RUG

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Classification of Exceptional EFTs

An introduction

Thesis BSc Physics

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Abstract:
The subjects of non-linear symmetries with corresponding Goldstone modes and effective field theories (EFTs) are introduced in this paper. Here, special attention is given to EFTs with enhanced soft limits and its subclass, exceptional EFTs. Furthermore, a systematic approach for finding a consistent set of non-linear symmetries is discussed leading to specific tree structures of Goldstone fields. The method is applied to the cases of a single scalar and a single vector Goldstone. This results in a full classification of theories where for the scalar the only exceptional EFTs correspond to the scalar Dirac-Born-Infeld theory and the special Galileon. The only vector-algebra that gives rise to a consistent theory is found to be that of a massive vector coupled to a special Galileon. The specific interaction terms are investigated and show strong correspondence to findings from other authors.

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1 Introduction

The starting point for many philosophical debates about science is mostly whether the framework we constructed is *true* or even *real*. Sometimes, this framework is referred to as a paradigm: a set of rules and observations on which almost everybody agrees as being important. Although this applies to a certain extend also to physics, physicists realize that a systematic switching between these paradigms is of great convenience. For example, when studying a system slow-moving particles, the physics we calculate is well approximated by Newtonian mechanics and relativistic theories would only complicate matters unnecessarily. The systematic approach in switching from relativity to classical mechanics is simply,

$$\frac{v}{c} \ll 1.$$

In other words, we can let $c \to \infty$ and thus computations which are initially complex or even unsolvable become simple. A similar mechanistic approach is what underlies the practice of effective field theory (EFT). The idea of EFTs [1] is choosing a scale of interest from the set of all scales which affect the physics, for example in terms of mass,

$$\frac{m}{M} \ll 1,$$

where m is the mass scale of interest similar to the low-speeds of interest, v, in the above discussion. As an example consider a Lagrangian of a real scalar field with higher order interactions represented by the sum [2],

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{g_n}{M^{n-4}}\frac{\phi^n}{n!},$$

where ϕ has the dimensionality of m such that the total mass dimension of every term is fixed to be m^4. Although the UV-fields do not appear explicitly in the Lagrangian, the UV-theory still determines the coefficients, g_n, in the EFT. Therefore, not being aware of the high energy behaviour we need to measure these coefficients to complete the theory. The EFT-logic now goes as following: all interactions with $n > 4$ are suppressed by M to some power. This means that up to some desired accuracy $n = k > 4$, we can neglect all other terms in the Lagrangian which have $n > k$. The fact that only finite set of interactions remains, greatly simplifies the task of computing the observables. For example, in the context of Quantum Field Theory (QFT) one can compute the observables for certain processes from the full Lagrangian if it is known and then take the appropriate limit to obtain the results for the scale of interest. However, the approach of first building a Lagrangian for this specific scale, like we did above, and then calculating the observables is more convenient. Sometimes up to the point where one calculation is possible and the other not. This is a 'top-down' approach. On the other hand, sometimes a high energy Lagrangian is not available and to make any prediction about the physics, an EFT can be set up. Here, the method for finding an useful EFT is choosing the most general Lagrangian with the relevant degrees of freedom and which obeys all symmetries. Naturally, this approach is called 'bottom-up'.

The symmetries which need to be satisfied by the theory can consist of, for example, the usual rotations and boosts[1] or some transformations from one field into another (internal). Usually, we think of these symmetries as being satisfied by the physical system for any energy. Such a symmetry is linearly realized by all states. Nevertheless, there are certain types of symmetry which are non-linearly realized by the lowest-energy state: the vacuum. This means that a field theory close to the vacuum does not 'see' the full

[1] Referred to as Lorentz transformations from hereon.

symmetry group but instead is invariant under some non-linear transformations in which the complete group is hidden. A consequence is that we can encounter massless particles corresponding to a non-linear symmetry. These are called Goldstones. As a sidenote: non-linear symmetries are also reffered to as 'broken' symmetries. This lets it seem that these symmetries can be disposed of or have no effect on the dynamics anymore. Since this is clearly not the case, this paper will adopt the former nomenclature of linear and non-linear. For internal symmetry groups, every distinct non-linear transformation corresponds to a Goldstone. When also spacetime symmetries are involved, the story becomes more complex: some of the Goldstone fields will seize to exist but leave their trace on the dynamics of the field or fields that are left. Regardless, we can build a Lagrangian which satisfies the non-linear symmetry by means of what is referred to as the coset construction. In a certain way, this is a 'top-down' method since knowledge about the full symmetry group is assumed.

In general, these Goldstone fields affect the observed physics in a profound manner. The scattering amplitude[2] for a transition involving the emission of a low-energy Goldstone mode vanishes. The phenomenon is known as Adler's zero [3]. It can be generalized to the concept of soft limits: when the amplitude scales as the Goldstone's momentum, p, to a certain power, σ, in the limit $p \to 0$. For Adler's zero $\sigma = 1$ and thus the amplitude is linear in momentum and indeed vanishes in the soft limit. For more special cases with $\sigma \geq 2$, the soft behaviour is *enhanced.* The existence of an enhanced soft limit requires non-linear *spacetime* symmetries in addition to internal ones. On the other hand, a soft limit can also be trivial when derivatives appear in the Lagrangian. For example, it is unsurprising that the kinetic term vanishes in a zero-momentum situation. Theories which give rise to enhanced soft limits and are least trivial, are reffered to as exceptional EFTs. One of these exceptional EFTs is the so called special Galileon.

A goal of this paper is to start from the non-linear symmetries and construct the most general Lie-algebraic structure that is allowed. After this, it is investigated which algebras specifically give rise to exceptional EFTs. The emphasis will lie on algebras with a vector Goldstone and more precisely on what different couplings there exist for a gauge vector and special Galileon scalar.

Firstly, we start by examining internal non-linear symmetries at themselves and try to give some intuition about what this physically means. Tied into this is the coset construction which yields the apparatus to set up a Lagrangian which is invariant under the symmetry in question. Secondly, the formalism is extended to non-linear spacetime symmetries and utilized to eliminate the redundant Goldstones. Here, there will also be given some explanation as to why this is a logical step to take. After these sections, soft limits and some relevant EFTs with enhanced soft limits are briefly reviewed. Subsequently, we connect the non-linear spacetime symmetry discussion with enhanced soft limits and thus give a hint on how to classify the corresponding theories. Using the background above, a systematic classification method is introduced involving 'inverse Higgs trees' and other algebraic arguments. Equipped with these tools, the case of a single scalar Goldstone is studied and we will see that it produces two exceptional EFTs. Then we take a step further and try to classify the exceptional EFTs arising from a vector Goldstone. As we will see, the vector carries some more complications but this is solved by coupling it to a special Galileon. Finally, the interactions corresponding to the algebra with the vector and special Galileon are investigated.

[2]Scattering amplitudes are probability amplitudes for a scattering of particles which can be calculated by looking at the Feynman diagrams corresponding to that scattering.

2 Non-linear Symmetries and Soft Limits

2.1 Non-linear Internal Symmetry and Goldstone's Theorem

We first consider the hypothetical situation where a man is standing on the top of a completely symmetric hill like in 1. That means that he can walk down the hill in any straight line and find himself at the lowest point after the same number of steps. Our man, in an effort of minimizing his action, desires to reach the lowest point but because of the hill's symmetry he has to choose randomly. Once he has reached his destiny, he realizes that the symmetry that manifested to him before he walked down, is no longer. Subsequently, he notices that he can still follow a path (the blue line in 1), comprised of an infinite number of lowest points, without climbing: in a circle around the mountain. This follows directly from the symmetry he observed at the top of the hill. The analogy tries to explain the concept of a non-linearly realized symmetry: although the hill will always exhibit the same symmetry, it is no longer manifest in one of the infinitely many ground states. Yet, it still tells you how to move from one vacuum state to another. Let us now translate this story into terms of field theory [4, 5]. Consider two scalar fields, ϕ_1 and ϕ_2, and a potential, U, invariant under global, internal $SO(2)$ rotation transformations on the fields:

$$U(\phi_1, \phi_2) = \lambda[\phi_1^2 + \phi_2^2 - a^2]^2. \tag{1}$$

It is important to point out that this potential, depicted in 1, includes mass terms of both fields.

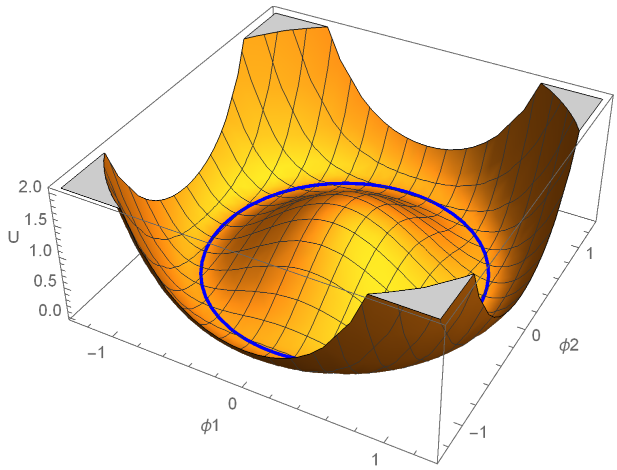

Figure 1: A plot of the potential function from (1) *where the blue circle indicates the vacuum.*

Now choose one of the infinitely many vacuum states: $\langle\phi_1\rangle_0 = a, \langle\phi_2\rangle_0 = 0$ and shift the fields such that $\phi_1' = \phi_1 - a, \phi_2' = \phi_2$. The potential as a function of these transformed fields takes the form:

$$U(\phi_1', \phi_2') = \lambda[\phi_1'^2 + \phi_2'^2 - 2a\phi_1']^2. \tag{2}$$

Because of the ϕ_1' term appearing in (2), it no longer exhibits the $SO(2)$ symmetry. Moreover, after expanding it becomes clear that the potential contains no mass terms for ϕ_2'. This statement is better justified when switching to polar coordinates defined by $\phi_1 = \rho\cos\theta$ and $\phi_2 = \rho\sin\theta$. An expansion about the vacuum in these coordinates similar to (2) yields:

$$\mathcal{L}(\rho,\theta) = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\rho+a)^2(\partial_\mu\theta)^2 - U(\rho+a). \tag{3}$$

Again we encounter a massless field, this time θ. These modes are known as Goldstone bosons. Now, when acting on the fields with a $SO(2)$ transformation, ρ is unchanged and $\theta \to \theta + \alpha$. This non-linear constant shift symmetry in the θ field shows that $SO(2)$ is indeed *non-linearly* realized. It is not specific for this theory that we encounter massless fields. On the contrary, it has been proven in full generality [6] and is referred to as Goldstone's theorem. It states that every non-linearly realized symmetry corresponds to a Goldstone boson.

Let us explain this more precisely in terms of the Lie algebra associated with the Lie group that governs the symmetry. The reader is referred to [7] for a refresher on the topics of Lie groups and algebras but let us quickly sketch the most relevant subjects for this paper. A symmetry group such as rotations in three dimensions can locally be described by a set of generators. One can go from the group to the generators by differentiation of the representations of the group and vice versa by exponentiation. In this framework, we think of the symmetry group as a Lie group and the Lie algebra then contains the associated generators.

