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Physics Bachelor's Thesis

Non-Relativistic Supersymmetry

Author:
Cian Hamilton

Supervisors:
Eric Bergshoeff
Johannes Lahnsteiner

Second Examiner:
Anupam Mazumdar

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Part I

Introduction

1 Introduction

1.1 Outline

Supersymmetry is an elegant principle which answers a lot of questions in particle physics. Supersymmetry proposes a symmetry between the two main classes of elementary particles, fermions and bosons. One of the main reasons the Large Hadron Collider at CERN was built was to test the predictions of supersymmetry. In its 10 years of operation the the LHC has not found any evidence for supersymmetry.

This paper is focused on supersymmetry in a very different context to the area of high-energy particle physics where it was originally proposed. The subject we are interested in is the non-relativistic case of supersymmetry. The non-relativistic case is relevant in the context of condensed matter physics, where supersymmetry has been observed as an emergent symmetry in some settings. [4–7]

In the following section we will explain why relativistic supersymmetry has become such a huge topic, and why we are specifically interested in non-relativistic supersymmetry.

1.2 What is Symmetry?

Symmetry is a deceivingly difficult concept to describe. However Feynman comes to our aide, defining symmetry as "something that you can do, so that after you are finished doing it, [the system] looks the same as it did before". In our case, we are not interested in the symmetry of physical objects (such as the rotational symmetry of a sphere), but of the symmetries of the *physical laws* of our universe.

1.2.1 Kinds of Symmetries

There are many different ways to categorise symmetries. For our purposes the two most important distinctions are if the symmetry is *continuous* or *discrete*, and if it is a *spacetime* or an *internal* symmetry. Continuous symmetries will be discussed in appendix A (we are not interested in discrete symmetries). The other important distinction is explained below:

- **Spacetime Symmetries**
  A spacetime symmetry is a symmetry which involves a transformation on the spacetime coordinates. For example, a spacetime translation on the coordinates is: $\mathbf{x}'^{\mu} = \mathbf{x}^{\mu} + \mathbf{a}^{\mu}$.

  Elements of the Lorentz group (rotations and boosts) are also spacetime symmetries. The group of the combination of translations and Lorentz transformations is called the **Poincare group**.

- **Internal Symmetries**
  Internal symmetries are not related to a change in the coordinates, but by "an equivalence between different fields at the same spacetime point."[1] In general they are given by: $\Phi^a(\vec{x}, t) \to M^a_b \Phi^b(\vec{x}, t)$.

  A specific example is the phase rotation of a field, $\phi \to e^{i\alpha}\phi$. If the transformation parameter (in this case it was α, and in general M^b_a) is a constant, then it is a *global* transformation, if it is a function of spacetime ($\alpha(\vec{x}, t)$or $M^b_a(\vec{x}, t)$), then it is a *local* transformation. We will be dealing with global transformations in this paper.

1.3 What is Quantum Field Theory and the Standard Model?

Quantum field theory is, according to David Tong, "the language in which the laws of Nature are written"[14]. More precisely, it is the quantisation of the classical field. It treats the fields themselves as fundamental, and not the particles. Most importantly, it is the product of the merging of quantum mechanics with special relativity.

The Standard Model is an example of a particular QFT, and (according to Lambert) it is also "one of the most successful and accurate scientific theories" [11]. The Standard Model unifies three of the four fundamental forces (not gravity) with an internal symmetry.

1.3.1 Effective Field Theories

Despite the Standard Model being a hugely successful theory, it is also an effective field theory, and so it does not give us the full picture. An effective field theory is an approximation to a more fundamental theory, valid only at certain energy ranges. Analogous to how a finite Taylor expansion is only a good approximation under certain magnitudes around the centre point, i.e. $\sin x \approx x - \frac{x^3}{3!}$ for $|x| < 1$. If we wander too far from this range we will start to get nonsensical answers. This does not mean that the approximation is useless, but that it is only valid within a certain range.

For the Standard Model, the valid range are the energies lower than the Planck energy ($\sim 10^{19} GeV$). So somewhere around this energy range the effects of the "new physics" become difficult to ignore.

1.4 Problems with the Standard Model

Apart from only being valid at certain energy ranges, the Standard Model also has other deep problems:

- **Quantum Gravity**
  The most glaring problem of the theory is that gravity at quantum scales is excluded. This means that as we approach smaller length scales and quantum effects start to become unignorable, our theory of gravity starts to break down and give predictions which are no longer physically meaningful.

- **Hierarchy Problem**
  The hierarchy problem is a technical problem related to the relative strengths of the fundamental forces. The hierarchy problem is closely related to finetuning and naturalness. See [12] for a more detailed explanation.

1.5 Beyond the Standard Model

There are a few options to generalise the Standard Model and find a more fundamental theory:

- **Experimental evidence**
  The LHC has confirmed the predictions of the Standard Model (the

most notable of which was the existence of the Higgs Boson) to high precision, but has not necessarily given the field much of a hint as to which direction to investigate further.

- **More general symmetries**
  Now that we are beginning to get an idea of how central the study of symmetry is to physics, it is unsurprising that physicists rely upon it as a guide to help us move towards a more fundamental theory. By generalising the symmetries of the Standard Model it is hoped that we can uncover a more fundamental theory. Supersymmetry is a result of this way of thinking.

- **Something completely different**
  Even if the Standard Model is extended, it is unknown how this will solve the main problem, which is describing quantum gravity. It may be that the only way to unify the four forces is by, at least largely, abandoning the Standard Model and forging a new path. Due to the wide ranging successes of the Standard Model, most physicists are understandably hesitant to follow this path.

1.6 Supersymmetry Emerges Unexpectedly

In more recent years, it has been shown that supersymmetry is not only relevant in high-energy particle physics. There are several settings in condensed matter physics where supersymmetry appears, not as a fundamental symmetry, but as an emergent one [4–7]. This has led some to argue that "even though the corresponding microscopic models do not exhibit it, SUSY emerges macroscopically." [4]

Condensed matter physics is inherently non-relativistic, and so there is now a growing interest in non-relativistic supersymmetry.

To reach non-relativistic supersymmetry, we will first start with the relativistic theory, and then take the limit as $c \to \infty$. So in the next part we begin the investigation into relativistic supersymmetry, and then we will be in a position to take the limit into the non-relativistic world.

Part II

Relativistic Supersymmetry

2 Evading the Coleman-Mandula Theorem

The Poincare group describes the symmetries of special relativity. It is built of two important subgroups, the Lorentz group, and the translation group. The Standard Model contains other symmetries as well, all of which are internal symmetries. The algebra of the Poincare group and the algebra of the internal symmetries are split "trivially", which means that they form a direct product i.e. the generators of these two algebras commute,

$$[P_\mu, T_\alpha] = 0 = [M_{\mu\nu}, T_\alpha]$$

Where P_μ is the generator of the translation group, $M_{\mu\nu}$ is the generator of the of Lorentz transformations, and T_α are the generators of any internal symmetry.

This tells us that the internal symmetries' generators do not interact or communicate with the Poincare generators.

Since the Poincare algebra is so central to our understanding of spacetime, it is natural to ask if our universe contains a fundamental symmetry that pairs with the Poincare symmetries non-trivially, i.e. a symmetry which has generators which do not all commute with the Poincare generators. This question was answered by Coleman and Mandula in their 1967 theorem, which states[1] that, under a few reasonable conditions:

the only interacting quantum field theories have a Lie algebra which is a direct product of the Poincare algebra with that of an internal symmetry (meaning the two classes of generators must commute).[11]

This theorem seems to suggest that the Poincare group is the most fundamental symmetry of the standard model, and that there is no other symmetry which can interact with it.

As is often the case with no-go theorems, by examining the assumptions that it makes, we can find ways to get around it. The key assumption of the Coleman-Mandula theorem is that the new symmetry obeys a Lie

[1] paraphrased

algebra. However, there are other kinds of algebras, and so it is possible that a symmetry that obeys a different kind of algebra might still be able to "mix" with the Poincare algebra.

