalent representations of Spin(8) that describe spinors with opposite eight-dimensional chirality. The inequivalent representations are denoted as $\mathbf{8_s}$ and $\mathbf{8_c}$.

Another important difference between the RNS-formalism and the GS-formalism is that there is only one sector. In the GS-formalism there is a preferred sign for the periodicity of the fermions to prevent breaking of the spacetime supersymmetry. Spacetime SUSY is unbroken when the fermionic zero-mode is kept. The required boundary conditions at the endpoints are:

$$S^{1a} \mid_{\sigma=0} = S^{2a} \mid_{\sigma=0} \tag{5.19}$$

$$S^{1a} \mid_{\sigma=\pi} = S^{2a} \mid_{\sigma=\pi} \tag{5.20}$$

where $a = 1, \ldots, 8$ are spinor indices and the superscripts $1, 2$ label the two fermionic world-sheet fields. The world-sheet mode expansions for the open-string fermionic fields that are consistent with the boundary conditions and equations of motion are:

$$S^{1a} = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} S_n^a e^{-in(\tau-\sigma)} \tag{5.21}$$

$$S^{2a} = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} S_n^a e^{-in(\tau+\sigma)} \tag{5.22}$$

Quantization imposes anti-commutation relations on the mode expansion coefficients $S_n^a$:

$$\left\{S_m^a, S_n^b\right\} = \delta_{m+n,0}\delta^{ab} \tag{5.23}$$

We want to construct closed-string states. Closed-string states can be constructed from tensor products of open-string states, by tensoring the left -and right-movers. The ground state for the open-string Hilbert space is demanded to be a representation of the *zero-mode* Clifford algebra:

$$\left\{S_0^a, S_0^b\right\} = \delta^{ab} \quad \text{where } a, b = 1, \ldots 8 \tag{5.24}$$

The ground-state for the open-string is a direct-sum representation of dimension 16 of the eight-dimensional vector -and spinor representations:

$$\mathbf{8_v} \oplus \mathbf{8_c} \tag{5.25}$$

Before constructing the Hilbert-spaces of Type IIA and Type IIB we need to mention what distinguishes these two theories. In a closed string theory, the fermionic fields $S^1$ and $S^2$ may belong to the same representation $\mathbf{8_s}$ or $\mathbf{8_c}$ of Spin(8). If they do, we have Type IIB string theory. If $S^1$ and $S^2$ belong to different representations we have Type IIA string theory. Hence, to construct the Type IIA spectrum we need to tensor the following open-string states:

$$(\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \tag{5.26}$$

while for Type IIB we consider the following tensor product:

$$(\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \tag{5.27}$$

Note that we have $16 \times 16 = 256$ states in the ground state of a closed string theory. The Bosonic sectors of Type IIA an Type IIB both contain states of the form $\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ corresponding to the NS-NS sector. The $\mathbf{1}$ is the dilaton field, $\mathbf{28}$ the Kalb-Ramond field and $\mathbf{35}$ the graviton. This we already mentioned at the beginning of section 5.2. The other part of the massless bosonic sectors, the R-R sectors, differ for Type IIA and Type IIB. Type IIA contains the states $\mathbf{8_s} \otimes \mathbf{8_c} = \mathbf{8_v} \oplus \mathbf{56_t}$ where $\mathbf{8_v}$ is a one-form and $\mathbf{56_t}$ is a three-form. The type IIA R-R sector contains *odd* gauge potentials.

The type IIB R-R sector contains the bosonic states $\mathbf{8_c} \otimes \mathbf{8_c} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+$. Here $\mathbf{1}$ is a zero-form gauge potential (i.e. a scalar), $\mathbf{28}$ a two-form potential and $\mathbf{35}_+$ denotes a four-form with *self-dual* field strength in ten dimensions. This fact introduces some complications in formulating an action for type IIB to be discussed below. Field strengths $F$ are constructed from gauge potentials by the action of the exterior derivative, so e.g. $F_5 = dC_4$ if the five-form is exact. To fix some notation, we will denote the R-R gauge fields as $C_0, C_2, C_4$ etc. Since the exterior derivative is a nilpotent operator all exact forms are closed. The opposite is only locally true which goes by the name of *Poincaré's lemma*. Whether a $p$-form field is exact over the entire manifold depends on its topological properties. Self-duality means $F_5 = \star_{10} F_5$ where $\star_{10}$ denotes the Hodge operator in ten dimensions. We conclude that the R-R sector of type IIB R-R contains *even* gauge potentials. The other combinations of tensor products form fermionic states which we will not further discuss.

### 5.2.2 dualities

The low-energy approximation to string theory gives a supergravity. One obtains a supergravity theory by making supersymmetry a local symmetry. Local supersymmetry means local Poincaré invariance. This is the gauge group of general relativity so we have obtained a theory with gravity. From the point of view of string theory we can view supergravity theories as a leading order approximation in $\alpha'$. Above we mentioned that the R-R sector of type IIB contains a four-form with self-dual field strength. This conditions introduces the difficulty of writing down an action that contains the correct amount of degrees of freedom. Usually, kinetic terms of gauge fields are proportional to the square of the field strength, so naively one would write down a term for the R-R four-form: $\int d^{10}x F_5 \wedge \star F_5$. However, such a term does not take into account the self-duality condition of the field strength. It therefore describes too many degrees of freedom (in fact, twice too many).

A simple resolution is to write down an action and impose the self-duality constraint by hand. We won't discuss how to construct this action but simply state the result. The Type IIB supergravity dynamics is described by:

$$S_{IIB} = S_{NS} + S_R + S_{CS} \tag{5.28}$$

$$\tilde{F}_5 = \star \tilde{F}_5 \tag{5.29}$$

corresponding to an action for NS-fields, R-fields and a Chern-Simons term:

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \Big( R + 4\partial_\mu \Phi \partial^\mu - \frac{1}{2} H_3 \wedge \star H_3 \Big) \tag{5.30}$$

$$S_R = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-G} \Big( F_1 \wedge \star F_1 + \tilde{F}_3 \wedge \star \tilde{F}_3 + \frac{1}{2} \tilde{F}_5 \wedge \star \tilde{F}_5 \Big) \tag{5.31}$$

$$S_{CH} = -\frac{1}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3 \tag{5.32}$$

$$F_{n+1} \equiv dC_n, \quad H_3 \equiv dB_2 \tag{5.33}$$

$$\tilde{F}_3 \equiv F_3 - C_0 H_3, \quad \tilde{F}_5 \equiv F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \tag{5.34}$$

Here $\frac{1}{\kappa^2} = \frac{2\pi}{(2\pi l_s)^2} = \frac{g_s^2}{16\pi G_{10}}$ is related to the ten-dimensional Newton's constant. The NS-action is described in *string-frame*. Via a Weyl-rescaling of the metric the action can be transformed to Einstein frame. Note that the R-action and CS-action do not contain the exponential factor of the dilaton. This is a consequence of the definition of the R-fields. By field redefinitions one can incorporate the dilaton factor.

Another important observation which is not obvious is that the Type IIB action has a global SL(2,ℝ) symmetry. This is important for the discussion of S-duality below. To make this symmetry manifest, the action needs to be rewritten. This is done by defining the following fields:

$$\tau \equiv C_0 - ie^{-\Phi} \tag{5.35}$$
$$G_3 \equiv F_3 - \tau H_3 \tag{5.36}$$

$\tau$ is referred to as the *axio-dilaton* field because the R-R zero-form has a continuous shift-symmetry $C_0 \to C_0 + const.$ in the supergravity theory (but not in the full string theory), just like an axion. Under global SL(2,ℝ) transformations $\tau$ transforms as a modular-parameter. This means that for a general SL(2,ℝ) transformation, $\tau$ transforms as:

$$A \in SL(2,\mathbb{R}) \to A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{where } a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1 \tag{5.37}$$
$$\tau' = A\tau = \frac{a\tau + b}{c\tau + d} \tag{5.38}$$

This is a Möbius transformation. This fact of the axio-dilaton features prominently in constructing examples where several swampland conjectures hold. In terms of these new fields the action can be rewritten such that the global SL(2, ℝ) symmetry is manifest:

$$S_{IIB} = \frac{1}{\kappa^2} \int d^{10}x \sqrt{-G} \Big( R - \frac{\partial\tau\partial\bar{\tau}}{2(\mathrm{Im}(\tau))^2} - \frac{G_3 \wedge \star G_3}{2\mathrm{Im}(\tau)} - \frac{\tilde{F}_5 \wedge \star \tilde{F}_5}{4} \Big)$$
$$- \frac{i}{8\kappa^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\mathrm{Im}(\tau)}$$

which is the action in Einstein frame.

**S-duality**

We are now in a position to discuss the basics of S-duality. S-duality is a strong-weak duality, relating strongly-interacting theories to their weakly-interacting dual theories. A well-known example is the electric-magnetic duality in Maxwell theory when the existence of magnetic monopoles is conjectured. The electric -and the magnetic gauge couplings are inversely proportional due to Dirac's quantization condition: $g_{el} \sim \frac{1}{g_{mag}}$. Thus, problems in a strongly electrically-interacting field theory can be treated in a weakly-interacting magnetic dual theory. In string theory, strong-weak duality is very similar. Type IIB string theory turns out to be self-dual under strong-weak transformations, i.e. $g_s \to \frac{1}{g_s}$ is a symmetry of Type IIB. This is closely related to the axio-dilaton being a modular parameter as we will now show.

Above we have discussed the SL(2,ℝ) in Type IIB supergravity. However, the full Type IIB superstring theory does not preserve this symmetry entirely: it is broken down via stringy effects to an infinite discrete subgroup SL(2,ℤ). This can be shown by noting that strings can couple to the NS-NS two-form and thus carry Kalb-Ramond charge. This is the so called fundamental-string or F-string. Strings can also be viewed as D1-branes which couples to an R-R two-form and hence such strings carry R-R charge. This is known as a Dirichlet-string or D-string. Because the NS-NS and R-R two-forms transform as a doublet under SL(2, $\mathbb{R}$) the F -and D-string also transform as a doublet under this group. According to Dirac's quantization condition the charges carried by the strings must be integer. Hence, we must restrict to a discrete symmetry group.

Under SL(2,$\mathbb{Z}$) $\tau$ still transforms as a modular parameter. We can choose a particular background geometry in which the R-R zero-form vanishes identically, $C_0 \equiv 0$ so that $\tau = -ie^{-\Phi}$. We can then consider a Möbius transformation for which $a = 0, b = -1, c = 1, d = 0$ so that:

$$\tau \to \tau' = -\frac{1}{\tau} = ie^{\Phi} \tag{5.39}$$

$$g_s \to g'_s = \frac{1}{g_s} \tag{5.40}$$

This is an example of an S-duality transformation showing that Type IIB is S-dual to itself. Note that this is only one element from the infinite duality group SL(2,$\mathbb{Z}$).

**T-duality**

The criticial dimension of closed bosonic string theory is 26. Imagine compactifying one of the 25 spatial coordinates on a circle of radius R, e.g. the 25th. One has to impose a periodicity condition on this coordinate of the following form:

$$X^{25}(\tau, \sigma + 2\pi) = x^{25}(\tau, \sigma) + 2\pi W \quad W \in \mathbb{Z} \tag{5.41}$$

W is called the winding number and it counts how many times the closed string winds the circle. The mode expansion of this compact coordinate is $X^{25}(\tau, \sigma) = x^{25} + 2\alpha' p^{25}\tau + 2RW\sigma + \dots$ where the dots indicate oscillator terms. Along the circle the momentum $p^{25}$ is quantized. This means that the wave function $\psi \sim e^{ip^{25}x^{25}}$ is single-valued along the circle. Then, momentum is quantized as:

$$p^{25} = \frac{K}{R} \quad K \in \mathbb{Z} \tag{5.42}$$

where $K$ is called the Kaluza-Klein (KK) excitation number or KK charge. We can decompose the 25th coordinate into left -and right moving modes as:

$$X_R^{25}(\tau - \sigma) = \frac{1}{2}(x^{25} - \tilde{x}^{25}) + (\alpha'\frac{K}{R} - WR)(\tau - \sigma) + \dots \tag{5.43}$$

$$X_L^{25}(\tau + \sigma) = \frac{1}{2}(x^{25} + \tilde{x}^{25}) + (\alpha'\frac{K}{R} + WR)(\tau + \sigma) + \dots \tag{5.44}$$

$$X^{25}(, \tau, \sigma) = X_R^{25}(\tau - \sigma) + X_L^{25}(\tau + \sigma) \tag{5.45}$$

To proceed, we need some general results in closed bosonic string theory such as the string mass and the eigenvalue of the zeroth Virasoro operator. One can arrive at the relation:

$$\alpha' M^2 = \alpha'\Big[\big(\frac{K}{R}\big)^2 + \big(\frac{WR}{\alpha'}\big)^2\Big] + 2N_L + 2N_R - 4 \tag{5.46}$$

where $N_R, N_L$ are number operators counting the number of right respective left-moving excitations. To arrive at the above relation we have to impose the *level-matching condition* $N_R - N_L = WK$. It turns out that when one simultaneously exchanges the KK excitation number and the string winding number together with the transformation $R \to \frac{\alpha'}{R}$, the equation above remains unchanged. The transformation of the circle of radius R to another circle of radius $\frac{\alpha'}{R}$ is a T-duality transformation. These transformations can nicely be written as a single transformation of the complete left -and right-moving modes of string:

$$X_L \to X_L \qquad X_R \to -X_R \tag{5.47}$$

Thus T-duality reverses the sign of the *entire* right-moving mode. The closed bosonic string theory is self-dual under T-duality transformations: compactifying on a circle of radius R leads to the same physics as compactifying on a circle of radius $\tilde{R}$. We can also consider a compactification of closed bosonic string theory on a product of circles, i.e. on a torus. This is simply a generalization of a circle compactification but it provides some insights that we will need for the compactification of heterotic string theory on a torus[5]. Instead of decomposing 26-dimensional Minkowski spacetime as $\mathbb{R}^{25} \times S^1$ we consider the decomposition $\mathbb{R}^{26-n} \times T^n$. This means that the metric factorizes as:

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu + G_{MN} dY^M dY^N$$
$$\text{where } \mu, \nu = 0, \dots, 26-n \text{ and } M, N = 1, \dots, n$$

into a metric on the non-compact spacetime and on the n-torus. The $n$ coordinates on the torus satisfy the circular periodicity condition:

$$Y^M(\tau, \sigma + \pi) = Y^M(\tau, \sigma) + 2\pi W^M \quad W^M \in \mathbb{Z}$$

Again, we can consider mode-expansions of the compact coordinates and the momenta are quantized along the $n$ circles. However, now that we have multiple momenta we have several left -and right-moving momenta $P_L, P_R$ that are not equal. By employing the periodicity conditions of the compact coordinate one can derive that:

$$P_L^M - P_R^M = 2W^M \quad \text{and} \quad P_L^M + P_R^M = K^M \tag{5.48}$$

so their difference is an even integer. The important point here is that a general momentum vector $P^M = (P_L^M, P_R^M)$ with $2n$-components, along one of the circles of the torus, lives on a *lattice*. This lattice has very special properties as we will see. To arrive at a similar relation for the mass as the one above again involves general results that we will not discuss. One thing we mention is that in this construction the level-matching condition is naturally modified to $N_R - N_L = W^M K_M$. Basically, the formula is generalized to a matrix equation. This also shows that the group of T-duality transformations is extended to an infinite discrete group. This is the group $O(n, n; \mathbb{Z})$. This group consists of $2n \times 2n$ matrices $A$ whose entries are integers that satisfy an "orthogonality-like" condition:

$$A^T \begin{bmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix} A = \begin{bmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix} \tag{5.49}$$

[5]This will serve as support of the weak gravity conjecture (see section 6.3).

The relations 5.48 of the left -and right-moving momenta are very interesting. We multiply the two relations and find that:

$$P^2 \equiv P_L^2 - P_R^2 = 2W^M K_M = \frac{1}{2}(N_R - N_L) \in 2\mathbb{Z} \tag{5.50}$$

We have defined the length here with respect to the metric on the torus. Thus, the length squared of a momentum vector is an even integer. This shows that the the momenta in toroidal compactifications of the bosonic string lie on an *even* lattice.

For the Type IIA and Type IIB superstring theories the T-duality symmetry group is slightly modified. This is related to the fact that Type IIB is a chiral theory while Type IIA is non-chiral. We will not discuss how the T-duality group is precisely modified but it turns out to be the subgroup $SO(n, n; \mathbb{Z})$ of $O(n, n; \mathbb{Z})$. The existence of T-duality seems to be deeply connected with the finiteness of the volume of moduli spaces, just like S-duality.

Another important result is that Type IIA and Type IIB are related to each other via T-duality. Compactify the coordinate $X^9$ on a circle. If we then perform a T-duality transformation we have $X_L^9 \to X_L^9$ ad $X_R^9 \to -X_R^9$. The world-sheet fermion $\psi^9$ must transform in the same way: $\psi_L^9 \to \psi_L^9$ and $\psi_R^9 \to -\psi_R^9$. It can be shown that this reverses the chirality of the right-moving R-sector ground state. The chirality of the left-moving and right-moving ground states is what distinguishes IIA and IIB but since a T-duality transformation only reverses the right-moving mode it follows that Type IIA on a circle of radius R is dual to Type IIB on a circle of radius $\tilde{R}$.

## 5.3 The moduli space of Ricci-flat metrics

Let us start with setting up the notation and summarizing some facts about Calabi-Yau manifolds [52]. Consider compactification of type II string theory on a Calabi-Yau threefold. A Calabi-Yau $n$-fold is a complex manifold of complex dimension $n$. A complex manifold is a manifold on which a $(1, 1)$-tensor field $\mathcal{J}$ is defined, known as the complex-structure. A complex manifold is coordinatized with holomorphic -and anti-holomorphic coordinates, giving rise to holomorphic indices $i$ and anti-holomorphic indices $\bar{j}$. A Calabi-Yau manifold is a specific type of Kähler manifold since it has vanishing first Chern class $c_1(CY) = 0$. The first Chern class is defined as the equivalence class:

$$c_1(CY) \equiv \left[\mathcal{R}/2\pi\right] \in H^2(CY; \mathbb{R}) \tag{5.51}$$

A Kähler manifold is a hermitian manifold, i.e. its metric is hermitian with respect to the complex-structure:

$$g(\mathcal{J}X, \mathcal{J}Y) = g(X, Y) \quad \text{with } X, Y \in TM \tag{5.52}$$

Furthermore, it is equipped with a closed Kähler form $dJ = 0$, which is a two-form. The Kähler form is related to the volume of the compactification manifold via:

$$V(CY) = \frac{1}{6}\int_{CY_3} J \wedge J \wedge J \tag{5.53}$$

for a CY three-fold. We have already mentioned that locally, the Kähler metric can be obtained from a Kähler potential $K$ by taking two derivatives. The Kähler potential is uniquely defined up to Kähler transformations:

$$K \to K + f(z) + f(\bar{z}) \tag{5.54}$$

where $f(z), f(\bar{z})$ are holomorphic respectively anti-holomorphic functions. Equivalently, one may define a Calabi-Yau manifold as a Kähler manifold that is Ricci-flat. This is a highly non-trivial statement and goes by the name of Yau's theorem. This means that the Kähler metric gives rise to a vanishing Ricci two-form. On a Kähler manifold the components of the Ricci tensor and Ricci two-form coincide so the metric gives rise to a vanishing Ricci tensor. Compactification on a Calabi-Yau manifold preserves a quarter of the original supersymmetry of the higher-dimensional theory. This is due to the fact that the Kähler metric on a Calabi-Yau manifold has SU(3) holonomy. Thus, when compactifying Type II, we go from 32 to 8 supercharges. This is phenomenologically interesting, as for example compactification of heterotic string theory on a Calabi-Yau manifold yields a theory $N = 1$ SUSY. Finally, a Calabi-Yau $n$-fold can be defined as a Kähler manifold on which there exists a nowhere-vanishing holomorphic $(n, 0)$-form often denoted $\Omega$.

Now, one can consider specific type of perturbations that preserve the Calabi-Yau condition, i.e. perturbations to the metric such that the perturbed metric also gives rise to a Ricci-flat Kähler manifold:

$$R(g) = R(g + \delta g) = 0 \quad \text{where } g + \delta g \text{ is hermitian} \tag{5.55}$$

Note that the perturbed metric is hermitian with respect to a *new* complex-structure. When neglecting terms beyond linear order in the metric perturbation, and upon gauge fixing to avoid including diffeomorphism equivalent perturbations, one finds the Lichnerowicz equation:

$$\nabla^m \nabla_m \delta g_{pq} + 2R_{pq}^{rs} \delta g_{rs} = 0 \tag{5.56}$$

Here, the summation is over holomorphic -and anti-holomorphic indices. The Lichnerowicz equation decouples into two deformation equations, one for the complex-structure $\mathcal{J}$ and one for the Kähler-form $J$. Deformations of the form $\delta g_{i\bar{j}}$ yields the *Kähler class deformations*:

$$\delta J = i\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \tag{5.57}$$

$\delta J$ is a harmonic $(1, 1)$-form if the Lichnerowicz equation is satisfied. Hence, the number of Kähler class deformations is equal to $h^{1,1}$. Such deformations are referred to as *Kähler moduli*. They span a parameter space known as the *Kähler moduli space*. We already mentioned that the perturbed metric $g + \delta g$ is hermitian with respect to a new complex-structure tensor field $\mathcal{J}$. Thus, we are looking for deformations $\mathcal{J} \to \mathcal{J} + \delta\mathcal{J}$ such that $g + \delta g$ is hermitian with respect to the complex-structure $\mathcal{J} + \delta\mathcal{J}$. This leads to the definition of *complex-structure deformations*. They can be defined with the aid of the nowhere-vanishing holomorphic $(3, 0)$-form $\Omega$:

$$\chi = \Omega_{ijk} \delta\mathcal{J}^k_{\bar{l}} dz^i \wedge dz^k \wedge d\bar{z}^{\bar{l}} \tag{5.58}$$

If the Lichnerowicz equation is satisfied it follows that $\chi$ is a harmonic $(2,1)$-form. Hence, we have $h^{2,1}$ *complex-structure moduli* and they span the *complex-structure* moduli space. Collectively, these two classes of moduli are known as *geometric moduli*. The total moduli space of Ricci-flat metrics is then decomposed as:

$$\mathcal{M} = \mathcal{M}_K \times \mathcal{M}_C \tag{5.59}$$

## 5.4 The complex-structure moduli space

It is an interesting fact that more is known about the parameter space of CY manifolds than about CY manifolds themselves. We consider the case of a CY three-fold. The complex-structure moduli space $\mathcal{M}_C$ is a Kähler manifold coordinatized by the complex-structure moduli $z^i, \bar{z}^i$ where $i = 1, \dots, h^{2,1}$. We define the norm of the holomorphic $(3,0)$-form $\Omega$ as:

$$\|\Omega\|^2 = i \int_{CY_3} \Omega \wedge \bar{\Omega} \tag{5.60}$$

This defines a Kähler potential on a local patch and it takes the form:

$$K = -\log \|\Omega\|^2 \tag{5.61}$$

The metric derived from it is known as the *Weil-Peterson* metric $g_{WP}$. $\mathcal{M}_C$ is not a smooth manifold but it contains singularities. To find the explicit form of $\Omega$ we have to chose a basis of homology three-cycles. A useful choice is a *symplectic* basis, in which the different cycles only intersect once:

$$\begin{aligned} A_I \cap A_J &= \emptyset \\ B_I \cap B_J &= \emptyset \\ A_I \cap B_J &= \delta_{IJ} \end{aligned}$$

DeRhams theorem states the duality between homology cycles and cohomology cycles so we can define a dual cohomology basis as:

$$\begin{aligned} \int_{CY_3} \alpha_L \wedge \alpha_J &= 0 \\ \int_{CY_3} \beta_L \wedge \beta_J &= 0 \\ \int_{CY_3} \alpha_L \wedge \beta^J &= \delta_I^J \end{aligned}$$

The basis is called "symplectic" because these properties are preserved under the action of the symplectic modular group $Sp(2h^{2,1}+2, \mathbb{Z})$. For higher-dimensional CY it is more difficult to find a proper basis. When we below generalize to CY $D$-folds we will assume such a basis can be found. Having a basis of homology three-cycles we can define *periods* of the holomorphic $(3,0)$-form $\Omega$:

$$\Pi^I(z) = \int_{A^I} \Omega \tag{5.62}$$

Such a period might vanish for some $I$. We then speak of a *conifold* singularity. The cycle $A^I$ for which this happens is called a *vanishing* cycle. Singularities signal the breakdown of the classical low-energy description. Hence, $\mathcal{M}_C$ contains quantum effects in its geometry. At these points physical quantities diverge, in this case the WP metric. We will come back to this later but we introduce here an associated notion. Suppose the period $\Pi^1$ vanishes. If $h^{2,1} = 1$, the subspace $\Pi^1 = 0$ is just a point. This point can be encircled by a closed loop. When the loop is traversed once, the basis only returns to itself up to a symplectic transformation. This phenomenon is known as *monodromy* and it seems to be very deeply connected to the swampland distance conjecture as we will see. $\Omega$ can also be expressed in terms of the periods $\Pi$ as a matrix equation:

$$\Omega = \Pi^T(z) \cdot \vec{\gamma} \tag{5.63}$$

Here we have adopted a combined notation: $\gamma_I = (\alpha_J, \beta^K)$ and $\Gamma_I = A_J \cap B^K$ so that:

$$\int_{\Gamma^J} \gamma_I = \delta_I^J \tag{5.64}$$

Note that the period becomes $\Pi(z) = \int_\Gamma \Omega$. For convenience of notation, we introduce an *intersection* matrix $\eta_{IJ}$, which is anti-symmetric in (complex) odd dimensions and symmetric in even dimensions:

$$\eta_{IJ} \equiv \int_{CY^3} \gamma_I \wedge \gamma_J \tag{5.65}$$

We now generalize to CY $D$-folds. The Kähler potential is:

$$K = -\log\Big[ - i^D \int_{CY_D} \Omega \wedge \bar{\Omega} \Big] \tag{5.66}$$

Expressing $\Omega$ in terms of the periods $\Pi(z)$ and the intersection matrix we get:

$$K = -\log\Big[ - i^D \Pi^T \eta \bar{\Pi} \Big] \tag{5.67}$$

Above we mentioned the existence of a conifold singularity. Such a singularity is at *finite* proper distance as measured by the WP metric. The physical explanation for this is that when approaching the conifold singularity, a *single* state becomes light. Since a CY compactification is a description in which massive modes have been integrated-out, we stumble upon an inconsistency when this state becomes light. In this case, the state becoming light is a BPS state arising from $D3$-branes wrapping *special Lagrangian cycles*. Besides the conifold, $\mathcal{M}_C$ contains more singularities, in particular *infinite distance* singularities. A point at infinite distance $P$, with respect to $Q$, is defined as the point for which *all* smooth paths to this point are infinitely long, i.e.:

$$d_\gamma(P, Q) = \int_\gamma ds \sqrt{g_{IJ}^{WP} \dot{x}^I \dot{x}^J} \to \infty \quad \forall \gamma : \mathbb{R} \to \mathcal{M}_C \tag{5.68}$$

where $s$ is an arc-length parameter. Points at infinite distances are characterised by two properties. The first is the obvious statement that the curve $\gamma$ needs to be connected to a *singular divisor* in $\mathcal{M}_C$. In this case, a singular divisor is just a curve in $\mathcal{M}_C$ connecting points for which the distance diverges with respect to a point $Q$. Second, points on such a singular divisor are characterised by monodromy.

More precisely, at such a point $P$, there exists a monodromy matrix of *infinite order*. The remarks made here will be made mathematically precise below. As a final remark, note that the conifold singularity is at finite distance and related to "falsely" integrating-out a single state. This might suggest that infinite distance singularities are related to "falsely" integrating-out *many* (in fact, infinitely many) states. This seems indeed the case as we will see later.

## 5.5 The swampland distance conjecture

Before we start with the formulation of the conjecture we need three other conjectures about the moduli space of a theory of quantum gravity [39, 2, 46].

**Conjecture 2.** *The moduli space of a consistent quantum gravity theory is parametrized by the vacuum expectation values of massless scalar fields.*

This is of course a very familiar statement in the context of string theory, where the string coupling is governed by the dilaton field: $g_s \sim e^{-\langle\Phi\rangle}$. A point in the moduli space corresponds to fixing the (gauge) couplings of the effective theory and thus to a choice of effective Lagrangian. Note that there must exist at least some massless fields in the effective theory otherwise there is no notion of an interacting theory. This provides the lower bound on the number of massless gauge fields mentioned in section 5.1. This conjecture also allows to introduce a metric, and hence the notion of proper distance, on the moduli space via the kinetic term of the moduli fields in the Lagrangian:

$$S \sim \int d^4x \sqrt{-\tilde{g}}(R + g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \dots) \tag{5.69}$$

Here $\tilde{g}$ denotes the metric on the non-compact spacetime that has Lorentzian signature, the dots indicate other fields in the effective theory and the $\phi^i$ are the moduli fields. Note that $g(\phi)$ has to be positive definite, i.e. it has to be Riemannian, otherwise the kinetic term has the wrong sign leading to ghost states. We assume here $V(\phi^i) = 0$, i.e. the moduli are not stabilized.

**Conjecture 3.** *The moduli space $M$ is non-compact. That is, fix $P \in \mathcal{M}$. Then, $\forall T > 0 \quad \exists \quad Q \in \mathcal{M}$, such that $d(P, Q) > T$. $d(P, Q) > T$ is the geodesic distance.*

This is supported by many examples but let us first mention a situation where it is violated. Consider a quantum field theory with a global U(1) symmetry, not coupled to gravity. As a manifold, U(1) is a circle and hence compact. This compact global symmetry can be spontaneously broken to a subgroup $H \subseteq G$. The quotient group $G/H$, which is the remaining set of transformations that act non-trivially on the vacuum state, is the moduli space. $G/H$ has finite diameter and hence the conjecture is violated. With a metric, the volume of the moduli space can also be defined:

$$V = \int_{\mathcal{M}} d\phi \sqrt{g(\phi)} \tag{5.70}$$

In all known examples in string theory in turns out to be finite [1].

**Conjecture 4.** *The volume of the moduli space of an effective field theory coupled to gravity is finite, despite being non-compact.*

This conjecture is closely related to the existence of dualities in string theory. The compactification moduli space of tori turns out to parametrize even self-dual lattices. The dimensionality of this space can be found by counting the number of degrees of freedom. On a $n$-torus we have the metric $G_{IJ}$ which is a positive-definite symmetric matrix but there can also be background fields. If we take into account a non-vanishing Kalb-Ramond field $B_{MN}$, which is an anti-symmetric matrix, then the total number of degrees of freedom is $n^2$. This is then the dimension of the moduli space $\mathcal{M}$ of even self-dual lattices. It turns out that such a space can be represented as the coset space:

$$\mathcal{M}_0 = O(n,n;\mathbb{R})\big/\big[O(n;\mathbb{R}) \times O(n;\mathbb{R})\big] \tag{5.71}$$

But points related via T-duality have to be divided out, i.e. we need to consider the quotient space:

$$\mathcal{M}_p = \mathcal{M}_0/O(n,n;\mathbb{Z}) \tag{5.72}$$

$\mathcal{M}_p$ denotes the *physical* moduli space, also known as the *Narain* moduli space. In the case of Type II theories compactified on an $n$-torus we mentioned that the T-duality group becomes the subgroup $\mathrm{SO}(n,n;\mathbb{Z})$ of $\mathrm{O}(n,n;\mathbb{Z})$. We then have the Narain moduli-space [1]:

$$\mathcal{M}_p = \big[SO(n,n;\mathbb{R})/SO(n;\mathbb{R}) \times SO(n;\mathbb{R})\big]/SO(n,n;\mathbb{Z}) \tag{5.73}$$

This extra quotient renders the volume finite. So, T-duality seems to correlate with the finiteness of the volume of the above moduli space. Another example involves S-duality and the axio-dilaton. If we regard S-duality as a postulate, then the axio-dilaton must be a modular parameter. To find the volume of its moduli space we need to integrate the metric over a *fundamental domain* $\mathcal{F}$ of $\tau$. This renders the volume finite [1, 47]:

$$V(\mathcal{M}_\tau) = \int_{\mathcal{F}} \frac{\partial\tau\partial\bar{\tau}}{2(\mathrm{Im}(\tau))^2} = \frac{\pi}{3} \tag{5.74}$$

This metric of the axio-dilaton can be read off from its kinetic term in the Type IIB action. However, even though the above examples are non-trivial, there seem to exist very simple counterexamples to the latter conjecture. However, as noted in [39], a volume divergence correlates with a cut off on the effective theory. Suppose we compactify a theory on a circle of radius $r$. The radius of the circle is a modulus and the line element on its moduli space is:

$$ds^2 = \left(\frac{dr}{r}\right)^2 \tag{5.75}$$

Without a cut off, the volume is infinite since:

$$V = \int_0^\infty \left(\frac{dr}{r}\right) \to \infty \tag{5.76}$$

Consider a cut off scale $\Lambda^{-1}$ of the effective theory. In this case the (regularized) volume is:

$$V = \int^{\frac{1}{\Lambda}} \left(\frac{dr}{r}\right) \sim -\log\Lambda \tag{5.77}$$

The volume is finite but in the limit $\Lambda \to 0$ the effective theory breaks down and the volume diverges again. Thus, the effective theory at a point $P \in \mathcal{M}$ is only properly defined in a finite region around the point. Now that we have established that we can move into non-compact directions, we can formulate the swampland distance conjecture [39]:

**The Swampland Distance Conjecture (SDC)**
*Consider two points $P, Q$ in the moduli space $\mathcal{M}$ of a consistent theory of quantum gravity. The points $P, Q$ can be connected by smooth curves $\gamma_i$. Suppose we fix $P$ and let $Q$ vary. Then, in the limit $d(P,Q) \to \infty$, i.e. the proper geodesic distance diverges, an infinite tower of states becomes light exponentially fast in the proper distance:*

$$\frac{m(P)}{m(Q)} \to e^{-\lambda \frac{d(P,Q)}{M_{pl}}} \quad \text{as} \quad d(P,Q) \to \infty \quad \text{where } \lambda \in \mathbb{R}^+$$

### General remarks

$m(P), m(Q)$ indicate the mass-scales of an infinite tower of particles or extended objects. If we start at $P$ (holding $P$ fixed) and move to $Q$ we get some infinite tower that becomes light. If we move from $Q$ (holding $Q$ fixed) to $P$ this tower becomes heavy but a *different* tower becomes light. This is an interesting symmetric property of the conjecture. It is important to note that the distance conjecture is a limiting statement about the moduli space. It does not specify when the exponential behaviour of the mass scales kicks in. The real number $\lambda$ is arbitrary. However, all known constructions have $\lambda \sim \mathcal{O}(1)$. This is elevated to a conjecture in the refined formulation of the SDC [40]. This refined formulation is based on a conjectured relation with another swampland conjecture, the weak gravity conjecture, which is the topic of the next chapter.

Naively one might think that the moduli space of a compact scalar, for example an axion, violates the conjecture since the moduli space is a circle. This would imply that quantum gravity forbids compact scalars which is of course not true. The conjecture should be understood as the statement that the moduli space of a periodic scalar is embedded in a larger moduli space that is non-compact. Furthermore, nothing is specified about the physical properties of the infinite tower, except its characteristic mass scale. It is not known what states make up the tower, what the structure of its mass spectrum is, and so on.

### Relations with other quantum gravity aspects

The SDC is intimately connected to a number of things. The infinite tower that arises according to the conjecture can be understood as a protection against exact global symmetries in a theory of quantum gravity. The conjecture is also closely related to dualities in string theory. Indeed, sometimes the SDC is illustrated with a similar duality-web as one can draw for the perturbative string theories (see figure 5.3). Finally, among other interesting things, the infinite distance emerges from integrating-out to one-loop order in perturbation theory an infinite tower of states. This induces a quantum correction to the metric on moduli space. So the infinite distance seems to emerge *purely* due to quantum effects[6]. The emergence of infinite

[6] One has to be careful. It is possible that the classical tree-level contribution dominates over the loop contributions and hence it is not purely a quantum phenomenon. See [3] for more details.

distances can arise in two very similar, but conceptually different manners. We will discuss both in this thesis.

