Mathematics General: Geometry Track

Bachelor Thesis

The Minimal Obstruction Problem of Ellipsoids into Balls.

A Combinatorial Approach

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Abstract

The problems of symplectic embeddings have found applications in as far-reaching areas as combinatorics. One aspect of this paper will be to describe the tools used to solve symplectic embedding problems. Those tools are Embedded Contact Homology (ECH) and ECH capacities. The ECH is defined via counts of J-holomorphic curves inside of a symplectization of a contact 3-manifold. The capacities are then derived from this homology. The main result is to reverify a minimal obstruction problem. Namely, finding those μ such that $E(1, \mu^2)$ symplectically embeds into $B^4(\mu)$. The solution to which appear as $\mu = g_{n+1}/g_n$, $n \in \mathbb{Z}_{\geq 0}$ where the g_n's are odd-indexed Fibonacci numbers. Through combinatorial methods a Diophantine equation will be derived, with solution set directly associated with solving the minimal obstruction problem. The Diophantine equation is solved via graph theory. The graph theory approach is very likely applicable to other minimal obstruction problems.

Contents

1 Introduction

The origin of the word symplectic is from Weyl who replaced the Greek root of complex, in describing a group of matrices, with the Latin, symplectic. Symplectic in this connotation thus makes sense as a pairing. What is being paired are coordinates on even-dimensional euclidean space $\mathbb{R}^{2n}$, like so $(x_1, y_1, ..., x_n, y_n) = ((x, y))$. With $\mathbb{R}^{2n}$ a canonical area/(symplectic) form can be associated $\sum_i dx_i \wedge dy_i$, this is the local picture for a symplectic manifold. Under consideration are maps that preserve this form, thus preserving area but as it happens to preserve volume as well. The idea for this thesis is to discuss the existence of maps that give more than just volume preservation, but need only a volume condition to be satisfied. The "more" is a so-called symplectic embedding. The tools used to study this phenomenon are invariants on symplectic manifolds. The relation between invariants imply relations between manifolds. Manifold for this introduction will simply be a subset of Euclidean space, see appendix A.1.

The idea, of a symplectic embedding, is to take one symplectic manifold and "push it" into a second while preserving the symplectic structure. This preservation needs to happen with respect to the structure of the second manifold. For more precise definitions see Section 2.

Why are symplectic embeddings so difficult? On the whole, conditions that concern backward operations in mathematics are easier to verify, whereas forward operations are much more difficult. For example, the image of an arbitrary intersection of sets is not necessarily the arbitrary intersection of images of those sets. Although, the pre-image always preserves intersections. A more technical example, the pullback of a vector field is always globally defined, whereas the pushforward vector field does not always exist globally. The difficulty is high enough that knowledge is concentrated on dimension 4 and in higher dimensions very little, about symplectic embeddings, is known.

The 4 dimensional (where $\mathbb{R}^4$ is identified with $\mathbb{C}^2$) and simple higher-dimensional examples are tied to the following sets, endowed with the canonical symplectic form.

Ball

$$B^{2n}(r) = \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n} \mid \pi|x|^2 + \pi|y|^2 \le r\}$$

Cylinder

$$Z^{2n}(R) = \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n} \mid \pi x_1^2 + \pi y_1^2 \le R\}$$

Ellipsoid

$$E(a, b) = \left\{(z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1\right\}$$

Polydisk

$$P(a, b) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \le a, \pi|z_2|^2 \le b\}$$

By the complexified cube we mean $C(a) = P(a, a) = D^2(a) \times D^2(a)$, $D^2(a)$ the disk of radius a. The most important volumes, coming from the canonical symplectic form, are given for the ellipsoid $E(a, b)$ by ab and the 4-ball $B^4(r)$ as r^2. When discussing embeddings we will encounter the interior of the ellipsoid Int $E(a, b)$ is the same as defined as above but with the non-strict inequality replaced by a strict inequality.

Problems of symplectic embeddings appear in ball packing problems, for example, Biran [1]. The question of symplectic embedding k disjoint balls $B^4(r)$ into $C^4(1)$ gave conditions further than volume constraints, which was unusual for the time, 1991. In the 1985 paper [2] Gromov proved a condition for a symplectic embedding of a ball $B^{2n}(r)$ into a cylinder $Z^{2n}(R)$, which exists if and only if $r \le R$. A result distinct from a volume constraint. Later, in 2009, McDuff and Schlenk [3] found a function describing all the obstructions to embedding an ellipsoid $E(1, a)$ into a ball $B^4(\lambda)$. The function turned out to have a property of independent interest, the graph contained an infinite staircase. Furthermore, in 2012 Frenkel and Muller [4] discovered by similar methods another infinite staircase. This appeared by describing the function that obstructs ellipsoid $E(1, a)$ from (symplectically) embedding into $C(\lambda)$. Finally, by combinatorial methods Cristofaro-Gardiner and Kleinman, in 2013 [5], re-verified the two previous staircases and added a third. The third dealt with the embedding problem of $E(1, a)$ into $E(2\lambda, 3\lambda)$.

On each graph of these functions there are points where the obstruction is minimal and these will be the main objects of study. The main result of this paper is to (re-)reverify these minimal obstructions for McDuff and Schlenk's staircase. This will be achievable via an ad hoc graph theory method on an equivalent formulation of the problem by Cristofaro-Gardiner and Kleinman.

The structure of the paper will begin with background material in Section 2 and a review of the problem in the next few paragraphs. Section 3 is devoted to attempting to explain the main tool to solve the problem, embedded contact homology. There is a further appendix devoted to more background on this subject matter, appendix A for review and appendix B supporting Section 2 and 3. Finally, in Section 4 we will follow the methodology of Cristofaro-Gardiner and Kleinman to do the re-verification.

We will now explain the problem of determining the conditions when a symplectic embedding can occur. More precisely, we will define the function associated to symplectically embedding one ellipsoid into another ellipsoid.

The minimal obstruction problem for ellipsoids is answering: when does the following occur?

$$\text{Int } E(a,b) \overset{s}{\hookrightarrow} E(c,d) \ \& \ ab = cd.$$

In words, when does a symplectic embedding occur between the interior Int $E(a,b)$ and $E(c,d)$ such that $ab = cd$. We say minimal because it turns out that the volume must always be preserved for symplectically embeddings so that volume equality is the minimum possible requirement. Note that the volume of the interior manifold is equal to the volume of the manifold.

Answers to this problem and less restrictive questions come from certain tool(s) of study. Firstly, for 3 dimensional contact manifolds the tool is the "ECH spectrum". The second, for (certain) 4 dimensional symplectic manifolds they are "ECH capacities".

The ECH spectrum associated to (Y, λ) a contact 3 dimensional manifold is a sequence $c_k(Y, \lambda)$:

$$0 = c_0(Y,\lambda) < c_1(Y,\lambda) \leq c_2(Y,\lambda) \leq \cdots \leq \infty.$$

The ECH capacities of a 4 dimensional symplectic manifold (M, ω), often with boundary, is a sequence $c_k(M, \omega)$:

$$0 = c_0(M,\omega) < c_1(M,\omega) \leq c_2(M,\omega) \leq \cdots \leq \infty.$$

Where ECH stands for Embedded Contact Homology, a homology on contact 3 dimensional manifolds. This homology is derived from the Embedded Contact Complex (ECC).

The axiomatic definition for symplectic capacities in Section 2.1 of Hofer-Zehnder [6] is given as follows. Let (M, ω) be a manifold of fixed dimension $2n$. A symplectic capacity is $(M,\omega) \mapsto c(M,\omega)$ a map of a symplectic manifold (M, ω) to a non-negative number or ∞, which satisfies the following properties:

(A1) Monotonicity: $c(M,\omega) \leq c(N,\tau) \Longleftrightarrow (M,\omega) \overset{s}{\hookrightarrow} (N,\tau)$

(A2) Conformality: $c(M,\alpha\omega) = |\alpha| c(M,\omega)$, for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

(A3) Weak Nontriviality: $c(B^4(1),\omega_0) > 0, \ \infty > c(Z^4(1),\omega_0)$.

With ω_0 the canonical symplectic form. Note that each element of the sequence of ECH capacities are individually examples of symplectic capacities.

The relationship between ECH capacities on (M, ω) and spectrum on (Y, λ) is this: if $Y = \partial M$ and $\omega|_Y = d\lambda$ and M is a Liouville domain then $c_k(M,\omega) = c_k(Y,\lambda)$ for all k.

Examples of computed ECH capacities:

ECH capacities for the ellipsoid, (Prop. 1.2, [7]).
The k-th term, $\mathcal{N}_k(a,b)$ is the $(k+1)$-th smallest element in the set $\{ma + nb \mid m,n \in \mathbb{Z}_{\geq 0}\}$. (Note that we count with repetitions.)

ECH capacities of the polydisk (Theorem 1.4, [7]).

$$c_k(P(a,b)) = \min\{ma + nb \mid (m,n) \in \mathbb{Z}^2_{\geq 0} \ (m+1)(n+1) \geq k+1\}.$$

Both the ellipsoid and polydisk are endowed with the canonical symplectic form.

From the monotonicity of symplectic capacities it is possible to determine when a symplectic embedding will *not* occur for a pair of two manifolds. But this is rather a large collection of examples and we are after the reverse implication.

As it so happens, McDuff proved (in [8]) that for ellipsoids ECH capacities are sharp. By sharpness we mean that *both* the monotonicity axiom holds and the axioms reverse implication. More formally,

$$\text{Int } E(a,b) \overset{s}{\hookrightarrow} E(c,d) \Leftrightarrow c_k(E(a,b)) \leq c_k(E(c,d)), \text{ for each } k \in \mathbb{Z}_{\geq 0}.$$

From the arguments on the Hofer conjecture along with the work of Muller, (Corr. 11, [9]) plus (Prop. 1.4, [4]), ECH also gives sharp obstructions to ellipsoids embedding into polydisks. Monotonicity of ECH capacities was proved in general by Hutchings, (Theorem 1.1, [7]).

We would like to study the ellipsoid embedding into ellipsoids with some more detail. By rescaling, the following occurs precisely when we already have Int $E(a,b) \overset{s}{\hookrightarrow} E(c,d)$

$$\text{Int } E(1,b/a) \overset{s}{\hookrightarrow} (1/a)E(c,d).$$

Equivalently, by replacing $E(1, b/a)$ with $E(a/b, 1)$ we would have a similar statement. So we only focus on the former and consider the problem for numbers a, b, c and d such that $|b/a| \geq 1$, and

$$\text{Int } E(1, b/a) \overset{s}{\hookrightarrow} (1/a)E(c, d)$$

holds true.

In the work of Frenkel and Muller (Lem. 2.4, [4]) an elementary number theory problem is to prove the identity $c_k(C(1)) = c_k(E(2, 1))$ for each $k \in \mathbb{Z}_{\geq 0}$. The proof allows for the unification of results in the ellipsoid case.

We define the following function on the numbers a, b, c and d with $|b/a| \geq 1$,

$$c_{a,b,c,d} := \inf\{\lambda \mid \text{Int } E(1, b/a) \overset{s}{\hookrightarrow} (1/a)E(\lambda c, \lambda d)\}.$$

The way to intuitively understand the above function is to repeat to oneself that we are searching for the scaling for a symplectic embedding to still occur, given real numbers a, b, c and d.

We can simplify this expression by saying that $(1/a)$ is enveloped by the scale factor λ. Moreover, there is no need for dependence on b *and* a so we choose one-parameter, say a. We will now restrict to $c = 1$ and $d = k/l$ for integers k, l,

$$c_{a,k,l} = \inf\{\lambda \mid \text{Int } E(1, a) \overset{s}{\hookrightarrow} E(\lambda, \lambda k/l)\}$$

There are three (k, l) that we will consider $(k, l) = (1, 1)$ from the primary paper McDuff and Schlenk [3], $(k, l) = (2, 1)$ in the joint work of Frenkel and Muller [4]. Also, by an alternative approach investigated in section 5, $(k, l) = (3, 2)$ from Cristofaro-Gardiner and Kleinman [5].

All these three have some special features in common, namely:

(i) The function $c_{a,k,l}$ is bounded below by a curve called the volume curve.

(ii) The graph of the function $c_{a,k,l}$, $G(c_{a,k,l})$, has a sub-graph $F(c_{a,k,l})$ consisting of an infinite number of alternating horizontal and sloped lines. Fig 1

(i) We certainly have $E(1, a) \overset{s}{\hookrightarrow} E(c_{a,k,l}, c_{a,k,l} \cdot (k/l))$ as $\text{Int } E(1, a) \overset{s}{\hookrightarrow} E(\lambda, \lambda k/l)$, for all $\lambda \geq c_{a,k,l}$. Thus, because of volume preservation,

$$\text{vol}(E(1, a)) \leq \text{ vol}(E(c_{a,k,l}, c_{a,k,l} \cdot (k/l))) \Rightarrow a \leq c^2_{a,k,l}(k/l) \Rightarrow c_{a,k,l} \geq \sqrt{\frac{a}{\text{vol}(E(1, k/l))}} = \sqrt{\frac{la}{k}}.$$

The volume curve is then $\lambda(a) = \sqrt{la/k}$.

(ii) The heights of the horizontal and the slopes of the lines carry with them an associated sequence.

For example the case $(k, l) = (1, 1)$ gives $E(k, l) = E(1, 1) = B(1)$ and the sequence associated to the staircase sub-graph $F(c_{a,k,l})$ are the odd-index Fibonacci numbers. The Fibonacci numbers are the integers recursively defined by $f_0 = f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ and by odd-index we mean $g_n = f_{2n-1}$, $n \geq 1$. We care about defining this derived sequence independently and write $g_0 = g_1 = 1$ and require the rule $g_{n+1} = 3g_n - g_{n-1}$.

The final section will go some way to explaining why the g_n's appear. The staircase looks somewhat like the following figure.

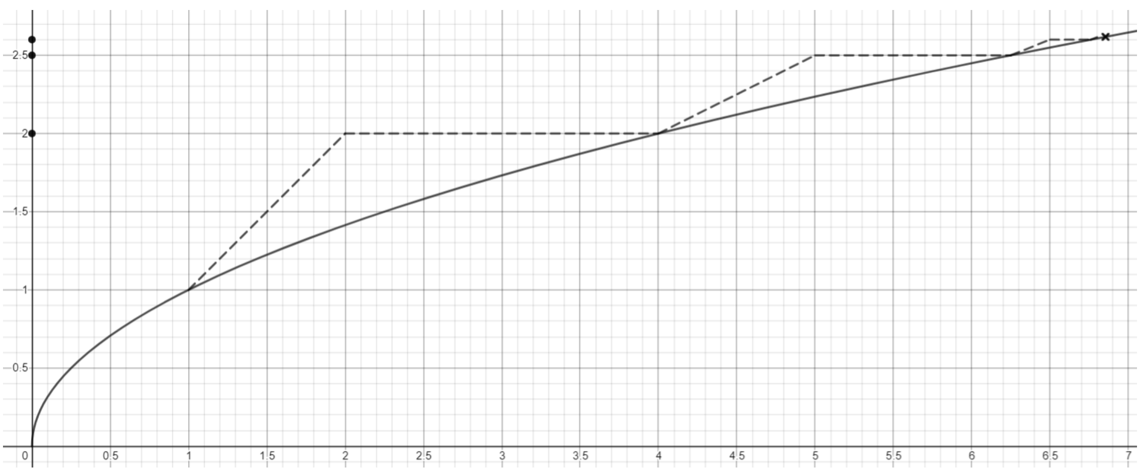

Figure 1: Fibonacci Stairs, λ vertical, a horizontal

The above graph was reverse engineered by knowing the associated sequence and taking sloped lines through origin. The lines then stop at the heights determined by the g_{n+1}/g_n, $n \in \mathbb{Z}_{\geq 0}$. Notice the points that are concave meetings of the horizontal and sloped lines on the sub-graph $F(c_{a,1,1})$. Morally these points should be called "interior corners", because they are interior when considering the shapes between the sub-graph and the volume curve. In this case the interior corners are $(g_{n+1}^2/g_n^2, g_{n+1}/g_n)$, because $\lambda = g_{n+1}/g_n$ the heights of the stairs and the volume curve $\lambda = \sqrt{a}$. The interior corners give a discrete set of solutions to the minimal obstruction problem with $a = 1$, $b = \mu^2$ and $c = d = \mu$. Thus, instead, to emphasize this property we will say the minimal obstruction points. The minimal obstruction points now coincide with the verification that will occur in this thesis.

As a reminder we search for those $\mu \geq 1$ such that

$$E(1, \mu^2) \overset{s}{\hookrightarrow} B(\mu).$$

Proving that $\mu = \mu_n = g_{n+1}/g_n$ for each $n \in \mathbb{Z}_{n\geq 0}$ verifies the discrete set of solutions to the minimal obstruction problem.

The reason why we chose to mention that ECH capacities of $E(2,1)$ and $C(1) = P(1,1)$ are equal is because we would like to have a similar function for embedding an ellipsoid into the complexified cube. The resulting staircase was studied by Frenkel and Muller in [4]. The next figure is part of that staircase.

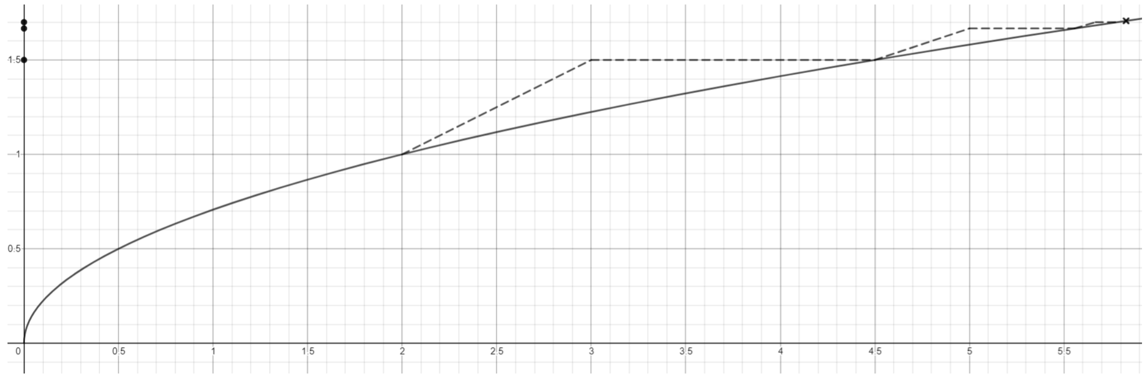

Figure 2: Pell Stairs, λ vertical, a horizontal

Again we can see the volume curve and the minimal obstructions points lying on it. This time the heights are defined via the Pell numbers, another recursively defined integer sequence.

The method applied to computing these staircases explicitly, developed by McDuff, will not be discussed in this thesis. What we will do instead is layout the foundation to compute the minimal obstruction points of the first of two staircases above. As we have mentioned already we will follow the combinatorial methods worked out by Cristofaro-Gardiner and Kleinman, wherein those authors also add a third staircase.

2 Symplectic, Contact and Holomorphic Structures

The purpose of the first half of this section is to summarize and give the basics of symplectic and contact geometry. The second half is dedicated to understanding J-holomorphic curves with relevant references publicized when we arrive there. The latter half of the story of ECH will be told in Section 3. Primary references for these first two subsections will be from Wendl's notes [10] and the book of Hofer and Zehnder [6]. Throughout, appendix A will be an invaluable resource.

2.1 Symplectic Geometry

To begin with symplectic geometry we must first define a symplectic vector space.

A real vector space V of dimension m can be equipped with a skew-symmetric bilinear form which we denote by Ξ. From skew-symmetry we may infer $\Xi(v, u) = -\Xi(u, v)$, $\forall u, v \in V$. As Ξ is bilinear $\Xi : V \times V \to \mathbb{R}$ and define $\hat{\Xi}$ such that $\hat{\Xi}(v)(u) = \Xi(v, u)$. Then, with $\hat{\Xi} : V \to V^*$, we interest ourselves in Ker $\hat{\Xi}$. In analogy to the Gram-Schmidt procedure for symmetric bilinear forms we have a "canonical" basis on V via Ξ (Theorem 1.1, [11]). Namely, $(u_1, ..., u_k, e_1, ..., e_n, f_1, ..., f_n)$ is a basis for V such that

$$\begin{aligned} &\Xi(v, u_i) = 0 \text{ for all } v \in V, \text{ and } i \in \{1, ..., k\}, \\ &\Xi(e_i, f_j) = \delta_i^j, \\ &\Xi(e_i, e_j) = \Xi(f_i, f_j) = 0 \text{ for all } i, j \in \{1, ..., n\}. \end{aligned}$$

The skew-symmetric form Ξ is customarily written in matrix form by

$$[\text{—}u\text{—}]\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \mathbb{1}\\ 0 & -\mathbb{1} & 0\end{bmatrix}\begin{bmatrix}|\\ v\\ |\end{bmatrix}.$$

The horizontal and vertical vectors are represented on the above basis. From this basis it follows that $u_1, ...u_k$ spans Ker $\hat{\Xi}$ and thus when $k = 0$ (Ker $\hat{\Xi} = \{0\}$) we have $m = 2n$. The map $\hat{\Xi}$ being injective is an algebraic condition that we impose on Ξ to make (V, Ξ) into a symplectic vector space (Def. 1.3, [11]). Injectivity is equivalent to bijectivity in this case by letting a functional Φ act on the basis $\{e_i, f_i\}$ and then choosing a suitable v such that $\hat{\Xi}(v) = \Phi$.

By following a procedure like above it is possible to identify (V, Ξ) any symplectic vector space with the standard symplectic vector space $(\mathbb{R}^{2n}, \Xi_0)$. Where on $(\mathbb{R}^{2n}, \Xi_0)$, $e_i = \underbrace{(0, ..., 1, ..., 0)}_{i\text{-th slot}}$, $f_i = \underbrace{(0, ..., 1, ..., 0)}_{(n+i)\text{-th slot}}$ so that Ξ_0 acts as

$$u^T\begin{bmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{bmatrix}v.$$

The question of determining which matrices preserve this basis is similar to asking which matrices preserve an orthogonal basis, precisely those orthogonal matrices. We call a matrix A symplectic if

$$A^T\begin{bmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{bmatrix}A = \begin{bmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{bmatrix}.$$

Using this discussion around symplectic vector spaces we can take a 2-form ω on an even dimensional manifold and use it is to define a symplectic manifold. Starting with a smooth manifold M on each tangent space T_pM we attach a skew-symmetric bilinear form ω_p, a 2-form at the point p. Requiring that $\hat{\omega}_p$ be injective for each $p \in M$ implies dim M = dim $T_pM = 2n$. The further condition we will impose on ω is for it to be closed ($d\omega = 0$), which in contrast to non-degeneracy is a differentiable (or analytical) condition. The non-degeneracy condition, $\hat{\omega}_p = \iota_\bullet\omega_p$ must be injective $\forall p \in M$ (ι interior multiplication), can be written more compactly as Ker $\iota_\bullet\omega \equiv \{0\}$.

We collect these conditions into a definition.

Definition 1. A *symplectic manifold* is a pair (M, ω) with M a $2n$-dimensional smooth manifold, and ω a 2-form satisfying the following properties:

1. Closure ($d\omega = 0$), and
2. Non-degeneracy, Ker $\iota_\bullet\omega \equiv \{0\}$.

The non-degeneracy condition can be rephrased to give a stronger feeling for its importance. A volume form on a manifold M is a top-form (degree equal to dim(M)) and is non-vanishing. We have that, ω is non-degenerate if and only if $(1/n!)\omega^n = (1/n!)\omega \wedge \cdots \wedge \omega$ (n-times) defines a volume form. With this volume form we are obviously able to define volume, although a more subtle implication is that every symplectic manifold is orientable.

