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Students' Understanding of Exponents and Exponential Functions at a Secondary School in the Northern Netherlands

Bachelor's Project Mathematics

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Abstract

The purpose of this study is to describe and analyze students' understanding of exponents and exponential functions within different existing frameworks on students' understanding, and to compare these frameworks. The study was conducted with 17 secondary school students of a school in the northern Netherlands with the use of a survey and semi-structured interviews. The results of this study suggest that students in the first level of understanding can solve tasks with exponents with positive integer base and power, but that they have difficulties with exponents with negative base or power. Students in the second level can solve tasks with a negative integer base and positive integer power, but have difficulties with a rational base and negative rational power. In the third level of understanding, students can work with exponents with rational base and negative rational power. The study suggests that participants in this study performed better than participants in previous studies. Notable cases of student misconceptions were students' who moved the minus sign of the power to the base or preceding the base, students inability to explain why the rules of exponentiation are true and students who composed growth factors incorrectly when they were given two coordinates.

Contents

1 Introduction

Learning about the concepts of exponential and logarithmic functions is an important part of mathematics at secondary school, because of applications in decay and population growth. I always had difficulties myself with fully grasping logarithms at school, and many students have difficulties mastering exponentiation and logarithms even at postsecondary level (Confrey & Smith, 1995). A study by Weber (2002a) revealed that university students struggle with understanding the rules of exponentiation and connecting them with the rules of logarithms.

I am tutoring students at the age of 17 and 18 in *Wiskunde A* and *Wiskunde B* at secondary schools for two years now, and in my experience as a tutor, I see that my students are having problems with the understanding of exponential and logarithmic functions too. A second problem arises here: I notice that I have difficulties with teaching them the concepts in a satisfactory manner.

In this thesis, I explore students' understanding of exponents and exponential functions at a secondary school in the northern Netherlands in order to get insights into students' reasoning and difficulties.

I use qualitative methods to collect data:

- Conducting a survey with the students to get a wide picture of their understanding of exponents and the exponential function.
- Interviewing the students to gain in-depth insight in their understanding of exponents and the exponential function.

I use *Educational Research: Planning, Conducting, And Evaluating Quantitative And Qualitative Research* by Creswell (2008) to analyze the data in a systematic way. This analysis initially consists of transcribing the data, and then identifying text segments and assigning code labels to the segments to build description and themes about students' understanding of exponents and exponential functions.

Research on the teaching and learning of exponents and exponential functions can enable more effective teaching methods, provide better understanding of student learning and support curriculum design.

2 Literature Review

I review the research literature on the structure of exponential functions, on teaching approaches to this function and on the development of an understanding of this function. I found eight studies which follow one of three major perspectives on the development of an understanding of exponential functions: the covariation approach by Confrey and Smith (1995), and the approaches by Weber (2002a) and Pitta-Pantazi, Christou and Zachariades (2007). I use those three perspectives to analyze the student responses to the survey and the interviews in order to gain an insight in these students' understanding of exponents and the exponential function. Summaries of the seven (Confrey and Smith (1995) is excluded) studies can be found in Appendix B.

2.1 The Perspective of Confrey and Smith (1995)

2.1.1 Splitting

A familiar approach to multiplication and division is considering these operations as repeated addition or subtraction. For example, calculating the product 5×3 by means of repeated addition involves adding the number 3 five times with itself: $5 \times 3 = 3 + 3 + 3 + 3 + 3$.

Confrey and Smith (1995) explore a second approach, which they have labeled *splitting*. A splitting structure is the structure of the group $(\mathbb{R}_+, \cdot)$. In $\mathbb{R}_+$under the operation of multiplication, multiplication is the basic operation and one is the identity element. This is in contrast to a counting structure. A counting structure is the structure of the group $(\mathbb{R}, +)$. In $\mathbb{R}$ under addition, zero is the identity element.

An example of splitting is the sharing of discrete items, which can be done without counting the number of elements in each group to see if the groups are equally sized. When a child deals out a pile of candies (Fischbein, Deri, Nello, & Marino, 1985), the groups can be created by distributing items one by one to each group, and moreover, the equivalence of the group sizes can be confirmed by visually recognizing similarity.

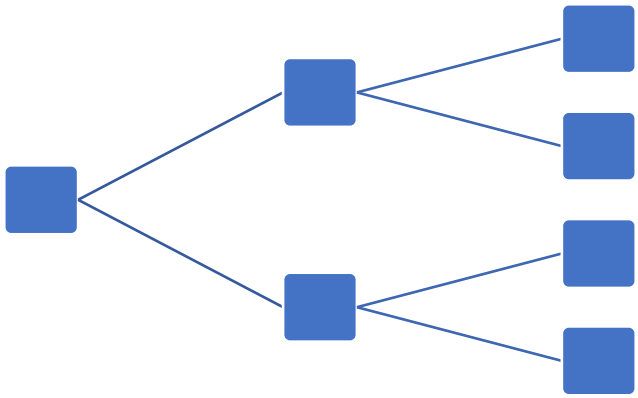


Figure 1. Example: Tree diagram of 2-splits (adapted from Confrey & Smith, 1995, p. 70)

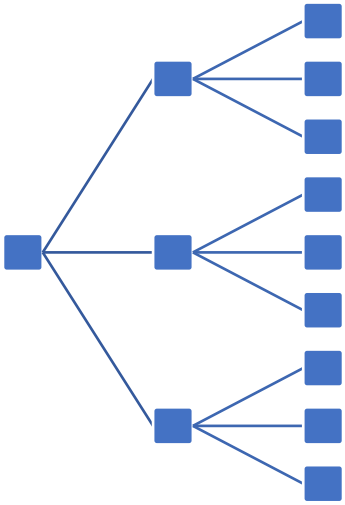


Figure 2. Example: Tree diagram of 3-splits (adapted from Confrey & Smith, 1995, p. 70)

Another example of a splitting structure is the tree diagram in Figure 1, where the movement in one direction is doubling and in the other direction is halving. The same for a geometric sequence, 16, 8, 4, 2, 1, 1/2, 1/4, 1/8, 1/16, etc.: one direction is doubling and the other direction is halving. We label these splits as 2-splits. In fact, any n-split is possible. An example of a structure consisting of 3-splits is given in Figure 2. In this figure, we end up with nine branches. Hence the overall structure is a 9-split, but it is composed of a 3-split followed by a 3-split. More generally, every n-split can be seen as the composition of prime splits.

A key process in splitting is 'reinitializing'. In Figure 1, after the first split, one can treat each product of the split as the basis for reapplication for the next split. We call this choosing of a new basis *reinitializing*.

In Figure 3, Confrey and Smith (1995) provide a list with parallel characteristics of splitting and counting structures (or worlds, in their terminology).

Characteristics of Counting and Splitting Worlds

Splitting	Counting
One is the origin	Zero is the origin
Splitting by n is the successor action	Adding one is the successor action
The unit (of growth) is n or n:1	The basic unit is one
Multiplication and division are basic operations	Addition and subtraction are the basic operations
One is the identity element	Zero is the identity element
Reinitializing to one	Reinitializing to zero
Commutativity applies to multiplication	Commutativity applies to addition
Ratio is used to describe the interval between two successive "whole" numbers	Difference is used to describe the interval between two successive "whole" numbers
Composite units are made by raising "splitting units" to a higher power	Composite units are formed by aggregating counts into larger groups
Parts (multiplicative) are created by n-rooting	Parts (additive) are created by n-splitting
Exponentiation is created as repeated multiplication	Multiplication is contructed as repeated addition
Distributivity is applied to exponentiation over multiplication	Distributivity is applied to multiplication over addition
Rate is the ratio per unit time	Rate is the difference per unit time

Figure 3. Parallel characteristics of counting and splitting worlds (Confrey & Smith, 1995, p.75)

Confrey and Smith argue that ratio is the central concept underlying the development of the splitting world. They use a primitive concept of ratio: they describe ratio as 'the invariance across situations'. They use the examples in Figure 4 to make this more clear:

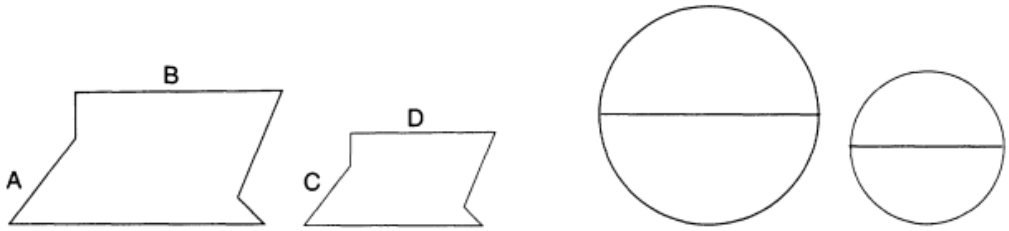


Figure 4. Ratio (Confrey & Smith, 1995, p.74)

Recognizing ratio is recognizing which is constant in the relationship between A and B compared to the relationship between C and D, or recognizing which is constant in the relationship between the circumference and the diameter of the large circle compared to the relationship between the circumference and the diameter of the small circle. They claim that this primitive notion of ratio based on visual similarity is an intuitive concept for children, and that the child's capability of doubling, halving, reinitializing, or in general repeat a splitting activity is the result of the capability to recognize ratio.

2.1.2 Covariation Approach to Exponential Functions

The *covariation approach to functions* (Confrey & Smith, 1991) has as its aim to make students build a mental image of a function. Confrey and Smith describe the covariation approach as "the *construction* of the domain as an ordered mathematical structure, the *construction* of the range, *and the identification of linked data points,* enough to allow one to complete the mapping" (Confrey & Smith, 1995, p. 79). In other words, the approach entails the student observing two data columns, recognizing patterns in each column individually and recognizing patterns between the values in the different columns.

This is another way to express the 'relationship' between two structured worlds than the *correspondence approach* which is more often used in curricula. The correspondence approach defines a function as a relation between domain and range, such that for any element in the domain there exists an element in the range. This approach is more abstract than the covariation approach and it usually states the relation as an algebraic rule (e.g. $f(x) = 5x + 3$). Another difference is that the covariation approach puts more emphasis on the examination of the operational structures of the range and domain.

Covariation can be associated to the construction of an isomorphism between $\mathbb{R}_+$under multiplication and $\mathbb{R}$ under addition. Like covariation, an isomorphism represents a relationship between two sets and their operational structure. A difference is that an isomorphism is a formal relationship, whereas covariation is more informal and more general.

The construction of an exponential function can be based on covariation between a counting and a splitting world (see Figure 3). In other words, an understanding of an exponential function can be built on recognizing the operational structures in the counting and splitting world and recognizing the relationship between these operational structures. This connection can be formally exceeded to an isomorphism between these two worlds. The basis for this isomorphism is provided by the operational equivalencies in Figure 3. Multiplication (repeated addition) in the counting world is linked to exponentiation (repeated multiplication) in the splitting world. The splitting unit then becomes the base of the exponential function. Likewise, a fractional part of the form $1/n$ of an element in the counting world is equivalent to n-rooting in the splitting world. The equivalence of fractional parts and rooting can be extended to all real numbers.

One implication of approaching exponential functions as an isomorphism between counting and splitting worlds is that the rules for logarithms can be understood in terms of

equivalencies between operational structures in both worlds; for example, we can see $\log(ab) = \log(a) + \log(b)$ as the equivalence of multiplication in the splitting world and addition in the counting world. Another implication is that the base of the exponential function, the splitting unit, can be understood more intuitively when seen as a constant ratio of a splitting world. For example, radioactive decay problems are often modelled with the use of the base e (Confrey & Smith 1995), whereas the base ½ is more intuitive since we are dealing with the half-life of a substance. Confrey and Smith make the conclusion that the exploration of counting and splitting worlds and the exploration of the isomorphism between the operational structures provides a foundation for an effective introduction of the exponential function.