### A remark on cut offs

The SDC signals the complete breakdown of quantum effective field theory due to the appearance of *infinitely* many light states. We could formulate this as an exponential drop of the cut off of the effective theory [53]:

$$\Lambda_{QG} = M_{pl} \exp\left\{-\lambda \frac{\Delta\phi}{M_{pl}}\right\} \tag{5.78}$$

This is really a quantum gravity cut off, not some cut off above which we should adopt a different effective description via integrating-in new states. Above, the picture emerged that the moduli space cannot be covered by a single effective field theory. There is also a cut off associated to this domain of validity but this is of course not a quantum gravity cut off. Consider an effective field theory defined at a particular region in moduli space and suppose we calculate an observable with this theory. As we move towards the boundary of validity of this theory, corrections induced by heavy states that were integrated-out from our initial theory, become more and more important and when we cross the boundary they dominate over the tree level correction. There, a different effective description should take over, that incorporates additional states that were previously light. Conversely, suppose we approach a point $Q$ in moduli space that is infinitely far away from a point $P$. According to our reasoning, the calculation of an observable with respect to the effective theory defined at $Q$ should become more accurate as we approach $Q$ while the calculation with respect to the theory at $P$ breaks down. However, according to the SDC, an infinite tower should appear since we have moved a very large distance to reach $Q$. Thus, we cannot approach the point $Q$ arbitrarily close. The situation is depicted in the figure 5.2.

## 5.5.1 A simple example

One can do a simple check for the SDC to gain some intuition [46]. Consider the familiar KK compactification of pure five-dimensional Einstein gravity on a circle. The five-dimensional line element is typically parametrized as:

$$G_{MN} = R^{-\frac{1}{3}} \begin{pmatrix} g_{\mu\nu} + RA_\mu A_\nu & RA_\mu \\ RA_\nu & R \end{pmatrix} \tag{5.79}$$

where $g_{\mu\nu}$ is the four-dimensional metric, $A_\mu \equiv G_{\mu 5}$ a four-dimensional vector-field and $R \equiv G_{55}$ a scalar-field. This indeed matches the 15 degrees of freedom of the metric in five dimensions. The notation here is very suggestive: $A_\mu$ turns out to be a Maxwell gauge-field and $R$ the scalar field parametrizing the size of the compactification circle. The four-dimensional Lagrangian contains the term:

$$S \sim \int d^4x \sqrt{-g} \Big[ \Big(\frac{1}{R}\Big)^2 dR^2 + \dots \Big] \tag{5.80}$$

where the dots indicate the Maxwell -and gravity part of the action. We read off the line-element on the moduli space:

$$ds^2 = \frac{dR^2}{R^2} \tag{5.81}$$

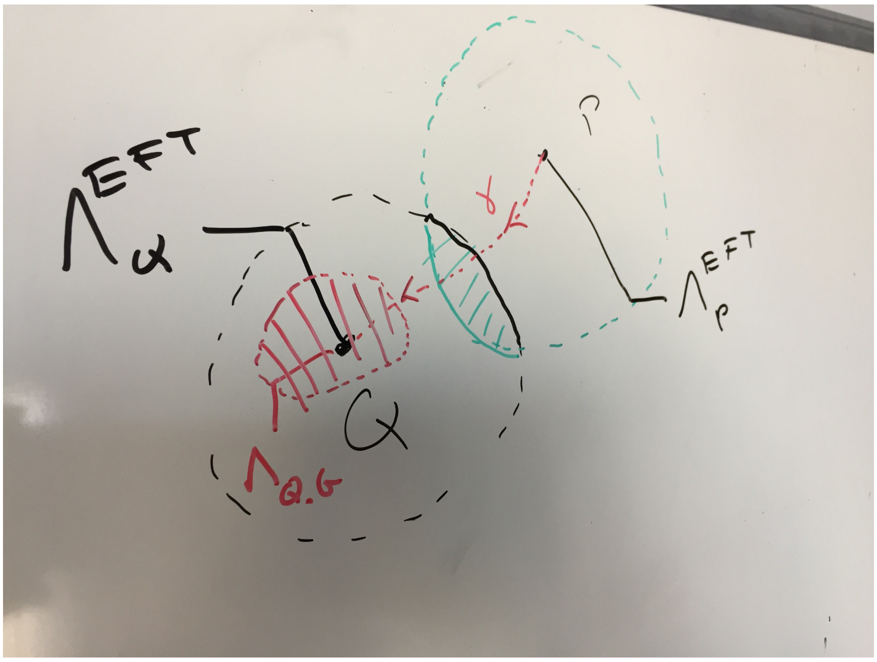



Figure 5.2: Two points $P, Q$ in the moduli space in whose vincinities (defined by $\Lambda_Q, \Lambda_P$) an EFT is well-defined. The distance conjecture limits how close we can approach a point by introducing a new cut off $\Lambda_{QG}$.

The length between a fixed point $p_0 \in \mathcal{M}$ and a varying point $p \in \mathcal{M}$ is:

$$d(p_0, p) \sim \int_{R_0}^{R} \frac{dR'}{R'} \sim \log(R) - \log(R_0) \tag{5.82}$$

We can take $R \to \infty$, corresponding to a point $p \in \mathcal{M}$ at infinity, so that the distance becomes: $d(p_0, p) \sim \log(R)$ or $e^{d(p_0,p)} \sim R$. Something that we not yet discussed in this thesis is that in KK compactifications we have an infinite tower of massive Kaluza-Klein states. Typically the massive modes are neglected by demanding that the characteristic size of the compact dimensions is sufficiently small so that the massive KK-modes can be integrated-out. This is because the KK-masses are inversely proportional to the size of the compact dimensions. To see this, one has to note that the five-dimensional equations of motion impose the condition that the fields inside the five-dimensional metric are eigenfunctions of the five-dimensional Laplace operator. The Laplace operator decomposes into a D'Alembertian and a one-dimensional Laplacian which is just a second derivative. Thus, the scalar-field $R$ obeys the equation of motion:

$$(\Box - \partial_y \partial^y)R = 0 \tag{5.83}$$

where $y \sim y + 2\pi R$ is the periodic coordinate along the circle of radius $R'$. We Fourier expand $R$ as:

$$R = \sum_{n=0}^{\infty} R_n(x^\mu) e^{\frac{i2\pi y}{R'} n} \tag{5.84}$$

Substituting this in the equation of motion we get:

$$(\Box - \frac{n^2}{R^{2\prime}})R_n = 0 \tag{5.85}$$

We can extract the mass term and see that it scales as:

$$M_{KK} \sim \frac{1}{R} \tag{5.86}$$

In the limit $R \to \infty$ the KK-masses scale as

$$M_{KK} \sim e^{-d(p_0,p)} \tag{5.87}$$

showing the exponential behaviour in the infinite-distance limit.

### Towers and string-dualities

In this case, an infinite tower of KK-modes becomes light exponentially fast in the proper distance. We know that T-duality in string theory exchanges KK-modes and winding-modes. If an infinite tower of KK-modes becomes light in a particular direction in moduli space, we expect that in another "T-dual" direction an infinite tower of winding-modes becomes exponentially fast light in the proper field distance. Indeed, this can be shown [2]. Infinite distances in moduli space are also related to weak-coupling limits. Thus, the tower becoming light is related to an S-dual tower becoming light in a strongly-coupled dual description. There are more dualities in string theory besides T-duality and S-duality and this suggests that there are many infinite towers that are dual to each other. This hints towards a very deep connection between the SDC and dualities in string theory. This situation is depicted in the figure 5.3. Starting from any point $P$ in the bulk of the moduli space we can move to points infinitely far away to a point where an infinite tower of states appears.

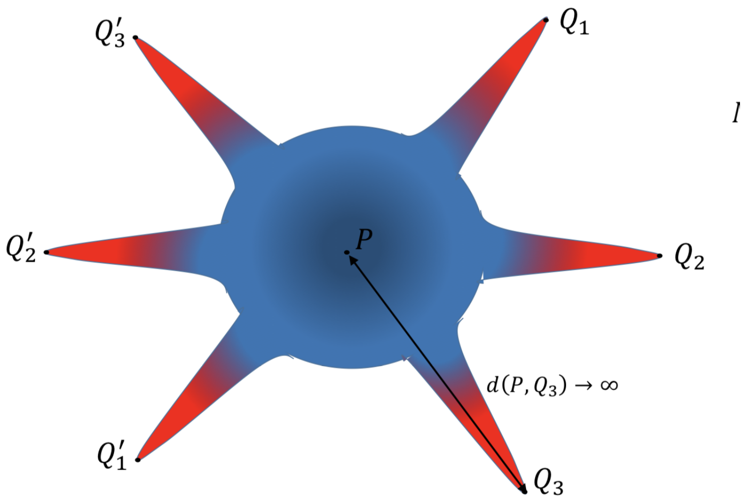
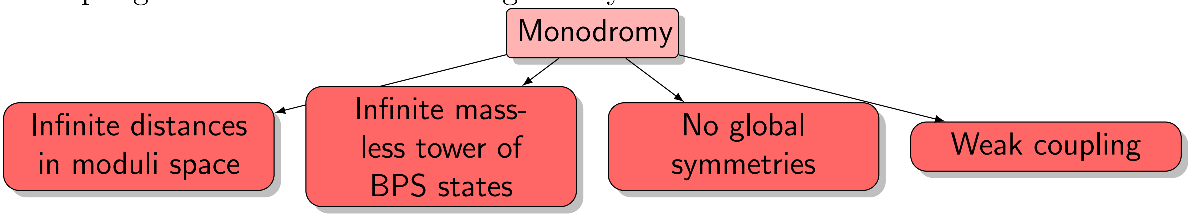



Figure 5.3: The moduli space of an effective field theory coupled to gravity. Primed points and unprimed points are separated an infinite distance from the point P. The infinite towers emerging at $Q_i'$ and $Q_i$ are dual. This picture is taken from [46] but it also appears in [2].

## 5.6 Validity in the complex-structure moduli space

In [3], a very interesting picture emerged, establishing a deep connection between the notion of *monodromy* to infinite distances, infinite massless towers of states, weak-coupling and the non-existence of global symmetries.




These connections were found in an investigation of the validity of the SDC in the complex-structure (CS) moduli space of Type IIB string theory on a CY $D$-fold. We review here the arguments given in [3] to justify the above picture. We also follow the notation of that reference. The idea is to find the structure of the Kähler potential around infinite distance singularities. This amounts to finding local expressions of the periods $\Pi$ which is established by a very powerful mathematical theorem due to Schmidt, known as the *Nilpotent orbit theorem* [54].

#### The geometry near points at infinite distance

To start, we already mentioned that periods can experience monodromy near conifold singularities. Periods also experience monodromy along paths encircling infinite distance loci/singular divisors. We introduce complex coordinates (moduli) $z^I$ such that the singular divisor satisfies $z^j = 0$ for some $j \in \{1, \ldots, h^{D,1}\}$. Now, suppose we encircle the singular divisor, i.e. consider a transformation $z^j \to e^{2\pi i} z^j$. The periods will experience monodromy:

$$\Pi(\ldots, e^{2\pi i} z^j, \ldots) = T_j \Pi(\ldots, z^j, \ldots) \tag{5.88}$$

where $T_j$ is a *monodromy matrix*. $j$ labels the divisor along which we consider the monodromy transformation. The collection of all such matrices in the entire $\mathcal{M}_C$ form the *monodromy group*. An important property of a monodromy matrix is that it is quasi-unipotent:

$$\left(T^{m_j+1} - 1\right)^{n_j+1} = 0 \quad \text{with } m_j, n_j \in \mathbb{Z}$$

where 1 denotes the identity matrix. This allows for a decomposition of any monodromy matrix:

$$T_j = T_j^{(s)} \cdot T_j^{(u)} \tag{5.89}$$

where $T^{(s)}$ is of *finite order* and $T_j^{(u)}$ is unipotent. $T^{(s)}$ turns out to be irrelevant upon a clever coordinate transformation and hence the monodromy matrix is entirely given by its unipotent part $T_j^{(u)}$. Unipotent means that there exists an integer $n_i$ such that $\left(T_j^{(u)} - 1\right)^{n_i} \neq 0$ but $\left(T_j^{(u)} - 1\right)^{n_i+1} = 0$. A simple crucial lemma is that when $T_j^{(u)} \neq 1$ it is of infinite order. If a monodromy matrix is of infinite order another lemma states its logarithm is some nilpotent matrix $N$:

$$N = \log T \tag{5.90}$$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Nilpotency means that there exists an integer $n_i \neq 0$ such that for $n_i + 1$ we have $N^{n_i+1} = 0$ is the zero-matrix. It is a mathematical fact, though highly non-trivial, that for CY three-folds $n_i \leq 3$. Now we can invoke the nilpotent orbit theorem. Near the point lying on the singular divisor, the period vectors $\Pi(z, \xi)$[7] take the form:

$$\Pi(z,\xi) = \exp\Big[\sum_i \frac{1}{2\pi i}(\log z^i) N_i\Big] \vec{A}(z,\xi) \tag{5.91}$$

This expression is exact and note that it is more general than the situation we are interested in. The sum indicates that there are multiple monodromy matrices meaning that the points at infinite distance lie on multiple singular divisors. This has also been studied. We will only be interested in the situation where all infinite distance points lie on a single divisor. We will therefore neglect the summation over $i$. "Nilpotent-orbit" refers to an approximation to the above expression. The vector $\vec{A}(z, \xi)$ turns out to be an analytic function of the complex-structure moduli $z^i$. Hence, its Taylor expansion exists:

$$\vec{A}(z,\xi) = \vec{a}_0 + \sum_{i=1}^{\infty} \vec{a}_i(\xi) z^i \tag{5.92}$$

We will only truncate this expression to zeroth order only. Then:

$$\Pi(,\xi) \approx \exp\Big[\sum_i \frac{1}{2\pi i}(\log z) N\Big] \vec{a}_0(\xi) \tag{5.93}$$

Note that the period vector diverges when we move to a point at infinite distance and that the exponential terminates at finite order because $N$ is nilpotent. Note also that this approximation is exponentially accurate. Since the Kähler potential is completely determined by the local expression of the period vectors, we now have an expression for the WP-metric near a point at infinite distance. We write the Kähler potential as:

$$e^{-K} = i \log\Big[\vec{\Pi}^T \eta \vec{\Pi}\Big] \tag{5.94}$$

where $\eta$ is the intersection matrix. The intersection matrix is preserved under the action of the monodromy matrix: $T^T \eta T = \eta$. Using this property, and a coordinate redefinition $t^i \equiv \frac{1}{2\pi i} z^i$ so that the point at infinite distance is now defined by $\mathrm{Im}\{t\} \to \infty$, we substitute the expression for $\vec{\Pi}$ to find:

$$e^{-K} \approx -i \vec{a}_0^T \eta \exp\Big[-2i\,\mathrm{Im}\{t\} N\Big] \vec{a}_0 + \mathcal{O}(e^{2\pi i t}) \tag{5.95}$$

By Taylor expanding the exponential around $\mathrm{Im}\{t\} = 0$ we can write this as:

$$e^{-K} \approx P(\mathrm{Im}\{t\}) + \mathcal{O}(e^{2\pi i t}) \tag{5.96}$$

Note that the degree of the polynomial is determined by the maximal integer power $d$ of the nilpotent matrix $N$ that does not annihilate the vector $\vec{a}_0$: $N^d \vec{a}_0 \neq 0, N^{d+1}\vec{a}_0 = 0$. Note that in any case, since the matrix $N$ is nilpotent, we have $d \leq n$ with $n$ the nilpotent index. We mentioned that for a CY three-fold $n = 3$

[7]The $\xi$ coordinates are essentially auxiliary. See section 2.2 of [].

and so we have $d \leq 3$. Taking the logarithm and deriving with respect to $t$ and $\bar{t}$ we obtain the WP-metric:

$$g_{t\bar{t}} = \partial_t \partial_{\bar{t}} K = \frac{d}{4(\operatorname{Im} t)^2} + \frac{c_1}{(\operatorname{Im} t)^3} + \cdots + \mathcal{O}(e^{2\pi i t}) \tag{5.97}$$

The first term appears for any complex-structure moduli space related to type IIB compactifications on CY three-folds. If we neglect the higher-order corrections in $\frac{1}{\operatorname{Im}\{t\}}$ we can compute the proper distance between points $P, Q$ connected by a geodesic $\gamma$:

$$d(P,Q) \approx \int_Q^P \sqrt{g_{t\bar{t}}} d\lambda = \frac{\sqrt{d}}{2} \log(\operatorname{Im} t)\,|_Q^P \tag{5.98}$$

Thus, the proper distance as measured by the WP-metric, defined in a local patch around a singular divisor, diverges as we move towards the point lying on the singular divisor $\operatorname{Im}\{t\} \to \infty$ that is characterized by the existence of a monodromy matrix of infinite order. This asymptotic structure of the WP-metric is also the reason why a tower of states eventually becomes light exponentially. It is highly non-trivial to determine the degree $d$ of the polynomial.

This asymptotic structure of the moduli space metric is also used to provide a link with the field space metric of $\alpha$-attractors [53], essentially relating the degree of the polynomial to the parameter $\alpha$. We will discuss this is the last section of this chapter.

### Monodromy and infinite distances

There is a deep mathematical conjecture relating monodromy to points lying on singular divisors, i.e. points at infinite distance. The advantage is that a geometrical quantity (infinite distance) can be related to an algebraic statement (monodromy) and the latter are easier to work with in these discussions. For a singular divisor, this statement is actually a theorem:

**Theorem 1.** *A point $P \in \mathcal{M}_{CS}$ is at infinite distance with respect to an arbitrary point in the bulk of $\mathcal{M}_{CS}$ if and only if $N\vec{a}_0 \neq 0$*

Note that if the polynomial 5.96 has vanishing degree $d = 0$ then $N\vec{a}_0 = 0$ and the point is at finite distance. Thus, we have that $d \in \mathbb{Z}$ is bounded by $0 < d \leq 3$ each value of $d$ labelling a "different" infinite distance singularity.

### A remark on global symmetries

Expression 5.96 is a very interesting result. At infinite distances in the complex-structure moduli space $\operatorname{Im}\{t\} \to \infty$ the Kähler potential is an exact polynomial in the imaginary part of $t$ up to exponentially suppressed corrections. In contrast, at infinity the real part of $t$ only enters in the exponentially suppressed part. This shows that at large values of $\operatorname{Im}\{t\}$ we have an approximate global shift-symmetry in $\operatorname{Re}\{t\}$, weakly-broken by exponentially suppressed corrections: $\operatorname{Re}\{t\} \to \operatorname{Re}\{t\} + a, a \in \mathbb{R}$. We can therefore identify $\operatorname{Re}\{t\}$ as an axion whose global shift-symmetry becomes exact at $\operatorname{Im}\{t\} = \infty$. We can now make the link between the statement that exact global continuous symmetries do not exist in quantum gravity. Precisely in the global-symmetry restoring limit $\text{Im}\{t\} \to \infty$ the distance conjecture tells us an infinite tower of states becomes exponentially light, signalling the total breakdown of local quantum field theory. We can therefore understand the distance conjecture as an obstruction to recovering exact global symmetries. However, note also that the distance conjecture not specifies when the exponential drop kicks in. If we knew this we could make some statement on to what extend we can have approximate global symmetries. This is an important question with respect to UV completions of large-field models of inflation since it would tell us the maximum field variation of the inflaton over which the potential is protected against Planck-suppressed operators. Obviously, these conclusions can only be drawn if the potential has at least degree one.

### The tower of states

Up to now we have only discussed the geometry near infinite distance points. Near such a point the distance conjecture tells us that some infinite tower of states becomes massless exponentially fast in the distance to that point with respect to some arbitrary point in the bulk moduli space. In the complex-structure moduli space we already mentioned a very well-known tower of states that becomes massless. These where the D3-branes wrapping special Lagrangian three-cycles. These charged states are BPS states and their mass is equal to the central charge. The formula for their mass is well-known and can be found in textbooks [47]:

$$M_q = |Z_q| = e^{\frac{K}{2}} \left| \int_{\Sigma_q} \Omega \right| \tag{5.99}$$

$\Sigma_q$ denotes a special Lagrangian three-cycle, $\Omega$ the nowhere-vanishing holomorphic three-form always present on CY-folds and $K$ the Kähler potential. Note that the state becomes massless when the cycle vanishes. The label $q$ has two interpretations. It labels which three-cycle the brane is wrapping and it denotes the charge under the corresponding gauge field. The second interpretation is valid because the charges are part of the gauge/vector-multiplet. These states are actually supersymmetric extremal black holes in four dimensions. This setting also allows for a connection with the weak gravity conjecture but we will defer this to the next chapter. The non-trivial thing to show is whether these charges are really realized by BPS states. We are not going to discuss this here but has been done in [3].

To make contact with the above we need to some massage a few earlier expressions. We Taylor expand the exponential in 5.95 to get:

$$e^{-K} = -i\vec{a}_0^T \eta \Big[ \sum_{n=0}^{d} \frac{1}{n!} (-2i\, \text{Im}\{t\} N)^n \Big] \vec{a}_0 + \mathcal{O}(e^{2\pi i t}) \tag{5.100}$$

Following the notation of [], we rewrite this as:

$$e^{-K} = i \sum_{n=0}^{d} \frac{1}{n!} (-2i\, \text{Im}\{t\})^n S_n(\vec{a_0}^T, \vec{a}_0) + \mathcal{O}(e^{2\pi i t}) \tag{5.101}$$

where $S_n(\vec{a_0}^T, \vec{a}_0)$ defines the inner product $\vec{a}_0^T \eta N^n \vec{a}_0$. In this notation, the BPS mass can be rewritten as:

$$M_q = e^{\frac{K}{2}} |S(\Pi, q)| \tag{5.102}$$

Now it is a matter of substitution to express the BPS mass in terms of the period vectors. One gets:

$$M_q = \frac{\sum_{l=0}^{d} \frac{1}{k!} t^k S_k(\Pi, q)}{\sqrt{i \sum_{n=0}^{d} \frac{1}{n!}(-2i \operatorname{Im}\{t\})^n S_n(\vec{a_0}^T, \vec{a}_0)}} + \mathcal{O}(e^{2\pi i t}) \tag{5.103}$$

We can simplify this expression by using the infinite distance limit $\operatorname{Im}\{t\} \to \infty$:

$$M_{\vec{q}} \approx \frac{\sum_k \frac{1}{k!} (\operatorname{Im} t)^k S_k(\vec{q}, \vec{a}_0)}{\sqrt{\frac{2}{d!}} (\operatorname{Im} t)^{\frac{d}{2}}} \tag{5.104}$$

Now we can derive the following for two points $P, Q$ infinitely far apart as measured by the WP metric[8]:

$$\frac{M_{\vec{q}}(P)}{M_{\vec{q}}(Q)} \approx \exp\left\{ -\frac{2}{\sqrt{d}} d(P, Q) \right\} \tag{5.105}$$

where we use expression 5.98 to relate $\operatorname{Im} t$ to the distance. In summary, we have seen that points at infinite distance in the complex-structure moduli space of type IIB string theory compactified on CY three-folds are related to monodromy of infinite order. At this infinite distance point an infinite tower of BPS states becomes light exponentially in the distance as measured by the WP-metric. This has been one of the most general tests so far of the distance conjecture. Also, at these infinite distance singularities we would restore global symmetries in the theory but this is obstructed by the infinite tower of states.

## 5.7 Troubles for large-field inflation?

At first sight one might think that the distance conjecture has some really dramatic implications for the fate of large-field inflation but this is not obvious at all. If we think of inflation as being an emergent phenomenon on the boundary of the complex structure moduli space, as in [53], it does not seem to be very constraining. It is not only the distance conjecture that questions models of inflation. The de Sitter conjecture brings the entire idea of single field slow roll inflation into problems [55]. We will first discuss this conjecture and then discuss the possible consequences for theories of inflation.

### The de Sitter conjecture

The de Sitter (dS) conjecture is a very radical statement[56, 46, 55, 57]. It forbids the existence of (meta)stable dS vacua in a theory of quantum gravity. It was motivated by the difficulty of constructing stable dS vacua in string theory. Indeed, the construction of stable dS vacua involves a lot of stringy technicalities. Furthermore, various no-go theorems are formulated of which [58] is the most well-known. The statement that dS vacua do not exist in a theory of quantum gravity suggests a particular condition on the effective $4d$ scalar potential $V(\phi)$ arising from a string compactification. Since dS vacua require a minimum at $V > 0$, it is tempting to

[8]The notation is slightly confusing: $d(P, Q)$ is the geodesic distance between $P, Q$ while $d$ denotes the degree of the polynomial expression of the Kähler potential.

conjecture something like $|\nabla V| > A$ with $A > 0$, forbidding local minima for any value of $\phi$. However, it does not forbid particular supersymmetric vacua. Supersymmetric vacua are characterized by $V(\phi_{min}) \leq 0$ and broken SUSY corresponds to $V(\phi_{min}) > 0$. Hence, we see that $|\nabla V| > A$ allows for SUSY vacua with flat directions. To exclude SUSY vacua with flat directions we let $A$ be a non-positive function on field space $A = A(\phi) \leq 0 \quad \forall\phi$. For example, we can set $A(\phi) = cV(\phi)$ with $c > 0$. We can now formulate the original dS conjecture.

**The de Sitter conjecture**
*The effective 4d scalar potential of a consistent theory of quantum gravity satisfies*:

$$|\nabla V| \geq cV \quad \text{where} \quad c \sim \mathcal{O}(1) \quad \text{in Planck units}$$

Before proceeding, let us clarify that in the multi-field case the absolute value of the gradient is computed with respect to the metric on the scalar field space, that is:

$$|\nabla V|^2 = G^{ab}\frac{\partial V}{\partial \phi^a}\frac{\partial V}{\partial \phi^b} \quad \text{with} \quad a, b = 1, \ldots, N \tag{5.106}$$

The precise value of the real number $c$ depends on the specific compactification data and cannot be determined a priori. However, in all known examples $c \sim \mathcal{O}(1)$. Note that when decoupling gravity the conjecture is satisfied trivially. Restoring the Planck mass and sending $M_{pl} \to \infty$ yields $|\nabla V| \geq 0$.

An interesting example to test the conjecture, and suggesting a connection with the distance conjecture, is for a SUSY vacuum with $V = 0$ and a flat direction. This yields the ill-defined expression $\frac{0}{0}$. To resolve this situation one deforms the theory by adding a mass term for $\phi$ to the potential. We get:

$$\frac{|\nabla V|}{V} = \frac{2}{|\phi|} \quad \text{for} \quad |\phi| \neq 0 \tag{5.107}$$

The conjecture is violated in the limit $|\phi| \to 0$. However, we can invoke the distance conjecture and conjecture that this is not allowed as an infinite tower of states becomes light exponentially fast in this limit. In particular, if we bound $|\phi| < 1$ in Planck units we recover the dS conjecture.

Note that the dS conjecture does not forbid quintessence models of dark energy. In quintessence models we have $V > 0$ while $\phi$ not being at a minimum. At late times, the potential of the quintessence field satisfies:

$$\nabla V \sim \mathcal{O}(V) \tag{5.108}$$

which is extremely small in our universe since the cosmological constant is extremely small, $\Lambda \sim 10^{-120}$ in Planck units. This is consistent with the dS conjecture since $c \sim \mathcal{O}(1)$.

**Some universal implications**

It is not clear how constraining the distance conjecture is with respect to inflaton field variations. It crucially depends on the magnitude of the number $\lambda$ appearing in the exponent and the number of efolds. On the other hand, the dS conjecture places very strong constraints on single-field slow-roll inflation.

The number of e-folds is related to the field displacement via:

$$\frac{dN}{d\phi} = \frac{1}{\sqrt{2\epsilon_V}} \tag{5.109}$$

$N$ denoting the number of e-folds. If we assume $\epsilon_V$ is independent of $N$ we find that $\Delta\phi \sim \sqrt{2\epsilon_V}N$. Now, $N = 50 - 60$ to solve the big-bang puzzles so we get $\Delta\phi < 5.6$ since the slow-roll parameter $\epsilon_V < 0.0044$, inferred from the bound on the tensor-to-scalar ratio $r < 0.007$ [9]. Whether this really problematic is not clear but this probably indicates some tension with the distance conjecture and depends on possible future observations of B-modes in the CMB. Note also that there is in principle no upperbound on the number of e-foldings so $\Delta\phi$ can be much larger.

The ds conjecture states that $\frac{|\nabla V|}{V} \to 0$ is forbidden but many inflationary potentials, for instance plateau potentials, violate this condition[9]. Also, according to observations we have $\frac{|\nabla V|}{V} < 0.09$ since $\epsilon_V < 0.0044$ while the conjecture requires this to be $\mathcal{O}(1)$.

We can combine the distance conjecture and the dS conjecture to derive the following. The slow-roll parameter $\epsilon_V$ and the dS conjecture are related as:

$$c < \sqrt{2\epsilon_V} \tag{5.110}$$

This implies that, independent of any observational bounds [59]:

$$60c < \Delta\phi \tag{5.111}$$

Both are claimed to be simultaneously $\mathcal{O}(1)$ but this is impossible according to the inequality above.

Recall that the Lyth bound relates superplanckian field variations to observable tensor perturbations. A large tensor-to-scalar ratio requires $\Delta\phi > M_{pl}$ and hence implies the appearance of a tower of light states. But recall also that a large amplitude of tensor perturbations implies a high characteristic energy scale of inflation $E_{inf} \sim r^{\frac{1}{4}} \to M_{pl}$. We need, however, $E_{inf} < M(P)$ where $P \in \mathcal{M}$ the point in field space where the tower appears, otherwise the effective description is totally nonsense. This shows that it is very hard to obtain large values of $r$ and satisfying $E_{inf} < M(P)$ simultaneously. In fact, for the refined distance conjecture, the tension is exponential since we have [40]:

$$E_{inf} < M \exp\left\{-\lambda \frac{\Delta\phi}{M_{pl}}\right\} \tag{5.112}$$

where it is conjectured that $\lambda \sim \mathcal{O}(1)$ always and the tower is fundamentally correlated with planckian distances. So there seems to be exponential tension between large-field inflation and the refined distance conjecture.

[9]Especially for many efolds.

### $\alpha$-attractors and the distance conjecture

With the machinery developed in sections 5.4 and 5.6 we can pretty easily derive a constraint on the number of efolds in $\alpha$-attractor models [53]. The emergent Kähler potential at the boundary of the complex structure moduli space is given by expression 5.96. From this expression we can derive the kinetic Lagrangian up to subleading powers in $\frac{1}{\mathrm{Im}\{t\}}$ and exponential corrections[10]:

$$\mathcal{L} = -\frac{d}{\phi^2}(\partial\phi)^2 \tag{5.113}$$

where $d$ is the degree of the polynomial in expression 5.96 and takes the values $d = 1, 2, 3$[11]. Recall that the integer $d$ characterizes the type of singularity. The mechanism behind the universal predictions of $\alpha$-attractors is due to a pole of second order in the kinetic term:

$$\mathcal{L} = -\frac{3\alpha}{4\phi^2}(\partial\phi)^2 \tag{5.114}$$

It is now very tempting to make the identification $d = 3\alpha$. We can understand the distance conjecture as an exponential drop of the initial cut off of the EFT as in expression 5.78. We can slightly rewrite expression 5.112 as[12]:

$$\frac{M_{pl}}{\lambda}\log\frac{M_{pl}}{H} \equiv \Delta\varphi_c > \Delta\varphi \tag{5.115}$$

where we defined a critical field displacement at which the EFT breaks down due to the tower of states. Recall that inflation in $\alpha$-attractors happens as the field moves away from the singularity at infinity. An important question is *how far* away from the boundary inflation starts as we certainly cannot start arbitrarily close to the boundary in canonical field space (i.e. the number of efolds cannot be infinite). It is therefore useful to rewrite the field displacement in terms of the number efolds before the end of inflation to check whether we can reach 60 efolds:

$$\Delta\varphi = \sqrt{\frac{3\alpha}{2}}\log N - \Delta\varphi_e \tag{5.116}$$

which we can easily rewrite using the above as:

$$\sqrt{\frac{3\alpha}{2}}\log N \leq \frac{M_{pl}}{\lambda}\log\frac{M_{pl}}{H} \longrightarrow N \leq \left(\frac{M_{pl}}{H}\right)^{\sqrt{\frac{2}{d}}\frac{M_{pl}}{\lambda}} \tag{5.117}$$

where we used $d = 3\alpha$[13]. Since the tensor-to-scalar ratio is upperbounded by $r < 0.07$ we can infer an upperbound on $H$ so that $\frac{M_{pl}}{H} > 3.7 \times 10^4$ and we see that the distance conjecture easily allows for 60 efolds of inflation. Since this bound is independent of $\alpha$ it holds for all models of $\alpha$-attractors.

[10] We change notation here from $\mathrm{Im}\, t \to \phi$. Note that the inflaton is *not* an axion in this case. The imaginary part of the field $T = \theta + i\phi$ is called a *saxion*. As a side remark, such a radial partner to the axion is conjectured to always appear in supersymmetric theories as a consequence of the moduli space being simply connected [39].

[11] $d = 0$ characterized a singularity at finite distance so we are not interested in that option.

[12] $\varphi$ denotes the canonical inflaton. Furthermore, we have replaced $E_{inf}$ by $H$.

[13] One can also re-express $\lambda$ in terms of $d$ and some rational number $p$ characterizing the tower of states [3, 53]. In fact, the dependence of $d$ cancels out.

# Chapter 6

# The weak gravity conjecture

It was conjectured already a few decades ago that quantum gravity seems to forbid exact continuous global symmetries [37, 60]. The arguments against this class of symmetries come into two flavours that are correlated and both are based on the semi-classical physics of black holes. The non-existence of global symmetries is of great importance for the swampland program and many more recent conjectures can be understood as trying to make this statement more quantitative, including the swampland distance conjecture of the previous chapter. The weak gravity conjecture (WGC) is also an example of an attempt of such a quantitative obstruction [44, 61]. In its most elementary form it states the existence of a particle for which gravity is the weakest force in any four-dimensional massless abelian gauge theory coupled to gravity. This is actually a very specific formulation of the conjecture and in some sense also a little imprecise.

We devote this chapter to a careful study of the arguments that have been put forward in favour of the WGC. Some of these are very similar in spirit as for why quantum gravity should have no global symmetries. Besides these semi-classical black hole arguments, which are long-distance or infra-red arguments[1], we can also collect case-by-case evidence for the WGC from a top-down stringy perspective. There have been many explicit constructions in various corners of the duality web of string theory to motivate the WGC [62, 63, 44]. We will discuss the most simple construction which is based on compactification of heterotic SO(32) string theory on a six-torus [44]. We proceed with discussions of its various generalizations -and refinements, in particular the generalization to higher-rank gauge fields and higher-dimensional gauge theories [44], as well as to U(1) product gauge groups [64], leading to the geometric convex-hull condition. The original WGC is not robust under compactifications and demanding this preservation under compactifications leads to the the lattice WGC (LWGC) [63]. The LWGC demands the existence of a particle for which gravity is the weakest force at every site of the (in general multi-dimensional) charge lattice, consistent with Dirac's quantization condition. This is an interesting generalization of a conjectured property of quantum gravity that dates back to the beginning of this millenium, the so-called completeness conjecture [65].

The WGC has rather dramatic implications for theories of large-field inflation, in

[1] We proceed with further motivation from the infra-red in chapter 9 along the lines of positivity bounds on the wilson coefficients of the lowest order field theory corrections.

particular for axion inflation because it is claimed that the WGC constrains the axion decay constant to be sub-planckian in order to keep perturbative control [66, 67, 27, 42, 43, 44]. Now, it has turned out to be non-trivial to apply the WGC in the axion setting and this has led to specific formulations of "axionic" conjectures [27, 2]. One reason for this difficulty arises because axions descending from string theory arise from intergration of $p$-form gauge fields over $p$-cycles in the compact space, yielding a zero-form field in the $4d$ effective theory. This is different from the usual "WGC setting", which are U(1) gauge fields and particles. The U(1) one-form gauge fields are replaced by zero-form axion fields and the point particles are replaced by a family of instantons. It is a priori not clear whether the WGC, in whatever formulation, can constrain some property of the instanton. However, string dualities will save the day [27]. A more simple reason follows directly from the generalization of the WGC to arbitrary dimensionality -and rank of the gauge fields. It follows that a WGC for a zero-form in any dimension only exists if there is a non-vanishing interaction between a dilatonic field and the gauge sector[63]. Eventually, we will arrive at a consistent WGC for both the single axion as the multi-axion case, thereby *possibly* constraining single-axion inflation as well as multi-axion inflation. We cannot completely rule-out most large-field models due to particular loopholes in the arguments.