To see why we obtain a volume form we return to symplectic vector spaces. In general, if we have (V, Ξ), (V, $2n$-dimensional) with Ξ non-degenerate then from the basis $\Xi = \sum_i e_i^* \wedge f_i^*$, $e_i^*, f_i^* \in V^*$. The wedge product ($\wedge$) is just the pointwise wedge product of the appendix. We care about computing $\Xi^n = \Xi \wedge \cdots \wedge \Xi$ (n-times).

$$\Xi^n = \left(\sum_i e_i^* \wedge f_i^*\right)^n = \sum_{i_j \text{ distinct}} e_{i_1}^* \wedge f_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge f_{i_n}^*$$
$$= \sum_{\sigma \in S_n} e_{\sigma(1)}^* \wedge f_{\sigma(1)}^* \wedge \cdots \wedge e_{\sigma(n)}^* \wedge f_{\sigma(n)}^*$$

as $e_{\sigma(i)}^* \wedge f_{\sigma(i)}^* \wedge e_{\sigma(j)}^* \wedge f_{\sigma(j)}^* = e_{\sigma(j)}^* \wedge f_{\sigma(j)}^* \wedge e_{\sigma(i)}^* \wedge f_{\sigma(i)}^*$ then

$$= \sum_{\sigma \in S_n} e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* = (n!) e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*.$$

Implying, $(1/n!)\Xi^n = e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*$ which is non-vanishing because $\{e_i, f_i\}$ is a basis for V. By taking $\Xi = \omega_p$, ω a symplectic form, and $V = T_pM$ we can smoothly vary the basis to create a frame $\{e_i, f_i\}$ and dual frame $\{\text{ev}_{e_i}, \text{ev}_{f_i}\}$. With ev coming from the appendix. Due to the arbitrarily chosen basis this result requires some scrutiny.

Given two symplectic manifolds we call φ a symplectic map between (M_1, ω_1) and (M_2, ω_2) if $\varphi^*\omega_2 = \omega_1$. We call φ a symplectomorphism if it is a symplectic map and a diffeomorphism. The following fundamental theorem by Darboux says that locally symplectic structures are symplectomorphic to a canonical structure.

Theorem 1 (Darboux, Theorem 1, [6]). *Suppose ω is a non-degenerate 2-form on a $2n$-dimensional manifold M. Then ω is closed if and only if at each $p \in M$ there is a chart $(U, \varphi) = (U, x_1, ..., x_n, y_1, ..., y_n)$, where φ has co-domain $U \subset M$, $\varphi(0) = p$ and*

$$\varphi^* \omega = \omega_0 = \sum_i dy_i \wedge dx_i.$$

One way to think about the significance of this theorem is that a symplectic manifolds can be characterized by being locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. We call, from now on, ω_0 the canonical symplectic form. From the equivalence between ev and d the exterior derivative we can follow the same argument as above to state that $\omega_0^n = (\sum_i dy_i \wedge dx_i)^n = n!(dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n)$. As the pullback is linear $(\varphi^*\omega)^n = \varphi^*(\omega^n) = n!(dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n)$. A chart on a symplectic manifold then has volume given by

$$\frac{1}{n!} \int_U \varphi^*(\omega^n) = \frac{1}{n!} \int_{\varphi(U)} (\varphi^{-1})^*(\varphi^*(\omega^n)) = \frac{1}{n!} \int_{\varphi(U)} \omega^n.$$

In this specific case, φ being a symplectic map, there is volume preservation between domain and co-domain of φ. Volume preservation implies that the volume of the domain cannot exceed the volume of the codomain for a symplectic map. The importance of this was already seen in the minimal obstruction problem of the introduction.

Definition 2 (Definition 1.7, [10]). A symplectic map φ is a *symplectic embedding* between (M_1, ω_1) and (M_2, ω_2) if $\varphi(M_1)$ is a symplectic submanifold of (M_2, ω_2) (a submanifold where restricting ω_2 to $\varphi(M_1)$ stays non-degenerate).

Equivalently, φ is a symplectic embedding when it is a symplectic map and an embedding. From the appendix we know this means φ has differential is everywhere injective and φ is a homeomorphism onto its image, as well as being symplectic. The second of these is vital to defining a topology on the image $\varphi(M_1)$ from the topology of M_2. The notation used to say that there exists a symplectic embedding between (M_1, ω_1) and (M_2, ω_2) is

$$M_1 \overset{s}{\hookrightarrow} M_2.$$

The existence of a symplectic embedding is a difficult problem in the general case and as was discussed in the introduction only simple cases in 4 dimensions have been understood fully. As a reminder, the goal of this thesis is to search for the discrete set of μ that solve the problem

$$E(1, \mu^2) \overset{s}{\hookrightarrow} B^4(\mu).$$

The above problem is a minimal obstruction problem because the volume of the domain is equal to the volume of the codomain.

2.2 Contact Geometry

The twin brother to symplectic geometry is contact geometry. We will require a basic understanding of contact geometry to define the complex for embedded contact homology. Although it should be remarked that contact geometry is a tool to be able to define invariants for the symplectic case, the one we really care about for applications.

To begin we introduce a star-shaped domain. These are compact subsets $U \subset \mathbb{R}^{2n}$ with boundary, such that there exists $p \in U$ where the boundary is transverse to the vector field $V = (x_i - x_i(p))(\partial/\partial x_i) + (y_i - y_i(p))(\partial/\partial y_i)$.

By way of example, the disk in the plane $\mathbb{R}^2$ is a star-shaped domain. The radial vector field centered at the disk's origin meets perpendicularly the tangent to the disk's boundary. Intuitively, this is what transverse means, at least in $\mathbb{R}^2$. At a point on the boundary the space of tangent vectors on $\mathbb{R}^2$ can be spanned by the tangent vector to the disk's boundary and the radial vector field at that point.

Let (M, ω) be any symplectic manifold. We say V is a Liouville vector field on (M, ω) if $\mathcal{L}_V \omega = \omega$, the Lie-derivative with respect to V preserves ω. By Cartan's formula, $\mathcal{L}_V \omega = d(\iota_V \omega) + \iota_V d\omega = d(\iota_V \omega)$ as ω is closed. Thus to show V is a Liouville vector field it amounts to show $d(\iota_V \omega) = \omega$.

A hypersurface, co-dimension 1 submanifold, Y of a symplectic manifold (M, ω) is of contact type if it is transverse to a Liouville vector field. In particular, the standard Liouville vector field is $V = x_i \partial/\partial x_i + y_i \partial/\partial y_i$ and so the boundary of a starshaped domain (with p the origin) is a hypersurface of contact type.

Definition 3 (Definition 2.2, [12]). A *Liouville domain* is a compact symplectic manifold with boundary (M, ω), along with a Liouville vector field V that points transversally out of the boundary.

Therefore, if (M, ω) is a Liouville domain, ∂M is of contact type.

Definition 4 (Definition 1.35, [10]). Let Y be a manifold of odd dimension $(2n-1)$ and let λ be a 1-form. Denote, what will be called, the *contact structure* on Y by $\xi = \text{Ker } \lambda$. If λ is such that $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on ξ then the pair (Y, λ), or more commonly (Y, ξ), is a *contact manifold*.

For our purposes we will only need to consider contact manifolds as the boundary of symplectic manifolds. Therefore, in the following let (M, ω) be a Liouville domain and $(Y = \partial M, \lambda)$ a contact manifold where $\omega = d\lambda|_Y$. We are now prompted to consider the flow of the Liouville vector field. Consider generally the general flow of some vector field X.

Definition 5 (Definition 14.9, [13])**.** The *(local) flow* $\varphi^t(x)$ of a vector field X is defined by the solution to the following initial value problem

$$\frac{d\varphi^t}{dt} = X\varphi^t,$$
$$\varphi^0 = \mathbb{1}.$$

The flow is valid in a local neighbourhood of the origin $(-\varepsilon, \varepsilon)$, for some small $\varepsilon > 0$.

The flow of a vector field in a sufficiently small neighbourhood of zero exists from the Picard-Lindelöf theorem. Now let $\varphi^t_V(x)$ be the flow of a Liouville vector field V and pick $\varepsilon > 0$ small such that the map

$$\Phi : (-\varepsilon, \varepsilon) \times Y \to M, \quad (t, x) \mapsto \varphi^t_V(x)$$

is an embedding. We can now readily give a symplectization of Y by $\mathbb{R} \times Y$. By symplectization we mean to take a contact manifold and add additional structure to create a symplectic manifold. There are other symplectization but we follow Hutchings and Taubes in [14].

Proposition 1. *One symplectization of a contact manifold Y is given by $\mathbb{R} \times Y$ where the symplectic form is $\omega = d(e^t\omega)$. This will be proved in the following two steps*

1. $(\varphi^t_V)^*\omega = e^t\omega$,
2. $\Phi^*\omega = d(e^t\hat{\lambda})$, *(where $\hat{\lambda} = \pi_2^*\lambda$ has been pulled back to $(-\varepsilon, \varepsilon) \times Y$ by canonical projection, π_2).*

Proof.

1. We use the derivative formula for the pullback of the flow,
$$\frac{d}{dt}[(\varphi^t_V)^*\omega] = (\varphi^t_V)^*(\mathcal{L}_V\omega) = (\varphi^t_V)^*\omega.$$
Assuming $(\varphi^t_V)^*\omega = f(t)\omega$ then $\varphi^0_V = \mathbb{1} \Rightarrow f(0) = 1$. By the above relation $f'(t) = f(t) \Rightarrow f(t) = e^t$. So we arrive at $(\varphi^t_V)^*\omega = e^t\omega$.
2. Note that because $\omega|_Y = d\lambda$ and the relationship for ω, then $\iota_V\omega|_Y = \lambda$. We will first prove that $\Phi^*(\iota_V\omega|_Y) = e^t\hat{\lambda}$. As $\Phi = \varphi^t_V \circ \pi_2$ then
$$\Phi^*(\iota_V\omega|_Y) = \pi_2^*(\varphi^t_V)^*(\iota_V\omega|_Y) = \pi_2^*(\varphi^t_V)^*(\omega(V, \cdot)|_Y)$$
$$= \pi_2^*(e^t\omega(V, \cdot)|_Y) = e^t\pi_2^*(\iota_V\omega|_Y) = e^t\pi_2^*\lambda = e^t\hat{\lambda}.$$
Therefore, as $d\Phi^*(\iota_V\omega|_Y) = \Phi^*d(\iota_V\omega|_Y) = \Phi^*\omega$ we must have $\Phi^*\omega = d(e^t\hat{\lambda})$. From now on though we write λ instead of $\hat{\lambda}$ on $(-\varepsilon, \varepsilon) \times Y$. We have a symplectic form $d(e^t\lambda)$ on $(-\varepsilon, \varepsilon) \times Y$ which we can (smoothly) extend to the whole of $\mathbb{R} \times Y$.

□

Later, when we will explain the ECC (embedded contact complex) this symplectization will be needed. Finally, for this section we need the definition of a Reeb vector field on a contact manifold.

Definition 6 (Definition 1.40, [10])**.** Let Y be a contact manifold and λ the contact 1-form. A *Reeb vector field* R has uniquely determined direction by two properties

$$d\lambda(R, \cdot) = 0 \quad \text{and} \quad \lambda(R) = 1.$$

It will turn out that the flows of the Reeb vector field will be very important in defining ECC. Actually, we will be interested in those closed loops γ satisfying $\dot{\gamma} = R\gamma$. Within the context of mechanics these are analogous to the classical solutions to the action functional. The analogous action function is the symplectic action, namely

$$\mathcal{A}(\gamma) = \int_\gamma \lambda.$$

Via this idea we could interpret ECH as an analogue of a physical theory, which turns out to be almost true through its relationship to monopole Floer homology. The last homology being a physically realizable theory to do with counting magnetic monopoles, found as solutions to the Seiberg-Witten equations, see Kronheimer and Mrowka [15].

2.3 Holomorphic Structures

This subsection will explain J-holomorphic curves and their relation to symplectic topology. Primarily the reference list consists of lecture notes of Hutchings and Taubes [16], separately, Wendl [10], and the book of McDuff and Salamon [17], along with other texts being referenced as all the details cannot be included. We are also heavily supported by appendix B.

The complexification of a vector bundle is the general case of a more specific notion we will encounter again and again in the forthcoming subsection. It is defined by $E_{\mathbb{C}} = E \otimes \mathbb{C} = \{v \otimes 1 + w \otimes i \mid v, w \in E\}$ (page 14, [18]). This is the first instance of a complex structure on a vector bundle. A complex structure i is an endomorphism of a bundle E, mapping the "real" part to "imaginary" part, and vice versa, for which $i^2 = -\mathbb{1}$. Globally i (or $\mathbb{J}_{std}$) is a block diagonal matrix with all blocks as $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Complex structures naturally arise from complex manifolds. Complex manifolds generalize smooth manifolds M of even dimension $2n$. The extra stipulation on M, being locally biholomorphic (holomorphic diffeomorphsim with holomorphic inverse) to $\mathbb{C}^n$ along with transition maps also being biholomorphic. Let M and N be two complex manifolds each with induced complex structures on their tangent bundles J, J'. We say $u : M \to N$ is holomorphic if it satisfies the following

$$du \circ J' = J \circ du. \tag{2.1}$$

Where du is an overuse of notation as here what is meant is the differential or pushforward of the map u. If ever du is meant as a 1-form such a situation will be indicated. The other symbol to explain is $\circ$ which is just the composition of maps.

Locally this situation can be modelled by $M = U \subset \mathbb{C}^m$ open and $N = \mathbb{C}^n$ which cause (2.1) to turn into the multidimensional analogue of the Cauchy-Riemann equations.

An almost complex structure J is map from the tangent bundle TM to TM, of a manifold M, which has the property $J^2 = -\mathbb{1}$ and is locally a complex structure. If an almost complex structure is induced by a collection of holomorphic coordinates then it is called integrable and is equivalent to a complex structure. When J, an almost complex structure, is attached to TM then (M, J) is called an almost complex manifold. It is a difficult question to answer whether an almost complex structure is integrable. A special case is for complex dimension 1 manifolds where all almost complex structures are integrable.

Definition 7. A *Riemann surface* Σ is a complex manifold of complex dimension 1.

It is then possible to use the induced complex structure j and write the pair (Σ, j). Presently we give the definition of a J-holomorphic curve in the particular case of the domain being a Riemann surface (Σ, j).

Definition 8. A map u from Riemann surface Σ to an almost complex manifold M with almost complex structure J must satisfy the non-linear partial differential equation

$$du \circ j = J \circ du,$$

to be called a *J-holomorphic curve.*

One object on bundles we make use of is called a bundle metric $\langle , \rangle$ on E. That being a smooth fiberwise inner product on a vector bundle E over a manifold M. A bundle metric on a complex vector bundle is called Hermitian if the inner product on each fiber is complex linear in the first entry and anti-complex linear in the second.

Let (M, ω) be a symplectic 4 dimensional manifold with an almost complex structure J on its tangent bundle. We say J is ω-tame if $\omega(v, Jv) > 0$ for all $v \neq 0$. We also say J is ω-compatible if $\omega(\cdot, J\cdot) = g(\cdot, \cdot)$ defines a (not necessarily symmetric) Riemannian metric on TM. All compatible almost complex structures are tame but the opposite is not true, unless J is also ω invariant i.e. $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$. Note that a Riemannian metric is an example of a bundle metric on the tangent bundle. A (smooth) manifold with a Riemannian metric is called a Riemannian manifold, distinct from a Riemann surface.

A Riemannian metric g induces a vector bundle isomorphism between TM and T^*M. In Calculus we have encountered the gradient vector field and we define a similar object by being the unique vector field to satisfy $g(\text{grad} f, \cdot) = df$.

With the pull-back already defined for forms it is also possible to define pullbacks on any vector bundle and construct the pull-back bundle. If $f : N \to M$ is a smooth map of manifolds and we have $E \xrightarrow{\pi} M$, with trivializations $\{\phi\}$ with trivial covering $\{U\}$, then f^*E is the pull-back bundle with trivializations $\{f^*\phi\}$ and trivial covering $\{f^{-1}(U)\}$. Specifically, we will need the pullback bundle u^*TM of the tangent bundle TM over M by a J-holomorphic curve, $u : (\Sigma, j) \to (M, J)$. Note that u^*TM is a bundle over Σ.

We define what it means to be a connection on the tangent bundle, the more general definition appears in the appendix B.1.

Definition 9. A (Koszul) *connection* (∇ on TM) is a bilinear map from $\mathcal{X}(M) \times \mathcal{X}(M)$ to $\mathcal{X}(M)$ that satisfies the following two properties:

$$\begin{aligned} \nabla_X(hY) &= h(\nabla_X Y) + Y(dh(X)), && \text{Leibnitz rule} \\ \nabla_{hX} Y &= h(\nabla_X Y). && C^\infty\text{-linear} \end{aligned}$$

Where $h \in C^\infty(M)$ and $X, Y \in \mathcal{X}(M)$.

A connection ∇ is called symmetric if M has Riemannian metric g and if $\nabla_X Y - \nabla_Y X = [X, Y]$ ($[\_, \_]$ the commutator bracket for vector fields). On a Riemannian manifold, M, there exists the fundamental Levi-Cita connection. This connection is symmetric and defined on a orthogonal frame $\{e_1, ..., e_n\}$ of TM (guaranteed by g) as such

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k.$$

The symbols Γ^k_{ij} are called the Christoffel symbols and symmetry implies $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all $1 \leq i, j, k \leq n$.

We want to be able to know more about the properties of J-holomorphic curves through analysis techniques of non-linear PDEs. We intend to work in a subspace of smooth maps between a Riemann surface (Σ, j) and some yet to be determined symplectic manifold (M, J). This subspace contains those smooth maps that will satisfy a non-linear PDE for which the homogeneous solutions will be J-holomorphic curves.

For now though we focus on a general complex vector bundle (E, J) of complex rank n. This vector bundle admits a holomorphic structure if Σ has an open covering $\{U_\alpha\}$ with complex-linear trivializations $E_{U_\alpha} \to U_\alpha \times \mathbb{C}^n$ whose transition maps are holomorphic. These transition maps are defined on open sets of Σ and map into $\mathrm{GL}(n, \mathbb{C})$. (2.4, [10])

For a smooth function $f \in C^\infty(\Sigma, \mathbb{C}) = C^\infty(\Sigma)$ we have

$$\begin{aligned} \overline{\partial} &: f \mapsto df + i df \circ j, \\ \partial &: f \mapsto df - i df \circ j. \end{aligned} \tag{2.2}$$

We call f holomorphic if $\overline{\partial} f = 0$ (and anti-holomorphic if $\partial f = 0$). To make sense of this formula it would be prudent to take $s + it$ complex coordinates on Σ. We have j acting by

$$j\frac{\partial}{\partial s} = \frac{\partial}{\partial t}, \quad j\frac{\partial}{\partial t} = -\frac{\partial}{\partial s}.$$

Then evaluate $\overline{\partial} f$ by the argument $\partial/\partial s$ to obtain,

$$\frac{\partial f}{\partial s} - i\frac{\partial f}{\partial t}.$$

With $\partial/\partial t$ as argument the result is similar. Then stipulating that $\overline{\partial} f = 0$ results again in the Cauchy-Riemann equations. There is an implicit complex anti-linearity happening in evaluating so defining the following map makes sense

$$\overline{\partial} : C^\infty(\Sigma) \to \Gamma(\overline{\mathrm{Hom}}(T\Sigma, \mathbb{C})).$$

The codomain is $\Gamma(\overline{\mathrm{Hom}}(T\Sigma, \mathbb{C}))$, smooth sections of complex anti-linear maps from $T\Sigma$ to $\mathbb{C}$.

If (E, J) has a holomorphic-structure, then define

$$\begin{aligned} \overline{\partial} &: \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}(T\Sigma, E)), \\ s &\mapsto ds + J \circ ds \circ j. \end{aligned}$$

This is in analogy to extending real valued differential k-forms to E-valued k-forms. Define $s \in \Gamma(E)$ to be holomorphic if $\overline{\partial} s = 0$, where an overuse of notation happens again as ds is the pushforward of s. The eventual application is for $E = u^*TM$ the pullback of the tangent bundle over M, a bundle over Σ.

The following will be a sharp spike in difficultly, it is then advisable to step back and explain where we are going. The purpose of introducing J-holomorphic curves is due to them being one half of the story of ECH. The other half coming from contact geometry. The essence of the following is to take a top-down approach to understand the space in which J-holomorphic curves reside. Although inherently infinite dimensional our goal will be is to realize this space as a finite dimensional smooth manifold. The dimension of which will follow, in part, from Riemann-Roch which we will encounter soon. This finite number will then be restricted further until the point at which the space becomes discrete.

Define the non-linear operator acting on a smooth function u from (Σ, j) to (M, J) by $\overline{\partial}_J u := du + J \circ du \circ j$. We care about J-holomorphic maps u so that we ask for these maps to satisfy $\overline{\partial}_J u = 0$, homogeneous solutions. These homogeneous solutions will be a subspace with extra conditions

$$\mathcal{M}_J = \{u \in C^\infty(\Sigma, M) \mid \partial_J u = 0 \ \& \ \dots\} \subset \overline{\partial}_J^{-1}(\{0\}).$$

Where ... are a replacement for extra conditions to be investigated soon. This will be the basis for the moduli-space of J-holomorphic curves.

Theorem 2. *The map $\overline{\partial}_J$ is a smooth section of the bundle $\mathcal{E} \xrightarrow{\hat{\pi}} \mathcal{B}$. Where $\mathcal{B} := C^\infty(\Sigma, M)$ and $\mathcal{E}$ has fibers $\hat{\pi}^{-1}(u) = \mathcal{E}_u := \Gamma(\overline{\mathrm{Hom}}(T\Sigma, u^*TM))$. Furthermore, the linearization, at a particular u, of this section is a map which acts on $\eta \in \Gamma(u^*TM)$ by*

$$D\overline{\partial}_J(u)\eta = \mathcal{D}_u\eta = \nabla\eta + J(u) \circ \nabla\eta \circ j + (\nabla_\eta J)du \circ j.$$

The linearization contains some important terms, 1. ∇ is a connection chosen on the manifold M, and the linearization is independent of this connection. 2. The symbol $J(u)$ is the pullback of the almost complex structure to u^*TM. 3. We are able to talk about $(\nabla_\eta J)$ as connections have a defined action on $(1,1)$-tensors, of which J is an example. For information about linearization and the section $\overline{\partial}_J$ itself see appendix B.3. Now to explain what this theorem is stating.

We start with a smooth curve $u : \Sigma \to M$ so that $u \in \mathcal{B}$. Then we pre-image by $\hat{\pi}$ and obtain the fiber $\hat{\pi}^{-1}(u) = \mathcal{E}_u$. This fiber is now the smooth sections of another bundle namely the bundle $\overline{\mathrm{Hom}}(T\Sigma, u^*TM)$ over Σ. Sections of $\mathcal{E}$ over $\mathcal{B}$ are maps taking smooth functions from Σ to M to complex anti-linear maps, from $T\Sigma$ to u^*TM. It is the case that $\overline{\partial}_J$ is a well-defined section of our bundle, where smoothness is implied by smoothness of J, as a map from TM to TM.

The technicalities of this theorem appear in the space $\mathcal{B}$ and the bundle $\mathcal{E}$. Both should be complete but as of now are not. Actually they need to be, the for now mysterious, Banach manifold and Banach space bundle, respectively. The reason why is, in part, the want to apply the infinite dimensional analogue of the inverse function theorem. This problem is solved via Sobolev completions. In the following, $W^{k,p}(\Omega)$ is the Sobolev completion of Ω. Where k is the class of differentiability and p references the space of L_p-integrable functions, see appendix B.4.

The following are the truer elements of this theorem

$$\begin{aligned} \mathcal{B}^{k,p} &= W^{k,p}(\Sigma, M) && \text{(base-space)} \\ \mathcal{E}^{k-1,p} & && \text{(total-space)} \\ \mathcal{E}_u^{k-1,p} &:= W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)) && \text{(fibers)} \end{aligned}$$

The smooth section

$$\overline{\partial}_J : \mathcal{B}^{k,p} \to \mathcal{E}^{k-1,p} : u \mapsto du + J \circ du \circ j,$$

(page 102, [10]). Where smoothness persists due to J being smooth. A map $\mathcal{B}^{k,p} \to \mathcal{E}^{k-1,p}$, decreases the class of differentiability by 1, which makes sense because of the way $\overline{\partial}_J$ acts.