2.1.3 Effectiveness of the Covariation Approach

Ellis, Özgür, Kulow, Dogan and Amidon (2016) and Ferrari-Escolá, Martínez-Sierra and Méndez-Guevara (2016) follow Confrey and Smith (1995) in their studies. Ellis et al. (2016) and Ferrari-Escolá et al. (2016) designed teaching sessions based on the covariation approach, and they described students' evolving understanding of exponential functions in steps. The research of Ellis et al. (2016) suggests that exploring exponential growth by coordinating covarying quantities promotes building an understanding of exponential functions, because coordinating covarying quantities can support students' ability to coordinate multiplicative growth in y with additive growth in x, which is important for understanding the nature of exponential growth. Moreover, the covariation approach played a significant role in the development of students' ability to express exponential relationships algebraically; in the study, the researchers observed that students who attended the sessions built a covariation view and a correspondence view (see p. 9 for covariation and correspondence), and that the gradual development of one view influenced the development of the other and vice versa. The other attempt to test the covariation approach for exponential functions was made by Ferrari-Escolá et al. (2016). Their research gave evidence that their teaching sessions encouraged students to develop covariational reasoning with an arithmetic and a geometric sequence. (See Appendix B for summaries of those research studies.)

2.1.4 What Can Contribute to Covariational Reasoning?

Ellis, Özgür, Kulow, Williams and Amidon (2015) identified three important factors that contributed to shifts in covariational reasoning of students aged 13-14: (1) successively removing the students' ability to calculate. This challenged students to mentally

coordinate the relationship between x- and y-values; (2) repeatedly expressing students' to the same mathematical activity; (3) inviting students to compare their actions with the actions they did in the other tasks to identify common elements.

Ellis et al. (2016) concluded that the exploration of continuous covariation was supported by their task involving a plant in a computer algebra system (in their case, GeoGebra) that grows exponentially, because it enables the student to manipulate the x- and y-values and to visualize the growth. Ferrari-Escolá et al. (2016) created a card game based on Stiffer's *Arithmetica integra* (1544) that encouraged students to develop discrete covariational reasoning. This was achieved by making the cards constitute an arithmetic and a geometric sequence (Ferrari-Escolá et al., 2016, p. 98). (See Appendix B for summaries of those studies.)

2.2 Weber's Conceptual Framework

A teaching approach other than the covariation approach of Confrey and Smith (1995) is provided by Weber (2002b). In order to design instructions for the students, Weber (2002a) proposed a set of four successive levels that students might make while learning exponential functions, based on Dubinsky's APOS theory (Dubinsky, 1991). These four levels (from basic to more advanced) were:

(1) An *action understanding* of exponents (computing b^x involves repeatedly multiplying by b x times).
(2) A *process understanding* of exponents. A process understanding arises when a student repeats an action and reflects upon it. According to Weber (2002a), a student with a process understanding can imagine the result of exponentiation without actually performing the computation. He or she can reason about properties of the exponential function and can reverse exponentiation to obtain logarithms; the domain is restricted to the natural numbers.
(3) *Exponential expressions as the result of a process.* At this level, an exponential expression represents the output of exponentiation. The number 2^3 can be perceived in two ways: it can be perceived as an external prompt to multiply the number two three times with itself, or it can be perceived as a mathematical object, representing the output of exponentiation.
(4) A *generalized understanding* of exponents. In this level, the student can interpret situations where the input of an exponential function is not restricted to the natural numbers.

Weber (2002b) noticed that an action understanding (the first level) of exponents overlaps with a procedural understanding (Sfard, 1991) of exponents; for example, viewing b^x as the procedure to multiply b by itself x times is understanding b^x as an algorithm. Moreover, Weber (2002b) noticed that understanding exponential expressions as the result of exponentiation (third level) overlaps with a structural understanding (Sfard, 1991) of exponents; namely, a student with a structural understanding treats a mathematical expression as an object in its own right (Sfard, 1991).

In Weber (2002b), the instruction students received was designed to promote a process understanding first by letting students write a computer program that performed exponents as repeated multiplication. Tall and Dubinsky (1991) suggest the use of a computer program to interiorize the steps of an operation into a process. Second, Weber (2002b) designed pen-and-paper exercises where students were encouraged to describe exponents and logarithms as mathematical objects, in the words of the authors: "to understand b^x as the number that is the product of x factors of b and $\log_b m$ as the number of factors of b that are in the number m" (Weber, 2002b, p. 1022)). This should promote a generalized understanding (Weber's fourth level), because the domain can then be extended to the rational numbers; for example, a student can than interpret $2^{1/2}$ as 'one half factor of 2'. Students who received the instruction designed by the researchers were better at performing computations, and they were better at recalling rules and explaining why they were true (Weber 2002b). Moreover, they could reconstruct rules by making use of a deep understanding of exponents and logarithms.

2.3 The Framework of Pitta-Pantazi, Christou and Zachariades

Instead of the four successive levels of understanding proposed by Weber (2002a), the research of Pitta-Pantazi et al. (2007) suggested three successive levels of understanding when it comes to exponents. Pitta-Pantazi et al. (2007) investigated the knowledge of exponential expressions of 202 high school students in Cyprus. The students were asked to sit a test, and interviews with students were taken, from which three levels of understanding emerged. Pitta-Pantazi et al. (2007) agree that Weber's (2002a) first level (action understanding) is identical with Sfard's (1991) procedural understanding, and they take this level of understanding as their first level: the preconceptual level.

The second level by Pitta-Pantazi et al. (2007), which is more advanced than the first level, is the conceptual level. According to Pitta-Pantazi et al. (2007), this level partially

overlaps with Weber's (2002a) level 2 (process understanding). A difference between the conceptual level as described by Pitta-Pantazi et al. (2007) and the process level as described by Weber (2002a) is that a student in the latter level can only compute exponents with integer powers, whereas a student ranked in the former level is capable of computing exponents with negative powers as well. According to Weber (2002a), students who understand exponents with negative integers as domain have a generalized understanding.

The third level that Pitta-Pantazi et al. (2007) identified is the restructured level. Students who are in this level view exponents as mathematical objects. More specifically, they understand and can work with exponents with rational powers and generalize this to real powers. Pitta-Pantazi et al. (2007) asserted that understanding exponents with rational powers requires a structural understanding (Sfard, 1991).

2.4 Problems With Teaching and Learning of Exponents

The data that the tests and the interviews provided in the research of Pitta-Pantazi et al. (2007) suggest that students that were ranked in the first level (preconceptual level) relied too much on prototype examples (central examples provided by teacher and textbook) where exponents are viewed as repeated multiplication, with base and power being integers. These examples do not help students with exercises where base or power are real numbers.

Other obstacles were identified by Cangelosi, Madrid, Cooper, Olson and Hartter (2013). Cangelosi et al. (2013) asked 904 university students, enrolled for calculus and algebra courses in the USA, to sit a test on exponents. From the tests, it was found that there are persistent errors among the students participating in the research. These errors frequently involved confusion about a minus sign as part of or preceding a base, and about a minus sign in the exponent. Cangelosi et al. (2013) argue that an underdeveloped understanding of additive and multiplicative inverses underlies these mistakes. Teachers and textbooks should make more use of the terms inverse and identity, and teachers should help students to recognize a connection between the algebraic structures of additive and multiplicative inverses.

2.5 Counting Versus Splitting Structures and the Restructured Level

As stated earlier, in the covariation approach to exponential functions (Confrey & Smith, 1995), an isomorphism is constructed between a counting and a splitting world ($\mathbb{R}$ under addition and $\mathbb{R}_+$under multiplication). One of the relationships between the operational structures of the two worlds is that a fractional part of the form $1/n$ of an element in the counting world is equivalent to n-rooting in the splitting world (see Figure 3, p. 8). For example, in a counting world with basic unit one that covaries with a splitting world with unit of growth nine, taking a half of one in the counting world is equivalent to taking the square root of nine in the splitting world, since a half *plus* a half equals one and a square root of nine *times* a square root of nine equals nine (see Figure 5). Hence, a student who is aware of the isomorphism between the two worlds, should relate taking $1/n$ of an element of the domain of an exponential function with taking the nth-root of the corresponding element in the range of the exponential function. According to Pitta-Pantazi et al. (2007), a student who understands exponents with rational powers is likely to be in the restructured level. Therefore, I hypothesize in my study that a student who sees the relationship between the counting structure of the domain of the exponential function and the splitting structure of the range of the exponential function to the extent of additive and multiplicative parts, will be in the restructured level. I use my data for a preliminary test of this hypothesis (see section 5.3, p. 31).

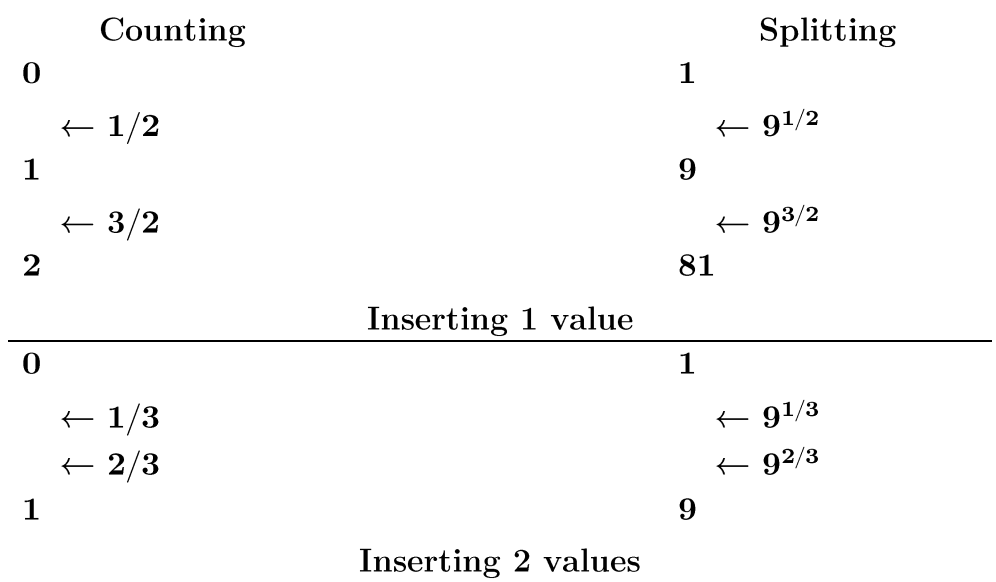


Figure 5. Interpolation in counting and splitting structures (adapted from Confrey & Smith, 1995, p. 82)

2.6 Concluding Statement of the Literature Review

The two perspectives on the development of an understanding of exponential functions that are found in the literature are the covariation approach [where the construction of exponential functions is based on an isomorphism of splitting (multiplicative) and counting (additive) structures] and the approaches by Weber (2002a) and Pitta-Pantazi et al. (2007). Ellis et al. (2016) and Ferrari-Escolá et al. (2016) showed that the covariation approach can be effectively taught. The majority of research on students' understanding and difficulties has been done in Cyprus (Pitta-Pantazi et al., 2007) and the USA (Cangelosi et al., 2013).

I am given the opportunity to collect data from secondary school students who have already been introduced to exponents and exponential functions. That gives a good opportunity to analyze students' understanding of this concept within the levels of understanding by Pitta-Pantazi et al. (2007) and Weber (2002a). The framework of Pitta-Pantazi et al. (2007) will serve as the framework of reference, because Weber's (2002a) study was still ongoing when published in conference proceedings. Furthermore, students' understanding of exponential functions as a covariation of splitting and counting worlds (Confrey & Smith, 1995) might be related to a level of understanding within the framework of Pitta-Pantazi et al. (2007). Table 1 presents my hypothesis of the levels of understanding exponents and exponential functions based on Confrey & Smith (1995), Weber (2002a) and Pitta-Pantazi et al. (2007). Following the table is an explanation of Table 1.

Table 1. Hypothesized levels of understanding exponents and exponential functions

Pitta-Pantazi et al. (2007)	**Weber (2002a)**	**Confrey & Smith (1995)**
Preconceptual level	Action understanding	
Conceptual level	Process understanding & exponential expression as the result of a process	
Restructured level	Generalized understanding	Understanding the relationship between splitting and counting structures

Pitta-Pantazi et al. (2007) argue that Weber's (2002a) action understanding matches their preconceptual level. I want to test with my research whether the conceptual level (Pitta-Pantazi et al., 2007), the process understanding and exponential expressions as the result of process (Weber, 2002a) match to a second level of understanding. I hypothesize that the restructured level overlaps with the generalized understanding (fourth level) of Weber (2002a), because in the fourth level of Weber (2002a), students can understand exponents with rational or real powers, just as in the restructured level. Moreover, in the previous paragraph I built a hypothesis that understanding the relationship between splitting and counting structures might match the restructured level.