The Standard Model includes two kinds of fields, fermionic and bosonic fields. The fermionic field obeys *anticommutation* relations, while the bosonic field obeys *commutation* relations. Because of this difference, if there would exist a symmetry between fermions and bosons, then the set of generators of the resulting algebra would not be closed under the normal bracket of a Lie algebra, but under a different bracket. This bracket does not necessarily have to satisfy anticomutivity, i.e. $[A, B] \neq -[B, A]$. The resulting algebra is called a Z_2 *graded algebra*. The Z_2 part of the name refers to the cyclic group of order 2, and its elements are the two classes of the generators, *even* and *odd*. The operation of this group is the new "graded" bracket. The even generators are the generators of the Poincare algebra, and the odd generators are those of the new symmetry which relates fermions to bosons; **supersymmtery**. The even generators act as the identity of the Z_2 group, and so by simply knowing how the cyclic group of order two behaves, we can determine how the two classes of generators mix:

$$[\text{Even}, \text{Even}] = \text{Even}$$

$$[\text{Even}, \text{Odd}] = \text{Odd}$$

$$\{\text{Odd}, \text{Odd}\} = \text{Even}$$

This tells us that

- two Poincare (even) generators will produce another (linear combination of) Poincare generator.
- a Poincare generator and a supersymmetric (odd) generator will give a supersymmetric generator.
- two supersymmetric generators will form a Poincare generator.

There is an indepth look at this last (and very remarkable) fact later in section 6.

3 A Supersymmetric Transformation

Let us now see what supersymmetry does at the level of the fields. As we have just learnt, supersymmetry is a relation between fermions and bosons. This means that a supersymmetric transformation will "mirror" bosons to fermions, and fermions to bosons. Different kinds of particles have different kinds of fields. These fields are described by different mathematical objects. A spin 0 particle (a boson since 0 is an integer), will be described by a complex scalar field (as both scalars and spin 0 particles are not affected by spin). Spin $\frac{1}{2}$ particles (fermions) are described by a spinor field, as both spin $\frac{1}{2}$ particles and spinors need to be rotated through 720°to reach their initial state, not 360°as is the case for vectors. A spinor is a Grassmann quantity. The most relevant fact about Grassmann quatities for our purposes is that they anticommute.

If we have a complex scalar field ϕ (describing a spin 0 boson), and a spinor field χ (describing a spin $\frac{1}{2}$ fermion), their infinitesimal suppersymmetric transformation rules must be vaguely of the form:

$$\delta\phi \propto \epsilon\chi \qquad\qquad \delta\chi \propto \epsilon\phi$$

where ϵ is the infintesimal parameter of the supersymmetry. By examining the variation of ϕ, we see that a scalar field is proportional to a spinor field. The only way for a spinor to be made equal to a scalar is by multiplying the spinor by a conjugate spinor. This means that, in this case, **the infinitesimal parameter of supersymmetry must be a spinor**[2]. For fields describing other kinds of fermions, the infinitesimal parameter may not neccessarily be a spinor, but will always be a Grassmann quantity. It is the fact that the infinitesimal parameter of supersymmetry must always be Grassmann that leads to the Z_2 graded algebra, and why the fermionic part has anticummutation relations.

4 Noether's Theorem

Emmy Noether was considered one of the greatest mathematicians of the time by her peers, and her most famous theorem is essential to the theory

[2]In this very specific case, it must actually be a conjugate spinor, which is denoted by a bar above the spinor, and so $\delta\phi \propto \bar{\epsilon}\chi$

of supersymmetry, and QFT in general. The theorem links two of the most fundamental parts of physics, symmetries and conservation laws.

Noether's theorem states that for every continuous symmetry there is a corresponding conserved current. By integrating this current over space (and not time), we can find the conserved charge. Not only does Noether's theorem tell us that the charges exist, but it gives us a systematic way to start from a symmetry and calculate the currents, and thus the charges.

Before we find the conserved currents we first vary the Lagrangian to see if the Lagrangian really is invariant under this symmetry. If the Lagrangian is invariant, the only terms left over will be a total derivative of the fields[3], i.e. $\delta\mathcal{L} = \partial_\mu F^\mu$.

To find the conserved currents we use the symmetries of each of the fields, and the F^μ which we found by varying the Lagragian. The form of Noether's theorem for field theory is given in 1

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta(\Phi_\alpha) - F^\mu \tag{1}$$

Where we are summing over all of the fields (index α), and $\delta(\Phi_\alpha)$ is the transformation of the field by the symmetry. The current j^μ can shown to be conserved ($\partial_\mu j^\mu = 0$) by imposing the equations of motion. The equations of motion are found using the Euler-Lagrange equation. Once this has been shown we can find the charge:

$$Q = \int d\vec{x}^3 j^0$$

Examples of corresponding symmetries and conserved quantities include:

- spatial translation $\Leftrightarrow$ conservation of linear momentum
- time translation $\Leftrightarrow$ conservation of energy.
- phase invariance of the field (e.g. $\delta\psi = e^{i\theta}\psi$) $\Leftrightarrow$ total particle number (particles $-$ antiparticles $= 0$)

Although at first glance supersymmetry might appear to be a discrete symmetry, it is characterised by an infinitesimal parameter, which can be continuously adjusted, and so it is a continuous symmetry. This concept is

[3]See appendix B for an explanation of this

made more clear by studying supersymmetry in a certain phase space such that a supersymmetric transformation acts as a rotation in this space. Due to the limited scope of this paper, we will not be considering this superspace formalism.

And so the obvious question to ask is what kind of conservation corresponds to a *super*symmetry?

5 Supersymmetry in Action: The Wess Zumino Model

5.1 Massless Wess Zumino Model

We start with the Lagrangian density of the massless, non-interacting Wess-Zumino model, and then integrate this over spacetime to give the action.

$$S_{WZ} = \int d^4x\big(- \partial_\mu \phi^* \partial^\mu \phi - \bar{\chi}_R \gamma^\mu \partial_\mu \chi_L - \bar{\chi}_L \gamma^\mu \partial_\mu \chi_R\big)$$

Where ϕ is a complex scalar field, and χ is a Marjorana spinor field split into its two irreducible components, χ_L and χ_R. The complex field ϕ describes a spin 0 particle, while χ_l and χ_R describes the two chirality states of a spin $\frac{1}{2}$ particle.

To show that this Lagrangian is invariant under a supersymmetric transformation, we will vary the Lagrangian, and see if there any terms leftover. As with all Lagrangian variations, we assume that the terms that are a total derivative of the fields will go to zero as explained in the appendix.

The precise variations for the Wess-Zumino model are given as,

$$\delta\phi = \bar{\epsilon}_L \chi_L \qquad \text{and} \qquad \delta\chi_L = \frac{1}{2}\gamma^\mu \epsilon_R \partial_\mu \phi$$

When this supersymmetric transformation is applied to the Wess-Zumino Lagrangian, all but four of the terms cancel out with each other. These four leftover terms are all given as total derivatives of the fields as shown.

$$\delta\mathcal{L} = \partial_\mu F^\mu = \partial_\mu \left(\bar{\epsilon}_R \Big(\frac{1}{2}\gamma^\nu \gamma^\mu \chi_R \partial_\nu \phi - \chi_R \partial^\mu \phi \Big) + \bar{\epsilon}_L \Big(\frac{1}{2}\gamma^\nu \gamma^\mu \chi_L \partial_\nu \phi^* - \chi_L \partial^\mu \phi^* \Big) \right)$$

Since we ignore the total derivative terms, we have shown that the system is supersymmetric, since the Lagrangian is invariant under a supersymmetry transformation (up to a to a total derivative).

To calculate the Noether current we follow the method given in the previous section, and we find:

$$j^\mu = -\bar{\epsilon}_L \gamma^\nu \gamma^\mu \chi_L \partial_\nu \phi^* - \bar{\epsilon}_R \gamma^\nu \gamma^\mu \chi_R \partial_\nu \phi$$

By integrating the Noether current[4] we obtain the Noether charges resulting from supersymmetry, i.e. the *supercharge*

$$Q_L = \int d\vec{x}^3 (-\gamma^\nu \gamma^0 \chi_L \partial_\nu \phi^*)$$
$$Q_R = \int d\vec{x}^3 (-\gamma^\nu \gamma^0 \chi_R \partial_\nu \phi)$$

For a given mass, the supercharge preserves the difference between the number of bosons and fermions with that mass.