In the context of Goldstone's theorem, let T^A be the generators of the full symmetry group G and a combination of the generators $T^A = T^a + T^\alpha$ which respectively generate the subgroup H and coset space G/H. The (left) coset is the set of elements that result from left-multiplication of a subgroup, H, by all elements of the complete group, G. All cosets of the group together form the coset space denoted by G/H. Immediately the commutation relations between these generators follow:

$$[T^a, T^b] = f^{ab}{}_c T^c, \quad [T^\alpha, T^a] = f^{\alpha a}{}_\beta T^\beta, \quad [T^\alpha, T^\beta] = f^{\alpha\beta}{}_a T^a + f^{\alpha\beta}{}_\gamma T^\gamma. \tag{4}$$

The subgroup H is defined to leave the vacuum state invariant, $T^a|0\rangle = 0$, whereas the generators of the coset space, the non-linear generators, do not, $T^\alpha|0\rangle \neq 0$. Each of these non-linear generators corresponds, according to Goldstone's theorem, to a massless field, ϕ_α. The simple example from (1) only has one generator corresponding to the rotation which is also the non-linear generator with corresponding massless mode, θ. Acting with an element of the coset space on the vacuum and working infinitesimally yields fluctuations about the vacuum:

$$|0\rangle \to e^{\phi_\alpha T^\alpha}|0\rangle = |0\rangle + \phi_\alpha T^\alpha|0\rangle + \mathcal{O}(\phi_\alpha T^\alpha)|0\rangle. \tag{5}$$

Notice that this infinitesimal change is not necessarily non-zero for every type of non-linear symmetry. This has implications on the number of massless fields as will be discussed in the next section. For now, let us assume they are non-zero. In (5) we have tacitly parameterized an element in the coset space in some neighbourhood of the identity of G. Similarly, an element of the full group can be written uniquely as $g = e^{\phi_\alpha T^\alpha} e^{\phi_a T^a}$. This convenient choice of parameterization [8] clearly shows how the ϕ_α fields transform under the action of the full group by means of:

$$g\, e^{\phi_\alpha T^\alpha} = e^{\phi'_\alpha T^\alpha} e^{\phi'_a T^a} \tag{6}$$

One can easily verify that this action satisfies the conditions for group multiplication. For any other field ψ which is a linear representation of the subgroup H, transformation under g is defined as:

$$g: \quad \psi \to \psi' = D(e^{\phi'_a T^a})\psi = e^{\phi'_a U^a}\psi. \tag{7}$$

Here the representation D is also written as an exponential with U representing T which will prove to be convenient later on. The non-linear realization of the group G is thus defined by (6) together with (7). Furthermore, these transformations reduce to linear ones if we consider actions of the subgroup H:

$$h\, e^{\phi_\alpha T^\alpha} = (h\, e^{\phi_\alpha T^\alpha} h^{-1})(h) = e^{\phi'_\alpha T^\alpha} e^{\phi'_a T^a}. \tag{8}$$

Using (4) in combination with (8) yields $\phi_\alpha \to D_2(h)\phi_\alpha$ and by identifying $e^{\phi'_a T^a} = h$ it naturally follows that $\phi_a \to D(h)\phi_a$. In general, we can put any non-linear realization of a compact, semi-simple internal group G, linear on the subgroup H, in this form. The latter statement is more apparent by interpreting ϕ_α and ψ as coordinates on the manifold corresponding to our Lie-group G. We refer to [8] for full details but the most important point now is that if we can find some set of coordinates (ϕ_α, Ψ) which transform linearly on H, then we can extract the standard coordinates by $(0, \psi) = e^{-\phi_\alpha T^\alpha}(\phi_\alpha, \Psi)$. Finally, the standard coordinates transform as in (6) and (7).

This formalism has a clear application in constructing Lagrangians which are invariant under the non-linear realization because it provides for the covariant building blocks: ψ. An ordinary derivative of the fields is in general not one of these blocks and therefore we need to introduce a modified type of derivative, the covariant derivative of the coset construction [9]. We insist that these covariant derivatives also follow (7). In order to discover their explicit form, let us first state the infinitesimal form of (6) and (7):

$$g\, de^{\phi_\alpha T^\alpha} = de^{\phi'_\alpha T^\alpha} e^{\phi'_a T^a} + e^{\phi'_\alpha T^\alpha} de^{\phi'_a T^a}, \tag{9}$$

$$d\psi' = de^{\phi'_a U^a}\psi + e^{\phi'_a U^a} d\psi. \tag{10}$$

At this point, the trick is to set $g = e^{-\phi_\alpha T^\alpha}$ which has as consequence that $\phi'_\alpha = 0, \phi'_a = 0, \psi' = \psi$. Of course, this is exactly the change of coordinates which yields the standard form if the initial set of coordinates is linear on H. Now it is clear that, since indeed $d\phi_\alpha$ and $d\psi$ transform linearly under H, their covariant counterparts are exactly constructed by the aforementioned trick. In other words, using $g = e^{-\phi_\alpha T^\alpha}$ yields the covariant derivatives which correspond to $d\phi'_\alpha$ and $d\psi'$. Let us now perform this trick:

$$e^{-\phi_\alpha T^\alpha} de^{\phi_\alpha T^\alpha} = d\phi'_\alpha T^\alpha + de\phi'_a T^a = \omega_\alpha T^\alpha + \omega_a T^a = ((\omega_\alpha)_\mu T^\alpha + (\omega_a)_\mu T^a)dx^\mu, \tag{11}$$

$$d\psi' = d\phi'_a U^a \psi + d\psi = d\psi + \omega_a U^a \psi. \tag{12}$$

(11) is known as the Maurer-Cartan form. Lastly, the covariant derivatives are straightforwardly written as:

$$D_\mu \phi_\alpha = (\omega_\alpha)_\mu, \tag{13}$$

$$D_\mu \psi = \partial_\mu \psi + (\omega_a)_\mu U^a \psi. \tag{14}$$

To summarize, there exist symmetry groups which leave the Lagrangian but not the vacuum invariant. This is called a non-linear realization of the group. The group can then be split into a subgroup which leaves the ground state invariant and the coset space which does not. Every element of this coset space induces fluctuations about the vacuum which correspond to the massless Goldstone bosons. By introducing the coset construction formalism, we can find all transformation laws and covariant objects to construct an invariant Lagrangian.

2.2 Spacetime extensions

In the previous section the coset construction was introduced building upon the ideas of Goldstone's theorem and symmetry 'breaking' in the sole context of internal symmetries. In this section, this formalism will be adjusted to allow also spacetime symmetries to be included [10, 11, 12]. That is, we extend G, initially only containing internal transformations, by Poincaré symmetries. It necessarily implies that the subgroup, H, contains in addition to the linear internals also the Lorentz group but not the full Poincaré group. This is due to the fact that translations themselves are a non-linear transformation in the spacetime coordinates. Therefore, we have to include momentum, P^μ, into the coset space G/H. This has several crucial consequences. Firstly, it expands the possibility of having only scalar, spin-0 generators and corresponding fields to ones that have any spin. Secondly, the number of Goldstone fields does not have to equal the dimension of the coset space anymore. Finally, the spacetime coordinates will now transform like the standard form introduced in the preceding section and we have to adjust our metric and invariant measure accordingly.

The spacetime extension of the coset construction is in practice rather straightforward. It is a matter of choosing the right parameterization of the coset space:

$$e^{x_\mu P^\mu} e^{\phi_\alpha T^\alpha}. \tag{15}$$

We can treat x_μ, the coordinate corresponding to translations, similar to $\phi_\alpha(x_\mu)$ in the procedure. Notice that x^μ are the spacetime coordinates of the linearly realized subgroup H. In other words, the modes $\phi_\alpha(x_\mu)$ can only propagate in the directions of the linear translations. The action of g can then be summarized by:

$$g: \quad x \to x',\ \phi_\alpha(x) \to \phi'_\alpha(x'),\ \psi(x) \to \psi'(x') = e^{\phi'_a(\phi_\alpha(x),g)U^a}\psi(x). \tag{16}$$

In (16) the argument of ϕ'_α in the last transformation is explicitly stated to avoid confusion about any x' dependence. Subsequently, the Maurer-Cartan form can be computed to find our covariant building blocks:

$$e^{-\phi_\alpha T^\alpha} e^{-x_\mu P^\mu} d(e^{x_\mu P^\mu} e^{\phi_\alpha T^\alpha}) = ((\omega_\nu)_\mu P^\nu + (\omega_\alpha)_\mu T^\alpha + (\omega_a)_\mu T^a)dx^\mu. \tag{17}$$

Analogous to using $(\omega^\alpha)_\mu$ instead of $\partial_\mu \phi^\alpha$, we have to consider $(\omega^\mu)_\nu dx^\nu$ instead of dx_μ. The objects which are in absence of the non-linear symmetry constructed from dx_μ are now built up from the Maurer Cartan components for translations. These objects include the invariant measure, dV, and the spacetime metric, $g_{\mu\nu}$, [10]:

$$dV = -iDet(\omega)d^4x, \qquad g_{\mu\nu} = \omega^\kappa{}_\mu \omega^\lambda{}_\nu \eta_{\kappa\lambda}. \tag{18}$$

In the beginning of this section we said that an important consequence of including spacetime symmetries was that not all non-linear symmetries corresponded to a massless mode. In other words, Goldstone's theorem is not valid anymore.

The secret which underlies this counting problem can be found in (17). If we would compute the Maurer Cartan form explicitly, we would come across commutators of the generators T^α and T^a with translations. Firstly, (4) is extended to:

$$[P^\mu, P^\nu] = 0, \quad [P^\mu, T^a] = f^{\mu a}{}_\nu P^\nu, \quad [P^\mu, T^\alpha] = f^{\mu\alpha}{}_a T^a + f^{\mu\alpha}{}_\beta T^\beta. \tag{19}$$

The third commutation relation in (19) allows for some ϕ_α to be linearly connected to the generator T^β in ω_β. Explicitly this is up to first order:

$$\omega_\beta = (\partial_\mu \phi_\beta - f_{\mu\beta}{}^\alpha \phi_\alpha)dx^\mu. \tag{20}$$

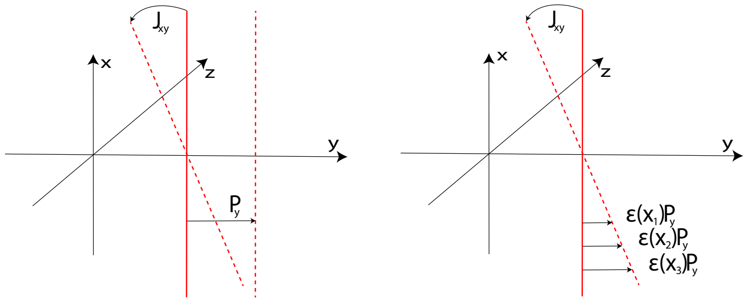

Figure 2: Two global spacetime symmetry transformations, P_y and J_{xy}, are shown in the left figure. On the right it is depicted how locally P_y can simulate a J_{xy} transformation.

Remember that the non-linear generators were defined such that they allowed for fluctuations about the vacuum (see (5)). The Lagrangian which consists of the covariant building blocks will act on the vacuum via these objects and result in a similar equation:

$$\omega_\beta T^\beta|0\rangle = (\partial_\mu \phi_\beta - f_{\mu\beta}{}^\alpha \phi_\alpha) dx^\mu T^\beta |0\rangle. \tag{21}$$

Notice that this is in linear order. From (21) we can see that when we include spacetime symmetries, some fluctuations can be non-existent by expressing one Goldstone field in terms of a derivative of the other. Clearly, this removes some of the fields (ϕ_α in this case) from our description. Setting one of the Maurer Cartan components to zero to relate two fields is called an Inverse Higgs Constraint (IHC) [12, 11]. In general you can eliminate a number of fields by this method which are called *inessential* and the ones that remain are then *essential*. The number essentials and inessentials are in total still equal to the number of non-linear generators but because the essentials will only appear in the physics, there are not as many Goldstones as non-linear generators.