After conducting a careful literature search (see Appendix A for the procedure of my literature search), I found no research on:

- Students' understanding of and difficulties with exponents and exponential functions at secondary schools in the Netherlands.
- The relation between students' understanding of exponential functions as a splitting structure (Confrey & Smith (1995)) and their level of understanding within the framework of Pitta-Pantazi et al. (2007).

3 Purpose and Research Questions

The purpose of this study is to analyze the present situation of the understanding of exponents and exponential functions of some students at a secondary school in the northern Netherlands. I explore their understanding within the framework of Pitta-Pantazi et al. (2007) (section 5.1, p. 23) and I compare the resulting levels with the levels proposed by Weber (2002a) (section 5.2, p. 28). Moreover, I investigate students' understanding of splitting and the isomorphism between additive and multiplicative structures in the context of exponential functions, and whether a student who sees this latter relationship may be in the restructured level of Pitta-Pantazi et al. (2007) (section 5.3, p. 31). Also, I investigate students' familiarity with the definition of the exponential function (section 5.4, p. 34). So the research question in this study is:

What is students' understanding of exponents and exponential functions at a secondary school in the northern Netherlands?

This question will be answered by reading the literature, by creating and grading the survey to get a wide picture of the levels understanding in this class, and by conducting interviews with students to gain more insight into each level.

4 Methodology

4.1 Participants

The participants of this study were 17 students 16 - 18 years of age of a secondary school in the northern Netherlands. All of the students attended the same mathematics course: Wiskunde B of the fifth year of the *voortgezet wetenschappelijk onderwijs* (vwo). The exponential function was introduced earlier in the curriculum.

4.2 Data Collection

The data was collected with the use of a survey for 17 students and interviews with 5 of the 17 students.

When collecting qualitative data, Creswell (2008) stresses the importance of obtaining permission to study the participants, of disclosing the purpose of the research to the participants, of protecting the anonymity of the participants and of disrupting the individuals at the research site as little as possible. Therefore, I asked the teacher for permission to observe the lessons, I asked students for permission to be interviewed, and I included consent forms to the survey and interviews (Appendices D and E). In the consent forms, I disclosed the purpose of the research and I guaranteed confidentiality. I protected the anonymity of the students by assigning numbers to them.

4.2.1 Surveys

The surveys were administered during class in a session of 20 minutes and they consisted of five questions (see Figure 6 for the survey questions). The students were not allowed to use a calculator. The questions given to students were translated from English into Dutch.

Q1. What does the function $f(x) = a^x$ mean to you?

Q2. Is 5^{17} an even number or an odd number? Why?

Q3. $b^x b^y$ can be simplified to what? Why?

Q4. A hacker creates a virus that destroys files of the computer memory it infects (repeating itself and thereby increasing the memory space which is occupied) and stops to exist when everything is destroyed. The owner of the computer that was infected by the virus observes that

- At 2:15 p.m., 1MB of memory is damaged
- At 3:15 p.m., 9MB of memory are damaged

(i) If the owner went to the supermarket and opened the computer one hour later, i.e., at 4:15 p.m., how many MB of memory would be lost? Justify your answer.

(ii) If the owner found the virus 30 minutes earlier than 3:15 p.m., that is, at 2:45 p.m., how many files would be destroyed at that time? Justify your answer.

Q5. Compare the quantities below using the symbols $<$, $=$ or $>$ and justify your answers:

(i) 23^8 , . . . , 23^{13}

(ii) 24^9 , . . . , 15^9

(iii) $(-12)^{13}$, . . . , $(-12)^{17}$

(iv) 0.5^{-15} , . . . , 0.6^{-15}

(v) $17^{(-3/5)}$, . . . , $15^{(-3/5)}$

Figure 6. Survey questions

Table 2. Source and level of the survey questions

Question	**Source**	**Level of understanding: (see Table 1, p. 15)**
Q1	Weber (2002b)	(Familiarity with exponential function)
Q2	Weber (2002b)	Conceptual
Q3	Weber (2002b)	Preconceptual if explained by referring to rule, conceptual if explained why

Q4	Adapted from Confrey and Smith (1995)	Restructured
Q5(i)	Pitta-Pantazi et al. (2007)	Preconceptual
Q5(ii)	Pitta-Pantazi et al. (2007)	Preconceptual
Q5(iii)	Pitta-Pantazi et al. (2007)	Conceptual
Q5(iv)	Pitta-Pantazi et al. (2007)	Restructured
Q5(v)	Pitta-Pantazi et al. (2007)	Restructured

The first three questions were taken from Weber (2002b). The research purpose for the first question was to explore students' familiarity with the exponential function. The second and third question were meant to investigate students' understanding within the framework by Weber (2002b). According to Weber (2002b), correctly answering the second question requires a process understanding of exponents, since the student is forced to reason what will happen without actually performing the computation. For the third question, explaining why $b^x b^y = b^{x+y}$ requires a student to understand exponentiation as the result of a process (Weber, 2002a) by expanding $b^x b^y$ to $\underbrace{b \cdots b}_{x\ times} \cdot \underbrace{b \cdots b}_{y\ times}$.

The fourth question is an adapted version of the virus problem mentioned in Confrey and Smith (1995, p. 69). My first supervisor and I reformulated it such that less time was needed to compute the answer, and such that it was adjusted to an international audience. In the exercise, the student is asked to extrapolate and interpolate values in either a counting world (when the student interprets that the virus grows linearly) or a splitting world (when the student interprets that the virus grows exponentially). Both ways are possible. In the case of exponential growth, the domain of the resulting exponential function is the time in hours, and the range is the amount of megabytes. If we take 2:15 p.m. as $t = 0$, then time t has a counting structure, and the amount of megabytes has a splitting structure (see Table 3). The time has zero as its origin and adding one is the successor action. The amount of megabytes has one as its origin and splitting by nine is the successor action. Halving the time is equivalent to square rooting the amount of megabytes.

Table 3. Counting and splitting structure Question 4

t (in hours)	MB
0	$9^0 = 1$
1/2	$9^{1/2} = 3$
1	$9^1 = 9$
2	$9^2 = 81$

I hypothesized that if a student relates halving in a counting world with square rooting in a splitting world, he or she should be in the restructured level (section 2.5, p. 14). In the case of Question 4, if a student relates halving the time with square rooting the amount of megabytes, he or she should be in the restructured level. If a student gives an incorrect solution, or if a student uses that the virus grows linearly, he or she is ranked in the preconceptual/conceptual level.

The fifth question is a selection of five of the twenty tasks created by Pitta-Pantazi et al. (2007) (Figure 7).

The performance of students in each group

Compare the exponents using the symbols <, = or >	Group 1 students' performance (%)	Group 2 students' performance (%)	Group 3 students' performance (%)
1. $23^8, \ldots, 23^{13}$	97.6	97.4	99.2
2. $(-12)^{13}, \ldots, (-12)^{17}$	33.3	75.4	97.4
3. $0.5^{25}, \ldots, 0.5^{31}$	14.3	49.3	62.1
4. $(-0.3)^{35}, \ldots, (-0.3)^{43}$	23.0	54.8	62.8
5. $23^{-8}, \ldots, 23^{-13}$	31.0	83.6	94.7
6. $(-12)^{-7}, \ldots, (-12)^{-9}$	39.3	55.3	66.7
7. $0.5^{-14}, \ldots, 0.5^{-23}$	40.3	57.1	61.2
8. $(-0.3)^{-15}, \ldots, (-0.3)^{-21}$	16.7	54.9	68.4
9. $24^9, \ldots, 15^9$	90.5	99.2	100
10. $(-14)^{19}, \ldots, (-25)^{19}$	28.6	75.4	94.7
11. $-17^{-9}, \ldots, 15^{-19}$	11.9	65.6	92.1
12. $(-12)^{-13}, \ldots, (-14)^{-13}$	39.3	52.6	69.0
13. $0.7^8, \ldots, 0.4^8$	61.9	67.2	84.2
14. $0.5^{-15}, \ldots, 0.6^{-15}$	26.2	42.6	94.7
15. $(0.3)^{19}, \ldots, (-0.5)^{19}$	31.0	62.3	84.2
16. $(-0.8)^{-17}, \ldots, (-0.7)^{-17}$	36.9	50.0	64.3
17. $24^{3/5}, \ldots, 15^{3/5}$	81.0	86.1	94.7
18. $17^{(-3/5)}, \ldots, 15^{(-3/5)}$	21.4	59.8	84.2
19. $0.7^{5/7}, \ldots, 0.4^{5/7}$	66.7	62.3	71.1
20. $0.3^{(-3/5)}, \ldots, 0.5^{(-3/5)}$	21.4	47.5	81.6

Figure 7. The twenty tasks and students' performance (Pitta-Pantazi et al., 2007, p. 305)

Pitta-Pantazi et al. (2007) recognized a hierarchy of tasks that were solved by more than 60% of the students (see Figure 8). Three groups can be identified. The tasks in the first row are successfully answered by more than 60% of students in all groups, the tasks in the

second row are successfully answered by more than 60% of students in the second and third group and the tasks in the third row are successfully answered by more than 60% of students in the third group only. For Pitta-Pantazi et al. (2007) Group 1 corresponds to the preconceptual level, Group 2 to the conceptual level and Group 3 to the restructured level.

Tasks solved by more than 60% of students in each class

Secondary school students		
Group 1	Group 2	Group 3
1, 9, 13, 17, 19	1, 9, 13, 17, 19 2, 5, 10, 11, 15	1, 9, 13, 17, 19 2, 5, 10, 11, 15 3, 4, 6, 7, 8, 12,14, 16, 18, 20

Figure 8. Hierarchy of tasks (Pitta-Pantazi et al., 2007, p. 306)

In discussion with my first supervisor, I picked five of the tasks of Figure 8 in order to investigate students' understanding within the three levels of Pitta-Pantazi et al. (2007). I did this by choosing:

- Two tasks (Task 1 and Task 9 of Figure 7) that were answered correctly by the vast majority of the students. Participants who answer these questions correctly should be at least in the preconceptual level.
- Task 2 of Figure 7 that was answered correctly by a low percentage of students in the first group and high percentages in the second and third group. Participants who answer this question correctly should be at least in the conceptual level.
- Two tasks (Task 14 and Task 18 of Figure 7) that were answered correctly by low percentages of the first and second group and a high percentage of the third group. Participants who answer these questions correctly should be in the restructured level.

4.2.2 Interviews

Based on the results of the surveys, I interviewed five students. I asked 7 students to participate in a interview about their train of thought in the survey. These students were picked in such a way that I would be interviewing students in a full spectrum of my interpretation of student levels of understanding of exponents and exponentiatial functions. This interpretation of student levels was based on the survey scores. Since two of the selected students appeared to be ill on the interview day, I interviewed only one student I ranked in the preconceptual level and only one student I ranked in the

restructured level. In particular, I interviewed Student 6, Student, 10, Student 11, Student 13 and Student 14. Based on their answers to the survey, I interpreted that Student 10 was at the preconceptual level, that Student 6, Student 13 and Student 14 were at the conceptual level and that Student 11 was at the restructured level.

The semi-structured interviews were in Dutch and took place at the students' school a week after the surveys so the students remembered how they were thinking while they were responding to the survey questions. The interviews were audiorecorded and lasted for approximately 10 minutes each. At the beginning of the interviews, the students were assured that their data would be kept confidential (Appendix E). The interview protocols can be found in Appendix C. The interview questions can be divided in two parts:

- First students were asked about the representation they used (symbolic/graphical) when they were thinking about the definition of the exponential function for their response to Question 1 of the survey.
- Then students were asked to explain their thinking behind specific questions of the survey. These were questions they solved in a different way than the majority of their fellow students, or these were questions where their reasoning in their survey responses was unclear. Students who correctly solved tasks about exponents with a minus sign in the power in the survey, were asked during the interview to give in-depth explanation about exponents with a minus sign in the power, in order for me to compare their explanations with the explanations that participants of Pitta-Pantazi et al. (2007) gave to similar interview questions. For example:

Interviewer:	How do you know that 23^{-8} is equivalent to $1/23^8$?
Anna*:	It's a rule, but . . . I can show you why. You can write, for example, exponents with base 2 in the following order, and you will get the equivalence: 2^2 is 4, $2^1 = 2$, $2^0 = 1$, $2^{-1} = 1/2$, because you divide each time by 2. Thus, $2^{-2} = 1/4$ or $1/2^2$. [Pitta-Pantazi et al., 2007, p. 307]

* Anna is a pseudonym for anonimity reasons

4.3 Data Analysis

Students' responses to survey Question 1 through 5 were analyzed. This was done by coding students' responses for each question. This resulted in 17 codes for Question 1 (for example, 'student draws graph' or 'student mentions a is growth factor') and approximately 5 codes per question for Question 2 through 5 (for Question 2 for example, the codes were 'student mentions repetition', 'student mentions odd exponent' etc.). Then, a grading scheme for the codes of Question 2 through 5 was set up; in other words, for each coded answer, it was being determined if the students' response provided evidence for a level of understanding of exponents and exponential functions (see Table 1, p. 15). I found that some responses did not provide enough evidence for a level of understanding. For example, when a student writes down the right symbol ($<$, $=$ or $>$) for Question 5 without providing any explanation, it cannot be said with certainty whether the student guessed or not. A table with individual survey results of the participants (anonymized) can be found in Appendix F.