5.2 Klein, Gordon, and Dirac

The Klein-Gordon Lagrangian is a relativistic description of a (complex) scalar field. Thus, it can describe the field of a spin 0 particle.

$$\mathcal{L}_{KG} = -\partial_\mu \phi^* \partial^\mu \phi - c^2 m_\phi^2 \phi^* \phi$$

The Dirac Lagrangian is a relativistic description of a spinor field. Thus, describing the field of a spin $\frac{1}{2}$ particle

$$\mathcal{L}_D = -\bar{\chi} \gamma^\mu \partial_\mu \chi + m_\chi c^{\bar{\chi}\chi}$$

[4] In fact, we do not integrate the entire current, but everything except the infinitesimal parameter ϵ

5.3 The Massive Wess-Zumino Model

We will now take the massless Wess-Zumino Lagrangian from equation 5.1 and include the mass terms to make it more interesting:

$$\mathcal{L}_{WZ} = -\partial_\mu \phi^* \partial^\mu \phi - m_\phi^2 c^2 \phi^* \phi - 2\bar{\chi}\gamma^\mu \partial_\mu \chi + 2m_\chi c \bar{\chi}\chi$$

With the mass terms included it is now more obvious that the Wess-Zumino Lagrangian is simply the addition of the Klein-Gordon and Dirac Lagrangians.

$$\mathcal{L}_{WZ} = \mathcal{L}_{KG} + \mathcal{L}_D$$

Part of the reason that it is so simple is because of the fact that there are no interactions between the two fields, which means that there are no terms which mix the fields together. This means that we can work on the Klein-Gordon and Dirac Lagrangian individually for simplicity, and then add the results back together at the end.

An important point to note is that to make this system supersymmetric the two masses must be equal, $m_\phi = m_\chi$. In more sophisticated variants of supersymmetry, this symmetry is spontaneously broken, meaning that the masses could be different. This is considered much more useful, since if there really were spin 0 particles with the mass of the electron, we would have discovered it by now.

Unsurprisingly, when we use the transformation rules of the massless Wess Zumino model on the *massive* model, the theory is no longer invariant. This is because we need to change the transformation rules to accommodate the new mass terms.

After we vary the massive Lagrangian with the suspersymmetric transformation rules for the massless theory (ignoring the total derivatives as usual) we are left with:

$$\delta\mathcal{L} = -mc\bar{\epsilon}\gamma^\mu \chi \partial_\mu \phi^* + mc\bar{\chi}\gamma^\mu \epsilon \partial_\mu \phi - m^2 c^2 \phi \bar{\chi}\epsilon - m^2 c^2 \phi^* \bar{\epsilon}\chi$$

We can fix this by adding extra mass terms to the massless transformation terms. In the leftover terms of the variation of the Lagrangian we see that the highest power of m is 2. This means that the new massive terms of the

transformation rules should contribute, at most, a term with a factor of m^2. As the mass term of the Klein-Gordon equation ($m_\phi^2 c^2 \phi^* \phi$) already has a factor of m^2, we know that the transformation rule of ϕ cannot have a mass term, as this would bring us over our limit of m^2.

Therefore, the additional mass term must be added to the variation of χ. Since $2m_\chi c\bar{\chi}\chi$ already contains a power of m, the additional mass term must have exactly one factor of m.

$$\delta\chi = \frac{1}{2}\gamma^\mu \epsilon \partial_\mu \phi \ + \ ?$$

By using dimensional analysis of the fields we can drastically reduce the number of potential terms to add to the transformation rules. We know the action must have a mass dimension of 0, and so the Lagrangian (density) must have a mass dimension of +3 (do not forget that we are working in 3 dimensions, not 4).

$$S = \int d\mathbf{x}^3 \mathcal{L}$$

By knowing that the Lagrangian must have a mass dimension of +3, and examining individual terms of the Lagrangian, we can ascertain the mass dimension of the fields and infinitesimal parameter. We find the mass dimensions to be:

$$[\phi] = \frac{1}{2} \qquad\qquad [\chi] = 1 \qquad\qquad [\epsilon] = -\frac{1}{2}$$

And so to add an additional term to the transformation of χ, our options are actually very limited. We know that the total mass dimension must be 1 ($[\delta\chi] = [\chi] = 1$), that it must include ϕ (it's a supersymmetric transformation after all), ϵ (the infinitesimal parameter of the variation) and m (to cancel out the terms which have factors of m in the Lagrangian variation). Since the dimensions of m, ϕ and ϵ add up to 1, we know that we do not need any other dimensionful factors (e.g. derivatives). Thus, the additional transformation rule term for χ is proportional to $m\phi\epsilon$.

$$\delta\chi = \frac{1}{2}\gamma^\mu \epsilon \partial_\mu \phi + V\phi\epsilon m \tag{2}$$

Where V is a constant which we can tune to make the Lagrangian invariant. Obviously, the variation of $\bar{\chi}$ has the Dirac conjugate of ther term, $V^\dagger \phi^* \bar{\epsilon} m$.

When we vary the Lagrangian using (2) and its conjugate for the transformation rules for χ and $\bar{\chi}$, we find that the Lagrangian is invariant for $V = V^\dagger = \frac{1}{2}c$. Thus, the new transformation rules for χ and $\bar{\chi}$ are given below, while the transformation rules for ϕ and ϕ^* remain unchanged.

$$\delta\chi = \frac{1}{2}\gamma^\mu \epsilon \partial_\mu \phi + \frac{1}{2}mc\phi\epsilon$$
$$\delta\bar{\chi} = -\frac{1}{2}\bar{\epsilon}\gamma^\mu \partial_\mu \phi^* + \frac{1}{2}mc\phi^*\bar{\epsilon}$$

6 What happens after two supersymmetric transformations?

More precisely, what is the commutator of two supersymmetric transformations? Since it is only under commutation that the generators of an algebra are closed, not composition.

On first thought, it might seem that two consecutive supersymmetric transformations would leave the system invariant, as the fermions would be "mirrored" to bosons, before being mirrored back to fermions (and vice versa for the bosons). It is true that the two transformations leave the fermions as fermions and the bosons as bosons, but there is also a different change to the system.

To show this, we will use this equality between the transformation of a field, and a Poisson bracket with the Noether charge[5]:

$$\delta\Phi = \{\bar{Q}\epsilon, \Phi\}_{PB}$$

To calculate the supersymmetric transformation of another supersymmetric transformation, we can embed one of these Poisson brackets within another one:

$$\delta_\epsilon(\delta_\beta\Phi) = \delta_\epsilon\big(\{\bar{Q}\beta, \Phi\}_{PB}\big) = \big\{\bar{Q}\epsilon, \{\bar{Q}\beta, \Phi\}_{PB}\big\}_{PB}$$

(In this case I have given the first supersymmetric transformation the infinitesimal parameter β, and the second ϵ). Since we are interested in the *commutator* of two supersymmetric transformations:

$$\Rightarrow [\delta_\epsilon, \delta_\beta]\Phi = \big\{\bar{Q}\epsilon, \{\bar{Q}\beta, \Phi\}_{PB}\big\}_{PB} - \big\{\bar{Q}\beta, \{\bar{Q}\epsilon, \Phi\}_{PB}\big\}_{PB}$$

[5] This is kind of the reverse of Noether's theorem, since it gives the symmetry rules for a given conserved charge.

And we can use the Jacobi identity as a shortcut:

$\big\{\bar{Q}\epsilon, \{\bar{Q}\beta, \Phi\}_{PB}\big\}_{PB} + \big\{\bar{Q}\beta, \{\Phi, \bar{Q}\epsilon\}_{PB}\big\}_{PB} + \big\{\Phi, \{\bar{Q}\epsilon, \bar{Q}\beta\}_{PB}\big\}_{PB} = 0$

To finally give us this equation:

$$[\delta_\epsilon, \delta_\beta]\Phi = -\big\{\Phi, \{\bar{Q}\epsilon, \bar{Q}\beta\}_{PB}\big\}_{PB}$$

Explicitly for the case of the Wess-Zumino model in the previous section, the commutator of two supersymmetric transformations on a complex scalar field[6] we find:

$$[\delta_\epsilon, \delta_\beta]\phi = \frac{1}{2}\big(\bar{\beta}_L\gamma^\mu\epsilon_R + \bar{\beta}_R\gamma^\mu\epsilon_L\big)\partial_\mu\phi \tag{3}$$

And as this is of the form $\tau^\mu\partial_\mu\phi$, where τ is a Lorentz vector, the commutator of two supersymmetric transformations is a **spacetime translation**. This is an extremely important fact for supersymmetry, and so we shall put it in a box:

The commutator of two supersymmetric transformations gives a spacetime translation

We saw a hint of this remarkable fact in the earlier discussion of Lie groups, when we saw that the anti-commutator of two odd (supersymmetric) generators produce a Poincare generator (spacetime translations are elements of the Poincare group). This result gives us more intuition of the fact that supersymmetry is a spacetime symmetry, not an internal symmetry, since by doing two supersymmetric transformations we get a movement through spacetime. This has led some people to say (in the loosest sense) that a supersymmetric transformation is the "square-root of a translation".

