university of groningen

faculty of science and engineering

mathematics and applied mathematics

Monotone Convergence in Complete Metric Spaces

Bachelor's Project Mathematics

Juli 2019

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Abstract

In this thesis, we prove a monotone convergence theorem for complete metric spaces. Afterwards we show how coincidence of betweenness relations can be used to interpret the monotone convergence theorem in complete metric spaces with some additional structure. We conclude with various examples.

Contents

Chapter 1

Monotone Convergence

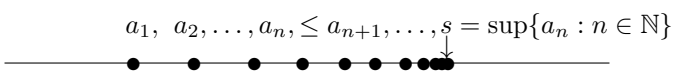
1.1 Introduction

In the book Understanding Analysis by Stephen Abbot, the reader is introduced to a theorem called the *Monotone Convergence Theorem* [1, p. 56]. We briefly refresh the reader on the statement of the theorem;

Definition 1. *We say a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is monotone if it is either increasing or decreasing.*

Theorem 1. ***(Monotone Convergence Theorem)*** *If a sequence of real numbers is monotone and bounded, then it converges.*

Intuitively, theorem 1 states that whenever a sequence extends in a direction, but can not extend beyond some fixed point, then the sequence will simply not have enough space to diverge. Below is given an example of a monotone and bounded sequence of real numbers.



This theorem is extremely useful and has found lots of applications throughout mathematics. For example, monotone convergence is an indispensable tool in the study of infinite series where it is used to prove various convergence tests [1]. The intuition behind monotonicity has found its way into domains other than real analysis as well. For instance, in the lecture notes Measure and Integration by Henk de Snoo and Henrik Winkler [2, p. 56], there is another *Monotone Convergence Theorem* which states:

Theorem 2. ***(Monotone Convergence Theorem)*** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $f_n, n \in \mathbb{N}$, and f be nonnegative measurable extended real-valued functions on Ω, such that*

$$f_n(\omega) \to f(\omega), \quad f_n(\omega) \leq f_{n+1}(\omega), \quad \omega \in \Omega$$

then

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu$$

A fine example indeed, and certainly one out of many. Monotone convergence is in the scope of copious amounts of papers including but not limited to [3, 4, 5]. But also more recent papers such as [6], illustrating that monotone convergence is still an active research topic. Due to its high impact on various branches of mathematics, it seems worthwhile to try and generalize the concept. In this thesis we investigate a possible generalization.

1.2 Closed Metric Intervals

In 1906, Maurice Fréchet introduced the notion of *écart* (semi-metric), and in 1914 Felix Hausdorff coined the term *metric space*. After that, it did not take long for Karl Menger to introduce metric spaces in geometry. In 1928 his *"Untersuchungen Über Algemeine Metrik"* [7] got published. In this publication he writes, among other things, about the points that are in between two points. A point x is said to be between a and b if we have $d(a, x) + d(x, b) = d(a, b)$. Nowadays, the set of all such points is called a *Closed Metric Interval* [8, p. 13]. Because Menger his text is in German, and most of his proofs are given verbally, the relevant statements are presented here to the reader in a more mathematically rigorous fashion.

Notation 1. *For a metric space (X, d) and the points $a, b \in X$ we introduce some notation for the following set.*

$$[a, b]_X^d := \{x \in X : d(a, x) + d(x, b) = d(a, b)\}$$

Here the sub- and superscript are used to remove any ambiguity revolving around the metric space in which the set is defined. When it is clear from the context which metric space we are referring to, we will simply write $[a, b]$. It can be verified that $[a, b]_{\mathbb{R}}^{|\cdot|}$ defined as above, is just the closed interval of real numbers between a and b (which is the sole reason for this choice of notation). Indeed, a closed metric interval is a generalization of a closed interval of real numbers. In section 1.3 we show how closed metric intervals are used to proof a *Monotone Convergence Theorem* in the context of metric spaces. In some sense, closed metric intervals behave very much like line segments. But, in some other sense, they are less well behaved, as will be illustrated.

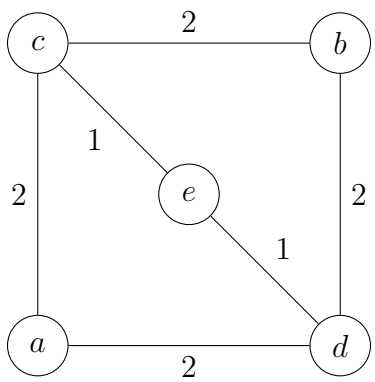
Lemma 1. *$x \in [a,b]$ if and only if $[a,x] \subseteq [a,b]$*

Proof. Suppose $[a,x] \subseteq [a,b]$, since $d(a,x) + d(x,x) = d(a,x)$ we have $x \in [a,x]$ so $x \in [a,b]$. Conversely, suppose $x \in [a,b]$. Then for all $y \in [a,x]$ we have:

$$\begin{aligned} d(a,b) &= d(a,x) + d(x,b) \\ &= d(a,y) + d(y,x) + d(x,b) \\ &\geq d(a,y) + d(y,b) \\ &\geq d(a,b). \end{aligned}$$

In particular $d(a,y) + d(y,b) = d(a,b)$ so $y \in [a,b]$. This tells us that indeed $[a,x] \subseteq [a,b]$, concluding the proof.

Note that $[a,b] = [b,a]$ in all metric spaces, and therefore we must find that also $x \in [a,b]$ if and only if $[x,b] \subseteq [a,b]$. This property of closed metric intervals is very much consistent with the intuition inherited from the real line. However, the stronger statement: "*$x, y \in [a,b]$ if and only if $[x,y] \subseteq [a,b]$*", does **not** hold in general as the following counter example demonstrates.



Consider the above collection of nodes and let the distance between two nodes be defined as the shortest path between them. Then we have that $[a,b] = \{a,b,c,d\}$ meaning $c, d \in [a,b]$, however $[c,d] = \{c,d,e\}$ is not a subset of $[a,b]$. Instead, we do have the following statement.

Lemma 2. *take $x, y \in [a,b]$, then $x \in [a,y]$ if and only if $y \in [x,b]$.*

Proof. Take $x, y \in [a,b]$, we show the two are equivalent.

$$\begin{aligned} & & d(a,x) + d(x,y) &= d(a,y) \\ &\iff & d(a,x) + d(x,y) + d(y,b) &= d(a,y) + d(y,b) \\ &\iff & d(a,x) + d(x,y) + d(y,b) &= d(a,b) \\ &\iff & d(x,y) + d(y,b) &= d(a,b) - d(a,x) \\ &\iff & d(x,y) + d(y,b) &= d(x,b) \end{aligned}$$

Theorem 3. *Let (X, d) be a metric space and take $a, b \in X$. The closed metric interval $[a, b]$ is a closed set.*

Proof. We show that $[a, b]^{\complement}$ is open. Suppose $x \in [a, b]^{\complement}$, we need to produce an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq [a, b]^{\complement}$. Let us show that $\varepsilon := d(x, [a, b])$ satisfies. First we verify that $d(x, [a, b]) > 0$. Observe that for $y \in [a, b]$ we have

$$\begin{aligned} 2d(x, y) &\geq |d(a, x) - d(a, y)| + |d(b, x) - d(b, y)| \\ &\geq |d(a, x) + d(x, b) - d(a, y) - d(y, b)| \\ &= |d(a, x) + d(x, b) - d(a, b)| \\ &> 0 \end{aligned}$$

so in particular, $d(x, y) \geq \frac{1}{2}|d(a, x) + d(x, b) - d(a, b)| > 0$. since $d(x, [a, b])$ is the greatest lower bound to $d(x, y)$, we must find that $d(x, [a, b]) > 0$ as desired. Lastly, we show that $B(x, d(x, [a, b])) \subseteq [a, b]^{\complement}$. suppose that $z \in B(x, d(x, [a, b]))$ then $d(x, z) < d(x, [a, b])$. This means that $z \in [a, b]^{\complement}$. ∎