The fact that some of these fields are related can be explained by thinking of a infinite string in conventional spacetime [13]. It reaches all the way out to infinite $-x$ and $+x$ in a space with coordinates x, y and z. Clearly, translating the string in x leaves it unchanged. Also, rotating it in the y, z plane has no effect. All other translations and rotations are non-linearly realized by the vacuum. Now, if we focus on the rotations in the x, y plane, generated by J_{xy}, and translations in the y direction, generated by P_y, we can easily infer that these are two distinct transformations that do not belong to H. See the left hand-side of 2. Therefore, they can be assigned to two Goldstone modes: ϵ for the translation and θ for the rotation. As already discussed, these modes depend on the H spacetime coordinates so in this case only x. Thus their fluctuations,

$$\theta(x) J_{xy}|0\rangle, \quad \epsilon(x) P_y|0\rangle. \tag{22}$$

The punchline of this story is that we can, by letting $\epsilon(x)$ depend on x, deform the translation-fluctuation such that it exactly looks like the rotation-fluctuation. This is depicted on the right hand-side of 2. In other words, pulling the string piece by piece (locally) to a certain position is the same as rotating the whole string. The other way around, local rotations and global translations, is completely analogous. In conclusion, we have now two dependent modes and thus not all fields corresponding to non-linear generators appear in the spectrum.

2.3 Effective Field Theories with Soft Limits

Non-linear symmetries affect the infrared behaviour of the theory by means of Adler's zero [3]. If we have some transition between two general states, $\alpha \to \beta$, then its amplitude when emitting a Goldstone will be proportional to the momentum of the Goldstone in the zero limit of this momentum. The concept of soft limits [14] is a generalization of Adler's zero: the amplitude is now proportional to any power of the momentum, denoted by σ. To be precise:

$$\lim_{p\to 0}\mathcal{A}(p)=\mathcal{O}(p^{\sigma}). \tag{23}$$

Of course, $\sigma=1$ corresponds to Adler's zero and if we have $\sigma\geq 2$ the amplitude vanishes even faster. The latter has the nomenclature *enhanced soft limit.* This enhancement is strictly bounded from above by $\sigma\leq 3$ which in this paper will also become clear from an algebraic perspective. It is important to realize that derivative terms, $\partial^{(m)}\phi^{(n)}$, in the Lagrangian will already imply a softening of the amplitude. In fact, in the trivial case, the amplitude will be proportional to the momentum to a power $(m+2)/(n+2)$ where already the kinetic term has been included. Thus, a soft limit degree is only non-trivial if the amplitude is softer than one would expect based on the derivative power counting [15]:

$$\sigma>\frac{m+2}{n+2}, \tag{24}$$

where already the kinetic term is taken into account. There is a subclass of theories with enhanced soft limits which is exceptional: they have the maximum soft degree for a given m/n. This results in a maximally restricted theory. For a complete overview of theories with soft behaviour the reader is referred to [14, 16]. However, a few EFTs with enhanced soft limits, reappearing later on in the article, will be discussed globally. We are not particularly interested in $\sigma=1$ and therefore start at $\sigma=2$. There are two theories of interest: the Dirac-Born-Infeld (DBI) [17, 18] and Galileon EFTs [19]. The BI Lagrangian, a precursor of the DBI EFT, was introduced by Born to deal with the divergent terms in quantum electrodynamics and was later generalized by Born and Infeld to also include the gravitational field [20]. In order to not let physical quantities diverge, they took a form of the Lagrangian analogous to the γ-factor of Einstein's relativity [17]. This is also reflected in the Lagrangian for the DBI which is a scalar generalization of the BI:

$$\mathcal{L}=-F^{d}\sqrt{1\pm\frac{\partial\phi\cdot\partial\phi}{F^{d}}}+F^{d}. \tag{25}$$

As we can see it has effectively one derivative per field. Since it has a maximal soft limit for this counting, the DBI is an exceptional EFT. Its action corresponds to a non-linear realization of a five-dimensional Poincaré algebra with the four-dimensional Poincaré as the linear realized subgroup. In terms of generators, this means that the coset needs to include both the Lorentz generator of the fifth dimension as well as translations in this dimension. If we define $\mu=4$ to be our fifth-dimension then the non-linear generators are thus P^4 and $J^{\mu 4}$. Notice that we could define the metric to be both $g_{44}=\pm 1$ for respectively a time- or space-like extra dimension. The sign we choose corresponds exactly to the $\pm$ in (25).

The Galileon action originates as an infrared modification of gravity [21]. There is a connection between this EFT and the DBI [22]. That is, the five-dimensional Poincaré group can be contracted to a non-relativistic version. In a way, this is analogous to the transition from special relativity to Galilean relativity. The resulting non-linear realization yields, from Galilean coordinate transformations, a well-known transformation rule for the Goldstone:

$$\pi\to\pi+b_{\mu}x^{\mu}+c. \tag{26}$$

Now the non-linear generators are B^{μ} and C responsible for respectively the x^{μ} and constant term in the transformation rule.

The Galileon has a hidden symmetry for a specific choice of coefficients [23]. It has an additional transformation, generated by $S^{\mu\nu}$:

$$\delta\pi = s_{\mu\nu}x^\mu x^\nu + \alpha s^{\mu\nu}\partial_\mu\pi\partial_\nu\pi. \tag{27}$$

As will become apparent later on in this paper, the quadratic term leads to an even softer limit: $\sigma = 3$. Because it is the only EFT known with this soft degree, it is automatically also an exceptional EFT.

2.4 How inessentials are essential

Before classifying any EFTs with soft limits, we must first find out how these two are connected. In the following, this relation will be clarified and we will use the opportunity to explicitly derive transformation rules with the apparatus of the coset construction.

First of all, it was shown in [14] that a polynomial of degree n in the transformation rule of the Goldstone generally leads to a soft limit of degree $\sigma = n+1$. Having already encountered two examples, the Galileon with $n = 1 \implies \sigma = 2$ and special Galileon $n = 2 \implies \sigma = 3$, this relation may not come as a surprise. The important thing about this observation in the context of non-linear spacetime symmetries is that polynomials naturally occur in the transformation rule for the Goldstone once we include inessential generators. We will illustrate this by an explicit example in which Q is the essential and K^μ is the inessential generator. As we discussed in section 2.2 demanding K^μ to be inessential requires a commutation relation between this generator and translations of:

$$[P^\mu, K^\nu] = g^{\mu\nu}Q + \text{linear generators}. \tag{28}$$

Let us start with parameterizing the coset space. We assign the coordinate θ to Q and ξ_μ to K^μ. In principle, this will look like (15) resulting in:

$$\gamma = e^{x_\mu P^\mu} e^{\theta Q + \xi_\mu K^\mu}.$$

However, there is a certain freedom in choosing this element allowing for a more convenient choice that is generally less stringent [24]:

$$\gamma = e^{x_\mu P^\mu} e^{\xi_\mu K^\mu} e^{\theta Q}. \tag{29}$$

Now acting with an element of g on this coset element yields the transformation rules. To remind the reader, multiplication of two exponentials is given by the Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\cdots}. \tag{30}$$

The first action which is considered, is that of the essential Q which of course looks like $e^{\alpha Q}\gamma$. By inspecting (30), it becomes apparent that the computation of this action hinges on the commutation relations between the three generators. Since Q is an essential, scalar generator we can let it commute with all other generators. Thus,

$$e^{\alpha Q}\gamma = e^{\alpha Q} e^{x_\mu P^\mu} e^{\xi_\mu K^\mu} e^{\theta Q} = e^{x_\mu P^\mu} e^{\xi_\mu K^\mu} e^{(\alpha+\theta)Q}. \tag{31}$$

Observe that $\alpha + \theta$ is now the transformed coordinate θ' as in (6). This induces the explicit non-linear transformation $\theta \to \theta + \alpha$ which is similar to the transformation rule for the first Goldstone that we encountered in section 2.1. Of course this is no coincidence: as is clear from the explicit computation, an algebra only involving scalar Goldstones will generally only involve linear shifts. These type of shifts with $n = 0$ lead to $\sigma = 1$ or in other words Adler's zero. In this context, Adler's zero is a natural consequence of the transformation rule for the Goldstone. However, it starts to become yet more interesting in the presence of inessentials and this is the situation right now. Focusing just on θ' one has not have to deal with any K^μ commutators other than (28) when calculating the action of the inessential on the coset element. In first order,

$$e^{\beta_\nu K^\nu}\gamma = e^{\beta_\nu K^\nu}e^{x_\mu P^\mu}e^{\xi_\mu K^\mu}e^{\theta Q} = e^{x_\mu P^\mu + [\beta_\nu K^\nu, x_\mu P^\mu]}e^{\beta_\mu K^\mu}e^{\xi_\mu K^\mu}e^{\theta Q} =$$
$$e^{x_\mu P^\mu + \beta_\mu x_\nu (g^{\mu\nu}Q)}e^{\beta_\mu K^\mu}e^{\xi_\mu K^\mu}e^{\theta Q} = e^{x'_\mu P^\mu}e^{\xi'_\mu K^\mu}e^{(\theta+\beta_\mu x^\mu)Q}. \tag{32}$$

It is apparent from (32) that the inessential generator induces a transformation rule of the Goldstone, $\delta_\beta\theta = \beta_\mu x^\mu$, that is of a polynomial form[3] with degree $n = 1$. By exercising this simple example, a similar result to the Galileon transformation, (26), is obtained. In fact, choosing the generators $Q = C$ and $K^\mu = B^\mu$ to be exactly those of the Galileon yields trivially the exact same transformation rule as (26). For higher order inessential generators this result can be generalized if the algebra satisfies certain simple conditions which will be discussed in the next section. It is then important to realize that when generalized [25] these extended symmetries alone do not produce non-trivial soft limits. For exceptional algebras, also field dependent terms must be present in the transformation rules and this requires non-vanishing commutators between non-linear generators.

3 A Systematic Approach to Classification

3.1 Inverse Higgs Tree

From section 2.2 we know that the IHC relates two different non-linear modes such that it eliminates the essentials in favour of any inessentials. It required the computation of the Maurer-Cartan form and setting one of its components to zero. There is an intuitive shortcut to this method [13]. Consider again the linear T^a and non-linear T^α generators. A non-linear realized spacetime symmetry allows for some fluctuations induced by non-linear generators to vanish. Mathematically, there exists solutions to:

$$\phi_\alpha T^\alpha|0\rangle = 0. \tag{33}$$

This is not to be confused with just a non-linear generator acting on the vacuum state because that would still be zero. The important point is that because not all ϕ_α are linearly independent there exist solutions to (33). The next step is then to act with momentum on this equation. We allow for the momentum to take its operator form $-\partial_\mu$. Then it follows naturally that:

$$0 = -P_\mu(\phi^\alpha T_\alpha)|0\rangle = -(P_\mu\phi^\alpha T_\alpha + \phi^\alpha[P_\mu, T_\alpha])|0\rangle = (\partial_\mu\phi^\alpha - f_{\mu\beta}{}^\alpha\phi^\beta)T_\alpha|0\rangle. \tag{34}$$

Notice that (34) and (21) are of similar form so this expression too allows for elimination. The commutator $[P_\mu, T_\alpha]$ is still given by (19) but here it acts on the vacuum so any linear generators appearing in the commutator will drop out. In fact, if the commutator of translations with the non-linear generator under consideration only contains linear generators, it cannot be expressed in terms of any other non-linear generators. This is exactly how we defined our essentials and therefore it is also a necessary condition for an essential generator G_0 to satisfy:

$$[P^\mu, G_0] = \text{linear generators}. \tag{35}$$

Let us now apply (34) to express an essential Goldstone mode in terms of an inessential:

$$\partial_\mu\phi^0 = f^0{}_{\mu\beta}\phi^\beta + \mathcal{O}(\phi^2). \tag{36}$$

[3]At least, when combined with the constant shift $\delta_\alpha\theta = \alpha$.