7 Summary of the Procedure

1. Starting with the Lagrangian and the supersymmetry transformation rules of its fields, I varied the Lagrangian and showed that it is invariant up to a total derivative of the fields.

2. Using this total derivative of the fields, I calculated the Noether current using Noether's theorem.

[6] To show the same fact for the spinor field requires the use of Fierz identities, which is beyond the scope of this thesis.

3. By using the equations of motion of the Lagrangian (calculated using the Euler-Lagrange equation), I checked that the Noether current is conserved.
4. By integrating the current over space I found the Noether charge.
5. To check that the charge is correct and everything is consistent, I used this calculation for each of the fields: $\delta\Phi = \{\bar{Q}\epsilon, \Phi\}_{PB}$
6. I then find the commutation relations between the members of the super-Poincare algebra.

8 Super-Poincare

As discussed in the 2 section, the group that describes special relativity is the Poincare group (consisting of Lorentz transformations and spacetime translations).

These are the non-vanishing parts of the Poincare algebra are given below: [10]

$$\begin{aligned} \left[J_{ij}, J_{kr}\right] &= 4\delta_{[i[k}J_{r]j]} & \left[J_{ij}, P_k\right] &= -2\delta_{k[j}P_{i]} \\ \left[J_{ij}, K_k\right] &= 4\delta_{k[j}K_{i]} & \left[K_i, K_j\right] &= 4J_{ij} \\ \left[K_i, P_j\right] &= -2\delta_{ij}P_0 & \left[K_i, P_0\right] &= -2P_i \end{aligned} \tag{4}$$

Where J_{ab} is the generator of spatial rotations in the a-b plane, P_μ is the generator of translations in the μ direction, and K_a is the generator of boosts in the a direction. The subscript brackets signify commutators of the indices, for example, $\delta_{k[j}K_{i]} = \frac{1}{2}(\delta_{kj}K_i - \delta_{ki}K_j)$

Boosts and rotations are both Lorentz transformations, and so are usually classified into the more general generator of Lorentz transformations, $M_{\mu\nu}$. However, the distinction will be important for the non-relativistic limit and so we have separated them.

As discussed earlier, supersymmetry provides a non-trivial spacetime extension to the Poincare group by having a graded Lie algebra. The supersymmetric extension of the Poincare algebra is called the super-Poincare algebra.

The additional commutation and anti-commutation relations that come from the supersymmetric generator are given below[7]:

$$\begin{aligned} \big[J_{ij}, Q\big] &= -\frac{1}{2}\gamma_{ij}Q \qquad \big[P_\mu, Q\big] = 0 \\ \big[K_i, Q\big] &= -\frac{1}{2}\gamma_i\gamma_0 Q \quad \big\{Q_\alpha, \bar{Q}^\beta\big\} = -\frac{1}{2}\big(\gamma^\mu\big)^\beta_\alpha P_\mu \end{aligned} \tag{5}$$

Together, relations 4 to 5 form the super-Poincare algebra.

9 Extended Supersymmetry

It is possible to add a second supersymmetry generator to the super-Poincare algebra, to obtain an algebra where there are two separate supersymmetry generators. The resulting theory is called $\mathcal{N} = 2$ supersymmetry. In general, we could add any number of supersymmetry generators, $Q_1, Q_2, ..., Q_\mathcal{N}$. To obtain a non-relativistic supersymmetric theory that is still a spacetime symmetry[8], we need *at least two supersymmetry generators*[9].[8] We shall study the simplest case, $\mathcal{N} = 2$ extended supersymmetry.

The simplest way to have an extended supersymmetry is by mixing the supercharges trivially, i.e.

$$\{Q_i, Q_j\} \propto \delta_{ij} P_\mu$$

where $i, j = \{1, 2\}$. This is equivalent to having multiple copies of supersymmetry, and the different supercharges not interacting.

To make our system more interesting, we can allow the different supercharges to interact non-trivially, by introducing a new generator:

$$\{Q_i^\alpha, Q_j^\beta\} \propto \epsilon_{ij}\mathcal{Z}^{\alpha\beta}$$

It is called the *central*[10] *extension*, and it commutes with all of the other generators,

$$[P, \mathcal{Z}] = [J, \mathcal{Z}] = [K, \mathcal{Z}] = [Q, \mathcal{Z}] = 0$$

[7] Note how the only anticommutator of the whole algebra is from Q with itself, since it is the only fermionic (odd) member of the algebra

[8] i.e. a theory which maintains the property, $\{Q, Q\} \propto P_\mu$

[9] We shall explain why this is so in section 11.2.1

[10] central in terms of group theory, i.e. it commutes with everything in the algebra

$\mathcal{Z}$ must be antisymmetric to preserve the symmetry of the anticommutator, i.e. when we exchange i with j *and* α with β, nothing changes.

The nonzero parts of the $\mathcal{N} = 2$ SuperPoincare Algebra

$$\left[J_{ij}, J_{kr}\right] = 4\delta_{[i[k}J_{r]j]} \qquad \left[J_{ij}, P_k\right] = -2\delta_{k[j}P_{i]}$$

$$\left[J_{ij}, K_k\right] = 4\delta_{k[j}K_{i]} \qquad \left[K_i, K_j\right] = 4J_{ij}$$

$$\left[K_i, P_j\right] = -2\delta_{ij}P_0 \qquad \left[K_i, P_0\right] = -2P_i$$

$$\left[J_{ij}, Q^k\right] = -\frac{1}{2}\gamma_{ij}Q^k \qquad \left[K_i, Q^k\right] = -\frac{1}{2}\gamma_i\gamma_0 Q^k$$

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = -\delta^{ij}\left[\gamma^\mu\gamma^0\right]_{\alpha\beta}P_\mu - \varepsilon^{ij}\varepsilon_{\alpha\beta}\mathcal{Z}$$

Part III

Non-Relativistic Supersymmetry

10 Introduction

The term *non-relativistic* is a very poor one, since the principle of relativity does not refer to the finite speed of light, but states that: *the laws of physics are the same in all admissible*[11] *reference frames.*

Will we show that when we take the "non-relativistic limit" of the theory of special relativity, we obtain Galilean mechanics. However, Galilean mechanics is just as relativistic a theory as special relativity, and so the term "non-relativistic limit" is a confusing misnomer. However, as is often the case with physics, this term is continued to be used for the sake of consistency, and we shall continue to use it to mean taking $c \to \infty$.

11 From Poincare to Galileo, and Dirac to Schrödinger

There are (at least) three different approaches to the non-relativistic limit of a field theory. These are done by manipulating:

- **The co-ordinate system**
- **The algebra**
- **The fields**

11.1 Manipulating the Coordinates

The starting point of making the coordinates compatible with a non-relativistic limit is to separate time and space. Until now we have been treating them equally, as different parts of the four vector, with time corresponding to x^0,

[11] The kinds of reference frames which the principle of relativity refers to depends on the theory. For special relativity and Galilean relativity, the reference frames must be inertial, and for general relativity this condition is relaxed to include non-inertial reference frames.

and the spatial coordinates being x^i. The factor of c is usually ignored in relativistic field theory, but now we will have to show it explicitly. To make the dimensions of the four vector equivalent (all of dimension of length), we use $x^0 \equiv ct$.