1.3 Monotone Convergence Theorem

Given that we are working with metric spaces, we have a clear definition of convergence. To achieve a monotone convergence theorem, we still need a notion of monotonicity. We will obtain this notion from the following partial order relation;

Theorem 4. *Let (X, d) be a metric space and fix $a \in X$, then the relation $\preceq_a$ defined by:*

$$x \preceq_a y \iff x \in [a, y] \quad \textit{for all } x, y \in X,$$

is a partial order relation

Proof. From lemma 1 we see that $x \preceq_a y$ if and only if $[a, x] \subseteq [a, y]$. Realizing that the subset relation is a partial order relation completes the proof. ∎

It may be clear to the reader that the partial order defined as above, is dependent on which $a \in X$ we fix. This gives rise to an entire family of partial order relations as Zhang et al. have pointed out in [9]. Take, for example $a, b \in X$ we can consider the corresponding partial order relations $\preceq_a$ and $\preceq_b$. Combining theorem 4 with lemma 2, gives that for $x, y \in [a, b]$ we have $x \preceq_a y \iff x \in [a, y] \iff y \in [b, x] \iff y \preceq_b x$. Remember that $[x, b] = [b, x]$. In particular, the order of x and y has been reversed when we changed between relations.

Definition 2. *Let (X, d) be a metric space and take $a \in X$, we say a sequence (x_n) is a-increasing if $x_n \preceq_a x_{n+1}$ for all $n \in \mathbb{N}$ and a-decreasing if $x_{n+1} \preceq_a x_n$ for all $n \in \mathbb{N}$. A sequence is a-monotone if it is either a-increasing or a-decreasing.*

Theorem 5. ***(Monotone Convergence Theorem)*** *Let (X, d) be a complete metric space and fix $a, b \in X$. If a sequence (x_n) takes its elements from $[a, b]$ and is a-monotone, then it converges.*

Proof. Suppose our sequence is a-increasing. Notice that $x_n \preceq_a x_{n+1}$ implies $d(a, x_n) \leq d(a, x_{n+1})$. Since (x_n) takes its elements from $[a, b]$ we know that $x_n \preceq_a b$ for all $n \in \mathbb{N}$. So the sequence $(d(a, x_n))$ is a sequence of real numbers that is increasing and bounded above by $d(a, b)$, thus, by theorem 1 it converges and as a consequence, it is Cauchy. Meaning that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } m, n \geq N \text{ implies } |d(a, x_m) - d(a, x_n)| < \varepsilon.$$

Assume without loss of generality that $n \leq m$, then we find that $x_n \preceq_a x_m$, meaning $x_n \in [a, x_m]$ so that $d(a, x_n) + d(x_n, x_m) = d(a, x_m)$. And therefore we find:

$$\begin{aligned} \varepsilon &> |d(a, x_m) - d(a, x_n)| \\ &= d(x_n, x_m). \end{aligned}$$

This tells us that (x_n) is a Cauchy sequence. Since our space is complete the sequence converges. Moreover, if the sequence is a-decreasing, e.g. $x_{n+1} \preceq_a x_n$, then by lemma 2 we have that the sequence is b-increasing, e.g. $x_n \preceq_b x_{n+1}$. Repeating the previous argument, with the roles of a and b interchanged, shows that the sequence converges. ∎

This monotone convergence theorem can, for example, be used as follows. Find a complete metric space that has your interest, then use the properties of the respective metric to compute what an increasing sequences looks like explicitly. If one is interested in $(\mathbb{R}, |\cdot|)$, then this theorem coincides with theorem 1. It is useful to know equivalent representations of closed metric intervals when tackling this problem. These will be discussed in the next chapter.

Chapter 2

Betweenness Relations

2.1 Preliminaries

In the previous chapter we discussed how betweenness is defined in metric spaces and how we can use it to construct a partial order relation that satisfies a monotone convergence theorem. More generally, betweenness relations can be defined in different contexts as well. The motivation for its usage in this thesis is the fact that each order relation induces a betweenness relation [9, Ex. 1] and each betweenness relation induces a family of partial order relations [9, Thm. 3]. Betweenness relations were first formalized by Pasch [10] and later investigated by many authors such as Hedlíková [11], Smiley [12], and Fishburn [13]. We will restrict discussion to the work of Smiley [12] who investigates when metric, algebraic, and lattice betweenness coincide. He uses the following definitions;

Definition 3. *For a metric space (X, d) and for any $a, x, b \in X$, the ternary relation $(axb)_M$ defined by:*

$$(axb)_M \iff d(a, x) + d(x, b) = d(a, b),$$

is called metric betweenness.

Definition 4. *Let X be a vector space over real or complex numbers. For any $a, x, b \in X$, the ternary relation $(axb)_A$ defined by:*

$$(axb)_A \iff x = \lambda a + (1 - \lambda)b, \quad \lambda \text{ a real number } 0 \leq \lambda \leq 1,$$

is called algebraic betweenness

Definition 5. *For a lattice $(X, \vee, \wedge)$ and for any $a, x, b \in X$, the ternary relation $(axb)_L$ defined by:*

$$(axb)_L \iff (a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b)$$

is called lattice betweenness

The work of Smiley is interesting in the context of our monotone convergence theorem, because it allows for multiple representations of monotonicity. Indeed, betweenness relations induce partial order relations. So whenever the betweenness relations coincide, so will the partial order relations. In particular, when metric betweenness coincides with algebraic, or lattice betweenness, we can decide on monotonicity using the algebraic or lattice structure of the sequence. Observe that for a metric space (X, d) we have written the closed metric interval of points between a and b as:

$$[a, b]_X^d = \{x : (axb)_M\}$$

If betweenness relations coincide, we may as well write:

$$[a, b]_X^d = \{x : (axb)_A\}, \quad \text{or} \quad [a, b]_X^d = \{x : (axb)_L\},$$

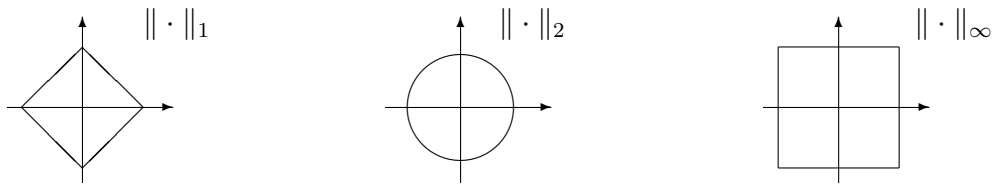
This gives the possibility for multiple representations of monotonicity. Observe that all three relations coincide in $(\mathbb{R}, |\cdot|)$.