The interviews were transcribed and translated into English, and provided an in-depth understanding of how the students were thinking when answering the survey. This provided examples of how students on different levels of understanding exponents and exponential functions think while working on those areas.

5 Results

5.1 Students' Levels of Understanding (Pitta-Pantazi et al., 2007)

As stated earlier, in this thesis I use the levels described by Pitta-Pantazi et al. (2007) as the point of reference. Based on Question 5 of the surveys, the participants were ranked in one of the three levels of Pitta-Pantazi et al. (2007) (Table 4). There were three cases in which the explanations of the participants did not provide enough evidence to put them in one particular level. In those cases they were placed in the category 'preconceptual or conceptual' or 'conceptual or restructured'.

Table 4. Participants' level of understanding

Level	**Number of students**
Preconceptual	2
Preconceptual or conceptual	2
Conceptual	5
Conceptual or restructured	1
Restructured	7

5.1.1 Reasoning of Students in the Preconceptual Level

Based on Question 5, two students were placed in the preconceptual level. The main characteristic of the students that were ranked in the preconceptual level in this study, is that they adequately reasoned for exponents with a positive integer base and power, but that they inadequately reasoned for exponents with a negative base or power. A few notable ways of reasoning stood out. One of the students explained why $(-12)^{13}$ was greater than $(-12)^{17}$ by writing down in the survey:

Student 7: $(-p)^a = -(p)^a$ [Survey]

So this student was confused about a minus sign as part of the base and preceding a base, which is one of the frequently made mistakes that Cangelosi et al. (2013) recognized. The equality Student 7 mentioned is true when the power is odd, but Student 7 stated the equality as a general rule.

Both students in the preconceptual level had difficulties with negative exponents: for Question 5(iv) and 5(v), Student 7 mentioned that he could not explain his answer. When Student 10 was asked to compare $17^{(-3/5)}$ and $15^{(-3/5)}$, she indicated that she did not know the logic behind the minus sign in the exponent, as shown in the following interview extract:

Interviewer: For Question 5(v), what was your train of thought there?
Student 10: Uhm, well, I thought, because they had the same exponent, and then 15 was a smaller base, so uhm, because it is negative, the base becomes larger, with a larger base the result was smaller, because...
Interviewer: How do you know that?
Student 10: Because it was a negative exponent; that's why.
Interviewer: Okay.

Student 10: Because yes, that is taught to us, but I don't know the logic behind it. [Interview]

The student did not seem to know about the rule $b^{-x} = 1/b^x$.

When Student 10 mentioned that a function of the form $f(x) = a^x$ is always convex increasing, she was asked in the interview to explain her statement, and she answered:

Student 10: Because it is positive, well if a is positive, and then because x… no matter how small x becomes it does not go below zero, so it always increases. [Interview]

She did not seem to take into account exponential functions with a base smaller than one, this might indicate that her overview of the function was quite limited.

My findings confirm Pitta-Pantazi et al.'s (2007) description of the preconceptual level, namely that students in this level can compute a^x, if a and x are positive integers.

5.1.2 Reasoning of Students in the Conceptual Level

The main characteristic of the students that were ranked in the conceptual level in this study, is that they adequately reasoned for exponents with a negative integer base and a positive integer power (in contrast to students in the preconceptual level), but that they inadequately reasoned for exponents with a rational base and a negative rational power. The next interview excerpt reflects students' adequate reasoning for exponents with a negative base. The student explained why $(-12)^{13}$ is greater than $(-12)^{17}$:

Student 13: (…) they are both negative because the exponent is odd, so…
Interviewer: And why do they become negative if you have an odd exponent?
Student 13: Because you have minus times minus times minus, so if you have an even number, you can turn the minus into a plus. [Interview]

The response of Student 13 shows that she interpreted the minus sign as being part of the base.

It stood out that multiple students in the conceptual level confused a minus sign in the power with a minus sign preceeding the exponent, even though they seemed not to have difficulties with exponents with a negative base since they all solved Question 5(iii). In the survey and interviews, students explained why 0.5^{-15} or 0.6^{-15} is larger (Question 5(iv)), for example:

Student 8:	Negative number because of odd exponent. Same power but smaller base, so less negative. [Survey]
Student 14:	It is a negative power, and there are no brackets around it, so then it becomes… a negative number? [Interview]
Student 15:	Base is larger, so eventually it will be more negative. [Survey]

Student 8 seemed to interpret the minus sign as being part of the base, because he mentioned the odd power (a negative base raised to an odd power becomes a negative number). Student 14 seemed to interpret the minus sign as preceding the exponent, because she did not detect any brackets, but her response is too vague for me to be certain.

Students' reasoning when explaining why $17^{(-3/5)}$ or $15^{(-3/5)}$ is larger (Question 5(v)) was the following:

Student 6:	$-\sqrt[3]{17^5}$ $-\sqrt[3]{15^5}$ left is $\sqrt{}$ of a larger number, so it remains larger. [Survey]
Student 8:	Higher base so more negative. [Survey]
Student 14:	Becomes a minus number so a larger number in the minus is smaller than a larger number in the minus. [Survey]

Not only did the students have difficulties with negative powers, at least two students were confused because of the rational base. In the survey, one student's explanation why 0.6^{-15} is larger than 0.5^{-15} was the following:

Student 12:	$0 < g < 1$ so the graph is decreasing. 0.6 is larger, so I guess in this way, but I do not know. [Here g is interpreted as being the base.] [Survey]

Student 12 seemed to picture the graph of a single exponential function. Since he compared two different bases, it seems to me that he treated the base of the function as a variable; but in the case of an exponential function, the power should be the variable. He did not seem to take into account the negative power, which is crucial when comparing the two exponents.

In the interview, Student 13 explained why she "guessed" that 0.5^{-15} is larger than 0.6^{-15}:

Student 13:	…it was closer to guessing.
Interviewer:	So it was a guess here.
Student 13:	Yes
Interviewer:	And why did you guess in this way, and not otherwise?

Student 13:	Uhm…
Interviewer:	In other words, why did you choose for the smaller base, because, for example, here you chose for the larger base? [points to exercise 5(ii)]
Student 13:	Because it is… uhm… a fraction.
Interviewer:	Okay
Student 13:	And if you multiply fractions, you end up with a smaller number. No, if you multiply something with a fraction, you get a smaller number, that was my reasoning, but… [pause] Yes. [Interview]

Student 13 seemed to know that multiplying a number between zero and one with itself yields a smaller number, but she did not seem to take into account the negative power, which, again, is crucial when comparing the two exponents.

My findings do not confirm Pitta-Pantazi et al.'s (2007) description of the conceptual level. Pitta-Pantazi et al.'s (2007) description of the conceptual level reads: "Students compute a^x, if a is a rational number (positive or negative), and x is a positive or negative integer" (Pitta-Pantazi et al., 2007, p. 308). My findings suggest that students who are able to work with exponents with a negative base (since they correctly solved Question 5(iii)) are not necessarily able to work with exponents with negative power, as we saw in this section. Moreover, some students ranked in my conceptual level had difficulties with a rational base, which is also in conflict with Pitta-Pantazi et al.'s (2007) description of the conceptual level. Since these students exceeded the preconceptual level (since they correctly solved Question 5(iii) with a negative base), this could indicate that the second level might be expanded to include difficulties with rational base.

5.1.3 Reasoning of Students in the Restructured Level

The main characteristic of the students that were ranked in the restructured level in this study, is that they could solve tasks with exponents with a negative or positive rational base and a negative or positive rational power.

All students in this level knew how to correctly apply the rule $b^{-x} = 1/b^x$. For Question 5(v), some students showed that they knew the rule $b^{x/y} = \sqrt[y]{b^x}$ (even though this is not necessary for solving the task).

In the surveys, no participant gave an explanation in the way participants in Pitta-Pantazi et al. (2007) gave for the rule $b^{-x} = 1/b^x$. Participants in Pitta-Pantazi et al. (2007) gave explanantions like:

Interviewer:	How do you know that 23^{-8} is equivalent to $1/23^8$?
Anna:	It's a rule, but . . . I can show you why. You can write, for example, exponents with base 2 in the following order, and you will get the equivalence: 2^2 is 4, $2^1 = 2$, $2^0 = 1$, $2^{-1} = 1/2$, because you divide each time by 2. Thus, $2^{-2} = 1/4$ or $1/2^2$. [Pitta-Pantazi et al., 2007, p. 307]

Due to illness of two of the interview participants, only one student was asked to give in-depth explanation about exponents with negative or zero powers. When asked why $(1/2)^{-1}$ is equal to 2^1, Student 11 answered:

Student 11:	(…) I think a half to the power minus one is… uhm… one divided by a half to the power one, so that's the same as one divided by a half, isn't it?
Interviewer:	And why does this rule hold?
Student 11:	Ah… I actually have no idea, that is something we learned and I don't remember that it has ever been explained. [Interview]

And later on in the interview with the same student:

Interviewer:	And why is every number to the power zero equal to one?
Student 11:	Uhm, that is explained to us, but I cannot remember why. But I know that we discussed it, uhm… what was it… uhm… I don't think I know that one. [Interview]

It is not clear to me if and how the teacher gave an explanation for exponents with negative or zero powers other than referring to a rule.

5.2 The Levels of Pitta-Pantazi et al. (2007) Compared to Weber's Levels (2002a)

In Table 1 (p. 15), I presented my hypothesis of the connection between the levels of students' understanding of exponents and exponential functions by Pitta-Pantazi et al. (2007) and by Weber (2002a). I hypothezised that students, who are ranked in the preconceptual level by Pitta-Pantazi et al. (2007), are within an action understanding by Weber (2002a). Students ranked in the conceptual level by Pitta-Pantazi et al. (2007) are within a process understanding and see exponential expressions as the result of a process by Weber (2002a). Students ranked in the restructured level by Pitta-Pantazi et al. (2007) are within a generalized understanding by Weber (2002a).

Table 5 summarizes by student (first column) the connection I found between the levels of understanding of exponents and exponential functions by Pitta-Pantazi et al. (2007) and Weber (2002a). Column 2 presents students' levels by Pitta-Pantazi et al. (2007) based on Question 5 of the survey. Column 3 presents students' levels by Weber (2002a) based on Question 2 and 3 of the survey.

Table 5. Students' performance for Question 2,3 and 5

Student	**Level Pitta-Pantazi et al. (2007) for Q5**	**Level Weber (2002a) for Q2 and Q3**	**Hypothesis confirmed**
1	restructured	action	no
2	restructured	process or higher	yes
3	conceptual/restructured	process or higher	yes
4	restructured	process or higher	yes
5	preconceptual/conceptual	action	yes
6	conceptual	process or higher	yes
7	preconceptual	process or higher	no
8	conceptual	process or higher	yes
9	restructured	process or higher	yes
10	preconceptual	action/process	yes
11	restructured	result of a process or higher	yes
12	conceptual	action	no
13	conceptual	action	no
14	conceptual	process or higher	yes
15	preconceptual/conceptual	process or higher	yes
16	restructured	process or higher	yes
17	restructured	process or higher	yes

My reflection on Table 5:

- For Question 3 of the survey, 15 students correctly mentioned the rule $b^x b^y = b^{x+y}$, but only one student gave an explanation for the rule:

> Student 11: b is multiplied x times with itself and subsequently multiplied y times with itself. So, you could also multiply b $x + y$ times with itself at once. [Survey]

In the interview, Student 11 provided insight into this explanation with an example:

Student 11:	Yes because actually it reads b to the power x and then you multiply that with b to the power y, so then… let me see. Uhm… Yes for example, if x is 3 and y is 5, then firstly you do b times b times b, times [‘times’ emphasized] b times b times b times b times b, and then you might as well just multiply b three plus five is eight times with itself. [Interview]

The student correctly explained the rule by expanding $b^x b^y$ to $\underbrace{b \cdots b}_{x\ times} \cdot \underbrace{b \cdots b}_{y\ times} = \underbrace{b \cdots b}_{x+y\ times}$.