The problems encountered to arrive at a axionic WGC are very similar to the ones encountered for a domain-wall WGC [68]. The importance for a domain-wall WGC is related to constraining axion monodromy inflation [69]. However, in this case we encounter even more difficulties and not all have been solved. To constrain axion monodromy, we first need to argue that we can extend the WGC to codimension-1 objects (two-branes) in four dimensions, charged under three-form gauge fields. The WGC can constrain the tension of these objects. This has been done and is very similar to deriving the axionic WGC, utilizing string dualities [68]. It does not stop here, because axion monodromy in string theory is related to a discrete gauge symmetry. The two-branes are charged under a gauged $\mathbb{Z}_N$-symmetry, forcing us to generalize the WGC to this setting [70]. This generalization is still under development [71]. In addition, and this is probably the most severe difficulty, the axion monodromy field theory formulation is one of a *massive* gauge three-form theory [69][2]. The ordinary WGC formulations are statements on massless gauge theories.

## 6.1 Folk theorems

In this section we discuss three conjectures that already existed before the concept of the string swampland was developed. The arguments of why it is thought that global symmetries do not exist is quantum gravity will feature prominently this chapter. Yet it is extremely hard to make these arguments rigorous and there do not seem to be any microscopic inconsistencies[3]. However, this is in line with the swampland philosophy. We want to prevent pathological situations in the long-distance theory via properties of the UV completion. Next, we will discuss the completeness conjecture. It may seem at first disconnected from the rest of this chapter but the most

[2]There is an extension of the WGC to massive vector gauge theories but this is a consequence of the distance conjecture [72].

[3]This is mostly due to the difficulties of proving why planckian black hole remnants are physically inconsistent. This is the so-called remnant problem [37].

rigorously established formulation of the weak gravity conjecture is a natural generalization of it. Finally, and also seeming a bit of an outsider, we briefly mention why it is thought that abelian gauge groups in a theory of quantum gravity should always be compact. All these three conjectures are related, as clearly pointed out by Seiberg and Banks in [73]. Prior to discussing any of these arguments, we will recapitulate some well-known facts about black hole solutions within classical GR and about abelian gauge theories [74, 75].

### Reissner-Nordström extremal black holes

Let us start with a general theorem within classical GR about any black hole. Black holes have a rather remarkable property: they are characterised by a very limited set of parameters. Which parameters are the relevant ones is determined by the field content of the theory. That is why one speaks of *a* no-hair theorem and not *the* no-hair theorem. It states the following [74]:

**Theorem 2.** *In an effective field theory with a U(1) gauge symmetry (Maxwell) coupled to general relativity, asymptotically flat black hole solutions that are non-singular outside the event horizon are fully characterised by the parameters mass, electric -and magnetic charge, and angular momentum.*

Quantum mechanically, black holes can carry additional hair. An example is discrete hair associated to a discrete gauge symmetry $\mathbb{Z}_N$ [76]. There is no long-range classically observable force associated to this gauge symmetry, yet it can be detected via the Aharonov-Bohm effect at spatial infinity. Another example is hair associated to massive higher-spin (i.e. $s \geq 2$) tensor fields [77, 78].

The Einstein equation can be solved exactly under the assumption of spherical symmetry and for the energy-momentum tensor of Maxwell theory. We assume we are in four spacetime dimensions. Hence, the metric ansatz -and energy-momentum tensor are:

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2$$
$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

The solutions will be electrically charged black hole solutions. They have the form:

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2 \quad \text{where} \quad \Delta \equiv 1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} \tag{6.1}$$

where $M$ is the mass of the black hole and $Q$ its electric charge. More precisely, the black hole mass is the Arnowitt-deser-Misner (ADM) mass measured at infinity where the spacetime is flat. The charge $Q$ is measured by measuring the electric flux through a sphere surrounding the black hole. In principle we could also add a magnetic charge in the equations. This would modify the third term in the expression for $\Delta$ to $\frac{G_N(Q^2+Q^2_{mag})}{r^2}$. The event horizon of the spacetime can be found by setting the radial component of the metric to zero (as in the case of the Schwartzschild solution):

$$\Delta(r) = 1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} = 0 \tag{6.2}$$

whose solution is:

$$r_\pm = GM \pm \sqrt{G_N^2 M^2 - G_N Q^2} \tag{6.3}$$

At this point three cases can be distinguished. The *extremal* case is the one for which $G_N M^2 = Q^2$. This condition is known as the extremality condition. It can be interpreted as an exact balance between the electric force and gravity. Two extremal black holes with like and equal electric charges will gravitationally attract each other but repel each other eletrically and there is an exact balance between the two effects.

### Hawking radiation

One of the most important characteristics of quantum gravity upon which most theorists agree is that black holes can evaporate via the emission of thermal Hawking radiation [79]. Black holes have a temperature. For the Schwartzschild black hole this temperature is:

$$T = \frac{1}{8\pi M G_N} \tag{6.4}$$

In classical general relativity, black holes cannot decay. Hence, their mass can only increase. However, objects with finite temperature emit thermal radiation suggesting the black hole mass should decrease. Hawking argued that near the event horizon the gravitational is strong enough to cause virtual pair-production leading the black hole to thermally radiate. The handwaving picture associated to this is that one particle of the pair crosses the horizon and the other is emitted as a real (on-shell) physical particle. Hawkings calculation works for a classical curved background geometry. Thus, the black hole has to be large enough, i.e. the size of its event horizon, so that gravity can be treated classically. We conclude that quantum mechanically, it is possible for black holes to evaporate or decay, a process known as Hawking evaporation.

In classical general relativity we have Hawkings area law, stating that the area of the event horizon is non-decreasing. This suggests an analogy with the second law of thermodynamics. In combination with the above temperature one can derive the relation:

$$S = \frac{A}{4G_N} \tag{6.5}$$

This is known as the Bekenstein-Hawking formula. It holds for any black hole in any dimension. Again, the area needs to be sufficiently large so that classical GR is valid.

### Brief remarks about monopoles

We will encounter a few times magnetic monopoles below. There exist various notions of magnetic monopoles of which Dirac's monopole is the most familiar [80]. Dirac's monopole is a point-like magnetic charge. The existence of only one magnetic monopole implies the quantization of electric (and magnetic) charge. This simply follows from Dirac's quantization condition which states:

$$\frac{q_m e}{4\pi} = \frac{1}{2} n \quad \text{with} \quad n \in \mathbb{Z} \tag{6.6}$$

Note that two magnetic monopoles interact much more strongly than electric monopoles: $q_m^2 \sim \alpha^{-2} e^2$ which is factor of $\sim 10^4$. Magnetic monopoles naturally appear in the context of spontaneous symmetry breaking of a simple non-abelian gauge group to a U(1). This is known as the 't Hooft-Polyakov monopole. More precisely, the object that appears is a soliton, which is a static finite energy solution to the classical equations of motion. There is a whole topological discussion associated to solitons, very similar to instantons, which we will not go into. The soliton is an extended object and in this case of symmetry-breaking has the interpretation of a magnetic monopole. The symmetry-breaking scale is associated to the mass or size of the magnetic monopole. Hence, we can interpret the magnetic monopole mass or soliton size as a cut off on the field theory above which the U(1) gauge group gets embedded into a non-abelian gauge group and the full gauge symmetry is restored. Outside the soliton "core" we have a U(1) gauge theory and inside the core the UV completion.

### 6.1.1 No global symmetries, completeness and compactness

We are now in a position to discuss the folk theorems and arguments that support them. This section is heavily inspired by [2, 46] We start with the non-existence of global symmetries.

**Conjecture 5.** *An effective field theory coupled to Einstein gravity with an exact continuous global symmetry $G$, that has a well-defined conserved charge, belongs to the swampland.*

#### 1. Troubles with entropy bounds?

Consider Einstein gravity with a global U(1) symmetry. We can construct charged black holes by colliding a bunch of particles charged under the global symmetry. Consider an observer outside the black hole horizon. Due to the no-hair theorem, the observer cannot determine the charge of the black hole because the U(1) charge is not imprinted on the black hole horizon. Now, classically one could argue that this is not problematic by pointing out that the charge is behind the horizon. Of course we cannot go into the black hole, determine the charge, and return. Instead, the observer can just assign an arbitrary charge to the black hole. The point is, we cannot associate any uncertainty to the observer's ability of determining the black hole charge because the amount of charge is just in some spacetime region from which the observer cannot return. The situation gets worse when we start to consider semi-classical gravity. In this case we need to take into account the Hawking evaporation of black holes. The black hole evaporates away but at the same time the global charge is not imprinted on the horizon due to the no-hair theorem. The observer has no idea what the initial charge of the black hole was. Roughly speaking, there is an infinite uncertainty associated to the observer, or equivalently an infinite entropy [81]. This is inconsistent since the black hole entropy is semi-classically well-defined via the Bekenstein-Hawking entropy formula and the entropy should be finite. Intuitively, this argument makes sense but it is clear that it is not really rigorous.

**2. Troubles with remnants?** [4]

Consider again the Hawking evaporation of the same charged black hole as above and let us look a little closer to the nature of Hawking radiation. Suppose the initial clump of U(1) charged particles have mass $m$. Then, as long as the Hawking temperature satisfies $T_H \ll m$, the black hole does not radiate any charged particle. Now, it turns out that the black hole will not do so until its mass is roughly equal to $M_{pl}$ [81]. However, for black holes with mass $M \sim M_{pl}$ Hawkings semi-classical computation breaks down and we have no idea what kind of quantum gravitational physics takes over. Furthermore, the black hole is too light and hence it is kinematically forbidden to emit any significant amount of charged particles to really discharge. This is resolved if we assume that the end point of the Hawking radiation is some very light object that is stable and contains approximately all the initial global charge. More precisely, we have an object with arbitrary large U(1) charge $Q$ and mass $M \sim M_{pl}$ corresponding to a size $R \sim l_{pl}$. This is called a *planckian remnant*. This situation should make us wonder whether it is really possible to store so much information (arbitrary large $Q$) on a Planck-sized domain. Moreover, theoretically we could create an infinite number of such remnants of any charge. An effective field theory with an infinite number of exactly stable bound states below any fixed mass scale seems inconsistent, yet it is hard to see why. As a side remark, note that remnants are naturally associated to the strongly-coupled regime of gravity and hence are strongly-coupled objects.

We have two infinities, one infinite uncertainty of how to determine the black hole charge and one infinite number of planckian remnants storing a huge amount of information. It seems natural to associate to this infinite number of remnants an infinite entropy, claiming that entropy provides the correlation between the two arguments against global symmetries. It is hard to establish such a connection rigorously.

**Conjecture 6.** *A theory of quantum gravity has a complete spectrum: in an effective field theory with a U(1) gauge group coupled to Einstein gravity, all charges are realized by physical states.*

This was first conjectured by Polchinski in 2003 [65] and sharpened by [73, 82]. The meaning of the conjecture is best illustrated with an example. Consider pure Maxwell theory:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{6.7}$$

This is a perfectly well-defined quantum field theory, upon adding a gauge-fixing term for proper path-integral quantization, with a U(1) gauge symmetry. Obviously, not all charged states appear. In fact, no charged states appear at all in this theory. What is meant with "completeness" is that all charges are *realized* by states. In a consistent theory of quantum gravity, charged states *must* appear. The argument is again based on black holes. In a U(1) gauge theory coupled to gravity we can construct Reissner-Nordström (RN) black holes of arbitrary charge Q. We know that black holes have an entropy:

$$S = \frac{A}{4G_N} \tag{6.8}$$

[4] This title is inspired by the paper of L.Susskind addressing the possible problems with planckian remnants [37].

Entropy must have an interpretation in terms of microstates. Since the black hole is charged, the microstates must also be charged. So at least some charged states appear in the theory. This argument assumes that the area of the black hole is large enough to be treated in semi-classical GR. Note that this conjecture makes no reference to the massess of these states. It could very well be that they are heavy and hence not part of the effective description. The completeness conjecture is closely related to the (sub)lattice weak gravity conjecture as we will see.

**Conjecture 7.** *The U(1) gauge group in a theory of quantum gravity is always compact.*

This conjecture can actually be interpreted as a lemma of the no global symmetries conjecture [73]. It is most important for our discussions on axion monodromy á-la Kaloper-Sorbo in chapter 8. Namely, the compactness of the gauge group implies the Dirac quantization of U(1) charge. If the U(1) is non-compact, i.e. it is a $\mathrm{U}(1)_{\mathbb{R}}$-symmetry, charge cannot be quantized.

## 6.2 The original formulations

The setting of the WGC is different in the sense that we are now dealing with gauge symmetries which are of course allowed in any theory of quantum gravity. Yet, the arguments for the WGC are very similar to the ones mentioned above against global symmetries. To see why, we consider a U(1) gauge theory with gauge coupling $g$ and we consider the limit $g \to 0$. In this limit we effectively obtain a global symmetry. The "localness" properties of the symmetry are negligible. This can be seen from the action of this theory:

$$S = \int d^4x \sqrt{-g}\Big(\frac{M_{pl}^2}{2}R - \frac{1}{4g^2}F_{\alpha\beta}F^{\alpha\beta} + \dots\Big) \tag{6.9}$$

where $F = dA$ and where the dots indicate field theory corrections about which more later[5]. In the limit $g \to 0$, the kinetic term of the gauge field diverges and hence we are left with a non-propagating gauge field. In other words, the U(1) gauge boson is no longer mediated in any interaction. Yet, the gauge symmetry restricts the form of the action and in this sense still acts as a symmetry. But this is very similar to the behaviour of a global symmetry and hence we cannot distinguish between the two.

This is a situation we want to prevent in consistent theories of quantum gravity so we need something that obstructs the continuous limit of vanishing gauge coupling. This is exactly what the WGC establishes. It states that gauge theories coupled to gravity cannot be arbitrarily weakly-coupled. At this point we can mention two fundamental formulations of the WGC and it will be very useful to think of all other formulations about to be discussed as generalizations of these two:

[5]These higher-derivative corrections modify the extremaliy bound for Reissner-Nordström black holes in a way that establishes a connection with positivity (see section 9.1).

**The electric weak gravity conjecture** [44]
*In a four-dimensional massless U(1) gauge theory coupled to gravity there exists an electrically charged particle whose mass satisfies the bound* $m_{el} \lesssim g_{el} q M_{pl}$.

**The magnetic weak gravity conjecture** [44]
*A four-dimensional massless U(1) gauge theory coupled to gravity has a cut off below the Planck-scale:* $\Lambda \lesssim g M_{pl}$

Already at this point we can raise numerous questions about the meaning or interpretation of these conjectures. In particular, the electric WGC refers to a particle but it leaves its properties (its mass and charge) unspecified. We will see that there is room for three possible states that might satisfy the electric WGC. But there is even a more subtle point: the state satisfying the electric WGC does not have to be a point particle at all, i.e. its mass may very well be super-planckian so that it is actually a black hole [6]. The magnetic WGC is a remarkable statement, especially from the point of view of QFT. It is not precisely clear where this cut off should be placed. However, there is growing consensus that it is related to the energy scale of an infinite tower of states. This suggests a connection with the distance conjecture.

### 6.2.1 Black hole arguments

We want to use our black hole arguments against global symmetries also as motivation for the electric WGC. We point out some subtleties in the arguments, mainly due to the physical differences between global -and gauge symmetries. See the review [2] for more details and references.

#### Are entropy bounds problematic?

As opposed to a global U(1) symmetry, with a gauge U(1) symmetry we cannot theoretically construct an infinite number of planckian remnants. The reason is that there is an upperbound on the number of species below a fixed mass scale $\Lambda$ [83, 84]:

$$N_{species} \sim \frac{\Lambda}{g M_{pl}} \tag{6.10}$$

Note that this bound is strictly only true for particle species, implying we are interpreting our remnants as particles. This is an assumption and we are now forced to consider the remnants below $M_{pl}$, otherwise we are not dealing with a particle but with a black hole. The notion of a particle collapses beyond $M_{pl}$. Setting $\Lambda = M_{pl}$ we get:

$$N_{species} \sim \frac{1}{g} \tag{6.11}$$

[6] Again, this is related to the modified extremality bound for electrically charged black holes. If the bound "weakens" due to higher-derivative corrections, black holes can always decay into smaller black holes. In this case, black holes are themselves the states required by the electric WGC.

There is another difference compared to the global U(1) case. There, the no-hair theorem guaranteed that there was no imprint of the charge on the horizon. For a black hole solution in a massless abelian gauge theory this is no longer true and the black hole carries continuous hair. The charge can be determined by measuring the electric field flux through a sphere surrounding the black hole. An observer outside the black hole can precisely determine the U(1) charge without any uncertainty. The "infinite entropy" argument runs afoul in this case. Yet, we can still ask the question whether the limit $g \to 0$ leads to an inconsistency with respect to entropy bounds since $N_{species} \to \infty$. In particular, it has been argued that this leads to a violation of the covariant entropy bound of to Bousso [85] but again, establishing this rigorously is very difficult. Let us point out why [2].

We consider the simplest case of the Bousso bound in flat space applied to a sphere with radius $R$ and we want to relate it to another entropy bound known as the Bekenstein bound. In this case, according to the Bousso bound, the entropy in the sphere is bounded by:

$$S \leq \frac{A(R)}{4} \tag{6.12}$$

The relation of the Bekenstein bound to gravity is not clear and it also holds in pure QFT. For matter with mass-density $M$ inside the sphere of radius $R$ it states:

$$S_{matter} \leq 2\pi M R \tag{6.13}$$

We now consider the limit $N_{species} \to \infty$ in the decoupling limit $M_{pl} \to \infty$. We can assume the matter is hardly interacting in the box. Naively applying Boltzmann's entropy formula tells us that $S_{matter} = k_b \ln N_{species}$ which diverges but only logarithmically. In fact, it has been shown that $S_{matter}$ is always finite, even if $N_{species} \to \infty$. Thus, in pure QFT the entropy cannot be arbitrarily high due to a large number of species. The story changes when we take $M_{pl} \neq 0$ finite. Gravity enters the game and we need to consider the covariant entropy bound. We will only state the final result. Via a combination of the Bekenstein -and Bousso bound one can arrive at the bound:

$$N_{species} \lesssim R^2 \tag{6.14}$$

Naively, it seems like we can always violate this for an appropriate choice of $N_{species}$. However, we cannot arbitrarily increase the number of species without also lowering the strong-coupling scale of gravity[7]:

$$\Lambda_{species} \sim \frac{1}{\sqrt{N_{species}}} \sim \frac{1}{R} \tag{6.15}$$

This is the statement of the species bound, a bound that will play a role in section 7.1. It can be understood perturbatively as a renormalization of $M_{pl}$ due to loop corrections of the many species to the graviton propagator. The point that is emphasized in the literature is that, since the above cut off corresponds to the zero-point energies of the quantum fields in the sphere, the species bound limits the number of species we can add to the theory to be less than or equal to $R^2$. It is thus impossible to make any conclusions about whether remnants are problematic with respect to entropy bounds.

[7] A priori it is not clear that we should associate the species scale with the scale at which new gravitational dynamics appears. We will later review the (non-perturbative) black hole argument supporting this claim [84]

### Getting rid of remnants - all black holes should be able to decay

Often, the electric WGC is formulated as the statement that all black holes should be able to decay via Hawking radiation. This statement is not entirely correct as we will discuss below but let us first mention a few other points. A black hole charged under a gauge symmetry is actually capable of losing its charge, as opposed to a black hole charged under a global symmetry where the no-hair theorem essentially prevents it. The requirement that a black hole is able to discharge itself places a constraint on the emitted particles. In other words, if we want black holes to discharge themselves, the theory better has an appropriate state in the Hilbert space realizing this decay. If we have a black hole of mass $M$ and charge $Q$ decaying into $n_i$ species of charged particles of charge $q_i$ and mass $m_i$ then conservation of energy -and charge imply:

$$M \geq \sum_i m_i \tag{6.16}$$

$$Q = \sum_i q_i \tag{6.17}$$

In terms of mass-to-charge ratios this implies:

$$\left(\frac{M}{Q}\right) \geq \frac{1}{Q}\sum_i m_i = \frac{1}{Q}\sum_i \frac{m_i}{q_i} q_i \geq \frac{1}{Q}\left(\frac{m}{q}\right)_{min}\sum_i q_i = \left(\frac{m}{q}\right)_{min} \tag{6.18}$$

Thus, the mass-to-charge ratio for the particle with *minimal* mass-to-charge ratio must not exceed that of the black hole. In particular, if we demand that extremal black holes ($\frac{M}{Q} = gM_{pl} \equiv 1$) should be able to decay then we demand the particle that minimizes the ratio to be *super-extremal*, i.e. its mass-to-charge ratio is $\frac{m}{q} \leq gM_{pl}$. Note that this reproduces the electric WGC and therefore it is tempting to think that the electric WGC is equivalent to demanding that extremal black holes should decay[8].

If we do not allow for this decay mode, we can imagine an extremely weakly coupled gauge theory with gauge coupling $g \sim 10^{-100}$. The black hole extremality bound implies that we can associate about $10^{100}$ different charges to a planckian black hole. In the limit $g \to 0$ we have infinitely many remnants and we are back to the problem that motivated us to forbid global symmetries in the first place.

How, i.e. by which mechanism, the black hole discharges depends essentially on the Hawking temperature of the black hole, $T_H \sim \frac{M_{pl}^2}{M}$ for a black hole of mass $M$. For hot -and light black holes, such that $T_H \gg m$ where $m$ is the mass of the Hawking quanta, the black hole can discharge via thermal pair-production of charged quanta on the horizon. The negatively charged particle falls into the black hole while the positively charged particle moves away to spatial infinity.

We can also consider the opposite case where we have a massive large -and cold black hole so that $T_H \ll m$. One might wonder if such a black hole can actually discharge. Luckily, this is possible due to a non-perturbative phenomenon in QFT,

[8]This is the subtle point that the electric WGC refers to a point particle while the decay of extremal black holes does not necessarily require such a particle.

called the Schwinger effect. The gauge field under which the black hole is charged sources an electric field outside the black hole. If the amplitude of the electric field exceeds a particular critical value, it can produce particle anti-particle pairs from the quantum vacuum. This is the Schwinger pair-production effect. It precisely happens in the regime of extremely cold charged black holes. The anti-particle decreases the positively charged black hole and the particle escapes again to spatial infinity.

Now that we know that there are mechanisms for the black hole to discharge and that the mass-to-charge ratio is essentially constrained to satisfy 6.18, the question remains *why* black holes should decay. This is the million dollar question and answering it would provide a proof of the electric WGC.

**Strong, mild and minimal**

It was a very specific choice to focus on the particle that minimizes the mass-to-charge ratio. Note that the mass-to-charge ratio can still be minimal even if $q$ is rather large. We could also consider the lightest particle in the spectrum but it is clear that this is a much stronger statement. In this case, the electric WGC would be a much stronger constraint on the effective theory. The case we focus on here (the WGC particle minimizes the mass-to-charge ratio) is referred to as the *mild* electric WGC. Demanding the WGC particle to be the lightest particle in the spectrum is known as the *strong* electric WGC. There exists one other possibility, namely, that the WGC state is the one with minimal charge. While it was originally thought that there existed counterarguments against this statement [44], it became clear that may equally well be valid [63]. These statements are rather important so we will collect them here [44].

**The strong weak gravity conjecture**
*Consider an effective field theory coupled to Einstein gravity with a U(1) gauge group. Then the lightest state in the Hilbert space of that theory satisfies the bound:*

$$\frac{m_{min}}{q} \leq \left(\frac{M}{Q}\right)_{EBH} \tag{6.19}$$

**The mild weak gravity conjecture**
*Consider the same data as above. Then the state in the Hilbert space of the theory that minimizes the mass-to-charge ratio, has a mass-to-charge ratio smaller than the ratio for extremal RN black holes:*

$$\left(\frac{m}{q}\right)_{min} \leq \left(\frac{M}{Q}\right)_{EBH} \tag{6.20}$$

**The minimal charge weak gravity conjecture**
*The bound should be satisfied by the particle carrying minimal charge under the gauge symmetry:*

$$\left(\frac{m}{q_{min}}\right) \leq \left(\frac{M}{Q}\right)_{EBH} \tag{6.21}$$

It is also clear that the strong WGC implies the mild WGC. Recently, counterexamples were constructed against the strong WGC [62]. Since the mild WGC cannot constrain effective theories dramatically some people have proposed the stronger statement known as the effective weak gravity conjecture [43], demanding that the state which satisfies the WGC should be part of the effective field theory. As a final comment, note that the equality sign in all these conjectures is somewhat mysterious. It is reminiscent of the BPS bound, where the mass equals the central charge, in supersymmetric theories. We come back to this below in section 6.3.

### Do we really need particles?

Extremal black holes saturate the bound $M \geq gQM_{pl}$ in a U(1) gauge theory coupled to gravity, if we truncate the theory to lowest order in curvature invariants and gauge field derivatives. Including field theory corrections in the Maxwell-Einstein theory gives additional terms like:

$$\mathcal{L} \supset \frac{a_1}{M_{pl}^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{a_2}{M_{pl}^4}(F_{\mu\nu}\tilde{F^{\mu\nu}})^2 + \frac{b_1}{M_{pl}^2}F_{\mu\nu}F^{\mu\nu}R + \frac{b_2}{M_{pl}^2}F_{\mu\nu}F^{\mu}_{\alpha}R^{\nu\alpha} + \dots \tag{6.22}$$

We can also construct extremal black hole solutions in the presence of these higher-dimensional (four-derivative) operators. When we include these higher-derivative corrections, extremal black holes no longer saturate the extremality bound. It has been shown that the extremality bound always "weakens". This shows that we do not need to require the existence of a super-extremal particle, as stated by the electric WGC, in order for all black holes to decay into smaller ones. In the presence of field theory corrections, black holes are themselves the state required by the electric WGC. That the bound always weakens is a direct consequence of positivity and will be discussed in chapter 9.

### An infinite number of gravitational bound states

The electric WGC is responsible for the name of the conjecture. For two particles of mass $m$, electric charge $q$ and separation $r$, the gravitational attraction is $\frac{m^2}{M_{pl}^2 r^2}$ while the electric repulsion is $\frac{|q|^2}{r^2}$. Requiring that gravity is the weakest force means $\frac{m^2}{M_{pl}^2} \leq |q|^2$. Thus, the particle required by the electric WGC should couple stronger to the U(1) force than to gravity. We could consider what happens when there do not exists particles for which gravity is the weakest force and see if we arrive at an infra-red pathology. This is similar in spirit to the problem with remnants in the sense that we will obtain an infinite number of stable bound states and it is not clear whether this is really inconsistent.

Suppose there is *no* state in the Hilbert space that does satisfy the WGC bound, i.e. the gravitational attraction wins over the repulsive gauge force. The net force between two such particles is attractive and a stable bound state can be formed. The mass-to-charge ratio decreases because some energy goes into binding energy and hence the mass of the bound state is less than the sum of the individual masses that formed the state. One can continue this process by adding more particles that violate the bound and form bound states so that the mass-to-charge ratio keeps decreasing. At some point this has to stop since the mass-to-charge ratio will correspond to those of extremal black holes. We end up with arbitrarily many exactly stable gravitational bound states because the extremal black holes cannot decay since there does not exist a particle that satisfies the mild WGC. This seems to justify why we need to demand black hole decay.

Note that the formulation of the WGC used here, namely, that the gauge repulsion should dominate over the gravitational attraction, is different from the original electric WGC that is a constraint on the spectrum of the theory. The formulation used in the above argument is also referred to as the "repulsive force conjecture". It coincides with the original WGC in the case where electromagnetism is the long-range gauge force and gravity the long-range attractive force but it becomes a different conjecture when there are additional long-range attractive forces mediated by massless scalar fields. Note also that the repulsive force conjecture can be formulated in non-gravitational theories and hence is not necessarily a conjecture on the quantum gravity swampland.

### 6.2.2 The magnetic WGC

The magnetic WGC , as its name suggests, is related to the electric WGC and can be understood as a "dual" statement. However, the magnetic WGC is a more powerful and constraining statement than the electric WGC. It is clear that the magnetic WGC directly constrains the EFT while the electric WGC might be satisfied by a state whose mass is outside the domain of the EFT.

Let us first establish the dual relationship with the electric WGC. This can be shown by appealing to the principle of naturalness. The electric WGC states:

$$m_{el} \lesssim g_{el} q M_{pl} \tag{6.23}$$

We can impose a similar condition on the dual magnetic monopole. Here one should note that the electric coupling is inversely proportional to the magnetic coupling according to Dirac's quantization condition

$$m_{mag} \lesssim g_m q_m M_{pl} \sim \frac{q_m}{g_{el}} M_{pl} \tag{6.24}$$

Now one needs to note that monopole mass serves as a cut off for the U(1) effective gauge theory. This is analogous to the naturalness example we discussed in section 3.1.1. The self-energy of the magnetic field around the monopole integrated down to distances $r_{mon} \equiv \Lambda^{-1}$ is linearly divergent:

$$\Delta m_m \sim \frac{q_m^2}{g_{el}^2} \Lambda \tag{6.25}$$

where the cut off is the inverse radius of the monopole[9]. According to naturalness we should demand $\Delta m_{mag} < m_{mag}$. Hence, combining this naturalness bound with the magnetic WGC bound we get:

$$\frac{q_m^2}{g_{el}^2} \Lambda \lesssim \frac{q_m}{g_{el}} M_{pl} \Rightarrow \Lambda_{WGC} \lesssim \frac{g_{el}}{q_m} M_{pl} \tag{6.26}$$

[9]We will discuss on page 104 why the size of the monopole can serve as an EFT cut off.

In this argument we have dropped some numerical factors which are irrelevant concerning naturalness arguments. In the original paper the monopole charge is set to unity and the bound is written as:

$$\Lambda_{WGC} \lesssim g M_{pl} \tag{6.27}$$

Some form of "new physics" should appear below the Planck scale. If the magnetic WGC bound is satisfied by a monopole of large charge, the cut off will be relatively far below the Planck scale. One should interpret this cut off $\Lambda$ properly. It means that when we approach the cut off, the theory is no longer a weakly-coupled local effective theory with a U(1) gauge boson. It *could* be that new fundamental Planck-scale interactions come into play but it is also possible that the U(1) gets embedded into a larger non-abelian Lie group. This is what we discussed above in the context of the t' Hooft-Polyakov magnetic monopole. Thus, the cut off not necessarily indicates a complete breakdown of the effective theory and hence is a priori unrelated to the scale at which quantum gravity effects become important. However, it will be argued below that the cut off imposed by the magnetic WGC is associated to the mass scale of an infinite tower of states. Note also that the magnetic WGC obstructs the appearance of global symmetries via $\Lambda_{WGC} \to 0$ as $g \to 0$. The limit $\Lambda \to 0$ is one in which the effective description breaks down.

**Beyond QFT**

The magnetic WGC clearly illustrates that the WGC is a statement beyond QFT. To see why, let us recapitulate some ingredients from QED in four dimensions. QED is an abelian gauge theory with gauge group U(1) and gauge coupling $e$, the electric charge. Its renormalizable Lagrangian is:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} \tag{6.28}$$

The photon propagator in momentum space is:

$$iD_{\mu\nu}(q) = -\frac{ie^2}{q^2}\Big(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\Big) \tag{6.29}$$

where $\xi$ is a gauge parameter. To renormalize the gauge coupling we consider fermion loop corrections to the propagator. The quantum-corrected propagator is:

$$iD^p_{\mu\nu}(q) = -\frac{ie^2}{q^2}g_{\mu\nu}\frac{1}{1+e^2\Pi(q^2)} + \mathcal{O}(q_\mu q_\nu) \tag{6.30}$$

Here we used the Ward identity $q_\mu \Pi^{\mu\nu} = 0$, which is a consequence of the U(1) gauge invariance, where $\Pi^{\mu\nu}$ is the vacuum polarization tensor which includes all 1-particle irreducible diagrams. Imposing the standard renormalization conditions so that the residue of the pole in the interacting propagator is the physical or renormalized coupling, we find [75]:

$$e_R^2 = \frac{e^2}{1+e^2\Pi(q^2=0)} \tag{6.31}$$

We are interested in the one loop correction, hence we compute:

$$(-1)\int \frac{d^4p}{(2\pi)^4} Tr\Big[i\gamma^\nu \frac{i}{\not{p}+\not{q}-m} i\gamma^\mu \frac{i}{\not{p}-m}\Big] \tag{6.32}$$

This is actually quite a nasty integral. Since we are dealing with a gauge theory, we need to regulate the integral with a regularization scheme that preserves the U(1) gauge symmetry. Dimensional regularization achieves this. The result is:

$$\Pi(q^2 = 0) = \frac{1}{12\pi^2} \ln\big(\frac{\Lambda}{m}\big) \tag{6.33}$$

Therefore the renormalized gauge coupling is:

$$e_R^2 \approx e^2 \Big(1 - \frac{e^2}{12\pi^2} \ln\big(\frac{\Lambda}{m}\big)\Big) \tag{6.34}$$

In QFT, coupling constants are functions of the renormalization scale $\mu$. This "running" of the coupling is governed by the Callan-Szymanzik equation. This defines the notion of a beta function of the theory and for QED it is (at one-loop):

$$\frac{de}{d\mu} = \beta(e) = (+)\frac{1}{12\pi^2} e_R^3 + \mathcal{O}(e_R^5) \tag{6.35}$$

The sign is positive so the interaction strength increases as a function of the energy. This gives rise to the so-called *Landau pole*: the energy scale where the coupling strength diverges and where the effective field theorist would place his UV cut off. Typically, the Landau pole appears somewhere way beyond the Planck scale[10]. Yet, this is too naive from the point of view of quantum gravity.

**The monopole is not a black hole**

Another argument arriving at the magnetic WGC is by demanding the magnetic monopole of minimal charge not to be a black hole. This is quite a deep statement, even though it might not be that obvious. A rather subtle point above was, in arguing for the existence of a cut off below the Planck scale, that we applied the naturalness principle to the divergence of the monopole mass. We have seen in section 3.1.1 that the mass of electrically charged point particles is also divergent. So why did we not apply the naturalness argument to this divergence? One simple reason is the following: the integration of the electric self-energy has to be cut off at the Compton wavelength. We have seen that in QED the mass of the electron is only logarithmically divergent so the mass correction will be very small. Hence, electrically charged particles cannot bound the cut off very well. But the question remains why magnetically charged objects *can* bound the cut off.

This is supported by a variety of arguments [86, 87]. One answer is that black holes are regarded as effective descriptions built out of more fundamental objects that are *not* black holes. After all, a black hole is a classical concept and hence we should regard it as a "long-distance" object with the expectation that it has substructure. Now, we "know" that a consistent UV completion has a compact U(1) gauge group. Therefore, magnetically charged black hole solutions exist and their substructure consists of fundamental magnetically charged objects quantized according to Dirac's condition. In particular, the extremality condition is:

$$M \geq Q_{mag} = \frac{2\pi}{g} n \quad n \in \mathbb{Z} \tag{6.36}$$

[10] One could argue that this is not the case since perturbation theory breaks down at some point.

For the minimally charged extremal black hole we have $M \geq \frac{2\pi}{g}$. If the electric theory is weakly coupled $g \ll 1$, the mass of the black hole is large compared to $M_{pl}$ and we expect it to be a consistent notion in the infra-red, i.e. we expect it to be described by an EFT. This might lead to the expectation, as do [44] that there also exists a minimally charged magnetic monopole that is *not* a black hole. In fact, we are forced to demand this if we want to avoid planckian remnants. Namely, if this minimally charged magnetic monopole is a black hole, it could in principle be labelled by any charge because a black hole is a classical notion. The minimally charged monopole, however, is labelled by a very precise charge, so it would be very surprising if this monopole is a black hole. Demanding that an object is not a black hole implies that its size exceeds its own gravitational, or Schwarzschild radius. If $\Lambda$ is a cut off on the EFT and the monopole has size $R \sim \Lambda^{-1}$ and mass $M_{mom} \sim \frac{\Lambda}{g^2}$ then we demand that:

$$R_G \sim \frac{M_{mon}}{M_{pl}^2} \sim \frac{2\Lambda}{g^2 M_{pl}^2} \lesssim \Lambda^{-1} \Rightarrow \Lambda \lesssim g M_{pl} \tag{6.37}$$

neglecting the $\mathcal{O}(1)$ numbers. However, avoiding remnants is subject to a number of counterarguments as we emphasized multiple times. Therefore, we provide a more solid argument, first discussed in [86]. Suppose again the gauge coupling is very small $g \ll 1$. A U(1) gauge theory containing both light, point-like magnetic -and electric charges is non-local. Non-locality implies the theory cannot be UV complete and hence must have a cut off. This cut off is set by demanding the fundamental magnetic charges to be heavy and large solitons. Local QFT is valid outside the soliton radius $R_{sol} \sim \frac{1}{\Lambda}$ with $\Lambda < M_{pl}$. The mass of the soliton is then $M \sim \frac{1}{R_{sol} g^2} > \frac{1}{R_{sol}}$. In the extreme weak coupling regime we must demand that the soliton is *not* a black hole. This means that its size is larger than its gravitational radius $2G_N M$. Hence, we derive the bound:

$$\Lambda < \sqrt{2} g M_{pl} \tag{6.38}$$

which is the magnetic WGC.