2.2 Metric- and Algebraic Betweenness

In the case of normed vector spaces, there is a condition that is both necessary and sufficient for the equivalence of algebraic and metric betweenness. It is defined as follows:

Definition 6. *A normed real or complex vector space $(X, \|\cdot\|)$ is strictly convex whenever for $a, b \in X$ both nonzero, it satisfies; $\|a\| + \|b\| = \|a + b\| \implies a = \gamma b$, for some real number $\gamma > 0$.*

A strictly convex space is a space for which the closed unit ball is a strictly convex set. In other words; for two distinct points x and y on the boundary of the unit ball, the line segment xy intersects the boundary only at the points x and y. For example, consider $\mathbb{R}^2$ with the Euclidean norm. In this case, the unit sphere is a circle, so any line connecting two points on this circle, will intersect the circle only at those two points. However for $\mathbb{R}^2$ with the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$, the space is not strictly convex. Below is a picture of the various unit spheres. One can see that, for $\|\cdot\|_1$ and $\|\cdot\|_\infty$, it is possible to draw lines between boundary points that intersect in more than two points.



Theorem 6. *A seminormed real or complex vector space $(S, \|\cdot\|)$ is strictly convex if and only if algebraic and metric betweenness coincide in S.*

Proof. Let S be a strictly convex seminormed real or complex vector space. We first show that $(axb)_A \implies (axb)_M$. Suppose $x = \lambda a + (1-\lambda)b$ for some $0 \leq \lambda \leq 1$, we require that $\|a - x\| + \|x - b\| = \|a - b\|$. And indeed, observe that:

$$\begin{aligned}\|a - x\| + \|x - b\| &= \|a - \lambda a - (1-\lambda)b\| + \|\lambda a + (1-\lambda)b - b\| \\ &= (1-\lambda)\|a - b\| + \lambda\|a - b\| \\ &= \|a - b\|.\end{aligned}$$

Next we show that $(axb)_M \implies (axb)_A$. For $x = a$ or $x = b$ the proof is trivial. When $a \neq x \neq b$, suppose that $\|a - x\| + \|x - b\| = \|a - b\|$. Then we see $\|a - x\| + \|x - b\| = \|(a - x) + (x - b)\|$. Using the strict convexity of S we find that $a - x = \gamma(x - b)$ for some $\gamma > 0$. Consequently we have that:

$$\begin{aligned} & a - x = \gamma(x - b), \\ \implies \quad & a + \gamma b = (1+\gamma)x, \\ \implies \quad & x = \frac{1}{1+\gamma}a + \frac{\gamma}{1+\gamma}b. \end{aligned}$$

Letting $\lambda = \frac{1}{1+\gamma}$ tells us that $(axb)_A$ holds true. So we can conclude that strict convexity implies the equivalence of algebraic and metric betweenness.

Conversely, assume that algebraic and metric betweenness coincide. We need to show the space is strictly convex. So suppose for nonzero a and b, we are given $\|a\| + \|b\| = \|a + b\|$ we are tasked with finding a $\gamma > 0$ so that $a = \gamma b$. From $\|a\| + \|b\| = \|a + b\|$ we know that $0 \in [a, -b]_S^{\|\cdot\|}$. So $(a0 - b)_M$ holds true. By the equivalence of metric and algebraic betweenness we also have $(a0 - b)_A$. Meaning $0 = \lambda a + (1-\lambda)(-b)$ for some $0 < \lambda < 1$. Remember that $0 \neq \lambda \neq 1$ because $a \neq 0 \neq b$. We find that $a = \frac{1-\lambda}{\lambda}b$. So letting $\gamma = \frac{1-\lambda}{\lambda}$ shows that S is strictly convex. This concludes the proof.

From this theorem we know that in strictly convex vector spaces we can write the closed metric intervals as $[a, b] = \{x : (axb)_A\} = \{\lambda a + (1-\lambda)b : 0 \leq \lambda \leq 1\}$. This means that the closed metric interval is just all the convex combinations of a and b. In this case, the closed metric interval is really just a line segment. Consequently, a monotone sequence is just a sequence of points on this line segment. Examples of strictly convex vector spaces are given by the L_p function spaces for $1 < p < \infty$. The fact that these function spaces are strictly convex will be proven in the next chapter.

We also know from this theorem, that when a vector space is not strictly convex, algebraic and metric betweenness do not coincide. For example, the function spaces

L_1 and L_∞ are not strictly convex. It follows that the closed metric intervals in these spaces are not simply given by line segments. And therefore, we can expect the existence of monotone sequences that are somewhat more extravagant than a progression on a line segment.

2.3 Metric- and Lattice Betweenness

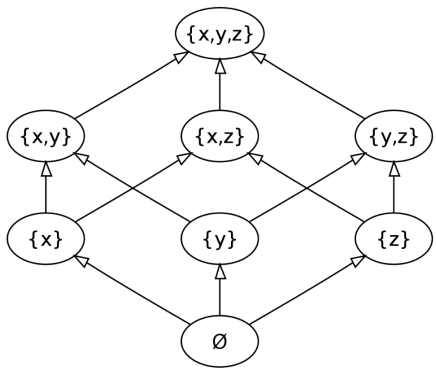
In his paper [12], Smiley also states necessary and sufficient conditions for metric and lattice betweenness to coincide. We will provide the statement but not prove it. First we write the definition of a lattice

Definition 7. *A lattice X is a partially ordered set equiped with two well defined binary operators $\wedge$ and $\vee$*

$$\wedge : X \times X \to X : (a, b) \mapsto a \wedge b$$
$$\vee : X \times X \to X : (a, b) \mapsto a \vee b$$

Where $a \wedge b$ is the greatest lower bound to both a and b. Likewise, $a \vee b$ is the least upper bound to both a and b.

An example of a lattice is $(\mathbb{R}, \min, \max)$. Another example is a collection of sets equiped with intersection and union where the partial ordering is given by the subset relation. The picture below shows such a lattice. The arrows indicate set inclusion.



Definition 8. *We say a lattice $(X, \wedge, \vee)$ is distributive if for all $a, b, c \in X$ we have:*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Lemma 3. *If X is a distributive lattice and $a, x, b \in X$, then:*

$$(axb)_L \iff a \wedge b \leq x \leq a \vee b$$

Proof. Suppose X is a distributive lattice. If we have $(axb)_L$ then by definition $(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b)$. So then we find $(a \vee b) \wedge x = x = x \vee (a \wedge b)$. This means $a \wedge b \leq x \leq a \vee b$. Reversing all steps provides the converse.

Theorem 7. *If (S, d) is a semimetric space which is also a lattice, then metric betweenness and lattice betweenness coincide in S if and only if:*

i) For evey $a, x, b \in S$ the inequalities $a \leq x \leq b$ imply that $(a, x, b)_M$.

ii) For every $a, b \in S$, $d(a, b) = d(a \wedge b, a \vee b)$ and $d(a, a \vee b) = d(b, a \wedge b)$.

Proof. This is theorem 2 of [12].

Chapter 3

Examples

In this chapter we will interpret the monotone convergence theorem explicitly in the context of specific complete metric spaces. As described earlier, it is useful to find equivalent representations of betweenness. These equivalent representations then may be used as a convenient criterion for monotonicity.

3.1 Strict Convexity of L_p for $1 < p < \infty$

In the previous chapter its proven that closed metric intervals reduce to line segments whenever the metric space is a strictly convex normed vector space. An example of such a space is L_p with $1 < p < \infty$. Before we can prove this we need the following lemma.

Lemma 4. *Let $\alpha, \beta \in \mathbb{R}$ be nonnegative and let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Then*

$$\alpha^{1/p}\beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q}.$$

Furthermore, the equality holds if and only if $\alpha = \beta$.