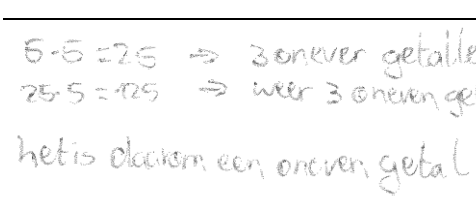
It is notable that Student 11 also performed very well for questions belonging to Pitta-Pantazi et al.’s (2007) conceptual and restructured level.

- There were four cases in which the result did not accord with the hypothesis. Student 1, Student 12 and Student 13 were ranked in the conceptual level or higher based on Question 5, but they had difficulties with solving Question 2 (Is 5^{17} an odd or an even number? Why?), a question that according to Weber (2002a) requires a process understanding. In the survey, Student 1 responded to the Question 2 in a way that is also representative for Student 12 and Student 13:

Student 1:	5^{17} has an odd base and an odd exponent, causing the answer to be odd. [Survey]

The odd exponent is not relevant in Question 2: an odd number raised to any natural number is odd. It has to be noted here that an inadequate understanding of odd numbers might be at the root of the incorrect reasoning, instead of an inadequate understanding of exponents.

- Student 7 was the fourth student who’s levels were not in accordance with the hypothesis. Student 7 had difficulties with solving tasks involving exponents with negative base or power (so he should be in the preconceptual level), but he showed in his response to Question 2 that he might be in the process level:



Student 7: $5 \cdot 5 = 25$ → 3 odd numbers → odd · odd = odd
$25 \cdot 5 = 125$ → again 3 odd numbers
therefore, it is an odd number. [Survey]

Even though the student noticed that an odd number times an odd number is an odd number, he did not explicitly finish his induction. In order to be ranked in the process level, a students should complete the induction. Therefore, the ranking in the process level is not without doubt.

The levels of the other 13 students were in agreement with the hypothesis. However, it has to be noted that the ranking in terms of the levels of Pitta-Pantazi et al. (2007) and Weber (2002a) is tentative due to the limited amount of questions in the survey.

5.3 Splitting and the Levels of Pitta-Pantazi et al. (2007)

Six students had difficulties with understanding a splitting structure and understanding the relationship between splitting and counting structures. The other 11 students showed that doubling the time in Question 4 is equivalent to squaring the amount of MB's, and that halving the time is equivalent to square rooting the amount of MB's, which might show students' understanding of the relationship between splitting and counting structures.

I hypothesized that students who solve problems within the splitting structure described by Confrey and Smith (1995) to the extent of additive and multiplicative parts, are in the restructured level of Pitta-Pantazi et al. (2007). In Table 6, the students are ranked in terms of the levels of Pitta-Pantazi et al. (2007). Students who correctly solved Question 4 using exponential growth are ranked in the restructured level. Students who solved Question 4 using linear growth, or who could not solve Question 4 are ranked in the preconceptual/conceptual level.

Table 6: Students' performance for Question 4 and 5

Student	**Level Pitta-Pantazi et al. (2007)**	**Splitting**	**Hypothesis confirmed**
1	restructured	restructured	yes
2	restructured	restructured	yes
3	conceptual/restructured	preconceptual/conceptual	yes
4	restructured	restructured	yes
5	preconceptual/conceptual	preconceptual/conceptual	yes
6	conceptual	preconceptual/conceptual	yes

7	preconceptual	preconceptual/conceptual	yes
8	conceptual	restructured	no
9	restructured	restructured	yes
10	preconceptual	preconceptual/conceptual	yes
11	restructured	restructured	yes
12	conceptual	restructured	no
13	conceptual	restructured	no
14	conceptual	preconceptual/conceptual	yes
15	preconceptual/conceptual	restructured	no
16	restructured	restructured	yes
17	restructured	restructured	yes

Student 8, 12, 13 and 15 did not confirm the hypothesis: they correctly solved Question 4 of the survey about the splitting structure, but they had difficulties with Question 5(iv) and 5(v) of the survey. These were the questions that indicated a restructured level. The levels of the other 13 out of 17 students were in agreement with my hypothesis.

Student 6 and Student 14 used an incorrect base when solving Question 4. Both students used base 8 instead of base 9. In the interviews, they were asked about their choice for this base:

Interviewer: How did you arrive at $8^1 + 1 = 9$?
Student 6: You already had *one*, and the difference between one and nine is eight. So to the power one, that is after an hour, so I thought after half an hour you had to raise to the power a half. [Interview]

Interviewer: Okay, and how did you get that eight?
Student 14: Uhm, that eight? Because eight MB is damaged in one hour. [Interview]

It seems that the two students use the difference [adding $9 - 1 = 8$ MB each hour] to compute the base (or splitting unit in terms of Confrey and Smith (1995)) instead of the ratio [increasing to $9/1 = 9$ times as much each hour]. So the students had an incorrect understanding of the splitting structure, even though they related halving the time with square rooting the amount of megabytes ("(...) I thought after half an hour you had to raise to the power a half", Student 6). It is noteworthy that based on Question 5, both students were in the conceptual level of Pitta-Pantazi et al. (2007).

It is notable that Student 7, one of the students ranked in the preconceptual level, was the only student who thought for Question 4 that that the growth was linear instead of exponential, as he wrote in the survey:

Student 7:	(i) 1 hour is equal to 8 MB, so 2 hours to 16 MB. 1 + 16 = 17 MB at 4:15 p.m. (ii) 1 hour is equal to 8 MB, so ½ hour to 4 MB. 1 + 4 = 5 MB at 2:45 p.m. [Survey]

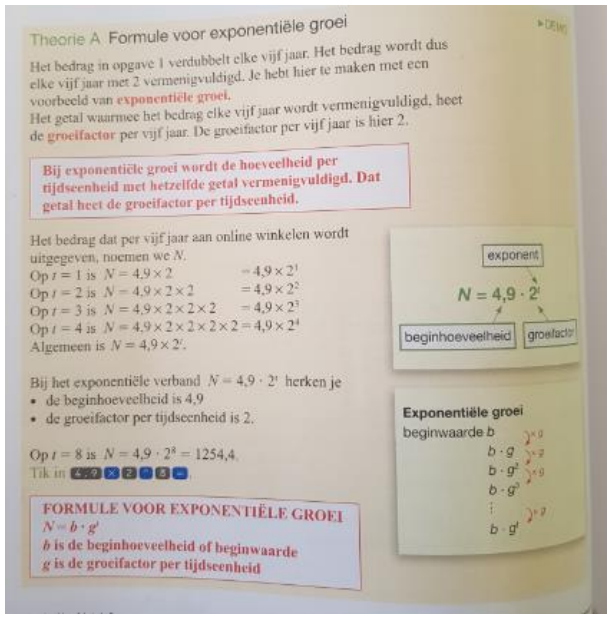
Question 4 suggests a splitting structure (the virus is "repeating itself") but since the exponential growth is not explicitly stated, Student 7's reasoning is correct. Since he did not recognize a splitting structure, I did not rank him in the restructured level.

One student solved Question 4 by using both the linear and exponential growth:

Interviewer:	For Question 4, how did you find out that there were two ways of solving?
Student 13:	Well, it was not stated that it was necessarily an exponential relationship.
Interviewer:	Yes.
Student 13:	So I thought now I am going to be smart… [laughs]
Interviewer:	Yes.
Student 13:	…and I will just show that, and I don't know whether I did it right, there is a linear relationship as well.
Interviewer:	Okay, and why did you think that there could also be a linear relationship?
Student 13:	Because there were just two quantities given… yes two quantities, so then you can do many things. [Interview]

The response of Student 13 illustrates the dual nature of Question 4: the exponential growth is not explicitly articulated, and with only two coordinates given, the student can also interpret the growth as being linear. Since Student 13 also recognized and could work with a splitting structure, she was ranked in the restructured level.

5.4 Student Familiarity With the Exponential Function



Theorie A Formule voor exponentiële groei

Het bedrag in opgave 1 verdubbelt elke vijf jaar. Het bedrag wordt dus elke vijf jaar met 2 vermenigvuldigd. Je hebt hier te maken met een voorbeeld van **exponentiële groei**.
Het getal waarmee het bedrag elke vijf jaar wordt vermenigvuldigd, heet de **groeifactor** per vijf jaar. De groeifactor per vijf jaar is hier 2.

Bij exponentiële groei wordt de hoeveelheid per tijdseenheid met hetzelfde getal vermenigvuldigd. Dat getal heet de groeifactor per tijdseenheid.

Het bedrag dat per vijf jaar aan online winkelen wordt uitgegeven, noemen we N.
Op $t = 1$ is $N = 4{,}9 \times 2 = 4{,}9 \times 2^1$
Op $t = 2$ is $N = 4{,}9 \times 2 \times 2 = 4{,}9 \times 2^2$
Op $t = 3$ is $N = 4{,}9 \times 2 \times 2 \times 2 = 4{,}9 \times 2^3$
Op $t = 4$ is $N = 4{,}9 \times 2 \times 2 \times 2 \times 2 = 4{,}9 \times 2^4$
Algemeen is $N = 4{,}9 \times 2^t$.

Bij het exponentiële verband $N = 4{,}9 \cdot 2^t$ herken je
- de beginhoeveelheid is 4,9
- de groeifactor per tijdseenheid is 2.

Op $t = 8$ is $N = 4{,}9 \cdot 2^8 = 1254{,}4$.
Tik in 4,9 × 2 ^ 8 =

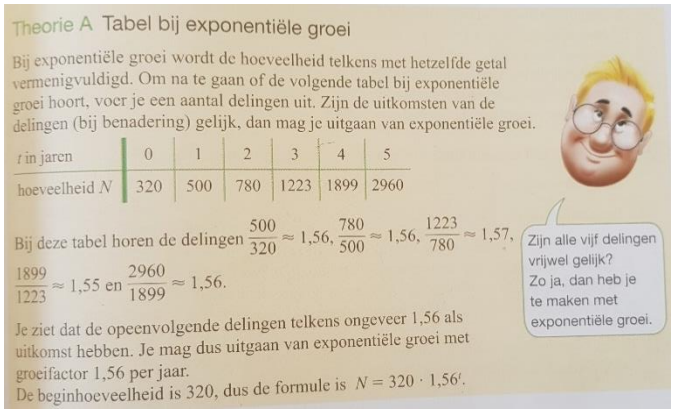
FORMULE VOOR EXPONENTIËLE GROEI
$N = b \cdot g^t$
b is de beginhoeveelheid of beginwaarde
g is de groeifactor per tijdseenheid

exponent
$N = 4{,}9 \cdot 2^t$
beginhoeveelheid
groeifactor

Exponentiële groei
beginwaarde b
$b \cdot g$
$b \cdot g^2$
$b \cdot g^3$
$\vdots$
$b \cdot g^t$

Figure 9. Course textbook introduction of the exponential function (Reichard & Dijkhuis, 2011)

In the textbook the students of this study used (Reichard & Dijkhuis, 2011) the exponential function is introduced as the 'formula for exponential growth' (Figure 9). The formula for exponential growth is given in the form $N = b \cdot g^t$. The terminology for the components of this formula are 'initial value' for b, 'growth factor' (per unit of time) for g and 'exponent' for t. Graphs are not introduced in the section, but tables are (Figure 10). The textbook states that in order to check if a table fits exponential growth, "you have to conduct a number of divisions. If the outcomes of the divisions are (approximately) equal, you can assume exponential growth." (Reichard & Dijkhuis, 2011, translation from Dutch).