### Remnants and the magnetic WGC

The magnetic WGC might provide some additional insights in the seemingly inconsistent situation of a large number of remnants. The magnetic WGC states that the limit $g \to 0$ implies $\Lambda \to 0$. We could then interpret the presence of an infinite number of remnants below some fixed mass scale as a breakdown of the EFT. This is supported from the top-down. In string theory one can send $g \to 0$ and obtain an infinite tower of states. One has to utilize a different effective description in this case.

### Perturbative corrections

Finally, there is one point we have not yet emphasized. We have only discussed the WGC at tree level, neglecting loop corrections to the mass of the WGC particle. The mass -and charge run with the renormalization scale and hence their ratio also runs. In spinor QED it is argued that the mass-to-charge ratio has to be evaluated at the physical mass of the WGC particle [44] because it is this mass that is inherent to the "black holes should decay" reasoning supporting the electric WGC. In the case of the magnetic WGC, we should evaluate the gauge coupling $g$ at the value of the WGC UV cut off $\Lambda_{WGC}$. Recall that the magnetic WGC is a consequence of naturalness. Interestingly, when considering the WGC beyond tree level in scalar QED, one can show that there is tension between the WGC and the principle of naturalness due to the scalar mass being quadratically divergent [64].

## 6.3 Explicit construction

We will discuss the example given in the original paper which provides evidence for the mild formulation of the WGC. This is based on heterotic superstring theory with gauge group SO(32), compactified on a six-torus. To "verify" the weak gravity conjecture we need its ingredients: U(1) gauge fields and extremal black holes. Let us discuss some of the ingredients of heterotic string theory for this purpose [47][11].

### 6.3.1 Even self-dual lattices

We want (to some extend) formalize the idea of a lattice. A *lattice* $\Gamma_{n_1,n_2}$ is defined as vector subspace. Below we will clarify what the subscripts mean. We assume it is a subspace of $\mathbb{R}^{(p,q)}$ equipped with a Lorentzian inner product. A lattice is defined as the set:

$$\Gamma = \{\sum_{i=1}^{N} n_i e_i, \quad n_i \in \mathbb{Z}\} \tag{6.39}$$

Here $e_i$ are basis vectors of the lattice vector space. We can define the *dual* lattice as

$$\Gamma^* = \{\sum_{i=1}^{N} n_i {e_i}^*, \quad n_i \in \mathbb{Z}\} \tag{6.40}$$

where $e_i^*$ denotes the dual basis to $e_i$, i.e. $e_i \dot{e}_j^* = \delta_{ij}$. The metric on the lattice and the dual lattice are each others inverses. A lattice is called *even* if for every two elements $x, y \in S$ their inner product $x \cdot y \in \mathbb{Z}$ and for every $x \in S$, $x^2$ is even. A lattice is self-dual if $\Gamma = \Gamma^*$ which is an equality between vector subspaces. The length squared of momentum vectors on tori is $P^2 = P_L^2 - P_R^2 \in 2\mathbb{Z}$ is even so these vectors define an even-lattice. The definition of length of such a $2n$-momentum vector also shows that the signature of the lattice must be of the form $((1)^{n_1}, (-1)^{n_2})$. What we really mean by signature is the values of the integer-pair $(n_1, n_2)$. Thus the notation $\Gamma_{n_1,n_2}$ specifies a lattice of signature $(n_1, n_2)$. The self-duality of the lattice can be seen as a consequence of the fact that the bosonic theory is self-dual under the T-duality group $O(n, n; \mathbb{Z})$. Under a T-duality transformation one has to simultaneously invert the metric on the lattice, and so one obtains the self-dual lattice.

An important question to ask regarding the bosonic construction of the heterotic string is for what signatures $(n_1, n_2)$ even self-dual lattices exist. Fortunately, it turns out that mathematicians have figured this out. The result is that $n_1$ and $n_2$ need to satisfy $|n_1 - n_2| \in 8\mathbb{Z}$. This indeed coincides with the heterotic string, where

[11] Relevant chapters include chapter 7 and chapter 11.

the right-moving sector is superstring theory of dimension 10 and the left-moving sector is bosonic string theory of dimension 26, so that the difference is 16.

### 6.3.2 Bosonic construction

There are two possible consistent constructions of the heterotic string: the fermionic construction and the bosonic construction. We only discuss the bosonic construction. We discussed in quite some detail the toroidal compactification of closed bosonic string theory. This was necessary precisely for the bosonic construction of the heterotic string. To construct a heterotic string we idenfity the 26 degrees of freedom of bosonic string theory as left-moving degrees of freedom and the 10 degrees of freedom of superstring theory as the right-moving degrees of freedom. We focus on the left-movers because the two constructions differ in this part.

In the bosonic construction we *only* use 26 left-moving bosonic coordinates. Thus we have $X_L^\mu(\tau+\sigma), \mu = 0, \dots, 9$ bosonic fields describing the spacetime part and $X_L^M(\tau+\sigma), M = 1, \dots, 16$ bosonic fields compactified on a sixteen dimensional torus $T^{16}$. We know that the momenta $P_L$ associated to these bosons is quantized along the compactified directions and we know that they form an even self-dual sixteen-dimensional lattice $\Gamma_{16}$. This means that we can write a general momentum vector $P_L$ as $P_L = \sum_i n_i e_i$. Now that we have constructed the heterotic string we can consider its toroidal compactification.

Since the heterotic string theory is a superstring theory, we have a ten-dimensional spacetime. Thus, we compactify the theory on a six-torus $T^6$. We make the decomposition: $M_{10} = \mathbb{R}^4 \times T^6$. Obviously, we need *both* left -and right-movers in the four-dimensional non-compact Minkowski spacetime to have a sensible closed string theory. Hence, we need to compactify $16+6$ of the left-moving degrees of freedom and 6 of the right-moving degrees of freedom. In "lattice-language", we need a lattice $\Gamma_{22,6}$ that describes 22 left-moving compact dimensions and 6 right-moving compact dimensions. This is a Narain lattice.

When compactifying a theory on a torus, its isometries become gauge symmetries in the non-compact theory. The compactification moduli space of heterotic string theory on a $n$-torus is:

$$\mathcal{M} = \Big(O(16+n, n, \mathbb{R}) / \big(O(16+n, \mathbb{R}) \times O(n, \mathbb{R})\big)\Big) / O(16+n, n; \mathbb{Z}) \tag{6.41}$$

where we have the additional quotient by $O(16+n, n; \mathbb{Z})$ due to T-duality. Now, the moduli space parametrizes Narain lattices and each dimension of the lattice contributes a U(1) gauge field. This means that in the case of compactifying on a six-torus we have at a generic point in the moduli space the gauge symmetry:

$$U(1)^{28} = U(1)^{22} \times U(1)^6 \tag{6.42}$$

so that we have 28 U(1) gauge fields. Thus, the allowed charges of these gauge fields lie along the Narain lattice $\Gamma_{L,R} = \Gamma_{22,6}$. We can proceed the discussion in terms of left -and right-moving charges. This means that the charges satisfy the condition analogous to the condition for momentum vectors (expression 5.48):

$$Q_L^2 - Q_R^2 \in 2\mathbb{Z} \tag{6.43}$$

Thus, a general 28-dimensional charge vector decomposes into a 22-dimensional -and 6-dimensional part:

$$\vec{Q} = \begin{pmatrix} Q_L \\ Q_R \end{pmatrix} \tag{6.44}$$

Next, we recall the mass formula for the heterotic SO(32) string states:

$$\frac{1}{4}\alpha' M^2 = \frac{1}{2}P_R^2 + N_R = \frac{1}{2}P_L^2 + N_L - 1 \tag{6.45}$$

In terms of left -and right-moving charges and in units in which $\alpha' = 4$ we get:

$$M^2 = \frac{1}{2}Q_R^2 + N_R = \frac{1}{2}Q_L^2 + N_L - 1 \tag{6.46}$$

The $N_L, N_R$ are number operators counting the number of left -and right-moving excitations. The fact that the argument will work is due to the (-1) on the left-hand side of the above formula. It is due to the tachyon of the left-moving bosonic string spectrum. Next, we need some results from black hole physics that we won't motivate. It turns out that supersymmetric BPS states, i.e. states that saturate the BPS bound, have $N_R = 0$ so that we get:

$$M^2 = \frac{1}{2}Q_R^2 = \frac{1}{2}Q_L^2 + N_L - 1 \tag{6.47}$$

This tower of states are referred to as the *Dabholkar-Harvey* tower. These states coincide with the extremal BPS black hole solutions of the heterotic SO(32) theory:

$$M_{EBH}^2 = \frac{1}{2}Q_R^2 \tag{6.48}$$

This demonstrates the equality sign in the WGC [88]. An extremal BPS black hole is marginally stable. When $N_L = 0$, we have non-supersymmetric states and the mass satisfies:

$$M^2 = \frac{1}{2}Q_L^2 - 1 = \frac{1}{2}Q_R^2 + N_R \tag{6.49}$$

The non-BPS extremal black hole solutions satisfy:

$$M_{EBH}^2 = \frac{1}{2}Q_L^2 \tag{6.50}$$

Now, such an extremal black hole can always decay since there is always a state of mass $M^2 = \frac{1}{2}Q_L^2 - 1 = \frac{1}{2}Q_R^2$ in the string spectrum because:

$$Q_L^2 - Q_R^2 = 2 \tag{6.51}$$

is consistent with the Narain lattice structure. By emitting such a state, the non-BPS extremal black hole moves away from extremality. This construction has demonstrated the validity of the WGC in the heterotic SO(32) theory. At this point we can clarify the *equality* sign in the WGC. We refer to this as the *sharp* WGC:

**The sharp weak gravity conjecture**[88]
*The equality sign of the weak gravity conjecture holds if and only if the states saturating the bound are BPS states. Equivalently, BPS extremal black holes are marginally stable.*

A very similar construction of heterotic SO(32) on a torus can prove the validity of the lattice weak gravity conjecture [63], a formulation we will discuss below.

## 6.4 More IR motivation

Besides the "extremal black holes should decay" argument, there are more long distance motivations for the validity of the WGC. We can also require the low-energy effective theory of an abelian gauge theory coupled to gravity to have a Wilsonian UV completion whose S-matrix is unitary, causal and analytic. This approach was initiated in [89]. Consider the effective theory of Einstein gravity coupled to a U(1) gauge theory with cut off $\Lambda = M_{pl}$:

$$\begin{aligned}\mathcal{L} = \frac{M_{pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a_1}{M_{pl}^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{a_2}{M_{pl}^4}(F_{\mu\nu}\tilde{F^{\mu\nu}})^2 + \dots \\ +\frac{b_1}{M_{pl}^2}F_{\mu\nu}F^{\mu\nu}R + \frac{b_2}{M_{pl}^2}F_{\mu\nu}F^{\mu}_{\alpha}R^{\nu\alpha} + \frac{b_3}{M_{pl}^2}F_{\mu\nu}F_{\alpha\beta}R^{\mu\nu\alpha\beta} + \dots \\ +c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + \dots\end{aligned}$$

where the dots indicate all other allowed higher-dimensional operators. In the absence of charged sources, these do not include higher-derivative operators such as for example $(\nabla_\mu F_{\alpha\beta})^2$ as these can be written in terms of the operators already written down. One might wonder how we can derive any information related to the WGC from this effective theory. The idea is that electromagnetic interactions, e.g. light-by-light scattering, contribute to the coefficients $a_i, b_i$. The magnitude of the contribution depends on the mass and the charge of the particles in the spectrum. The contributions are the largest for the particle that has the largest charge-to-mass ratio which is precisely the relevant quantity for the WGC.

Only in three dimensions can a rigorous bound be derived because gravity is non-dynamical [89]. In four dimensions, it is hard to make sense of the analyticity arguments because non-convergence properties of the photon-photon scattering amplitude. We return to this point of view to the WGC in chapter 9.

## 6.5 Generalizations

### 6.5.1 Discrete gauge groups

We already know that there do not exists exact *continuous* global symmetries in a consistent theory of quantum gravity. But there exists a stronger conjecture, proposed by Seiberg and banks [73], stating that all symmetries in a theory of quantum gravity are gauged. Thus, also discrete symmetries are local. If this is true, the WGC needs to be generalized to discrete gauge theories coupled to gravity as well.

The continuous global symmetries are forbidden in a theory of quantum gravity because it gives rise to infinitely many planckian remnants. However, whether such an argument works for a discrete global symmetry, say $\mathbb{Z}_k$, is not obvious. This is related to the fact that the group $\mathbb{Z}_k$ is of finite order for $k$ finite while the order of continuous groups is uncountably infinite. If we take the integer $k$ sufficiently large then we are able to obtain many planckian remnants, but for $k$ a small integer this is less clear. Yet, in string theory also discrete symmetries are gauged and therefore it is generally thought to be true.

Now, discrete gauge symmetries $\mathbb{Z}_k$ with gauge coupling $g$ arise upon breaking a continuous gauge symmetry $G$ with gauge coupling $g$. If we take the limit $g \to 0$ for this $\mathbb{Z}_k$ gauge symmetry we obtain a global $\mathbb{Z}_k$ symmetry, which is forbidden at least for sufficiently large $k$. Hence, a generalization of the WGC should exist to prevent us from taking this limit in the discrete case. We consider the case where the $\mathbb{Z}_k$ arises from breaking a U(1) gauge symmetry with coupling $g$. Suppose the WGC is violated in the theory with unbroken U(1). Then we can form bound states with arbitrary large charge $q$ under U(1) as we have discussed before. $k$ of these states are distinguished by their discrete charge under $\mathbb{Z}_k$. These states are stable unless there exists a particle on which gravity acts weaker than the discrete gauge force. If the order $k$ of the discrete group is large enough we would have many stable states with respect to the discrete gauge group and this might lead to some seemingly pathological situation. For small $k$ the situation is remains less clear.

An important consequence for inflationary model building is that, while the original WGC cannot constrain axion monodromy inflation, it is argued that this generalization can. The reason is that in the string embedding of the EFT of axion monodromy, monodromy corresponds to a discrete gauge symmetry $\mathbb{Z}_k$ [90]. We will discuss this low energy effective field theory of axion monodromy in chapter 8.

### 6.5.2 Multiple U(1) gauge groups

Up to now we have looked at the WGC applied to a particle charged under a *single* U(1) gauge group. Something very interesting happens when we generalize the mild WGC to particles charged under *multiple* U(1) abelian gauge groups [64]. It is known that extending the strong WGC to multiple gauge groups is problematic, hence we focus on the mild conjecture. Consider a product of $N$ U(1) gauge groups:

$$U(1)^N \equiv \prod_{a=1}^{N} U(1)_a \tag{6.52}$$

Consider particle species labelled by $m_j$ charged under multiple U(1)'s so they carry charges $q_{aj}$. We define a charge vector $\vec{q_j} \equiv q_{aj}$. This allows to define a mass-to-charge vector:

$$\vec{z_j} \equiv \vec{q_j}\frac{M_{pl}}{m_j} \tag{6.53}$$

How should we extend the WGC? The most naive option would be to require that there exists one species $j$ whose charge-to-mass vector satisfies $|\vec{z_j}| > 1$. However, this only allows extremal black holes that are charged in this direction of charge space to decay. Extremal black holes charged in directions orthogonal to the $j$th direction are not able to decay. One could go a step further and require that in *each* orthogonal direction $j$ of charge space there exists a species $j$ that satisfies the bound $|\vec{z_j}| > 1$. This means we require the existence of $N$ particles, each charged under a *single* U(1), satisfying $|\vec{z}| > 1$. It might come as a surprise that this is still not a strong enough requirement to satisfy the WGC bound. This can be seen as follows.

Consider an extremal black hole with charge vector $\vec{Q}$, mass $M$ and charge-to-mass vector $\vec{Z} = \frac{M_{pl}}{M}\vec{Q}$. Choose the normalization such that $|\vec{Z}| \equiv 1$ which defines an N-dimensional unit-ball in charge space. Extremal black holes lie on the edge of this unit-ball. The WGC demands that such an object decays. Suppose it decays into $i$ different species, the number of each species $i$ denoted by $n_i$. We invoke energy -and charge conservation to impose the conditions:

$$\vec{Q} = \sum_i n_i \vec{q_i} \qquad M > \sum_i n_i m_i \tag{6.54}$$

Define the quantity $\sigma_i$ to be the fraction that decays into the $n_i$ species, i.e. $\sigma_i = \frac{m_i}{M} n_i$. We rewrite the above conditions in terms of this mass fraction:

$$\vec{Z} = \sum_i \sigma_i \vec{z_i} \qquad \sum_i \sigma_i < 1 \tag{6.55}$$

These conditions define what is known as a *subunitary* weighted average of $\vec{z}_i$. It has a very nice geometrical interpretation [64, 27]. The requirement that multi-charged extremal black holes can decay shows we need to require that in every *rational* direction of charge space there must exist a state (possibly a multi-particle state) for which gravity is the weakest force. Here "rational-direction" means a ray in charge space that intersects a lattice site that is consistent with Dirac's quantization condition. This is equivalent to the statement that the charge-to-mass vectors $\vec{z_i}$ span an object known as the *convex-hull*. As long as the convex-hull spanned by the charge-to-mass vectors contains the N-dimensional unit ball, the WGC is satisfied and extremal black holes charged under multiple gauge groups can decay. Thus we have the conjecture, known as the convex-hull condition (CHC):

> **The convex-hull conndition**[64]
> *In an EFT with multiple U(1) gauge groups coupled to gravity, the WGC states that the convex hull of the charge-to-mass vectors of the particle species in the theory must contain the unit-ball.*

Typically, the interior of the unit ball is referred to as the "black-hole region", its boundary as the "extremal boundary" and the exterior as the "superextremal region".

A very important observation, in particular regarding multi-axion models of inflation, is that the WGC becomes a stronger constraint in the presence of multiple gauge groups. By this we mean what we mentioned above: the constraint $|\vec{z}_j| > 1$ in every direction $j$ of charge space is not sufficient for extremal black holes to decay. This is related to the fact that the total charge of black holes charged under multiple U(1)'s adds in quadrature, so for example $Q_{tot} = \sqrt{\frac{Q_1^2}{g_1} + \frac{Q_2^2}{g_2} + \cdots + \frac{Q_N^2}{g_N}}$.

We look at the following example in [64] with $N = 2$ and two particles charged under these groups. Denote the charge-to-mass vectors as $\vec{z}_1, \vec{z}_2$. The convex-hull condition states that the convex-hull spanned by the vectors $\vec{z}_1, \vec{z}_2$ of the two particles must contain the unit-disk. This implies the following geometric condition on the vectors $\vec{z}_i$, which can be derived from the conditions defining a subunitary weighted average of these vectors:

$$(|\vec{z}_1|^2 - 1)(|\vec{z}_2|^2 - 1) > (1 + |\vec{z}_1 \cdot \vec{z}_2|)^2 \tag{6.56}$$

Suppose $|\vec{z}_1| = |\vec{z}_2| = |\vec{z}|$ and $\vec{z}_1 \perp \vec{z}_2$. We derive from the above relation:

$$|\vec{z}| > \sqrt{2} > 1 \tag{6.57}$$

Hence, we see that the WGC becomes a stronger condition compared to a single U(1) which only required $|\vec{z}| > 1$. We could further generalize this to $N$ U(1)'s

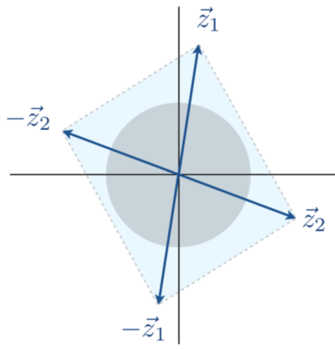


Figure 6.1: This picture is taken from [64]. We see that the CH of the two vectors contains the unit-disk and therefore the WGC is satisfied.

with $N$ charged particles under these U(1)'s. Suppose all vectors are orthogonal $\vec{z}_1 \perp \vec{z}_2 \perp \ldots \vec{z}_N$ and have equal magnitude $|\vec{z}_1| = |\vec{z}_2| = \cdots = |\vec{z}_N| = |\vec{z}|$. Suppose further that each particle is charged under a single U(1). It is a mathematical result that in this case the convex-hull is a *cross-polytope* of circumradius $|\vec{z}|$. A *circumscribed* N-polytope of the N-ball is a N-polytope that contains the N-ball. It turns out that the largest N-ball contained in the cross-polytope has radius $R = \frac{|\vec{z}|}{\sqrt{N}}$. For the convex-hull to contain the unit-ball we need to require that $|\vec{z}| > \sqrt{N}$. This is the proper requirement for consistency with the WGC in the case of $N$-particles charged under $N$ U(1)'s. It is this factor of $\sqrt{N}$ that cancels the $\sqrt{N}$-enhancement of N-flation, preventing super-planckian trajectories of the collective field.

## 6.6 The generalized weak gravity conjecture

It was already "loosely" conjectured in the original work [44] that the electric WGC could be generalized to gauge fields of arbitrary rank in arbitrary dimensions. Specifically, we generalize the conjecture to $p$-form gauge fields that couple electrically to $(p-1)$-dimensional objects in $d$-dimensions. The magnetic dual to this electric object is a $(d-p-1)$-dimensional object. The electric WGC becomes a statement on the tensions of these objects. Although loosely, this generalization is natural from a stringy perspective as string theory naturally contains higher-dimensional extended objects charged under higher-rank gauge fields and there do exist corresponding charged black brane solutions in arbitrary dimensions so that we can the invoke remnant argument.

**The "loose" generalized weak gravity conjecture**[44]
*Consider a p-form gauge theory in d-dimensions. Then there exist electrically -and magnetically charged extended objects whose tensions satisfy the bounds:*

$$T_{el} \lesssim M_{pl}^d g \quad \text{and} \qquad T_{mag} \lesssim \frac{M_{pl}^d}{g} \tag{6.58}$$

Here $M_{pl}^d$ is the $d$-dimensional Planck mass and $g$ denotes the charge density of the extended object. Such a conjecture would allow extremal black $p$-branes to decay. There has been a lot of discussion to what extend this generalization makes sense, in particular for the $p = 0, d = 4$ case [63, 27, 66]. Note that this refers to some bound on the tension of instantons. Instantons are objects that couple to zero-form fields such as axions. In this case, instantons play the role of the WGC particle while axions play the role of the U(1) gauge fields. The quantity that is upper-bounded is the tension of the instanton which corresponds to its action.

The generalized WGC would have striking implications for inflationary models. Already in the original WGC paper the generalized conjecture was applied to extra-natural inflation. A *closed string* axion, sometimes also referred to as a fundamental axion, descending from a $p$-form gauge field upon integration over a non-trivial $p$-cycle in the compact space, is a zero-form that couples electrically to an instanton. By dimensional counting the mass dimension of the charge density $g$ is $p + 1 - \frac{d}{2}$. For $p = 0$ and $d = 4$ we get $-1$. An axion decay constant $f$ has mass dimension $+1$ so we infer that $g \sim \frac{1}{f}$. The instanton tension is interpreted as the instanton action $S_{inst}$. One thus gets the bound:

$$S_{inst} \lesssim \frac{M_{pl}}{f} n \tag{6.59}$$

where $n$ denotes the instanton number/charge. Via a generalization of the completeness conjecture we can always set this equal to unity[12]. Recall that for large-field displacements we have to require that the axion decay constant exceeds the Planck mass. To protect the flatness of the potential we also had to require that the instanton action is larger than unity so that we can neglect higher-instanton contributions to the inflationary action. We see that the loose generalized WGC predicts that for parametrically large axion decay constants higher harmonics contribute to the axion potential and hence spoil its flatness. Thus, there is at least tension between a parametrically large axion decay constant and the generalized WGC. This is confirmed in many controlled string compactifications.

Even though the above generalized WGC seems reasonable and it is supported by top-down constructions, there are actually quite some subtleties we have to deal with to place this conjecture, and in particular its four-dimensional zero-form case, on a more firm footing. For one reason, as will become clear below in the discussion below, the zero-form WGC seems only to make sense in the presence of a scalar field coupled to the U(1) sector. This is due to the fact that the conjecture above is "loose" in the sense that the coefficient in front depends on the spacetime dimensionality.

### 6.6.1 The precise generalized electric WGC

The most significant work firmly establishing the generalized electric WGC is due to the work in [63]. They constructed higher-dimensional black brane solutions, charged under arbitrary $p$-form gauge fields and in the presence of a dilaton field that couples to the gauge sector. This construction explicitly reveals the $\mathcal{O}(1)$ number

[12] This is actually the lattice weak gravity conjecture (see section 6.6.3).

in front of the electric WGC. Specifically, the coefficient depends on the number of dimensions and on the strength of the coupling of the dilaton to the gauge sector. We consider the action:

$$S = \frac{1}{2\kappa^2{}_d}\int d^dx\sqrt{-g}(R_d - \frac{1}{2}(\partial\phi)^2) - \frac{1}{2e^2_{p;d}}\int d^dx\sqrt{-g}e^{-\alpha_{p;d}\phi}F^2_{p+1} \tag{6.60}$$

Here $\kappa_d$ is related to the $d$-dimensional Planck mass as $\frac{1}{\kappa_d^2} = \frac{1}{8\pi G_d} = M_d^{d-2}$, $R_d$ the $d$-dimensional Ricci scalar and $\alpha_{p;d}$ is the scalar $p$-form coupling in $d$-dimensions. $F_{p+1} = dA_p$ is a field strength. Next, magnetically charged black brane solutions of this action are obtained. Via electric-magnetic duality one obtains electrically charged black p-brane solutions whose extremality bound is:

$$\gamma e^2_{p;d}Q^2 \leq \kappa^2 T^2 \tag{6.61}$$

Here $Q$ is the electric charge of the extremal black p-brane and $T$ its ADM-tension. For $d = 4, p = 0$ the RN extremality bound is recovered. The factor $\gamma$ is defined as

$$\gamma_{p;d}(\alpha) \equiv \Big[\frac{\alpha^2}{2} + \frac{p(d-p-2)}{d-2}\Big]^{-1} \tag{6.62}$$

Having obtained the extremality bound for electrically charged black $p$-branes, we find the generalized electric WGC by demanding this object to decay. Specifically, we demand the existence of super-extremal particles and $p$-branes so that any extremal black hole or black p-brane can evaporate:

**The precise generalized weak gravity conjecture**[63]
*There exists a charged object with tension $T_p$ and quantized charge $q$ that satisfies the bound*

$$\Big[\frac{\alpha^2}{2} + \frac{p(d-p-2)}{d-2}\Big]T_p^2 \leq e^2_{p;d}q^2 M_d^{d-2} \tag{6.63}$$

A few remarks should be made about this conjecture. The first is that the derivation of this result only holds for $1 \leq p \leq d-3$. In particular, when we neglect the dilaton coupling $\alpha \equiv 0$, the statement is degenerate for $p = 0$ (axions) and for $p = d-2$ (strings in $d = 4$). For $p = d-1$ (domain walls), the equation cannot be solved and it is automatically satisfied. The phenomenological importance for the $p = 0$ was emphasized already above. The case of domain walls is of equal importance because it might allow for a constraint on field variations generated by axion monodromy. This is a very subtle issue and we will discuss this in chapter 8.

Secondly, it turns out that the generalized WGC in $d$-dimensions is a stronger condition than the generalized WGC in $(d-1)$ dimensions. Consider a gravitational theory with a $p$-form gauge field that satisfies the generalized WGC in $d$-dimensions (note that there is no scalar field). Compactification of the gravitational part on a circle of radius $R$ gives a theory in $(d-1)$-dimensions containing a $p$-form -and a $(p-1)$-form gauge field. We also have KK-modes, charged under the gauge symmetry, descending from the $d$-dimensional metric. Compactification on a circle comes with a modulus which is the radius of the circle. Now, we can consider two cases. The volume modulus can be stabilized (massive) or remain unstabilized. If it is massless then $\alpha$ changes in such a way that the bound remains equally strong in the

$(d-1)$ dimensional theory. If the volume modulus obtains a potential and becomes massive, the bound becomes weaker in the lower-dimensional theory, and thus is satisfied. At long distances we can integrate-out the modulus and we are effectively left with an Einstein-Maxwell action which is the original setting of the electric WGC. Obviously, this is a weaker bound since $\alpha = 0$. It seems to be the case that the WGC in higher dimensions is a stronger statement than the statement in lower dimensions.

### 6.6.2 Behaviour under circle compactifications

The electric generalized WGC is not robust under compactifications. However, *how* it precisely fails under compactifications is rather subtle. The above action 6.60 can be compactified on a circle in two ways. It can be compactified where the $p$-form in the lower-dimensional theory descends from a $p$-form in the higher-dimensional theory, *preserving* the rank. Or, it can be compactified to one lower dimension and also *reduce* the rank of the form. It turns out that the extremality bound given above remains unchanged in both cases. This shows that the generalized electric WGC is preserved under any toroidal compactification.

However, there remains a subtle inconsistency which can be seen as follows. Suppose we first compactify a pure gravity theory on a circle of radius $R$. We have seen that the degrees of freedom of the metric in the higher-dimensional theory are split into a graviton, a scalar and a U(1) gauge field. This "emergent" U(1) field is often called a KK-photon. The KK-photon gauge coupling is:

$$\frac{1}{e_{KK}^2} = \frac{1}{2} R^2 M_d^{d-2}$$

The volume modulus also couples to the KK-photon, giving rise to the following scalar-Maxwell coupling:

$$\alpha_{KK} = \sqrt{\frac{2(d-1)}{d-2}}$$

It turns out that the numerical factor $\gamma_{1,d}(\alpha_{KK})$ in the generalized WGC in this case becomes $\gamma_{1,d}(\alpha_{KK}) = \frac{1}{2}$. Substituting these results into the generalized WGC gives the bound:

$$m^2 \leq \frac{q^2}{R^2}$$

The mass $m_0$ of the particle in the higher-dimensional theory, giving rise to a KK-tower in the lower-dimensional theory, is related to the mass of the states in the KK-tower via:

$$m_0^2 = m^2 - \frac{q^2}{R^2} \tag{6.64}$$

The important observation is that massless particles in the higher-dimensional theory (e.g. the graviton propagating in five dimensions) give rise to KK-modes that *saturate* the WGC bound. Massive higher-dimensional particles generate KK-modes that violate the bound. So we conclude that a five-dimensional pure gravity theory generates KK-modes that saturate the WGC bound, if the volume modulus does not obtain a mass. If it does, the bound becomes weaker. The fact that the bound is only saturated for a single KK-photon suggests that we may encounter problems when additional photons come into play. The way to test whether the generalized WGC remains true in the multi U(1) case is to construct black hole solutions charged two U(1)'s: One "fundamental" $U(1)_F$ and one "KK" $U(1)_{KK}$ arising upon compactification. This is rather technical and discussed in great detail in [63]. Now, the new extremality bound arises from dimensional reduction of black strings charged under gauge fields. The form of the bound depends on the rank of the gauge field to which the black string couples. If the black string is coupled to a one-form gauge field, it is:

$$M^2 \geq \gamma e_d^2 M_d^{d-2} Q_F^2 + \frac{1}{R^2}\Big(Q_{KK} - \frac{\theta}{2\pi} Q_F\Big)^2 \tag{6.65}$$

Here $\theta$ is an axion. Demanding the existence of a super-extremal object we obtain the generalized electric WGC. Note the characteristic quadrature-addition of the different U(1) charges. If the black-string is coupled to a two-form gauge field it is:

$$\kappa_d M \geq \sqrt{\gamma} e_d |Q_F| + \sqrt{\gamma_{KK}} e_{KK} |Q_{KK}| \tag{6.66}$$

Note that we now have a different addition of the charges. As the structure of the addition of the charges is related to the convex-hull condition, this shows that the geometric structure of the convex-hull spanned by the charge-to-mass vectors is modified. Also note that putting $Q_{KK} = 0$ or $Q_F = 0$ in either of the bounds yields the previous extremality bound 6.61. It can be shown that for the bound 6.66, when we consider the mixing between the KK-photon and the photon arising from a two-form in $d$-dimensions, the generalized electric WGC in $d$-dimensions implies the convex-hull condition in $(d-1)$-dimensions. The interesting case is the one in which the KK-photon mixes with a photon arising from a $d$-dimensional one-form. We will highlight the main insights obtained in this situation.

We suppose the generalized electric WGC is satisfied in the $d$-dimensional theory. The charge space is two-dimensional because we have two U(1)'s. We define a charge-to-mass vector for a particle with mass $m$ charged under both U(1)'s in the $(d-1)$-dimensional theory as:

$$\vec{z} \equiv \frac{1}{m}\Big(\sqrt{\gamma} e_d M_d^{\frac{d-2}{2}} Q_F, \frac{1}{R}\big[Q_{KK} - \frac{\theta}{2\pi} Q_F\big]\Big) \tag{6.67}$$

where $d \equiv D-1$. Indeed, if we consider the length-squared of this vector and set it equal to unity we see that this definition is consistent with the extremality bound 6.65. The black-hole region corresponds to the unit-disk $|\vec{z}|^2 = |\vec{z}_{KK}|^2 + |\vec{z}_F|^2 < 1$, and $|\vec{z}| = 1$ to the extremal boundary. Because of our assumption that the generalized WGC is satisfied in the $d$-dimensional theory, there exists a particle charged under a single $U(1)_F$ of mass $m_0$ and $Q_F \equiv q$ that satisfies:

$$\vec{z}_0 \equiv e_D M_d^{\frac{d-2}{2}} \sqrt{\gamma} \frac{|\vec{q}|}{m_0} \geq 1 \tag{6.68}$$

To this particle there is an associated KK-tower in the $(d-1)$-dimensional theory. The relation between $m_0$ and the masses of the KK-modes is slightly altered because we now compactify a gauge theory coupled to gravity on a circle and not a pure gravity theory. We get the mass spectrum:

$$m_n^2 = m_0^2 + \frac{1}{R^2}\big(n - \frac{q\theta}{2\pi}\big)^2 \tag{6.69}$$

where $Q_{KK} = n$ is the KK-charge. Since we have an infinite tower of states we also need to define infinitely many charge-to-mass vectors $\vec{z}_n$ for every KK-mode. These are appropriately defined as:

$$\vec{z}_{(n)} \equiv \frac{1}{\sqrt{m_0^2 R^2 + \left(n - \frac{q\theta}{2\pi}\right)}} \left( m_0 R e_d M_d^{\frac{d-2}{2}} \sqrt{\gamma} \frac{|\vec{q}|}{m_0}, \left(n - \frac{|\vec{q}|\theta}{2\pi}\right) \right) \tag{6.70}$$

Now, recall that the convex-hull condition states that the convex-hull spanned by all these charge-to-mass vectors must contain the unit-disk. It is shown that the vector 6.70 satisfies $\left|\vec{z}_{(n)}\right| > 1$ for every $n \in \mathbb{Z}$ but from our discussion of the convex-hull condition in a previous section we know that such a condition is generally insufficient. This is indeed the case here. One can show that the distance from the line-segment $\left|\vec{z}_{(n+1)} - \vec{z}_{(n)}\right|$ can be less than unity for some set of values of $n$. This means that this line intersects the unit disk at some points so that this object is not entirely contained inside the convex-hull. [63] derives the interesting bound:

$$(m_0 R)^2 \geq \frac{1}{4z_0^2(z_0 - 1)} + \frac{n_0(1 - n_0)}{z_0^2} \quad \text{where} \quad n_0 \equiv \frac{q\theta}{2\pi} \mod 1 \tag{6.71}$$

showing that whatever particle we pick in the $d$-dimensional (i.e. the choice of $z_0$) that satisfies the generalized electric WGC, we can always violate the convex-hull condition in the $(d-1)$-dimensional theory by using a suitable choice of the value for the radius $R$ of the compactification circle. Note the interesting point that we have not stabilized the radius of the circle, so it is a modulus. This means that for the generalized WGC to hold, the convex-hull condition must be satisfied in the *entire* moduli-space of the circle compactification.