Proof. Observe that for $t \geq 1$ we have that

$$t^{1/p} \leq \frac{t}{p} + \frac{1}{q}$$

This holds because for $t = 1$ the equality is satisfied and in each case, the derivative of the left-hand side is strictly less than the derivative of the right-hand side for $t > 1$. Substituting α/β or β/α depending on their size proves the result. Since we only obtain equality for $t = 1$ we know we only obtain equality for $\alpha = \beta$. ✐

To prove strict convexity, besides this lemma, we will need to use Hölder's inequality. It is stated as follows

Theorem 8. ***(Hölder's inequality)*** *Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. If $f \in \mathcal{L}^p(\Omega)$ and $g \in \mathcal{L}^q(\Omega)$, then $fg \in \mathcal{L}^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. First consider the case $1 < p < \infty$ if either $\|f\|_p = 0$ or $\|g\|_q = 0$, then the result is a straightforward verification. If we have $\|f\|_p \neq 0 \neq \|g\|_q$ then we can apply Lemma 4 to

$$\alpha = \left(\frac{|f(\omega)|}{\|f\|_p}\right)^p, \quad \beta = \left(\frac{|g(\omega)|}{\|g\|_q}\right)^q.$$

This results in

$$\frac{|f(\omega)g(\omega)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}\left(\frac{|f(\omega)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(\omega)|}{\|g\|_q}\right)^q.$$

Integrating both sides leads to

$$\int_\Omega \frac{|f(\omega)g(\omega)|}{\|f\|_p \|g\|_q} \, \mathrm{d}\mu \leq \frac{1}{p} + \frac{1}{q} = 1,$$

which leads to the result. If $p = 1$ then $q = \infty$ and we have

$$|f(\omega)g(\omega)| \leq |f(\omega)| \|g\|_\infty \quad \text{almost everywhere.}$$

Integrating this expression provides the result. The case $p = \infty$ and $q = 1$ is treated similarly. ✐

Armed with Lemma 4 and Hölder's inequality, one can prove the Minkowski inequality. Which is the triangle inequality for the norm $\|\cdot\|_p$. To prove strict convexity, we consider the equality case of the triangle inequality. This will induce equality for Hölder's inequalty, which in turn induces equality almost everywhere for the inequality in Lemma 4.

Theorem 9. *Take any measure space $(\Omega, \mathcal{A}, \mu)$ and take $1 < p < \infty$. Then the normed vector space $(L_p(\Omega), \|\cdot\|_p)$ is strictly convex.*

Proof. Let $1 < p < \infty$ and take $f, g \in L_p(\Omega)$ both non zero such that $\|f\|_p \neq 0 \neq \|g\|_p$. Suppose that $\|f\|_p + \|g\|_p = \|f + g\|_p$, to demonstrate strict convexity, we need to produce a real number $\gamma > 0$ such that $f = \gamma g$. Choose q such that $1/p + 1/q = 1$. We now find that $\|f + g\|_p^{p-1} = \|(f + g)^{p-1}\|_q$. This is true because

$$\|f + g\|_p^{p-1} = \left(\int_\Omega |f + g|^p \, \mathrm{d}\mu\right)^{1-1/p} = \left(\int_\Omega |f + g|^{(p-1)q} \, \mathrm{d}\mu\right)^{1/q} = \|(f + g)^{p-1}\|_q.$$

Since we assumed $\|f+g\|_p = \|f\|_p + \|g\|_p$, we can multiply both sides of this equation by $\|f+g\|_p^{p-1}$ to obtain the following:

$$\begin{aligned}
\|f+g\|_p^p &= (\|f\|_p + \|g\|_p)\,\|f+g\|_p^{p-1} \\
&= (\|f\|_p + \|g\|_p)\,\|(f+g)^{p-1}\|_q \\
&\geq \|f(f+g)^{p-1}\|_1 + \|g(f+g)^{p-1}\|_1 \\
&\geq \|(f+g)(f+g)^{p-1}\|_1 \\
&= \|f+g\|_p^p.
\end{aligned}$$

Where the first inequality is obtained from Hölders inequality. The second inequality comes from the triangle inequality. The discussion above shows that equality holds for both the Hölder inequality and the triangle inequality. In particular we have

$$\|f\|_p\|(f+g)^{p-1}\|_q = \|f(f+g)^{p-1}\|_1, \tag{$\blacklozenge$}$$

and

$$\|g\|_p\|(f+g)^{p-1}\|_q = \|g(f+g)^{p-1}\|_1, \tag{$\bigstar$}$$

and

$$\|f(f+g)^{p-1}\|_1 + \|g(f+g)^{p-1}\|_1 = \|(f+g)(f+g)^{p-1}\|_1. \tag{$\spadesuit$}$$

Using ($\blacklozenge$) we see that:

$$\int_\Omega \frac{|f||f+g|^{p-1}}{\|f\|_p\|(f+g)^{p-1}\|_q}\,\mathrm{d}\mu = 1 = \frac{1}{p}\int_\Omega \frac{|f|^p}{\|f\|_p^p}\,\mathrm{d}\mu + \frac{1}{q}\int_\Omega \frac{|(f+g)^{p-1}|^q}{\|(f+g)^{p-1}\|_q^q}\,\mathrm{d}\mu.$$

The left hand side is obtained by dividing both sides of ($\blacklozenge$) by $\|f\|_p\|(f+g)^{p-1}\|_q$. The right hand side is obtained by seeing that the integrals both equal 1 and that $1/p + 1/q = 1$. Now, from Lemma 4 we know that:

$$\frac{|f||(f+g)^{p-1}|}{\|f\|_p\|(f+g)^{p-1}\|_q} \leq \frac{1}{p}\left(\frac{|f|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|(f+g)^{p-1}|}{\|(f+g)^{p-1}\|_q}\right)^q.$$

But since their integrals are equal, we have to conclude that the left-hand side and the right-hand side are equal almost everywhere. Again from Lemma 4, we now know that

$$\frac{|f|}{\|f\|_p} = \frac{|(f+g)^{p-1}|}{\|(f+g)^{p-1}\|_q} \quad \text{almost everywhere.}$$

Repeating the same argument, but now for ($\bigstar$), we obtain a similar result. Combining the two gives us:

$$\frac{|f|}{\|f\|_p} = \frac{|g|}{\|g\|_p} \quad \text{almost everywhere.}$$

Since we are dealing with complex valued functions we find:

$$f = \frac{\|f\|_p}{\|g\|_p} e^{i\theta} g \quad \text{almost everywhere, for } \theta \in \mathbb{R}.$$

Substituting this result into (♠) yields

$$\begin{aligned}
& \|f(f+g)^{p-1}\|_1 + \|g(f+g)^{p-1}\|_1 = \|(f+g)(f+g)^{p-1}\|_1 \\
\iff \quad & \left|\frac{\|f\|_p}{\|g\|_p} e^{i\theta}\right| \|g(f+g)^{p-1}\|_1 + \|g(f+g)^{p-1}\|_1 = \left|\frac{\|f\|_p}{\|g\|_p} e^{i\theta} + 1\right| \|g(f+g)^{p-1}\|_1 \\
\iff \quad & \left|\frac{\|f\|_p}{\|g\|_p} e^{i\theta}\right| + |1| = \left|\frac{\|f\|_p}{\|g\|_p} e^{i\theta} + 1\right|.
\end{aligned}$$

This last equation therefore has to hold, but this can only happen if $\theta = 2\pi K$ for $K \in \mathbb{Z}$. We now have to conclude

$$f = \frac{\|f\|_p}{\|g\|_p} g \quad \text{almost everywhere.}$$

Letting $\gamma = \frac{\|f\|_p}{\|g\|_p}$ yields $f = \gamma g$ in $L_p(\Omega)$, which is what we set out to prove. ✐