Fact: $\overline{\partial}_J^{-1}(\{0\})$ is independent of k and p. This follows from the space of solutions having a C^∞ topology that is equivalent to a $W^{k,p}$ topology under conditions of elliptic regularity (page 99 and Section 2.5, [10]). It is possible through the inverse function theorem on Banach manifolds to carry out the linearization from the appendix. Thus, for $u \in \overline{\partial}_J^{-1}(\{0\})$

$$\begin{aligned} \mathcal{D}_u : W^{k,p}(u^*TM) &\to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)), \\ \eta &\mapsto \nabla\eta + J(u) \circ \nabla\eta \circ j + (\nabla_\eta J)du \circ j. \end{aligned}$$

Where again an arbitrary connection ∇ on M has been chosen.

This map is identical to $D\overline{\partial}_J(u)$ (the linearized map) under C^∞ being dense in $W^{k,p}$. If $\mathcal{D}_u$ has a bounded right inverse and is surjective then Ker $(\mathcal{D}_u)$ is finite dimensional. Surjectivity is necessary for transverse intersections with the zero section, and also implies that the Coker $(\mathcal{D}_u) := Y/\mathrm{Im}(\mathcal{D}_u)$ has dimension 0. Via the linearization we may obtain a local understanding of the set $\overline{\partial}_J^{-1}(\{0\})$ (and restrictions upon it) from Ker $(\mathcal{D}_u)$. Thus studying the kernel of these operators imply local properties on the moduli space of J-holomorphic curves.

An operator $D : X \to Y$ of Banach spaces X, Y is said to be a Fredholm operator if Ker (D) and Coker $(D) := Y/\mathrm{Im}(D)$ are finite dimensional. In general, the index of a Fredholm operator (called the Fredholm index) is defined by (page 103, [10])

$$\text{ind } (D) := \dim \text{Ker}(D) - \dim \text{Coker}(D).$$

We will need the Chern number associated to a 2-cycle in the following theorem. If M is a symplectic manifold and $A \in H_2(M)$ and $c_1(E)$ the first Chern class on a vector bundle E over M, appendix B.1. Then,

$$c_1(A) = \langle c_1(E), A \rangle = \int_M c_1(E) \wedge \mathrm{PD}(A)$$

where PD stands for the (homological) Poincaré dual of appendix B.2.

We now state the promised theorem, Riemann-Roch. The importance of this theorem, along with Fredholm theory, is in allowing the step from infinite dimensional to finite dimensional moduli spaces. Thus, everything that follows about reducing the dimension of the moduli space is only possible because of consequences of Riemann-Roch.

Theorem 3 (Riemann-Roch, Theorem 3.2.2, [10])**.** *Let (Σ, j) be a Riemann surface and (M, J) an almost complex manifold of dimension $2n$. For any $u \in \bar{\partial}_J^{-1}(\{0\})$, $\mathcal{D}_u$ is a Fredholm operator with index*

$$\text{ind } (\mathcal{D}_u) = n\chi(\Sigma) + 2\langle c_1(TM), [u] \rangle.$$

Where $[u] := u_*[\Sigma] \in H_2(M)$, $c_1(TM) \in H^2(M)$ is the first Chern class of the complex vector bundle (TM, J) and $\chi(\Sigma)$ is the Euler characteristic of Σ.

From surjectivity $\text{ind}(\mathcal{D}_u) = \dim \text{Ker}(\mathcal{D}_u)$, appropriate as this index will come to be the local dimension of the moduli space of J-holomorphic curves.

2.4 Further Properties of J-holomorphic Curves

This leads us on to defining further properties of J-holomorphic curves with the direction of portraying the relevance of the embedded quantifier in "embedded contact homology" (ECH).

We will first need more properties of J-holomorphic curves. We will now put extra conditions on our Riemann surface (Σ, j), those being closed, oriented and connected whereas keeping (M, J), an almost complex manifold, the same.

We say that a J-holomorphic curve u is multiply covered if it factors through a branched covering map with degree strictly greater than 1. Such a (continuous) map is $\varphi : \Sigma \to \Sigma'$ with (Σ', j') a distinct closed, oriented and connected Riemann surface. By factors through we mean that there exists a J-holomorphic curve $u' : \Sigma' \to M$ such that

$$u = u' \circ \varphi.$$

The degree of a branched covering is an integer $\deg(\varphi) \in \mathbb{Z}$ with $\varphi_*[\Sigma] = \deg(\varphi)[\Sigma']$. Here $[\Sigma]$ denotes the fundamental class of the surface Σ.

Branch covers of Riemann surfaces may be characterized as follows. It is a fact that $\deg(\varphi) \geq 0$, so begin with $\deg(\varphi) = 0$ which occurs if and only if φ is constant. The case $\deg(\varphi) = 1$ happens if and only if φ is a biholomorphism. Finally the situation $\deg(\varphi) \geq 2$ arises if and only if φ is locally a covering map of degree $k \in \{2, ..., \deg(\varphi)\}$, (2.14, [10]).

There is a more general theory to do with ramification points that we are unable to achieve in this thesis and so instead leave a reference for further reading [19].

We repeat a result from the appendix here for convenience as it will be used in a moment.

Corollary 1 (Corollary 2.59, [10])**.** *If $u : \Sigma \to M$ is smooth and J-holomorphic and not constant then*

$$\text{Crit}(u) := \{z \in \Sigma \mid du(z) = 0\}$$

contains at most a countable number of points.

If u is not multiply covered then it is called *simple.* If u is a simple curve then u is somewhere injective, meaning there exists $z \in \Sigma$ such that

$$u^{-1}(\{u(z)\}) = \{z\} \quad \text{and} \quad du(z) \neq 0.$$

This follows from Proposition 2.5.1 of McDuff and Salamon [17]. A non-trivial fact is that the implication holds the other way, (Theorem 2.117, [10]). Furthermore, it closely follows from Corollary 1 that if u is somewhere injective then it is almost everywhere injective. By almost everywhere it is meant that the complement to the set of somewhere injective points is of negligible measure, which is in this case says the set is at most countable.

Let u and u' be two J-holomorphic curves from closed connected Riemann surfaces $(\Sigma, j), (\Sigma', j')$ respectively. We call u and u' equivalent if there exists an associated biholomorphism $\phi : \Sigma' \to \Sigma$ such that

$$u \circ \phi = u'.$$

These should naturally be called the same *curve* due to images in (M, J) having the same structure which in essence is what we care about for applications. Let $A \in H_2(M)$ determined by u ($[u] := u_*[\Sigma] = A$) and g be a chosen integer genus. Then define (Def. 4.1, [10]) the moduli space of J-holomorphic curves to be

$$\mathcal{M}_g(A, J) = \{(\Sigma, j, u)\}/\sim,$$

with (Σ, j) any closed connected Riemann surface of genus g. The equivalence relation $u \sim u'$ holds precisely when there exists a biholomorphism as above between u and u'.

The moduli space $\mathcal{M}_g(A, J)$ is not a manifold but a more general object called an *orbifold*, not defined here. Orbifolds have a type of "expected" dimension or virtual dimension which can be interpreted as a Fredholm index. Indeed the Fredholm index of our earlier linearized operator, sufficiently adjusted for the situation at hand, which accounts for a correction term. In our case (Def. 4.2, [10])

$$\text{ind}(u) := \text{vir-dim } \mathcal{M}_g(A, J) = (n-3)(2-2g) + 2c_1(A) = (n-3)\chi(\Sigma) + 2c_1(A)$$

having substituted $\chi(\Sigma)$ for $2-2g$. This formula is remarkably similar to the Fredholm index through Riemann-Roch. Under further assumptions on u, Fredholm regularity, we can say that the moduli space near u is a smooth manifold of dimension equal to $\text{ind}(u)$ (Theorem 4.43, [10]), proved using Riemann-Roch.

We also require the notion of intersection of distinct J-holomorphic curves and self-intersections of a curve u with itself. Given are $(\Sigma_0, \Sigma_1, \Sigma_2)$ three oriented Riemann surfaces with J-holomorphic curves u_i (let $u := u_0$), $u_i : \Sigma_i \to M$. We denote the intersection number of u_1 and u_2 as

$$\delta(u_1, u_2) := \#\{(z_1, z_2) \in \Sigma_0 \times \Sigma_1 \mid u_1(z_1) = u_2(z_2)\}$$

and the self-intersection number of u as

$$\delta(u) := \frac{1}{2}\#\{(z_1, z_2) \in \Sigma \times \Sigma \mid u(z_1) = u(z_2),\ z_1 \neq z_2\}.$$

These numbers are finite in the cases where the disjoint union $u_1 \sqcup u_2$ is a simple curve, in the first, and when u is a simple curve, in the second. The following results are exclusively for 4-dimensions and so we set $\dim M = 4$, (page 605, appendix E.1, [17]). Distinct points (z_1, z_2), $(z_1 \neq z_2)$ give rise to x if $u(z_1) = u(z_2) = x$, we call these x non-injective points of u.

Positivity of intersections (Theorem 2.6.3, [17]) is a result first noticed by Gromov and then later proved by Gromov and McDuff. It gives a lower bound ($\delta(u_1, u_2)$) on the intersections of homology classes A_1 and $A_2 \in H_2(M)$ represented by u_1 and u_2, respectively. The following theorem formalizes a second result that will be of great use to us, the adjunction inequality.

Theorem 4 (Theorem 2.6.4, [17])**.** *Let (M, J) be an almost complex 4 dimensional manifold and (Σ, j) a closed Riemann surface (connected or not) and $u : \Sigma \to M$ a simple J-holomorphic curve. Denote by $A \in H_2(M; \mathbb{Z})^*$ the homology class represented by u. Under these circumstances*

$$2\delta(u) \leq A \cdot A - c_1(A) + \chi(\Sigma),$$

where equality holds precisely when the intersections are transverse. To wit, for any x, a non-injective point of u for $z_1 \neq z_2$, the pushforward of tangent spaces $u_*T_{z_1}\Sigma$ and $u_*T_{z_2}\Sigma$ have direct sum T_xM.

The term $A \cdot A$ is the homological intersection number of A with itself, see appendix B.2. The inequality becomes a formula in the case when the intersections are transverse. A corollary of McDuff follows swiftly,

Corollary 2 (Corollary E.1.7, [17])**.** *Let M, Σ, u and A, be as in the previous theorem, thereupon*

$$0 \geq A \cdot A - c_1(A) + \chi(\Sigma)$$

with equality precisely when u is an embedded J-holomorphic curve.

To make sweeping analogues later it will be best to define the index

$$I(A) := c_1(A) + A \cdot A,$$

for $A \in H_2(M)$. When discussing A represented by a J-holomorphic curve, A will be replaced by $[u]$ and $I(u) := I([u])$. For now the significance of the index is the following. Applying $n = 3$ to the definition of $\text{ind}(u)$ we obtain

$$\text{ind}(u) = -\chi(\Sigma) + 2c_1([u]).$$

*The homology of 2-cycles under integer coefficients, the universal coefficient theorem (Section 2 Theorem 8, [20]) that makes this equivalent to the homology of 2-cycles with real coefficients.

Along with the adjunction formula (assuming transverse intersections) we have

$$\mathrm{ind}(u) = I(u) - 2\delta(u).$$

We would like to investigate the implications of this formula, when imposing conditions on u. Necessarily u must be a simple J-holomorphic so that $\delta(u) < \infty$. If now we impose a further condition that $I(u) = 0$ then $\delta(u)$ must be negative or zero. The situation of negative self-intersections is rich in connections with algebraic geometry that we do not investigate here, see [21] for more information. Alternatively, when $\delta(u) = 0$ then two things happen, 1. the dimension of the moduli space is locally 0 and 2. u is embedded. This is of special interest as it allows for counts to be computed. Before discussing these counts we need to understand compactness results.

A measurement on J-holomorphic curves is called the energy defined by,

$$E(u) := \int_\Sigma u^*\omega.$$

We would like to restrict to those curves with uniformly bounded energy. This bound leads to compactness results in moduli spaces and to well-defined counts. What is being counted are curves with index $I(u) = 0$, which after considering the multiply covered case implies that the space of interest is 0-dimensional. The result in question, stated imprecisely, is the following

Theorem 5 (Gromov, [2])**.** *The moduli space of curves of energy bounded by some constant E (modulo reparametrization) can be compactified by adding in stable curves of total energy bounded by E.*

The stable curves mentioned are part of an effect called bubbling. By identifying $\mathbb{C}$ with modulus 1 complex numbers we can form the Riemann sphere $\mathbb{C}P^1$. By addition of enough bubbles ($\mathbb{C}P^1$) to the domain of a sequence of J-holomorphic curves, unwanted behaviour from reparametrization can be controlled. The consequence of adding bubbles produces stable curves.

Taubes, in defining his Gromov-Witten invariant [22], also counted index 0. However, no longer counting J-holomorphic curves, but instead J-holomorphic currents. A J-holomorphic current is $\mathcal{U} = \{(u_k, d_k)\}$ a collection of J-holomorphic curves u_k each with their own multiplicity d_k. The corresponding compactness result is for spaces of J-holomorphic currents.

Note: the energy of a current $E(\mathcal{U}) = \int \mathcal{U}^*\omega := \sum_k d_k E(u_k)$.

Theorem 6 (Section 2.4, [16])**.** *Let (M, ω) be a compact symplectic 4 dimensional manifold, possibly with boundary, and let J be an ω-compatible almost complex structure. Let $\{\mathcal{U}_n\}_{n\geq 1}$ be a sequence of J-holomorphic currents (possibly with boundary in ∂M) such that $\int \mathcal{U}_n^*\omega$ has an n-independent upper bound. Then there is a subsequence which converges as a current and as a point set to a J-holomorphic current $\mathcal{U} \subset M$ (possibly with boundary).*

There are two types of convergence in the theorem, firstly by converging as a current it is meant that for any 2-form η, $\lim_{n\to\infty} \int \mathcal{U}_n^*\eta = \int \mathcal{U}^*\eta$. To converge as a point set means to converge in the following metric defined on compact subsets of $C_1, C_2 \subset M$,

$$d(C_1, C_2) = \sup_{x\in C_1} \inf_{y\in C_2} d(x, y) - \sup_{y\in C_2} \inf_{x\in C_1} d(x, y),$$

(pages 22-23, [16]).

The purpose of giving this theorem is that later we will work with the classification of low (ECH) index curves. The low index will imply independence from multiply covered curves and the theorem has the consequence of finite moduli space.

3 Morse and (Embedded) Contact Homologies

This section will outline Morse and Contact homologies, along with explaining the inner workings of ECH to a reasonable degree of detail. Finally, the ECH spectrum and capacities will be defined as well as finding the ECH capacities associated to ellipsoids.

3.1 Morse Theory

To give motivation for defining a homology via counts we divert our attention to the elegant subject of Morse Theory. The sequel is based on the notes of Rubermann [23].

Definition 10 (Definition 1.1, [23])**.** A *Morse function* on a smooth Riemannian manifold (M, g) is a map $f : M \to \mathbb{R}$ such that the only critical points of f are non-degenerate.

By critical points we mean those p such that $df_p \equiv 0$, where p is non-degenerate if $\text{Hess}_p(f)$ is invertible. The Hessian is defined for the Levi-Cita connection ∇ by $\text{Hess}(f)(X,Y) = g(\nabla_X \text{grad}(f), Y)$. The Hessian is self-adjoint and thus has real spectrum. With this spectrum we can define the dimension of the negative eigenspaces, denoted $\text{ind}_p(f)$ for index of f at p, which will correspond to local maxima. We will assume that f is self-indexing, meaning for $p, q \in \text{Crit}(f)$, it is the case that $\text{ind}_p(f) > \text{ind}_q(f)$ implies $f(p) > f(q)$. We write $p \succ q$ when $\text{ind}_p(f) > \text{ind}_q(f)$ and $p, q \in \text{Crit}(f)$. We need to study the downward flow, those $\gamma : \mathbb{R} \to M$ such that

$$\dot{\gamma}(t) = -\text{grad}_{\gamma(t)} f. \tag{3.1}$$

The reason why we choose the downward flow is because these γ decreases the index and we care about positive difference of index "ind".

Define the moduli space $\mathcal{M}(p,q)$ for $p, q \in \text{Crit}(f)$ to be solutions to (3.1) such that

$$\lim_{t \to -\infty} \gamma(t) = p \quad \& \quad \lim_{t \to \infty} \gamma(t) = q.$$

As f decreases along flow lines this space is empty save for $f(p) > f(q)$.

By generic choice of metric g it is possible to guarantee the Morse-Smale transversality property: $\mathcal{M}(p,q)$ is a smooth manifold of dimension $\text{ind}_p(f) - \text{ind}_q(f)$, for $p \succ q \in \text{Crit}(f)$. A way to think about this is the intersection of the descending space at p with the ascending space at q.

By reparametrizing by elements of $\mathbb{R}$, an $\mathbb{R}$-action is induced on $\mathcal{M}(p,q)$, thus we would like to consider curves to be the same under this reparametrization, so form $\hat{\mathcal{M}}(p,q) = \mathcal{M}(p,q)/\mathbb{R}$.

Choosing an arbitrary orientation on the negative eigenspaces of the Hessian at p orients $\mathcal{M}(p,q)$ and $\hat{\mathcal{M}}(p,q)$. Specifically, we can count $\hat{\mathcal{M}}(p,q)$ when the moduli space is (compact) oriented and of dimension 0, which necessarily requires $\text{ind}_p(f) = \text{ind}_q(f) + 1$. The count is with signs, giving $\#\hat{\mathcal{M}}(p,q) \in \mathbb{Z}$ (those signs given by the way the ascending and descending spaces intersect).

Critical points will be generators for the chain complex. Let $\langle p \rangle$ denote the generator $p \in \text{Crit}(f)$. The group $C_k(M,f)$ is a freely generated $\mathbb{Z}$-module on the generators with $\text{ind}_p(f) = k$. The differential ∂ associated to the chain complex takes as coefficients, $(\partial \langle p \rangle, \langle q \rangle) = \#\hat{\mathcal{M}}(p,q)$, for q such that $\text{ind}_q(f) = k - 1$.

A slightly different, but equivalent construction, is given by Milnor in his paper on the h-cobordism theorem [24]. In which the Morse complex is proved to be isomorphic to a cellular complex from the CW decomposition of M with Morse function f. This cellular complex has $\partial^2 = 0$, thus it makes sense to discuss Morse Homology. A direct proof is non-trivial and was given by Schwarz in [25].

The purpose of this discussion is to form a basic feeling for how a Morse type homology or more generally a Floer type homology can be constructed and the philosophy behind such a construction.

This idea is to define a Morse theory or more generally a Floer theory through counting the number of points in a 0-dimensional smooth compact manifold. Our goal will be to verify that we are indeed counting over the "correct space" and then form a differential around that count, so that the embedded contact complex can be defined.

3.2 Contact Homology

Firstly, we need to investigative an aspect of contact type homology, in which we rely on notation and definitions in a brief sketch [26] and a more in depth introduction [27].

Let Y be a closed oriented 3-manifold, with contact 1-form λ such that $\lambda \wedge d\lambda > 0$ (λ is non-degenerate). A contact structure is the oriented 2-plane field $\xi = \text{Ker}(\lambda)$ a sub-bundle of TY.

As defined in Section 2, a Reeb vector field is the vector field R that has uniquely determined direction by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. The flow of the Reeb vector field can have additional requirements making the flow into a Reeb orbit. These orbits exist in dimension 3 due to the proof of the Weinstein conjecture by Taubes [28]. A Reeb orbit is a closed orbit of the Reeb vector field i.e. (modulo reparametrization)

$$\gamma : \mathbb{R}/T\mathbb{Z} \to Y, \quad T > 0 \quad \text{satisfying} \quad \dot{\gamma}(t) = R(\gamma(t)).$$

With homology class $[\gamma] := \gamma^*[\mathbb{R}/T\mathbb{Z}] \in H_1(Y)$.

We can define a symplectic map (preserving $d\lambda$) on $\gamma^*\xi$ through the linearized return map, namely

$$\Psi_\gamma := d(\varphi_T)_y|_{\xi_y} : \xi_y \to \xi_y.$$

Where φ denotes the flow of the Reeb vector field and $y \in \text{Im}(\gamma)$. It is necessary to state that the eigenvalues of Ψ_γ are independent of point y. If Ψ_γ has all real eigenvalues then γ is called hyperbolic. Otherwise, γ is called elliptic.

The action functional is defined by

$$\mathcal{A} : C^\infty(\mathbb{S}^1, Y) \to \mathbb{R} : \gamma \mapsto \int_\gamma \lambda.$$

From Lemma 1 of [27] we have that all the critical points of the action functional precisely correspond to Reeb orbits. The general idea is to define a Morse theory, more commonly Floer theory, for this "morse function". As in Morse theory we need to study the Hessian of $\mathcal{A}$ at critical points which correspond to the linearized Reeb flow Ψ_γ near a periodic orbit. A closed Reeb orbit is called non-degenerate if Ψ_γ has no eigenvalue equal to 1. This means that γ is non-degenerate precisely when γ is a non-degenerate critical point of $\mathcal{A}$, modulo reparametrization. We will need that all our critical points are non-degenerate so we give the following lemma.

Lemma 1 (Lemma 2, [27])**.** *For any contact structure ξ on M there exists a contact form λ for ξ such that all the closed orbits of R are non-degenerate.*

The significant implication is that we may now choose a generic λ, for which all Reeb orbits are non-degenerate, to define our contact homology. Let γ be a non-degenerate closed Reeb orbit of period T. Fix a symplectic trivilization τ for ξ along $\gamma^\dagger$. A path $\varphi_t : \xi_p \to \xi_{\varphi_t(p)}$ $(t \in [0,T])$, is represented by a path of symplectic matrices Ψ_γ, with $\Psi_\gamma(0) = \mathbb{1}$ and $\det(\Psi_\gamma(T) - \mathbb{1}) \neq 0$. A number $t \in [0,T]$ is called a crossing if $\det(\Psi_\gamma(t) - \mathbb{1}) = 0$. Denote the kernel by $E_t = \mathrm{Ker}(\Psi_\gamma(t) - \mathbb{1})$ and the set of crossings by Cross_γ. A crossing form $\Gamma(\Psi_\gamma, t)$ is a quadractic form on E_t defined by

$$\Gamma(\Psi_\gamma, t)v = d\lambda(v, \dot{\Psi}_\gamma v), \text{ for } v \in E_t.$$

A quadratic form can be orthogonally diagonalized (originally due to Jacobi) giving a diagonal matrix of 1s, -1s and 0s. A quadratic form is non-degenerate when there are no 0s on the diagonal, and a crossing t is called regular when this occurs. The signature is an invariant for quadratic forms (by Sylvester's law of inertia) and is defined by (n_0, n_-, n_+) the number of 0s, -1s, and 1s respectively. The signature *sign* of a non-degenerate quadractic forms will correspond to the number n_+. Now if the path ψ_γ has regular crossings it is possible to define:

$$\mathrm{CZ}_\tau(\gamma) := \mathrm{CZ}_\tau(\Psi_\gamma) = \frac{1}{2}\mathrm{sign}\ \Gamma(\Psi_\gamma, 0) + \sum_{t \in \mathrm{Cross}\setminus\{0\}} \mathrm{sign}\ \Gamma(\Psi_\gamma, t).$$

The above definition turns out to be invariant under homotopy, proved by Robbin and Salamon (Lem. 3, [27]). So under a small perturbation $\tilde{\Psi}$ that has only regular crossings, Ψ and $\tilde{\Psi}$ have the same index. This can always be arranged and we call $\mathrm{CZ}_\tau(\tilde{\Psi})$ the Conley-Zehnder index, (Def. 4, [27]) .