### 6.6.3 Resolution - the lattice conjectures

Here we will discuss the lattice WGC's which arise upon demanding the generalized electric WGC to be robust under some classes of dimensional reduction. We will proceed along the lines of [63] but only emphasizing the details and important insights.

To solve the problem, [63] proposed a modification of the WGC motivated by the following considerations. First, note the last remark of the previous section. Recall that the swampland distance conjecture indicates that the validity of an effective field theory correlates with a finite diameter in moduli space. Hence, we really should introduce a UV cut off $\Lambda_d < \frac{1}{R_{min}}$ in the higher-dimensional theory. In this case, the convex-hull condition will be satisfied in a restricted region of moduli-space.

Equivalently, we could set the UV cut off at a very high scale but finite. To keep our effective description valid we incorporate the charged states that have become light. If we demand these particles to "assist" in satisfying the convex-hull condition, we need them to satisfy $|\vec{z}| > 1$, i.e. we demand these extra particles to be additional "weak-gravity particles" in the higher-dimensional theory, because this will make the convex-hull "larger" in the lower-dimensional theory. Note that the bound 6.71 is independent of the mass of the KK-modes but only depends on properties of the state that gave rise to these KK-modes, i.e. its charge under $U(1)_F$ and its mass

$m_0$. Now, suppose some of our extra particles in $d$-dimensions have equal charge $q$. Then, only the lighter particle assists in solving the problem. Hence, only the extra states with *distinct* charge will assist in satisfying the convex-hull condition. The point is, as long as our UV cut off is not infinitely far away, we have to incorporate only a finite number of extra charged states in our effective description in $s$-dimensions, implying that we can always choose a minimum radius such that the convex-hull condition is violated.

Thus, the only way out to satisfy the convex-hull condition in the lower-dimensional theory is to incorporate *infinitely* many particles, all satisfying the WGC in $d$-dimensions and all having *distinct* charges under $\mathrm{U}(1)_F$. Demanding this in the higher-dimensional theory is known as the *Lattice Weak Gravity Conjecture*.

**The Lattice Weak Gravity Conjecture (LWGC)**[63]
*Consider a d-dimensional U(1) gauge theory coupled to gravity and compactify it on a circle. Then, to satisfy the convex-hull condition in the (d-1)-dimensional theory, we need that for every point $\vec{q}$ on the charge lattice $\Gamma$, there exists a single-particle state of mass $m$ and charge vector $\vec{q}$ with charge-to-mass vector $\vec{z}$ satisfying the bound*:

$$\frac{m}{M_{pl}}\vec{q} \geq \left(\frac{M}{M_{pl}}\vec{Q}\right)$$

This formulation establishes that imposing the generalized electric WGC in higher-dimensions is preserved under (circle, and hence toroidal) compactifications to lower dimensions. Note that this is a natural extension of the completeness conjecture. This demanded the existence of states at every point on the charge lattice consistent with Dirac's quantization condition. Now, we conjecture the additional requirement, that at each such lattice site there must exist a particle for which gravity is the weakest force. The LWGC is the strongest formulation of the electric WGC so far.

A few additional remarks need to be made about the LWGC. First and most importantly, it has been shown that the LWGC, even though it is supported by a few string theory constructions, is a too strong requirement and can be violated if a U(1) gauge theory coupled to gravity is compactified on other manifolds than a circle or torus [62]. What happens is that not all lattice sites are populated by a particle for which gravity is the weakest force. However, and this will again lead to a more refined conjecture, the LWGC is satisfied on a proper subset of lattice sites [62, 61] of the full lattice. This sublattice WGC implies that the strong electric WGC is falsified. Secondly, we require single-particle states because otherwise we cannot define associated KK-towers in the lower-dimensional theory.

Finally, and also very important for the following discussions in the next chapter, we need to require the existence of infinitely many charged particles. An EFT coupled to an infinite tower of charged particles implies the existence of a UV cut off. If we take the gauge coupling $e \to 0$ this infinite tower of WGC particles becomes light since every point on the charge lattice has a state satisfying $m_{el} \lesssim eqM_{pl}$. We see that in the weak coupling limit the EFT breaks down which is reminiscent of the magnetic WGC and we see it is connected to an infinite tower of states. Note that we now do not need to argue with monopoles or naturalness. Note also the similarity with the swampland distance conjecture, where also an infinite tower of light states emerges. It seems to be the case that weakly coupled gauge theories are inconsistent theories of physics as the theory breaks down in the weak coupling limit. However, this is not true since we could switch to another effective description, one or more dimensions higher, that is weakly coupled. For example, consider a four-dimensional theory with an infinite tower of KK-modes that are light so that this description breaks down. Then, switching to a five-dimensional description resolves the problem. We now slightly reformulate the LWGC:

**The sublattice Weak Gravity Conjecture (sLWGC)**[62]
*Consider an abelian gauge theory coupled to gravity with a charge lattice* $\Gamma$. *Then there exists a proper sublattice* $\Gamma_{sub} \subseteq \Gamma$ of the same dimensionality as the full lattice such that $\forall\ \vec{q} \in \Gamma_{sub}$, $\exists$ a (possibly an unstable resonance and/or multi-particle) state of mass $m$ that satisfies:

$$\frac{1}{m}|\vec{q}| \geq M_{pl}$$

The sLWGC also demands the existence of an infinite tower of charged particles that obey the WGC bound. The sLWGC is supported by a lot of explicit string constructions. In fact, one can almost derive the sLWGC from modular invariance in perturbative string theory. It is the most rigorously established formulation of the electric WGC. The sLWGC has also been extended to non-abelian gauge groups in [61]. Finally, let us comment on the WGC particle being an unstable resonance. Often in explicit examples, at many lattice sites the superextremal particles are unstable. This is no problem as long as the theory is weakly-coupled since in this case the decay width is narrow and unstable particles are just narrow resonances. For strongly-coupled theories the decay width is very broad, the sLWGC needs to be reformulated in terms of stable states.

## 6.7 Troubles from axionic models

Now that we have established the generalized electric WGC and that its internal consistency under dimensional reductions requires an infinite tower of super-extremal states with distinct charge in the higher-dimensional theory, we can proceed with resolving the problems related to the degenerate cases. These cases are relevant for constraining the models of large-field inflation discussed in chapter 4. In the literature two constructions exist to formulate an axionic WGC. One uses euclidean solutions of the Einstein-Maxwell action known as gravitational instantons [66]. This is an EFT point of view. The other uses string dualities to map the WGC imposed on a massless U(1) gauge theory (the usual setting) to an instanton charged under a zero-form [27]. Here we will discuss the stringy perspective.

### 6.7.1 An electric zero-form WGC

Establishing a clear formulation of the WGC for axions coupling to instantons has been the topic of a lot of debate in the literature. We discuss here the very clever approach of [27], utilizing T-duality to map axions and instantons to particles and U(1) gauge fields. This will involve a few technicalities of string theory. We then want to extend the zero-form WGC to multiple axions, namely, to a zero-form convex hull condition. Therefore, we first want to point out how one should define the concept of field displacement in a multi-dimensional periodic moduli space [91, 92].

#### How to define "field-range"?

In large-field models with a single axion the diameter of the field space is clearly defined. The field space is $S^1$ with diameter $2\pi f$. Thus, field range is a concept completely determined in terms of the axion decay constant. In multi-axion models the field space has more structure and it is not immediately clear how we can properly define the notion of a diameter. But even before doing that, we should define the fundamental domain in the multi-field case. Suppose we have $N$ periodic scalars $c_i$ with periodicity $\mathcal{Q}^i_j c_i \equiv \mathcal{Q}^i_j c_i + 2\pi$ induced by instanton effects. $\mathcal{Q}$ is an $N \times N$-matrix whose entries are all integers. As an aside, it is always possible to canonically normalize the kinetic term but at the expense of a non-integral charge matrix $\mathcal{Q}$. In the non-canonical $a_i$ basis the action is:

$$\mathcal{L} = \frac{1}{2} K^{ij} \partial a^i \partial a^j - \sum_{i=1}^{N} \Lambda_i^4 \Big(1 - \cos \mathcal{Q}^i_j a^j\Big) \tag{6.72}$$

The metric $K^{ij}$ on the moduli space $\mathcal{M}$ is Kähler in a SUSY context but here it is, besides being Riemannian, completely arbitrary. Note that the periodicities define hyperplanes in $\mathcal{M}$. Following Mcallister and collaborators, the multi-field fundamental domain is defined as the intersection of all periodic identifications [91]:

$$\mathcal{M}_F \equiv \mathcal{M}/\Gamma_1 \cap \cdots \cap \mathcal{M}/\Gamma_N \tag{6.73}$$

This field space has an invariant quantity which can serve as the definition of the diameter $\mathcal{D}$. This quantity is just the periodicity of the instanton-induced potential. However, below we will use a different, perturbative measure for field range, in terms of $\mathcal{Q}^T\mathcal{Q}$. More precisely, the eigenvalues of this matrix correspond to the square of the inverse decay constants since close to $c^i = 0$, where $c^i$ denote the canonical axion fields[13], it is equal to the mass matrix. This definition of field range coincides with the one for a single axion field. Thus we have:

$$\mathcal{Q}^T\mathcal{Q}\vec{c} = \lambda\vec{c} \Longleftrightarrow \lambda = \frac{1}{f^2} \tag{6.74}$$

It can be shown, using very similar arguments we will discuss below, that the definition of field range in terms of the diameter $\mathcal{D}$ is upper-bounded by the multi-axion WGC to be $\mathcal{D} < 2\pi$ [67].

[13]The concept of field range is only physically relevant for canonically normalized fields.

### Heuristics of the mapping

We won't go into all of the technical details, but rather describe the heuristics to arrive at the desired result of [27]. The starting point is type IIB string theory. In this theory we have the ten-dimensional RR two-form $C_2$ that couples to a $D1$-brane (D-string). We compactify this theory on a six-manifold and integrate the two-form over a two-cycle in the compact space. Specifically, we write $C_2 = \sum_{i=1}^{b^2} a_i \omega^i$ in some harmonic two-form basis and define the axions as:

$$a_i = \int_{\Sigma_j} C_2 \tag{6.75}$$

Note that these axion fields are non-canonical. The potential is generated via instanton effects. The instanton effects are in this case euclidean D1-branes wrapping the same two-cycles from which we generate the axions. The potential takes the familiar form:

$$\sum_{i=1}^{N} \Lambda_i^4 \Big(1 - \cos \mathcal{Q}^i_j a^j\Big) \tag{6.76}$$

At this level the charge matrix $\mathcal{Q}$ still has integer entries. Next, we diagonalize the kinetic-mixing matrix. Then the entries of the charge matrix $\mathcal{Q}$ become non-integer. The matrix $D$ that diagonalizes the kinetic-mixing matrix $K^{ij}$ is related to the charge matrix $\mathcal{Q}$ via:

$$\mathcal{Q} = \frac{1}{\sqrt{2}\pi g_s} D^{-1} \vec{q} \tag{6.77}$$

where $q^i_j \equiv \int_{\Sigma_j} \omega^i$ are the (integer) instanton charges, as can be seen from the definition of our non-canonical axions and the expansion of the RR two-form. Note that the instanton charge is "two-cycle" dependent. Now T-duality enters the game. We perform a compactification of one of the spatial directions along a circle of radius $R$. This setup is T-dual to type IIA in which one spatial direction is compactified on a circle of radius $\tilde{R} = \frac{\alpha'}{R}$. Under T-duality, our euclidean D1-brane is mapped to a euclidean D2-brane in the type IIA setup. The euclidean D2-brane wraps the same two-cycle as the D1-brane, giving us massive particles instead of dynamical instantons. What has happened to the RR two-form $C_2$? Interestingly, it has mapped to the type IIA three-form $C_3$ of which one component is along the circle of radius $\tilde{R}$. This is precisely a U(1) gauge field. Equivalently, we have "lifted" the $C_2$ axion to a one-form gauge field. Hence, it is T-duality that provides a mapping from zero-form gauge fields and instantons to one-form gauge fields an massive particles. The type IIB instanton action is related to the type IIA particle mass by:

$$m = \frac{S_{inst}}{2\pi \tilde{R}} \tag{6.78}$$

This is derived by demanding the instanton action to coincide with the action of a point particle of mass $m$ running in a loop of radius $\tilde{R}$.

However, there is an additional complication. We want to construct charged black hole solutions in the type IIA setting and then demand the existence of a super-extremal particle to arrive at the electric WGC. But to construct black hole solutions in type IIA we need an isotropic background. This might be obstructed by the Wilson line, which is the component of the type IIA three-form along the circle of radius $\tilde{R}$:

$$W = \int_{S^1} A_{(1)} \tag{6.79}$$

The problem is that this Wilson line in the type IIB setting might induce anisotropies in the type IIA background upon T-duality. However, we can smear out all anisotropies in the type IIA background by considering the limit $\tilde{R} \to \infty$, or equivalently, $R \to 0$. Now, this limit is equivalent to the strong coupling limit $g_s^A \to \infty$ of type IIA. This follows from the relation between the string couplings of type IIA and type IIB. By T-duality we have that $R\tilde{R} = \alpha'$ and therefore:

$$\frac{2\pi R}{(g_s^A)^2} = \frac{2\pi \tilde{R}}{(g_s^B)^2} \Longrightarrow g_s^A = \frac{\sqrt{\alpha'}}{R} g_s^B \tag{6.80}$$

We can also relate the Planck masses of the different theories:

$$M_{pl}^A = \frac{M_{pl}^B \sqrt{\alpha'}}{\tilde{R}} \tag{6.81}$$

To overcome this final subtlety, we look at the duality-web in figure 5.1 and we see that type IIA at strong string-coupling corresponds to M-theory. Luckily, the one-form gauge fields and massive particles are not mapped onto higher-rank gauge fields and extended objects in this setting. The only difference is that M-theory is an eleven-dimensional theory, implying that after compactification on a six-manifold we need to construct $5d$ extremal black hole solutions. Higher-dimensional Reissner-Nördstrom solutions of GR are very well-known. The couplings of M-theory are related to those of type IIA via:

$$\frac{2\pi r}{g_M^2} = \frac{1}{g^2} \tag{6.82}$$

Here, $r$ is the radius of the circle on which we compactify one of the M-theory dimensions and $g_M, g$ denote the gauge couplings in M-theory respectively type IIA. The type IIA gauge coupling is given by:

$$g = \frac{1}{(2\pi)^2 M_{pl}^A g_s^A \sqrt{\alpha'}} \tag{6.83}$$

The extremal charged $5d$ black hole solutions satisfy the extremality condition:

$$M_{5d} = g_M |\vec{n}| \sqrt{\frac{3}{2}} M_{pl}^{(5)} \tag{6.84}$$

Here $|\vec{n}|$ denotes the length of the charge vector. This requires some explanation. The T-duality map lifts the axions to U(1) gauge fields. Therefore, the $4d$ effective action in type IIA contains the term:

$$S \supset \int d^4x \sqrt{-\tilde{g}} \frac{1}{4g^2} K_{ij} F^{(i)}_{\alpha\beta} F^{(j)\alpha\beta} \tag{6.85}$$

T-duality is a transformation that does not act on the compact space and hence the moduli space metric is identical to the one in the $4d$ effective action derived from type IIB. Note that the spacetime metric does change, as indicated by the tilde. The M-theory action is basically the same except that it lives one dimension higher:

$$S \supset \int d^5x \sqrt{-\tilde{g_{5D}}} \frac{1}{4g_M^2} K_{ij} F^{(i)}_{\alpha\beta} F^{(j)\alpha\beta} \tag{6.86}$$

Thus we see that, for our $5d$ multi-charged black hole solutions, the length of the charge vectors $\vec{n}$ is measured with respect to the moduli space metric $K_{ij}$, $|\vec{n}|^2 = K_{ij}n^i n^j$. Since we know how the gauge couplings in the different theories are related we can derive the type IIA extremality bound. It reads [27]:

$$M_{4d} = g|\vec{n}|\sqrt{\frac{3}{2}} M_{pl}^{(A)} \tag{6.87}$$

Demanding the existence of a super-extremal particle we obtain the electric WGC. Via the relations 6.80 and 6.81 we can formulate the bound in the original type IIB setting of axions and instantons. Therefore, by ruling-out particular regions in the parameter space of M-theory with the $5d$ electric WGC, we rule out the corresponding portions of the type IIB parameter space related to axions and instantons.

### The conjecture

We first focus on the electric WGC for the axion fields and instantons. From the type IIA electric WGC 6.87 we can easily arrive at the single axion electric WGC. By using the relations between all parameters we find that the electric WGC demands the existence of an instanton whose action satisfies:

$$S_{inst} < \frac{M_{pl}^B}{f} \tag{6.88}$$

which is indeed the good old electric WGC we already intuitively arrived at via the loose generalized electric WGC. It is now interesting to see what form the convex-hull condition takes in the instanton-axion setting. How are the U(1) charges determined for the type IIA particles? Every particle has its corresponding instanton in the type IIB setting and every instanton charge depends on the two-cycle wrapped by the euclidean D1-brane. Thus, the charge of the particle with mass $m_i$ arising from instanton $k$ is: $Q_k^i = gq_k^i$.

The convex-hull condition in type IIA states that the charge-to-mass vectors $\vec{v}_k \equiv g\vec{q}_k \frac{M_{pl}^A}{m}$ must contain the ball in the space of charges as measured by the kinetic-mixing matric $K^{ij}$. Note that we do not refer to the *unit*-ball. This can be seen from the type IIA extremality condition 6.87. The condition is slightly relaxed due to the numerical prefactor of $\sqrt{\frac{3}{2}}$. We need to demand that the charge-to-mass vectors contain the ball of radius $\sqrt{K^{ij}n_i n_j} = \sqrt{\frac{2}{3}}$ so that type IIA extremal black holes can decay.

It is now easy to arrive at the type IIB convex-hull condition by substitution of all the type IIB parameters in the convex-hull condition for type IIA. It states that the convex-hull of $\frac{\vec{Q}_k}{S_{inst}}$ contains the ball of radius $\sqrt{\frac{2}{3}}$.

### 6.7.2 Evading the electric WGC

Before going into discussions of the consequences derived from the WGC for large-field inflation, which will be very dramatic, we anticipate some subtleties. Namely, it has been suggested that the electric WGC can be evaded by some simple, but clever tricks.

#### Loophole 1 - higher harmonics are not problematic

There is problem with the argument given above against single-axion inflation. It turns out that the higher instanton contributions to the potential can be suppressed *even* if the instanton action is extremely small, $S_{inst} \ll 1$ [86, 63][14]. This means that we can realize large-field inflation and at the same time be compatible with the WGC. This is known in the literature as the *small action loophole*. To see that the contributions of higher-instanton harmonics are negligible, we need to look at the instanton expansion in more detail. The loophole can be exploited when an axion descends from a gauge-field integrated over a cycle in the compact space, i.e. extra-natural inflation [94]. Suppose there is one extra dimension in the form of a circle of radius $R$ and that we integrate a five-dimensional one-form gauge field along this circle. Thus, the axion is given by:

$$\theta(x^\mu) = \oint_S d^5x A(x^\mu, x^5) \tag{6.89}$$

Assuming the $5D$ gauge theory is perturbative, the potential is:

$$V(\theta) = \frac{3(-1)^F}{4\pi^2} \frac{1}{(2\pi R)^4} \sum_{n\in\mathbb{Z}} c_n e^{-2\pi n R m_5} \mathrm{Re}[e^{\frac{in\theta}{f}}] \tag{6.90}$$

Here $F$ is the fermion-number operator. The instanton harmonics are suppressed due to the form of the coefficients $c_n$:

$$c_n(2\pi R m_5) = \frac{(2\pi R m_5)^2}{3n^3} + \frac{2\pi R m_5}{n^4} + \frac{1}{n^5} \tag{6.91}$$

A detailed calculation can be found in appendix A of [93]. The interesting point is that even when $m_5 = 0$ corresponding to $S_{inst} = 0$, which seems to be the worst case scenario from the naive point of view where $S_{inst} \ll 1$ signals a breakdown of the instanton expansion, higher terms are still suppressed due the $\frac{1}{n^5}$ prefactor.

#### Loophole 2 - parametric suppression

Loophole 1 demonstrates clearly that we can have large-field variations, although in a very specific model, while higher instanton-harmonics are not parametrically suppressed. Loophole 2 will demonstrate that it is even possible to achieve parametric suppression of the higher harmonics while at the same time generating an inflationary potential for the axion in any model [27, 67]. Loophole 2 arises if we consider more than one instanton. For simplicity we consider two instantons and a single axion. The corresponding action, for a non-canonical axion, is:

$$\mathcal{L} = \Lambda_1^4 e^{-S_1}\left(1 - \cos\frac{\phi}{f}\right) + \Lambda_2^4 e^{-S_2}\left(1 - \cos\frac{k\phi}{f}\right) \quad k \in \mathbb{Z} \tag{6.92}$$

[14] Appendix A of [93] contains some details of the relevant computation.

Suppose that instanton 2 with action $S_2$ satisfies the zero-form WGC so that $S_2 < \frac{M_{pl}}{f}k$. Consequently, instanton 1 should account for the large-field potential and dominate over the second instanton term ensuring perturbative control. The first requirement means $f > M_{pl}$ and $S_1 > 1$. The second requires, assuming that the UV-scales $\Lambda_1 \sim \Lambda_2$, the hierarchy $S_2 > S_1$. Note that loophole 2 can be evaded if the strong WGC is true, i.e. the super-extremal particle is demanded to be the lightest state of the spectrum. We know now that the strong WGC is violated in some examples. Hence, it is possible to utilize loophole 2 to generate large-field variations. However, in practice one encounters some severe difficulties [67].

**Evading the loopholes with the magnetic WGC**

We see that the electric WGC can impossibly constrain the field range to be sub-planckian. However, the magnetic WGC still forbids super-planckian decay constants in extra-natural inflation [86, 42]. The magnetic WGC states the existence of a UV cut off below the Planck scale. We then demand the size of the compactification manifold, i.e. the radius of the circle, to be *larger* than the inverse of the UV cut off $\Lambda^{-1}$ of the magnetic WGC $R > \Lambda^{-1} = R_{mon}$. This is a constraint on the axion decay constant since:

$$f^2 = \frac{1}{2\pi R g_5^2} = \frac{1}{(2\pi R g_4)^2} \quad \text{where} \quad \frac{1}{g_4^2} = \frac{2\pi R}{g_5^2} \tag{6.93}$$

Then a super-planckian decay constant $f > M_{pl}$ can be achieved if $g_4 \ll 1$ and $\frac{1}{R} \ll M_{pl}$, i.e. when the compactification scale is sub-planckian. However, such weak-coupling and such a compactification scale are forbidden by the magnetic WGC. The latter is forbidden by the argument that the magnetic monopole with the least charge is forbidden to be a black hole. Thus, consistency with the magnetic WGC demands:

$$1 < 2\pi R\Lambda < 2\pi R g_4 M_{pl} = 2\pi R \frac{M_{pl}}{2\pi R f} = \frac{M_{pl}}{f} \tag{6.94}$$

In this way the constraint $\frac{M_{pl}}{f} > 1$ is derived without considerations of the size of the instanton action and the small-action loophole can be circumvented.

## 6.8 Constraining inflationary models

We would now like to constrain several models of axion inflation via the electric WGC for axions. In the next section we will focus on constraints from the WGC for $\alpha$-attractors and more generally, pole-inflation, and in the next chapter we will make various attempts of constraining axion monodromy.

**1. The simplest argument against N-flation**

We first specialize to the case of extranatural N-flation because it allows for a simple argument to rule it out. Above we mentioned that the total charge of a black hole charged under multiple U(1)'s adds in quadrature. The extremality condition is:

$$Q_{tot} = Q\sqrt{\frac{1}{g_1} + \frac{1}{g_2} + \cdots + \frac{1}{g_N}} \leq \frac{M}{\sqrt{2}M_{pl}} \tag{6.95}$$

where for simplicity we have assumed that $Q_i \equiv Q$, i.e. the black hole has charge $Q$ under every U(1). An important requirement for the following argument to work is that we consider a magnetically charged black hole. From our discussion of *diagonal* N-flation in section 4.2.1, we know that the total field displacement is:

$$\Delta\Phi^2 = \sum_{i=1}^{N}(\Delta\phi_i)^2 = \sum_{i=1}^{N} f_i^2 \equiv f_{tot}^2 \tag{6.96}$$

If the axions arise from higher-dimensional gauge fields, such as in the extranatural inflation model, we can find an expression for the sum over all axion decay constants. Suppose that the N axions arise from the integration of N five-dimensional Maxwell fields along a circle of radius R. Then we would get:

$$f_{tot}^2 = \frac{1}{(2\pi R)^2}\left(\frac{1}{g_1^2} + \frac{1}{g_2^2} + \cdots + \frac{1}{g_n^2}\right) \tag{6.97}$$

We have seen this relation above but then for the simple case of a single U(1). Note the similar form between the total black hole charge and the total decay constant. We want the black hole to decay. This can happen in a variety of ways but we will discuss the simplest case [42]. It can emit a N-charged monopole with charge-vector $\vec{q} = (q, q, \ldots, q)$. By emitting this state the orientation of the charge vector of the black hole remains the same but its magnitude decreases. We now invoke the naturalness argument from above:

$$m_{mon} > \frac{q_{mon}^2}{g_{el}^2}\Lambda \Rightarrow m_{mon} > \Lambda q_{mon}^2 \sum_{i=1}^{N}\frac{1}{g_i^2} \tag{6.98}$$

where the arrow indicates the generalization of the naturalness requirement for a monopole charged under multiple U(1)'s. To demand that extremal black holes can decay, this monopole state should also obey the WGC bound:

$$m_{mon}^2 < q_{mon}^2 M_{pl}^2 \sum_{i=1}^{N}\frac{1}{g_i^2} \tag{6.99}$$

To circumvent the small-action loophole we demand that the radius of the circle from which we obtained the axions satisfies $2\pi R\Lambda > 1$. Using the definition of $f_{tot}^2$, the bounds on the monopole mass and the radius of the circle, one can show that:

$$\boxed{f_{tot} < \frac{M_{pl}}{q_{mon}}} \tag{6.100}$$

This shows that for the case of diagonal N-flation, super-planckian decay constants are forbidden by WGC. This argument can be generalized to the case where the black-hole emits an arbitrarily charged monopole and even further to "off-diagonal N-flation". This very similar to the argument above and we will not discuss that generalization here but refer to [42]. We have discussed axions arising from one-form gauge fields in five dimensions but this argument can also be generalized to higher-rank gauge fields in arbitrary dimensions.


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**2. Utilizing the zero-form conjecture**

The zero-form conjecture also forbids N-flation. This argument is very general but not robust against the small-action loophole. One considers the convex-hull condition in type IIB stating that the convex-hull generated by the instantonic charge-to-mass ratios $\frac{\mathcal{Q}_k}{S_{inst}}$ contains the ball of radius $\sqrt{\frac{2}{3}}$. This can be reformulated if we demand the following. We require the instanton action to satisfy $S^k_{inst} > \sqrt{\frac{3}{2}}$ for every $k$. Equivalently, this demands $e^{-S_{inst}} < e^{-\sqrt{\frac{3}{2}}}$, declaring a regime of perturbative control over the higher instanton harmonics. On the boundary of the perturbative regime we have $S^k_{inst} = \sqrt{\frac{3}{2}}$ for every $k$. In this case the convex-hull condition becomes the ordinary statement that the vectors $\mathcal{Q}_k$ must contain the unit-ball. This convex-hull condition can be formulated in a clever way so that ruling out N-flation becomes almost trivial. Recall that the convex-hull condition can be formulated as a subunitary weighted average. These are the conditions $\sum_k \alpha_k = 1$ and $\sum_k \alpha_k \mathcal{Q}_k = \rho\vec{c}$ where $\vec{c}$ is a unit vector and $\rho > 1$. What is new is that it is claimed that $\rho$ and $\alpha$ are unique numbers [27].

To investigate the validity of N-flation we consider the eigenvalues of the field space metric $\mathcal{Q}^T\mathcal{Q}\vec{c} = \lambda\vec{c}$. The eigenvalues equal the inverse axion decay constants squared: $\lambda_k = \frac{M^2_{pl}}{f^2_k}$. From the eigenvalue equation we find that $\lambda = \mathcal{Q}^T\mathcal{Q} = \sum_k \tilde{\alpha}^2_k$ where $\tilde{\alpha}_k \equiv \mathcal{Q}^i_k c^i$. We can define coefficients $\alpha_j$ that normalize the $\tilde{\alpha}_k$'s:

$$\sum_k \alpha_k = \frac{\sum_k \tilde{\alpha}^2_k}{\sum_j \alpha_j} = 1 \tag{6.101}$$

From the eigenvalue equation for the field space metric we obtain:

$$\sum_k \tilde{\alpha}_k \mathcal{Q}_k = \rho\vec{c} \quad \text{with} \quad \rho \equiv \frac{\sum_k \tilde{\alpha}^2_k}{\sum_k \tilde{\alpha}_k} > 1 \tag{6.102}$$

Thus $\lambda = \sum_k \tilde{\alpha}^2_k > 1$ and therefore:

$$f_k < \frac{M_{pl}}{\sqrt{\lambda_k}} \quad \forall k \tag{6.103}$$

excluding super-planckian field variations.

**3. Troubles for pole inflation**

We have seen that pole inflation or $\alpha$-attractors cannot be ruled out by the swampland distance conjecture. However, the WGC can, and this is discussed in great detail in [95]. For single-field models this boils down to the following. In disk variables $\Phi, \bar{\Phi}$ the Kähler metric is:

$$K_{\Phi\bar{\Phi}} = \frac{3\alpha}{(1-\Phi\bar{\Phi})^2} \tag{6.104}$$

The radial mode is $\Phi\bar{\Phi} \equiv \phi^2$ and its axionic partner has decay constant:

$$f^2(\phi) = \frac{6\alpha\phi^2}{(1-\phi^2)^2} \tag{6.105}$$

It is then claimed that the slow-roll conditions imply $f^2(\phi) \gtrsim \mathcal{O}(1)$ violating the zero-form WGC. In [95], in the spirit of N-flation, an attempt was made to relax the WGC constraints via the assistance of additional scalar fields. Unfortunately, the situation does not improve with respect to the single field case.


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# Chapter 7

# The swampland emerges

It has become clear that the string-swampland is an area of physics dominated by speculation. Therefore, it would be worth the effort to try to physically motivate various ideas and place the swampland conjectures on a more firm footing. The first step in this direction was already taken by Ooguri and Vafa in 2006 in the context of the distance conjecture [39]. They suggested that the appearance of infinite distances in field space is a consequence of falsely integrating-out infinitely many states by showing that it can never be generated via integrating-out a single particle.

Consider the simple scalar field theory:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\alpha)^2 + \frac{1}{2}m^2(\phi)\alpha^2 \tag{7.1}$$

where $\phi$ is a modulus which controls the mass of the other scalar field $\alpha$. Assume that the theory 7.1 has a cut off $\Lambda \ll m(\phi)$ so that we can integrate-out the field $\alpha$. We may assume that that the vev of $\phi$ is slightly perturbed $<\phi> + \delta\phi$ and expand the mass term around this perturbation generating several interaction terms:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\alpha)^2 + \frac{1}{2}m^2(\langle\phi\rangle)\alpha^2 + m(\langle\phi\rangle)\frac{\partial m}{\partial\phi}\mid_{\phi=\langle\phi\rangle}\delta\phi\alpha^2 + \dots \tag{7.2}$$

where the dots indicate higher n-point interactions. We assume the theory is weakly-coupled so $m(\langle\phi\rangle)\frac{\partial m}{\partial\phi}\mid_{\langle\phi\rangle}\ll 1$. The cubic interaction introduces loop corrections to the modulus propagator and this causes wavefunction renormalization. We compute:

$$i\mathcal{M} = \frac{(-i)^2}{2}\big(m\partial_\phi m\big)^2\int\frac{d^4k}{(2\pi)^4}\frac{i^2}{\big((p-k)^2 - m^2 + i\epsilon\big)\big(k^2 - m^2 + i\epsilon\big)} \tag{7.3}$$

The factor of $\frac{1}{2}$ is a symmetry factor. Imposing the standard renormalization conditions following from the Källen-Lehman representation, i.e. the physical mass of $\alpha$ is defined as the simple pole in the propagator at $k^2 = m^2$ whose residue is the wave-function renormalization factor $\delta Z$, we find that the modulus kinetic term receives the following quantum correction:

$$\delta Z = \Big(\Big(\frac{\partial m}{\partial\phi}\Big)^2\frac{1}{8\pi^2}\big(\frac{2\pi}{3\sqrt{3}} - 1\big)\Big) \tag{7.4}$$

We now calculate the quantum correction to the proper distance between two points in field space:

$$d(\phi_1,\phi_2) = C\int_{\phi_1}^{\phi_2} d\phi\Big(\frac{\partial m}{\partial\phi}\Big) = C\big(m(\phi_2) - m(\phi_1)\big) \tag{7.5}$$

where $C$ is the numerical factor of the quantum correction. Suppose now that the field-space point $\phi_1$ is such that $m(\phi_1) = 0$ and the effective description breaks down. Assuming the classical contribution generates convergent proper distances, this does not generate an infinite distance singularity since $m(\phi_2)$ is finite. Thus, integrating-out a single particle does not generate infinite distances in field space. One can do a similar calculation for a Dirac fermion yielding the same conclusion[1].

It is now very natural to suggests that integrating-out many states will do the job. To treat this situation in EFT we must have a finite number of particles, i.e. we need to impose a UV cut off $\Lambda_{UV}$ somewhere. In the next section we argue that this scale coincides with Dvali's species scale and this scale also coincides with the scale at which gauge theories become strongly coupled. Thus we have $\Lambda_{UV} \sim \Lambda_{species} \sim \Lambda_{Landau}$ where $\Lambda_{species}$ and $\Lambda_{Landau}$ indicate where gravity respectively gauge forces become strong.

## 7.1 The species-scale and integrating-out towers

The species bound was originally motivated as a possible solution to the hierarchy problem. The species bound states that in a theory containing a large number of particle species, the scale at which gravity becomes strongly-coupled is lowered by a factor $\frac{1}{\sqrt{N}}$. Before we state it we first mention the bound on the number of species because it is closely related to the species bound.

**Species bound on the Planck mass** [83]
*Consider a discrete gauge theory, with discrete gauge group $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 = \mathbb{Z}_2^N$, coupled to a theory of gravity. Suppose we have N-species sectors $\Phi_{j_k}, j = 1, \ldots N, k \in \mathbb{Z}$ each containing at least one particle carrying unit $\mathbb{Z}_2$-charge and each sector transforming separately under a $\mathbb{Z}_2$, i.e:*

$$\begin{gathered}\Phi_{i_k} \to -\Phi_{i_k} \quad \textit{with} \quad i = 0, \ldots, N-1 \\ \Phi_{j_k} \to \Phi_{j_k} \\ \textit{for } i \neq j \textit{ and } k \in \mathbb{Z}\end{gathered}$$

Denote the mass-scale associated to the heaviest sector with $\Lambda$. In this case the Planck scale obeys the bound:

$$N\Lambda^2 \lesssim M_{pl}^2 \tag{7.6}$$

To be more precise, the bound above is a large-N approximation and we are neglecting logarithmic contributions. If $N \sim 10^{32}$, which could be that many copies of the Standard Model species, the Planck scale is lowered to the electroweak scale of $\mathcal{O}(\text{TeV})$, solving the hierarchy problem. It is illuminating to go through the proof of the above bound. Consider the initial data described above. We can construct macroscopic[2] black holes carrying discrete charge. From each sector we throw one

[1] The fermionic calculation is very interesting. it shows that the field space metric may diverge yet the proper distance remains always finite.