Here, also higher order terms can be included without disobeying (33). Of course, (36) can only have a first order non-zero contribution if $[P^\mu, G^\beta] = f^{\mu\beta}{}_0 G^0$. Indeed, this formalism is similar to the IHC. Furthermore, in contrast to the IHCs it is impossible to calculate any higher order, field dependence. The reason, however, for introducing this strategy, is its natural ability to generalize. Namely, act with another copy of translations, P_ν, on (36). Then, if there is a non-zero translational commutator for the T_β associated with ϕ^β ($f^\beta{}_{\mu\alpha} \neq 0$), it can be expressed in terms of yet another inessential Goldstone,

$$\partial_\mu \partial_\nu \phi^0 = f^\beta{}_{\mu\alpha} f^0{}_{\nu\beta} \phi^\alpha + \mathcal{O}(\phi^2). \tag{37}$$

Schematically[4], this looks like $[P, [P, G_2]] = [P, G_1] = G_0$ meaning that we can 'trace' back some inessential to the essential generator via a path of inessentials. At this stage there is however no reason to assume that the first and second order inessentials are irreducible representations of the Lorentz group. As a matter of fact, it is the decomposition at every order that gives rise to a tree structure, in [26] appropriately called the *inverse Higgs tree.*

The most convenient choice for denoting a Lorentz irreducible representation is by the eigenvalues, (j_1, j_2). This notation is due to the fact that we can 'decouple' the Lorentz algebra into the algebra of $SU(2) \otimes SU(2)$. In other words, *locally* there exists a correspondence [27],

$$SO(3,1) \simeq SU(2) \otimes SU(2), \tag{38}$$

where $SO(3,1)$ stands for the Lorentz group with three spatial dimensions and one temporal. Therefore, the representation theory of the Lorentz group is determined by that of $SU(2) \otimes SU(2)$. This is convenient since the irreps of the latter are well-known.

Observe that the group $SU(2)$ is the double cover of the group of three dimensional proper rotations, $SO(3)$. This means that two elements of $SU(2)$ are mapped to exactly one in $SO(3)$. The irreps of $SO(3)$ are characterized by their spin so to each irrep in $SU(2)$ an eigenvalue of half-integer or integer spin j is assigned. Thus the irreps for the two $SU(2)$ algebras and hence also the Lorentz group are denoted by two of these eigenvalues: j_1 and j_2. To acquire some insight into this notation, an overview of some fields with their corresponding notation is given:

$$\begin{array}{cc} (0,0) & \text{scalar,} \\ (\frac{1}{2},0),(0,\frac{1}{2}) & \text{Weyl-spinors,} \\ (\frac{1}{2},\frac{1}{2}) & \text{four-vector,} \\ (1,0),(0,1) & \text{anti-symmetric traceless tensor,} \\ (1,1) & \text{symmetric traceless tensor.} \end{array} \tag{39}$$

We can see that the degrees of freedom for any of these fields corresponds to $(2j_1 + 1)(2j_2 + 1)$. For example, a symmetric traceless tensor has in total 9 independent entries which indeed is equal to $(2 + 1)(2 + 1) = 9$.

Although the subject of Lorentz representation theory can be discussed in much more detail and rigor, we will only use the fact that a given direct product of two representations is decomposed by the usual Clebsch-Gordan series. This is completely analogous to the decomposition of quantum mechanical spins. To make this precise, consider the following example:

$$(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (0,0) \oplus (1,0) \oplus (0,1) \oplus (1,1). \tag{40}$$

This means that two spin $s = 1$ fields decompose into one $s = 0$ scalar field, two $s = 1$ fields and one $s = 2$ fields.

[4] In this paper a schematic commutation relation is just one without structure constants or even indices if not necessary, only emphasizing the proportionality.

Now we return to the inverse Higgs tree again. Realizing that momentum corresponds to $(\frac{1}{2}, \frac{1}{2})$, we can infer that by acting with the momentum on a general field with spin s we obtain one $s-1$, two s and one $s+1$ fields at first order similar to (40). This can be repeated at any order and thus the tree grows.

At this point, we only demanded $[P, G_n] = G_{n-1} + ...$: a generator at order n must always contain a generator of order $n-1$ in its commutator. However, in the ellipses we may still find any other generator. This results in a confusing web of interconnections which makes any concrete statement about the structure impossible. We can make a consistent choice of generators that contains only generators which lie on a specific route to the essential. For example, we choose the generators of the $s-1$ second order irrep such that one commutes into a $s-1$ first order and another one into a s first order generator and so on. The generators that lie on a path are linearly independent meaning that we will find an unique combination of generators in one commutator. This also forbids the presence of an end-point generator of a specific path in one of the commutator of translations with other generators belonging to that path. If this was not the case and some generator G^p of path p included the end-point of the same p then acting a finite number of times again with the commutator would result in a combination of generators including G^p ultimately spoiling the linear independence. A slight hinge of this argument is that we could introduce more than one essential generator, say a spin-0 and spin-2, which are connected to the same inessential, in this case some spin-1. This does not adhere to the linear independence requirement and also the assumption about the end-points will not work. Excluding this situation and working only with scalar and vector[5] essentials, the commutators for a generator belonging to a specific path p and tree i are simplified to:

$$[P, G^{(i,n,p)}] = G^{(i,n-1,p)} + \text{linear generators}. \tag{41}$$

This is due to the specific basis that can be chosen based on the assumption of linear independence of the generators lying on one path and that an end-point of a path can never occur in a commutator. Also, in case of multiple trees i each belonging to the essential G^i, the trees decouple.

Observation (37) gave rise to this tree structure but also restricts it. Without altering the connection between the modes, we can interchange μ and ν on the right hand-side. This unsurprisingly also implies that the left hand-side and thus $f^{\beta}{}_{\mu\alpha} f^{0}{}_{\nu\beta}$ is symmetric in μ and ν. We already stated that this expression amounted to $[P_\mu, [P_\nu, G_2]]$ and naturally the symmetry holds here. This also follows from another, ultimately equivalent, condition: the Jacobi identity. A Jacobi identity is a consistency condition on the commutator operator belonging to the algebra. If we have three elements A, B and C in some algebra the Jacobi identity tells us that $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$. Since translations commute, the Jacobi identity on (P_μ, P_ν, G_2) trivially reproduces our symmetry statement. The relevance for the inverse Higgs tree is that it reduces the number of independent generators at second order and higher. By symmetry it relates different paths to the essential. An equivalent way of viewing this, is by Taylor expanding the essential spin s irrep up to second order. This sort of expansion makes sense from the perspective of (37): by integrating the partials out on the left hand-side, the right hand-side will contain a polynomial with the same symmetry properties as the expansion. Take the scalar as example,

$$\phi = c + c_\mu x^\mu + c_{\mu\nu} x^\mu x^\nu. \tag{42}$$

From this, it is obvious that all coefficients $c_{\mu_1 \ldots \mu_n}$ must be completely symmetric at any order: $c_{[\mu_1 \ldots \mu_n]} = 0$. In the general case, the coefficients will contain a number of

[5] Although not included in this paper, this also holds for fermions

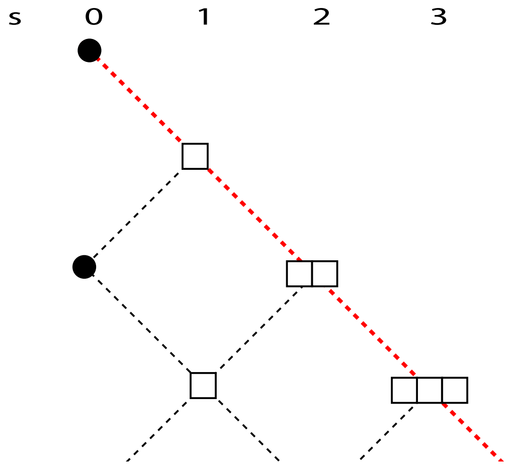

Figure 3: The inverse Higgs tree for an essential scalar Goldstone. The spin corresponding to the generator is denoted by 's'. On the red line only the symmetric traceless generators lie.

symmetric indices equal to the order. Not all generators we came up with, 'live' in the Taylor expansion of the essential. To be precise, the initial degeneracy of $(1, 4, 6, 4, 1)$ reduces to $(1, 2, 4, 2, 1)$ by this argument.

If we had a scalar generator corresponding to an essential scalar Goldstone, the same symmetry argument from (37) implies, as discussed, that second order generators are completely symmetric in μ and ν. For example at second order only the traceless symmetric matrix $S^{\mu\nu}$ and the trace $g^{\mu\nu}T$ survive this restriction. These correspond to the spin 0 and 2 Lorentz irreps whereas the spin 1 irrep, an anti-symmetric tensor $T^{\mu\nu}$ vanishes from the tree. Notice that the general $s-2$ and $s-1$ irreps are trivially not included in the tree of a scalar because these simply have no meaning; they do not exist. Of course, also at higher order this restriction plays a role and only allows for diagonals in the diagram. Moreover, all the nodes that disappear from the tree cannot be connected and so generators which could in principle be included by diagonals from some generator are also not present if that generator disappears. For example, at first order in the scalar, there exists no $s = 0$ generator so at second order no $s = 1$ can be included. Notice that the nonexistence of $s = 0$ generators at first order in a scalar tree is a priori excluded by the simple argument that a scalar is not in the decomposition of an essential scalar. More precisely, $(\frac{1}{2}, \frac{1}{2}) \otimes (0, 0) = (\frac{1}{2}, \frac{1}{2})$. A part of the scalar tree is shown in the diagram 3. For a vector essential, the first order unsurprisingly does not look like the second order of the scalar tree. We can of course introduce an anti-symmetric tensor here because (37) and generalizations only applies to order ≥ 2. For now, all three first orders survive but this will change. Moreover, the second and higher orders in the tree will also have a short lifespan in this paper. We will encounter this later on. To conclude the consequences of (37) and its generalizations, we extract also some information about the transformation rules it implies. As was discussed in section 3, the transformation rules of a Goldstone are intimately linked to the existence of the inessentials. Moreover, the existence of any coordinate dependence in the rules is due to the non-zero commutation relations between the inessentials and translations. This is why (37) allows us to immediately see any possible x^μ-dependence. As was explained, the expression is similar to a 'double' commutator with translations, $[P, [P, G_2]]$ and thus its generalizations are simply obtained by commuting it sequentially with more copies of translations. Now, for a transformation rule which is explicitly computed by acting on it with a non-linear inessential, G_n, of order n, also higher order terms of the BCH-formula (30) can be present. This means that the higher order commutators appear in the transformation rule. An inessential of order n, can at most have a non-zero commutator with n copies of translation generators. Because of the inverse Higgs tree, we will always end up with the essential so a $n+1$ commutator will always vanish. The non-vanishing higher order commutators have a crucial consequence for the transformation rules. Every momentum is accompanied by its field, x^μ, and therefore the n copies of translations in some higher order commutator is accompanied by x^n. To make this precise, let $\delta_n \phi^0$ be the infinitesimal change due to the n^{th} order generator. Then:

$$\delta_n \phi^0 = c_n x^n + \ldots . \tag{43}$$

Notice that the right hand-side has to be of the same Lorentz representation as the Goldstone. The form of (43) is of close resemblance to the terms in the scalar's Taylor expansion, (42). As is clear from (43), the 0^{th}-order generator which is of course the generator corresponding to the essential Goldstone, induces a constant shift. This is not surprising for this result was already found in section 3. Also, the non-vanishing commutators between any non-linear generators will induce additional, field dependent transformations which are indicated by ... in (43). Turning to soft limits, in absence of ... a transformation with (43) does not correspond to any non-trivial behaviour. It would amount to introducing $n+1$ derivatives per field as the building blocks such that $\delta \partial^{(n+1)} \phi = \partial^{(n+1)}(c_n x^n) = 0$ and thus respecting the symmetry. These correspond to non-trivial soft limits of $\sigma = n+1$. Following the line of thought of section 2.3, the enhanced soft limits arise when we do not have the $n+1$ derivatives per field and such terms can arise from the coset construction if (43) has field dependence[6]. Thus, although the commutators with translations allow for systematic approach for finding spacetime dependence in the transformation rules, interesting soft limits can only be found when also considering commutators between other generators. We will come back to this point in the classification section.