Let's see what a Lorentz transformation of the spatial coordinates looks like,

$$\delta x^i = \Lambda^i_\mu x^\mu = \Lambda^i_0 x^0 + \Lambda^i_j x^j = \Lambda^i_0 (ct) + \Lambda^i_j x^j$$

As we are keeping in mind the fact that soon we'll be taking the limit as $c \to \infty$, we are alarmed by the positive power of c in the Lorentz boost term. We want to make sure to get rid of the factor of c without losing the term as a whole, since we want to have a non-relativistic sytem which still has boost. We define $\Lambda^a_0 \equiv \frac{1}{c}\lambda^a$. This gives us a non-relativistic spatial transformation which has both boosts and rotations:

$$\delta x^i = \frac{1}{c}\lambda^i (ct) + \Lambda^i_j x^j = \lambda^i t + \Lambda^i_j x^j$$

A transformation on the time coordinate gives us:

$$\delta x^0 = \delta(ct) = \Lambda^0_\mu x^\mu = \underbrace{\Lambda^0_0}_{=0} x^0 + \underbrace{\Lambda^0_i}_{\equiv \frac{1}{c}\lambda_i} x^i = \frac{1}{c}\lambda_i x^i$$

$$\Rightarrow \delta t = \frac{1}{c^2}\lambda_i x^i \Rightarrow \underbrace{\delta t = 0}_{c \to \infty}$$

This shows us that in a non-relativistic theory, **time is absolute**, and not relative.

11.2 Manipulating the Algebra

Working with the algebra is the most fundamental way to get to non-relativistic supersymmetry

This is the roadmap that we will use to navigate to our final goal; the $\mathcal{N} = 2$ super-Bargmann algebra.

$$\begin{tikzcd}
\text{Poincare} \arrow[r, "\mathcal{Z}"] \arrow[d, "c\to\infty"] & \text{Extended Poincare} \arrow[r, "\text{2 Supercharges}"] \arrow[d, "c\to\infty"] & \mathcal{N} = 2 \text{ Super-Poincare} \arrow[d, "c\to\infty"] \\
\text{Galilean} & \text{Bargmann} & \mathcal{N} = 2 \text{ SuperBargmann}
\end{tikzcd}$$

In the relativistic part of this paper we have found the entire top line of this chart, the Poincare, extended Poincare, and $\mathcal{N} = 2$ super-Poincare algebra.

It is well known that by taking the non-relativistic limit of special relativity (which obeys the Poincare algebra), we obtain classical mechanics (corresponding to the Galilean algebra).

To show that the Galilean algebra is the non-relativistic limit of the Poincare algebra is very simple. It just involves explicitly including the factors of c in the Poincare algebra, and then taking the limit $c \to \infty$. To make sure we differentiate between the members of the algebra before and after the factors of c have been included, we will redefine them as [10]:

$$K_i \to cG_i \qquad P_0 \to \frac{1}{2c}H$$

And once we take the limit we do indeed obtain the Galilean algebra:

$$\begin{aligned} &[J_{ij}, J_{kr}] = 4\delta_{[i[k}J_{r]j]} && [J_{ij}, P_k] = -2\delta_{k[j}P_{i]} \\ &[J_{ij}, G_k] = 4\delta_{k[j}G_{i]} && [G_i, H] = -4P_i \end{aligned}$$

Note that two Lorentz boosts do not commute, while two Galilean boosts do.

The Bargmann algebra is the Galilean algebra with an additional generator added to it. This generator corresponds to an internal symmetry. In fact, this new generator is the generator we introduced in section 9.

It is called the central[12] extension, and it commutes with all of the other generators. Thus, if the central extension is ignored in the Bargmann algebra, we obtain the Galilean algebra.

However, instead of extending the Galilean algebra with the central extension to obtain the Bargmann algebra, we shall extend the Poincare, and then take the limit again.

These are the redefinitions we make to the Poincare generators to extend the Poincare algebra with Z, [10]

$$K_i \to cG_i \qquad P_0 \to cZ + \frac{1}{2c}H$$

[12] central in terms of group theory, i.e. it commutes with everything in the group

to obtain the Bargmann algebra:

$$[J_{ij}, J_{kr}] = 4\delta_{[i[k}J_{r]j]} \qquad [G_i, H] = -P_i$$

$$[J_{ij}, P_k] = -2\delta_{k[i}P_{j]} \qquad [J_{ij}, G_k] = 2\delta_{k[j}G_{i]}$$

$$[G_i, P_j] = -2\delta_{ij}Z$$

11.2.1 The $\mathcal{N} = 2$ SuperBargmann Algebra

To get to the destination on our algebra map, we will do as we have just done with the Extended-Poincare→Bargmann procedure; redefine the generators, and then take the limit. This time however, we will be starting from the $\mathcal{N} = 2$ Super-Poincare Algebra.

We will be using the same redefinitions as before, added to the new redefinitions of the supercharges. However, before we can do that, we must make linear combinations of the supercharges instead[13]. These linear combinations are given as:

$$Q^{\pm}_{\alpha} \equiv Q^1_{\alpha} \pm \epsilon_{\alpha\beta}Q^2_{\beta}$$

So now we redefine Q^+ and Q^- instead of Q^1 and Q^2. These redefinitions, along with those of the other generators are given below:

$$Q^- \to \sqrt{c}Q^- \qquad Q^+ \to \frac{1}{\sqrt{c}}Q^+$$

$$K_i \to cG_i \qquad P_0 \to cZ + \frac{1}{2c}H$$

$$Z \to -cZ + \frac{1}{c}H$$

After substituting these redefinitions into the Super-Poincare we take the limit and are left with the Super-Bargmann.

[13] due to the Inonu-Wigner contraction

The purely fermionic anti-commutation relations are the most challenging ones. However, all three of the relations are very similar, and so I shall show how to calculate $\{Q_+, Q_-\}$ as an example.

$$
\begin{aligned}
&\{\hat{Q}^+_\alpha, Q^-_\sigma\} = \{Q^1_\alpha + \epsilon_{\alpha\beta}, Q^1_\sigma - \epsilon_{\sigma\rho}Q^2_\rho\} \\
&= \{Q^1_\alpha, Q^1_\sigma\} + \epsilon_{\alpha\beta}\{Q^2_\beta, Q^1_\sigma\} - \epsilon_{\sigma\rho}\{Q^1_\alpha, Q^2_\rho\} - \epsilon_{\alpha\beta}\epsilon_{\sigma\rho}\{Q^2_\beta, Q^2_\rho\} \\
&= -(\gamma^\mu\gamma^0)_{\alpha\sigma}P_\mu + \epsilon_{\alpha\beta}(\epsilon_{\beta\sigma}\epsilon^{21})Z - \epsilon_{\sigma\rho}(\epsilon_{\alpha\rho}\epsilon^{12})Z - (\gamma^\mu\gamma^0)_{\beta\beta}\delta_{\alpha\sigma}P_\mu - (\gamma^\mu\gamma^0)_{\sigma\alpha}P_\mu \\
&= -2(\gamma^\mu\gamma^0)_{\alpha\sigma}P_\mu - 2\delta_{\alpha\sigma} = -2(\underbrace{-(\gamma^0\gamma^0)_{\alpha\sigma}}_{-\delta_{\alpha\sigma}}P_0 + (\gamma^i\gamma^0)_{\alpha\sigma}P_i + \delta_{\alpha\sigma}P_0) \\
&= -2(\gamma^i\gamma^0)_{\alpha\sigma}P_i
\end{aligned}
$$

Where we have used the following identities:

$$
\begin{aligned}
&\epsilon_{\alpha\beta}\epsilon_{\sigma\rho} = \delta_{\alpha\beta}\delta_{\beta\rho} - \delta_{\alpha\rho}\delta_{\beta\sigma} \\
&(\gamma^\mu\gamma^0)_{\beta\beta} = \mathrm{tr}(\gamma^\mu\gamma^0) = 2\eta^{\mu 0} \\
&\epsilon_{\beta\alpha}\epsilon_{\beta\sigma} = \delta_{\alpha\sigma}
\end{aligned}
$$

In this case, the redefinitions of the generators do not actually alter the equation, and so we are left with this as our final answer:

$$\left\{Q^+_\alpha, Q^-_\sigma\right\} = -2(\gamma^i\gamma^0)_{\alpha\sigma}P_i$$

Once the other commutatation and anti-commutation rules have been found, we are left with the **$\mathcal{N} = 2$ SuperBargmann Algebra:**

$$
\begin{aligned}
&\left[J_{ij}, P_k\right] = -2\delta_{k[i}P_{j]} && \left[J_{ij}, G_k\right] = 2\delta_{k[j}G_{i]} \\
&\left[G_i, P_j\right] = -\delta_{ij}Z && \left[G_i, Q^+\right] = -\frac{1}{2}(\gamma_i\gamma_0)Q^- \\
&\left[J_{ij}, Q^\pm\right] = -\frac{1}{2}\gamma_{ij}Q^\pm && \left[G_i, H\right] = -P_i \\
&\left\{Q^+_\alpha, Q^+_\sigma\right\} = 2\delta_{\alpha\sigma}H && \left\{Q^-_\alpha, Q^-_\sigma\right\} = 4\delta_{\alpha\sigma}Z \\
&\left\{Q^+_\alpha, Q^-_\sigma\right\} = -2(\gamma^a\gamma^0)_{\alpha\sigma}P_i
\end{aligned}
$$