Now that we have proven that the normed vector space $(L_p(\Omega), \|\cdot\|_p)$ is strictly convex for $1 < p < \infty$, we can combine this result with the discussion from the previous chapter. We now know that algebraic and metric betweenness coincide in L_p. This means that the closed metric interval is a line segment. So for, $f, g \in L_p(\Omega)$, we have that $[f, g] = \{x : x = \lambda f + (1-\lambda)g, \text{ for some } 0 \leq \lambda \leq 1\}$. Well, we can actually solve for what this λ is in each case of x. Under the assumption that $f \neq x \neq g$, and since we satisfy $\|f - x\|_p + \|x - g\|_p = \|f - g\|_p$ we find by strict convexity that:

$$f - x = \frac{\|f - x\|_p}{\|x - g\|_p}(x - g).$$

This in turn implies

$$x = \frac{\|x - g\|_p}{\|f - g\|_p} f + \frac{\|f - x\|_p}{\|f - g\|_p} g.$$

So that we find, for this particular x, its corresponding $\lambda = \frac{\|x-g\|_p}{\|f-g\|_p}$. Suppose we are given an f-monotone sequence (x_n) in $[f, g]$, then using theorem 3, 5, and 6, we see this sequence converges to some convex combination of f and g. More precisely

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\|x_n - g\|_p}{\|f - g\|_p} f + \lim_{n\to\infty} \frac{\|f - x_n\|_p}{\|f - g\|_p} g.$$

3.2 Continuous Functions with L_∞ Norm

From the Cauchy criterion for uniform convergence [1, Thm. 6.2.5] combined with the continuous limit theorem [1, Thm. 6.2.6], we know that the space of continuous functions defined on some set $A \subseteq \mathbb{R}$ is complete with the sup norm. So indeed, it is a candidate for our monotone convergence theorem. Since the norm $\|\cdot\|_\infty$ is not strictly convex, we know that metric betweenness does not coincide with algebraic betweenness. So besides all the convex combinations, we have more functions that sit in between. Smiley proved in [12, Thm. 5] necessary and sufficient condition for metric betweenness in the metric space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$. We will apply this equivalent representation to the monotone convergence theorem, but first we give some intuition on what betweenness looks like for continuous functions. In general, we have the following theorem.

Theorem 10. *we can rewrite the closed metric interval $[a,b]$ as follows:*

$$[a,b] = \bigcup_{0\leq\lambda\leq 1} (B[a, \lambda d(a,b)] \cap B[b, (1-\lambda)d(a,b)])$$

Where $B[x,\varepsilon]$ is defined as the closed metric ball of radius ε centered at x.

Proof. In case $a = b$ the statement is trivial. Otherwise, take $x \in [a,b]$, then by definition $d(a,x) + d(x,b) = d(a,b)$. This means that $0 \leq d(a,x) \leq d(a,b)$. And consequently, $0 \leq \frac{d(a,x)}{d(a,b)} \leq 1$. Let $\lambda = \frac{d(a,x)}{d(a,b)}$, then we have:

$$d(a,x) = \lambda d(a,b), \quad \text{and} \quad d(x,b) = (1-\lambda)d(a,b)$$

So that indeed $x \in \bigcup_{0\leq\lambda\leq 1} (B[a; \lambda d(a,b)] \cap B[b; (1-\lambda)d(a,b)])$.

Conversely, suppose there exists $0 \leq \lambda \leq 1$ such that

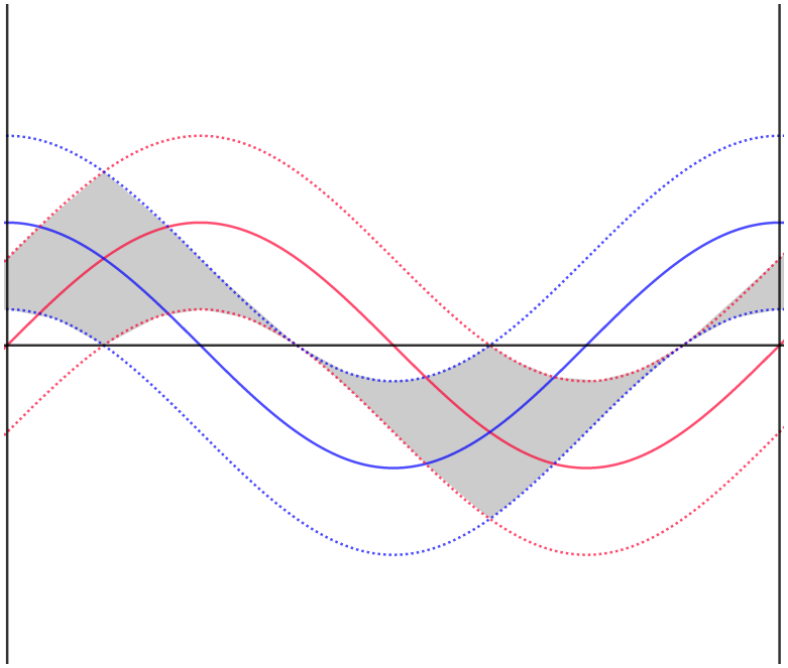
$$d(a,x) \leq \lambda d(a,b), \quad \text{and} \quad d(x,b) \leq (1-\lambda)d(a,b)$$

Then $d(a,x) + d(x,b) \leq \lambda d(a,b) + (1-\lambda)d(a,b) = d(a,b)$ the triangle inequality tells us now that $d(a,x) + d(x,b) = d(a,b)$ so that $x \in [a,b]$ as desired. ∎

This way of rewriting $[a,b]$ shows we can construct the closed metric interval by taking closed metric balls around a and b of such a radius that they touch. For the norm $\|\cdot\|_2$ this would give two touching discs. We care about the points in which the two discs intersect, and for the strict convex norm $\|\cdot\|_2$ this will be a unique point. Varying the radius will produce the line of convex combinations. In the case of the sup norm, which is not strictly convex, we know there is at least one radius for which the closed metric balls intersect in more than one point. The theorem above allows us to cut the closed metric interval in thin slices and

examine these cross sections to at least get some idea of what the interval looks like.

Let us take for example the functions $\sin(x), \cos(x) \in \mathcal{C}[0, 2\pi]$. We will consider the cross section of the closed metric interval $[\sin, \cos]^{\|\cdot\|_\infty}_{\mathcal{C}[0,2\pi]}$ at $\lambda = 1/2$. That is, $B[\sin, 1/2\|\sin - \cos\|_\infty] \cap B[\cos, 1/2\|\sin - \cos\|_\infty]$. Below we have plotted the sine and cosine function.



Above and below these functions, there are dotted lines indicated at the fixed distance $\pm 1/2\|\sin - \cos\|_\infty$ from the respective functions. If a function g is within the strip around the sine function for example, then we know $\|\sin - g\|_\infty \leq 1/2\|\sin - \cos\|_\infty$. The area where these strips intersect is shaded. If a function $g \in \mathcal{C}[0, 2\pi]$ lands in this shaded area, we have $g \in B[\sin, 1/2\|\sin - \cos\|_\infty]$ and $g \in B[\cos, 1/2\|\sin - \cos\|_\infty]$. This implies $g \in [\sin, \cos]$. Indeed, one can imagine exotic functions that are still in between the sine and cosine function. However, doing so is not in the scope of this thesis.