Theorie A Tabel bij exponentiële groei

Bij exponentiële groei wordt de hoeveelheid telkens met hetzelfde getal vermenigvuldigd. Om na te gaan of de volgende tabel bij exponentiële groei hoort, voer je een aantal delingen uit. Zijn de uitkomsten van de delingen (bij benadering) gelijk, dan mag je uitgaan van exponentiële groei.

t in jaren	0	1	2	3	4	5
hoeveelheid N	320	500	780	1223	1899	2960

Bij deze tabel horen de delingen $\frac{500}{320} \approx 1{,}56$, $\frac{780}{500} \approx 1{,}56$, $\frac{1223}{780} \approx 1{,}57$, $\frac{1899}{1223} \approx 1{,}55$ en $\frac{2960}{1899} \approx 1{,}56$.

Je ziet dat de opeenvolgende delingen telkens ongeveer 1,56 als uitkomst hebben. Je mag dus uitgaan van exponentiële groei met groeifactor 1,56 per jaar.
De beginhoeveelheid is 320, dus de formule is $N = 320 \cdot 1{,}56^t$.

Zijn alle vijf delingen vrijwel gelijk? Zo ja, dan heb je te maken met exponentiële groei.

Figure 10. Table for exponential growth (Reichard & Dijkhuis, 2011)

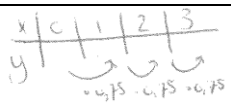
According to Student 13, the teacher often uses graphical representations of functions. She answered when asked why she drew a graph: "Because, Mr. Smit*, my teacher, draws a graph as well when we talk about a particular formula."

- Five students used a graphical representation (two verbal and three drawn) when describing the formula $f(x) = a^x$, one student used a tabular representation. The other eleven students described the function only verbally.

Table 7. Representation of the exponential function in Question 1 and 4

Student	**Q1 (codes)**	**Example Q1**	**Q4 (codes)**	**Example Q4**	**Level Pitta-Pantazi et al. (2007) (based on analysis of survey responses to Q5)**
1	Verbal description	"function with exponential growth"	Formula with initial value	"$f(x) = 1 \times 9^x$"	restructured
2	Verbal description		Formula with initial value		restructured
3	Verbal description		Formula with initial value		conceptual or restructured
4	Verbal description		Formula with initial value		restructured
5	Verbal description		Splitting	"$1 \cdot 9 = 9$ so growth factor is 9 MB per hour. $9 \cdot 9^1 = 81$"	preconceptual or conceptual
6	Verbal description		Splitting		conceptual
7	Verbal description		Linear	"1 hour is equal to 8	preconceptual

* Mr. Smit is a pseudonym for anonymity reasons

				MB, so 2 hours to 16 MB. 1 + 16 = 17 MB at 4:15 p.m.”	
8	Verbal description		Formula without initial value	“$V(t) = 9^t$”	conceptual
9	Graphical description		Formula with initial value		restructured
10	Verbal graphical description	“the graph is convex increasing”	Formula without initial value		preconceptual
11	Verbal graphical description		Formula with initial value		restructured
12	Graphical description		Splitting		conceptual
13	Graphical description		Splitting & linear		conceptual
14	Verbal description		Splitting		conceptual
15	Tabular description	 “repeatedly the same amount”	Splitting		preconceptual or conceptual
16	Verbal description		Splitting		restructured
17	Verbal description		Splitting		restructured

- From Table 7 it stands out that all six students who used the textbook definition of the exponential function with explicit initial value were, based on Question 5, in the restructured level. In this research, this means that they were able to apply the rules $b^{-x} = 1/b^x$ and $b^{x/y} = \sqrt[y]{b^x}$.

- It may be remarkable that both students (Student 6 and Student 14) who used base 8 instead of 9 for Question 4 mentioned in the interview that they did not know how the graph of an exponential formula looks like.

- When in the interviews Student 14 is asked to elaborate on Question 1 (What does the function $f(x) = a^x$ mean to you?), she seemed to confuse the notation of the exponential function with the notation of the power function. In line 4 and 5 of the interview extract below, Student 14 described a function with x being the base and a being the power (which is the power function), but she also stated that one of the two symbols is a growth factor.

1	Interviewer:	And have you seen it before? [the function $f(x) = a^x$]
2	Student 14:	Yes, yes.
3	Interviewer:	And with what… to what was it related then?
4	Student 14:	Yes, most of the time it was the other way around, so then the x was below
5		and the… the a as… power.
6	Interviewer:	And what were the x and the a then?
7	Student 14:	Factors of growth.
8	Interviewer:	Okay. Both?
9	Student 14:	No, oh god [laughs], the x I think. Or no… that's a good question. Uhm.. the
10		power? Or the… I don't know, I have no idea. [Interview]

The notation of the power function seemed to be more familiar to her than the notation of the exponential function ("most of the time it was the other way around", line 4). She seemed to state that one of the symbols of the power function is the growth factor, but she did not know whether it is the base or the power (line 9-10).

6 Discussion

The research question for my study was: What is students' understanding of exponents and exponential functions at a secondary school in the northern Netherlands?

In order to answer the research question, I explored students' understanding within the levels of Pitta-Pantazi et al. (2007). Moreover, I compared the levels of Pitta-Pantazi et al. (2007) with the levels proposed by Weber (2002a), and I investigated students' understanding of the relationship between splitting and counting structures (Confrey & Smith, 1995). More specifically, I compared the three frameworks by hypothesizing that students, who are ranked in the preconceptual level by Pitta-Pantazi et al. (2007), are within an action understanding by Weber (2002a). Students ranked in the conceptual level by Pitta-Pantazi et al. (2007) are within a process understanding and see exponential expressions as the result of a process by Weber (2002a). Students ranked in the restructured level by Pitta-Pantazi et al. (2007) are within a generalized understanding by Weber (2002a). Moreover, I built a hypothesis that understanding the relationship between splitting and counting structures might match the restructured level. Also, I investigated students' familiarity with the definition of the exponential function.

In Table 8, the levels of understanding of the participants in this study are given.

Table 8. Students' levels of understanding

Level Pitta-Pantazi et al. (2007)	**#Students**	**Features of understanding**	**Misconceptions**
Preconceptual	2	positive integer base and power	negative base or power
Conceptual	5	negative integer base and positive integer power	rational base and negative rational power
Restructured	7	rational base and negative rational power	

Based on students' responses to the questions of Pitta-Pantazi et al. (2007), only two students were ranked in the preconceptual level. The main characteristic of students' performance in this level, is that they could solve tasks with exponents with positive

integer base and power, but that they had difficulties with exponents with negative base or power. For example, they understood why 23^{13} is larger than 23^{8}, but it's not evident from their responses that they understood why $(-12)^{13}$ is larger than $(-12)^{17}$.

Five participants were ranked in the conceptual level. Students in this level correctly solved tasks with a negative integer base and positive integer power: they understood why $(-12)^{13}$ is larger than $(-12)^{17}$. Students that were ranked in the conceptual level in my study had difficulties with exponents with rational base and negative rational power. Some students were confused because of the rational base, for example when reasoning why 0.5^{-15} should be larger than 0.6^{-15}. But it stands out that at least 4 out of 17 students confused a minus sign in the power with a minus sign preceding the base. For example, multiple students thought that $17^{(-3/5)}$ was a negative number. It has to be noted that Pitta-Pantazi et al. (2007) rank students who can work with exponents with both negative rational base and/or power are in the conceptual level. However, in this study we only used a task with a negative integer base to test for the conceptual level. The reason for this was that we also included in the survey other tasks of similar ranking from Weber (2002). This study revealed that students who could work with negative powers could work with negative bases as well, but that the opposite is not always true: multiple students could work with exponents with negative base but had a misunderstanding of negative powers. Moreover, some students ranked in my conceptual level had difficulties with a rational base, which is also in conflict with Pitta-Pantazi et al.'s (2007) description of the conceptual level. Since these students exceeded the preconceptual level (since they correctly solved a task with a negative base), this could indicate that the second level might be expanded to include difficulties with rational base.

Finally, seven students showed they could work with exponents with rational base and negative rational power. Especially, they showed that they could correctly apply the rules $b^{-x} = 1/b^{x}$ and $b^{x/y} = \sqrt[y]{b^{x}}$. For example, students could show why $17^{(-3/5)}$ is smaller than $15^{(-3/5)}$ by applying these rules. No student gave an explanation for these rules.

Pitta-Pantazi et al. (2007) investigated students' understanding of exponents among 202 secondary school students in Cyprus. Exponential expressions were introduced to the students two months prior to their research. In the study by Pitta-Pantazi et al. (2007), 25.7% of the participants was ranked in the preconceptual level, 61.9% was ranked in the conceptual level and 12.4% was ranked in the restructured level. In this study, based on tasks from Pitta-Pantazi et al. (2007), 11.8% of the participants was ranked in the preconceptual level, 29.4% was ranked in the conceptual level and 41.1% was ranked in

the restructured level (the response of 17.6% of the participants did not provide enough evidence for a certain ranking). It stands out that the participants in this study performed better than the participants in the study by Pitta-Pantazi et al. (2007). This might be due to the small sample of the study or because students in this study had more experience with working with exponential expressions (they were at a point in the curriculum where they were already introduced to differentiating and integrating exponential functions).

Cangelosi et al. (2013) found that students at a university in the United states have difficulties with exponents with negative power. For example, in their study, students interpreted 2^{-3} as $2^{1/3}$ or $2/(1/3)$ or $3/2$. Mistakes of this nature did not occur in this study. But, in the study by Cangelosi et al. (2013), some students moved the negative sign in the power to precede the base and erroneously wrote $2^{-3} = -2^3$. This mistake occurred multiple times in this study.

I wanted to test with my research whether the conceptual level (Pitta-Pantazi et al., 2007), the process understanding and exponential expressions as the result of process (Weber, 2002a) match to the second level of understanding. 13 out of 17 students confirmed my hypothesis about the relationship between the conceptual level, a process understanding and exponential expressions as the result of process. The ranking of the 4 out of 17 students who disconfirmed the hypothesis is not without doubt: the data of these students did not provide watertight evidence for a ranking in terms of the levels of Weber (2002a). This might be due to other difficulties that may have interfered. For example, sometimes it was not clear whether an inadequate understanding of odd numbers was at the root of incorrect reasoning, instead of an inadequate understanding of exponents. Overall, the ranking is tentative due to the limited amount of questions in the survey.

Weber (2002a, 2002b) investigated students' understanding of exponents among 15 university students enrolled in a traditional precalculus course in the United States. Exponential expressions were introduced to the students three weeks prior to the research. In Webers' (2002a, 2002b) study, only 6 out of 15 participants recalled the rule $b^x b^y = b^{x+y}$, and none of them could explain why the rule was true. Only 3 out of 15 correctly understood that an odd number raised to any natural number is odd. In this study, 15 out of 17 participants recalled the rule $b^x b^y = b^{x+y}$, and only Student 11 could explain why the rule was true by expanding $b^x b^y$ to $\underbrace{b \cdots b}_{x\ times} \cdot \underbrace{b \cdots b}_{y\ times}$. Thirteen out of 17 students seemed to understand why 5^{17} is odd. So overall, participants in this study were in a higher level of understanding than the 15 participants in Weber's (2002a, 2002b) study.

I also investigated students' understanding of splitting and the relationship between splitting and counting structures (Confrey & Smith, 1995). Six out of 17 students had difficulties with understanding a splitting structure and understanding the relationship between splitting and counting structures. These difficulties involved, among others, composing an incorrect base (or splitting unit in terms of Confrey and Smith (1995)). The other 11 students seemed to show that they understand the relationship between splitting and counting structures by relating doubling in a counting world with squaring in a splitting world, and relating halving in a counting world with square rooting in a splitting world.

Thirteen out of 17 students confirmed my hypothesis about the relationship between the restructured level and understanding the relationship between splitting and counting structures. The remaining 4 out of 17 students showed that they might understood the relationship between splitting and counting structures (because they related halving in a counting world with square rooting in a splitting world), but they were ranked in a lower level than the restructured level.

Students used different kinds of representations when describing the exponential function:

- Verbal description (11 students)
- (Verbal) graphical description (5 students)
- Tabular description (1 student)

All six students who explicitly used the textbook definition of the exponential function $N = b \cdot g^t$ for the splitting problem, were also able to apply the rules $b^{-x} = 1/b^x$ and $b^{x/y} = \sqrt[y]{b^x}$.