Since $\{Q^+_\alpha, Q^+_\sigma\}$ gives a spacetime translation (H is a time translation), it seems as though Q^- is uneccessary, and we can reduce our system to a consistent $\mathcal{N} = 1$ supersymmetric theory without it. This is incorrect. The algebra has a curious asymmetry for the Galilean boosts (G_i), and the supercharges, $\left[G_i, Q^+\right] \propto Q^-$, while $\left[G_i, Q^-\right] = 0$. This means that we cannot remove Q^-, since it is required by the existence of Q^+. We can remove Q^+ and keep Q^-, but then our supersymmetry ceases to be a spacetime symmetry, and is only an internal symmetry, since $\left\{Q^-, Q^-\right\} \propto Z$. Thus, an $\mathcal{N} = 2$ theory of supersymmetry is required to ensure that supersymmetry remains a spacetime symmetry[14] after taking the non-relativistic limit.

The fact that we need two supercharges to obtain a non-relativistic theory of supersymmetry is not actually that surprising. In the relativistic theory, space and time are both considered to be on equal footing and that they are both just components of spacetime. However in a non-relativistic theory space and time are considered fundamentally different things, and are treated as such. So it is clear that one supercharge in a non-relativistic theory could not give a full spacetime translation, so we need two as they will be spilt into the time translation, $\{Q^+, Q^+\} \propto P_0$, and the spatial translation, $\{Q^+, Q^-\} \propto P_i$.

11.3 Manipulating the Fields

11.3.1 Klein-Gordon to Schrödinger

Starting with the Klein-Gordon Lagragian

$$\begin{aligned}\mathcal{L} =& \partial_\mu \phi^* \partial^\mu \phi + m^2 c^2 \phi^* \phi \\ =& -\frac{1}{c^2} \partial_t \phi^* \partial_t \phi + \partial_i \phi^* \partial^i \phi + m^2 c^2 \phi^* \phi\end{aligned}$$

A mode with energy E oscillates in time proportionally to e^{-iEt} [12]. Since we are interested in the non-relativistic limit we ignore the kinetic energy and approximate $E \approx mc^2$. Thus, we redefine $\phi \to e^{-imc^2 t}\phi$. Due to the complex conjugation ϕ^* will have the opposite sign in the exponential,

[14] although we are requiring that supersymmetry is a spacetime symmetry, there are other papers which don't require this property (for an example, see [9]), and they would be fine with $\mathcal{N} = 1$ super-Bargmann algebra.

$\phi^* \to e^{+imc^2 t}\phi^*$.

$$\mathcal{L} = -\frac{1}{c^2}(+imc^2\phi^* + \partial_t\phi^*)(-imc^2\phi + \partial_t\phi) + \partial_i\phi^*\partial^i\phi + m^2c^2\phi^*\phi$$

Due to this introduced phase factor the divergent terms will cancel. We now take the limit $c \to \infty$ which causes the double time derivative to vanish, leaving us with:

$$\mathcal{L} = im\phi\partial_t\phi^* - im\phi^*\partial_t\phi + \partial_i\phi^*\partial^i\phi$$

By using the Euler-Lagrange equation we can find the equations of motion of this system. Varying with ϕ^* gives us this familiar equation[15]:

$$i\partial_t\phi = -\frac{1}{2m}\partial_i\partial^i\phi$$

This is not a very surprising result, since the Klein-Gordon equation describes relativistic spin 0 particles, while the Schödinger equation describes non-relativistic spin 0 particles. An important point to note is that ϕ is a classical field, not a wavefunction as we are used to seeing with the Schödinger equation.

11.3.2 Dirac to Schrödinger

To find the non-relativistic limit of the Wess-Zumino Lagrangian, we will need to first find the non-relativistic limit of the Dirac Lagrangian

$$\mathcal{L} = \bar{X}\Big(\frac{1}{c}\gamma^0\partial_t + \gamma^i\partial_i - mc\Big)X$$

In this case $\bar{X}$ signifies a *Dirac conjugate*, $\bar{X} \equiv iX^\dagger\gamma^0$

Using the same reasoning as with the Klein-Gordon limit, we redefine the fields:

$$\chi \to e^{-imc^2 t}\chi \qquad \bar{\chi} \to e^{+imc^2 t}\bar{\chi} \tag{6}$$

Before the two exponentials with opposite sign cancel out, the time derivative of the first term means that the product rule introduces a fourth term

[15] Varying with ϕ will give us the complex conjugate of the same equation.

before this can happen:

$$\mathcal{L} = \bar{X}\left(\frac{1}{c}\gamma^0\partial_t + \frac{1}{c}\gamma^0(-imc^2) + \gamma^i\partial_i - mc\right)X$$

To immediately lose a lot of terms, we introduce these redefinitions and equalities:

$$\begin{aligned} &X \equiv X_+ + X_- && \hat{X} \equiv X_+ - X_- \\ &X_\pm \equiv \frac{1}{2}\big(I \pm i\gamma^0\big) \\ &X + \hat{X} = 2X_+ && \hat{X} = i\gamma_0 X \end{aligned}$$

By noticing a few of the properties of these $X_\pm$, we can lose a lot of the terms straight away. $\bar{X}_\pm X_\mp = \frac{1}{4}\bar{X}(I \pm i\gamma_0)(I \mp i\gamma_0)X = \frac{1}{4}\bar{X}(I - (i\gamma_0)^2)X = 0$. Similarly $\bar{X}_\pm\gamma^i\partial_i X_\pm = 0$.

We had four terms, and when we expand them into $X_\pm$, each term becomes four new terms, producing sixteen in total. By using these properties, we are left with five terms.

We now redefine $X_+ \to \frac{1}{\sqrt{c}}\chi_+, \quad X \to \sqrt{c}\chi_-$, and take the non-relativistic limit, $c \to \infty$.

On taking the limit, one of these terms will go to 0, and nothing will happen to the rest since they are independent of c.

$$\mathcal{L} = i\bar{\chi}_-\partial_t\chi_- + \bar{\chi}_+\gamma^i\partial_i\chi_- + \bar{\chi}_-\gamma^i\partial_i\chi_+ - 2m\bar{\chi}_+\chi_+$$

Now our Lagrangian has four terms. We can use the Euler-Lagrange equation to find the equations of motion. Note that the Lagrangian has only one derivative of χ_+, while there are two for χ_-. In this case, this means that we will obtain nicer equations of motion by varying χ_+ and its conjugate. These equations of motion are:

$$\bar{\chi}_+ = -\frac{1}{2m}\partial_i\bar{\chi}_-\gamma^i \qquad \chi_+ = \frac{1}{2m}\gamma^i\partial_i\chi_-$$

These equations of motion can be used to express the Lagrangian purely in terms of χ_-, and when we do this, we see that two of the four terms cancel, leaving us with:

$$\mathcal{L} = i\bar{\chi}_-\partial_t\chi_- + \frac{1}{2m}\bar{\chi}_-\partial_i\partial^i\chi_-$$

When we find the equations of motion, we are finally left with the equation:

$$i\partial_0\chi_- = -\frac{1}{2m}\partial_i\partial^i\chi_-$$

This shows that the non-relativistic limit of the Dirac Lagrangian is the Schrödinger Lagrangian, which gives us the Schrödinger equation.

This result is actually more subtle than it appears, since the Schrödinger Lagrangian (and equation) describe spin 0 particles, while the Dirac Lagrangian describes spin $\frac{1}{2}$ particles. Did we change the particle's spin when we took the limit? No, of course not. What has happend is that the spinor field χ_- is a solution to the Schrödinger equation *component wise*. In fact, what we have found is more accurately described as the Pauli equation, since that is the formulation of the Schrödinger equation for spin $\frac{1}{2}$ particles. However, in the absence of an external electromagnetic field, the Pauli equation is equivalent to the Schrödinger equation if the Schrödinger equation of a spinor is viewed component wise.