Example 3.1[Section 2.3, [26]]
Consider the boundary of the ellipsoid $\partial E(a,b)$ with $a, b > 0$, real numbers, a/b irrational. Naturally, as $Y = \partial E(a,b) \subset \mathbb{C}^2$ there exists coordinates $z_j = x_j + iy_j$, $j = 1,2$ and contact form

$$\lambda = \frac{1}{2}\sum_{j=1}^{2}(x_j dy_j - y_j dx_j).$$

The Reeb vector field is

$$R = \frac{2\pi}{a}\frac{\partial}{\partial\theta_1} + \frac{2\pi}{b}\frac{\partial}{\partial\theta_2},$$

where $\partial/\partial\theta_j = x_j\partial/\partial y_j - y_j\partial/\partial x_j$, for $j = 1,2$.

We need to check that $\lambda(R) = 1$ and $d\lambda(R, \cdot) = 0$. Start with the normalization,

$$\lambda(R) = \frac{2\pi}{2a}(x_1^2 + y_1^2) + \frac{2\pi}{2b}(x_2^2 + y_2^2) = \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} = 1.$$

With the last equality following from the condition on $Y = \partial(E(a,b))$. For the second condition of Reeb vector fields,

$$d\lambda(R, \cdot) = \frac{2\pi}{2a}(x_1dx_1 + y_1dy_1) + \frac{2\pi}{2b}(x_2dx_2 + y_2dy_2) = \frac{\pi}{2a}d(|z_1|^2) + \frac{\pi}{2b}d(|z_2|^2) = \frac{1}{2}d\left(\frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b}\right) = 0.$$

Which uses the same condition again and follows from d applied to a constant being zero.

As a/b is irrational there are precisely two closed Reeb orbits, both are circles. Begin by parametrizing one closed Reeb orbit by

$$\gamma(t) = (\sqrt{a/\pi}\exp(2\pi it/a), 0), \ t \in [0,a].$$

The contact structure along γ, $\gamma^*\xi$, is spanned by the vectors fields associated to the other angle, $\partial/\partial x_2$ and $\partial/\partial y_2$, for all $t \in [0,a]$. This is one example of a symplectic trivialization τ of ξ along γ mentioned earlier. Although not

†This trivilization is an isomorphism between $\gamma^*\xi$ and $[0,1] \times \mathbb{R}^2$, $\tau : (t,v) \to \tau_t(v)$ where $(\gamma \circ \tau)^* d\lambda = \omega_0$ the canonical symplectic form on $\mathbb{R}^2$. For a more general and insightful definition, see page 83 of [29] obligatory to the context we calculate the action of γ. By the parametrization,

$$\mathcal{A}(\gamma) = \int_\gamma \lambda = \int_0^a \gamma^*\lambda = \frac{1}{2}\int_0^a \gamma^*(x_1 dy_1 - y_1 dx_1)$$
$$= \frac{a}{2\pi}\int_0^a (\cos(2\pi t/a)d(\sin(2\pi t/a)) - \sin(2\pi t/a)d(\cos(2\pi t/a))) = \frac{a}{2\pi}\int_0^a \frac{2\pi}{a}dt = a.$$

Continuing with the example at hand, the linearized Reeb flow $d\varphi_t : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ is given by a rotation of $2\pi t/b$. This rotation which has eigenvalue t whenever t is an integer multiple of b. Thus,

$$\text{Cross}_\gamma = \{lb \mid l \in \mathbb{Z}, lb \in [0,a]\}$$

and $\#\text{Cross}_\gamma = \lfloor a/b \rfloor + 1$. Again as a/b is irrational, a is not a crossing. Furthermore, γ is a non-degenerate closed Reeb orbit, as are all of its covers e.g. the k-th cover of γ is $\gamma^k = \gamma + \cdots + \gamma$ (k-times). We calculate the Conley-Zehnder index of γ through the quadratic form $\Gamma(\Psi_\gamma, t)$, with $t \in \text{Cross}_\gamma$. Given $t \in \text{Cross}_\gamma$ with $\Psi_\gamma(0) = \mathbb{1}$, we have

$$\dot{\Psi}_\gamma(t) = \frac{2\pi}{b}\begin{bmatrix} -\sin(2\pi t/b) & -\cos(2\pi t/b) \\ \cos(2\pi t/b) & -\sin(2\pi t/b) \end{bmatrix} \quad \text{(reminder } t = lb, \text{ for some } l)$$
$$= \frac{2\pi}{b}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{2\pi}{b}J_{\text{std}},$$

J_{std} the standard almost complex structure in 2-dimensions. Then we have the standard euclidean bundle metric (up to scale factor) for our quadractic form. As $d\lambda|_{\xi_{\gamma(t)}} = dx_2 \wedge dy_2$

$$d\lambda(v, \dot{\Psi}_\gamma v) = \frac{2\pi}{b}dx_2 \wedge dy_2(v, J_{\text{std}}v) = \frac{2\pi}{b}\langle v, v\rangle_{\text{std}}, \text{ for every } v \in E_t.$$

Therefore the signature must be 2 (counting the necessarily two 1s on the diagonal) for all crossings $t \in \text{Cross}_\gamma$. The Conley-Zehnder index with respect to the trivialization τ is now

$$CZ_\tau(\Psi_\gamma) = 1 + \sum_{t\ \in \text{Cross}\setminus\{0\}} 2 = 2\left\lfloor \frac{a}{b} \right\rfloor + 1.$$

This example is not only important to our needs but is the usual situation for computing the Conley-Zehnder index. Indeed, there commonly exists a rotation angle θ of the linearized Reeb orbit around γ with respect to a trivialization τ. This θ appears inside the floor function, upon replacing a/b. A final remark, the Conley-Zehnder index acting over a k-fold cover γ^k is multiplicative with the multiplicity appearing inside the floor argument. What will come to be so important is that formulas we will explain will not depend on the trivialization. This will mean that invariants can be defined. The explanation to introduce this "primary" contact homology is not a substitute for Hutching's embedded contact homology which we will get to very soon.

3.3 Embedded Chain Complex

In the continuation, we will assume everything as we have for the previous subsection, Y a closed oriented contact 3 dimensional manifold with λ a non-degenerate contact 1-form. The Floer theory in ECH is dependent on generators of the embedded chain complex, which are initially called orbit sets $\alpha = \{(\alpha_i, m_i)\}$. Orbit sets are finite sets of pairs α_i a Reeb orbit with a natural number multiplicity m_i, where the orbits are distinct. The orbits sets are generators if $m_i = 1$ whenever α_i is hyperbolic. The action functional now becomes

$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda.$$

From the linearity of the sum, we observe that the critical points correspond to the orbits sets. Furthermore, we fix a homology class in Y, $\Gamma \in H_1(Y)$ for which the generators must satisfy the identity $\sum_i m_i[\alpha_i] = \Gamma$. The $[\alpha_i]$ terms signify the class of all those maps that homotopically equivalent to α_i, otherwise known as the homology class of α_i. As in Morse theory we need to construct "flow-lines" between critical points. These will be, sufficiently restricted, J-holomorphic curves.

However, this is merely the set up we need to discuss how to create the chain complex $\text{ECC}(Y, \lambda, \Gamma)$, how the differential ∂ is defined and how to attack defining the ECH capacities.

To begin with we will need more background. Firstly, this is not the right environment for defining J-holomorphic curves thus we must take a symplectization of Y. As was done in Section 2 we take $\mathbb{R}_s \times Y$ (s is the $\mathbb{R}$-coordinate) as our 4 dimensional manifold with symplectic form $\omega = d(e^s\lambda)$. As a reminder on Y we have the Reeb vector field R, with direction uniquely determined by $\lambda(R) = 1$ and $d\lambda(R, \cdot) = 0$, and the contact structure $\xi = \text{Ker}(\lambda)$.

Definition 11. We call, the almost complex structure, J *symplectization-admissible* if

> $J(\partial/\partial s) = R$,

> $J(\xi) = \xi$ and J rotates ξ positively i.e. $d\lambda(v, Jv) \geq 0,\ v \in \xi$,

> J is $\mathbb{R}$-invariant.

The space of symplectization-admissible almost complex structures is contractible and depends on the contact structure ξ. We fix such a symplectization-admissible J that is ω-compatible, with ω as above.

If, γ is an embedded Reeb orbit, then, $\mathbb{R} \times \gamma$ is an embedded J-holomorphic curve in $\mathbb{R} \times Y$. These curves will be called trivial cylinders and come to play and important role when decomposing J-holomorphic curves.

Consider J-holomorphic curves $u : (\Sigma, j) \to (\mathbb{R} \times Y, J)$, (Σ, j) a compact Riemann surface with finitely many points removed, called punctures. We make sure to identify u with u' that are parametrized by a biholomorphism ϕ, $u = u' \circ \phi$, as before.

Let γ be a, possibly multiply covered, Reeb orbit then u has a positive end at γ when: there exists a neighbourhood of a puncture with coordinates $(\sigma, \tau) \in (\mathbb{R}/T\mathbb{Z}) \times [0, \infty)$, $j(\partial/\partial\sigma) = \partial/\partial\tau$, such that

$$\lim_{\sigma\to\infty} \pi_{\mathbb{R}}(u(\sigma, \tau)) = \infty, \quad \& \quad \lim_{\sigma\to\infty} \pi_2(u(\sigma, \cdot)) = \gamma.$$

A negative end is defined similarly but with $\lim_{\sigma\to-\infty}$. We assume that all punctures correspond to positive or negative ends. This is the similarity to Morse theory where the flow lines between critical points of a function have become J-holomorphic curves between sets of Reeb orbits. The Reeb orbits are critical points of the action functional, in analogy to a morse function, and instead of the downward gradient flow we require u to satisfy the J-holomorphic PDE, $du \circ j = J \circ du$. This is the culmination of one factor this thesis has attempted to achieve. The following will now be analogy to previous work.

With parallel in the discussion of dimension of moduli spaces of J-holomorphic curves, Section 2, we also define an index that turns out to be a dimension. Let J be generic and u a somewhere-injective J-holomorphic curve in $\mathbb{R} \times Y$ with positive ends at Reeb orbits $\gamma_1, ..., \gamma_m$ and negative ends at Reeb orbits $\delta_1, ..., \delta_n$. Then the moduli space of J-holomorphic curves near u is a manifold of dimension:

$$\mathrm{ind}(u) = -\chi(\Sigma) + 2c_1(u^*\xi, \tau) + \sum_{i=1}^{m} \mathrm{CZ}_\tau(\gamma_i) - \sum_{j=1}^{n} \mathrm{CZ}_\tau(\delta_j).$$

There are some terms to explain here. As before $\chi(\Sigma)$ is the Euler characteristic of the domain of u. The second term on the right hand side is analogous to the first Chern class, called the relative first Chern class. Relative because it is with respect to the symplectic trivialization τ on the complex dimension 1 bundle $u^*\xi$. It is defined by counting the zeroes (with sign) of a generic section ψ, of $u^*\xi$, which is non-vanishing at the ends and constant with respect to the chosen trivialization. The relative first Chern class is elaborated on page 9 of [30]. The Conley-Zehnder term, the third term, will denoted by $\mathrm{CZ}^{\mathrm{ind}}_\tau(\gamma, \delta)$, with $\gamma = (\gamma_1, ..., \gamma_m)$ and $\delta = (\delta_1, ..., \delta_n)$ ordered lists of Reeb orbits.

It turns out that even though each term in the formula for "ind" is dependent on the trivialization τ, "ind" itself is independent.

Given are orbit sets $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$. We can define $\mathcal{M}(\alpha, \beta)$ as the set of somewhere-injective J-holomorphic curves in $\mathbb{R} \times Y$ with positive ends α and negative ends β. As a quick aside, only important for later, as J has been chosen to be $\mathbb{R}$-invariant the set $\mathcal{M}(\alpha, \beta)$ will have an $\mathbb{R}$ action, translation by some real number s.

Upon assuming u was a somewhere-injective curve we had

$$c_1(A) = \chi(\Sigma) + A \cdot A - 2\delta(u).$$

To analogize, we pick $u \in \mathcal{M}(\alpha, \beta)$. Now we will have a relative adjunction formula (which picks up a new term)

$$c_1(u^*\xi, \tau) = \chi(\Sigma) + Q_\tau(u) + w_\tau(u) - 2\delta(u).$$

Appearing again are the trivialization τ of ξ over Reeb orbits α_i and β_j. There again is the relative first Chern class. By the work of Seifring [31] the self intersection number $\delta(u)$ is still defined and finite in the symplectization case.

The analogue term to $A \cdot A$ is $Q_\tau(u)$, but instead of representing classes in singular homology we will need to determine representatives of a relative singular homology. That homology which we desribe now.

Given ECH generators $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ the following identity must hold

$$\sum_i m_i[\alpha_i] = \sum_j n_j[\beta_j] \in H_1(Y)$$

for there to exist J-holomorphic curves between α and β. As before we denote this class by Γ.

Denote by $H_2(Y,\alpha,\beta)$ the set of two-chains Z in Y such that $\partial Z = \sum_i n_i[\alpha_i] - \sum_j m_j[\beta_j]$ where we modulo by the equivalence relation $Z \sim Z'$, $[Z - Z'] = 0 \in H_2(Y)$. Relative thus refers to the dependence on the generators α and β. Under translating an element Z in $H_2(Y,\alpha,\beta)$ by an element W in $H_2(Y)$ to produce $W + Z$, we find that there is no change in the differential. Indeed, $\partial(W + Z) = \partial W + \partial Z = 0 + \partial Z = \partial Z$. This is what it means for $H_2(Y,\alpha,\beta)$ to be an *affine space* over $H_2(Y)$. By construction every J-holomorphic curve $u \in \mathcal{M}(\alpha,\beta)$ defines a class $[u] \in H_2(Y,\alpha,\beta)$.

Given is $Z \in H_2(Y,\alpha,\beta)$ for which we would like to compute the relative intersection pairing $Q_\tau(Z) \in \mathbb{Z}$. We represent classes by embedded oriented surfaces and count (with sign) the number of intersection points between the representing surfaces. In the symplectization case we choose two embedded, save at the boundary, oriented surfaces $S, S' \subset [-1,1] \times Y$, representing Z. We require that

$$\partial S = \partial S' = \sum_i m_i \cdot \{1\} \times \alpha_i - \sum_j n_j \cdot \{-1\} \times \beta_j$$

and S, S' intersect transversally, save at the boundary. The relative intersection pairing of Z will be the count of the intersection of Int S with Int S' (by counting we mean, again, in an algebraic sense with signs). The problem with this definition is that it is not well-defined under choosing slightly different surfaces with different boundary intersections.

We will need to count the intersection by first establishing the boundary behaviour, this behaviour will depend on the trivilization τ. We wish for the projections $\pi_2(S)$ and $\pi_2(S')$ to be both embeddings near the boundary. Furthermore, both images must be a transverse slice to any α_i or β_j as rays. These slices must not intersect nor rotate (by which we mean $d\lambda(J\cdot,\cdot) = 0$ on the slices), with respect to τ. Under this condition, defined in full detail over pages 11-12 in [30], the count of Int $S \cap$ Int S' will be the integer we call $Q_\tau(Z)$. That integer is dependent on α, β, Z and τ. In the case of a J-holomorphic curve u, $Q_\tau([u]) = Q_\tau(u)$.

The other term in the relative adjunction formula $w_\tau(u)$ is called the asymptotic writhe which, in very simplistic terms, is another count. This count is of slices of somewhere-injective J-holomorphic curves. These slices produce braids that can first be identified with $\mathbb{S}^1 \times D^2$ and then identified with the 3-Torus $\mathbb{T}^3$. Anymore details on braids and writhes takes us too far from our goal and so we rely on pages 10-11 of [30], and Section 5 of [16].

We are almost ready to put together the formulas we have gathered, before that we must tackle the ECH index. For a particular $Z \in H_2(Y,\alpha,\beta)$ we define the ECH index to be

$$I(\alpha,\beta,Z) = c_1(u^*\xi,\tau) + Q_\tau(Z) + \mathrm{CZ}^I_\tau(\alpha,\beta).$$

Again, $Q_\tau(Z)$ is the relative intersection pairing and the Conley-Zehnder term is

$$\mathrm{CZ}^I_\tau(\alpha,\beta) = \sum_i \sum_{k=1}^{m_i} \mathrm{CZ}_\tau(\alpha_i^k) - \sum_j \sum_{l=1}^{n_j} \mathrm{CZ}_\tau(\beta_j^l).$$

If γ is a Reeb orbit and k a positive integer then the k-fold cover of γ is γ^k, the remark earlier about CZ being multiplicative applies now. It should be remarked upon that the two Conley-Zehnder terms $\mathrm{CZ}^{\mathrm{ind}}_\tau(\alpha,\beta)$ and $\mathrm{CZ}^I_\tau(\alpha,\beta)$ are quite different. Namely, the former is a sum over the Conley-Zehnder indicies of Reeb orbits themselves and the latter sums over the Reeb orbits and all their iterates, up to multiplicity.

When $u \in \mathcal{M}(\alpha,\beta)$ then write $I(u) = I(\alpha,\beta,[u])$. The relative ECH index is additive over pairs of orbit sets (α,β) and (β,γ) with $Z \in H_2(Y,\alpha,\beta)$ and $W \in H_2(Y,\beta,\gamma)$, explicitly

$$I(\alpha,\beta,Z) + I(\beta,\gamma,W) = I(\alpha,\gamma,Z+W).$$

When we choose Z and W to be represented by J-holomorphic curves this is called gluing and thus we call the ECH index, additive under gluing. We will also need the fact that the Fredholm index is additive under gluing.

We analogize, for the final time, to the symplectic situation where we found that combining the index, Fredholm index and adjunction formula together gave an (in)equality. We have in the symplectization case

Theorem 7 (Theorem 4.15, [30]). *Let α,β be orbit sets, $u \in \mathcal{M}(\alpha,\beta)$ J-holomorphic, then*

$$\mathrm{ind}(u) \leq I(u) - 2\delta(u).$$

Specifically, $\mathrm{ind}(u) \leq I(u)$ *with equality if and only if u is embedded.*

This theorem follows from the previous formulas

$$\begin{aligned} \mathrm{ind}(u) &= -\chi(\Sigma) + 2c_1(u^*\xi,\tau) + \mathrm{CZ}^{\mathrm{ind}}_\tau(\alpha,\beta) \\ c_1(u^*\xi,\tau) &= \chi(\Sigma) + Q_\tau(u) + w_\tau(u) - 2\delta(u) \\ I(u) &= c_1(u^*\xi,\tau) + Q_\tau(u) + \mathrm{CZ}^I_\tau(\alpha,\beta) \end{aligned}$$

and Lemma 4.20 of [30] which implies an upper bound on the asymptotic writhe

$$w_\tau(u) \leq \mathrm{CZ}_\tau^I(\alpha, \beta) - \mathrm{CZ}_\tau^{\mathrm{ind}}(\alpha, \beta).$$

The following propositions about the classification of low ECH index curves is dependent on combinatorial conditions not stated here but can be found over 7.1 and 7.2 (Definition 7.11) of [14]. These conditions are related to the irrational sloped line $y = \theta x$ approximated by integer lattice paths from above and below. Where θ is the angle appearing in the Conley-Zehnder index.

Notation: For $u \in \mathcal{M}(\alpha, \beta)$ a J-holomorphic curve there is a decomposition into u_0 and u_1 written $u = u_0 \cup u_1$. The u_0 curve contains components that map only to $\mathbb{R}$-invariant cylinders and the curve u_1 contains *no* component with this property.

Let $u \in \mathcal{M}(\alpha, \beta)$ have the decomposition $u = u_0 \cup u_1$, we call u admissible if, (1) u_1 is embedded and its image does not intersect u_0 and, (2) the combinatorial conditions hold. A remark on (1), this is where the embedded part of embedded contact homology comes from. A general remark, the admissible term is used to describe curves that are not of concern in the proof of that the differential for the embedded contact complex squares to zero.

Proposition 2 (Proposition 7.14, [14])**.** *Given that* $u \in \mathcal{M}(\alpha, \beta)$ *has decomposition* $u = u_0 \cup u_1$*. Then:*

(1) $\mathrm{ind}(u_1) \leq I(u_1) - 2\delta(u_1)$.

(2) $I(u_1) \leq I(u) - 2[u_0] \cdot [u_1]$

These propositions above and below are changed from the original to fit the thesis and not lead to an unnecessary amount of explaination. In essence the purpose of Proposition 1 is as a utility to prove Proposition 2 as we will see now.

Proposition 3 (Proposition 7.15, [14])**.** *Suppose that* J *is generic and* $u \in \mathcal{M}(\alpha, \beta)$ *has decomposition* $u = u_0 \cup u_1$*. Then:*

(1) $I(u) \geq 0$ *with equality implying* $u = u_0$

(2) If $I(u) = 1$*, then* u *is admissible and* $\mathrm{ind}(u_1) = 1$.

The proof of the proposition includes an incisive use of $\mathbb{R}$-action. Decompose u_1 into $v_1, ..., v_k$ each with covering multiplicity d_i. Note that $\mathrm{ind}(v_i) \geq 1$, $i = 1, ..., k$, from J being generic. Take u_1' to be the union over i and over d_i $\mathbb{R}$-translates of each v_i. Note u_1' is still in $\mathcal{M}(\alpha, \beta)$. The Fredholm index is additive and so $\mathrm{ind}(u_1') = \sum_{i=1}^k d_i \mathrm{ind}(v_i)$. The ECH index is the same for both u_1 and u_1' as the translation does not change the relative homology class. By applying Proposition 1 (1) to u_1' and Proposition 1 (2) to $u_0 \cup u_1'$ we obtain the following relation

$$\sum_{i=1}^{k} d_i \mathrm{ind}(v_i) \leq I(u) - 2\delta(u_1) - 2[u_0] \cdot [u_1'].$$

This formula then implies (1) and (2) of Proposition 2.

What this implies is that by identifying a curve u with the collection $\{(v_i, d_i)\}$ we can work with currents instead of curves. Moreover, when $I(u) = 1$ we will have that the current is somewhere-injective meaning that our differential need not make reference to multiply covered curves.

Finally, we can actually define ECH, with $\mathbb{Z}/2\mathbb{Z}$ coefficients. It is also possible to define ECH with integer ($\mathbb{Z}$) coefficients but we are not concerned with this case. As before let Y be a closed oriented 3 dimensional manifold with non-degenerate contact form λ, and homology class $\Gamma \in H_1(Y)$. Subsequently, the chain complex $\mathrm{ECC}(Y, \lambda, \Gamma)$ is defined. Create the freely generated $\mathbb{Z}/2\mathbb{Z}$-module on generators α under the condition $\sum_i m_i[\alpha_i] = \Gamma$.

To define the differential ∂ on the chain complex $\mathrm{ECC}(Y, \lambda, \Gamma)$ first define the subspace of index 1 J-holomorphic curves,

$$\mathcal{M}_1(\alpha, \beta) = \{u \in \mathcal{M}(\alpha, \beta) \mid I(u) = 1\}.$$

If α, β are generators, we define $\langle \partial\alpha, \beta \rangle$, meaning the coefficient of β in the sum $\partial\alpha$, as

$$\#_2 \mathcal{M}_1(\alpha, \beta)/\mathbb{R}.$$

The $\#_2$ denotes the count is to happen modulo 2. The equivalence relation $u + s \sim u$, $s \in \mathbb{R}$ is being used to identify curves. Furthermore, by representing a J-holomorphic curve as its equivalent current $\{(u_i, d_i)\}$ of covering cylinders we call two curves the same if they have the same multiplicities d_i.