[2] This essentially means a black hole that can Hawking-evaporate, i.e. its mass satisfies $M > M_{pl}$

particle carrying unit $\mathbb{Z}_2$-charge into the black hole. Hence, the net discrete charge of the black hole is $N$. Consider now its Hawking-evaporation. As long as the Hawking temperature satisfies $T_H \ll \Lambda$ the black hole won't discharge since the emission of discrete charged quanta is exponentially suppressed by the factor $e^{-\frac{T_H}{\Lambda}}$. After some time we have $T_H \sim \Lambda$ and the black hole is able to discharge. By conservation of charge, the black hole has to return the initial information (charge) stored in the $N$ particles we threw in. The black hole mass formula tells us that at this point in time its mass is:

$$M_{BH} \sim \frac{M_{pl}^2}{\Lambda} \tag{7.7}$$

By conservation of energy, the amount of particles the black hole can emit at this point is:

$$n_{max} \sim \frac{M_{pl}^2}{\Lambda^2} \tag{7.8}$$

These particles must contain the information of all the initial $\mathbb{Z}_2^N$-charge, so we have that $n_{max} \sim N$. Therefore, we see that $M_{pl}$ is upper-bounded by:

$$N\Lambda^2 \lesssim M_{pl}^2 \tag{7.9}$$

Having established this bound we can proceed to discuss Dvali's species bound. The species bound is a new *fundamental cut off* in the effective theory of $N$ species-sectors, at which it is argued that gravity becomes strongly-coupled. There is a strong non-perturbative argument supporting this claim. The argument is very similar to the argument above, employing black hole physics. Consider a black hole of initial size $R_{BH} \sim \Lambda_G^{-1}$, initial mass $M_{BH} \sim \frac{M_{pl}^2}{\Lambda_G}$ and Hawking temperature $T_H \sim \Lambda_G$. $\Lambda_G$ is the same scale as $\Lambda$ in 7.9. The subscript $G$ obviously stands for "gravitational". Thus, we can emit particles from the $N$-species. We assume that the black hole does not carry the discrete charge of the $N$-species because this is not necessary for the argument. Its evaporation-rate is:

$$\frac{dM_{BH}}{dt} = -NT_H^2 \tag{7.10}$$

We can integrate this equation and express the Hawking-temperature in terms of the black hole mass:

$$T_H = \frac{M_{pl}^2}{M_{BH}} \tag{7.11}$$

so that we get:

$$t_{BH} \approx \frac{1}{N}\int_{M=0}^{M=M_{BH}} dM_{BH}\frac{M_{BH}^2}{M_{pl}^4} \tag{7.12}$$

This integral is trivial and we get:

$$t_{BH} \approx \frac{M_{pl}^2}{N\Lambda_G^3} \tag{7.13}$$

up to a numerical factor of $\mathcal{O}(1)$. Now, a point that is not always emphasized in the literature is that if we assume that the bound 7.9 is *saturated* we get:

$$t_{BH} \approx \frac{1}{\Lambda_G} \tag{7.14}$$

Thus, we arrive at the conclusion that in a theory containing $N$ different species-sectors, the black hole evaporation time is of the same order as its size. This can impossibly be a (semi)-classical black hole. One reason is that such a black hole cannot have a well-defined Hawking temperature because $T_H$ itself changes on time scales $\Lambda_G^{-1}$. Another reason is that the evaporation time is extremely short, even shorter than the light-crossing time of the black hole. Therefore, the conclusion is that at the scale $\Lambda_G$, which coincides with the mass scale of the heaviest species when the bound 7.9 is saturated, new gravitational dynamics has to enter the theory. This means the fundamental gravitational cut off is not the Planck scale but the species scale:

$$\Lambda_G \lesssim \frac{M_{pl}}{\sqrt{N}} \tag{7.15}$$

Now that we have argued that this cut off is related to new gravitational dynamics, we want to mention that it also is a perturbative feature of QFT. When one considers loop-corrections to the graviton propagator in which the many species run in the loop, the perturbative expansion breaks down at a premature scale. This scale is precisely the species scale $\Lambda_G \sim \frac{M_{pl}}{\sqrt{N}}$, see section 3.2 of [84] for the argument. One can arrive at the same gravitational cut off (but a priori unrelated) in $\mathbb{Z}_N$ gauge theories with large periodicity (i.e. large $N$).

We now want to relate this discussion to the (sub)lattice WGC. We start with the most simple situation in which we can apply the LWGC: a four-dimensional U(1) gauge theory coupled to gravity with gauge coupling $e$. We can compute the one-loop beta function with loop-corrections induced by the infinite tower of WGC particles demanded by the LWGC. We assume the WGC tower consists of massive charged scalar particles, i.e. we consider scalar QED:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^\star D^\mu\phi \tag{7.16}$$

We denote the U(1) Landau-pole scale by $\Lambda_{UV}$. The beta function easily generalizes by summing over the WGC particles with U(1) charge $q_i$ and mass $m_i$:

$$\frac{1}{e^2(\Lambda_{UV})} = \frac{1}{e^2} - \sum_i \frac{b_i}{8\pi^2}q_i^2 \ln\frac{\Lambda_{UV}}{m_i} \tag{7.17}$$

with $b_i$ some beta function coefficient of $\mathcal{O}(1)$ and $e$ the low-energy gauge coupling. We are now going to make a parametric estimate. We will approximate all numerical factors and logarithms as $\mathcal{O}(1)$ factors[3]. Suppose we consider the limit of $\Lambda_{UV}$ in which the UV gauge coupling $e(\Lambda_{UV})$ goes to zero, i.e. we consider:

$$\frac{1}{e^2} \sim \sum_i^{\mathcal{Q}} q_i^2 \sim \mathcal{Q}^3 \tag{7.18}$$

Note that we sum to some state in the tower that has maximal U(1) charge $\mathcal{Q}$. This maximal charge is determined by the fact that a theory obeying the (sub)LWGC must have a UV cut off because in the limit of zero gauge coupling the infinite tower

[3]This is dangerous since one can have large logarithms in these kind of calculations. However, it is argued that this is not problematic in appendix A of [61]

becomes massless and every QFT based on a finite number of particles breaks down. Therefore $\mathcal{Q}$ is determined to by:

$$\mathcal{Q} \sim \frac{\Lambda_{UV}}{eM_{pl}} \tag{7.19}$$

Combining the last two expressions gives:

$$\Lambda_{UV} \sim e^{\frac{1}{3}} M_{pl} \tag{7.20}$$

At this scale, the U(1) gauge theory becomes strongly-coupled. Now, lets compare this with the effect of a tower of LWGC particles on the species-scale. To be extremely clear, the WGC particles are forming the $N$ different species-sectors. This is of course an assumption. It could very well be that there are many neutral species to which the WGC does not apply. These particles enter into the species bound but do not contribute to the one-loop beta function and hence invalidate the argument. Making this assumption, the number of species below the species cut off $\Lambda_G$ satisfies (due to the LWGC):

$$N(\Lambda_G) \gtrsim \frac{\Lambda_G}{eM_{pl}} \tag{7.21}$$

This implies the species bound:

$$M_{pl}^2 \gtrsim \frac{\Lambda_G^3}{eM_{pl}} \Longleftrightarrow \Lambda_G \lesssim e^{\frac{1}{3}} M_{pl} \tag{7.22}$$

which is exactly the same strong-coupling scale as for the U(1) gauge theory of 7.20. We come to the conclusion that abelian gauge theories coupled to gravity satisfying the (sub)LWGC have the property that gravitational -and gauge interactions become strongly-coupled at parametrically the same scale. This is a property of four-dimensional abelian gauge theories. In higher-dimensional -and non-abelian gauge theories this is no longer true. However, a similar statement survives in these cases, which is that the calculations of perturbative loop corrections to the photon -and graviton propagator break down at parametrically the same scale if the theory satisfies the LWGC.

## 7.2 The distance conjecture emerges

In chapter 5 we mentioned the conjecture that there do not exist coupling constants[4] in a theory of quantum gravity. All coupling constants are dynamical, i.e. they are functions of the moduli fields. We can therefore consider these coupling functions from a perturbative QFT viewpoint and consider loop-corrections to the moduli propagators from the infinite tower of states that appear at large distances in field space according to the distance conjecture. When flowing into the infra-red, by integrating-out the infinite tower of states, and under the assumption motivated by the LWGC that the loop expansions of gravity and gauge theory break down at parametrically the same scale, it can be shown that the quantum corrected field space metric generated infinite distances [96]. In fact, this field space metric has

[4] That is, all couplings are functions of scalar fields.

the same field dependence as the one for $\alpha$-attractors. This suggests that cosmological attractors, and inflation in general, is an emergent phenomenon, emerging by integrating-out dynamical fields in the ultraviolet.

We consider some trajectory in moduli space towards a point $P$ where an infinite tower of states appears. This tower of states may be fermions or scalars. We will assume that they are scalars but the result for fermions is exactly the same. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}K(\phi)(\partial\phi)^2 + \sum_i \left(\frac{1}{2}(\partial\alpha_i)^2 + m_i(\langle\phi\rangle)\partial_\phi m\alpha_i^2\right) \tag{7.23}$$

Note that the kinetic term is non-canonical. This follows from assuming that the tower becomes light linearly in the modulus field value $\phi$ as $\phi \to 0$ [5]. Namely, this requires a field redefinition. We want to consider the high energy behaviour of the quantum corrections $p^2 \gg m_i^2$ for all $i$ because this allows us to extract the strong coupling scale $\Lambda$, i.e. the scale where the loop expansion breaks down, and so we can make contact with the previous section. Without going to all the details we just state the resulting quantum correction to the $\phi$-propagator here:

$$\delta Z(p^2 \gg m_i^2) \sim \sum_{i|m_i<p} \frac{1}{K(\phi_0)}(\partial_\phi m_i)^2 p^{d-4} \tag{7.24}$$

where $\phi_0$ is the vev of $\phi$. The strong coupling scale $\Lambda$ is defined by the value of $p$ such that $\lambda_\phi(p) \sim 1$ where:

$$\lambda_\phi(p) \equiv \sum_{i|m_i<p} \frac{1}{K(\phi_0)}(\partial_\phi m_i)^2 \tag{7.25}$$

Note that we sum over only those scalars whose masses $m_i < p$. This effectively imposes the cut off on the theory. From 7.25 it is clear that the strong coupling scale depends on the vev of the modulus $\Lambda = \Lambda(\phi_0)$. Indeed, $\langle\phi\rangle$ determines the interaction strength between $\alpha$ and $\delta\phi$ in 7.23. Similary, the species bound determines when gravity becomes strongly coupled and we can infer the scale at which loop corrections to the graviton propagator diverge. Thus, it defines $\lambda_G(p) \sim 1$. In $d$-dimensions the species bound tells us that:

$$\Lambda_G^{d-2} = \frac{M_{pl}^{d-2}}{N(\Lambda_G)} \tag{7.26}$$

We can evaluate $\lambda_\phi$ at the species scale $\Lambda_G$. This provides the key relation[6]:

$$\lambda_\phi(p = \Lambda_G) = \frac{M_{pl}^d}{K(\phi_0)\Lambda_G}\frac{1}{N(\Lambda_G)} \sum_{i|m_i<\Lambda_G} (\partial_\phi m_i)^2 \sim \frac{M_{pl}^{d-2}}{K(\phi_0)\Lambda_G^2}\frac{\Lambda_G^2}{\phi^2} \tag{7.27}$$

[5] This also follows from the cut off scale $\Lambda_{UV}$ being a quantum gravity cut off and we expect exotic new physics there. But we do not impose a cut off on the theory a priori, instead we try to extract this strong coupling scale.

[6] The approximation assumes a particular structure of the tower of states, namely, that most particles have masses near the cut off $\Lambda_G$.

Now, according to the previous section, it seems natural to assume that all physics becomes strongly coupled at parametrically the same scale. This would imply we can set $\lambda_\phi(\Lambda_G) \sim 1$ to find that:

$$K(\phi_0) \sim \frac{M_{pl}^{d-2}}{\phi^2} \tag{7.28}$$

This moduli space metric generates logarithmically divergent proper distances, precisely as required by the distance conjecture. Furthermore, the masses of the states in the tower become exponentially fast light as a function of the geodesic distance. Note that it also shows that the modulus field interacts gravitationally with particles in the tower.

Note that this also reproduces the same structure as the field space metric of $\alpha$-attractors, agreeing with the idea that inflation emerges as a phenomenon at the boundary of moduli space.

# Chapter 8

# The Kaloper-Sorbo-Lawrence mechanism

Recently, Kaloper, Sorbo and Lawrence (KSL) developed in a series of papers a field theory formulation of large-field inflation that is natural and consistent with the non-existence of global symmetries in quantum gravity [97, 98, 99, 69, 100]. Based on earlier work in the context of relaxation of the cosmological constant problem and the strong-CP problem in QCD [101, 102], they considered the mixing of a topological four-form with an axion. The action they proposed was [99]:

$$S_{KSL} = \int d^4x\sqrt{-g}\Big(\frac{M_{pl}^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{48}F_{\mu\alpha\beta\gamma}^2 + \frac{m}{24}\phi\frac{\epsilon^{\mu\alpha\beta\gamma}}{\sqrt{-g}}F_{\mu\alpha\beta\gamma}\Big) + \\ + \frac{1}{6}\int d^4x\sqrt{-g}\nabla_\mu\Big(F^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma} - m\phi\frac{\epsilon^{\mu\alpha\beta\gamma}}{\sqrt{-g}}A_{\alpha\beta\gamma}\Big) + S_{branes}$$

Here $\nabla$ denotes the covariant derivative on the curved manifold, $\phi$ denotes a compact scalar field $\phi \equiv \phi + 2\pi f$ and hence is an axion field, $F$ the topological four-form with $F = dA = 4\partial_{[\mu}A_{\alpha\beta\gamma]}$ the three-form gauge potential, $\epsilon^{\mu\alpha\beta\gamma}$ the totally anti-symmetric tensor-density and $m$ is at this point just a parameter which will turn out the be the mass of the axion. The KSL action is gauge invariant under $A \to A + d\Omega$ where $\Omega$ is a two-form, up to boundary terms. Note that the periodicity $f$ of the axion is at this level undetermined. However, it will turn out to be constrained in an interesting way.

The structure of the KSL action arises in compactifications on products of tori of eleven-dimensional supergravity which is the low-energy limit of M-theory. In this context, the parameter $m$ has an additional interesting interpretation. It can be explicitly shown that $m$ is related to the internal magnetic four-form flux flowing inside the compact extra dimensions. In the presence of membranes, the four-form flux can fluctuate due to the nucleation of membranes. This induces "jumps" of the form $\Delta m = e$ with $e$ the membrane charge, implying that in the 4D theory the axion mass can fluctuate. Now, fluxes are quantized in string theory via a generalized Dirac quantization condition. This implies that the mass of the axion is also quantized[1].

We will refer to the first line of the action as the *bulk* action which describes the

[1] Note that this also ameliorates the problem of fine-tuning because now the inflaton mass is a landscape parameter.

local dynamics. The terms on the second line are boundary terms for the three-form gauge field $A$, known as a *Gibbons-Hawking-York* terms. The boundary terms are not very essential for the discussion in this chapter. In the case of gravity, it is well-known that one has to add such a boundary term otherwise the variation of the Einstein-Hilbert action will give the Einstein field equation plus a non-vanishing boundary term (if the spacetime has a boundary). Similarly, one has to add such a term for the four-form in order for the field-momentum conjugate to the three-form gauge field $A$ to be fixed at the boundary. When we include membranes, as we have done above, this boundary term will give the correct variation in the four-form flux across the membrane. The membrane action contains a term:

$$S_{brane} = \frac{e}{6}\int_W d^3\sigma\sqrt{h}\epsilon^{abc}\partial_a x^\mu \partial_b x^\nu \partial_c x^\lambda A_{\mu\nu\lambda} \tag{8.1}$$

Here we integrate over the world-volume $W$ of the membrane with $\sigma^a, a = 0, 1, 2$ the coordinates on W, $e$ denotes the membrane charge and $h$ the induced metric on the world-volume. However it is somewhat more enlightening to write the membrane action as:

$$S_{brane} = \frac{ne}{6}\int d^4x\sqrt{-g}A_{\mu\nu\lambda}J^{\mu\nu\lambda} + T_2^{(n)}\int_{\Sigma_3} dV_3 \tag{8.2}$$

The external source $J^{\mu\nu\lambda}$ is localized and conserved. This is the action of a bound state of membranes with total charge $ne$ and tension $T_2^{(n)}$. The second term is the brane self-action. The KSL model is a $4d$ effective theory and therefore these membranes are three-dimensional objects occupying some spatial region. Membranes in an effective field theory are different from membranes appearing in a string theoretic UV completion of the KSL model since in string theory such objects appear naturally (e.g. D-branes). Such a membrane is charged under some gauge field that upon compactification becomes the $4d$ three-form gauge field $A$.

As we will see very explicitly, the mass for the axion is generated via this mixing but let us mention here the physical reason behind this [98, 97]. Suppose there was no coupling between the axion and the four-form, i.e. $m = 0$, and consider the local dynamics of the three-form gauge field $A$. Varying the bulk action with respect to $A$ gives:

$$\nabla_\mu\left(\sqrt{-g}F^{\mu\alpha\beta\gamma}\right) = 0 \tag{8.3}$$

Thus, the field strength is locally constant and by anti-symmetry proportional to the Levi-Civita tensor:

$$F^{\mu\alpha\beta\gamma} = F_0\epsilon^{\mu\alpha\beta\gamma} \tag{8.4}$$

At this point $F_0 \in \mathbb{R}$ can take any value. However, following Polchinski and Bousso, we will argue that it is quantized [102]. Now, the variation of the membrane action with respect to $A$ shows that the presence of membranes induces a "jump" in the value of the four-form field strength across the membranes[2]:

$$\Delta F^{\mu\alpha\beta\gamma} = \sqrt{-g}\epsilon^{\mu\alpha\beta\gamma} \tag{8.5}$$

[2]The localized source $J^{\mu\nu\lambda}(x)$ is:

$$J^{\mu\nu\lambda} = \frac{e}{6}\int_W d^3\sigma\sqrt{h}\epsilon^{abc}\delta^{(4)}(x-\sigma)\partial_a x^\mu \partial_b x^\nu \partial_c x^\lambda$$

Suppose the action was $\mathcal{L} = F^{\mu\alpha\beta\gamma}F_{\mu\alpha\beta\gamma} + A_{\mu\nu\lambda}J^{\mu\nu\lambda}$. If we assume the 2-brane is flat and static, i.e. $x^\mu = \sigma^\mu$, the equation of motion tells us that $\partial_\mu F^{\mu\alpha\beta\gamma} = -\frac{e}{6}\delta(z)\epsilon^{\alpha\beta\gamma z}$. Hence, we see that

Equivalently, $F_0$ "jumps" across the membrane:

$$\Delta F_0 = \frac{e}{Z} \tag{8.6}$$

where $Z$ is some internal volume factor. It is important to note that the quantization of $F_0$ does not refer to these "jumps". When $m \neq 0$, the axion and the four-form mix and we note that the four-form is no longer locally constant. Varying the bulk action with respect to the three-form now gives:

$$\nabla_\mu F^{\mu\alpha\beta\gamma} = m\epsilon^{\mu\alpha\beta\gamma}\nabla_\mu \phi \tag{8.7}$$

demonstrating the four-form varies from point to point. The axion experiences this four-form background as local inertia. This local inertia is equivalent to a mass-term for the axion in the action[3].

What about quantum corrections? The mass of the inflaton is protected against radiative corrections due to a *discrete gauge* shift-symmetry. At the level of the action above this is not obvious. In fact, the action has a *global* shift-symmetry $\phi \to \phi + const.$ up to a total derivative. However, as we will see below, we can understand the global shift-symmetry as intrinsically gauged, due to an a observation by Dvali [101]. In some cases, global -and gauge symmetries can be physically equivalent. Because the symmetry is local, and not global, the KSL model is consistent with the conjecture that quantum gravity breaks all continuous global symmetries and hence might have a consistent UV completion in a full theory of quantum gravity. Furthermore, we also do not have to require the axion decay constant $f$ to be super-planckian in order to achieve super-planckian field variations[4]. This is consistent with the constraints of the WGC[5]. The symmetry is discrete due to the quantization of four-form flux.

The potential generated via axion-4-form mixing actually has an additional feature. In was realized in [99] that this potential generated *monodromy* for the axion and hence the model could be understood as a field theory formulation of axion monodromy. Axion monodromy is a particular realization of large-field inflation in string theory where monodromy is employed to yield super-planckian field variations while having at the same time $f < M_{pl}$. Therefore, we can think of axion monodromy as a consistent UV completion of the KSL model. This "monodromy feature" of KSL is a consequence of the quantization of four-form flux. However, at a more fundamental level, the quantization of flux depends on the topology of the gauge group, in particular whether the gauge group is a compact -or non-compact manifold. Only if it is compact, flux will be quantized. Whether the gauge group is compact or non-compact depends on the UV completion but if we follow the patterns in string theory the gauge group seems to be always compact and hence flux is always quantized [73, 82].

membranes localized at $z = 0$ separate two vacua with constant values of $F_0$ differing by an integer multipe of $e$.

[3]This must be the case due to Lorentz invariance.

[4]We will see in section 8.1.

[5]There are actually a lot of subtleties due to the WGC related to the nucleation of membranes. We will discuss this in section 8.3.

## 8.1 Local shifts and quantization of fluxes

In this section we aim to explain why we can think of the global shift symmetry of the KSL action as a gauge symmetry. Above, we formulated the KSL action as an axion-four-form theory. The KSL action can be reformulated as a renormalizable massive three-form gauge theory. In fact, this theory is *dual* to the axion-four-form theory in the sense that after a chain of field redefinitions we obtain the other theory. Consider the action for a free axion $\phi$ in four dimensions:

$$\mathcal{L} = (\partial\phi)^2 \tag{8.8}$$

Obviously, this action as an exact continuous global shift-symmetry $\phi \to \phi + c$ with $c \in \mathbb{R}$. Furthermore, there is only one propagating degree of freedom. The point Dvali emphasizes is that this global symmetry can be understood as a gauge symmetry [101]. To see why, we want to give the axion a mass while at the same time preserving the shift-symmetry and the number of *propagating* degrees of freedom. The latter requirement forms the non-trivial part of this discussion. It is well-known that bringing the theory into the Higgs phase gauges the shift-symmetry. Adding a spin-1 gauge vector field $A_\mu$ to the above action, we obtain:

$$\mathcal{L} = (\partial_\mu\phi - mA_\mu)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.9}$$

Indeed, the shift-symmetry is gauged:

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu\alpha(x) \\ \phi &\to \phi + m\alpha(x) \end{aligned}$$

However, we have increased the number of propagating degrees of freedom by two (the transverse polarizations of the vector field). This obstructs the interpretation of viewing the parameter $m$ as a mass for $\phi$. What we have achieved above is giving the vector a mass in a gauge invariant way, instead of the axion. To see where things go wrong, note the role played by $\phi$. Above $\phi$ is Stüeckelberg field for the gauge field $A_\mu$, meaning that it restores the original gauge invariance from before spontaneous symmetry breaking. What we want is to add a Stüeckelberg field for $\phi$ *itself* that does not add propagating degrees of freedom.

The punchline is that, in four dimensions, this demands the Stüeckelberg field to be a three-form gauge field $A_{\alpha\beta\gamma}$. A three-form gauge field does not have propagating degrees of freedom in four dimensions due to its gauge invariance and anti-symmetry. Thus, we will Higgs the gauge three-form but it is not the three-form that gets massive but the axion. The discussion proceeds most easily in terms of a two-form $b_{\alpha\beta}$. A two-form is Hodge-dual to a scalar in four dimensions[6], so by duality we can translate the analysis to the axion. We consider the action of a free two-form $b_{\alpha\beta}$:

$$\mathcal{L} = m^2 h_{\alpha\beta\gamma}h^{\alpha\beta\gamma} \tag{8.10}$$

[6]Hodge duality works at the level of field strengths. Thus, we do not mean that $\phi$ is dual to $b_{\alpha\beta}$ but that $d\phi = \partial_\mu\phi$ is dual to $db$. Indeed, a one-form field strength is dual to a two-form field strength in $d = 4$.

where $h = db = 3\partial_{[\alpha}b_{\beta\gamma]}$ is the field strength associated to $b$[7]. This is just the two-form generalization of the free axion theory. We have the global shift-symmetry $b_{\alpha\beta} \to b_{\alpha\beta} + \Omega_{\alpha\beta}$ where $\Omega_{\alpha\beta}$ is a constant two-form. The goal is to gauge this global shift-symmetry of $b$. This is achieved by coupling the three-form gauge field $A_{\alpha\beta\gamma}$ to $b_{\alpha\beta}$, realizing that $A_{\alpha\beta\gamma}$ does not add any extra propagating degrees of freedom[8]:

$$\mathcal{L} = -\frac{m^2}{12}(h_{\alpha\beta\gamma} - A_{\alpha\beta\gamma})^2 - \frac{1}{48}F_{\mu\alpha\beta\gamma}F^{\mu\alpha\beta\gamma} \tag{8.11}$$

where $F_{\mu\alpha\beta\gamma} = 4\partial_{[\mu}A_{\alpha\beta\gamma]}$. Note that this Lagrangian describes the Higgs phase of $A_{\alpha\beta\gamma}$. Furthermore, this Lagrangian has the gauge symmetries:

$$b_{\alpha\beta} \to b_{\alpha\beta} + \Omega_{\alpha\beta}(x) \tag{8.12}$$

$$A_{\alpha\beta\gamma} \to A_{\alpha\beta\gamma} - 3\partial_{[\alpha}\Omega_{\beta\gamma]} \tag{8.13}$$

Thus, we have gauged the global shift-symmetry of $b$ via the three-form gauge field $A$ while having not added any propagating degrees of freedom. Due to the latter, while we have "Higgsed" the three-form $A$, the *two-form* $b$ has become massive.

Via the two-form-scalar duality, we expect that the axion obtains a mass in the same way, while its shift-symmetry can be understood as intrinsically gauge. Indeed, this will bring us to the massive three-form gauge theory formulation of KSL. To see this, we think of the field strength $h$ as being independent of the gauge potential $b$ giving rise to it. Thus, we think of $h$ as a fundamental form itself. However, it is not any gauge field but constrained by the Bianchi identity:

$$\epsilon^{\mu\alpha\beta\gamma}\partial_{[\mu}h_{\alpha\beta\gamma]} = 0 \tag{8.14}$$

Constraints are implemented in actions via Lagrange multipliers. Hence, denoting the Lagrange multiplier by $\phi$, we obtain the action[9]:

$$\mathcal{L} = -\frac{m^2}{12}(h_{\alpha\beta\gamma} - A_{\alpha\beta\gamma})^2 - \frac{1}{48}F_{\mu\alpha\beta\gamma}F^{\mu\alpha\beta\gamma} + \frac{m}{6}\phi\epsilon^{\mu\alpha\beta\gamma}\partial_{[\mu}h_{\alpha\beta\gamma]} \tag{8.15}$$

Indeed, as the notation suggests, the Lagrange multiplier field is the axion field $\phi$. The parameter $m$ appears for dimensional reasons but it will turn out to be the mass for the axion. The action 8.15 is the massive three-form gauge theory formulation of the KSL action. To recover the axion-4-form theory, we can integrate-out the field $h$ via its equation of motion[10]. We find that $h_{\alpha\beta\gamma} = A_{\alpha\beta\gamma} + \frac{1}{m}\epsilon_{\alpha\beta\gamma\rho}\partial^\rho\phi$. Putting this back into 8.15 we get, up to a boundary term:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{48}F_{\mu\alpha\beta\gamma}F^{\mu\alpha\beta\gamma} + \frac{m}{24}\phi\epsilon^{\mu\alpha\beta\gamma}F_{\mu\alpha\beta\gamma} \tag{8.16}$$

[7]Note that the two-form field b is dimensionless. Furthermore, we have the gauge symmetry $b_{\alpha\beta} \to b_{\alpha\beta} + 2\partial_{[\alpha}R_{\beta]}$.

[8]Note that, strictly speaking, the two-form $b$ acts as a Stüeckelberg field for $A$. However, we could also interpret the three-form $A$ as the Stüeckelberg field for $b$.

[9]The Bianchi identity is a "scalar-constraint" so the Lagrange multiplier implementing this constraint is itself a scalar too.

[10]"Integrating-out" and elimination via the classical equations of motion are *exactly* the same here since $h$ is an auxiliary field. We could define a variable $\tilde{h}_{\alpha\beta\gamma} \equiv h_{\alpha\beta\gamma} - (A_{\alpha\beta\gamma} + \frac{1}{m}\epsilon_{\alpha\beta\gamma\rho}\partial^\rho\phi)$ so that the action depends quadratically on the field $\tilde{h}$. Hence, we can perform the path integral over $\tilde{h}$ exactly and we can integrate-out $h$ exactly.

which is just the KSL action without gravity and a slight change of notation $\mu \to m$. Now, this action has a global shift-symmetry $\phi \to \phi + \phi_0$ but we have just seen that it arises from a gauge symmetry of a two-form upon a duality transformation. Thus, the global shift-symmetry of the scalar is dual to a local shift-symmetry of the dual two-form. We now want to perform second duality transformation[11]. We now think of $A$ and $F$ as independent fields and enforce the condition $F = dA$ via a Lagrange multiplier field $q$[12]. This $q$ is a very interesting variable as we will see. We add the following term to the bulk action[13]:

$$\frac{q}{24}\epsilon^{\mu\alpha\beta\gamma}(F_{\mu\alpha\beta\gamma} - 4\partial_{[\mu}A_{\alpha\beta\gamma]}) \tag{8.17}$$

Since the four-form contains only one independent component (it is not propagating) we can integrate it out via its equation of motion: $F_{\mu\alpha\beta\gamma} = \epsilon_{\mu\alpha\beta\gamma}(q + m\phi)$. Putting it back into the axion-4-form action we get:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{48}F_{\mu\alpha\beta\gamma}F^{\mu\alpha\beta\gamma} - \frac{1}{2}(q + m\phi)^2 + \frac{1}{6}\epsilon^{\alpha\beta\gamma\mu}A_{\beta\gamma\mu}\partial_\alpha q \tag{8.18}$$

where the last term arises upon an integration by parts. The boundary term cancels exactly with the Gibbons-Hawking boundary term for the four-form. In this dual formulation it is obvious that the axion is massive. The duality transformation we have performed essentially replaces the four-form $F$ with its Hodge dual $q$. If we vary this action with respect to the three-form $A$ we find that $\partial_\alpha q = 0$. It is not difficult to see that varying the membrane action with respect to $A$ yields the analogous equation of the "jump" in four-form flux orthogonal to the membrane: $\Delta q \mid_{\vec{n}}= ne$.

### Quantization of q

Now that we have seen explicitly in the equations of motion that $q$ has essentially replaced $F$, we infer that also $q$ is quantized since $F$ is quantized. More precisely, the classical integration constant $F_0$ is quantized in units of the membrane charge $e$: $q = en, \quad n \in \mathbb{Z}$. The quantization of $q$ implies that the shift-symmetry in the KSL action cannot be continuous but must be discrete. We now have that:

$$\phi \to \phi + 2\pi f \tag{8.19}$$

$$q \to q - 2\pi m f \quad n \in \mathbb{Z} \tag{8.20}$$

which implies the consistency condition:

$$2\pi m f = en \tag{8.21}$$

so that the axion is a compact scalar with periodicity $\phi \equiv \phi + 2\pi\frac{e}{m}n$[14]. Let us briefly explain why the quantization of four-form fluxes (or $q$) is natural from a

[11] This seems an odd step since the Bianchi identity is trivially satisfied for a four-form in four dimensions. However, it will be a useful thing to do.

[12] We can also interpret the variable $q$ as the conjugate field momentum to the gauge field $A$. This follows from direct computation from 8.16.

[13] Note that the Lagrange multiplier field $q$ has mass dimensions 2.

[14] We could also work the other way around: assume that $\phi$ is a compact scalar with known periodicity and derive that four-form flux must be quantized.

stringy perspective. Bousso and Polchinski conjecture that the classical integration constant in the expression $F^{\mu\alpha\beta\gamma} = F_0\epsilon^{\alpha\beta\gamma\mu}$ is quantized: $F_0 = \frac{e}{Z}n$ [102]. In string theory we have a generalized Dirac quantization condition, stating for a four-form $F_{(4)}$ that:

$$\int_K F_{(4)} = \frac{2\pi}{e}n \quad n \in \mathbb{Z} \tag{8.22}$$

where we integrate over a four-dimensional space $K$. This quantization condition arises by demanding the membrane amplitude to be single-valued. Since we have for every gauge field both magnetic -and electric extended objects that couple to the gauge field it is natural to expect that the Hodge dual to $F_{(4)}$, which is a zero-form field in $D = 4$, is quantized. Quantization now implies:

$$\int_p F_{(0)} = F(p) = \frac{e}{Z}n \tag{8.23}$$

where $p$ is just a point and integration over a point is the same as evaluating the zero-form at that point. This quantization condition of zero-forms takes exactly the same form as the conjectured quantization of the constant $F_0$.

Let us make a few comments on the axion-4-form action 8.18. The effective potential $\frac{1}{2}(q + m\phi)^2$ has a minimum at $\phi_{min} = -\frac{q}{m}$. By fixing the four-form background in the dual formulation, $q = q_0 \neq 0$, we introduce a non-vanishing vev and the discrete gauge shift-symmetry is spontaneously broken. Another point worth remarking is that $q$ can be arbitrarily large, allowing the axion to traverse super-planckian distances in field space while $f < M_{pl}$. The potential generated is the simple $m^2\phi^2$-potential and is observationally known to lead to a too large value of $r$. However, field theory corrections to the action 8.18, 8.15 will save the day by suppressing $r$ below 0.1. One might also wonder about non-perturbative corrections to the potential generated by gauge -and gravitational instantons. Indeed, we unavoidably have to consider these corrections since we want to couple the inflaton-field eventually to a gauge sector to reheat the universe. These interactions take the form $\frac{\phi}{f}Tr(F \wedge F)$ for the coupling with a gauge sector. Instantons generate the familiar "cosine-potential":

$$V(\phi) = \sum_n \Lambda^4_{dyn} \cos\left(\frac{\phi}{f}n\right) \tag{8.24}$$

which we recognize as the inflationary potential for natural inflation. Here it is to be interpreted as a correction to the effective potential $\frac{1}{2}(m\phi+q)^2$ and we need to worry about whether it is suppressed or not. The dynamical scale $\Lambda^4_{dyn} = M^4_{uv}e^{-nS_{inst}} \sim M^4_{uv}e^{-n\frac{1}{g}}$ where $g$ is the gauge-coupling. As long as the gauge theory is not strongly-coupled ($g \ll 1$) during inflation, the instanton corrections will be suppressed and inflation will be driven by the quadratic potential. The instanton potential induces sinusoidal modulations on top of the $m^2\phi^2$-profile for small values of $\phi$. For particular values of the parameters $m, f$ such modulations can be related to the existence of domain walls. We will come back to this point in section 8.3 since domains walls/membranes might spoil the monodromy scenario according to the WGC. Similarly, gravitational instanton corrections take the form $\Lambda = M_{uv}e^{-\frac{M_{pl}}{f}}$ which are suppressed when $f < M_{pl}$. Maintaining slow-roll places stronger constraints on the instanton corrections.

## 8.2 Field theory corrections and monodromy k-inflation

Up to now we have only studied the KSL model to lowest order in the fields and their derivatives. We now want to study the more general situation in which we include higher dimensional operators parametrizing the UV physics. We will work in the massive three-form gauge theory formulation of the KSL model, i.e. the action 8.15, and later make the duality transformation to the axion-4-form theory, which essentially amounts to a field redefinition. Since massive three-form gauge theories are indeed renormalizable, higher dimensional operators arise upon integrating-out UV degrees of freedom. Examples of such degrees of freedom are moduli fields, KK modes and string states. The structure of non-renormalizable interactions is constrained by the discrete gauge shift-symmetry of the renormalizable part, forbidding polynomial corrections in the field $\phi$. We can distinguish the following classes of allowed higher-dimensional operators. Denoting the UV-scale by $M$, the first class is:

$$\delta\mathcal{L}_1 = \sum_{n=2}^{\infty} c_n \frac{F^{2n}}{M^{4n-4}} \tag{8.25}$$

These operators correct the potential $V(\phi)$. This is because when we integrate-out $F$ by virtue of its equation of motion we get:

$$\delta\tilde{\mathcal{L}}_1 = \sum_{n=2}^{\infty} c_n \frac{(m\phi + q)^{2n}}{M^{4n-4}} \tag{8.26}$$

which we can rewrite as:

$$\delta\tilde{\mathcal{L}} = m^2\phi^2 \sum_{n=2}^{\infty} c'_n \frac{(m\phi)^n}{M^{4n}} = V_0(\phi) \sum_{n=2}^{\infty} c'_n \Big(\frac{V_0(\phi)}{M}\Big)^n \tag{8.27}$$

where the absorption of $q$ into the Wilson coefficients $c_n$ redefines the Wilson coefficients. Are these corrections dangerous? As long as we have that $|F| < M^2$ they are not. This translates into a constraint on the field amplitude $|\phi|$. Since $F_{\alpha\beta\gamma\sigma} = \epsilon_{\alpha\beta\gamma\sigma}(q + m\phi)$ we have that $|\phi| < \frac{M^2}{m}$. This can also be seen by considering the corrections induced to the eta-parameter:

$$\Delta\eta = \sum_n a_n \frac{V_0^n(\phi)}{M^{4n}} \tag{8.28}$$

so that slow-roll is maintained if $V_0(\phi) < M^4$ with $V_0 = m^2\phi^2$. Note that this is compatible with large-field variations as long as $m \ll M$. The correct normalization to the amplitude of scalar perturbations $A_s$, as we have mentioned already many times, requires $m \sim 10^{-5} M_{pl}$. Note then that we can easily achieve the hierarchy $M_{pl} \ll |\phi| < \frac{M^2}{m}$.