The non-relativistic version of the massive Wess-Zumino Lagrangian is obtained simply by adding the non-relativistic Klein-Gordon Lagrangian to the non-relativistic Dirac Lagrangian, both derived in the previous sections:

$$\mathcal{L} = i\phi\partial_t\phi^* + \frac{1}{2m}\partial_i\phi^*\partial^i\phi + 2i\bar{\chi}\partial_t\chi + \frac{1}{m}\bar{\chi}\partial_i\partial^i\chi$$

The Galilean transformation rules of the Dirac field are given in appendix C

11.4 The non-relativistic supersymmetric transformation rules of the fields

We split the bosonic field, fermionic field, and the infinitesimal parameter into their real and imaginary components:

$$\phi \equiv \phi_1 + i\phi_2$$
$$\chi \equiv \chi_1 + i\chi_2$$
$$\epsilon \equiv \epsilon_1 + i\epsilon_2$$

The spinors (χ and ϵ) are both Dirac spinors, and their separated real components (χ_i, ϵ_i) are Majorana spinors.

We can now find the transformation of the real fields:

$$\delta\phi = \delta\phi_1 + i\delta\phi_2$$
$$\delta\phi = \bar{\epsilon}\chi = (\bar{\epsilon}_1 - i\bar{\epsilon}_2)(\chi_1 + i\chi_2) = \bar{\epsilon}_1\chi_1 + \bar{\epsilon}_2\chi_2 + i(\bar{\epsilon}_1\chi_2 - \bar{\epsilon}_2\chi_1)$$

By separating the result into its real and imaginary components, we can find the transformation rules for the real scalar fields, ϕ_1 and ϕ_2.

$$\delta\phi_1 = \bar{\epsilon}_1\chi_1 + \bar{\epsilon}_2\chi_2 \qquad \delta\phi_2 = \bar{\epsilon}_1\chi_2 - \bar{\epsilon}_2\chi_1 \tag{7}$$

For the Dirac field we split it into real and imaginary parts, use the redefinitions of 6 for the Dirac field and the complex scalar field, and then take the limit as $c \to \infty$.

$$\begin{aligned}\delta\chi &= \delta\chi_1 + i\delta\chi_2 \\ &= \frac{1}{2}\gamma^\mu\epsilon\partial_\mu\phi + \frac{1}{2}m\phi\epsilon \\ &= \frac{1}{2}\gamma^\mu(\epsilon_1 + i\epsilon_2)\partial_\mu(\phi_1 + i\phi_2) + \frac{1}{2}m(\phi_1 + i\phi_2)(\epsilon_1 + i\epsilon_2)\end{aligned}$$

And then separate it into Majorana parts:

$$\begin{aligned}\delta\chi_1 &= \frac{1}{2}\gamma^\mu\epsilon_1\partial_\mu\phi_1 - \frac{1}{2}\gamma^\mu\epsilon_2\partial_\mu\phi_2 \\ \delta\chi_2 &= \frac{1}{2}\gamma^\mu\epsilon_2\partial_\mu\phi_1 + \frac{1}{2}\gamma^\mu\epsilon_1\partial_\mu\phi_2\end{aligned} \tag{8}$$

We now follow the process done in [13], and redefine the two fermionic and bosonic parts as:

$$\begin{aligned}\epsilon_+ &\equiv \frac{1}{\sqrt{2c}}(\epsilon_1 + \gamma_0\epsilon_2) & \epsilon_- &\equiv \sqrt{\frac{c}{2}}(\epsilon_1 - \gamma_0\epsilon_2) \\ \chi_+ &\equiv \frac{1}{\sqrt{2c}}(\chi_1 + \gamma_0\chi_2) & \chi_- &\equiv \frac{1}{c\sqrt{2c}}(\chi_1 - \gamma_0\chi_2)\end{aligned} \tag{9}$$
$$\phi_i \to c\phi_i$$

To obtain the non-relativistic supersymmetric transformation rules is simply a matter of substituting the redefinitions (equations 9) into the relativistic transformation rules for the bosonic fields (7), and for the fermionic fields (8), and then taking the limit $c \to \infty$.

$$
\begin{aligned}
\delta\phi_1 &= \bar{\epsilon}_+\chi_+ + \bar{\epsilon}_-\chi_- \\
\delta\phi_2 &= \bar{\epsilon}_-\gamma_0\chi_- - \bar{\epsilon}_+\gamma_0\chi_+ \\
\delta\chi_+ &= \frac{1}{2}\gamma^0\epsilon_+\partial_t\phi_1 + \frac{1}{2}\gamma^i\epsilon_-\partial_i\phi_1 + \frac{1}{2}\epsilon_+\partial_t\phi_2 + \frac{1}{2}\gamma^{i0}\epsilon_-\partial_i\phi_2 \\
\delta\chi_- &= \frac{1}{2}\gamma^i\epsilon_+\partial_i\phi_1 + \frac{1}{2}\gamma^{i0}\epsilon_+\partial_i\phi_2
\end{aligned}
$$

When we found the algebra, we had two supercharges, Q^+, and Q^-. The transformation rules given here are equivalent to the result of the sum of the operation of both supercharges, i.e. $\delta\Phi = \bar{\epsilon}(Q^+ + Q^-)\Phi$.

We now want to see if we can reproduce the results of the $\mathcal{N} = 2$ super-Bargmann algebra using the transformation rules we have just obtained. In other words, we want to check if the transformations of the fields agree with the results of the algebra. The key parts of the $\mathcal{N} = 2$ super-Bargmann algebra are the purely fermionic parts:

$$
\begin{aligned}
\left\{Q^+, Q^+\right\} &\propto H \\
\left\{Q^+, Q^-\right\} &\propto P_i \\
\left\{Q^-, Q^-\right\} &\propto Z
\end{aligned}
\tag{10}
$$

To check these results on the level of the fields requires the commutator of two supersymmetric transformations of a field, in the same way as we did calculation 3.

$$
\begin{aligned}
[\delta_{\epsilon_+}, \delta_{\beta_+}]\phi =& \delta_{\epsilon_+}(\delta_{\beta_+}\phi) - \delta_{\beta_+}(\delta_{\epsilon_+}\phi) \\
=& \delta_{\epsilon_+}(\bar{\beta}_+\chi_+ - i\bar{\beta}_+\gamma_0\chi_+) - \delta_{\beta_+}(\bar{\epsilon}_+\chi_+ - i\bar{\epsilon}_+\gamma_0\chi_+) \\
=& \bar{\beta}_+(I - i\gamma_0)(\delta_{\epsilon_+}\chi_+) - \bar{\epsilon}_+(I - i\gamma_0)(\delta_{\beta_+}\chi_+) \\
=& \frac{1}{2}\Bigg((\bar{\beta}_+\gamma^0\epsilon_+\partial_t\phi_1 + \bar{\beta}_+\epsilon_+\partial_t\phi_2 - i\bar{\beta}_+\epsilon_+\partial_t\phi_1 - i\bar{\beta}_+\gamma_0\epsilon\partial_t\phi_2) \\
& - (\bar{\epsilon}_+\gamma^0\beta + \partial_t\phi_1 + \bar{\epsilon}_+\beta_+ + \partial_t\phi_2 - i\bar{\epsilon}_+\beta_+\partial_t\phi_1 - i\bar{\epsilon}_+\gamma_0\beta_+\partial_t\phi_2)\Bigg) \\
=& \bar{\beta}_+\gamma^0\epsilon_+\partial_t\phi_1 + \bar{\beta}_+\gamma^0\epsilon_+\partial_t(i\phi_2) \\
=& \bar{\beta}_+\gamma^0\epsilon_+\partial_t\phi
\end{aligned}
$$

Where we have used the following identities:

$$\begin{aligned}\bar{\epsilon}\beta =& \bar{\beta}\epsilon \\ \bar{\epsilon}\gamma^{\mu}\beta =& -\bar{\beta}\gamma^{\mu}\epsilon \\ \bar{\epsilon}\gamma^{i}\gamma^{0}\beta =& -\bar{\beta}\gamma^{i}\gamma^{0}\epsilon\end{aligned}$$

This result is identical to 3, except that here it is a translation through time only, not spacetime, due to the different nature of space and time in the non-relativistic case (as already discussed at the end of section 11.2.1). The other two commutators of supersymmetric transformations are calculated in the same way as the previous one and are given below:

$$[\delta_{\epsilon_+}, \delta_{\beta_-}]\phi = \bar{\beta}_-\gamma^i\epsilon_+\partial_i\phi \tag{11}$$

$$[\delta_{\epsilon_-}, \delta_{\beta_-}]\phi = 0 \tag{12}$$

We expected the spatial translation result of equation 11 due to the fact that the anticommutator of the algebra (shown above in 10) gives the generator of spatial translations.