3.2 Invariance of Kinetic Term

The restrictions we found for the inverse Higgs tree were solely based on the action of momentum on the non-trivial zero fluctuations. In this section, invariance of the kinetic term is demanded which has profound implications for the generators that can be included in the tree.

First of all, terms as (43) have to be of the same Lorentz representation as the Goldstone. For example, the first order version for a scalar looks like $b_\mu x^\mu$. However, higher order generators induce higher order powers of x^μ and for second order, there is a generator which leads to a term $\delta\phi = c x_\mu x^\mu$ with some constant c. Explicitly, the kinetic term transforms under this change as $\delta(\partial_\mu \phi \partial^\mu \phi) = 4c x_\mu \partial^\mu \phi$. This particular term does clearly not leave the kinetic term invariant; even not up to a total derivative. Therefore, the generator that contributes this term in the transformation rule cannot exist in the tree. This constraint can be generalized to higher orders. Again we start with a generic term from (43) but now we let it take a more general form:

$$\delta_n \phi^0 = s_{\mu_1 \ldots \mu_n} x^{\mu_1} \ldots x^{\mu_n} + \ldots . \tag{44}$$

In the previous section it was already concluded that the s parameters are fully symmetric. Let us now work with the second order case but notice that the results can

[6] In absence of field dependence, terms a lower derivative power counting can still follow from Wess-Zumino [28] terms.

be generalized easily. At second order, the infinitesimal change is simply $s_{\mu\nu}x^\mu x^\nu$. A straightforward calculation of the corresponding change for the kinetic term shows:

$$\delta(\partial_\mu\phi\partial^\mu\phi) = 4s_{\mu\nu}x^\nu\partial^\mu\phi = \partial^\mu(4s_{\mu\nu}x^\nu\phi). \tag{45}$$

This last step is only possible if the trace of the parameter is zero: $s_{\mu\nu}g^{\mu\nu} = s_\mu{}^\mu = 0$. This way the derivative on the right hand-side of (45) has only a non-zero contribution for the multiplication with ϕ and thus the equality holds. The necessary condition is that the kinetic term is invariant up to a total derivative and (45) clearly shows that only traceless parameters can be considered. This on its turn means that only traceless generators can be included in the tree and those are which lie on the spin $s = n$ order diagonal. This is also exactly the reason why the traceful option, $cx_\mu x^\mu = cg_{\mu\nu}x^\mu x^\nu$, obviously does *not* satisfy the invariance condition. To conclude, all branches of the scalar tree are pruned except for one consisting of the traceless, symmetric inessentials. These lie on the red line in 3. This necessarily implies that any soft limit of degree σ must have exactly $\sigma - 1$ inessentials in its tree since only one and not more than one generator at each order n can lead to x^n in the transformation rule.

The tree can be slightly more complicated if the essential is a vector. A similar expression for the infinitesimal change as (44) can be constructed but now we have to be aware that the index of the essential vector, A_μ, does not have to satisfy the symmetry condition:

$$\delta_n A_\mu = u_{\mu\nu_1\ldots\nu_{n-1}}x^{\nu_1\ldots\nu_{n-1}} + \ldots . \tag{46}$$

In other words, two possibilities for the parameter u emerge: one where it is fully symmetric and one where it is only symmetric in all the ν indices leaving one anti-symmetric pair. In general, the result of the scalar discussion will also apply here: only the traceless generators can be included if the kinetic term is to be invariant up to a total derivative. However, we can discuss the parameters in more detail if we specialize to a $U(1)$ gauge vector.

This gauge field possibly requires some introduction. In Quantum Electrodynamics (QED) a vector field A_μ is encountered which is associated with the photon [2]. The two degrees of freedom for a photon are the transversal polarization directions. However, the degrees of freedom of the vector are in total four and need to be reduced to the two of the photon. The two reductions are facilitated by the specific equations of motion and the action of the symmetry group corresponding to QED: the $U(1)$ group. The latter can be written as:

$$A_\mu \to A_\mu + \partial_\mu\lambda. \tag{47}$$

It tells us that vectors which are equivalent up to a (47) transformation, are physically the same vectors. It holds for any function $\lambda(x)$ that converges to zero fast enough when $x \to \infty$. Around $x = 0$ we can Taylor expand the derivative function $\partial_\mu\lambda(x)$ as following [29]:

$$\partial_\mu\lambda(x) = \partial_\mu\lambda(x)|_{x=0} + \frac{1}{1!}\partial_\mu\partial_{\nu_1}\lambda(x)|_{x=0}x + \frac{1}{2!}\partial_\mu\partial_{\nu_1}\partial_{\nu_2}\lambda(x)|_{x=0}x^{\nu_1}x^{\nu_2} + \ldots . \tag{48}$$

The evaluation at $x = 0$ means that we can write every λ term as a constant. Taking the derivative of this Taylor expansion results in a familiar form for the change in the vector:

$$\delta A_\mu = \sum_{n=0}^{\infty} l_{\mu\nu_1\ldots\nu_n}x^\mu x^{\nu_1}\ldots x^{\nu_n}. \tag{49}$$

The constants $l_{\mu\nu_1\ldots\nu_n}$ are completely symmetric and not Lorentz irreps since they consist of traceless *and* traceful parts. The transformations in the infinite sum look similar to those of (46) and can also be induced by inserting non-linear generators in the coset construction, $L^{\mu\nu_1\cdots\nu_n}$. However, it must be emphasized that their physical meaning is distinct: the generators corresponding to the gauge transformations collectively eliminate a degree of freedom in a single field whereas the inessentials determine the dynamics of the essential Goldstone and correspond to fields which can be eliminated themselves. Although the two types of generators are different, once one of them is included in the tree, it excludes the presence of the other. In other words, a $U(1)$ gauge field has no symmetric generators which are not gauge generators in its tree.

The only non-linear generators that can be found in the tree of the $U(1)$ gauge vector are those with one anti-symmetric pair of indices. For example, at first order there is only the completely anti-symmetric matrix $N^{\mu\nu}$. As we will see in section 4.2 however, this branch of the $U(1)$ gauge vector tree also dies off. Luckily, this discussion becomes vastly more interesting by discarding gauge symmetries and its two degrees of freedom vectors.

3.3 Finishing Touch

The systematic approach found in [26] which is summarized in this paper concludes with a rather straightforward, nevertheless important step. A consistent algebra must satisfy the Jacobi identities which we discussed earlier on in this paper. The procedure starts with writing all commutators between non-linear generators in its most general form. On the right hand-side of these commutation relations, all possible generators of the algebra are included such that Lorentz invariance is satisfied. Then you systematically work out the Jacobi identities which relate the coefficients on the right hand-side of the commutators. In general, there are only few, distinct choices for these coefficients. Each corresponds to a possible algebra. Finally, the actual classification begins by investigating each of these possible algebras. The main goal is, of course, to see whether some of these algebras correspond to exceptional EFTs.

4 Classification of Exceptional EFTs

First of all, we will classify the theories arising from a single scalar Goldstone after which the vector case will be discussed. Both of these cases include the Poincaré group with a Lie-algebra consisting of the Lorentz generator ($M^{\mu\nu}$) and the translations generator (P^μ) and commutation relations:

$$\begin{aligned}
[M^{\mu\nu}, M^{\kappa\lambda}] &= i(g^{\mu\lambda}M^{\nu\kappa} + g^{\nu\kappa}M^{\mu\lambda} - g^{\mu\kappa}M^{\nu\lambda} - g^{\nu\lambda}M^{\mu\kappa}),\\
[M^{\mu\nu}, P^{\kappa}] &= i(g^{\nu\kappa}P^{\mu} - g^{\mu\kappa}P^{\nu}),\\
[P^\mu, P^\nu] &= 0.
\end{aligned} \tag{50}$$

Moreover a generator in the inverse Higgs tree is denoted by G_n where n indicates the order. Naturally, G_0 is the essential. Furthermore, some general tensor B of any Lorentz representation has the following schematic commutation relations with the Lorentz generator:

$$[M, B] \propto B. \tag{51}$$

This commutator simply tells us that any tensor commutes with Lorentz transformations into itself. The specific form depends on the Lorentz representation of B. Trivial examples are translations P and M itself which are shown above.

4.1 Single Scalar Goldstone

From the preceding section we know that by choosing the right basis, in the commutator of some generator in the tree, only generators appear which lie above. Furthermore, it was concluded that the tree elements must be completely symmetric and traceless. The former observation restricts the commutation relation of the essential, $G_0 = Q$, to the only possibility:

$$[P^\mu, Q] = iaP^\mu. \tag{52}$$

The first order (for convenience denoted by $G_1 = V^\mu$) in the scalar tree can in addition to the necessary essential, also have the linear Lorentz generators in its commutator:

$$[P^\mu, V^\nu] = i(b^1 g^{\mu\nu} Q + cM^{\mu\nu} + d\epsilon^{\mu\nu\kappa\lambda} M_{\kappa\lambda}). \tag{53}$$

As we have seen, the decomposition of a product of two vectors, $(\frac{1}{2}, \frac{1}{2})$, yields two anti-symmetric matrices in addition to the trace and the traceless symmetric one. Remember that M is also an anti-symmetric tensor. The fact that there are two of this type in the decomposition is roughly speaking why we can add the term $\epsilon^{\mu\nu\kappa\lambda} M_{\kappa\lambda}$. At last, since we run out of generators, the higher-order commutators, $G^n = G^{\mu_1 \cdots \mu_n}$, all have the general form:

$$[P^\mu, G^{\nu_1 \cdots \nu_n}] = ib^n g^{\mu(\nu_1} G^{\nu_2 \ldots \nu_n)}. \tag{54}$$