A second class of operators is constructed from gauge invariant combinations of the form $h - A$, suppressing Lorentz indices, with $h = db$ and $F = dA$ like above:

$$\delta\mathcal{L}_2 = \sum_{n=2}^{\infty} \frac{d_n}{M^{2n-4}} (h_{\alpha\beta\gamma} - A_{\alpha\beta\gamma})^{2n} \tag{8.29}$$

It is not entirely obvious whether this class of operators is dangerous. However, it can be argued that they are suppressed in the UV-limit $m^2 \to 0$ where $m$ is the mass of the three-form gauge field and it also acts as a coupling parameter between the Stüeckelberg field $b$ and $A$ [69]. At this point we are going to make two assumptions. The first is that we assume the massive three-form gauge theory can be UV-completed in a renormalizable QFT or a consistent theory of quantum gravity like string theory, which has proper UV behaviour. This assumption, in combination with the gauge invariance, will allow us to use the Goldstone-boson-equivalence theorem (GBET) for three-form gauge fields. The GBET states the equivalence between two S-matrices. It is a little simpler to explain in the context of an abelian U(1) gauge theory. In this case it states that the S-matrix elements for $n$ scalars is the same as for $n$ longitudinal gauge bosons, up to terms of $\mathcal{O}(\frac{m}{E})$ with $E$ the energy of the scattering process. In addition to the existence of a renormalizable UV completion, we demand that the S-matrix elements do not diverge when taking the limit $m^2 \to 0$. The generalization to a three-form gauge theory is as follows: the scattering of $n$ longitudinal three-form gauge bosons, which is the only propagating degree of freedom, is equivalent to the scattering of $n$ two-form gauge bosons. By gauge invariance and dimensional analysis we have operators coming in the form $m(h - A)$. Since the $m^2 \to 0$ limit is assumed to be non-divergent, all higher-dimensional operators will have redefined wilson coefficients of the form:

$$d_n = \frac{d'_n}{M^{2n}} m^{2n} \tag{8.30}$$

so that in the limit $m^2 \to 0$ this class of operators is highly suppressed. We can make very similar arguments for the third class of operators:

$$\delta\mathcal{L}_3 = \sum_{n,m} \frac{e_{n,m}}{M^{2n+2m-4}} (h_{\alpha\beta\gamma} - A_{\alpha\beta\gamma})^{2n} F^m \tag{8.31}$$

where again we can redefine the wilson coefficients $e_{n,m} = \frac{e'_{n,m}}{M^{2n}} m^{2n}$ by virtue of the argument given above.

It is enlightening to perform a duality transformation on the operator classes 8.29 and 8.31 by using that $h_{\alpha\beta\gamma} = A_{\alpha\beta\gamma} + \frac{1}{m}\epsilon_{\alpha\beta\gamma\mu}\partial^\mu\phi$. This transforms 8.29 and 8.31 into higher-derivative corrections:

$$\delta\tilde{\mathcal{L}}_2 = \sum_{n>2} \frac{d'_n}{M^{4n-4}} (\partial\phi)^{2n} \tag{8.32}$$

$$\delta\tilde{\mathcal{L}}_3 = \sum_{n>1,m>1} \frac{e'_{n,m}}{M^{4n+4m-4}} (\partial\phi)^{2n} F^{2m} \tag{8.33}$$

We can perform another transformation by using that $F_{\alpha\beta\gamma\sigma} = \epsilon_{\alpha\beta\gamma\sigma}(q + m\phi)$. Now that we have constructed the allowed operators and performed the chain of duality transformations, we are in principle in a position to write down the most general effective field theory for axion monodromy. However, in [100] the effective field theory is more "refined". [100] applies *naive-dimensional-analysis* (NDA) to refine the Wilson coefficients, include an overall normalization of the effective Lagrangian and to divide by additional factorials and powers of $\frac{1}{2}$ to account for symmetry factors of loop diagrams corresponding to physical S-matrix elements. Interestingly, this framework lowers the strong-coupling scale $M$ to $M_s = \frac{M}{\sqrt{4\pi}}$, yielding a small regime in which the theory is strongly coupled *yet* under control as long as we consider energies $E < M$. Thus, the most general effective field theory of $\phi$ and $q$, dual to the axion-4-form theory is:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(m\phi+q)^2 - \sum_{n>2} c_n \frac{(m\phi+q)^n}{n!(\frac{M^2}{4\pi})^{n-2}} - \sum_{n>1} d'_n \frac{(\partial\phi)^{2n}}{2^n n!(\frac{M^2}{4\pi})^{2n-2}} - \\ - \sum_{n>1,m>1} e'_{n,m} \frac{(m\phi+q)^m}{2^n n! m!(\frac{M^2}{4\pi})^{2n+m-2}} (\partial\phi)^{2n}$$

It is convenient to define $\varphi \equiv \phi + \frac{q}{m}$. Since the variable $q$ is in principle arbitrary up to the quantization constraint, $\varphi \gg M_{pl}$ because nothing restricts $q$ from being extremely large. Note that this is consistent with $f < M_{pl}$. With this field redefinition it is easy to see that we obtain:

$$\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \sum_{n>2} c_n \frac{m^n\varphi^n}{n!(\frac{M^2}{4\pi})^{n-2}} - \sum_{n>1} d'_n \frac{(\partial\varphi)^{2n}}{2^n n!(\frac{M^2}{4\pi})^{2n-2}} - \\ - \sum_{n>1,m>1} e'_{n,k} \frac{m^k\varphi^k}{2^n n! k!(\frac{M^2}{4\pi})^{2n+k-2}} (\partial\varphi)^{2n}$$

since $q$ is locally constant, $\partial q = 0$. We can formally write this theory as:

$$\mathcal{L} = \mathcal{K}(\varphi, X) - V_{eff}(\varphi) = \frac{M^4}{16\pi^2}\mathcal{K}(\frac{4m\pi\varphi}{M^2}, \frac{16\pi^2 X}{M^4}) - \frac{M^4}{16\pi^2}V_{eff}(\frac{4m\pi\varphi}{M^2}) \tag{8.34}$$

with $X \equiv -\frac{1}{2}(\partial\phi)^2$ and where the normalization of the variables and the total Lagrangian reflects the NDA framework. The functions $\mathcal{K}, V_{eff}$ are to be understood as asymptotic series and hence are divergent but well-approximated by finitely many terms. Note that this is the action for *k-inflation*, possibly mixed with potential-driven (chaotic) inflation. The higher-derivative operators become important in the strongly-coupled regime, $E > M_s$ or $m\varphi > M_s^2 \equiv \frac{M^2}{4\pi}$. In fact, we are forced by observations to consider the strongly-coupled regime since in the weakly-coupled regime the above theory reduces to simple quadratic chaotic inflation for which $r \sim 0.2$ is too large. This is easy to see since for $m\varphi \ll \frac{M^2}{4\pi}$ the field theory corrections to the potential -and two-derivative kinetic term are small and hence we effectively have:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 \tag{8.35}$$

More precisely, we need to demand that during the last 60 e-folds of inflation we are in the strongly-coupled regime since this is the regime in which the anisotropies in the CMB temperature were created. We can in principle distinguish two phases of strongly-coupled inflation. In the first, the higher-derivative operators $X^p, p > 2$ are dynamically suppressed and inflation is potential-driven. The action reduces to:

$$\mathcal{L} = -\frac{1}{2}\mathcal{Z}_{eff}^2(\partial\varphi)^2 - \frac{M^4}{16\pi^2}V_{eff}(\frac{4m\pi\varphi}{M^2}) \tag{8.36}$$

It is interesting to look at the constraints slow-roll imposes. Computing $\epsilon_V, |\eta_V|$ we find:

$$\epsilon_V \sim \left(\frac{4\pi M_{pl} m}{M^2}\right)^2 \ll 1 \tag{8.37}$$

$$\eta_V \sim \left(\frac{4\pi M_{pl}^2 m}{M^2}\right)^2 \ll 1 \tag{8.38}$$

both imposing the same constraint on the inflaton mass $m$. We can trivially rewrite this as $m \ll \frac{M_s}{M_{pl}}$. This reveals an interesting compatibility condition. Demanding slow-roll and strongly-coupled inflation simultaneously implies super-planckian field variations $\varphi \gg M_{pl}$. Since initially $|\varphi| \gg M_{pl}$ inflation proceeds from strong-coupling to weak-coupling. The tensor-to-scalar ratio is suppressed due to *flattening* of the potential in this first scenario. One way of flattening the potential discussed in [100] is via wave-function renormalization. However, this flattening of the potential is only achieved if the duration of the weak-coupling regime is bounded by a particular number of e-folds. This translates into a condition on the inflaton mass.

## 8.3 Axion monodromy and the WGC

It is often taken for granted that the Kaloper-Sorbo theory allows for a UV completion in quantum gravity. There is still, however, a lot of possible tension with both the electric -and magnetic WGC. In this section we want to emphasize a few caveats which deserve more attention in the literature.

### Monodromy and membranes

Why is the KSL model suggested as (an/the) effective field theory formulation of axion monodromy? We have emphasized that $q$ obeys the quantization condition $q = ne, \quad n \in \mathbb{Z}$ and that this breaks the continuous shift-symmetry of the action to the discrete subgroup $\phi \to \phi + f$ with $f = n\frac{e}{m}$. The emergent low-energy effective axion potential ((see figure 8.1)) generated upon integrating-out the four-form is:

$$V(\phi) = \frac{1}{2}m^2\left(\phi + n\frac{e}{m}\right)^2 \tag{8.39}$$

where we have substituted the quantization condition for $q$. The symmetry of the action is $\phi \to \phi + \frac{e}{m}, n \to n-1$. The integer $n$ labels the different quadratic branches of the effective potential. As the axion moves over its fundamental period, the potential changes non-periodically, signalling monodromy. An obvious but important insight is that the multi-branched structure of the potential is a consequence of gauge redundancy. Membranes can mediate tunneling of $\phi$ between different branches of the potential via their nucleation. We will refer to these membranes as KS membranes as they "originate" from gauging the axion symmetry.

To appreciate the above observation, suppose the axion couples to a gauge theory[15] and the axion potential is not generated via axion-4-form mixing[16]. Consider

[15] We assume this coupling so that we generate the typical instanton potential but it is not a necessary assumption. Gravitational instantons are always present and generate similar potentials for the axion.

[16] This setup can also be understood in terms of a four-form field strength [103]. However, the three-form potential does not gauge any symmetry and so this really differs from the Kaloper-Sorbo

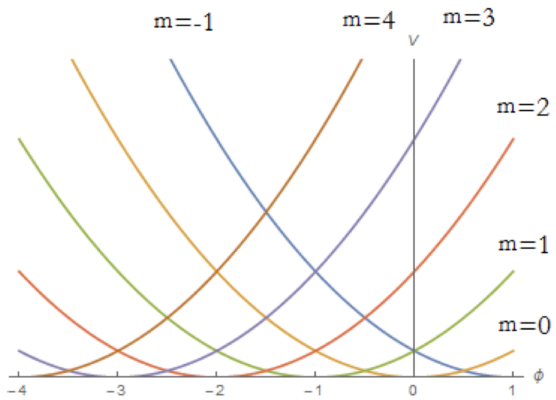


Figure 8.1: The monodromy potential as a consequence of gauging the axion shift symmetry by a three-form à la Kaloper-Sorbo. This picture is taken from [70].

the first instanton correction to the quadratic potential:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + M^4 e^{-S_{inst}}\left(1 - \cos\frac{\phi}{f}\right) \tag{8.40}$$

When $M^4 e^{-S_{inst}} > m^2 f^2$, local minima are introduced at small values of $\phi$. The field $\phi$ can then move to the next local mininum at a slightly lower value of the potential via quantum-mechanical tunneling. This tunneling is also achieved via the nucleation of a bubble/membrane but it is created by an instanton. The potential-barrier through which $\phi$ tunnels can be interpreted as a neutral membrane or as a domain wall, specified by its tension. We will refer to such membranes as effective membranes. They are intrinsically field-theoretic and are also referred to as field-theoretic bounces. The tunneling between these local minima at small $\phi$ is the analogous phenomenon to the tunneling between different branches of the potential as in figure 8.1. One has to clearly distinguish between membranes that arise from these "wiggles" on top of the quadratic potential from ones whose nucleation mediate the tunneling of $\phi$ between different branches of the monodromy potential. In

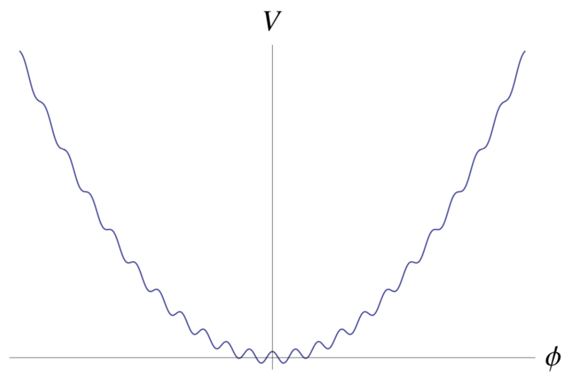


Figure 8.2: Sinusoidal modulations on top of a quadratic potential $m^2\phi^2$. The quadratic part of the potential does not originate from the Kaloper-Sorbo mechanism. This picture is taken from [103].

mechanism.

the case of $M^4 e^{-S_{inst}} > m^2 f^2$, the tension of the domain wall can be calculated in the effective theory and we do not need a UV completion to determine it. It is equal to $T \sim M^2 f$ where $M$ is the strong-coupling scale of the gauge theory to which $\phi$ couples. Thus, this scale plays a crucial rule: if it is small, the tension is small and the sinusoidal modulations are negligible and consequently the field easily slow-rolls classically down to the global minimum. In the case of KS membranes, even though one might have small wiggles, i.e. $e^{-S_{inst}} < m^2 f^2$, slow-roll inflation might still be spoiled via the nucleation of KS membranes.

Note that this seems to suggest that from the EFT point of view, i.e. considering the effective membranes, the electric WGC is automatically satisfied, as it demands $T < gM_{pl}$. This is exactly what slow-roll also demands. Thus, we cannot bound the effective membrane tension in this way. We can, however, constrain their tension with the magnetic WGC. The tension of KS membranes is not calculable within the effective theory and hence requires UV input. In this case we can constrain the tension via the electric WGC.

#### Axion monodromy inflation

In the context of inflation, if the membrane nucleation rate is very high, tunneling to lower local vacua is likely and the inflaton will move quickly to the global minimum of the potential. This means that the number of e-foldings is highly suppressed. Now, membrane nucleation is a non-perturbative process and hence expected to be exponentially suppressed. Coleman had already calculated the probability of tunneling in the field theory context under various simplifying assumptions a while ago. He considered the tunneling of a scalar field, not coupled to gravity, from a local vacuum to the global vacuum. The probability for this process is:

$$P \sim \exp\left\{-\frac{27\pi^2 T^4}{2(\Delta V)^3}\right\} \tag{8.41}$$

where $\Delta V$ is the difference in potential energy between the local -and global minimum. When the tension is constrained to be very small, the nucleation process is not exponentially suppressed. What does the electric WGC imply? Recall that we could not properly define a generalized WGC in $d = 4$ for two-branes (or in general for $p = d - 1$), unless a scalar field coupled to the gauge field. Another way to circumvent this degenerate case is using T-duality twice to alter the codimension of the two-brane. One can then derive that the tension of KS two-branes is bounded by [68]:

$$T \lesssim 2\pi f m \tag{8.42}$$

The bound one can derive is expression 8.44.

Now, it has been noted that this application of the formulas of Coleman (and DeLuccia) is too naive, as the tunneling process in axion monodromy models is not analogous to false vacuum decay[17]. It is a process of tunneling from a non-stable state to another non-stable state. In [70] these differences were taken into

[17] There are more differences related to the relaxation of some of the assumptions Coleman made, e.g. the so-called thin wall approximation.

account. However, the result relevant for inflation only differed from previous constraints derived in [27] from the electric WGC by a numerical factor of 8.

Besides the tunneling calculation there are more aspects to refine. It has been shown that in the string theory setting monodromy is associated to the discrete gauge symmetry $\mathbb{Z}_k$. The two-branes that naturally couple to the three-forms are charged under this discrete symmetry. Thus, we need a generalization of the WGC to discrete gauge symmetries. Furthermore, axion monodromy in the KSL setting is a *massive* three-form gauge theory. The WGC has been generalized to three-forms using the very same argument as establishing the axionic WGC. However, there does not exist a WGC for massive gauge fields[18]. This illustrates that there is still a lot to investigate.

We can bound the tension of the effective membranes via the magnetic WGC. If we neglect the problem that the 3-form gauge field couples magnetically to the dual $(-2)$-brane[19] which is ill-defined, we can demand that the monopole of least charge is not a black brane and show that the magnetic WGC becomes the statement that [103]:

$$\Lambda \lesssim g^{\frac{1}{3}} M_{pl}^{\frac{1}{3}} \tag{8.43}$$

where $g$ is the gauge coupling. It is not at all clear whether this formulation makes any sense[20], but if we suppose it does, one can derive the following bound on the axion field displacement [103]:

$$\frac{\Delta\phi}{M_{pl}} \lesssim \left(\frac{M_{pl}}{m}\right)^{\frac{2}{3}} \left(\frac{2\pi f}{M_{pl}}\right)^{\frac{1}{3}} \tag{8.44}$$

which allows for $\Delta\phi > M_{pl}$ as long as $f$ is not too small. Interestingly, the KS membranes and the effective membranes satisfy the same constraint but the first follows from the electric WGC while the second from the magnetic WGC.

It is clear that it is not obvious at all how severe the constraints coming from the WGC are.

## 8.4 $\alpha$-attractors and the Kaloper-Sorbo mechanism

Can we somehow formulate pole-inflation or $\alpha$-attractors in the language of Kaloper-Sorbo? If this is possible, it would result in a multi-field monodromy model which is protected against quantum gravitational corrections due to two discrete gauge

[18]This was also pointed out in section 3.6 of [69]. In [72], M.Reece claims there is a WGC for massive one-form gauge theories following from the distance conjecture.

[19]This is because the dual magnetic brane to an electric brane coupling to a $(p+1)$-form gauge field in $d$ dimensions is a $(d-(p+4))$-brane.

[20]The argument given in [103] is that in string compactifications the WGC is satisfied by lowering the cut off of the EFT (which is the KK scale $M_{KK}$), not by providing objects whose tension satisfies $T \lesssim gM_{pl}$. This is precisely what the magnetic WGC also does and therefore the electric WGC is automatically satisfied.

shift symmetries following from the dual two-form longitudinal modes of the four-form field strengths. However, the dual two-forms to the scalars should be non-abelian since the symmetry group of $\alpha$-attractors is the non-abelian coset group $SO(2,1)/SO(2)$. The literature on non-abelian forms is mathematically very challenging and there has not appeared yet a "physics translation".

**A naive approach**

We can generate models of $\alpha$-attractors by considering field theory corrections to the massive three-form U(1) gauge theory. $\alpha$-attractors are necessarily multi-field models as the underlying scalar manifold is hyperbolic. We will consider the simplest case of $\alpha$-attractors which is a two-field model:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \sinh^2\phi(\partial\pi)^2 - V(\phi,\pi) + \dots \tag{8.45}$$

Starting point is a generalization of the massive three-form gauge theory. We now have two U(1) three-form gauge fields with their associated field strengths:

$$\mathcal{L} = -\frac{1}{48}F^{(1)}_{\alpha\beta\gamma\sigma}F^{(1)\alpha\beta\gamma\sigma} - \frac{m_1^2}{12}(h^{(1)}_{\alpha\beta\gamma} - A^{(1)}_{\alpha\beta\gamma})^2 - \frac{1}{48}F^{(2)}_{\alpha\beta\gamma\sigma}F^{(2)\alpha\beta\gamma\sigma} -$$

$$-\frac{m_{(2)}^2}{12}G\Big(\frac{F^{(1)}}{M^2}\Big)(h^{(2)}_{\alpha\beta\gamma} - A^{(2)}_{\alpha\beta\gamma})^2$$

Here $h^{(1)} = db^{(1)}, h^{(2)} = db^{(2)}, F^{(1)} = dA^{(1)}, F^{(2)} = dA^{(2)}$ where the b's are two-forms and $G$ is an arbitrary dimensionless function of its argument $F^{(1)} = \epsilon_{\alpha\beta\gamma\mu}F^{(1)\alpha\beta\gamma\mu}$. The $F$-dependence of $G$ is what we would like to solve for. Note that it must depend on the field strength $F^{(1)}$ since this is ultimately dependent on $\phi$ after a chain of duality transformations. We also assume a quadratic dependence on the field strengths. This will generate quadratic potentials for the scalar field as we have seen above. One could generalize this to an arbitrary $F^{(i)}$-dependence and Dvali already derived how the potential is related to the scalar potential:

$$V^{(i)}(\phi) = -\int \mathrm{inv}\Big(\frac{dK^{(i)}}{dF^{(i)}}\Big)d\phi \quad i = 1,\dots,\#\text{scalars} \tag{8.46}$$

where the "inv"-operator denotes the *inverse function* of the argument. For example, in the single three-form case the quadratic potential in the KSL model is generated by the function $K = \frac{F}{M^2}$.

Next, we dualize the action to the axion-4-form theory. We now have to impose the Bianchi-identity twice, introducing two scalar fields $\phi$ and $\pi$ as Lagrange multipliers. This adds the terms $\frac{m_{(1)}}{6}\phi\epsilon^{\alpha\beta\gamma\sigma}\partial_{[\alpha}h^{(1)}_{\beta\gamma\sigma]}$, $\frac{m_{(2)}}{6}\pi\epsilon^{\alpha\beta\gamma\sigma}\partial_{[\alpha}h^{(2)}_{\beta\gamma\sigma]}$ to the Lagrangian. We now integrate-out $h^{(i)}$ via the equations of motion. For $i=1$ we get the usual expression $h^{(1)}_{\alpha\beta\gamma} = A^{(1)}_{\alpha\beta\gamma} + \frac{1}{m_{(2)}}\epsilon_{\alpha\beta\gamma\mu}\partial^\mu\phi$ but for $i=2$ we find:

$$h^{(2)}_{\alpha\beta\gamma} = A^{(2)}_{\alpha\beta\gamma} + \frac{1}{m_{(2)}}\Big(G\big(\frac{F^{(1)}}{M^2}\big)\Big)^{-1}\epsilon_{\alpha\beta\gamma\mu}\partial^\mu\pi \tag{8.47}$$

Substituting it back into the action and focussing only on the $\pi$-part:

$$\mathcal{L}_\pi = 2G^{-1}(\partial\pi)^2 - \frac{1}{48}F^{(2)}_{\alpha\beta\gamma\sigma}F^{(2)\alpha\beta\gamma\sigma} + \frac{m_{(2)}}{24}\pi\epsilon^{\alpha\beta\gamma\sigma}F_{\alpha\beta\gamma\sigma} - \frac{1}{6}\epsilon^{\alpha\beta\gamma\sigma}\epsilon_{\beta\gamma\sigma\mu}\pi\partial_\alpha G^{-1}\partial^\mu\pi$$

Integrating by parts the last term and using the identity $\epsilon^{\alpha\beta\gamma\sigma}\epsilon_{\beta\gamma\sigma\mu} = -6\delta^\alpha_\mu$ we find:

$$3G^{-1}(\partial\pi)^2 - \frac{1}{48}F^{(2)}_{\alpha\beta\gamma\sigma}F^{(2)\alpha\beta\gamma\sigma} + \frac{m_{(2)}}{24}\pi\epsilon^{\alpha\beta\gamma\sigma}F_{\alpha\beta\gamma\sigma} \tag{8.48}$$

We thus have the following algebraic equation for $G$:

$$\sqrt{3}G^{-\frac{1}{2}} = \sinh\phi \tag{8.49}$$

If we impose the condition $F^{(1)} = dA^{(1)}$ with a Lagrange multiplier $q_1$ we have that $F^{(1)}_{\alpha\beta\gamma\mu} = \epsilon_{\alpha\beta\gamma\mu}(m_1\phi + q_1)$ or when inverting this $\phi = -\frac{1}{m_1}(\frac{1}{24}F^{(1)} + q_1)$. Substituting this and taking the inverse on both sides we can solve for $G$ in terms of $F^{(1)}$. We could generalize this procedure to arbitrary many fields and a very large class of potentials. Such an action would be something like:

$$\mathcal{L} = \sum_i K^i(\frac{F}{M^2}) + \sum_i \tag{8.50}$$

**Non-abelian forms**

An interesting question is how this hyperbolic geometry of $\alpha$-attractors manifests itself in the three-form gauge theory. This could not be revealed above for the obvious reason that we considered a U(1) gauge theory. The field space geometry of $\alpha$-attractors is hyperbolic. The kinetic term is invariant under the action of the Möbius group, or equivalently, under the action of the coset space:

$$\mathbb{H}^2 \cong SO(2,1)\big/SO(2) \tag{8.51}$$

whose complex double-cover is the coset space $SU(1,1)\big/U(1)$. We can interpret this coset space as arising from a symmetry breaking pattern $SO(2,1) \to SO(2)$ whose associated goldstone bosons ($\phi$ and $\pi$) non-linearly realize the global $SO(2,1)$ symmetry in the low-energy effective theory. The action of the global $SO(2,1)$ on the goldstones is an enhanced shift-symmetry. The two-derivative $\alpha$-attractor action is:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \sinh^2\phi(\partial\pi)^2 - V(\phi,\pi) + \ldots \tag{8.52}$$

where the scalar potential weakly breaks this enhanced shift-symmetry so that the model is technically natural. While this generalized shift-symmetry protects the model against perturbative QFT corrections, it is a global symmetry and hence this is an effective field theory lying in the swampland. In the context of this chapter, we might try to cure this problem à la Kaloper-Sorbo, in which the simple shift-symmetry is a manifestation of a discrete gauge symmetry. However, in the Kaloper-Sorbo framework, there is no breakdown of a symmetry. Instead, we have a U(1) abelian three-form gauge theory where a two-form stüeckelberg potential realizes the longitudinal mode of the U(1) three-form. Upon dualizing this stüeckelberg two-form and integrating it out we obtain the axion-4-form U(1) gauge theory. Furthermore, what makes it hard to write down a gauge theory is that the underlying symmetry of $\alpha$-attractors is non-abelian. To make use of the Kaloper-Sorbo framework, we need to formulate a theory for a non-abelian three-form that is gauge-invariant under $SO(2,1)$ transformations, i.e. we need to gauge Möbius transformations.

# Chapter 9

# Positivity

The aim of the swampland program is to identify the conditions that separate the landscape of effective field theories that can consistently be coupled to quantum gravity from those that cannot. This program is based on the assumption that the space of quantum effective field theories, not coupled to gravity, are consistent within themselves. Of course, we might need to impose anomaly conditions but it turns out one has to impose more consistency conditions on the space of effective field theories. It was discovered, around the same time that the (string) swampland program developed, that not every Lorentz invariant local quantum field theory admits a wilsonian UV completion whose unitary S-matrix satisfies the standard analyticity postulates [45].

We have already seen an example of this in 5.1: the Euler-Heisenberg Lagrangian always comes with positive Wilson coefficients for the leading-order higher-derivative corrections. "Wrong sign" coefficients are related to superluminal propagation of fluctuations in non-trivial backgrounds. For instance, turning on a constant background electric field modifies the dispersion relation for plane waves. Subluminal propagation then requires that the sign of the leading operator is positive.

In this chapter we aim for the following: we want to investigate the recently established connections between positivity and the weak gravity conjecture [89, 104, 105, 106]. It has been claimed in some of those references that the WGC has been proven using positivity arguments [1]. Initially, we had a second aim namely, to study the relationship between positivity and soft limits. It has become clear that via postulating certain soft behaviour[2] of the scattering amplitude, together with standard factorization properties at the pole, one can reconstruct entire classes of EFT's [107, 108]. Since the degree of softness correlates with the amount of symmetry, "too much" softness forbids certain higher-derivative operators, which are typically demanded to be strictly positive by positivity bounds. Thus, scattering amplitudes cannot be arbitrarily soft. Unfortunately, this was already studied in [104] for a shift-symmetric fermion.

[1] We will not describe this in detail these proofs but comment in the end of the chapter on one attempt.

[2] Softness of the amplitude was always viewed as a consequence of postulated symmetries. The philosophy is precisely the other way around here but it is useful to keep in mind that the softer the amplitude, the more symmetry the theory has.

Note that we specifically refer to wilsonian UV completions, meaning that we UV complete a theory by integrating-in new degrees of freedom at energies $\Lambda$ at which for example scattering amplitudes computed in our EFT violate perturbative unitarity[3]. With these new degrees of freedom we reconstructed a weakly coupled field theory at energies $E > \Lambda$. Positivity violating EFT's might still UV complete via mechanism known as classicalization, in which a EFT "self-completes"[109].

## 9.1 Positivity and the weak gravity conjecture

What has positivity to do with the WGC? In the chapter on the WGC we mentioned that the states satisfying the electric form of the WGC do not necessarily have to be particle states. Black holes could play the role of the WGC state, as long as field theory corrections to the black hole extremality bound always weaken the bound. This implies that all charged black holes will be able to decay which proves the electric form of the WGC. However, for the extremality bound to weaken, a certain combination of Wilson coefficients has to satisfy a positivity constraint [106, 44]. We will now review this argument.

Consider four-dimensional Einstein-Maxwell theory corrected by higher-dimensional operators parametrizing the physics above the lightest charged particle threshold. Thus, the spectrum of the effective theory consists solely of electrically neutral massless particles, the photon and the graviton. In the spirit of wilsonian effective field theory we write down everything allowed by diffeomorphism invariance and U(1) gauge invariance:

$$\begin{gathered}\Delta\mathcal{L} = a_1(F_{\mu\nu}F^{\mu\nu})^2 + a_2(F_{\mu\nu}\tilde{F^{\mu\nu}})^2 + a_3(\nabla_\mu F_{\rho\sigma})(\nabla^\mu F^{\rho\sigma}) + \\ + a_4(\nabla_\mu F_{\rho\sigma})(\nabla^\rho F^{\mu\sigma}) + a_5(\nabla_\mu F^{\mu\nu})(\nabla^\rho F_{\rho\nu}) + a_6 F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu} + \\ + b_1 F_{\mu\nu}F^{\mu\nu}R + b_2 F_{\mu\nu}F^{\mu}_{\alpha}R^{\nu\alpha} + b_3 F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma} + \\ + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\end{gathered}$$

These are the leading (fourth derivative) higher-derivative corrections. The expression for $\Delta\mathcal{L}$ is highly redundant since not all of these operators are independent and we assume the absence of charged sources. Operators of the form $\nabla_\mu F_{\rho\sigma}$ are equivalent via the Bianchi identities to operators already written down. The absence of charged sources implies via the equations of motion that terms like $\nabla_\mu F^{\mu\nu}$ vanish in the unperturbed solution. This means that such terms contribute only at higher order in the Wilson coefficients in the corrected equations of motion. Variation of the operator $F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}$ shows that it equals $\frac{1}{2}(F_{\mu\nu}F^{mu\nu})^2$ and so the coefficient $a_6$ can be absorbed into $a_1$.

Now, the philosophy is very simple but the calculations are somewhat tedious[4]. One wants to calculate the modified extremality bound due to these field theory corrections. Via spherical symmetry one can make an ansatz for the modified metric which can be related to the energy-momentum tensor via Einstein's equation. The higher-derivative corrections induce corrections to the energy-momentum tensor

[3] This happens in (non-renormalizable) field theories whose interactions are governed by couplings with negative mass dimension.

[4] See appendix A of [106] for some more details.

which can be computed by varying $\Delta\mathcal{L}$ with respect to the inverse metric. One also needs to vary $\Delta\mathcal{L}$ with respect to the gauge field $A_\mu$ since the corrections modify Maxwell's equations and therefore the contribution of the term $F^{\mu\nu}F_{\mu\nu}$ to $T_{\mu\nu}$.

The modified extremality bound derived in [106] reads:

$$\frac{1}{\sqrt{2}}\frac{M}{Q} = 1 - \frac{2}{5q^2}\Big[2c_2 + 8c_3 + 2b_2 + 2b_3 + 7a_1 - 2a_3 - a_4\Big] \tag{9.1}$$

where we set $M_{pl} = 1$. Proving the electric WGC amounts to proving the positivity of the combination of these coefficients:

$$2c_2 + 8c_3 + 2b_2 + 2b_3 + 7a_1 - 2a_3 - a_4 > 0 \quad (?) \tag{9.2}$$

In this case we have for every extremal black hole another black hole whose ratio $\frac{M}{Q}$ is smaller than the original ratio which is required for the instability of this object. This is illustrated in the figure below. Recently, the above statement was proven by

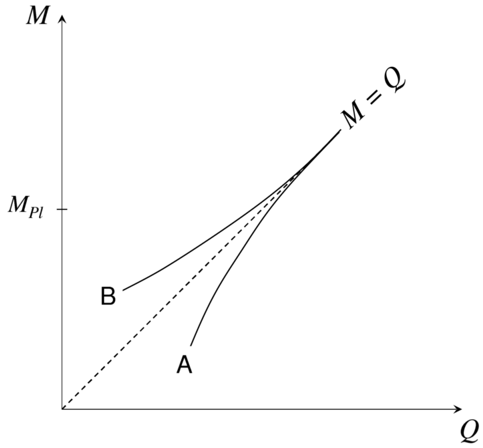


Figure 9.1: This picture is taken from [106]. Higher-derivative corrections may force black holes themselves to be super-extremal, i.e. they may be labelled by points on curve A. In this case black holes can play the role of the state demanded by the electric WGC and every extremal black hole can decay into a super-extremal one.

several authors to be true. Probably the most convincing method is the "S-matrix proof" of [104]. Above we already mentioned some redundancy in the description. The simplified expression reads:

$$\begin{aligned}\Delta\mathcal{L} = a_1(F_{\mu\nu}F^{\mu\nu})^2 + a_2(F_{\mu\nu}\tilde{F^{\mu\nu}})^2 + b_1F_{\mu\nu}F^{\mu\nu}R + b_2F_{\mu\nu}F^{\mu}_{\alpha}R^{\nu\alpha} +\\ +b_3F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma} + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\end{aligned}$$

We want to express physical observables in terms of field-redefinition invariant quantities. We achieve this by eliminating all dependence on the spacetime curvature in favour of the field tensor $F_{\mu\nu}$ and its Hodge dual $\tilde{F}_{\mu\nu}$ in $\Delta\mathcal{L}$. Recall that we can eliminate the Riemann tensor via the Gauss-Bonnet term $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ since it is a boundary term. Next, we consider the traceless part of the Riemann tensor which is known as the Weyl -or conformal tensor $W_{\mu\nu\rho\sigma}$. In four spacetime dimensions it reads:

$$W_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \big(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\big) + \frac{1}{3}g_{\rho[\mu}g_{\nu]\sigma}R \tag{9.3}$$

The Weyl tensor vanishes in three dimensions and it is undefined in $d = 0, 1, 2$. We can use the Weyl tensor te re-express the operator with coefficient $b_3$ as:

$$F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma} = F_{\mu\nu}F_{\rho\sigma}W^{\mu\nu\rho\sigma} + F_{\mu\rho}F^{\rho}_{\nu}R^{\mu\nu} - \frac{1}{3}F^{\mu\nu}F_{\mu\nu}R \tag{9.4}$$

Next, we use the energy-momentum tensor for Maxwell theory $T_{\mu\nu} = -F_{\mu\nu}F^{\rho}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$ to eliminate the dependence on the Ricci tensor -and scalar in the higher-derivative corrections via the tree-level Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2T_{\mu\nu}$. It is not hard to show that this yields:

$$R^2 = R_{\mu\nu}R^{\mu\nu} = (F_{\mu\nu}F^{\mu\nu})^2 \tag{9.5}$$

This substitution of the pure Einstein-Maxwell equations of motion into the higher-dimension operators is a special case of a more general field-redefinition of $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ that is second order in derivatives of the metric and the gauge field $A_\mu$. At this point we have reduced the operator basis from 8 to 4 operators. There is one additional useful identity in $4d$ that we can use to further reduce the expression:

$$2(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}\tilde{F^{\mu\nu}})^2 = 4F_{\mu\rho}F^{\rho}_{\nu}F^{\mu}_{\sigma}F^{\nu\sigma} \tag{9.6}$$

We end up with three independent operators with field-redefinition invariant Wilson coefficients:

$$\Delta\mathcal{L} = \alpha_1(F_{\mu\nu}F^{\mu\nu}) + \alpha_2(F_{\mu\nu}\tilde{F}^{\mu\nu}) + b_3F_{\mu\nu}F_{\rho\sigma}W^{\mu\nu\rho\sigma} \tag{9.7}$$

where:

$$\alpha_1 = a_1 - \frac{b_2}{2} - b_3 + c_2 + 4c_3 \tag{9.8}$$

$$\alpha_2 = a_2 - \frac{b_2}{2} - b_3 + c_2 + 4c_3 \tag{9.9}$$

We can rewrite the positivity requirement (9.1) in the simpler way [105, 104]:

$$2\alpha_1 - b_3 > 0 \quad (?) \tag{9.10}$$

## 9.2 Remarks on causality

Below we will discuss how causality manifests itself in field theory via the analytic properties of the S-matrix. Here we will review how causality can be expressed in terms of the propagation of fields. Consider the simple free Klein-Gordon scalar field theory defined on a translationally-invariant background. The propagator is defined as:

$$D(x-y) \equiv \langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{1}{2E_{\vec{p}}}e^{-ip\cdot(x-y)} \tag{9.11}$$

The commutator of two fields is:

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \tag{9.12}$$

and therefore we have that:

$$\langle 0| [\phi(x), \phi(y)] |0\rangle = D_{ret}(x-y) - D_{adv}(x-y) \tag{9.13}$$

which are the retarded -and advanced Green's functions respectively. If we want to implement causality in the theory, we need to demand that measuring the field value $\phi$ at $x$ cannot influence the measurement of the field value $\phi$ at $y$ if $x$ and $y$ are space-like separated. This amounts to the statement that $[\phi(x), \phi(y)] = 0$ for separations $(x-y)^2 < 0$ in the mostly-minus signature. Indeed, for space-like separations we can always do a Lorentz boost $x - y \to -(x-y)$ so that the RHS of 9.11 vanishes. Then, since the commutator vanishes, the retarded Green's function vanishes and therefore no signals can travel in the past-direction. The philosophy is the same for interacting theories: when two arbitrary local operators $\mathcal{O}(x), \mathcal{O}(y)$ satisfy $[\mathcal{O}(x), \mathcal{O}(y)] = 0$ for space-like separations, there is no acausal propagation of signals due to the Källen-Lehmann spectral representation of the interacting propagator.