Students in this study performed well compared to the studies by Pitta-Pantazi et al. (2007) and Weber (2002a, 2002b). In this study, the most notable deficiencies in students' understanding of exponents and exponential functions were:

- students' who moved the minus sign of the power to the base or preceding the base,
- students inability to explain why the rules of exponentiation are true,
- students who composed the growth factor in a splitting problem incorrectly.

Reflecting on my findings, I would like to recommend that in the teaching and learning of exponents and exponential functions, more attention should be paid to these deficiencies. In my practices as a tutor, I will at least try to give more intuitive interpretations of the rules; for example, by writing exponents with base 2 in the following order, and

emphasizing the pattern: 2^2 is 4, $2^1 = 2$, $2^0 = 1$, $2^{-1} = 1/2$, $2^{-2} = 1/4$, because you divide each time by 2. It seems to me that it is acceptable to claim that having an intuitive understanding of negative exponents might even prevent students to move the minus sign of the power to the base or preceding the base, but this could be studied in the future. Also, it would be interesting to investigate whether the covariation approach to exponential functions would prevent the three aforementioned deficiencies.

Limitations of this study are the small number of respondents in the survey and the small number of questions. The results of this study do not form generalizations. This study focused only on students 16-18 years of age attending a *vwo* Wiskunde B course in the northern Netherlands. It is possible that the results of the study might have been different if the participants were of other ages, attended other mathematics courses or attended other secondary schools.

Other limitations of this study are the lack of triangulation from different data sources (that is, letting the data of one source confirm the data of another source). In this study, students filled out a survey and participated in interviews. If additional data collection methods were used or if for example also the teacher would have been interviewed, the accuracy of the results in this study might be better guaranteed because the different sources might have supported eachother. Moreover, if there would have been time for it, member checking would have supported the accuracy of this study. Member checking involves showing the participants the results of the study and asking them about the accuracy and representativeness of the results.

This was a pilot study to shed light on the hypothesis, and in future I would like to get a large sample to test the hypothesis.

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Appendix A: Procedure of the Literature Search

I searched for the literature as described by Creswell (2008). Creswell (2008, p. 81) identifies five steps in conducting a literature review:

- *Identify key terms* to use in your search for literature.
- *Locate literature* about a topic by consulting several types of materials and databases, including those available at an academic library and on the Internet.
- *Critically evaluate and select the literature* for your review.
- *Organize the literature* you have selected by abstracting or taking notes on the literature and developing a visual diagram of it.
- *Write a literature review* that reports summaries of the literature for inclusion in your research report.

In order to identify key terms, Creswell (2008) suggests to start with thinking of a preliminary working title, so I wrote down *teaching and learning of exponential and logarithmic functions*. Important key terms in this title are 'exponential' and 'logarithmic'. Variations of these key terms are 'exponents' and 'logarithms'.

I first searched the Mathematics Education Database (a database suggested by my supervisor) for papers by entering the key terms 'exponent*' or 'logarithm*'. I filtered the results for English articles. This yielded 83 articles. Then I searched the online database of the Educational Resources Information Center (suggested by Creswell (2008)) for abstracts containing the key terms 'exponent*' or 'logarithm*'. This yielded 31 articles.

As suggested by Creswell (2008), I prioritized articles of refereed journals over conference papers and books, and I did not search for nonreviewed articles on websites. In order to determine whether the literature I found was of good quality, from the articles I found I selected only articles from journals that were graded by Törner and Arzarello (2012). Törner and Arzarello (2012) have classified journals in mathematics education by evaluating recognition of the journal, review process and quality standards, editors and editorial board, and citations to the journal. I also searched the catalog of the University of Groningen Library for online available journals from the list of Törner and Arzarello (2012). I located articles in these journals by entering the key terms 'exponent*' or 'logarithm*' in the search engines of the journal databases. After removing doubles and removing articles by screening the abstracts and titles of all of the articles I had found so

far, I selected 14 articles. I also searched the university library catalog for 'handbook mathematics education', selected 2 books based on title and searched their indexes for key terms, but this did not yield useful information.

After full text screens of the papers I had found, I added 4 papers from conference proceedings that played central roles in the papers. Finally, when it became clear that I am going to study the present situation of the understanding of exponents by students of a secondary school in the northern Netherlands, I removed papers that focused on teachers' knowledge, textbooks or logarithms. I ended up with 8 articles.

I organized the articles by abstracting them (see Appendix B). The abstracting involved describing the research problem, the research questions and hypotheses, the data collection procedure and the findings of each study. I created a literature map: this was a drawing that displayed groups of related topical areas or "families of studies" (Creswell, 2008).

For end-of-text and within-text references, I used the *Publication Manual of the American Psychological Association, 6th edition* (APA, 2010), the most popular style guide in educational research (Creswell, 2008). I wrote a thematic review of the literature (Creswell, 2008), where I identified themes based on the related topical areas in the literature map, and I cited literature to document the themes.

Appendix B: Summaries of Literature

Cangelosi, R., Madrid, S., Cooper, S., Olson, J., & Hartter, B. (2013). The negative sign and exponential expressions: Unveiling students' persistent errors and misconceptions. *The Journal of Mathematical Behavior, 32*(1), 69-82. doi:10.1016/j.jmathb.2012.10.002

Research problem

Algebra is an important basis for mathematics, and good algebra skills are essential for a career in science and engineering. But there has been done only few educational research on the simplification of algebraic expressions, especially on the role of the minus sign in the simplification of exponentials.

Hypothesis & research questions

The researchers hypothesized that as students progress with their studies, they kept on making the same mistakes when manipulating exponential expressions. When written in a research question: do students keep on making the same mistakes with the manipulation of exponential expressions as they progress through their studies? And if so, what are these mistakes, and how can we categorize them?

Data collection procedure

The participants were students of 2 universities in the USA who were enrolled for algebra and calculus courses (college algebra, pre-calculus, and first- and second-semester calculus mathematics courses). 904 students were asked to sit a test on exponential expressions; the student responses to these tests were analyzed by coding the answers as right or wrong and with statistics persistent errors were identified. 18 of the 904 students participated in semi-structured interviews such that the researchers could gain a better understanding of the errors. Field notes of the interviews were taken, and the interviews were videotaped if possible. The data from the interviews were analyzed according to qualitative methods, more specifically the data were collapsed using conceptual cross-case matrix analysis.

Findings

From the tests, it was found that there are indeed persistent errors among the students participating in the research. These errors frequently involved confusion about a minus sign as part of or preceding a base, and about a minus sign in the exponent. The

researchers noticed that most of the students relied only on an operational understanding when solving equations, and not on a structural understanding.

An operational understanding (Sfard, 1991) consists only of using inverse *operations*. A student with an operational understanding would for example solve the equation $x + a = b$ by using the inverse operation *subtracting with* a on both sides and solve the equation $ax = b$ by using the inverse operation *divide by* a on both sides. A structural understanding puts emphasis on the concepts of the inverse and the identity. A student with a structural understanding would solve the equation $x + a = b$ by adding the additive inverse $-a$ on both sides, and he or she would notice that on the left hand side, a and $-a$ cancel each other out resulting in the additive identity zero. In the same way, he or she would solve $ax = b$ by multiplying both sides with the multiplicative inverse $1/a$, and he or she would notice that on the left hand side, a and $1/a$ cancel each other out resulting in the multiplicative identity one. The researchers suggest that an underdeveloped understanding of additive and multiplicative inverses underlies these mistakes. Teachers and textbooks should make more use of the terms inverse and identity, and teachers should help students to recognize a connection between the algebraic structures of additive and multiplicative inverses.

Ellis, A. B., Özgür, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An Exponential Growth Learning Trajectory: Students' Emerging Understanding of Exponential Growth Through Covariation. *Mathematical Thinking and Learning, 18*(3), 151-181. doi:10.1080/10986065.2016.1183090

Research problem

A robust understanding of exponential functions is important in secondary school but many students have difficulties when working with exponential functions. This suggests that there are challenges with the teaching of exponential functions. Therefore investigating students' initial and evolving understanding of exponential functions is substantiated.

Research questions

An explicit research question is not stated. The authors write: "This article reports on the results of two teaching experiments investigating students' understanding of exponential growth within the context of covarying quantities" (p. 152), so one might deduce that the

researchers wanted to find an answer to the question: how does students' understanding of exponential functions develop within the context of covarying quantities?

Data collection procedure

The structure of the research was as follows: in order to answer the research question, the researchers developed a hypothetical learning trajectory (a trajectory describing students' evolving understanding) within the concept of covariation (Confrey & Smith, 1995). This learning trajectory was tested in a first teaching experiment and then revised. Then a second teaching experiment was conducted for another revision of the learning trajectory. Then, by reviewing the data of the first teaching experiment for a second time, the final learning trajectory was determined, which was called the Exponential Growth Learning Trajectory (EGLT).

The participants of the first study were three eight-graders (13-14 years). Lessons of one hour were videotaped over a period of 12 days. The participants did not encounter a formal course unit on exponential functions prior to the experiment.

The participants of the second study were eight students who had just finished eight grade (14 years). The lessons were part of a university-sponsored summer session. Lessons of 90 minutes were videotaped over a period of 9 days. Half of the students had had a exponential growth unit and the other half had not, but the students who had encountered exponential growth only described exponential expressions as representing repeated multiplication, and when asked, they could not quantify the nature of exponential growth.

Both experiments took place in the USA, and the sessions were taught by members of the research group. The data consisted of verbal and nonverbal actions of the students, and their written work and drawings. Two researchers coded the data independently, according to the coding scheme described by Strauss & Corbin (1990). Their codes were being compared, refined and being recoded.

Findings

A visual representation of the EGLT can be found on page 160. In the EGLT, the development of prefunctional reasoning precedes the development of both a covariation approach and a correspondence approach. The development of the covariation view influenced the development of the correspondence view, and vice versa.

The research suggests that exploring exponential growth by coordinating covarying quantities promotes building an understanding of exponential functions. Moreover, the

covariation approach played an important role in the development of students' ability to express exponential relationship algebraically. The exploration of continuous covariation can be supported by a task involving a plant in GeoGebra (or another computer algebra system) that grows exponentially, because it enables the student to manipulate the x- and y-values and to visualize the growth. It appears that a strong functional reasoning is not necessary for students when they are being introduced into exponential functions.

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. The Journal of Mathematical Behavior, 39, 135-155. doi:10.1016/j.jmathb.2015.06.004

Research problem

Exponential functions are an important topic at secondary school. The National Governor's Association Center for Best Practices recommend that students can understand exponential growth as a quantity changing with a constant rate when unit intervals are being compared.

Research questions

(a) What conceptual stages can be identified in students' exploration of exponential growth as they investigate a scenario with continuously co-varying quantities; and (b) What factors contributed to students' conceptual change over time?

Data collection procedure

In order to answer the research question, the researchers developed a hypothetical learning trajectory (a trajectory describing students' evolving understanding) within the concept of covariation (Confrey & Smith, 1995). This learning trajectory was tested in this teaching experiment and then revised. Then a second teaching experiment was conducted for another revision of the learning trajectory. Then, by reviewing the data of this teaching experiment for a second time, the final learning trajectory was determined, which was called the Exponential Growth Learning Trajectory (EGLT).

The participants of the study were three eight-graders (13-14 years). Lessons of one hour were videotaped over a period of 12 days. The participants did not encounter a formal course unit on exponential functions prior to the experiment.

The experiment took place in the USA, and the sessions were taught by members of the research group. The data consisted of verbal and nonverbal actions of the students, and their written work and drawings. Two researchers coded the data independently, according to the coding scheme described by Strauss & Corbin (1990). Their codes were being compared, refined and being recoded.

Findings

Answering research question (a), the researchers found three conceptual shifts, described in a table on p. 142. The students started with the view that exponentiation represents repeated multiplication, and they shifted (1) to coordinating repeated multiplication for y with additive growth in x. Next, they shifted (2) to coordinating constant ratios of y_1 and y_2 for corresponding constant Δx, where Δx is a natural number; then they shifted (3) to coordinating constant ratios of y_1 and y_2 for corresponding constant Δx, where Δx is a rational number. Answering research question (b), three important factors that contributed to these shifts were: (1) successively removing the students' ability to calculate. This challenged students to mentally coordinate the relationship between x- and y-values. (2) repeatedly express students' to the same mathematical activity. (3) invite students to compare their actions with the actions they did in the other tasks to identify common elements.