Here, the notation $(\nu_1, \nu_2 \ldots \nu_n)$ emphasizes that all of these indices must be completely symmetric since only completely symmetric tensors are included in the tree. Moving on the commutation relations between two non-linear generators, it can be written in the most general form where G^m is taken to be the generator of highest order:

$$\begin{aligned}
[G^{\mu_1 \cdots \mu_m}, G^{\nu_1 \cdots \nu_n}] = i \sum_{k=k_{min}}^{n} e_k^{(m,n)} \prod_{q=1}^{k} g^{\mu_q \nu_q} G^{\mu_{k+1} \cdots \mu_m \nu_{k+1} \cdots \nu_n} \\
+ if^m \prod_{q=1}^{m-1} g^{\mu_q \nu_q} P^{\mu_m} \quad (only\, if\; m = n+1) \\
+ \prod_{q=1}^{m-1} g^{\mu_q \nu_q} (h_0^m M^{\mu_m \nu_m} + h_1^m \epsilon^{\mu_m \nu_m \kappa\lambda} M_{\kappa\lambda}) \quad (only\, if\; m = n).
\end{aligned} \tag{55}$$

In all of the above expressions a few constants occur: $a, \ldots, f, h$ which will be determined by the Jacobi identities in a moment. For a finite tree with largest order Z, the first term in (55) cannot contain any generators which are of the order greater than Z. This is reflected in the lower bound of the sum which we take $k_{min} = \frac{1}{2}(m + n - Z)$. Now, we present the different scenarios allowed by Jacobi identities for every tree size Z. We will consider three cases: $Z = 1, 2$ and > 2. For $Z = 0$, there does obviously not exist any inverse Higgs relation and for the choice $a = 0$ it reduces to the most elementary algebra discussed multiple times in this paper. However, not yet encountered is the case where $a \neq 0$. This corresponds to the scalar generator for dilations. It is an additional spacetime symmetry also known as scaling symmetry. The corresponding transformations can be thought of as zooming in or out on a specific region and thus rescaling the spacetime coordinate $x \to \lambda x$ [30]. From the standpoint of the coset construction, when this dilatation is the non-linear generator and thus has for example $a = 1$, it necessarily induces a scaling of the spacetime coordinate:

$$g\,\gamma = (e^{\alpha D})(e^{xP} e^{\theta D}) = e^{(1+\alpha)xP} e^{(\theta+\alpha)Q}, \tag{56}$$

where we can now identify the scaling $\lambda = 1 + \alpha$. The scaling symmetry of spacetime is part of the larger conformal symmetry group. In essence, all conformal transformations are defined to preserve angles. Naturally, rescaling the coordinates has no effect in this respect but there are additional *special conformal transformations* which are generated by a vector K^μ which also adheres to this defining property.

4.1.1 $Z = 1$

For this case we can include some inessential vector in the scalar's (Q) tree: V^μ. It was shown in [16] that after satisfying all Jacobi identities, the algebra simplifies to:

$$\begin{aligned}
[P^\mu, V^\nu] &= i(g^{\mu\nu} Q + u M^{\mu\nu}),\\
[P^\mu, Q] &= -iuP^\mu,\\
[V^\mu, V^\nu] &= -ivM^{\mu\nu},\\
[V^\mu, Q] &= i(vP^\mu + uK^\mu).
\end{aligned} \tag{57}$$

There are three distinct choices for the coefficients u and v. The first and simplest one leads to the algebra where $u = v = 0$ which allows for only one non-vanishing commutator: $[P^\mu, V^\nu] = ig^{\mu\nu}Q$. This particular choice is compatible with the non-linear realized five-dimensional Galileon algebra where we identify $Q = C$ and $V^\mu = B^\mu$. Secondly, we let u vanish and v be compensated for by rescaling the non-linear generators. It leads to:

$$[P^\mu, V^\nu] = ig^{\mu\nu}Q, \quad [P^\mu, Q] = 0, \quad [V^\mu, V^\nu] = \mp iM^{\mu\nu}, \quad [V^\mu, Q] = \pm iP^\mu. \tag{58}$$

By identifying the non-linear generators with $Q = P^4$ and $V^\mu = M^{\mu 4}$ we can see that this algebra coincides with five-dimensional Poincaré which was already discussed in section 3.3. The $\pm$ in (58) indicate the nature of the extra dimension. The $+$ corresponds to an extra spatial dimension with the metric $g^{44} = +1$ and $-$ to the a fifth time-like dimension with $g^{44} = -1$. The belonging EFT is that of the DBI.

Finally, the choice $u \neq 0$ leads to the conformal algebra with $Q = D$ and $V^\mu = K^\mu$ regardless what value v takes. By rescaling and redefining non-linear generators if necessary the commutation relations read:

$$[P^\mu, K^\nu] = ig^{\mu\nu}Q + iM^{\mu\nu}, \quad [P^\mu, D] = -iP^\mu, \quad [K^\mu, K^\nu] = 0, \quad [K^\mu, D] = iK^\mu. \tag{59}$$

In summary, for $Z = 1$ three distinct algebras emerge of which two give rise to an EFT with an enhanced soft limit of degree $\sigma = 2$. This is due to the fact that there is one inessential in the tree which induces linear x-dependence in the transformation rule. Moreover, the choice $u = 0$, $v \neq 0$ results in the exceptional EFT corresponding to the DBI action. The lack of non-vanishing commutators between non-linear generators in the Galileon algebra stops it from being exceptional because it removes any field dependence in the transformation rule. Furthermore, when $u \neq 0$ we obtain the conformal algebra. The resulting EFT, in contrast to the other two, does not have an enhanced soft limit. The soft limit is spoiled by the non-zero commutator between the essential dilatation generator and translations [31, 32].

4.1.2 $Z = 2$

Now, a generator is added to the second order of the tree. This needs to be, as already discussed, a symmetric traceless tensor, $S^{\mu\nu}$, of the spin $s = 2$ Lorentz representation. The algebra is determined by the general expressions given in the pages above. After imposing Jacobi identities, we also restrict ourselves to the case of a vanishing translation commutator of the essential. The last restriction allows for Adler's zero and thus soft limits and excludes dilatations. This exclusion, moreover, is natural since conformal algebras cannot be complemented with a second order generator [33] which is the case of interest. For full details we refer the reader to [16]. The non-trivial commutation relations between non-linear generators will then have the following schematic form:

$$\begin{aligned}
&[V, V] = 0, \quad [V, Q] = 0, \quad [P, V] = Q,\\
&[S, S] = aJ, \quad [S, P] = V, \quad [S, V] = aP.
\end{aligned} \tag{60}$$

Notice that in disposing of the dilatation option, we also set the commutators between first and zeroth order generators to zero. In other words, the only $Z = 2$ algebras which allow for soft limits are extensions of the Galileon algebra. This $Z = 1$ algebra forms a subalgebra: all generators forming the subset commute into each other. If a vanishes, only the commutators necessary for IHCs are non-zero. This, as emphasized multiple times, means that we have only shifts in the transformation rule and no field dependence. The exceptional EFT can therefore only arise for the non-zero choice of a. Would we compute the transformation rules corresponding to this algebraic structure, we obtain (27) of the special Galileon concluding that for $Z = 2$, the special Galileon is the exceptional EFT.

4.1.3 $Z > 2$

The $Z = 2$ case generalizes to higher order. As we have seen, in addition to the Jacobi identities, it demands the absence of dilatations which results in the Galileon subalgebra of $Z = 1$. The same occurs for higher Z: a subalgebra arises with all vanishing commutators between two non-linear generators apart from the highest order:

$$[G^Z, G^Z] = aM, \quad [G^Z, G^Z - 1] = aP. \tag{61}$$

However, for higher orders $Z > 2$ the Jacobi identity involving the three highest order non-linear generators becomes problematic. Would $a \neq 0$ but 1,

$$\begin{aligned} 0 = [G^Z, [G^{Z-1}, G^{Z-2}]] + [G^{Z-2}, [G^Z, G^{Z-1}]] + [G^{Z-1}, [G^{Z-2}, G^Z]] \\ = 0 + [G^{Z-2}, P] + 0 = G^{Z-3}. \end{aligned}$$

Indeed, this requires us to cut the branch leaving only generators with $Z' < Z - 3$. We can keep pruning the tree until we reach a length of branch for which $[P, G^{Z-2}] = [P, Q] = 0$. In other words, $Z = 2$ is the maximum length for which non-vanishing commutators between non-linear generators exist. Therefore, the only algebras which can lead to exceptional EFTs must have $Z \leq 2$ [14].

4.2 A Vector Goldstone

Again the requirements of section 3 will be assumed which are to some extent similar to the scalar. In contrast to the scalar, the vector can also have not only fully symmetric tensor in its tree but also ones with an antisymmetric pair. The simplest example of this was given to be the first order anti-symmetric tensor $T^{\mu\nu}$. We will denote the essential vector generator by Q^μ and its associated Goldstone mode by A^μ. In analogy to the scalar discussion, we will first set up the general structure of the algebra by considering the usual requirements and Jacobi identities.

First of all, we will consider no generators other than the essential and those of the Poincaré. Except for the commutation relations which were already specified in (50) and (55), the only possible form allowed by Lorentz invariance is:

$$[P^\mu, Q^\nu] = aM^{\mu\nu}, \quad [Q^\mu, Q^\nu] = bM^{\mu\nu} + cN^{\mu\nu}. \tag{62}$$

Notice that for an *essential* vector generator no scalar is included in its translation commutator. The $N^{\mu\nu}$ generators can at this point be symmetric or anti-symmetric. Also, in (62) a sum over multiple N is possible but as we shall see this term will not even be possible anyway. Let us first fix the constant a by the Jacobi identity consisting of two copies of momentum and the non-linear generator,

$$\begin{aligned} 0 &= [P^\mu, [P^\nu, Q^\lambda]] + [Q^\lambda, [P^\mu, P^\nu]] + [P^\nu, [Q^\lambda, P^\mu]] \\ &= a[P^\mu, M^{\nu\lambda}] + 0 - a[P^\nu, M^{\mu\lambda}] = a[P^\lambda, M^{\mu\nu}]. \end{aligned} \tag{63}$$

From this we can conclude safely that $a = 0$. The Jacobi identity which includes one translation and two copies of non-linear generators will now determine the constants b and c if we additionally assume that N is subject to inverse Higgs ordering so $[P, N] \neq 0$,

$$\begin{aligned} 0 &= [P^\mu, [Q^\nu, Q^\lambda]] + [Q^\lambda, [P^\mu, Q^\nu]] + [Q^\nu, [Q^\lambda, P^\mu]] \\ &= [P^\mu, bM^{\nu\lambda} + cN^{\nu\lambda}] + 0 + 0 \\ &= b[P^\mu, M^{\nu\lambda}] + c[P^\mu, N^{\nu\lambda}]. \end{aligned} \tag{64}$$

This immediately restricts us to $b = c = 0$ and thus all commutators vanish apart from the Poincaré ones. Indeed, if extending the algebra with other generators this still holds and therefore the generators M, P and Q will form a subalgebra. Before considering the transformation rules and their implications we will first look at the case where we include a first order tensor $G^1 = N^{\mu\nu}$ in the Higgs tree. Notice that we already excluded the traceful parts for first order in the previous section. We will not show the full calculation of the Jacobi identities for this algebra but instead start by considering a schematic version of the Jacobi identity with one essential, one first order and one translation generator,

$$\begin{aligned} 0 &= [P, [G^0, G^1]] + [G^1, [P, G^0]] + [G^0, [G^1, P]] \\ &= [P, [G^0, G^1]] + 0 - [G^0, G^0 + P] = [P, [G^0, G^1]]. \end{aligned} \tag{65}$$

There exist only two generators in the algebra which commute with P^μ: another copy of momentum, P^ν, or the essential Q^μ. This leads to the only possibility $[G^0, G^1] = P + Q$. Notice that again higher order generators in the tree can never be included in this type of commutator since it will violate the tree structure. To complete the algebra involving G^1, we need the commutator between two copies of G^1. The Jacobi identity involving two copies of G^1 and one translation that fixes the commutator, requires a more tedious calculation but let us outline the important observations. Firstly, consider the fact that because of inverse Higgs ordering the first order generators commute into the essential under translations yielding in total two versions of $[G^1, G^0]$. This forbids the presence of any generators other than the Lorentz generator, M, and G^1 to be in $[G^1, G^1]$. We can conclude that again there exists a subalgebra but now with M, P, Q and N. The most important consequence of the subalgebras is that the action of elements corresponding the subalgebra cannot be altered by extending the algebra. In other words, the transformation rules due to generators from a subalgebra always remain intact.