Remark. The space of curves $\mathcal{M}_1(\alpha,\beta)/\mathbb{R}$ is discrete and by identifying the curves with their respective currents the previous compactness results of Taubes can be used to argue that the space, as a set, is finite, this means that ∂ is well-defined. This is not a trivial proof, but we will go one step further. In showing that $\partial^2 = 0$ took the collaboration of both Hutchings and Taubes over the course of two papers [14], [32]. This was very difficult to prove in part because of the possibility of multiply-covered J-holomorphic curves. Given both of these it is possible to define the homology of the chain complex, $\mathrm{ECH}(Y,\lambda,\Gamma)$. The proof that this homology is independent of the almost complex structure J is at present only possible by the isomorphism given by Taubes, over 5 papers the first of which is [33]. This isomorphism is between embedded contact homology and Seiberg-Witten theory.

We are principally avoiding the details over the past remark due to being way beyond the scope of this thesis. However, it would be remiss to not at least show the objects of study. The following are canonically isomorphic as relatively graded $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$-modules

$$\mathrm{ECH}_*(Y,\lambda,\Gamma) = \hat{\mathrm{HM}}^{-*}(Y, s_\xi + \Gamma)$$

The homology modules are direct summands of the graded modules where the grading comes from the ECH index which is the reason for the use of the term relative. Where s_ξ is a so-called $\mathrm{spin}_\mathbb{C}$-structure is determined completely by the contact structure ξ on Y. The right hand side, $\hat{\mathrm{HM}}^*$, is the Monopole (Seiberg-Witten) Floer cohomology which is dual of Monopole Floer homology, a Floer theory on the Seiberg-Witten Functional. This has no dependence on J and thus ECH is independent of J. Indeed all that the ECH "sees" about Y is the contact structure ξ.

For the beginings of Seiberg-Witten theory including the definition of $\mathrm{spin}_\mathbb{C}$-structure see Taubes and Hutchings notes [34], or the notes of Moore [35]. For more details on Seiberg-Witten Floer homology and the Seiberg-Witten Functional see Kronhemier and Mrowka [15].

Example 3.2
Returning again to the case of the ellipsoid $E(a,b)$, b/a irrational. The contact manifold is $Y = \partial E(a,b)$ with contact 1-form $\lambda = (1/2)\sum_{j=1}^{2} x_j dy_j - y_j dx_j|_Y$. Due to contractibility of the sphere $\mathbb{S}^3$, and Y being homotopic to $\mathbb{S}^3$, $H_1(Y) = H_1(\mathbb{S}^3) = 0$. Therefore, for any $\Gamma \in H_1(Y)$ we have $\Gamma = 0$. In the case $Y = \mathbb{S}^3$ along with $\xi = \mathrm{Ker}\lambda$ it is possible compute the Seiberg-Witten Floer Cohomology [15]. With Taubes isomorphism we obtain

$$\mathrm{ECH}_\bullet(Y,\lambda,0) = \begin{cases} \mathbb{Z} & \bullet = 0,2,4,... \\ 0 & \text{otherwise} \end{cases}.$$

So that the differential ∂, which decreases the degree by 1, must be identically 0.

We are of course interested in the ellipsoid because of the sharpness of ECH capacities. We will now give the outline to computing those capacities.

3.4 Filtered Embedded Contact Homology

We will follow the paper [7] by Hutchings et al. describing the construction of ECH capacities. Let $\alpha = \{(\alpha_i, m_i)\}$ be a generator for ECC. Consider the symplectic action of α

$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda.$$

The ECH differential, for any generic (symplectization) admissible J, strictly decreases the action. That is to say if $\langle \partial\alpha, \beta\rangle \neq 0$ then $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$. Hence, for real positive L, we can define a version of ECH called filtered ECH

$$\mathrm{ECH}^L(Y,\lambda,\Gamma).$$

This is defined as the homology of the sub-complex $\mathrm{ECC}^L(Y,\lambda,\Gamma,J)$. The sub-complex is finitely generated as a $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$-module on generators α with the restriction $\mathcal{A}(\alpha) < L$.

Filtered ECH is proved to be independent of generic (symplectization) admissible J in [36], however, it still dependent on a change of contact 1-form λ.

When $L < L'$ the derived sub-complexes are such that the first is included in the latter, after choice of J, this causes an induced map in homology

$$\iota_* : \mathrm{ECH}^L(Y,\lambda,\Gamma) \to \mathrm{ECH}^{L'}(Y,\lambda,\Gamma)$$

which is turns out to be independent of J. It is possible to retrieve the original homology by direct limit

$$\mathrm{ECH}(Y,\lambda,\Gamma) = \varinjlim \mathrm{ECH}^L(Y,\lambda,\Gamma).$$

Understanding the direct limit is not stictly necessary for this paper, although for the curious reader it is recommended to follow (there is nothing to prove) Exercise 14 of [37].

Upon scaling filtered ECH by a positive real number an isomorphism exists

$$s : \mathrm{ECH}^L(Y, \lambda, \Gamma) \xrightarrow{\sim} \mathrm{ECH}^{cL}(Y, c\lambda, \Gamma).$$

Under the scaling there exists a realizable (symplectization) admissible almost complex structure for $c\lambda$, which the isomorphism is independent of.

Before arriving at ECH capacities we must first discuss the ECH spectrum of a contact manifold. Let Y be a closed oriented three dimensional manifold with non-degenerate contact form λ. The restriction $\Gamma = 0$ appears in the following because otherwise the capacities would not be well-defined.

Definition 12 (Definition 3.1, [7])**.** For each positive integer k define

$$\tilde{c}_k(Y, \lambda) = \inf\{L \mid \dim \iota_L(\mathrm{ECH}^L(Y, \lambda, 0)) \geq k\},$$

where ι_L denotes the canonical inclusion $\mathrm{ECH}^L(Y, \lambda, 0) \hookrightarrow \mathrm{ECH}(Y, \lambda, 0)$. The collection $\{\tilde{c}_k(Y, \lambda)\}_{k=1,2,...}$ is the *full* ECH *spectrum* of (Y, λ).

The notion of dimension here is the number of distinct generators of the image of filtered ECH in the full homology. Then in words, the k-th element of the spectrum is given by the least (positive) L such that the filtered ECH for this L still has k generators.

There are two things to be remarked upon, firstly by definition

$$0 \leq \tilde{c}_1(Y, \lambda) \leq \tilde{c}_2(Y, \lambda) \leq \cdots \leq \infty.$$

Secondly, the scaling isomorphism implies that if c is a positve real number then

$$\tilde{c}_k(Y, c\lambda) = c \cdot \tilde{c}_k(Y, \lambda).$$

Now for the full ECH capacities.

Definition 13 (Definition 3.7, [7])**.** Let (M, ω) be a four dimensional Liouville domain with boundary Y. If k is a positive integer, define

$$\tilde{c}_k(M, \omega) := \tilde{c}_k(Y, \lambda)$$

where λ is the contact form on Y with $d\lambda = \omega|_Y$. The collection $\{\tilde{c}_k(M, \omega)\}_{k=1,2,...}$ are the *full* ECH *capacities* of (M, ω).

Lemma 3.8 of [7] asserts that the full ECH capacites of (M, ω) are independent of the contact form λ. Now which we can move to the full ECH capacities for the ellipsoid.

Proposition 4 (Proposition 3.12, [7])**.** *The full* ECH *capacities of an ellipsoid $E(a, b)$ are given by*

$$\tilde{c}_k(E(a, b)) = \mathcal{N}_k(a, b)$$

Where $\mathcal{N}_k(a, b)$ is the $(k + 1)$-th smallest element in the set $\{ma + nb \mid m, n \in \mathbb{Z}_{\geq 0}\}$.

As was the case in contact homology the Reeb vector field corresponding the contact form $\lambda = (1/2) \sum_{j=1}^{2} x_j dy^j - y^j dx^j|_Y$ is

$$R = \frac{2\pi}{a} \frac{\partial}{\partial \theta_1} + \frac{2\pi}{b} \frac{\partial}{\partial \theta_2},$$

with $\partial/\partial\theta_j = x_j \partial/\partial y_j - y_j \partial/\partial x_j$, for $j = 1, 2$.

Suppose that a/b is irrational, then there are only two embedded Reeb orbits $\gamma_1 = \{z_2 = 0\}$ and $\gamma_2 = \{z_1 = 0\}$, which are two circles on the boundary. These orbits are both elliptic and non-degenerate, furthermore, the action is $\int_{\gamma_1} \lambda = a$ as computed previously. Similarly, the action of γ_2, $\int_{\gamma_2} \lambda = b$. The contact form $\lambda|_{\partial E(a,b)}$ is non-degenerate and all ECH generators have the form $\{(\gamma_1, m), (\gamma_2, n)\}$ for $m, n \in \mathbb{Z}_{\geq 0}$. It was already discussed that as $H_1(\partial E(a, b)) = 0$ then any choice of homology class will result in $\Gamma = 0$. By parametrizing differently, for example the γ_1 orbit with multiplicity m, meaning parametrizing $\gamma_1^m = \gamma_1 + \cdots + \gamma_1$ (the m-fold cover of γ_1). One parametrization is

$$\gamma_1^m(t) = (\sqrt{ma/\pi} \exp(2\pi it/ma), 0), \quad t \in [0, ma].$$

Then γ_1 has action

$$\mathcal{A}(\gamma_1^m) = \frac{1}{2} \cdot \frac{ma}{\pi} \int_0^{ma} \frac{2\pi}{ma} dt = ma.$$

We can similarly obtain $\mathcal{A}(\gamma_2^n) = nb$ under a comparable parametrization.

The action is linear and so for a particular generator

$$\mathcal{A}(\{(\gamma_1, m), (\gamma_2, n)\}) = am + nb.$$

As remarked upon before, by Taubes isomorphism only the even homology groups are non-zero and so the chain complex differential vanishes for any J. The dimension, by that we mean the number of generators, for the image of $\text{ECH}^L(\partial E(a,b), \lambda, 0)$ in $\text{ECH}(\partial E(a,b), \lambda, 0)$ is now computed. The finite set of generators for filtered ECH is the following

$$\text{gen}_L := \{(m,n) \in \mathbb{Z}^2_{\geq 0} \mid ma + nb < L\}.$$

From L being positive $\#\text{gen}_L \geq 1$. Thus, the least (positive) L such that $\#\text{gen}_L \geq k$ is the $(k+1)$-th smallest element in $\{ma + nb \mid m, n \in \mathbb{Z}_{\geq 0}\}$.

It is possible to compute the full ECH capacities in the case for a/b rational by approximating above and below by real numbers.

This ends our discussion of ECH and ECH capacities. Through counting J-holomorphic curves in the symplectization of a contact manifold Y we were able to define invariants on Y. One might wonder about the applications of ECH. So far the most role ECH has played outside of symplectic embeddings is in its isomorphism to other homologies, Heegaard-[38] and Monopole Floer Homology. Although both of these homologies are defined in every dimension, whereas ECH is a strictly 3-dimensional theory making them very attractive. The capacities ECH define are invaluable to this thesis and so makes ECH very appealing for its implications in symplectic embeddings.

4 Counting Lattice Points

The purpose of this section is to give a flavour for how one method is applied to computing the discrete set of solutions to the minimal obstruction problem, see the introduction, for $a = 1$, $b = \mu^2$ and $c = d = \mu$. A combinatorial approach is taken that unifies the two previous staircase examples and adds a third.

4.1 Enumerative Combinatorics

Enumerative combinatorics deals with the cardinality of finite sets with special properties. One interesting example of such a set is integer dilations of polytopes intersected with lattice points. Lattice points in dimension d are $\mathbb{Z}^d$. A polytope is precisely the convex hull of a finite set of vertices (smallest convex set containing said vertices) considered as vectors. A polytope $\mathcal{P}$ has dimension d when the dimension of the affine span (all those points reached by $x + \lambda(y - x), x, y \in \mathcal{P}, \lambda \in \mathbb{R}$) of the polytope, is d. A dilation of a polytope is the dilation of the underlying set.

A point in the intersection is either a lattice point of the original polytope or a rational point where the least common denominator of the coordinates, in lowest terms, divides the dilation. To encapsulate this information, a counting function is introduced. Let $\mathcal{P}$ be a polytope of dimension d then

$$\mathcal{L}_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^d), \ \forall t \in \mathbb{Z}_{>0},$$

is its counting function.

We only consider positive integer dilations but there is way of assigning a value to $\mathcal{L}_{\mathcal{P}}(t)$ for $t = 0$ when the interior of the polytope contains the origin. We require this in the following anyway so that it can be argued that the assigned value *is* the true value.

Example 4.1
Take $\mathcal{P} = [0, \gamma]$, $(\gamma \geq 1)$ then $\mathcal{L}_{\mathcal{P}}(t) = \#\{t\mathcal{P} \cap \mathbb{Z}\} = \lfloor t\gamma \rfloor$, the form of this function is dependent on γ.
Case 1: γ is an integer, then $\lfloor t\gamma \rfloor = t\gamma$ for each t, i.e. $\mathcal{L}_{\mathcal{P}}(t)$ is polynomial of degree 1.
Case 2: γ is rational, writing $\gamma = p/q$, $(p \geq q)$ with $\gcd(p,q) = 1$. The symbol $\lfloor t\gamma \rfloor$ is now dependent on whether q divides t. Indeed, let $t \equiv r \bmod q$ so that $\lfloor t\gamma \rfloor = p(t-r)/q + \lfloor r\gamma \rfloor$, which is almost a polynomial except that its coefficients are periodically changing. Further, note that the least integer dilation t of $\mathcal{P}$ that makes $t\gamma$ an integer is exactly q.
Case 3: γ is irrational, here $\lfloor t\gamma \rfloor$ has no right to be a polynomial, or anything close for that matter.

To attempt to formalize the implication of this example it will be important to make additional definitions. A polytope is called integral if it has lattice points as vertices, and called rational if those vertices are rational coordinates, otherwise the polytope is irrational. Moreover, a formal sum of powers of t with coefficients $a_i(t)$ is called a quasi-polynomial with deg $= d$

$$\sum_{i=0}^{d} a_i(t) t^i.$$

Where the $a_i : \mathbb{N} \to \mathbb{Z}$ are periodic with finite period, for each i.

The period of a quasi-polynomial is the least common integer between the periods of the coefficient functions denoted P. The denominator of a polytope $\mathcal{P}$ is the least integer D such that $D\mathcal{P}$ is integral. Period collapse is the situation where $P < D$, i.e. the period is less than the denominator.

As found in Case 1 where γ was an integer the counting function was a polynomial, this generalizes to all polytopes with integral lattice points as proved originally by E. Ehrhart ([39]). This theory is actually called Ehrhart theory. For Case 2, γ being rational, generalizes to all rational polytopes having a quasi-polynomial for their counting function. In some interesting situations the period of the counting function is equal to 1 so that the quasi-polynomial is actually a polynomial, thus $D > P = 1$ implies period collapse in this situation. Although, notice that for the example we had $D = P = q$ so period collapse does not occur in the example. For Case 3, when vertices are irrational the theory is burgeoning [40] here counting functions have been found to be polynomials and quasi-polynomials. Morally the denominator of an irrational polytope should be infinite. Consider trying to applying the definition and arriving at the conclusion that there exists no integer large enough to make an irrational polytope integral. By this presumption every irrational polytope with a (quasi-) polynomial counting function exhibits, an extreme form of, period collapse. Which makes this case especially interesting for combinatorialists.

Example 4.2
Let $\mathcal{P} = \mathcal{T}(1,1)$, the triangle with vertices $(0,0), (0,1), (1,0)$, its counting function gives the $(t+1)$-th triangle number for each $t \in \mathbb{Z}_{\geq 1}$, i.e. $\mathcal{L}_{\mathcal{T}(1,1)}(t) = (1/2)(t+1)(t+2)$. This is a polynomial in t and a stepping stone to examples of period collapse.

From now on we will work only in dimension 2 ($d = 2$), so all our polytopes will be polygons, and all our polygons will be triangles. With this restriction it is time to reflect on how one can connect this theory to symplectic embeddings. The work discussed in previous chapters comes to the forefront here. Recall the ECH capacities of ellipsoids $E(u,v)$, $\mathcal{N}_k(u,v)$, $u, v \in \mathbb{R}^2_{>0}$. Furthermore, define $\mathcal{T}(u,v)$ to be the following triangle.

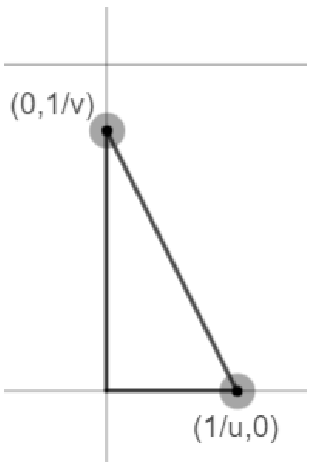

Figure 3: Triangle $\mathcal{T}(u,v)$

Introduce a new counting function,

$$L_{u,v}(t) = \#\{k \ : \ \mathcal{N}_k(u,v) \leq t\}$$

(the number of those k for which $\mathcal{N}_k(u,v) \leq t$) for $t \in \mathbb{Z}_{>0}$. We will show that the two counting functions are equal, i.e $L_{u,v}(t) = \mathcal{L}_{\mathcal{T}(u,v)}(t)$, $\forall t \in \mathbb{Z}_{>0}$.

Let, $ux + vy = mu + nv$, $m, n \in \mathbb{Z}_{\geq 0}, x, y \in \mathbb{R}_{\geq 0}, u, v \in \mathbb{R}_{>0}$ and some arbitrary $t \in \mathbb{Z}_{>0}$. The following sets are equal because of the definition of $\mathcal{N}_k(E(u,v))$,

$$\{k \mid \mathcal{N}_k(E(u,v)) \leq t\} = \{(m,n) \in \mathbb{Z}^2_{\geq 0} \mid mu + nv \leq t\}.$$

The next equality follows from the identity we called forth,

$$\{(m,n) \in \mathbb{Z}^2_{\geq 0} \mid mu + nv \leq t\} = \{(x,y) \in \mathbb{Z}^2_{\geq 0} \mid ux + vy \leq t\}.$$

Thus, as the sets are finite

$$L_{u,v}(t) = \#\{k \mid \mathcal{N}_k(E(u,v)) \leq t\} = \#\{(x,y) \in \mathbb{Z}^2_{\geq 0} \mid ux + vy \leq t\}.$$

Note that the last expression is precisely the number of integer points in the t-scaling of $\mathcal{T}(u,v)$ and so

$$L_{u,v}(t) = \mathcal{L}_{\mathcal{T}(u,v)}(t).$$

Then as $t \in \mathbb{Z}_{>0}$ was arbitrary the above equality holds for all $t \in \mathbb{Z}_{>0}$. We can now reach the interesting equivalence between symplectic embeddings and combinatorics via sharpness of ECH capacities for the ellipsoid, see introduction.

Lemma 2 (Sharpness for Combinatorics)**.**

$$E(a,b) \overset{s}{\hookrightarrow} E(c,d) \;\Leftrightarrow\; \mathcal{L}_{\mathcal{T}(a,b)}(t) \geq \mathcal{L}_{\mathcal{T}(c,d)}(t),\; \textit{for each } t \in \mathbb{Z}_{>0}$$

Proof. Given sharpness

$$E(a,b) \overset{s}{\hookrightarrow} E(c,d) \;\Leftrightarrow\; c_k(E(a,b)) \leq c_k(E(c,d)),\; \text{for each } k \in \mathbb{Z}_{\geq 0}.$$

A necessary condition for the right hand side of the implication to hold is

$$\{k \;:\; c_k(E(c,d)) \leq t\} \subset \{k \;:\; c_k(E(a,b)) \leq t\},\; \text{for each } t \in \mathbb{Z}_{>0}.$$

This is from $c_k(E(a,b)) \leq t$ being a weaker condition than $c_k(E(c,d)) \leq t$. By induction on t sufficiency follows. As each set is finite this is also equivalent to

$$\#\{k \;:\; c_k(E(c,d)) \leq t\} \leq \#\{k \;:\; c_k(E(a,b)) \leq t\}, \qquad \text{for each } t \in \mathbb{Z}_{>0}, \text{ equivalently}$$
$$L_{a,b}(t) \leq L_{c,d}(t), \qquad \text{for each } t \in \mathbb{Z}_{>0}.$$

Moreover, by the above argument

$$\mathcal{L}_{\mathcal{T}(c,d)}(t) \leq \mathcal{L}_{\mathcal{T}(a,b)}(t),\; \text{for each } t \in \mathbb{Z}_{>0}.$$

So that the chain of equivalences imply the Lemma. □

Even though this equivalence between symplectic embeddings and the inequality of counting functions is a tool unto itself. So far, though, practical applications seem to imply a need to prove equality of counting functions. For this we will introduce another definition, two polytopes are said to be Ehrhart equivalent if they have the same counting function.

4.2 Combinatorical Proofs of Symplectic Embeddings

It will be necessary to repeatedly refer back to a diophantine equation (an equation that requires solutions to be integers). To that end define

$$f_{k,l}(p,q) = kp^2 - (l+k+1)pq + lq^2 + 1$$

and seek the positive integer solutions to

$$f_{k,l}(p,q) = 0. \tag{4.1}$$

The following figure shows the positive part of the underlying curve to this equation when $k = l = 1$.

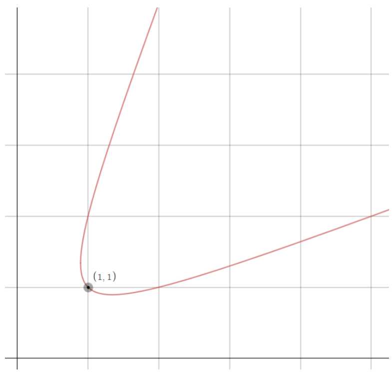

Figure 4: A (positive) portion of the underlying equation $x^2 - 3xy + y^2 + 1$

The following Theorem is due to Cristofaro-Gardiner and Kleinman that relates the solutions of (4.1) to Ehrhart equivalence. Restrict to $(k,l) \in \{(1,1),(2,1),(3,2)\}$, the only positive integers for which both k and l both divide $k+l+1$. There are counterexamples to the theorem when not restricting for instance when $(k,l) = (3,1)$, Remark 1.6, [5].

Theorem 8 (Theorem 1.3, [5])**.** *Fix $(k,l) \in \{(1,1),(2,1),(3,2)\}$ and triangles $\mathcal{T}(q/kp, p/ql)$ and $\mathcal{T}(1/k, 1/l)$, the two triangles are Ehrhart equivalent if and only if (4.1) holds for these (p,q,l,k), $(p,q) \in \mathbb{Z}^2_{\geq 0}$ as well as $\gcd(kp^2, lq^2) = 1$.*

To make this theorem useful to understanding points on the staircases we will also need to identify what exactly are the positive integers solutions of (4.1) for each pair (k,l). These will turn out to be exactly related to the recursive sequences found for past staircases.

The equation (4.1) may at first seem arbitrary but what has recently become significant in this area is leading and sub-leading asymptotics [41]. By leading asymptotics it is meant the coefficient of the highest power of a (quasi-)polynomial. The associated coefficient to the t^2 term of $\mathcal{L}_{\mathcal{T}(u,v)}(t)$ is the area of $\mathcal{T}(u,v)$

$$\text{Area}(\mathcal{T}(q/kp, p/ql)) = \frac{1}{2kl} = \text{Area}(\mathcal{T}(1/k, 1/l)).$$

The coefficient of t in the counting function is the sub-leading asymptotic and corresponds to the "affine perimeter". Theorem 8 then is stating the necessary and sufficient condition for Ehrhart equivalence is equality of affine perimeter. As it happens fixing $(k,l) \in \{(1,1),(2,1),(3,2)\}$ will mean that the counting functions will be polynomials.