Let us continue with the equivalent representation of metric betweenness given by Smiley. He uses the following Lemma:

Lemma 5. *In the metric space $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ the relation $(a0b)_M$ holds if and only if there is a point t_0 on $[0, 1]$ at which each of the functions $|a(t)|, |b(t)|$, and $|a(t) - b(t)|$ attains its maximum value and the relation $(a(t_0)0b(t_0))_M$ holds.*

Proof. If a or b equal zero, the statement holds trivially. Assume $a \neq 0 \neq b$. Suppose $(a0b)_M$ holds. Let $t_0 \in [0, 1]$ be a point at which the function $|a(t) - b(t)|$

attains its maximum value, then:

$$\begin{aligned}\|a\|_\infty + \|b\|_\infty &= \|a-b\|_\infty \\ &= |a(t_0) - b(t_0)| \\ &\leq |a(t_0)| + |b(t_0)| \\ &\leq \|a\|_\infty + \|b\|_\infty\end{aligned}$$

This shows that there exists a $t_0 \in [0,1]$ such that $|a(t)|, |b(t)|$, and $|a(t) - b(t)|$ attain their maximum and the relation $(a(t_0)0b(t_0))_M$ holds.

Conversely, if we have such a t_0, then we see

$$\|a-b\|_\infty = |a(t_0) - b(t_0)| = |a(t_0)| + |b(t_0)| = \|a\|_\infty + \|b\|_\infty$$

This completes the proof. ✐

Theorem 11. *In the metric space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$ the relation $(axb)_M$ holds if and only if there is a point t_0 on $[0,1]$ at which each of the functions $|a(t)-x(t)|, |x(t)-b(t)|$, and $|a(t)-b(t)|$ attains its maximum value and the relation $(a(t_0)x(t_0)b(t_0))_M$ holds.*

Proof. Take $a, x, b \in \mathcal{C}[0,1]$ and let $a' = a - x$ and $b' = b - x$. Then we have:

$$\begin{aligned}(a'0b')_M &\iff \|a'\|_\infty + \|b'\|_\infty = \|a' - b'\| \\ &\iff \|a-x\|_\infty + \|x-b\|_\infty = \|a-b\|_\infty \\ &\iff (axb)_M\end{aligned}$$

Furthermore, we have from Lemma 5 that $(a'0b')_M$ holds if and only if there is a t_0 in $[0,1]$ at which $|a-x|, |x-b|$, and $|a-b|$ attain their maximum value and $(a(t_0)x(t_0)b(t_0))_M$. This completes the proof. ✐

We will use this equivalent representation to obtain information on the properties of monotone sequences in the metric space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$.

Lemma 6. *Given a sequence (x_n) in the metric space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$, the relation $(ax_nb)_M$ holds for all $n \in \mathbb{N}$ if and only if there exists a point $t_0 \in [0,1]$ such that for all $n \in \mathbb{N}$ the functions $|a-x_n|, |x_n-b|$ and $|a-b|$ attain their maximum value and $(a(t_0)x_n(t_0)b(t_0))_M$ holds.*

Proof. Suppose we have $(ax_nb)_M$ for all $n \in \mathbb{N}$, then by theorem 11 we have a value $t_0 \in [0,1]$ at which each of the functions $|a-x_0|, |x_0-b|$ and $|a-b|$ attain their maximum and $(a(t_0)x_0(t_0)b(t_0))_M$ holds. Now for $n \neq 0$, we see the following

$$\begin{aligned}|a(t_0) - x_n(t_0)| + |x_n(t_0) - b(t_0)| &\leq \|a-x_n\|_\infty + \|x_n - b\|_\infty \\ &= \|a-b\|_\infty \\ &= |a(t_0) - b(t_0)|\end{aligned}$$

Using the triangle inequality on $|a(t_0) - b(t_0)|$ shows that this t_0 works for all n. The converse is trivial. ✐

Theorem 12. ***(Monotone Convergence Theorem)*** *Take $a, b, x_n \in \mathcal{C}[0,1]$ for all $n \in \mathbb{N}$ and suppose that there exists a $t_0 \in [0,1]$ such that:*

- *For all $n \in \mathbb{N}$ each of the funtions $|a - x_n|, |x_n - b|, |x_n - x_{n+1}|$ and $|a - b|$ attain their maximum at t_0.*
- *The sequence $(x_n(t_0))$ is monotone on $[a(t_0), b(t_0)]$.*

Then the sequence (x_n) converges uniformly.

Proof. Indeed, if we have a t_0 that satisfies the requirements, then from the monotonicity of the sequence $(x_n(t_0))$ on $[a(t_0), b(t_0)]$, we see that $(a(t_0)x_n(t_0)b(t_0))_M$ and $(a(t_0)x_n(t_0)x_{n+1}(t_0))_M$ hold. Combining this result with theorem 11 it can be seen that both $(ax_nb)_M$ and $(ax_nx_{n+1})_M$ hold for all $n \in \mathbb{N}$. This is of course a different way of saying that $x_n \in [a, b]$ and $x_n \in [a, x_{n+1}]$ for all $n \in \mathbb{N}$. From theorem 5 we know the sequence converges in the sup norm. This completes the proof. ✐

Notice how the requirement imposed on the sequence (x_n) are equivalent to this sequence being in $[a, b]$ and being a-monotone. We have already proven one direction of this equivalence. The converse is a consequence of Lemma 6.

3.3 Convergence of Sets in Measure Theory

In this section we will apply the monotone convergence theorem in the context of measure theory. We will describe how a σ-algebra can be seen as a complete metric space and hence is a candidate for applying the monotone convergence theorem. For a finite measure space $(\Omega, \mathcal{A}, \mu)$ define the function d_μ as follows:

$$d_\mu : \mathcal{A} \times \mathcal{A} \to \mathbb{R} : (A, B) \mapsto \mu(A \triangle B).$$

Here $\triangle$ denotes the symmetric difference. It turns out that this function is a semi metric on $\mathcal{A}$. We can turn this into a metric space by considering the equivalence relation $\sim$ defined by $A \sim B \iff \mu(A \triangle B) = 0$. Intuitively, two sets are equivalent if they are equal almost everywhere. Let us denote $\widetilde{\mathcal{A}} := \mathcal{A}/\sim$ and denote $\widetilde{A}$ for the equivalence class of A. We will use d_μ canonically;

$$d_\mu : \widetilde{\mathcal{A}} \times \widetilde{\mathcal{A}} \to \mathbb{R} : (\widetilde{A}, \widetilde{B}) \mapsto \mu(A \triangle B).$$

Theorem 13. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, then the space $(\widetilde{\mathcal{A}}, d_\mu)$ is a metric space.*

Proof. Notice that we require $\mu(\Omega) < \infty$ to ensure all distances are finite. Suppose that $d_\mu(\widetilde{A}, \widetilde{B}) = \mu(A \triangle B) = 0$, then A and B belong to the same equivalence class. Obviously d_μ is symmetric. Now note that,

$$\begin{aligned}\mu(A \triangle B) &= \mu(A \triangle X \triangle X \triangle B) \\ &= \mu(A \triangle X \setminus X \triangle B) + \mu(X \triangle B \setminus A \triangle X) \\ &\leq \mu(A \triangle X) + \mu(X \triangle B)\end{aligned}$$

So that triangle inequality is satisfied. ✐

Before we look for equivalent notions of betweenness in this metric space, we should verify that the metric space is complete. Otherwise we can not guarantee that the limit of our monotone convergence theorem exists inside $\widetilde{\mathcal{A}}$. Oxtoby proves in [14, p. 44], that the metric space $(\widetilde{\mathcal{A}}, d_\mu)$ is complete. He does so as follows.