For the sake of clarity, let us utilize the apparatus of the coset construction to find out what the transformation rule induced by the essential looks like. We now consider an empty tree so no inessentials appear in the coset element which would nevertheless have no effect on the result as concluded above.

$$e^{q_\mu Q^\mu} \gamma = e^{q_\mu Q^\mu} e^{x_\mu P^\mu} e^{A_\mu Q^\mu} = e^{x_\mu P^\mu} e^{(q_\mu + A_\mu) Q^\mu}. \tag{66}$$

As is clear from (66), the transformation for the essential vector does not differ much from an essential, internal scalar: both are constant shift symmetries. Also, in good comparison with the scalar case, these shift symmetries forbid the presence of a mass term in the Lagrangian. It must be emphasized that this is true only for *non-linear* symmetries and that this will not always hold for gauge symmetries.

Now a problem arises: the absence of a mass term implies a massless Goldstone which has at most two degrees of freedom. Although the equations of motion can dispel one of the unwanted degrees of freedom, the third degree of freedom associated with its longitudinal mode needs to accounted for as well. This problem is not unique and we have already seen in section 3 an example of a massless vector field that had an additional constraint to decrease its degrees of freedom: the $U(1)$ gauge field. Moreover, in this discussion we introduced a way of constructing an algebra consisting of $U(1)$ gauge symmetries in addition to the non-linear ones. The price we had to pay was in some sense infinite: all the symmetric generators were occupied to be gauge generators by means of (49). The gauge transformations have a different role than the actual inessentials: the latter have a physical effect on the dynamics of the field whereas the former do not. Therefore, the generators corresponding to the gauge symmetry will not be bothered and we will only discuss the non-linear generators that can still be included: the ones with one anti-symmetric pair of indices.

Let the vector field under investigation be a $U(1)$ gauge field. At first order the only non-linear generator is $N^{\mu\nu}$ which thus denotes the anti-symmetric tensor. A more detailed study of Jacobi identities [34] removes the presence of Q^μ in all commutators apart from that with translations. From (46) we know that the corresponding transformation is exactly:

$$\delta A^\mu = n^{\mu\nu} x_\nu. \tag{67}$$

Notice that whereas (46) took into account some possible field dependence, in (67) this is not the case. Up to first order, the only transformation is a shift proportional to x. This linear shift in x is similar to that of the scalar Galileon and thus we can call a vector theory invariant under (67) a *vector Galileon*. Because of the subalgebra that is formed, this rule will moreover not alter in the presence of higher-order generators.

Following [34] the Maurer Cartan components are computed and by the IHC, the tensor field $B^{\mu\nu}$ is eliminated in favour of A^μ. As discussed in sections 2.1 and 2.2 we can now construct the building blocks of the EFT by means of the covariant derivatives. In this case that is $\partial_\lambda F_{\mu\nu} = \partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu$. There are now double time-derivatives appearing in the Lagrangian which lead to Ostrogradski instability [35]. In short, this leads to an unstable Hamiltonian which is not bounded from below and hence facilitates negative energies. Obviously, building blocks which give rise to this instability cannot be considered. This means that we have to look for other terms to find a sensible theory for a vector Galileon. For the scalar case, the additional terms to the covariant coset objects are responsible for its non-trivial soft behaviour. An example would be $\partial_\mu \phi \partial^\mu \phi \Box \phi$. We see that the number of derivatives per field is smaller than two, which is the upper bound for a soft limit $\sigma = 2$ (see section 2.3). The presence of terms proportional to second derivatives in the scalar model suggests that they should also be present in the vector analogue. However, as was shown in [36] there exist only terms which are at most linear in the double derivatives of the vector field. This rules out the existence of a vector Galileon.

More dramatically, there are no non-linear generators left which can be included in the tree. Therefore, a $U(1)$ gauge vector cannot possess any non-linear symmetries.

4.2.1 Coupling to a special Galileon scalar

This seems to be the end of story for Goldstone vectors. However, there still exists a possibility. At the start of this section, it was concluded that because of the shift symmetry no mass term could be included in the Lagrangian. This type of transformation can still not be avoided when taking the vector as essential. It *can* be circumvented if the tree is extended inversely by an essential scalar. In other words, the vector has a non-zero commutator with translations of the form,

$$[P^\mu, Q^\nu] = i g^{\mu\nu} \Phi. \tag{68}$$

This relation, however, does not imply that the vector disappears from the spectrum as would be the case for just considering an essential scalar. We can see this by first looking at the algebra without extension. In particular, the algebra with one inessential, symmetric and traceless generator $Z^{\mu\nu}$. It corresponds to the starting point of the geometrical approach found in [37]. Here a D-dimensional brane is embedded into a $2D$-dimensional space which can be seen as a complex version of Minkowski spacetime.[7] For $D = 4$, the isometry group governing the eight-dimensional complex space, is a complexified version of the Poincaré group. The forms which transform invariantly under rotations and translations in complex space are given by [8],

$$ds^2 = \eta_{\mu\nu}(dX^\mu dX^\nu + dA^\mu dA^\nu), \qquad \omega = \eta_{\mu\nu} dX^\mu \wedge dA^\nu. \tag{69}$$

Here A^μ with $\mu = 0, ..., D-1$ has the role of a coordinate and together with X^μ indeed adds up to $2D$. We can derive the metric tensor on this space from the former. To compare, for regular Minkowski spacetime we would consider $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ yielding $\eta_{\mu\nu}$ as the metric tensor. The latter is the Kähler form where the $\wedge$-symbol defines an exterior product. For more information about this product, the reader is referred to [38]. Here, we will only use the fact that the evaluation of some function $f_i(x)$ evaluates in the following manner: $df_1 \wedge ... \wedge df_n = \det(\frac{\partial f_i}{\partial x_j})dx_1 \wedge ... \wedge dx_n$.

The two invariant objects define two tensors which transform covariantly on the brane. In other words, they form the building blocks of the EFT on the brane which non-linearly realizes the complete symmetry. To see this, consider the following parameterization for the embedding with the coordinates on the brane, x^μ:

$$X^\mu = x^\mu, \qquad A^\mu = A^\mu(x). \tag{70}$$

With such a parameterization, we can identify $dX^\mu = dx^\mu$ and $dA^\alpha = \frac{\partial A^\alpha}{\partial x^\mu}dx^\mu$ which leads to,

$$ds^2 = \eta_{\alpha\beta}(\delta^\alpha_\mu \delta^\beta_\nu + \frac{\partial A^\alpha}{\partial x^\mu}\frac{\partial A^\beta}{\partial x^\nu})dx^\mu dx^\nu = (\eta_{\mu\nu} + \partial_\mu A_\alpha \partial_\nu A^\alpha)dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu \tag{71}$$

The covariant tensor which follows from this is the induced metric $g_{\mu\nu}$. The Kähler form after identifying $f^\alpha = x^\alpha$ and $f^\beta = A^\beta(x)$,

$$\omega = \eta_{\alpha\beta} f^\alpha \wedge f^\beta = \eta_{\alpha\beta} \det(\partial_{\mu,\nu} f^{\alpha,\beta})dx^\mu \wedge dx^\nu = (\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu \wedge dx^\nu. \tag{72}$$

On the right hand-side one can recognize a covariant tensor which corresponds to the field strength, F. In summary, the two building blocks which respect the non-linearly realized complexified Poincaré group on the brane are:

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu A_\kappa \partial_\nu A^\kappa, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{73}$$

The treatment in [37] demands a vanishing Kähler form on the brane such that $\partial_\mu A_\nu = \partial_\nu A_\mu$ which allows for the identification of $A_\mu = \partial_\mu \phi$. This is equivalent to an IHC involving an essential scalar and inessential vector and thus only a longitudinal mode remains. Ultimately, it leads to ϕ being the special Galileon which then describes fluctuations on the brane. However, we can approach the implementation of the scalar also in a different way.

This approach involves a massive vector so we will quickly review gauge symmetries that apply to massive vectors. The procedure is known as the Stückelberg formalism [39]. In summary, Stückelberg wrote down a general Lagrangian for the massive vector field:

$$\mathcal{L} = -\partial_\mu A_\nu \partial^\mu A^\nu + m^2 A_\mu A^\mu. \tag{74}$$

[7] A brane is an object with space dimensions. For example, the surface of the earth is a 2-brane embedded into the (3+1) spacetime.

[8] Note that this corresponds to [37] when taken $\alpha = 1$.

This particular choice at itself contributes a negative term in the Hamiltonian. By adding a supplementary scalar field in the Lagrangian with same mass this problem is resolved,

$$\mathcal{L} = -\partial_\mu A_\nu \partial^\mu A^\nu + m^2 A_\mu A^\mu + \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2. \tag{75}$$

This trick would be worthless without any additional constraints because the total degrees of freedom is now five. Now, we follow a similar path to the QED case: by introduction of gauge symmetries on both fields, two degrees of freedom are removed. These are:

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \qquad \phi \to \phi + m\Lambda. \tag{76}$$

Finally, we have thus provided a massive vector with the right degrees of freedom (three) and which does not lead to an unhealthy term. In practice we will only use these gauge transformations.

The massive vector field and the corresponding Stückelberg scalar are introduced in the following manner:

$$A^\mu = \partial_\mu \phi + m\tilde{A}^\mu. \tag{77}$$

By plugging (77) into (73) we will have some $\tilde{A}$-dependence in $g_{\mu\nu}$. However, by considering small mass m and letting F absorb m, the covariant tensors become,

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \partial_\kappa \phi \partial_\nu \partial^\kappa \phi, \qquad F_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu. \tag{78}$$

The former is equivalent to the induced metric for only the special Galileon and it can be shown that indeed ϕ transforms accordingly. However, now we can also construct an EFT with the additional objects $F_{\mu\nu}$. This makes it possible to couple the two fields in terms of the two corresponding covariant objects,

$$g^{\kappa\lambda} F_{\lambda\nu} g^{\nu\rho} F_{\rho\kappa} = -F_{\mu\nu}F^{\mu\nu} - 2F_{\mu\nu}F^{\mu\rho}\partial^\nu \partial_\sigma \phi \partial_\rho \partial^\sigma \phi + \mathcal{O}(F^2\phi^4). \tag{79}$$

The first term on the right hand-side of (79) resembles the kinetic term for the vector. Additionally, we find two interactions from which the former is most interesting. Integrating it by parts twice with respect to the two ∂_σ leads to two total derivative and three distinct interaction terms in the Lagrangian. More precisely,

$$\int F_{\mu\nu}F^{\mu\rho}\partial^\nu \partial_\sigma \phi \partial_\rho \partial^\sigma \phi = S_1 - S_2 - \int F_{\mu\nu}F^{\mu\rho}\partial^\nu \Box \phi \partial_\rho \phi + \\ \int \Box F_{\mu\nu}F^{\mu\rho}\partial^\nu \phi \partial_\rho \phi + \int \partial_\sigma F_{\mu\nu}\partial^\sigma F^{\mu\rho}\partial^\nu \phi \partial_\rho \phi. \tag{80}$$

Interestingly, the latter of these terms corresponds exactly to the operator found in [40] by the soft bootstrap method. We can see this by rewriting the two-component notation [41] of [40] on the left hand-side of (81) in terms of the four-vector notation adopted in this paper:

$$g_1 \partial_\mu F_+{}^{\alpha\beta} \partial^\mu F_-{}^{\dot{\alpha}\dot{\beta}} \sigma^\nu_{\alpha\dot{\alpha}} \partial_\nu \phi \sigma^\rho_{\beta\dot{\beta}} \partial_\rho \phi = g_1 \partial_\sigma F_{\mu\nu} \partial^\sigma F^{\mu\rho} \partial^\nu \phi \partial_\rho \phi. \tag{81}$$

The $F_\pm$ denotes respectively the positive and negative helicity of the photon. Notice that for the algebra we are considering, only the special Galileon has a soft limit of $\sigma = 3$ and since the vector has no inessentials in its tree, it has $\sigma = 0$. These particular soft limits also agree with the observations of [40]. Therefore, from the perspective of the extended complexified Poincaré algebra non-linearly realized on the brane, the specific couplings found in [40] arise naturally from the algebra's covariant tensors. In addition to this particular interaction, two other terms emerge from (80)[9]. Let us discuss these.