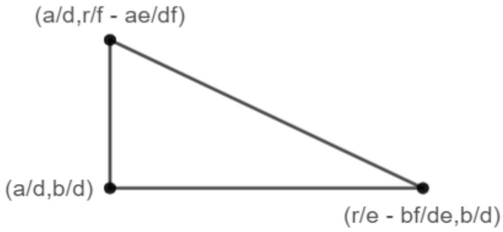

Figure 5: General Triangle

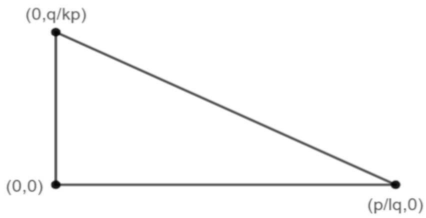

Figure 6: Specific Triangle

Before that though we will need to explain what the counting functions are for the triangles above. Let $\gcd(kp^2, lq^2) = 1$, now Theorem 2.10. of Beck and Robins [42] implies, for $a = b = 0$, $e = kp^2$, $f = lq^2$ and $r = pq$ (Fig. 5 to Fig. 6)

$$\begin{aligned} L_{\mathcal{T}(q/kp,p/ql)}(t) &= \frac{1}{2kl}t^2 + \frac{1}{2}t\left(\frac{q}{kp} + \frac{p}{lq} + \frac{1}{klpq}\right) \\ &+ \frac{1}{4}\left(1 + \frac{1}{kp^2} + \frac{1}{lq^2}\right) + \frac{1}{12}\left(\frac{kp^2}{lq^2} + \frac{lq^2}{kp^2} + \frac{1}{lkp^2q^2}\right) \\ &+ s_{-tpq}(kp^2, 1; lq^2) + s_{-tpq}(lq^2, 1; kp^2). \end{aligned} \tag{4.2}$$

Where the final line contains two *Fourier-Dedekind* sums, given in general

$$s_n(a,b;c) = \frac{1}{c}\sum_{i=1}^{c-1} \frac{\xi_c^{ni}}{(1-\xi_c^a)(1-\xi_c^b)}.$$

With ξ_c the c-th root of unity. The counting function $L_{\mathcal{T}(1/k,1/l)}(t)$ follows similarly by taking $p = q = 1$. If we were given that the two functions are equal then at least the coefficients in front of t and t^2 must be equal. For t^2 this is immediate as the coefficients are both $1/kl$. However, the equation resulting from equating coefficients of t gives something more interesting. Namely,

$$\frac{q}{kp} + \frac{p}{lq} + \frac{1}{klpq} = \frac{1}{k} + \frac{1}{l} + \frac{1}{kl}.$$

Multiplying on both sides by $klpq$ gives

$$lq^2 + kp^2 + 1 = lpq + kqp + pq,$$

and rearranging is exactly (4.1)

$$lq^2 + kp^2 - (l+k+1)pq + 1 = 0.$$

This implies that Ehrhart equivalence is necessary. Now for the sufficiency we rely on the computation by Cristofaro-Gardiner and Kleinman. The authors work with the Fourier-Dedekind sums of $L_{\mathcal{T}(q/kp,p/ql)}(t)$ via discrete Fourier series. Along the way Beck and Robins Theorem 8.8 (Rademacher reciprocity) is stated incorrectly, here it is corrected. The correction is the change from a cyclic permutation of (a,b,c) to what is stated below.

Lemma 3 (Rademacher reciprocity)**.** *Let* $n = 1, 2, ..., (a + b + c) - 1$*. Then*

$$s_n(b, c; a) + s_n(a, c; b) + s_n(a, b; c) =$$

$$-\frac{n^2}{2abc} + \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{ca} + \frac{1}{bc}\right) - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right).$$

An alternative formulation for $n = 0$, Corollary 8.7 of Beck and Robins, also corrected.

Lemma 4.

$$s_0(b, c; a) + s_0(a, c; b) + s_0(a, b; c) =$$

$$1 - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right).$$

From these two lemmas and previous work done on the Fourier-Dedekind sums sufficiency can be proven on a case by case basis. A crucial step in the argument, impossible without the correction, is based on the fact that for all a, b and n, $s_n(a, b; 1) = 0$. Which follows from the term containing an empty sum that is conventionally 0. Thus, $s_n(a, 1; b) + s_n(b, 1; a) + s_n(a, b; 1) = s_n(a, 1; b) + s_n(b, 1; a)$ which is the type of term inside the counting functions that Cristofaro-Gardiner and Kleinman obtain.

In the special case $(k, l) = (1, 1)$ these lemmas imply that the counting function is precisely that for the triangle in example 4.2, namely $(1/2)(t + 1)(t + 2)$. As the triangles, $T(q/p, p/q)$, we are concerned with have a non-trivial denominator these are examples of triangles that exhibit period collapse.

Now we will expose the sequences used to unify the approach. That approach being, to calculate the values of functions

$$c_{a,k,l} = \inf\{\lambda \mid E(1, a) \overset{s}{\hookrightarrow} B^4(\lambda)\},$$

for $(k, l) \in \{(1, 1), (2, 1), (3, 2)\}$. Recursively define a sequence dependent on k and l, $r(k, l)_0 = r(k, l)_1 = 1$

$$r(k, l)_{2n+1} = \frac{k + l + 1}{k} r(k, l)_{2n} - r(k, l)_{2n-1},$$
$$r(k, l)_{2n} = \frac{k + l + 1}{l} r(k, l)_{2n-1} - r(k, l)_{2n-2}.$$

With $(k, l) = (1, 1)$

$$r(1, 1)_{2n+1} = 3r(1, 1)_{2n} - r(1, 1)_{2n-1},$$
$$r(1, 1)_{2n} = 3r(1, 1)_{2n-1} - r(1, 1)_{2n-2}.$$

Which is the same recursive definition for both even and odd terms and also the recursive definition of the odd-indexed Fibonacci numbers so that $r(1, 1)_n = g_n$, mentioned in the introduction. We also define the sequence paramount to the minimal obstruction problem for $a = 1, b = \mu$, and $c = \mu, d = (k/l)\mu$. Define,

$$a(k, l)_n = \begin{cases} \frac{kr(k,l)^2_{n+1}}{lr(k,l)^2_n}, & n \text{ even} \\ \frac{lr(k,l)^2_{n+1}}{kr(k,l)^2_n}, & n \text{ odd.} \end{cases}$$

Again for $(k, l) = (1, 1)$, both even and odd terms are the same and equal to $r(1, 1)^2_{n+1}/r(1, 1)^2_n = g^2_{n+1}/g^2_n$. These are special because when $\mu = \sqrt{a(1, 1)_n}$ we have the discrete set of solutions to the minimal obstruction which was the goal to solve.

4.3 Solving the Diophantine Equation

We want to solve (4.1) in the special case of $(k, l) = (1, 1)$ which contains all the information about triangles $\mathcal{T}(q/p, p/q)$ equivalent to $\mathcal{T}(1, 1)$. Then by sharpness for combinatorics, Lemma 2, implies a symplectic embedding

$$E\left(\frac{q}{p}, \frac{p}{q}\right) \overset{s}{\hookrightarrow} B^4(1).$$

By scaling up and down by p/q means the next symplectic embedding is equivalent to the previous

$$E\left(1, \frac{p^2}{q^2}\right) \overset{s}{\hookrightarrow} B^4\left(\frac{p}{q}\right).$$

Therefore, if the $a_n(1, 1)$ really do solve our problem then $(p, q) = (g_{n+1}, g_n)$ for some $n \in \mathbb{Z}_{\geq 0}$. We would then expect $(p, q) = (g_{n+1}, g_n)$ to solve $f_{1,1}(p, q) = 0$. What we will show is that in some specific way these are the only solutions, to the Diophantine equation at least. Proposition 4.1, [5] solves (4.1) in more generality and strength than is really needed for what we want to solve. However, we will follow the structure but not the content.

The equation we care about solving is $x^2 - 3xy + y^2 + 1 = 0$ which we now rearrange to

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 1 = 0.$$

The primary solution to this equation is $(x, y) = (1, 1)$ as $1 - 3 + 1 + 1 = 0$. More solutions will be found via applying matrices g to $(1, 1)$ such that

$$g^T \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix} g = \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}.$$

If those solutions are also to be integers then we need g to also have integer entries. The matrices that we care about will have an interpretation on the curve itself as horizontal and vertical transformations that are the basis for matrices g. Explicitly,

$$\sigma = \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad \tau = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}.$$

These matrices are involutions which imply that they canonically split $\mathbb{R}^2$ into $\mathbb{Z}^2$ and $\mathbb{R}^2 \setminus \mathbb{Z}^2$. As σ fixes the vertical coordinate it is a horizontal transformation and τ is a vertical transformations because it fixes the horizontal coordinate. Now to show σ and τ satisfy (4.3).

$$\varphi_\sigma(A) = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} \\ \frac{3}{2} & -\frac{7}{2} \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 + \frac{3}{2} \\ -\frac{3}{2} & \frac{9}{2} - \frac{7}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix},$$

and,

$$\varphi_\tau(A) = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & \frac{3}{2} \\ \frac{3}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} - \frac{7}{2} & -\frac{3}{2} \\ -3 + \frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}.$$

In the following, assume $(x, y) \in \{(p, q) \mid f_{(1,1)}(p, q) = 0\}$, this is for convenience to lower the density of notation. What we will now attempt is to find all positive integer solutions of $f_{(1,1)}(p, q)$ by creating a graph. This graph will be the limit of a sequence of graphs all having the same properties and the limit will have vertices as precisely the solutions we care about. First, for a definition and brief conversation about graphs.

Definition 14. A *graph* $G = (V, E)$ is a pair. The first of the pair is V a set containing vertices, v_i, and the second E a subset of the power set of V containing only two point sets, edges $e_{i,j} = \{v_i, v_j\}$. We will use the convention $V(G)$ for the set of vertices of the graph and $E(G)$ for the set of edges. A graph is said to be connected if between each pair of vertices there exists a trail of edges connecting them.

An important distinction to make is the graph G as an abstract object and G being embedded into another space X. In the latter we have that $V \subset X$ and $E \subset \text{Lines}(X)$, where $\text{Lines}(X)$ is the set of paths for each pair of points in $X \times X$. We will restrict the edges to being the "shortest" path, one for each pair of points, which is found through choosing a metric on X. We are about to take $X = \mathbb{R}^2$ so we choose the Euclidean metric.

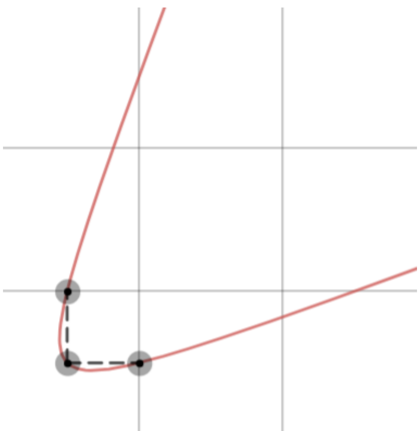

Figure 7: First example, Graph G_0

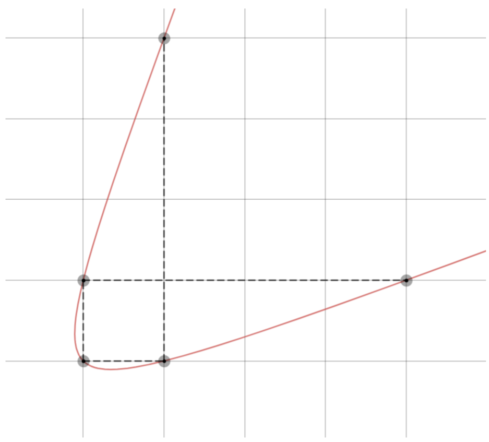

Figure 8: Basis for the induction, Graph G_1

The first example of a graph we will encounter is $G_0 = (V, E)$ with $V(G_0) = \{v_0, v_1, v_2\}$ and $E(G_0) = \{e_{0,1}, e_{0,2}\}$. To embed this graph in $\mathbb{R}^2$, we need to choose points for the v_i. Let $v_0 = (1, 1)$, $v_1 = (2, 1)$, and $v_2 = (1, 2)$ Fig 7. Then, $e_{0,1}$ and $e_{0,2}$ are the straight lines connecting, v_0 to v_1 and v_0 to v_2, respectively. This graph is connected because, v_1 is connected to v_0 by $e_{0,1}$, v_1 is connected to v_2 via $e_{0,1}$ and $e_{0,2}$, and v_0 is connected to either v_1 from $e_{0,1}$ or v_2 from $e_{0,2}$. A fact from graph theory is that adding a vertex to a graph along with an edge connecting it to any point of the connected part of the graph results in another connected graph.

The sets $\{\pi_1(v_2) > x\}$ and $\{\pi_2(v_1) > y\}$ (π_i projection onto the i-th coordinate, $i = 1, 2$) are the sections along the curve from v_0 to v_1 and v_1 to v_2, respectively. The set $\{\pi_1(v_2) > x\}$ is contained entirely in $\mathring{\square}(\{(1, 1), (1, 2), (0, 1), (0, 2)\})$ and, similarly, $\{\pi_2(v_1) > y\}$ is entirely contained in $\mathring{\square}(\{(1, 1), (2, 1), (1, 0), (2, 0)\})$. Therefore, neither $\{\pi_1(v_2) > x\}$ nor $\{\pi_2(v_1) > y\}$ contain integers. The operation $\mathring{\square}$ takes points as its arguments and makes a set, the interior of the square made out of said points.

We will now induct to form a connected graph that will have precisely all the integer solutions to $f_{1,1}(x, y) = 0$ as vertices.

Proof.

(1) **Basis**. Form the graph $G_1 = (V, E)$ with $V(G_1) = V(G_0) \cup \{\sigma(v_2), \tau(v_1)\}$ and label $v_3 = \sigma(v_2)$ and $v_4 = \tau(v_1)$ so that $V(G_1) = \{v_0, v_1, v_2, v_3, v_4\}$, depending on the new labeling write $E(G_1) = E(G_0) \cup \{e_{2,3}, e_{1,4}\} = \{e_{0,1}, e_{0,2}, e_{2,3}, e_{1,4}\}$ Fig 8. As we have seen, neither $\{\pi_1(v_2) > x\}$ nor $\{\pi_2(v_1) > y\}$ contain integers. By the property of τ preserving solutions and being a vertical transformation, $\{\pi_1(v_4) > x\} = \tau(\{\pi_2(v_1) > y\})$. Furthermore, the property that τ is an involution implies $\{\pi_1(v_4) > x\}$ contains no integers. It follows in almost the same way that $\{\pi_2(v_3) > y\} = \sigma(\{\pi_1(v_2) > x\})$ contains no integers. Here we have added two vertices, both with edges connecting them to G_0. Therefore, G_1 must be connected by the fact above.

(2) **IH**. Assume $G_n = (V, E)$ has $V(G_n) = \{v_0, v_1, v_2, ..., v_{2n-1}, v_{2n}\}$ and

$$E(G_n) = \{e_{0,1}, e_{0,2}, e_{2,3}, e_{1,4}, ..., e_{2n-2,2n-1}, e_{2n-3,2n}\}.$$

Also assume that G_n is connected and neither $\{\pi_1(v_{2n}) > x\}$ nor $\{\pi_2(v_{2n-1}) > y\}$ contain integers.

(3) **Induction Step**. Form the graph $G_{n+1} = (V, E)$ with $V(G_{n+1}) = V(G_n) \cup \{\sigma(v_{2n}), \tau(v_{2n-1})\}$ and label $v_{2n+1} = \sigma(v_{2n})$ and $v_{2n+2} = \tau(v_{2n-1})$ so that $V(G_{n+1}) = \{v_0, v_1, v_2, ..., v_{2n-1}, v_{2n}, v_{2n+1}, v_{2n+2}\}$. Dependent on the new labeling, write

$$E(G_{n+1}) = E(G_n) \cup \{e_{2n,2n+1}, e_{2n-1,2n+2}\} = \{e_{0,1}, e_{0,2}, e_{2,3}, e_{1,4}, ..., e_{2n-2,2n-1}, e_{2n-3,2n}, e_{2n,2n+1}, e_{2n-1,2n+2}\}.$$

By the hypothesis, neither $\{\pi_1(v_{2n}) > x\}$ nor $\{\pi_2(v_{2n-1}) > y\}$ contain integers. Then, the property of τ preserving solutions and being a vertical transformation, $\{\pi_1(v_{2n+2}) > x\} = \tau(\{\pi_2(v_{2n-1}) > y\})$. Furthermore, the property that τ is an involution implies $\{\pi_1(v_{2n+2}) > x\}$ contains no integers. It follows in almost the same way that $\{\pi_2(v_{2n+1}) > y\} = \sigma(\{\pi_1(v_{2n}) > x\})$ contains no integers. In this step we have added two vertices to G_n, both with edges connecting to G_n making G_{n+1} connected by the fact already discussed.

(4) **Conclusion**. For each $n \in \mathbb{Z}_{\geq 1}$, G_n is connected and neither $\{\pi_1(v_{2n+2}) > x\}$ nor $\{\pi_2(v_{2n+1}) > y\}$ contain integers.

□

From the above analysis of G_0 we have, for each $n \in \mathbb{Z}_{\geq 0}$, G_n is connected and neither $\{\pi_1(v_{2n+2}) > x\}$ nor $\{\pi_2(v_{2n+1}) > y\}$ contain integers. As $V(G_n) \subset V(G_{n+1})$ and $E(G_n) \subset E(G_{n+1})$ the limit of these graphs is well-defined. We call the limit G_∞. By construction G_∞ is connected and $V(G_\infty) = \{(x, y) \mid f_{(1,1)}(x, y) = 0\} \cap \mathbb{Z}^2$.

We claim that as σ and τ contain the recurrence relation for the odd-index Fibonacci numbers along with initial solution $(g_1, g_0) = (1, 1)$ the vertices of G_∞ are

$$V(G_\infty) = \{(g_{n+1}, g_n)\}_{n \geq 0} \cup \{(g_n, g_{n+1})\}_{n \geq 1}.$$

This characterizes all the positive integer solutions to (4.1) for $(k, l) = (1, 1)$.

Isolated Solutions to the Minimal Obstruction Problem

From considering the relationship between symplectic embeddings and counting functions we can now calculate $c_{a(1,1)_n,k,l}$. For now though we stay general and simplify by letting a_n and r_n denote by $a(k, l)_n$ and $r(k, l)_n$, respectively. With (k, l) fixed in $\{(1, 1), (2, 1), (3, 2)\}$ as before. Need to show that $c_{a_n,k,l}$ are points on the sub-graph that are the discrete set of solutions to the minimum obstruction problem.

From the definition of a_n, $(l/k)a_n = (r_{n+1}/r_n)^2$ for n even and $= (l^2/k^2)(r_{n+1}/r_n)^2$ for n odd. So what needs to be shown is

$$c_{a(k,l)_n,k,l} = \begin{cases} \frac{r_{n+1}}{r_n} & n \text{ even} \\ \frac{lr_{n+1}}{kr_n} & n \text{ odd} \end{cases}.$$

We will show the above for $(k, l) = (1, 1)$. We take $(p, q) = (g_{n+1}, g_n)$ which we have proved satisfy the Diophantine equation. Thus we must have that $T(q/p, p/q)$ and $T(1, 1)$ are equivalent, as long as $p = g_{n+1}$ and $q = g_n$ are relatively prime by applying Theorem 8. We now prove this last requirement by induction.

(i) **Basis**. We have $g_0 = 1$ is relatively prime to $g_1 = 1$ by choice of $(x, y) = (1, 0)$,

$$g_0 \cdot x + g_1 \cdot y = g_0 \cdot 1 + g_1 \cdot 0 = 1 \cdot 1 + 1 \cdot 0 = 1.$$

(ii) **IH**. Assume g_{n-1} and g_n are relatively prime.

(iii) **Induction Step**. By assumption there exist integers (x, y) such that

$$g_{n-1} \cdot x + g_n \cdot y = 1.$$

As $g_{n+1} = 3g_n - g_{n-1}$ take $(x', y') = (y + 3x, -x)$ so that we have

$$g_n \cdot x' + g_{n+1} \cdot y' = g_n \cdot y + g_n \cdot (3x) - 3g_n \cdot x + g_{n-1} \cdot x = g_{n-1} \cdot x + g_n \cdot y = 1.$$

(iv) **Conclusion**. Thus, g_n and g_{n+1} are relatively prime for all $n \geq 0$.

5 Conclusion

The final conclusion from the previous section is that $c_{a_n,1,1} = g_{n+1}/g_n$. As $a_n = g_{n+1}^2/g_n^2$ we have then verified a family of symplectic embeddings, parametrized by $n \in \mathbb{Z}_{\geq 0}$, namely

$$E\left(1, \frac{g_{n+1}^2}{g_n^2}\right) \overset{s}{\hookrightarrow} B\left(\frac{g_{n+1}}{g_n}\right).$$

The ellipsoid and ball above have the same volume, g_{n+1}^2/g_n^2, so that the points on the sub-graph of $c_{a,1,1}$ are $(g_{n+1}^2/g_n^2, g_{n+1}/g_n)$. These are then the discrete set of solutions to the minimal obstruction problem, which was the verification that this thesis aimed for.

We will now discuss how we have come to this solution. The most difficult problem lay in understanding the tool of ECH capacities. In essence, when taking the capacities for granted, the level of this thesis becomes merely of an advanced undergraduate. Therefore, we focus on the penultimate story.

Let $a, b > 0$ such that a/b is irrational and $Y = \partial E(a,b)$. The set Y is described by a line $x/a + y/b = 1$, $(x,y) = (\pi|z_1|^2, \pi|z_2|^2)$ variables in $\mathbb{R}^2_{\geq 0}$. By giving a topology on the line and pre-imaging back to $\mathbb{C}^2$ we can give Y the topology of a smooth (closed) manifold. With the smoothness comes a contact 1-form λ giving rise to the contact structure ξ and the Reeb vector field R (although all we care about are R's orbits). Through λ we can define $\mathcal{A}$, the symplectic action, with critical points as Reeb orbits. We defined a way to turn Y into a symplectic manifold $\mathbb{R} \times Y$, and by following Morse theory we were motivated to study the "flow lines" in $\mathbb{R} \times Y$ that asymptotically approach cylinders defined by Reeb orbits. The flow lines are defined not via downward gradient flow but by a souped-up version of the Cauchy-Riemann equations that describe J-holomorphicity via an almost complex structure J on $\mathbb{R} \times Y$. By the Riemann-Roch theorem, the space of J-holomorphic curves has a local finite dimension which under further conditions can make this space discrete. It is the count of sufficiently identified J-holomorphic curves of a specific type that defines the embedded contact complex and, under a lot of work, the embedded contact homology. From here the full ECH spectrum for Y was defined through filtered ECH and the minimum number of generators needed to generate each filtered ECH module. The full ECH capacities were defined via the full ECH spectrum whenever Y is the boundary of a Liouville domain and $\omega = d\lambda|_Y$, which is the case as $Y = \partial E(a,b)$. Finally, by approximating rational ellipsoids by irrationally parametrized ones we can define the ECH spectrum for $E(a,b), a, b \in \mathbb{Q}_{>0}$ and so imply when a symplectic embedding exists from the sharpness of ECH capacities for ellipsoids.

The symplectic embeddings that we investigated were for the ellipsoid into ball problem, with the extra condition of equal volume or minimal obstruction condition. In view of furthering this research, consider investigating how to apply the graph theory method of the past section to $(k,l) \in \{(2,1),(3,2)\}$ to find the discrete solutions to the other minimal obstruction problems.

A Background Geometry

A.1 Smooth Manifolds

Definition 15 (Definition 5.1, [13]). A topological space M is *locally Euclidean of dimension* n if every point p in M has a neighbourhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a chart, U a coordinate neighbourhood, and ϕ a coordinate map on U. The chart (U, ϕ) is said to be (centered) about p if U is a neighbourhood of p and $\phi(p) = 0$.

Note : We can always arrange for a chart to be "(centered) about" p. When we say smooth in the following, and in the rest of this thesis, what is meant is infinitely many times differentiable (C^∞). If smooth is omitted at any moment it is because it is assumed. If a circumstance requires a class less than smooth, such a situation will be indicated.

Definition 16 (Definition 5.2, [13]). A *topological manifold* is a Hausdorff, second countable, locally Euclidean space of some dimension n.

Definition 17 (Definition 5.5, [13]). Two charts (U, ϕ), (V, ψ) of topological manifolds are said to be *smooth compatible* if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$
$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V),$$

are smooth.