Theorem 14. *The metric space $(\widetilde{\mathcal{A}}, d_\mu)$ is complete.*

Proof. Let $(\widetilde{S}_n)$ be a Cauchy sequence in $\widetilde{\mathcal{A}}$. We show that $(\widetilde{S}_n)$ converges by proving it has a convergent subsequence. For each integer i there is a natural number n_i such that $d_\mu(\widetilde{S}_n, \widetilde{S}_m) < 1/2^i$ for all $n, m \geq n_i$. We may assume that $n_i < n_{i+1}$. Putting $F_i = S_{n_i}$ we have $d_\mu(\widetilde{F}_i, \widetilde{F}_j) < 1/2^i$ for all $j > i$. Define

$$H_i = \bigcap_{j=i}^{\infty} F_j, \quad \text{and} \quad S = \bigcup_{i=1}^{\infty} H_i.$$

By construction, these sets are in $\mathcal{A}$. Now its claimed that both $S \triangle H_k$ and $H_k \triangle F_k$ are contained in $\bigcup_{i=k}^{\infty}(F_i \triangle F_{i+1})$.

- $S \triangle H_k \subseteq \bigcup_{i=k}^{\infty}(F_i \triangle F_{i+1})$: Suppose $x \in S \triangle H_k$, since $H_k \subseteq S$ we only consider $x \in S \setminus H_k$. That is; $x \in \bigcup_{i=1}^{\infty} H_i$ and $x \notin H_k$. So there is a minimal index k' such that $x \in H_{k'}$. By minimality, we find $x \notin H_{k'-1}$. This means $x \in F_{k'}$ and $x \notin F_{k'-1}$. This also means that $k'-1$ is the maximal index i for which $x \notin H_i$. In other words, $k' - 1 \geq k$. We conclude that $x \in F_{k'} \triangle F_{k'-1}$ and hence $x \in \bigcup_{i=k}^{\infty}(F_i \triangle F_{i+1})$

- $H_k \triangle F_k \subset \bigcup_{i=k}^{\infty}(F_i \triangle F_{i+1})$: Suppose $x \in H_k \triangle F_k$, since $H_k \subseteq F_k$ we only consider $x \in F_k \setminus H_k$. That is $x \in F_k$ and $x \notin \bigcap_{j=k}^{\infty} F_j$. Find the smallest $k' > k$ such that $x \notin F_{k'}$. Then $x \in F_{k'-1}$ with $k' - 1 \geq k$. Meaning that $x \in F_{k'} \triangle F_{k'-1}$ so that $x \in \bigcup_{i=k}^{\infty}(F_i \triangle F_{i+1})$.

We can now deduce the following

$$\begin{aligned} d_\mu(\widetilde{S} \triangle \widetilde{S}_{n_k}) &= \mu(S \triangle F_k) \\ &\leq \mu(S \triangle H_k \cup H_k \triangle F_k) \\ &\leq \mu\left(\bigcup_{i=k}^{\infty} (F_i \triangle F_{i+1})\right) \\ &\leq \sum_{i=k}^{\infty} \mu(F_i \triangle F_{i+1}) \\ &< \sum_{i=k}^{\infty} 1/2^i = 1/2^{k-1} \end{aligned}$$

This means that $(\widetilde{S}_n)$ has a subsequence that converges to $\widetilde{S} \in \widetilde{\mathcal{A}}$. Meaning the metric space $(\widetilde{\mathcal{A}}, d_\mu)$ is complete. ✐

Now that we know the space is complete, we can start thinking about equivalent representations of metric betweenness. Since there is a natural lattice structure on $\mathcal{A}$ we canonically inherit this lattice structure on $\widetilde{\mathcal{A}}$. In stead of the subset relation we now have the "subsets almost everywhere" relation.

Theorem 15. *On the metric space $(\widetilde{\mathcal{A}}, d_\mu)$, the relation;*

$$\widetilde{A} \sqsubseteq \widetilde{B} \iff \mu(A \setminus B) = 0$$

is a partial order relation.

Proof. Its clear that $\mu(A \setminus A) = 0$ so for all $\widetilde{A}$ we see that $\widetilde{A} \sqsubseteq \widetilde{A}$. Now suppose we have $\widetilde{A} \sqsubseteq \widetilde{B}$ and $\widetilde{B} \sqsubseteq \widetilde{A}$. Then we certainly have $\mu(A \setminus B) = 0 = \mu(B \setminus A)$ so that $\mu(A \setminus B) + \mu(B \setminus A) = \mu(A \triangle B) = 0$. This must means that $\widetilde{A} = \widetilde{B}$. Lastly, suppose that $\widetilde{A} \sqsubseteq \widetilde{B}$ and $\widetilde{B} \sqsubseteq \widetilde{C}$, then we see that

$$\begin{aligned} \mu(A \setminus B) + \mu(B \setminus C) &= \mu((A \setminus C) \cup (A \cap C \setminus B) \cup (B \setminus A \cup C)) \\ &\geq \mu(A \setminus C) \end{aligned}$$

Its clear that we now find $\mu(A \setminus C) = 0$ so that $\widetilde{A} \sqsubseteq \widetilde{C}$. ✐

Indeed, this shows that $\widetilde{\mathcal{A}}$ is a partially ordered set. If we equip this partially ordered set with the operators $\wedge$ and $\vee$ defined as follows:

$$\widetilde{A} \wedge \widetilde{B} := \widetilde{A \cap B}, \quad \text{and} \quad \widetilde{A} \vee \widetilde{B} := \widetilde{A \cup B},$$

we find that, $(\widetilde{\mathcal{A}}, \wedge, \vee)$ is a distributive lattice. We can apply theorem 7 to see that metric and lattice betweenness coincide. Or alternatively, we can see the two coincide by the following computation.

Theorem 16. *Suppose that $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{A}}$, then we can identify the interval between $\widetilde{A}$ and $\widetilde{B}$ by: $[\widetilde{A}, \widetilde{B}] = \{\widetilde{X} \in \widetilde{\mathcal{A}} : \widetilde{A} \wedge \widetilde{B} \sqsubseteq \widetilde{X} \sqsubseteq \widetilde{A} \vee \widetilde{B}\}$.*

Proof. Take $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{A}}$, and suppose $\widetilde{X} \in [\widetilde{A}, \widetilde{B}]$, then by assumption

$$\begin{aligned}\mu(A \triangle B) &= \mu(A \triangle X \triangle X \triangle B),\\ &= \mu(A \triangle X \setminus X \triangle B) + \mu(X \triangle B \setminus A \triangle X),\\ &= \mu(A \triangle X) + \mu(X \triangle B).\end{aligned}$$

Here the last equality follows directly from $\widetilde{X} \in [\widetilde{A}, \widetilde{B}]$. As a consequence;

$$\begin{aligned}\mu(A \triangle X \setminus X \triangle B) &= \mu(A \triangle X),\\ \mu(X \triangle B \setminus A \triangle X) &= \mu(X \triangle B).\end{aligned}$$

We may rewrite it the following way respectively.

$$\begin{aligned}\mu(A \cap B^{\complement} \cap X^{\complement}) + \mu(A^{\complement} \cap B \cap X) &= \mu(A \cap X^{\complement}) + \mu(A^{\complement} \cap X),\\ \mu(B \cap A^{\complement} \cap X^{\complement}) + \mu(B^{\complement} \cap A \cap X) &= \mu(B \cap X^{\complement}) + \mu(B^{\complement} \cap X).\end{aligned}$$