Firstly, $\Box F_{\mu\nu}$ can be simplified by considering Maxwell's equations. In presence of any sources, these can be written as $\partial_\mu F^{\mu\nu} = J^\nu$ [10]. By making use of the Bianchi

[9] The total derivatives found in S_1 and S_2 are ignored since they do not affect the equations of motion.

[10] The conventional μ_0 is absorbed by J^ν

identity $\partial_{(\mu}F_{\nu\lambda)} = 0$ we derive the following,

$$\Box F_{\mu\nu} = \partial^\lambda\partial_\lambda F_{\mu\nu} = -\partial^\lambda(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}) = \partial_{[\mu}J_{\nu]}. \tag{82}$$

In absence of any interactions, $J_\nu = 0$ of course and thus the contribution to the Lagrangian $\Box F_{\mu\nu}$ vanishes. As we will see there is another, more general approach which removes this term.

Secondly, we observe the presence of $\Box\phi$ in the last term to discuss. This signals a proportionality to the equation of motion for ϕ up to first order. A way to remove higher-order derivatives in the EFT like this one is by means of a field redefinition [42]. The redefinition of the field can be found such that the lowest order (kinetic) term produces an equivalent but opposite contribution to the Lagrangian that cancels exactly the term or terms that we want to remove. This comes at a small price: overall, a number of higher order terms are also created in the process. Let us now make this concrete by executing this procedure. First notice that the term proportional to $\partial^\mu\Box\phi$ can be made proportional to the desired $\Box\phi$ by partial integration. The result, apart from a total derivative, is product of the l'Ambertian of the scalar and a function of the fields $f = \partial^\nu(F_{\mu\nu}F^{\mu\rho}\partial_\rho)$. A redefinition of the form, $\phi \to \phi - f$, leads to the following expansion of the kinetic term,

$$\begin{gathered}\mathcal{L}_{\phi,\text{Kin}} = -\int \frac{1}{2}\partial_\sigma\phi\partial^\sigma\phi \to -\int \frac{1}{2}\partial_\sigma(\phi - f)\partial^\sigma(\phi - f) = \\ -\int \frac{1}{2}[\partial_\sigma\phi\partial^\sigma\phi - 2\partial_\sigma\phi\partial^\sigma f + \mathcal{O}(f^2)] = -\int \frac{1}{2}\partial_\sigma\phi\partial^\sigma\phi + S_3 - \int \Box\phi f + \int \mathcal{O}(f^2).\end{gathered} \tag{83}$$

On the right hand-side, after integrating by parts with respect to ∂^σ, exactly the same product $\Box\phi f$ appears that arose from manipulating (80). Moreover, the redefinition on the kinetic term yields also a higher order term $\mathcal{O}(f^2) = \mathcal{O}(F^4\phi^2)$. Of course, when applying this redefinition consistently throughout the total Lagrangian, also other even higher order terms pop up. In line with the EFT rhetoric, these form perturbative corrections to the leading order interactions of the form $F^2\phi^2$.

We can apply exactly the same method for eliminating the contribution which carries a $\Box F_{\mu\nu}$. This time however, its connection to the kinetic term is yet more obvious: it is not the proportionality to the l'Ambertian we have to look at but the $F^{\mu\rho}$ itself. We can simply redefine $F_{\mu\rho} \to F_{\mu\rho} + 2\Box F_{\mu\nu}\partial^\nu\phi\partial_\rho\phi$ to acquire the following form of the kinetic term of the vector field,

$$\mathcal{L}_{A_\mu,\text{Kin}} = -\int \frac{1}{4}F_{\mu\rho}F^{\mu\rho} \to -\int \frac{1}{4}(F_{\mu\rho}F^{\mu\rho} + 4\Box F_{\mu\nu}F^{\mu\rho}\partial^\nu\phi\partial_\rho\phi + \mathcal{O}(F^2\phi^4)). \tag{84}$$

The middle term on the right hand-side of (84) is exactly a negative copy of the one found in (80). So again by redefining the field we can remove a term of the order $F^2\phi^2$ in favour of higher order contributions.

We can therefore conclude that the interactions found in [26] are equivalent to the results of [40] up to and including order $F^2\phi^2$,

$$\int g^{\kappa\lambda}F_{\lambda\nu}g^{\nu\rho}F_{\rho\kappa} = \mathcal{L}_{A_\mu,\text{Kin}} + \int \partial_\sigma F_{\mu\nu}\partial^\sigma F^{\mu\rho}\partial^\nu\phi\partial_\rho\phi + \mathcal{L}_{\text{Higher Order}}. \tag{85}$$

Thus, the non-linear symmetry presented in [26] fully explains these couplings associated with a massive vector and a special Galileon scalar with the particular soft limits of respectively $\sigma = 0$ and $\sigma = 3$.

5 Conclusion

In this paper, an overview is given of the necessary ingredients to classify exceptional EFTs. These include internal non-linear symmetries and Goldstones and its extension to spacetime. We find out that the symmetry group governing the Lagrangian could be realized non-linearly by the vacuum. For the internal case, every non-linear generator induces a massless mode, the Goldstone field, in the spectrum as stated in Goldstone's theorem. Subsequently, the coset construction is introduced which, by a convenient choice of parameterization, allows us to write down the covariant building blocks for the Lagrangian of the EFT. One can extend this formalism to spacetime in a straightforward manner by adding the translation generator and its corresponding field, spacetime coordinates, to the coset. Now, Goldstone's theorem is not valid anymore and by means of IHCs we are able to eliminate the inessential Goldstones.

The next ingredient is the understanding that for a general transition, the scattering amplitude vanishes when emitting a soft Goldstone. More precisely, the amplitude in the soft limit is proportional to the momentum of the Goldstone to a certain power, the soft degree. The classical example of Adler's zero exhibits soft degree one but there exist theories which give rise to enhanced soft limits where two and three are allowed. A soft limit is trivial if the interactions have enough derivatives. From the theories which lead to enhanced soft limits, we can specialize to ones that are least trivial: exceptional EFTs which posses the most restricted dynamics. Two of these are briefly reviewed: the DBI and the special Galileon. In addition we discuss the Galileon which has an enhanced soft limit but is not exceptional. In contrast to Adler's zero, enhanced soft limits require the presence of IHCs. This is illustrated in section 2.4 by explicitly computing some transformation rules which we linked to the Galileon ones.

Equipped with this knowledge, we start the systematic approach to classification found in [26] by shining light on the IHCs in a different way. Although this method is not able to calculate the full transformation rule, it allows for the construction of the inverse Higgs tree. This tree is a structured way to think about IHCs. After constraining the structure further by some algebraic observations, invariance of the kinetic term removes all generators but one at every order in the scalar case. The vector had more options but after specializing to $U(1)$ gauge vectors we realize that some of the possible generators in the tree corresponded to gauge transformations excluding their possible role as non-linear ones. Finally, it is in section 3.3 that, in order to complete the classification also the commutators between non-linear generators need to be computed in order to specify the resulting EFTs.

The actual classification starts in section 4 with the single scalar Goldstone. Firstly, the most general algebra is written down for this scalar. It consists of the trivial Lorentz algebra, the commutators with translations based on the discussed method and lastly, a very general form of the commutation relation between any two non-linear generators. Three types of trees were discussed which include respectively one, two and more than two inessentials with potential soft limit degree of two, three and larger than three. For the first case, there are three possible algebras which give rise to the DBI, Galileon and an EFT associated with dilations. As is discussed in section 4.1.1 the latter does not exhibit any soft behaviour whereas to former two have enhanced soft limits. Furthermore, one possibility is compatible with an exceptional EFT, the DBI. The subsequent case of two inessentials has only one possibility that is physically relevant: the special Galileon. However, this is one more than an algebra which involves more than two inessentials. It is argued that branches of this length must be cut off in order for the algebra to be consistent. A sequential pruning of the tree then results in one of the cases that were classified above.

The story of the vector Goldstone is more tragic. We start with an algebra without any inessentials and show that it must remain a subalgebra in the presence of any higher order generators. Also, the algebra with the addition of an inessential is a subalgebra. The importance of these subalgebras lies in the fact that the transformation rules they induce are not altered by any extensions of the tree. This observation immediately rules out the existence of a mass term in any algebra containing the essential: the transformation rule for the essential is a constant shift. Since this situation arises also in QED for the field associated with the photon, we apply the same techniques and let it correspond to a $U(1)$ massless gauge field. The only non-linear generator at first order compatible with gauge transformations leads, however, to an inconsistent theory. It is concluded that no non-linear realizations are possible for a vector Goldstone as essential. The way around this problem is to extend the tree upwards with an essential scalar which can only correspond to a special Galileon algebra. It leads to coupling of the special Galileon scalar to a massive vector field by means of the Stückelberg formalism. The specific couplings that emerge from this theory are investigated in more detail. It turns out that aside from some higher order terms, they are equivalent to the results of [40] obtained by the soft bootstrap method and also give rise to the same soft limits. The conclusion of this paper is, therefore, that the non-linear symmetry presented by [26] fully explains the interactions for a massive vector coupled to a special Galileon scalar which where found in [40].

The classification method outlined in this paper, is completely general and can therefore be applied to a number of Goldstone representations such as the scalar and vector discussed above. Not highlighted here, but an example of another type of representation would be fermions. Furthermore, the method allows to classify theories with multiple Goldstones and even couplings between different irreps of which the specific example of a vector-scalar is studied in this paper. Also, the linearly realized symmetry can be chosen different from Poincaré without spoiling the mechanism. Therefore, this approach is well-suited for any further research involving classifications of EFTs involving a wide variety of fields and symmetries.

Finally, the theory underlying the vector and scalar coupling discussed in this paper might have interesting properties and consequences. These have to be investigated and can be a well-suited subject for future work. Furthermore, further research might reveal the relevance of this coupling in other areas of research. For example, not only the scalar Galileons appears in the study of IR modification of gravity but also their vector analogues are under investigation in this field. It would be interesting to see what role the vector-scalar theory discussed in this paper can play regarding this subject.

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