Note : Compatibility holds vacuously over $U \cap V = \emptyset$. The two maps are inverses of each other and so are collectively called the transition map, and the plural is left for the collection of transition maps on a manifold.

Definition 18 (Definition 5.6, [13]). A *smooth atlas* on a locally Euclidean topological space M is a collection $\{(U_\alpha, \phi_\alpha)\}$ of pairwise smooth compatible charts that cover M, ($M = \bigcup_\alpha U_\alpha$).

An atlas is called maximal if it is not contained in a larger atlas, in fact there is a unique maximal atlas on every topological manifold (Prop. 5.10, [13])

Definition 19 (Definition 5.9, [13]). A *smooth manifold* is a topological manifold with a maximal smooth atlas. A manifold is said to have dimension n if its connected components have dimension n.

There is an obvious extension to the Cartesian product of smooth manifolds. Let M and N be smooth manifolds, with maximal smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$, respectively. The maximal atlas for $M \times N$ can be obtained by disjoint union of smooth compatible charts with the atlas $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$.

Definition 20 (Definition 6.1, [13]). Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be *smooth* at $p \in M$ if there exists a chart (U, ϕ) about p for which $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth at p. The function f is smooth if it is smooth at every point $p \in M$.

Note : The function f is not assumed to be continuous, however the definition implies that f cannot be otherwise.

Definition 21 (Definition 6.5, [13]). Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is smooth at $p \in N$ if there are charts (V, ψ) about $F(p)$ and (U, ϕ) about p such that $\psi \circ F \circ \phi^{-1}$: $\phi(F^{-1}(V) \cap U) \to \psi(F(F^{-1}(V) \cap U))$ is smooth at $\phi(p)$. The continuous map F is said to be *smooth* (between N and M) if it is smooth at every point $p \in N$.

A diffeomorphism F between two manifolds is a bijective smooth map $F : N \to M$ whose inverse F^{-1} is smooth. In particular transition maps are (local) diffeomorphisms. It is also possible to re-write the definitions for chart as a pair (U, ϕ), U still an open set but instead ϕ would be a diffeomorphism onto its image.

It is a fact that $f : M \to \mathbb{R}^n$ is smooth if and only if each of its component functions $f_1, ..., f_n : M \to \mathbb{R}$ are smooth (Prop. 6.13, [13]). Furthermore, the composition of smooth functions is smooth. (Prop. 6.9, [13])

Partial derivatives of maps from manifolds into Euclidean space or into other manifolds are necessary to define. Let M be an n-dimensional smooth manifold, $f : M \to \mathbb{R}$ a smooth map and $(U, \phi) = (U, x^1, ..., x^n)$ a chart on M. The components of ϕ are x^i defined by $r^i \circ \phi$ with r^i the standard coordinates on $\mathbb{R}^n$, for $i = 1, .., n$. Then,

$$\frac{\partial f}{\partial x^i} = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}$$

are equal as functions on $\phi(U)$.

In the special case that $f = \phi$ we have $\frac{\partial x^j}{\partial x^i} = \delta_{ij}$, where δ_{ij} is the Kronecker delta which is 0 for $i \neq j$ and 1 for $i = j$.

Definition 22 (Definition 6.23, [13])**.** Let $F : N \to M$ be a smooth map, and let $(U, \phi) = (U, x^1, ..., x^n)$ and $(V, \psi) = (V, y^1, ..., y^m)$ be charts on N and M respectively such that $F(U) \subset V$ (this condition allows us to skip the continuity assumption present throughout). Denote by

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

the i-th component in the chart (V, ψ). Then the matrix $[\partial F^i / \partial x^j]$ is called the *Jacobian matrix.*

When taking $M = N$ and $F = \mathbb{1}_M$ the Jacobian matrix becomes $[\partial y^i / \partial x^j]$. This is the Jacobian matrix of the transition map $\psi \circ \mathbb{1}_M \circ \phi^{-1} = \psi \circ \phi^{-1}$. Thus, the partial derivatives of the transition map are

$$\frac{\partial(\psi \circ \phi^{-1})^i}{\partial r^j} = \frac{\partial y^i}{\partial x^j}.$$

Equal as functions on $\phi(U \cap V)$. We will call a manifold that has transition maps with positive Jacobian determinant orientable. An example of an orientable manifold is the circle and the classical non-orientable surface is the möbuis strip.

This extension of partial derivatives to manifolds allows us to consider vectors on a manifold locally as linear combinations of partial derivative operators in the respective local coordinates given by the chart on the manifold. Furthermore, we define an operation on a function such that the operated function applied to a partial derivative will compute the derivative of the function. Indeed, $\mathrm{ev}_f(\partial/\partial x^i) = \partial f / \partial x^i$ where ev stands for "evaluate the function f by the following derivative" taken as the argument. In particular consider ev_{x^j}, as we have already seen that $\mathrm{ev}_{x^j}(\partial/\partial x^i) = \delta_{ij}$. This seems remarkably similar to the dual basis of a vector space and, in fact, it is!

A.2 Vector Bundles

Given a onto map $\pi : E \to M$, the inverse image $\pi^{-1}(\{p\})$ (for $p \in M$) is called the fiber at p. Alternative notation, $\pi^{-1}(\{p\}) = E_p$ with $\pi^{-1}(U) = E_U$. For two onto maps $\pi : E \to M$, $\pi' : E' \to M$ (same co-domain M), $\phi : E \to E'$ is called *fiber-preserving* if $\phi(E_p) \subset E'_p$ for all $p \in M$.

Let $\pi : E \to M$ be a smooth onto map between manifolds E and M. We call $\pi : E \to M$ *locally trivial vector bundle of rank* r if

(i) each fiber E_p has the structure of a vector space of dimension r,

(ii) for each $p \in M$ there exists U an open neighbourhood of p and $\phi : E_U \to U \times \mathbb{R}^r$ a smooth fiber-preserving map which when restricted to any fiber E_q, $q \in U$ is a vector space isomorphism to $\{q\} \times \mathbb{R}^r$.

We call U a *trivializing open set* (for E) and ϕ a *trivialization* (of E over U).

A collection $\{(U, \phi)\}$, $\{U\}$ an open cover of M, is called a local trivialization (for E), and $\{U\}$ is called a trivializing open cover (of M for E).

A smooth (real) vector bundle of rank r is the triple (E, M, π) consisting of (i) E (total space), (ii) M (base space), both smooth manifolds, and (iii) a smooth onto map $\pi : E \to M$ locally trivial of rank r. A common, though not completely correct, saying to describe this triple is "E is a vector bundle over M".

With $\pi : E \to M$ a smooth vector bundle of rank r, a chart $(U, \psi) = (U, x^1, ..., x^n)$ on M and

$$\phi : E_U \xrightarrow{\sim} U \times \mathbb{R}^r, \quad \phi(e) = (\pi(e), c^1(e), ..., c^r(e))$$

is a trivialization for E over U. Then

$$(\psi \times \mathbb{1}) \circ \phi = (x^1, ..., x^n, c^1, ..., c^r) : E_U \xrightarrow{\sim} U \times \mathbb{R}^r \xrightarrow{\sim} \psi(U) \times \mathbb{R}^r \subset \mathbb{R}^n \times \mathbb{R}^r$$

is a diffeomorphism of E_U onto its image, thus a chart map on E. Explicitly the chart would be $(E_U, (\psi \times \mathbb{1}) \circ \phi)$.

A section of a vector bundle $E \xrightarrow{\pi} M$ is a map in the reverse direction that when composed with π gives the identity on M. Moreover, $M \xrightarrow{s} E$ with $\pi \circ s = \mathbb{1}_M$ is a section of E, when s is a smooth section it is a section and a smooth map of manifolds.

We denote $\Gamma(E)$ for the space of smooth global sections on the vector bundle. This is a vector space over $\mathbb{R}$ and a module over $C^\infty(M, \mathbb{R})$ (smooth functions from M to $\mathbb{R}$). When a smooth section is just as stated it should be inferred that it is global, otherwise local will be the attached quantifier along with a neighbourhood U. If needed $\Gamma(E, U)$ denotes smooth local sections on U.

A smooth frame of E over M is a collection of smooth sections $s_1, ...s_r$, defined on all of M, that satisfy the properties for being a basis on every fiber (similarly for a *local* smooth frame).

Given a basis on each fiber any smooth section can be decomposed onto the smooth frame, fiberwise (Prop. 12.10, [13]). Consider the case of transition maps on the bundle E. These "glue" together the pre-images by π of overlapping charts on M. Let $(E_{U_\alpha}, (\psi_\alpha \times \mathbb{1}) \circ \phi_\alpha)$ and $(E_{U_\beta}, (\psi_\beta \times \mathbb{1}) \circ \phi_\beta)$ be two charts on E then the transition map is

$$[(\psi_\alpha \times \mathbb{1}) \circ \phi_\alpha] \circ [(\psi_\beta \times \mathbb{1}) \circ \phi_\beta]^{-1} : \psi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^r \to \psi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

and explicitly (with a smooth frame $\{s_1, ..., s_r\}$)

$$(p, c^1, ..., c^r) \mapsto (\psi_\beta^{-1}(p), c^1, ..., c^r) \mapsto (\psi_\alpha \circ \psi_\beta^{-1}(p), \underbrace{c^i s_i}_{\text{sum}}) \mapsto (\psi_\alpha \circ \psi_\beta^{-1}(p), d^1, ..., d^r)$$

In the following, and almost everywhere else in this thesis, Einstein summation convention is in use, i.e. whenever two indices appear in an expression sum over the index. For a particular p the c_i's are real numbers and so are d_j's. There is a transformation taking each c_i to d_j by way of g^i_j and summation over the upstairs with the downstairs indices, $d_j = g^i_j c_i$. This is the so-called Einstein summation convention in practice. As we have this for each i and j then $[g^i_j]$ must be a $(r \times r)$-matrix of real numbers. By preserving the frame (which amounts to a pointwise basis) $[g^i_j] \in \mathrm{GL}(r, \mathbb{R})$ so that we can compute inverses. We can now retract the dependence on a point p to obtain $[(g_{\alpha\beta})^i_j]$ a $(r \times r)$-matrix of real valued functions, alternatively a $(r \times r)$-matrix of real entries -valued function. Note that $g_{\alpha\beta}$ is restricted to only act locally between overlapping charts, we call these maps gluing co-cycles. Properties of gluing co-cycles follow, $g_{\alpha\alpha} = \mathbb{1}$, $g_{\alpha\gamma}g_{\gamma\beta} = g_{\alpha\beta}$, $g^{-1}_{\alpha\beta} = g_{\beta\alpha}$. Where as the matrix valued function is non-singular at every point then there must exist a well-defined pointwise inverse. The collection $\{g_{\alpha\beta}\}$ determine how the bundle E over M can transform over charts.

We can also define a complex vector bundle by replacing $\mathbb{R}$ with $\mathbb{C}$ in the above. Furthermore, we will need what it means for (E, π, M) and (E', π', M') to be isomorphic (as vector bundles), that is a pair $(\tilde{\phi}, \phi)$. A smooth map $\tilde{\phi} : E \to E'$ that descends to a smooth map $\phi : M \to M'$ and is also a fiberwise linear isomorphism.

A.2.1 Examples of Vector Bundles

Now to apply the ideas of the last paragraph of the previous subsection. Begin by identifying vectors v at a point $p \in M$ by taking the coordinates from a chart $(U_\alpha, \phi_\alpha) = (U_\alpha, x^1, ..., x^n)$ about p and writing

$$v = v_i(p) \frac{\partial}{\partial x^i}\Big|_p.$$

The collection of vectors identified like this form the tangent space of M at p, T_pM.
The vector bundle $E = TM = \{(p, v) | v \in T_pM\}$ is called the tangent bundle over M with projection $\pi(p, v) = p$. Each tangent space has the same vector space dimension, the same dimension as the manifold M, say n, so that the bundle is of rank n. One smooth frame over a neighbourhood of p are the partial derivatives with respect to the coordinate chart $\{\partial/\partial x^1, ..., \partial/\partial x^n\}$. The transformation between overlapping charts $(U_\alpha, \phi_\alpha) = (U_\alpha, x^1, ..., x^n)$ and $(U_\beta, \phi_\beta) = (U_\beta, y^1, ..., y^n)$ is given by the Jacobian matrix of the transition maps

$$\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}.$$

Given the Einstein summation convention this formula looks almost like cancelling fractions, of course this not what is happening here. In this case $\{g_{\alpha\beta}\}$ are the Jacobians of the transition maps between overlapping charts, (U_α, ϕ_α) and (U_β, ϕ_β), on M.

By revisiting the evaluation map we could identify co-vectors w at a point $p \in M$ by taking the coordinates from a chart $(U_\alpha, \phi_\alpha) = (U_\alpha, x^1, ..., x^n)$ about p and writing

$$w = w^i(p) \mathrm{ev}_{x_i}|_p.$$

With this collection of co-vectors we similarly form the cotangent space of M at p, T^*_pM. Note that the evaluation of a coordinate on a partial derivative at p gives a real number. The cotangent bundle then follows from writing $E = T^*M = \{(p, w) | w \in T^*_pM\}$, with projection $\pi(p, w) = p$.

A smooth frame for the cotangent fibres in a neighbourhood of p is the evaluation of each of the coordinates in the chart $\{\mathrm{ev}_{x^1}, ..., \mathrm{ev}_{x^n}\}$. The transformation happens in the opposite way

$$\mathrm{ev}_{y^j} = \frac{\partial y^j}{\partial x^i} \mathrm{ev}_{x^i}.$$

Therefore the gluing co-cycles are the inverses for the Jacobian of transition maps. This has a deeper significance in the general bundle theory where E^* is the dual bundle to E by inverting all of the gluing cocycles.

Up until now we have avoided talking what the evaluation map *is*. To try to understand it further pick a chart $(U, \phi) = (U, x^1, ..., x^n)$. By definition, $\mathrm{ev}_f(\partial/\partial x^i) = \partial f/\partial x^i$ and because $\mathrm{ev}_{x^j}(\partial/\partial x^i) = \delta^j_i$ we necessarily obtain

$$\mathrm{ev}_f = \frac{\partial f}{\partial x^i} \mathrm{ev}_{x^i}.$$

The above seems remarkably familiar to the multivariate chain-rule in calculus. In fact the usual notation for ev is d so that the above expression becomes

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

A.3 Differentiable Forms

We will need to define d on more just than functions, but before that we will need to introduce k-forms. Returning for a moment to the notion of smooth sections: $\Gamma(TM) =: \mathcal{X}(M)$ vector fields on M, $\Gamma(T^*M) =: \Omega^1(M)$ with $C^\infty(M) =: \Omega^0(M)$. Thus, for now our evaluation map acts like $\text{ev} : \Omega^0(M) \to \Omega^1(M)$ or $d : \Omega^0(M) \to \Omega^1(M)$.

The element dx^i is called a (differential) 1-form. Given another 1-form dx^j we may take, what is called, the wedge product $\wedge$. This is given locally by a chart $(U, \phi) = (U, x^1, ..., x^n)$ and $v_1, v_2 \in \mathcal{X}(M, U)$

$$dx^i \wedge dx^j(v_1, v_2) = \det \begin{bmatrix} dx^i(v_1) & dx^j(v_1) \\ dx^i(v_2) & dx^j(v_2) \end{bmatrix} = dx^i(v_1)dx^j(v_2) - dx^j(v_1)dx^i(v_2),$$

but can be extended globally.

Three properties drop out for the wedge product from the determinant. First, $dx^i \wedge dx^j = 0$ whenever, $i = j$. Second, $dx^i \wedge dx^j = -dx^j \wedge dx^i$. Finally, because the determinant is linear this procedure could be done on an arbitrary 1-form expanded on a smooth frame. A k-th wedge is a wedge product of a k number of 1-forms.

Now to define the k-th exterior power of M. This is the bundle $E = \wedge^k T^*M$, the set of pairs (p, α) where α belongs to the real span of all k-th wedge products of 1-forms. The sections $\Omega^k(M) := \Gamma(\wedge^k T^*M)$ are called (differentiable) k-forms. A frame using a chart $(U, \phi) = (U, x^1, ..., x^n)$ on M is given by

$$\{dx^I\}_I = \{dx^{i_1} \wedge \cdots \wedge dx^{i_k}\}_I$$

with I a multi-index running over all k-tuples $(i_1, ..., i_k)$ such that $1 \leq i_1 < \cdots < i_k \leq n$. An alternative definition of the wedge product appears for a k-form ω, l-form η in the frame above,

$$\omega \wedge \eta = \omega_I \eta_J dx^I \wedge dx^J.$$

Where the sum is over the disjoint union of I and J. Furthermore, $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

From this we can now define how d acts on k-forms, from now on the map that satisfies the previous "chain-rule" and satisfies the following extra properties will be called the exterior derivative (Prop. 19.1, [13]).

1. $d \circ d = 0$,
2. $d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^k \omega \wedge d\alpha$, ω a k-form.

The operation satisfying the above two conditions and $df = (\partial f / \partial x^i) dx^i$ is unique.

The first example of a co-chain complex (a sequence of modules, or vector spaces, with a differential increasing the index that also squares to zero) will be the following

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} \Omega^{n-1}(M) \xrightarrow{d_{n-1}} \Omega^n(M) \to 0.$$

In almost all cases each d_i is just denoted d. The above construction is called the de Rham complex. Note : all k-forms with $k > n$ are naturally 0.

We will encounter many counterparts to co-chain complexes called, just, chain complexes (sequences of modules, or vector spaces, with a differential decreasing the index, that also squares to zero).

Those k-forms ω with $d\omega = 0$ are called closed. If there exists a $(k-1)$-form η with $d\eta = \omega$, we call ω exact. Of course, all exact forms are closed which means $\text{Im } d_{i-1} \subset \text{Ker } d_i$ and the exact sequence of cohomology groups $H^i_{\text{dR}}(M)$ can be formed by "moduloing out" all closed forms that are not exact. More precisely, we have

$$0 \to H^0_{\text{dR}}(M) \xrightarrow{\tilde{d}_0} H^1_{\text{dR}}(M) \xrightarrow{\tilde{d}_1} \cdots \xrightarrow{\tilde{d}_{n-2}} H^{n-1}_{\text{dR}}(M) \xrightarrow{\tilde{d}_0} H^n_{\text{dR}}(M) \to 0,$$

with $\tilde{d}_i$ the induced differentials. It will not be necessary to keep the decoration dR, nor $\tilde{d}_i$, and so we write instead $H^\bullet(M) := H^\bullet_{\text{dR}}(M)$, and d, in the sequel.

A further linear operator on k-forms takes $v \in \mathcal{X}(M)$ a vector field on M and ω a k-form, such that $\iota_v \omega$ is a $(k-1)$-form. For $(k-1)$ vector fields $v_2, ..., v_k$, ι_v acts like so

$$\iota_v \omega(v_2, ..., v_k) = \omega(v, v_2, ... v_k).$$

We call the map that does the above and satisfies the next two properties interior multiplication (Prop. 20.8, [13])

1. $\iota_v \circ \iota_v = 0$,
2. $\iota_v(\omega \wedge \alpha) = \iota_v\omega \wedge \alpha + (-1)^k\omega \wedge \iota_v\alpha$, ω a k-form.

The operation satisfying the above two conditions is unique.

Due to the linearity of whichever form ι_v is being applied is C^∞ function linear. There is also an operation called the Lie-derivative of a form with respect to a vector field. The Lie-derivative has many equivalence definitions. The one presented here we will take to be a definition, Cartan's formula,

$$\mathcal{L}_X(\omega) = d\iota_X\omega + \iota_X d\omega.$$

Where X is a vector field and ω a k-form, for some k.

Further, linear operations on forms include pushforwards and pullbacks. A map $F : N \to M$ lifts to map $F_{*,p}$ (pushforward at p) from the tangent space of N at a point $p \in N$ to the tangent space of M at $F(p)$. The action on a vector X_p

$$F_{*,p}(X_p)(f) = X_p(f \circ F) \quad \text{(or in terms of evaluation)} \quad = \text{ev}_{f\circ F}(X_p) = d(f \circ F)(X_p).$$

Locally F_* is the Jacobian of F, also known as the differential.

Let $v_1, ..., v_k$ be vectors at $p \in N$, then the pullback on a k-form ω is

$$(F^*\omega_p)(v_1, ..., v_k) = \omega_p(F_{*,p}(v_1), ..., F_{*,p}(v_k)),$$

(the dependence of the pullback and the vectors on a point p is dropped for convenience).

The pullback is linear over the wedge product i.e. $F^*(\eta \wedge \omega) = F^*\eta \wedge F^*\omega$ (Prop. 18.11, [13]), and commutes with the exterior derivative $F^*d\omega = d(F^*\omega)$ (Prop. 19.5, [13]). To distinguish, the pullback is the same term used to refer to * acting on a function f that in turn acts on another function ϕ by $f^*\phi = \phi \circ f$, a contravariant operation on maps.

A.4 Miscellanea

A lot of geometry is not immediately necessary to understand the background material. That is why we push the less critical definitions to this subsection. The first of those less critical definitions is a submanifold (analogous to subset, subspace and so on). Firstly we need to define an immersion and an embedding

Definition 23. A smooth map $\varphi : N \to M$ is called an *immersion* at p if $\varphi_{*,p}$ (the pushforward at p) is injective. Furthermore, φ is called an *embedding* if it is an immersion for all $p \in N$ and a homeomorphism onto its image.

Definition 24. We will call N a *submanifold*, of dimension n, of a manifold M of dimension m when $n \leq m$, if N is itself a manifold, and the inclusion map $i : N \to M$ $(i(p) = p)$ is an embedding.

The condition homeomorphism-onto-its-image allows for the image of an embedding to have a topology induced from the ambient manifold, making a submanifold. During this discussion of related topics to submanifolds it would be wise to define when two submanifolds intersect transversally as this appears many times in the course of the thesis.

Definition 25. Let N_1 and N_2 be two submanifolds of a manifold M. N_1 and N_2 are said to *intersect transversally* when

$$\text{for every } p \in N_1 \cap N_2 \quad T_pN_1 + T_pN_2 = T_pM.$$

It is the case that if N_1, N_2 intersect transversally their intersection will only contain isolated points. This means its dimension will be 0. A necessary condition for N_1 and N_2 to intersect transversally is if they have compatible dimension, the sum of their dimensions is the dimension of the manifold M. In our special case M is of dimension 4 and the submanifolds we have investigated in Section 2 had dimension 2 so that intersection and most importantly self-intersection could occur.

A hypersurface N of M is a co-dimension 1 submanifold. One example of a hypersurface is the boundary of a manifold with boundary. To define this we can begin again with defining topological manifolds not by local homeomorphisms to $\mathbb{R}^n$ but instead considering a different codomain. The codomain we consider is the left half-plane $\mathbb{H}^n$. The left half-plane is $\mathbb{H}^n = \{(x^1, ..., x^n) \in \mathbb{R}^n \mid x^n \leq 0\}$ (which we attach the subspace topology to) and we call those manifolds locally homeomorphic to the half-plane, manifolds with boundary. In this appendix, up until now, we have actually been talking about manifolds without boundary which we now call closed. The construction is now precisely the same as in appendix A.1 but substituting $\mathbb{R}^n$ with $\mathbb{H}^n$.

The boundary of a manifold (with boundary) M is the set of those points p for which every neighbourhood of p has homeomorphic image in $\mathbb{H}^n$ always intersecting with the ambient space, $\mathbb{R}^n$. This is the intuitive notion of boundary and for topological reasons it is difficult to prove that is right one but we take it as our definition in any case. The notation for the boundary of M is ∂M. The interior of M, Int M, is defined as $M \setminus \partial M$.

B J-holomorphic curves

B.1 Connections and Chern Class

An algebraic tool will be needed in the following which we state now. Let V and W be a finite dimensional vector spaces and let W^* be the dual vector space of W then

$$W^* \otimes V \cong \text{Hom}(W, V).$$

Where $X \otimes Y$ can be defined via the universal property (making it unique) of being the "filter" for any bilinear map of the Cartesian product of vector spaces $X \times Y$.