Which tells us that:

$$\begin{aligned}\mu(A \cap B \cap X^{\complement}) &= \mu(A \cap B \setminus X) = 0,\\ \mu(A^{\complement} \cap B^{\complement} \cap X) &= \mu(X \setminus A \cup B) = 0.\end{aligned}$$

This means that $\widetilde{A \cap B} \sqsubseteq \widetilde{X} \sqsubseteq \widetilde{A \cup B}$. Which shows the first inclusion. Realizing that all steps we took are invertible provides the converse. ✐

Theorem 17. ***(Monotone Convergence Theorem)*** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and take $\widetilde{A}, \widetilde{B}, \widetilde{X}_n \in \widetilde{\mathcal{A}}$ for al $n \in \mathbb{N}$. Suppose that both*

$$\widetilde{A} \wedge \widetilde{B} \sqsubseteq \widetilde{X}_n \sqsubseteq \widetilde{A} \vee \widetilde{B},$$

and

$$\widetilde{A} \wedge \widetilde{X}_{n+1} \sqsubseteq \widetilde{X}_n \sqsubseteq \widetilde{A} \vee \widetilde{X}_{n+1},$$

hold for all $n \in \mathbb{N}$, then the sequence $(\widetilde{X}_n)$ converges.

Proof. This is an immediate consequence of Theorem 16 and Theorem 5. ✐

A consequence of this theorem is the familiar property that for a sequence of nested sets A_n, we have that $\mu(A_n)$ converges. Namely, suppose that $A_n \subseteq A_{n+1}$ for all n. Then its not hard to see we satisfy both

$$\widetilde{\emptyset} \wedge \widetilde{\cup_{i=1}^{\infty} A_i} \sqsubseteq \widetilde{A_n} \sqsubseteq \widetilde{\emptyset} \vee \widetilde{\cup_{i=1}^{\infty} A_i}$$

and

$$\widetilde{\emptyset} \wedge \widetilde{A}_{n+1} \sqsubseteq \widetilde{A}_n \sqsubseteq \widetilde{\emptyset} \vee \widetilde{A}_{n+1}$$

So by theorem 17, the sequence (A_n) converges in the semimetric d_μ to some set X. To see the limit of this sequence, one may reexamine the proof of the completeness of the d_μ metric to see that the sequence (A_n) converges to $\bigcup_{i=1}^{\infty} A_i$. So that from the reverse triangle inequality we have:

$$\left|\mu(A_n) - \mu\left(\bigcup_{i=1}^{\infty} A_n\right)\right| \leq \mu\left(A_n \triangle \bigcup_{i=1}^{\infty} A_i\right) \to 0$$

This means that the sequence of real numbers $(\mu(A_n))$ converges as well. Obviously, a similar result follows for decreasing sequences. This is a well known result that is frequently used in measure theory. However, notice that theorem 17 is more general than only convergence of nested sets.

For example, in the Borel measure space $(\mathbb{R}, \mathcal{B}, m)$ with Lebesgue measure, we can consider the following sets.

$$A = [0,1] \quad B = [1,2], \quad \text{and} \quad X_n = \left[\frac{n}{n+1}, 1 + \frac{n}{n+1}\right].$$

Then $A \cap B = \{1\}$ and $A \cup B = [0,2]$ so clearly the sequence X_n lies in $[A, B]$ as for each $n \in \mathbb{N}$ we have:

$$\{1\} \subseteq \left[\frac{n}{n+1}, 1 + \frac{n}{n+1}\right] \subseteq [0,2].$$

Furthermore, the sequence (X_n) is increasing because we satisfy:

$$A \cap X_{n+1} = \left[\frac{n+1}{n+2}, 1\right] \subseteq \left[\frac{n}{n+1}, 1 + \frac{n}{n+1}\right] \subseteq \left[0, 1 + \frac{n+1}{n+2}\right] = A \cup X_{n+1}.$$

So we know from theorem 17 that the sequence (X_n) converges. In particular, we see that $m(X_n \triangle B) \to 0$ as $n \to \infty$. Since:

$$\begin{aligned} m(X_n \triangle B) &= m\left(\left[\frac{n}{n+1}, 1 + \frac{n}{n+1}\right] \triangle [1,2]\right) \\ &= m\left(\left[\frac{n}{n+1}, 1\right) \cup \left(1 + \frac{n}{n+1}, 2\right]\right) \\ &= \frac{2}{n+1} \end{aligned}$$

Intuitively, we have taken some interval of length 1 and translated it from A towards B along the real line. This sequence (X_n) is an example of a convergent sequence of sets which are not nested.

Chapter 4

Suggestions for Further Research

When trying out the monotone convergence theorem on various metric spaces, it was observed that the shift map preserves metric betweenness. In particular, Let Σ be defined as follows:

$$\Sigma := \{(x_n) : x_n \in \{0, 1\} \text{ for all } n \in \mathbb{N}\}$$

That is, the collection of sequences of zeros and ones. For $x, y \in \Sigma$, define the metric d as follows:

$$d(x, y) := \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}$$

Then the metric space (Σ, d) is complete. Furthermore, the shiftmap σ defined by:

$$\sigma : \Sigma \to \Sigma : (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$$

is a chaotic map, as is discussed in [15]. To see why this map preserves metric betweenness we need the following theorem.

Theorem 18. *For $a, x, b \in \Sigma$, we have $(axb)_M$ if and only if $x_n \in \{a_n, b_n\}$ for all $n \in \mathbb{N}$.*

Proof. Suppose that $x_n \notin \{a_n, b_n\}$ for some index $n \in N$. Let n_0 be such an index. Then we see that $|a_{n_0} - x_{n_0}| + |x_{n_0} - b_{n_0}| - |a_{n_0} - b_{n_0}| = 2$. In particular we see

$$\sum_{n=0}^{\infty} \frac{|a_n - x_n| + |x_n - b_n| - |a_n - b_n|}{2^n} \geq \frac{2}{2^{n_0}} > 0.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{|a_n - x_n|}{2^n} + \sum_{n=0}^{\infty} \frac{|x_n - b_n|}{2^n} \geq \frac{1}{2^{n_0 - 1}} + \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n}.$$

We conclude $(axb)_M$ can not hold. The converse is trivial. ∎

Its clear that if $x_n \in \{a_n, b_n\}$ for all $n \in \mathbb{N}$, then we certainly have $x_n \in \{a_n, b_n\}$ for all $n \in \mathbb{N} \setminus \{0\}$. So then $(axb)_M$ implies $(\sigma(a)\sigma(x)\sigma(b))_M$. In other words, the shiftmap preserves betweenness relations. The fact that betweenness relations are preserved under a chaotic map hints that they might have applications in Dynamical Systems theory. To add to this suspicion, we point out that when studying chaotic systems, it is important to be able to accurately measure the current state of a system. After all, by definition of chaos, distinct points will lie further apart as time progresses. If one is able to distinguish between the points in space perfectly, then a Dynamical System is deterministic, however since measuring equipment can only measure up to certain significant digits, some degree of randomness is introduced. Indistinguishablity is studied in the book [16] by J. Recasens. The author points out that T-indistinguishability operators will turn into a metric under certain conditions. It would be interesting to investigate interactions of metrics, indistinguishability operators and chaos in the context of (Stochastic) Dynamical Systems. Partly because we've seen in section 3.3 that betweenness translates neatly into measure theory. Partly because metric betweenness is preserved by the shift map. And partly because indistinguishabilty operators can coincide with a metric.

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