The exploration of continuous covariation can be supported by a task involving a plant in GeoGebra (or another computer algebra system) that grows exponentially, because it enables the student to manipulate x- and y-values and to visualize the growth.

Ferrari-Escolá, M., Martínez-Sierra, G., & Méndez-Guevara, M. E. (2016). "Multiply by adding": Development of logarithmic-exponential covariational reasoning in high school students. *The Journal of Mathematical Behavior, 42*, 92-108. doi:10.1016/j.jmathb.2016.03.003

Research problem

Covariational reasoning plays a fundamental role in the learning process of a function. The research has as its aim to extend the understanding of covariational reasoning of exponential and logarithmic functions in high school students.

Research questions

(1) What are the mental actions, involving the discrete and dynamic perspectives of logarithm-exponential covariational reasoning, developed by the students during the tasks in the teaching experiment? (2) What levels of logarithm-exponential covariational reasoning are achieved by the students?

Data collection procedure

The participants were 15 students of a technical high school in Mexico. Seven of them were men, eight were women, and they were aged 17-18. The teaching experiment consisted of five sessions of two hours, one of the researchers was the instructor. The students worked together in five groups of three students, three of the groups were videotaped (focusing on the students) and all of the groups were audiotaped. Video, audio and worksheets of the students were analyzed with qualitative methods. The teaching experiment was a card game based on Stiffer's *Arithmetica integra* (1544) and on the ideas of Confrey & Smith (1995) on discrete covariational reasoning. The data was analyzed within the framework of the five levels of the development of continuous covariational reasoning as described by Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002) to see what levels of reasoning the students achieved. These five levels are described in terms of five mental actions. The researchers reformulated the five mental actions before the data analysis such that they were applicable to the discrete situation of the experiment.

Findings

The card game encouraged students to develop covariational reasoning with an arithmetic and a geometric sequence. This evidence was given by interpreting the development of the participants' covariational reasoning in terms of the five levels described by Carlson et al (2002). Moreover, even though the experiment was designed to support discrete reasoning, chunky and smooth continuous covariational reasoning was observed. Chunky continuous covariational reasoning happens when one treats variation as being discrete but one does not consider the gaps as being necessarily empty; for example filling a bottle with water and comparing discrete amounts of added water with the change of height of the water surface. Smooth continuous variation happens when one treats variation as being continuous; for example filling a bottle with water and comparing a continuous addition of water with the change of height of the water surface.

Pitta-Pantazi, D., Christou, C., & Zachariades, T. (2007). Secondary school students' levels of understanding in computing exponents. *The Journal of Mathematical Behavior, 26*(4), 301-311. doi:10.1016/j.jmathb.2007.11.003

Research problem

Exponents are important concepts in many mathematics courses. Little research has been done on student understanding and learning of exponents.

Research questions

The authors write: "the aim which inspired this research was to describe the levels of students' understanding of the concept of exponents, and to analyze this understanding within the framework of conceptual change and the influence of students' prototype concepts and interpretations." (p. 304) So the research question might be: What are the levels of students' understanding of exponents within the conceptual change theory, and what is the influence of students' prototype concepts?

Data collection procedure

Participants were 202 secondary school students on Cyprus. Data was collected using a test where students were asked to compare pairs of exponents by filling in the right sign ($<$, $=$ or $>$) The test was analyzed with descriptive statistics and latent class analysis (Bentler, 2004). This yielded three basic groups of students ordered by test performance. From each group, ten students were interviewed to give explanations of their answers of the test. Interviews lasted one hour and they were videotaped and transcribed.

Findings

The researchers could divide the students in three successive levels of understanding: a preconceptual level, a conceptual level and a restructured level. The researchers suggest that students in the preconceptual relied too much on prototype examples (central examples provided by teacher and textbook) where exponents are viewed as repeated multiplication, with base and power being integers. These examples do not help students with exercises where base or power are real numbers.

Weber, K. (2002a). Students' understanding of exponential and logarithmic functions. *Second International Conference on the Teaching of Mathematics* (pp. 1–10). Crete, Greece: University of Crete.

Research problem

Although exponential and logarithmic functions are important in college mathematics courses, many students have difficulties with these concepts. Moreover, there has been done little research on how the understanding of exponential and logarithmic functions develops.

Research questions

No research questions have been stated. The author writes: “The purpose of this study is to describe a theory of how students might develop their understanding of these topics and to analyze students' understanding of these concepts within the context of this theory” (p. 2), so one might deduce that the researcher wanted to find an answer to the questions: (1) What might be a theory of the development of students’ understanding of exponential and logarithmic functions? (2) What is the level of students’ understanding of these functions within the context of this theory?

Data collection procedure

Before collecting the data, a set of mental constructions that students might make while learning exponential and logarithmic functions were formulated, based on Dubinsky's APOS theory (Dubinsky, 1991). The data then was analyzed within the context of the set of mental constructions.

The participants were 15 students of a university in the USA. They were enrolled in a precalculus course and after three weeks of being taught exponential and logarithmic functions, they volunteered to be interviewed. Students were asked for properties of exponential and logarithmic functions and to perform calculations with these functions. Open-ended questions were also asked, to investigate students’ understanding of exponential and logarithmic functions.

Findings

A set of four successive levels that students might make while learning exponential and logarithmic functions were proposed, based on Dubinsky's APOS theory (Dubinsky, 1991). These four levels were: (1) an action understanding of exponents (computing b^x involves repeatedly multiplying by b x times), (2) a process understanding of exponents (to view exponentiation as a function, to reason about properties of this function and to reverse exponentiation to obtain logarithms), (3) exponential expressions are the result of a process (an exponential expression represents the output of exponentiation) and (4) a

generalized process understanding of exponents (interpreting situations where the input is not restricted to the natural numbers).

The data then was analyzed within the context of the set of mental constructions.

The data showed that most students had not progressed beyond an action understanding of exponents, i.e. they did not view exponentiation as a process.

Weber, K. (2002b, October). *Developing students' understanding of exponents and logarithms.* Paper presented at the 24th Annual Meeting of the North American Chapter of Mathematics Education, Athens, Georgia. Retrieved from https://files.eric.ed.gov/fulltext/ED471763.pdf

Research problem

Exponential and logarithmic functions are important topics in many mathematics courses. Little research has been done on student understanding of these topics.

Research questions

The author writes: "The purpose of this paper is to describe instruction designed to teach students the concepts of exponents and logarithms and to report a pilot study in which we tested the effectiveness of this instruction" (p. 1019). So the research question might be: how effective is our instruction on exponents and logarithms?

Data collection procedure

The instruction students received was designed to promote a process understanding (Dubinsky, 1991) first, and then to promote reification (acquiring a structural understanding (Sfard, 1991), reformulated by the researchers as "to understand b^x as the number that is the product of x factors of b and $\log_b m$ as the number of factors of b that are in the number m" (p. 1022)).

The participants were 15 students who had received the instruction of the researchers, and 15 students who had received traditional instruction by another instructor. The students were enrolled in a college algebra and trigonometry course at a university in the USA. After three weeks of being taught exponential and logarithmic functions, the participants volunteered to be interviewed. Students were asked for properties of exponential and logarithmic functions and to perform calculations with these functions. Open-ended

questions were also asked, to investigate students' understanding of exponential and logarithmic functions.

Findings

Students who received the instruction designed by the researchers were better at performing computations, and they were better at recalling rules and explaining why they were true. They could reconstruct rules by making use of a deep understanding of exponents and logarithms.

Appendix C: Interview Protocols

Student 6

For Question 1, how were you thinking when writing your response? You wrote down 'growth factor', why did you write that down? Some students sketch a graph, why didn't you do that?

For Question 4, how were you thinking? How did you arrive at $1 + 8^1 = 9$?

How did you think for Question 5(vi)? How did you think for Question 5(v)? For question (vi) and (v), they both have a minus sign in the power, but for Question (iv) the numbers you wrote down are positive and for (v) they are negative because you have this minus sign here, why did you make this one positive and this one negative?

Could you solve the integral $\int_0^1 \left(\frac{x^2}{e^{2x^3}}\right) dx$ and think loudly while solving? Does the integral seem familiar to you?

Student 13

For Question 1, how were you thinking when writing your response? Why did you sketch a graph? Do you find it useful?

For Question 4, how did you find out that there were two ways of solving?

For Question 5(iii), how did you find out that the numbers are negative?

Could you solve the integral $\int_0^1 \left(\frac{x^2}{e^{2x^3}}\right) dx$ and think loudly while solving? Does the integral seem familiar to you?

Student 11

For Question 1, how were you thinking when writing your response? Why did you write this down in particular? Some students sketch a graph too, why didn't you do that?

For Question 3, how did you arrive at your explanation?

For Question 5(iv), why is $(1/2)^{-15}$ equivalent to 2^{15}? Why does that rule hold?

What is according to you the meaning of $17^{1/5}$?

Could you solve the integral $\int_0^1 \left(\frac{x^2}{e^{2x^3}}\right) dx$ and think loudly while solving? Does the integral seem familiar to you?

Student 10

For Question 1, how were you thinking when writing your response? Why is the function convex increasing? Some students sketch a graph too, why didn't you do that?

For Question 4, how were you thinking when writing your response?

For Question 5(iv), why do you tell me that the number with the smaller base is the larger number?

Could you solve the integral $\int_0^1 \left(\frac{x^2}{e^{2x^3}}\right) dx$ and think loudly while solving? Does the integral seem familiar to you?

Student 14

Could you elaborate on Question 1?

For Question 4, how were you thinking when writing your response?

For Question 5(iv), how were you thinking when writing your response?

Could you solve the integral $\int_0^1 \left(\frac{x^2}{e^{2x^3}}\right) dx$ and think loudly while solving? Does the integral seem familiar to you?

Appendix D: Survey Informed Consent Form

Dear Pupils,

The purpose of the following questions is to establish your current level of understanding about some of the topics that were covered in the course. Please don't research the answers just tell us what you know.

This study is in no way connected to an evaluation of your abilities as a pupil. Your responses will have no effect on your course grade but it is important that you make a good effort in answering the questions. As you work on these questions, please show all your work, or indicate that you do not know how to respond to the question.

We will use this information strictly confidentially. Information about you as an individual will NOT be enclosed to your teachers. The results of this research study may be used in reports, presentations, and publications, but the researchers will identify you as an anonymous pupil.

There are **20** minutes allotted and **five** questions for you to answer.

Date: ____ - ____ - _____

Appendix E: Interview Informed Consent Form

Dear Student:

The purpose of this research study is to establish your current level of understanding about some of the topics that were covered in the course.

All information obtained in this study is strictly confidential.

The results of the study may be used in reports, presentations, and publications, but the researchers will not identify you. In order to maintain confidentiality of your records, an identification number will be assigned to all of your research artifacts and your name will not be used. No identifying information will appear on data (survey, interview, or other artifacts) or on the bachelor's thesis.

Participation is voluntary and you may withdraw at any time. Your decision to participate or not will NOT affect your relationship with your school, your teachers, or your grades in any way. This study is in no way connected to an evaluation of your abilities as a student. Information about individuals will not be disclosed to your school or your teachers.

The digital, de-identified data will be stored in a secure environment that will be made available to the other researchers for further study.

There are benefits related to the opportunity of discussing your experiences as student. Your participation benefits the larger community because your experiences may bring important changes in how some mathematical topics are taught at schools.

I have read the information given above.

Please sign under the proper column:

	YES	NO
I agree to be interviewed.	_____	_____
I agree my interview to be audio recorded.	_____	_____

Print Name: _____Date: _____

Appendix F: Table With Individual Survey Results

In the table below, the performance of each student (first column) can be found for each question of the survey (column 2 up to 10).

Legend:

	Correct understanding
	Incorrect understanding
	Not enough evidence to grade
+	Explains Question 3
L	Uses only linear growth for Question 4

Student	**Q2**	**Q3**	**Q4(i)**	**Q4(ii)**	**Q5(i)**	**Q5(ii)**	**Q5(iii)**	**Q5(iv)**	**Q5(v)**
1									
2									
3									
4									
5									
6									
7			**L**	**L**					
8									
9									
10									
11		+							
12									
13									
14									
15									
16									
17									