The fact that two negative[16] supersymmetric transformations gives 0 in equation 12 should not surprise us. The algebra tells us that two Q^- give an internal symmetry, which is trivially represented for massless[17] particles.

12 Conclusion

We started with the Wess-Zumino model, a description of a relativistic, non-interacting supersymmetric system. After investigating a few of its properties in the relativistic case, we took the non-relativistic limit in two main areas, the algebra, and the fields. The non-relativisitic limit of the algebra brought us to the $\mathcal{N} = 2$ super-Bargmann algebra. The non-relativistic limit of the fields showed us that the Wess-Zumino model (which is built of the Klein-Gordon and Dirac Lagrangians) is equivalent to two Schrödinger Lagrangians, one in terms of a bosonic field, the other in terms of a fermionic

[16] Referring to the transformation of Q^-

[17] For massive particles on the other hand, we would expect something proportional to the mass, since the central extension corresponds to the mass of the particles.

field. We finished the paper by showing that the the two areas where we took the non-relativistic limit (the algbera, and the fields) agree with each other.

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Part IV

Appendix

A Lie Algebra

To understand the inner working of supersymmetry and why it is such an exciting area of study, it is crucial to understand the basic ideas behind Lie algebra. Here I shall give a tiny look into the principle of Lie algebra, with our goal of understanding supersymmetry in mind. For a much more thorough and general study, I would recommend [Group Theory in Nutshell]

Marius Sophus Lie had a genius and simple idea in the late 1800s which takes advantage of the nature of continuous groups. The basic point is that any element of a continuous group can be divided into an infinite number of infinitesimal elements, and by acting each element one after another, we can recover the original group element. This is a hugely powerful technique as it allows us to take advantage of the infinitesimal nature of the group elements, in an analogous way that calculus takes advantage of the nature of infinitesimal changes.

An infinitesimal group element can be described by the identity added to an infinitesimal "nudge" $I + A$. This nudge encodes properties of the specific group which we are concerned with. A great example of the concept is the rotation group, and for the sake of simplicity, specifically the group of 2-dimensional rotations.

A defining property of rotations is that the lengths of vectors are the same before or after the rotation, i.e. $\vec{u}^T\vec{u} = \vec{u}'^T\vec{u}'$, where $\vec{u}'$ is the rotated vector, $\vec{u}' \equiv R\vec{u}$. This condition forces the rotation matrix to be unitary, $R^T R = I$. We apply this condition to an infinitesimal rotation, $(I+A)^T(I+A) = I \Rightarrow$

$I + A + A^T = I \Rightarrow A^T = -A$ where we ignored the A^2 term since A is an infinitesimal and so we only care about it in the first order.

Thus, the infinitesimal nudge of the rotation group is antisymmetric. Since we are concerned with two dimensional rotations, A is a 2 by 2 matrix, and all 2 by 2 antisymmetric matrices are proportional to:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Where we say J is the *generator* of the 2 dimensional rotation group.

A rotation of θ can be divided into N parts, and then acted one after another N times, to reproduce the rotation by θ. Once we take the limit as N goes to infinity, we can describe the small components as infinitesimal rotations, $\lim_{N\to\infty}(I + \frac{\theta}{N}J)$. This takes advantage of the infinitesimal (identity plus a nudge) nature. Now to act this infinitesimal part N times to reproduce the original rotation by θ:

$$\lim_{N\to\infty}\left(I + \frac{\theta}{N}J\right)^N$$

This expression can quickly be recognised as equivalent to $e^{\theta J}$. By using a Taylor expansion of this, we obtain:

$$\begin{aligned} e^{\theta J} = \sum_{n=o}^{\infty}\frac{(\theta J)^n}{n!} &= \Bigg(\underbrace{\sum_{k=o}^{\infty}(-1)^k\frac{\theta^{2k}}{(2k)!}}_{=\cos\theta}\Bigg)I + \Bigg(\underbrace{\sum_{k=o}^{\infty}(-1)^k\frac{\theta^{2k+1}}{(2k+1)!}}_{=\sin\theta}\Bigg)J \\ &= (\cos\theta)I + (\sin\theta)J \\ &= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{aligned} \tag{13}$$

We have obtained the rotation matrix for any angle θ, all through infinitesimal rotations. No messy trigonometry or vector components necessary.

In this example we had one generator, J. In general, there can be any number of generators forming a closed set. The generators are not closed under the same operation as the group from which they originate, i.e. $J_1 \odot J_2 \notin \mathcal{J}$, while $g_1 \odot g_2 \in G$. Generators of Lie algebra are closed under

commutation, i.e. $[J_1, J_2] = J_1 J_2 - J_2 J_1 \in \mathcal{J}$. This bracket must satisfy the Jacobi identity, i.e. $[J_1, [J_2, J_3]] + [J_2, [J_2, J_1]] + [J_3, [J_1, J_2]] = 0$, and the other usual properties of commutators.

In the context of n dimensional rotations, the generators are the n by n antisymmetric matrices, and we know that the product of two anti-symmetric matrices is not another antisymmetric matrix, but we can show that the commutator of two antisymmetric matrices *is* antisymmetric,

$$[J_1, J_2]^T = (J_1 J_2 - J_2 J_1)^T = J_2^T J_1^T - J_1^T J_2^T = [J_2, J_1] = -[J_1, J_2]$$

$$\Rightarrow [J_1, J_2] \in \mathcal{J}$$

Another important difference between Lie groups and Lie algebra is that the linear combinations of algebra members produces another member of the algebra, i.e.

$$\sum_{i=0}^{N} a_i J_i \in \mathcal{J}$$

While this is obviously not the same for groups. You cannot add different factors of rotation matrices to produce another rotation matrix.

B Invariance of a Lagrangian

To show invariance of the Lagrangian the variation must be equal to 0 or a total derivative of the fields.

This is due to Stokes' theorem. Roughly speaking, Stokes' theorem states that the integral of the derivative of a function over a volume[18] is equal to the integral of the function over the surface of that volume, i.e. $\int_\Omega d\omega = \int_{\partial\Omega} \omega$.

With regards to our case, this means that

$$\int_{\text{4D Space}} \delta\mathcal{L} = \int_{\text{4D Space}} \partial_\mu F^\mu = \int_{\text{boundary surface}} F^\mu$$

[18]It is important to note the confusing language used when discussing volumes and surfaces in different dimensions. We use the word "surface" to describe the $N-1$ dimensional boundary of the N dimensional "volume" which we are referring to. In our case, the "volume" is 4 dimensional, and so the surface is 3 dimensional.

As always, we are integrating the Lagrangian (density) over all of 4 dimensional space, the boundary of which is the 3 dimensional surface at infinity. We make the assumption that the fields converge to 0 arbitrarily far away, and so the integral of the fields over this infinitely far away surface vanishes.

C Galilean transformation rules for Dirac field

$$X_+ \to \frac{1}{\sqrt{c}}\chi_+ \qquad X_- \to \sqrt{c}\chi_-$$

$$\delta\chi_- = \lim_{c\to\infty}(\delta X) = \lambda^{ij}x_i\partial_j\chi_- - \lambda^i t\partial_i\chi_- - \frac{1}{4}\lambda^{ij}\gamma_i\gamma_j\chi_- - im\lambda^i x_i\chi_-$$

$$\begin{aligned}\delta\chi_+ =& \underbrace{\delta\chi_+ + c\delta\chi_-}_{\sqrt{c}\delta X} - \underbrace{c\lim_{c\to\infty}\left(\frac{1}{c}\delta\chi_+ + \delta\chi_-\right)}_{c\delta\chi_-} \\ =& \lambda^{ij}x_i\partial_j\chi_+ - \lambda^i t\partial_i\chi_+ - \frac{1}{4}\lambda^{ij}\gamma_i\gamma_j\chi_+ - \frac{1}{2}\lambda^i\gamma_i\gamma_0\chi_- - im\lambda_i x_i\chi_+\end{aligned}$$

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