Causality expressed as commuting observables at spacelike separations, together with locality, also ensures a Lorentz invariant perturbative S-matrix. In the Schrödinger picture states transform under time evolution $|\Psi\rangle_S \to U(t)\,|\Psi\rangle_S$, where $U(t) = e^{iHt}$, while operators are stationary $\mathcal{O}_S \to \mathcal{O}_S$. In the Heisenberg picture of quantum mechanics the operators time evolve according to $\mathcal{O}_H \to U^{-1}(t)\mathcal{O}_S U(t)$ while states do not transform $|\Psi\rangle \to |\Psi\rangle$. These pictures of quantum mechanics are convenient when we consider non-interacting systems. For interactions, we can consider the Hamiltonian $H = H_0 + H_{int}(t)$ where the interacting part of the Hamiltonian is to be understood as a tiny perturbation to the free Hamiltonian for which the dynamics can be exactly solved. It is now convenient to use a mixture of the Heisenberg -and Schrödinger picture known as the interaction picture. Operators time evolve according to the free Hamiltonian $H_0$ while the time evolution of states is governed by the interacting Hamiltonian $H_{int}(t)$:

$$|\Psi\rangle_I \equiv e^{iH_{int}t}\,|\Psi\rangle_S \tag{9.14}$$

$$\mathcal{O}_I(t) \equiv e^{-iH_0 t}\mathcal{O}_S e^{iH_0 t} \tag{9.15}$$

The point of these definitions is that the equation of motion governing the time-evolution of the ket $|\Psi\rangle_I$ in the interaction picture is a Schrödinger equation:

$$i\frac{\partial\,|\Psi\rangle_I}{\partial t} = H^I_{int}\,|\Psi\rangle_I \tag{9.16}$$

where $H^I_{int}$ means the interaction Hamiltonian in the interaction picture. To make contact with the S-matrix, the ansatz made for the state $|\Psi\rangle_I$ to solve (9.16) is:

$$|\Psi(t)\rangle_I = U(t,t_0)\,|\Psi(t_0)\rangle \tag{9.17}$$

The operator $U(t)$ is a unitary time evolution operator. Substituting this into (9.16) gives:

$$i\frac{dU}{dt} = H^I_{int}U \tag{9.18}$$

The solution is a time-ordered exponential, known as the Dyson series:

$$U(t,t_0) = T\Big[\exp\Big\{-i\int_{t_0}^{t} dt' H^I_{int}(t')\Big\}\Big] \tag{9.19}$$

The time-ordering appears because interaction Hamiltonians evaluated at different times do not necessarily commute. Below, we will define the S-matrix in a little more detail but here it suffices to define it as the time evolution operator evaluated at asymptotic times $S \equiv U(-\infty, \infty)$:

$$S = 1 + \sum_{j=1}^{\infty} \frac{(-i)^j}{j!} \int dt_1 \dots dt_j T \Big[ H^I_{int}(t_1) \dots H^I_{int}(t_j) \Big] \tag{9.20}$$

Locality implies that interactions take place at a spacetime point:

$$H_{int}(t) = \int d^3x \mathcal{H}_{int}(t, x) \tag{9.21}$$

So the only reason that the S-matrix is not Lorentz invariant is due to the time-ordering of events. For spacelike separations we can always perform a Lorentz boost to a frame in which the time-ordering of events is the opposite. However, if we require causality $[\mathcal{O}(x_1), \mathcal{O}(x_2)]$, $|x_1 - x_2| < 0$, this makes any time-ordering irrelevant at spacelike separations.

Causality expressed as $[\mathcal{O}(x_1), \mathcal{O}(x_2)]$ is obviously useful and important. Unfortunately, this requirement is formalism-dependent. When an off-shell formulation exists we can use the above commutator argument as a way of implementing causality. In the case of local QFT's we have such an off-shell formulation. However, perturbative string theory does not have an off-shell formulation. Therefore, it would be very satisfying to have a formalism-independent notion of causality and locality. This is achieved by defining locality and causality via (assumed) analytic properties of the S-matrix in certain kinematic variables.

## 9.3 The S-matrix and unitarity

The consequences of unitarity for scattering amplitudes are a little simpler than those coming from analyticity. The most important consequences of unitarity are the optical theorem and the existence of singularities in amplitudes. Let us first define the S-matrix. The S-matrix is the most important physical observable in a quantum field theory. It allows for the computation of cross-sections and decay rates of all kinds of processes. Indeed, when we evaluate Feynman diagrams we are computing S-matrix elements. The S-matrix is defined as:

$$S \equiv 1 + iT \tag{9.22}$$

Here $T$ describes the interacting part of a scattering process. It is called the transfer matrix. The S-matrix is postulated to be unitary $S^\dagger S = SS^\dagger = 1$, essentially as a consequence of unitary time evolution in quantum mechanics. On a more formal note, the S-matrix is defined via the construction of its elements. One constructs so-called asymptotic states, known as *in* -and *out* states, defined in the Heisenberg picture of quantum mechanics. The asymptotic states are defined at $t = \pm\infty$, constructed in a very particular way and are *assumed* to be non-interacting. Thus, for initial -and final asymptotic states $|i\rangle, |f\rangle$, we define S-matrix elements as:

$$S_{\alpha\beta} = \lim_{t_{1,2} \to \pm\infty} \langle f | U(t_1, t_2) | i \rangle = \delta_{\alpha\beta} + iT_{\alpha\beta} \tag{9.23}$$

from which extract the definition of the scattering amplitude $\mathcal{M}$:

$$T_{\alpha\beta} \equiv (2\pi)^4\delta^{(4)}(p_f - p_i)\mathcal{M}_{\alpha\beta} \tag{9.24}$$

The delta function appears to implement conservation of four-momentum. The unitarity of the S-matrix implies:

$$i(T^\dagger - T) = T^\dagger T \tag{9.25}$$

Computing the overlap between the in -and out states and inserting a complete set of states $\sum_n |n\rangle\langle n| = 1$ we obtain:

$$i(2\pi)^4\delta^{(4)}(p_f - p_i)\big[\mathcal{M}^\star(f \to i) - \mathcal{M}(i \to f)\big] =$$
$$\sum_n \int d\Pi_n (2\pi)^4\delta^{(4)}(p_i - p_n)(2\pi)^4\delta^{(4)}(p_f - p_n)\mathcal{M}^\star(f \to n)\mathcal{M}(i \to n)$$

where we used that $\langle f| T^\dagger |i\rangle = (\langle i| T |f\rangle)^\star$ and $\int d\Pi_n$ denotes a relativistic phase-space integral. Cancelling the overall momentum-conserving delta function we get:

$$\mathcal{M}(i \to f) - \mathcal{M}^\star(f \to i) = i\sum_n \int d\Pi_n (2\pi)^4\delta^{(4)}(p_i - p_n)\mathcal{M}^\star(f \to n)\mathcal{M}(i \to n) \tag{9.26}$$

which is the generalized optical theorem. An important special case is one for which $|i\rangle = |f\rangle \equiv |2\rangle$ are identical two-particle states. In this case we find a relation between the imaginary part of the 2-2 scattering amplitude and the total cross section:

$$\mathrm{Im}\{\mathcal{M}(|2\rangle \to |2\rangle)\} = s\sigma_{tot}(s) \tag{9.27}$$

$$\sigma_{tot}(s) \equiv \frac{1}{4\sqrt{s}|\vec{p}|}\sum_n \int d\Pi_n (2\pi)^4\delta^{(4)}(p_{|2\rangle} - p_n)|\mathcal{M}(|2\rangle \to n)|^2 \tag{9.28}$$

This is the optical theorem for massless particles. For massive particles we would obtain:

$$\mathrm{Im}\{\mathcal{M}(|2\rangle \to |2\rangle)\} = s\sqrt{1 - \frac{4m^2}{s}}\sigma_{tot}(s) \tag{9.29}$$

We can derive some more concrete bounds. This follows from the expansion of the total cross section into partial waves. We can write the forward 2-2 scattering amplitude as:

$$\mathcal{M}(\theta) = 16\pi\sum_{j=0}^{\infty}(2j+1)a_j p_l(\cos\theta) \tag{9.30}$$

where the $p_l$ are Legendre polynomials of the first kind and the coefficients $a_j$ are called partial waves. The Legendre polynomials of the first kind obey an orthogonality condition:

$$p_l(\cos\theta) = \int_{-1}^{1} d(\cos\theta)p_j(\cos\theta)p_{j'}(\cos\theta) = \frac{2}{2j+1}\delta_{jj'} \tag{9.31}$$

and have the property:

$$p_l(1) = 1 \quad \forall l \in \mathbb{Z} \tag{9.32}$$

The cross section can be written as:

$$\sigma(2 \to 2) = \frac{1}{32\pi s} \int_{-1}^{1} d(\cos\theta) |\mathcal{M}(\theta)|^2 \tag{9.33}$$

which upon substitution of the partial wave amplitude expansion and the orthogonality property gives:

$$\sigma(2 \to 2) = \frac{16\pi}{s} \sum_{j=0}^{\infty} |a_j|^2 (2j+1) \tag{9.34}$$

By the optical theorem this implies:

$$\mathrm{Im}\{\mathcal{M}(2 \to 2\} = 2\sqrt{s}|\vec{p}_i| \sum_n \sigma(2 \to n) \tag{9.35}$$

For forward 2-2 scattering one can derive the bounds:

$$0 \leq |a_j|^2 \leq 1 \tag{9.36}$$

$$|a_j|^2 \leq \mathrm{Im}\{a_j\} \tag{9.37}$$

$$0 \leq \mathrm{Im}\{a_j\} \leq 1 \tag{9.38}$$

$$-\frac{1}{2} \leq \mathrm{Re}\{a_j\} \leq \frac{1}{2} \tag{9.39}$$

These bounds indicate the scale at which perturbative unitarity is violated in scattering processes. The partial wave expansion of scattering amplitudes is a key ingredient in the proof of certain analytic properties which we will discuss below, most notably for the Froissart bound.

## 9.4 Analyticity and the S-matrix

### 9.4.1 Analytic properties - preliminary remarks

Analyticity of S-matrix elements together with the optical theorem implies dispersion relations. Dispersion relations are integral relationships between the imaginary -and real parts of an analytic function and were historically first derived in the context of X-ray studies. This resulted in the well-known Kramer-Kronig relations. To derive such relations, we need to postulate that scattering amplitudes as functions of complex Mandelstam invariants possess certain analytic properties. Recall that an analytic function $f(z) = u(x,y) + iv(x,y)$ is one whose real -and imaginary part obey the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{9.40}$$

and that these partial derivatives are continuous. Roughly speaking, the analytic properties we want to postulate are those that allow us to use Cauchy's contour deformation theorem, stating that:

$$\oint_{\gamma_1} f(z) dz = \oint_{\gamma_2} f(z) dz \tag{9.41}$$

for two homotopic closed contours $\gamma_1, \gamma_2$ and a function $f(z)$ that is analytic in the interior -and on both contours. Furthermore, we want to apply Cauchy's integral formula, telling us we can reconstruct an entire analytic function within a contour if we know the function on the contour:

$$f(z_0) = \oint_\gamma \frac{dz}{2\pi i} \frac{f(z)}{z - z_0} \tag{9.42}$$

with $z_0$ a point inside the contour. Even better, Cauchy's formula allows us to reconstruct every derivative of $f(z)$:

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{k+1}} \tag{9.43}$$

An analytic function admits a Laurent series expansion in an annular region $r < |z - z_0| < R$ around the (isolated) singular point $z_0$:

$$f(z) = \sum_n c_n z^n \tag{9.44}$$

where the coefficients $c_n$ are calculated from Cauchy's integral formula (9.43). Finally, we would like to be able to use Cauchy's residue theorem:

$$\oint_\gamma f(z) dz = 2\pi i \sum_n \mathrm{Res}(z_n) \tag{9.45}$$

where we sum over finitely many residues of isolated singularities[5] inside $\gamma$.

Now, how do these properties translate into properties of the S-matrix? We write scattering amplitudes as functions of independent Mandelstam variables $\mathcal{A}(s,t)$ where $s, t$ are related to $u$ via $s + t + u = 4m^2$ on-shell, for 2-2 scattering of particles of the same mass. These are Lorentz invariants and so the S-matrix is postulated to be a Lorentz invariant object. For scattering scalars and massless particles with spin, the amplitude is a Lorentz scalar. However, when scattering massive spinning particles, the amplitude transforms non-trivially under the Poincaré group. More precisely, for spin $j$, they transform in the $(2j+1)$-dimensional unitary irreducible representation of the little group $SO(3)$. For massless spinning particles, with little group $ISO(2)$, the little group transformations (which are phases $e^{i\theta(W,p)\sigma}$) cancels out and the amplitude behaves as a Lorentz scalar. The quantum number $\sigma$, denoting the helicity, behaves as an internal quantum number[6].

We focus on forward elastic 2-2 scattering. Forward scattering is a restriction on the kinematics, namely, it remains the same after the interaction. Elasticity implies

[5]Recall that isolated singularities come in three flavours: removable, poles and essential singularities. Removable singularities have vanishing residues. For poles of order $k$ we have a formula for computing residues. For essential singularities we cannot compute the residue algorithmically but instead we need to study directly the Laurent series of the function.

[6]We make these remarks here because an essential ingredient that enters into the derivation of dispersion relations is crossing symmetry. For 2-2 scattering of massless scalars in the forward limit, crossing symmetry simplifies to the statement $\mathcal{A}(-s) = \mathcal{A}(s)$. However, it becomes more involved for massive/masless fermions/bosons with spin [110]

that even the internal quantum numbers are preserved in the process. In the forward limit $\theta \to 0$, or equivalently $t \to 0$, the amplitude is a function of the CoM energy only $\mathcal{A}(s,t) \to \mathcal{A}(s)$. We allow the CoM energy $s$ to take complex values. In order to derive dispersion relations we want to deform contours so we need scattering amplitudes to be analytic inside and on those contours. It is therefore postulated that S-matrix elements are analytic inside most of the complex plane, *except* on -and near the real axis. Here, it has poles and branch cuts. These poles and branch cuts have physical interpretations. For example, at the pair-production threshold energy $s = 4m^2$ the amplitude becomes imaginary. This is the starting point of a branch cut, connecting the branch point $s = 4m^2$ and the point at infinity $\infty$. If $m = 0$, we see that branch cuts extend towards the origin $s = 0$. This is typically resolved by deforming the theory at low energy, i.e. giving the massless particle a small mass so that branch cuts do not extend to the origin $s = 0$. This is not without subtleties and we will address them later. For now we neglect those issues and assume the amplitude to be analytic around the origin.

Besides s-channel pair-production, we can also have u-channel pair-production in the forward limit with its associated branch cut. It is also assumed that the S-matrix has as few singularities as possible. However, some singularities are completely physical and hence unavoidable. For instance, we can have singularities associated to the (s-channel) formation of bound states at $s < 4m^2$ and/or for the exchange of off-shell particles.

### 9.4.2 Analytic properties of the S-matrix

It is not hard to see that the appearance of an imaginary part of the amplitude comes together with a branch cut. If $\tilde{s}$ is the threshold energy for pair-production of some particle species, we have that for $s < \tilde{s}$ and $s \in \mathbb{R}$ the (forward) amplitude $\mathcal{A}(s)$ is real and the amplitude obeys the so-called Schwartz reflection principle

$$\mathcal{A}(s) = \mathcal{A}(s^\star)^\star \tag{9.46}$$

For $s \in \mathbb{C}$ slightly complex and $s > \tilde{s}$ we have that:

$$\mathrm{Re}\{\mathcal{A}(s + i\epsilon)\} = \mathrm{Re}\{\mathcal{A}(s - i\epsilon)\} \tag{9.47}$$

$$\mathrm{Im}\{\mathcal{A}(s + i\epsilon)\} = -\,\mathrm{Im}\{\mathcal{A}(s - i\epsilon)\} \tag{9.48}$$

Thus, there is a discontinuity along the real axis, starting at $s = \tilde{s}$. This analytic property is not shared by all 2-2 forward scattering amplitudes. In particular, massive spin-$\frac{1}{2}$ fermions alter the analytic structure of the amplitude via their polarizations.

Application of Cauchy's integral formula gives us an integral representation of the forward 2-2 scattering amplitude along a convenient contour $\Gamma$:

$$\frac{1}{k!}\mathcal{A}^{(k)}(s = 0) = \oint_\Gamma \frac{ds}{2\pi i}\frac{\mathcal{A}(s)}{s^{k+1}} \tag{9.49}$$

More generally, we consider a point $s_0 \ll \Lambda$ where $\Lambda$ is the cut off of some effective theory. By analyticity we can Laurent (Taylor) expand around this complex energy:

$$\mathcal{A}(s) = \mathcal{A}(s_0) + (s - s_0)\mathcal{A}'(s_0) + \frac{1}{2!}(s - s_0)^2\mathcal{A}''(s_0) + \dots \tag{9.50}$$

and Cauchy's integral formula gives us the coefficients, if $s = s_0$ is the only singularity inside the contour $\Gamma$. There might be more poles in the low-energy regime and we should take into account their corresponding residues:

$$\frac{1}{k!}\mathcal{A}^{(k)}(s_0) + \sum_{allpoles} Res\Big[\frac{\mathcal{A}(s)}{(s-s_0)^{k+1}}\Big] = \oint_\Gamma \frac{ds}{2\pi i}\frac{\mathcal{A}(s)}{(s-s_0)^{k+1}} \tag{9.51}$$

For now we neglect these extra residue contributions which essentially boils down to choosing the point $s = s_0$ to satisfy $s_0 \gg m_i^2$ where $m_i^2$ denote the locations of IR singularities. To compute (9.49) and derive a dispersion relation, the idea is to deform this contour. Since $s_0 \ll \Lambda$ we can express the scattering amplitude $\mathcal{A}(s)$ within the EFT $\mathcal{L} = \sum_i c_i \frac{\mathcal{O}_i}{\Lambda^i}$. Indeed, the amplitude evaluated in the EFT should match the computation of the amplitude in the full theory at low energies (this is precisely what matching is). The RHS of (9.49) runs over all values of $s$ including $s \gg \Lambda$. The idea now is to prove that the integral on the RHS is strictly positive in any QFT whose S-matrix is unitary and analytic, i.e. in any unitary, local and causal field theoretic UV completion of our EFT at hand. This implies positivity bounds on the Wilson coefficients of our EFT. The typical structure of the complex s-plane and the contours we consider look like this:

To compute such Cauchy integrals we need more analytic properties. We want the S-matrix to be local. Locality is implemented in the S-matrix via a property known as polynomial boundedness. It is demanded that S-matrix elements do not grow faster than a certain polynomial in $s, t, u$. More precisely, in the limit $s \to \infty$ we require the amplitude cannot increase faster than $|s|^2$: $\lim_{s\to\infty} \frac{\mathcal{A}(s)}{s^2} \to 0$. The integrand in the Cauchy integral (for $s = 0$) therefore satisfies:

$$\frac{\mathcal{A}(s)}{s^3} \to 0 \quad \text{as} \quad |s| \to \infty \tag{9.52}$$

This is the lowest degree polymomial in $s$ that is compatible with locality. Obviously, polynomials of higher degree in $s$ are more convergent. The properties polynomial boundedness (locality) and unitarity (the optical theorem) together imply the Froissart bound which states that amplitudes are exponentially bounded at large CoM energies:

$$\mathcal{A}(s) \lesssim \sqrt{s}\ln s \quad \text{as} \quad s \to \infty \tag{9.53}$$

The above relation holds in four spacetime dimensions but the generalization is simple:

$$\mathcal{A}(s) \lesssim \sqrt{s}\ln^{\frac{4-d}{2}} s \tag{9.54}$$

Before proceeding, let us mention that "polynomial boundedness" is not robust against quantum gravity effects because theories of quantum gravity can exhibit non-local effects. The partial wave expansion of the amplitude no longer converges the Froissart bound is violated.

In the absence of quantum gravitational effects, the Froissart bound guarantees healthy behaviour of the scattering amplitude at large $s$. Therefore, the integrand vanishes along the semi-circular curves with infinite radius. Thus, we only need to compute the integral along the branch cuts:

$$\frac{1}{2}\mathcal{A}''(s=0) = \int_{cuts} \frac{ds}{2\pi i}\frac{\mathcal{A}(s)}{s^3} \tag{9.55}$$

where we can evaluate at $s = 0$ by assumption. Along a branch cut, the amplitude is a discontinuous function and makes a jump $Disc(\mathcal{A}(s)) = \mathcal{A}(x + i\epsilon) - \mathcal{A}(x - i\epsilon)$ where $\epsilon$ is a positive infinitesimal and $x$ denotes the real part of $s$. We assume that the amplitude satisfies the Schwartz reflection principle:

$$\mathcal{A}(s^\star) = (\mathcal{A}(s))^\star \tag{9.56}$$

Then, the integral along the branch cuts only depends on this discontinuity. Since $Disc(\mathcal{A}(s)) = 2i\,\mathrm{Im}\{\mathcal{A}(s)\}$ we have:

$$\frac{1}{2}\mathcal{A}''(s=0) = \frac{1}{\pi}\int_{cuts} \frac{Disc(\mathcal{A}(s))}{s^3} ds = \frac{1}{\pi}\int_{cuts} \frac{\mathrm{Im}\{\mathcal{A}(s)\}}{s^3} ds \tag{9.57}$$

Note that by the optical theorem we can relate the discontinuity to the total cross section for 2-2 scattering, which is strictly positive:

$$\mathcal{A}''(s=0) = \frac{2}{\pi}\int_{cuts} \frac{s\sigma(s)}{s^3} ds > 0 \tag{9.58}$$

There is one additional property of the S-matrix that can be invoked: crossing-symmetry. Crossing-symmetry is a non-perturbative symmetry of the S-matrix and follows from the LSZ reduction formula. In terms of S-matrix elements it states that for spin-0 particles:

$$\langle \ldots \phi(p)|\, S\, | \ldots \rangle \,|_{p=-k} = \langle \ldots |\, S\, | \phi^\star(k) \ldots \rangle \tag{9.59}$$

meaning that the amplitude for the scattering process with particle $\phi(p)$ in the final state is equal to the amplitude for anti-particle $\phi^\star(-p)$ in the initial state. In the case of the forward 2-2 scattering of identical particles (scalars) and in terms of the Mandelstam variables, crossing symmetry translates into the statement $\mathcal{A}(s) = \mathcal{A}(-s)$. Crossing-symmetry becomes less trivial when the scattering process involves particles with non-trivial polarizations, and when the particles are fermions we have the possible additional complication of an extra minus sign due to their anticommuting properties. This sign might spoil the positivity of the dispersion relation (9.60). Proceeding with the scalars, the integral along the cuts can be written as:

$$\mathcal{A}''(s=0) = \frac{4}{\pi}\int_{\tilde{s}}^{\infty} \frac{s\sigma(s)}{s^3} ds > 0 \tag{9.60}$$

where $\tilde{s} = m^2$ is the threshold energy for the lightest multi-particle state. Expression (9.60) is called a twice-subtracted dispersion relation, "twice" referring to the second derivative of $\mathcal{A}(s)$ with respect to $s$. It gives a relation between the deep IR and the deep UV. The LHS can be calculated within the effective theory at hand while the RHS is an integral over an arbitrary large energy interval. This relation is derived for spinless particles. At this point it is not clear whether such a dispersion relation can be generalized to particles with (arbitrary) spin. We also have assumed that the massless limit is smooth and does not spoil the analysis. Furthermore, we have neglected other possible low-energy poles.

As a simple example, consider the effective theory of an abelian higgs model:

$$\mathcal{L} = (\partial\pi)^2 + \frac{c}{M_h^4}(\partial\pi)^4 + \ldots \tag{9.61}$$

where $M_h$ is the Higgs mass and the contact interaction $(\partial\pi)^4$ is generated by integrating-out the Higgs at tree-level. The total amplitude of 2-2 scattering of two identical scalars is the sum of three channels. At tree-level in the effective theory, the amplitude is:

$$\mathcal{M}(s,t,u) = \frac{c}{M_h^4}(s^2+t^2+u^2) \tag{9.62}$$

which in the forward limit becomes $\mathcal{A} = \frac{2c}{M_h^4}s^2$ and so $\mathcal{A}''(s=0) = \frac{4c}{M_h^4}$. By the relation (9.60) $c$ is positive.

### Proving the WGC

When one wants to derive positivity bounds in theories containing a massless graviton in the spectrum, one needs to consider the 2-2 forward elastic scattering of gravitationally interacting particles with spin. Scattering of gravitationally interacting particles is singular in the forward limit:

$$\mathcal{M}(s,t\to 0) \sim -\frac{s^2}{M_{pl}^2 t} + \mathcal{O}(s) \tag{9.63}$$

This is the Coulomb singularity. To prove the electric WGC one has to regulate this singularity, as is done in [104]. Let us first briefly mention that positive dispersion relations for particles with any spin can be derived. In the case of 2-to-2 scattering of massless spinning particles the amplitude is dressed with indices:

$$\mathcal{M}(p_{1,a_1}^{\sigma_1}, p_{2,a_2}^{\sigma_2} \to p_{3,a_3}^{\sigma_3}, p_{4,a_4}^{\sigma_4}) \tag{9.64}$$

We consider forward elastic 2-2 scattering so this implies that $p_1^{\sigma_1} = p_3^{\sigma_3}$ and $p_2^{\sigma_2} = p_4^{\sigma_4}$. The elasticity implies $a_1 = a_3, a_2 = a_4$. Under a little group transformation, the phases cancel out and the amplitude behaves as a Lorentz scalar. Thus, the helicity index simply labels the external particles, just like the other internal quantum numbers $a$. Spinning particles have polarization vectors or spinors and these might introduce unconventional non-analyticities in the complex $s$-plane [110]. In [104] the dispersion relation was generalized to particles with arbitrary spin.

# Chapter 10

# Conclusion

We have seen that an understanding of the quantum gravity swampland requires a fair amount of knowledge from several areas of physics, most notably string theory and black hole physics. Furthermore, besides being relevant for questions in cosmology, the swampland program has a wide range of applicability in particle physics. With respect to cosmology, we have only focussed on the consequences for large-field inflation and a little bit on the initial conditions problem. However, the swampland ideas also find application in the eternal inflation scenario and in studies of the nature of dark matter and its interactions. For instance, in many models dark matter particles are modelled as massive vectors, so-called dark photons, whose masses vary orders of magnitude between different models. The sublattice WGC, however, implies that large portions from the allowed regions of the parameter space of dark photon masses are inconsistent [72].

It has also become clear that the swampland program is still in its infancy and that the conclusions we draw from the various conjectures are under heavy debate. It is in the authors opinion very likely that the answers the swampland conjectures provide to questions in cosmology and particle physics today, might change radically over time. Even when neglecting all possible subtleties related to the swampland conjectures, a complete rigorous answer to whether for instance large-field inflation can or cannot be realized in quantum gravity, could not be given. Nevertheless, let us come back to our two main research questions listed in the introduction and summarise the answers that today are thought to be valid or at least in the right direction.

1. *What are the consequences of the swampland conjectures for various large-field models of inflation?*

   The best, and most reliable, answer comes from the electric WGC in its zero-form formulation. In the case of natural inflation, which is based on a single axion and there are no potentially dangerous loopholes, there are convincing arguments that $f > M_{pl}$ is forbidden. Recently, however, it has been claimed that $f > M_{pl}$ is possible [111]. A distinction is made between an effective axion decay constant $f_e$ and a UV decay constant $f$. Since we model inflation in EFT, it is $f_e$ that matters and not $f$. One can then obtain $f_e \gg M_{pl}$ while $f < M_{pl}$. This illustrates that it is still an open question of whether or not $f < M_{pl}$ is entirely ruled out by the swampland conjectures. For extra-natural inflation we could evade the constraint from the electric WGC but the model can still be ruled-out via the magnetic WGC. The zero-form conjecture, or rather the convex-hull condition, brings multi-axion inflation (N-flation) in danger. However, this is subject to a possible loophole. In practice, it is extremely difficult to use this loophole to evade the WGC constraint. Extra-natural N-flation can be ruled out via the magnetic WGC.

With the same level of rigorousity we can study the implications for pole inflation and $\alpha$-attractors. The constraint coming from the distance conjecture is not as severe as the one from the WGC and in fact does not prevent $\alpha$-attractors from reaching 50-60 e-foldings of inflation. The evolution of pole inflation into pole-N-flation does not ameliorate the situation: it is in conflict with the sublattice WGC [112]. When we come to axion monodromy inflation in the framework of Kaloper-Sorbo, however, it is not at all clear how "worse" the situation really is. We lack a proper generalization of the WGC to massive discrete gauge theories.

Up to today, based on the WGC it is impossible to give a model independent answer whether large-field inflation in all its generality belongs to the swampland. From the point of view of the distance conjecture it (unfortunately) depends on your personal taste. Some authors state the distance conjecture as $\Delta\phi < 1.2M_{pl}$ or $\Delta\phi \lesssim \mathcal{O}(1)M_{pl}$ and then it becomes obvious that large-field inflation is in the swampland [40, 55]. However, as stated above, when we formulate the distance conjecture as it was originally proposed and study its implications, it allows for enough e-foldings in $\alpha$-attractors. The de Sitter conjecture does not directly indicate a possible problem with large-field inflation but it forbids single-field slow-roll inflation.

2. *Can we prove or derive the swampland conjectures from a set of well-established physical ideas/principles?*

   It is clear that we can construct explicit stringy settings in which various swampland conjectures hold. The difficulty of coming up with consistent counterexamples also suggests that we are on the right track. But this is not satisfying enough and it would be great if we could physically motivate the swampland conjectures, otherwise one could argue it is just mathematics.

   In the last years and months there has been a lot of improvement in this direction, starting with the emergence proposal. In chapter 7 we have seen that the assumption that the loop expansions of the photon -and graviton propagators break down at parametrically the same scale, which is justified by the WGC, and the assumption that a tower of states becomes massless implies a specific structure of the moduli space metric that generates infinite distances. The emergence proposal, which states that all physics in the infrared emerges from UV dynamics, could be a more general underlying principle of the swampland program.

   Finally, the ideas of positivity that developed parallel to the swampland program, have shown to be very useful for the swampland program. The ideas of positivity are on a firm footing and it is therefore great to see that, even though one could have certain objections against the proofs, it has been shown that these constraints imply the electric WGC.

## 10.1 Further lines of research

In the author's view, we will never get a definite answer to research question 1 if we have not properly answered question 2. However, let us nevertheless first mention a further research direction more closely related to question 1 and next sketch some possible new research lines for question 2.

1. *Extending the WGC to massive gauge theories*

   It would be a very interesting question whether it is possible to extend the (electric) WGC to massive gauge theories. This is equivalent to the question of whether black holes can be charged under massive gauge fields. If they can, we can use the standard "all extremal black holes should decay" argument to avoid pathologies. There are some hints that the WGC indeed can be extended to massive theories, as noted in [69]. In [113] it is claimed that when throwing a charged particle into a black hole it takes a time $t \sim \frac{1}{m}$ with $m$ the mass of the gauge field for the black hole to discharge. The uncertainty principle $\Delta E \Delta t \lesssim 1$ implies that for shorter time scales the gauge field mass is experimentally unobservable. The electric WGC constrains the massless gauge theory on these time scales. It is therefore naturally to expect that the WGC constrains the massive gauge theory at least in the limit $m \to 0$. If we have such a "massive" WGC we can constrain the Kaloper-Sorbo framework of axion monodromy.

   This is not the entire story as the WGC also needs to be generalized to discrete gauge theories and to domain walls, i.e. two-branes in $d = 4$.

2. *Can we formulate $\alpha$-attractors in the Kaloper-Sorbo language*

   It would be interesting to see how the hyperbolic geometry of $\alpha$-attractors manifests itself in the three-form gauge theory language of Kaloper-Sorbo. Conceptually this is easy but it probably very challenging on a mathematical level. But besides being interesting, this question might lack relevance. The only thing one seems to achieve is that $\alpha$-attractors is made consistent with the non-existence of global symmetries.

3. *Application of the swampland program to fundamental questions in particle physics*

   The three fundamental fine-tuning or naturalness problems in the standard model are the hierarchy problem, the strong-CP problem and the cosmological constant problem. Recently, the WGC has been applied to the first one in [114]. The best solution to date for the strong-CP problem is the introduction of the global $U(1)_{PQ}$ symmetry which spontaneously breaks and hence dynamically explains why $\Theta$ is extremely small. The goldstone boson is the axion with its global PQ shift symmetry. Quantum gravity breaks all global symmetries so this solution cannot be fundamental, i.e. the PQ solution is in the swampland. We should look for landscape consistent solutions to the strong-CP problem.

4. *Further applications of positivity*

   Positivity bounds have proven to be succesful in the context of the WGC. Is it possible to use them similarly as support for other swampland conjectures? For example, would it be possible to derive a positivity bound such that the cosmological constant always satisfies $\Lambda < 0$ in an EFT of quantum gravity so that dS space is forbidden?

   Positivity also puts an upperbound on the softness degree of scattering amplitudes. In the last 5 years it has become clear that soft theorems -and limits are related to asymptotic symmetries. Does this suggest that there is some relation between positivity and asymptotic symmetries?

5. *Proving the swampland conjectures rigorously*

   The ultimate goal is to prove the swampland conjectures. A promising approach is the emergence proposal, partially discussed in chapter 7. This has shown that infinite moduli space distances emerge from integrating-out towers of states that couple to the moduli and that the infinite tower of states becomes exponentially fast light as a function of the geodesic distance in the moduli space. Yet, this does not prove rigorously that large-moduli limits always yield infinite towers of light states.


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