It is possible to apply this isomorphism fiberwise over two vector bundles E_1, E_2 creating the new vector bundle $\text{Hom}(E_1, E_2) := E_1^* \otimes E_2$, where the dual of a vector bundle was encountered in appendix A.2. In particular, by taking $E_1 = E_2 = E$, $E^* \otimes E =: \text{End}(E)$ this defines the fiber-preserving linear maps from E to itself.

From this tool it is possible to consider extensions of forms that are E (total space)-valued. The whole story is almost the same as in appendix A.1 but instead define $\Omega^k(M, E) = \Gamma(\wedge^k T^*M \otimes E)$ to be E-valued k-forms.

There is also an extension of the covariant exterior derivative for E-valued forms, namely

$$d^E : \Omega^k(M, E) \to \Omega^{k+1}(M, E), \quad \text{with}$$
$$d^E(\mu \wedge \omega) = d^E\mu \wedge \omega + (-1)^k \mu \wedge d\omega$$

where $\mu \in \Omega^k(M, E)$ and ω is some other (ordinary) form. In this case, $d^E \circ d^E$ may not be zero.

Definition 26 (9.1.1, [43])**.** A (Koszul) *connection* (∇) is a bilinear map from $\mathcal{X}(M) \times \Gamma(E)$ to $\Gamma(E)$, that satisfies the following two properties:

$$\nabla_X(h\sigma) = h(\nabla_X\sigma) + dh(X) \otimes \sigma$$
$$\nabla_{hX}\sigma = h(\nabla_X\sigma).$$

Where $h \in C^\infty(M)$, $\sigma \in \Gamma(E)$ and $X \in \mathcal{X}(M)$.

(9.2.3.1, [43]) A Koszul connection on a vector bundle $\pi : E \to M$ gives rise to a uniquely defined covariant exterior derivative d^E for which the identity $d^E\sigma = \nabla_\bullet\sigma \in \Gamma(T^*M \otimes E) = \Omega^1(M, E)$, holds for all $\sigma \in \Gamma(E)$. Now to define how d^E acts on $\eta \in \Omega^k(M, E)$: For $X_0, ...X_k \in \mathcal{X}(M)$, we obtain

$$d^E\eta(X_0, ..., X_k) = (-1)^i\nabla_{X_i}(\eta(X_0, ..., \hat{X}_i, ..., X_k)) + (-1)^{i+j}\eta([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k)\mathbb{1}_{\{i<j\}}$$

the ˆ denotes removal, and the bracket $[\_, \_]$ is the commutator bracket of vector fields. For the case $k = 0$, $d^E\eta(X_0) = \nabla_{X_0}\eta$, $\eta \in \Omega^0(M, E) = \Gamma(E)$.

(9.2.4, [43]). Let $\{s_1, ..., s_r\}$ be a smooth frame of a rank r vector bundle E. Then, as before $d^E s_i \in \Omega^1(M, E)$ for each i, there exists a matrix $[\omega_i^j] \in M_r(\Omega^1(M, E))$ for which

$$d^E s_i = s_j \otimes \omega_i^j.$$

The matrix of 1-forms $[\omega_i^j]$, for each frame, are called the connection forms. Given a section $\sigma \in \Gamma(E, U)$, with U a neighbourhood of $p \in M$ and a local frame $\{s_1, ..., s_r\}$, then at a point p

$$\sigma_p = (\sigma^i s_i)_p.$$

Thus, locally, the covariant exterior derivative applied to any section σ is

$$d^E\sigma = s_j \otimes \sigma^i\omega_i^j + s_i \otimes d\sigma^i.$$

It is possible to put further restrictions on the connection to turn it into a orthogonal/unitary connection where the connection form takes values in the set of traceless skew-symmetric/skew-Hermitian endomorphisms of E. A Hermitian vector bundle which is a complex vector bundle with a holomorphic structure and a Hermitian metric. A Hermitian connection ∇ is a connection on a Hermitian vector bundle E such that ∇ is compatible with the Hermitian metric $\langle , \rangle$ in the following way:

$$d\langle\xi, \zeta\rangle = \langle\nabla\xi, \zeta\rangle + \langle\xi, \nabla\zeta\rangle, \quad \text{for all } \xi, \zeta \in E.$$

(9.3.1, [43]) We can now define the curvature of a connection form. By definition, $d^E(d^E\sigma) \in \Omega^2(M, E)$ and is $C^\infty(M)$-linear so that given a local frame and a section $\sigma = \sigma^i s_i$ then

$$d^E(d^E\sigma) = s_j \otimes (R_i^j\sigma^i),$$

with $R = [R^j_i] \in \Omega^2(M, \mathrm{End}(E))$ and
$$R^j_i = \omega^j_k \wedge \omega^k_i + d\omega^j_i.$$
When wanting to move across local frames gluing co-cycles are needed. Let g be a gluing co-cycle on the intersection of two local neighbourhoods we have
$$\hat{R} = g^{-1}Rg \quad \text{and} \quad \hat{\omega} = g^{-1}dg + g^{-1}\omega g,$$
thus R transform tensor-like whereas ω does not. Meaning ω transforms dependent on the gluing co-cycle, whereas R does not. This fact when considering simultaneously diagonalizable matrices means
$$\mathrm{tr}(\hat{R}) = \mathrm{tr}(g^{-1}Rg) = \mathrm{tr}(R)$$
so that $\mathrm{tr}(R)$ is globally defined.

A further useful property is the Bianchi identity
$$dR^j_i = R^j_k \wedge \omega^k_i - \omega^j_k \wedge R^k_i,$$
(9.3.6, [43]).

Given that the trace is a linear operator $d(\mathrm{tr}(R)) = \mathrm{tr}(dR)$ and from the Bianchi identity
$$\mathrm{tr}(dR^j_i) = dR^i_i = R^i_k \wedge \omega^k_i - \omega^i_k \wedge R^k_i = 0,$$
we see that $d(\mathrm{tr}(R)) = 0$. The last equality following from the fact that R^k_i is a 2-form and so no sign appears under interchange of the wedge product. We now have that $\mathrm{tr}(R)$ is closed and it can be shown that it defines a cohomology class by showing that difference choices of ∇ make $\mathrm{tr}(R)$ differ by an exact form (Theorem 1.35 (1), [18]). This means that $[\mathrm{tr}(R)] \in H^2(M)$.

In the real case define the first chern class to be $c_1(E) = [\mathrm{tr}(R)]$. Locally R is skew-symmetric so that all along the diagonals have to be zeros, implying a zero trace and thus $c_1(E)$ vanishes. In the complex case define $c_1(E) = (1/2\pi i)[\mathrm{tr}(R)]$. Here R is locally skew-Hermitian meaning R is a purely imaginary matrix and a purely imaginary trace. Dividing by i implies $c_1(E)$ is real. (Theorem 1.35 (2)/(3), [18]).

B.2 Singular Homology and Poincaré Duality

Definition 27. A *chain complex* $C_\bullet$ is a descending chain of algebraic objects (vector spaces/modules/groups) with a differential ∂ such that $\partial_k \circ \partial_{k+1} = 0$ for each k (usually denoted $\partial^2 = 0$).
$$\cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \to 0$$

Elements of C_k are called k-chains. Define subsets of k-chains by $Z(C_k) := \mathrm{Ker}\ \partial_k$, called cycles, and subsets of $(k+1)$-chains by $B(C_k) := \mathrm{Im}\ \partial_{k+1}$, called boundaries, . As $\partial_k \circ \partial_{k+1} = 0$ for each k, then $\mathrm{Im}\ \partial_{k+1} \subset \mathrm{Ker}\ \partial_k$ for each k, it makes sense to take the quotient by $\mathrm{Im}\ \partial_{k+1}$, a sub-object. Homology is then defined as the quotients and the k-th homology is
$$H_k := Z(C_k)/B(C_k).$$

We encounter multiple examples of homology in this thesis, we now show the build up to homology classes and induced maps in (singular) homology. The primary reference here is Hatcher's book, [44].

A n-simplex Δ^n is the convex hull of $(n+1)$ points $v_0, ..., v_n$ such that $v_1 - v_0, ..., v_n - v_0$ are linearly independent. By identifying two n-simplices under an orientation preserving homeomorphism we can consider the equivalence class of Δ^n by $[v_0, ..., v_n]$. The faces of n-simplices are $(n-1)$-simplices and the equivalence classes are $[v_0, ..., \hat{v}_i, ..., v_n]$, where v_i is removed.

Now we will define singular homology. A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$, possibly not nicely embedded. Let $C_n(X)$ denote the free $\mathbb{Z}$-module with generators as singular n-simplicies. Singular n-chains are sums $\sum_i n_i\sigma_i$, $n_i \in \mathbb{Z}$ with $\sigma_i : \Delta^n \to X$. The differential here is
$$\partial_n : C_n(X) \to C_{n-1}(X),$$
$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,...,\hat{v}_i,...,v_n]}.$$
This differential squares to zero from lemma 2.1, [44].

We may write $B_n(X) := \mathrm{Ker}\ \partial_n$ for the set of (singular) boundaries and $\mathrm{Im}\ \partial_{n+1} =: Z_n(X)$ the set of (singular) cycles. We can now define the n-th singular homology groups as
$$H_n(X) := Z_n(X)/B_n(X)$$
(page 108, [44]).

There is a notion that plays a significant throughout this thesis, that being homotopy.

Definition 28 (page 3, [44])**.** A *homotopy* between two continuous functions $f, g : X \to Y$ is a family of continuous functions $h_t : X \to Y$ such that $h_0 = f$ and $h_1 = g$, $(x, t) \mapsto h_t(x)$ from $X \times [0, 1] \to Y$.

Two functions f and g are called homotopic, if there exists a homotopy between them.

We call two spaces homotopy equivalent if there exist $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to $\mathbb{1}_Y$ and $g \circ f$ homotopic to $\mathbb{1}_X$. A fact called homotopy invariance (Corollary 2.11, [44]) is that homotopy equivalent spaces have isomorphic homology groups, we use this in Section 3.4.

Remark. As we are on the topic of homotopy, in this thesis "generic" $J/\lambda/\tau$ refers to a time-dependent collection $J_t/\lambda_t/\tau_t$ which give a homotopy to standard $J_{\text{std}}/\lambda_{\text{std}}/\tau_{\text{std}}$. The philosophy around this is that the spaces of almost complex structures, contact forms, symplectic trivializations are contractible meaning there will always exist a homotopy to the standard structure in that space. Furthermore, if the statement of a proposition/theorem mentions generic there is a further proof that needs to be made that the result is independent of homotopy. Then the standard structure can be taken and thus the result is independent of that structure.

Now for the induced map in (singular) homology. A map f between spaces X and Y induces a map $f_\#$ in the chain complex. The map $f_\# : C_n(X) \to C_n(Y)$ is defined via the composition $\Delta^\bullet \to X \to Y$, $f_\# := f \circ \sigma$. The map $f_\#$ acts linearly, $f_\#(\sum_i n_i\sigma_i) = \sum_i n_i f_\#\sigma_i = \sum_i n_i f \circ \sigma_i$.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \xrightarrow{\partial_{n-1}} & \cdots \\
 & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\
\cdots & \xrightarrow{\partial_{n+2}} & C_{n+1}(Y) & \xrightarrow{\partial_{n+1}} & C_n(Y) & \xrightarrow{\partial_n} & C_{n-1}(Y) & \xrightarrow{\partial_{n-1}} & \cdots
\end{array}$$

(page 111, [44]).

In the above figure we need to show that each square commutes, i.e. $f_\#\partial = \partial f_\#$.

$$\begin{aligned}
f_\#\partial(\sigma) &= f_\# \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \\
&= \sum_i (-1)^i \underbrace{f \circ \sigma}_{f_\#\sigma}|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \\
&= \partial f_\#(\sigma).
\end{aligned}$$

Due to this commutation $f_\#$ becomes a well-defined map which now induces the chain map $f_* : H_n(X) \to H_n(Y)$. Then f_* is well-defined because of the following claim, $f_\#$ takes cycles to cycles (1) and boundaries to boundaries (2). We have $f_\#\partial = \partial f_\#$. Let α be a cycle, then $\partial\alpha = 0$, and

$$\partial(f_\#\alpha) = f_\#(\partial\alpha) = f_\#(0) = 0.$$

The first equality follows from the commutation relation and the last equality from the linearity of $f_\#$. The preservation of boundary follows directly from the commutation relation: $f_\#\partial\beta = \partial f_\#\beta$ for all β.

The two basic properties of the chain map are: (1) $(fg)_* = f_*g_*$, $\Delta \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$. With $(fg)_* : H_n(X) \to H_n(Z)$, and (2) $\mathbb{1}_* = \mathbb{1}$.

Singular homology appears multiple times so we attempt to better understand it via familar objects, submanifolds. The lecture notes of Nicolaescu [45], specifically Section 7.3 on intersection theory and background from Section 7.2, will be our primary reference in the following.

Let M be a smooth oriented n-dimensional manifold without boundary. Before discussing intersection theory we have to explain a relevant detail associated to de Rham cohomology missed from appendix A.1, namely Poincaré duality.

We attempt to explain this by stating the following pairing for $\Omega^k(M) \times \Omega_c^{n-k}(M)$. The subscript c on the second coordinate denotes $\bullet$-forms with compact support meaning those $\bullet$-forms ω with $\text{Cl}(M \setminus \text{Ker}(\omega))$ is compact. For a smooth n-dimensional manifold M, the pairing is the following (Section 7.2.2, [45])

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \eta.$$

The pairing is a map into $\mathbb{R}$ and equal to the above if $\omega \in \Omega^k(M)$ and $\eta \in \Omega_c^{n-k}(M)$, otherwise the pairing is 0. Fixing ω forces the map to be $D = D^k : \Omega^k(M) \to (\Omega_c^{n-k}(M))^* : \omega \mapsto \int_M \omega \wedge \bullet$ (with codomain the dual of $\Omega_c^{n-k}(M)$). We would like to show that this map is independent of cohomologous elements. As this is a linear map we only need show that $D(\omega)(d\rho) = 0$ for $\rho \in \Omega^{n-k-1}(M)$ and some closed k-form ω,

$$D(\omega)(d\rho) = \int_M \omega \wedge d\rho = \int_M (\omega \wedge d\rho + (-1)^k d\omega \wedge \rho) = (-1)^k \int_M d(\omega \wedge \rho) \underbrace{=}_{\text{Stokes}} (-1)^k \int_{\partial M} \omega \wedge \rho = 0,$$

as M is a manifold without boundary. Stokes refers to Stokes Theorem (Theorem 23.12, [13]) which says $\int_M d\omega = \int_{\partial M} \omega$ whenever ω an $(n-1)$-form and M a manifold of dimension n. After a similar computation there is an induced map also $D : H^k(M) \to (H_c^{n-k}(M))^*$ where $H_c^\bullet(M)$ is the cohomology generated from the co-chain complex $\Omega^\bullet(M)$. In other words, the pairing $\langle \bullet, \bullet \rangle : H^k(M) \times H_c^{n-k}(M) \to \mathbb{R}$ is well-defined. Poincaré duality is then the induced dual map $(H^k(M))^*$ to $H_c^{n-k}(M)$ under the isomorphism between the second dual of a finite vector space and itself.

Our plan is to define a new homology which we can then identify with singular homology, and further with de Rham cohomology via Poincaré duality. These identifications will lead to defining the intersection number associated to compatible dimension cycles in singular homology.

Let M be a smooth oriented n-dimensional manifold. A k-dimensional cycle in M is a pair (S, ϕ) such that S is a compact (oriented) k-dimensional manifold without boundary and $\phi : S \to M$ a smooth map. Denote $\mathcal{C}_k(M)$ for the set of pairs (S, ϕ), (Def. 7.3.1, [45]).

We will define an equivalence relation of $\mathcal{C}_k(M)$, $\sim_c$.

$$(S_1, \phi_1) \sim_c (S_2, \phi_2) \;\Leftrightarrow\; \text{there exists } \Sigma \text{ and } \Phi : \Sigma \xrightarrow{\text{smooth}} M \text{ such that } \begin{cases} \partial\Sigma & = (-S_1) \sqcup S_2 \\ \Phi|_{S_i} & = \phi_i, \; i = 1, 2 \end{cases}$$

(Def. 7.3.2, [45]).

Where we have used the notation $-S_1$ is the same manifold as S_1 with the opposite orientation, and also $\sqcup$ for disjoint union. We say two cycles satisfying the equivalence relation are cobordant, side note Φ is called the cobordism of cobordant cycles. The set $\mathcal{C}_k(M)$ modulo cobordant cycles will be $\mathcal{Z}_k(M)$, and $[S, \phi]$ the equivalence class of (S, ϕ) in $\mathcal{Z}_k(M)$.

Further properties of cycles include trivial, meaning there exists Σ a $(k+1)$-dimensional compact manifold with boundary such that $\partial\Sigma = S$, and there exists $\Phi : \Sigma \to M$ smooth map such that $\Phi|_S = \phi$. Under the further assumption that M is connected the trivial cycles form the unit equivalence class $[0]$, the notation for which will become clear in a moment.

Moreover, a cycle (S, ϕ) is degenerate if and only if (S, ϕ) is cobordant to (S', ϕ') such that ϕ' is constant on each of the connected components of S'.

If given (S_1, ϕ_1), (S_2, ϕ_2) cobordant cycles then $((-S_1) \sqcup S_2, \phi_1 \sqcup \phi_2)$ is a trivial cycle. To see this take Σ and Φ as in the definition of the equivalence relation. From assuming $\sim_c$ is actually an equivalence relation, an addition of classes can be defined like so

$$[S_1, \phi_1] + [S_2, \phi_2] := [S_1 \sqcup S_2, \phi_1 \sqcup \phi_2].$$

Under the claim that this addition is well-defined (and commutative) along with $[0]$ the unit element turns $\mathcal{Z}_k(M)$ into an Abelian group, with inverses $-[S, \phi] = [-S, \phi]$. To define $\mathcal{H}_k(M)$, we form the subgroup generated by all degenerate cycles and take the quotient of $\mathcal{Z}_k(M)$ by this subgroup, (Prop. 7.3.4, [45]).

To each cycle there is a linear functional in $(H^k(M))^*$ defined for $\omega \in H^k(M)$ as $\int_S \phi^*\omega$. Then from the discourse on Poincaré duality we may identify this map with an element of $H_c^{n-k}(M)$. We use δ_S to symbolize the Poincaré dual of the cycle (S, ϕ).

There are four properties of $\delta_\bullet$ to make sure our map is well-defined on the group $\mathcal{H}_k(M)$. Firstly, $\delta_\bullet$ is the same element, up to exact form, over cobordant cycles. It is trivial (i.e. the 0 element in $H_c^{n-k}(M)$) on trivial cycles. The dual is additive over disjoint unions, with additive inverse on oppositely oriented cycles. So that now $\delta : \mathcal{H}_k(M) \to H_c^{n-k}(M)$ is well-defined and which we will call the homological Poincaré duality, (Prop. 7.3.10, [45]).

We now define invariants associated to pairs of cycles. Given M a smooth connected oriented n-dimensional manifold M and S a k-dimensional submanifold of M, the inclusion map $\iota : S \hookrightarrow M$ is smooth and (S, ι) defines a k-cycle in M.

Given a $(n-k)$-cycle (T, ϕ), T an oriented manifold, we can define the intersection number when S is transversal to ϕ. Transversallity will occur when (a) $\phi^{-1}(S)$ intersection with T is finite and (b)

$$\phi_*(T_xT) + T_{\phi(x)}S = T_{\phi(x)}M \text{ (direct sum)}.$$

Define the intersection number of S and T at x, denoted $\mathrm{i}_x(S, T)$. This number is equal to 1 when the tangent spaces $\phi_*(T_xT)$ and $T_{\phi(x)}S$ have the same orientation with respect to the tangent space $T_{\phi(x)}M$ and -1 if they have opposite orientations, with respect to $T_{\phi(x)}M$. Then by finiteness condition we define the "full" intersection number of T and S by

$$S \cdot T = \sum_{x \in \phi^{-1}(S)} \mathrm{i}_x(S, T),$$

(Def. 7.3.11, [45]).

B.3 Cauchy-Riemann Type Operators

Keeping in the general framework of Section 2 it will be necessary to first prove that $\overline{\partial}$ satisfies a Leibnitz rule. That being, for any smooth section $v \in \Gamma(E)$ and $f \in C^\infty(\Sigma)$ we have

$$\overline{\partial}(fv) = f\overline{\partial}(v) + (\overline{\partial}f) \otimes v.$$

Proof.

$$\begin{aligned}
\overline{\partial}(fs) &= d(fv) + J \circ d(fv) \circ j \\
&= df \otimes v + f(dv) + J[df(j\bullet) \otimes v + f(dv(j\bullet))] \\
&= f(dv) + J(f(dv(j\bullet))) + df \otimes v + J(df(j\bullet) \otimes v) \\
&= f[dv + J(dv(j\bullet))] + [df + J(df(j\bullet))] \otimes v \\
&= f[dv + J \circ dv \circ j] + [df + J \circ df \circ j] \otimes v \\
&= f\overline{\partial}(v) + (\overline{\partial}f) \otimes v.
\end{aligned}$$

□

Given a holomorphic structure on E, a collection of holomorphic gluing co-cycles $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$. By natural identification $\mathrm{GL}(n, \mathbb{C}) \cong \mathrm{GL}(2n, \mathbb{R})$ there is a standard complex structure $\hat{\imath}$. To say $g \in C^\infty(\Sigma, \mathrm{GL}(n, \mathbb{C}))$ is holomorphic it means the following expression is zero

$$\overline{\partial}_{\hat{\imath}}(g) = dg + \hat{\imath} \circ dg \circ j.$$

As g is matrix valued function then it can be thought of instead as a matrix of smooth functions. Let g_k^j be the kj-th function of the matrix, then the above formula descends to the holomorphic operator $\overline{\partial}$, which by the above condition must be zero.

To prove that our section is well-defined will need to prove that it transforms tensor-like, i.e. $g^{-1} \circ \hat{\overline{\partial}} \circ g = \overline{\partial}$. To that end choose a smooth frame $\{v_1, ..., v_n\}$ and act on the transformed frame. Then

$$\hat{\overline{\partial}}(g_k^j v_j) = \hat{\overline{\partial}}(g_k^j) \otimes v_j + g_k^l \hat{\overline{\partial}}(v_l) = g_k^l \hat{\overline{\partial}}(v_l) = g_k^l \overline{\partial}(v_l).$$

Therefore, there is no need to distinguish between $\overline{\partial}$ and $\hat{\overline{\partial}}$, and our section is now globally defined. Note that this depended primarily on the fact that the components of the transition map were holomorphic. Therefore, ∂ (the anti-holomorphic operator) would not be globally defined as there would be dependence on the transition map.

From our discussion so far, including Section 2, it seems prudent to generalize the types of operators we are encountering. Indeed,

Definition 29 (Definition 2.41, [10]). A *Complex-Linear Cauchy-Riemann type operator* on $(E, J) \to (\Sigma, j)$ is a complex linear map

$$D : \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}(T\Sigma, E)),$$

satisfying the Leibnitz rule

$$D(fv) = (\overline{\partial}f)v + f(Dv), \qquad \text{for all } v \in \Gamma(E), f \in C^\infty(\Sigma).$$

For brevity we shall write $\mathbb{C}$-linear C-R type operator for Complex-Linear Cauchy-Riemann type operator. Fix $(E, J) \to (\Sigma, j)$ and D, D' are $\mathbb{C}$-linear C-R type operators. Then for some $v \in \Gamma(E)$

$$(D - D')(fv) = (\overline{\partial}f)v + f(Dv) - (\overline{\partial}f)v - f(D'v) = f(D - D')(v)$$

so that $D' - D$ is C^∞-linear. Similar to the case for (Koszul) connections we can decompose $D' - D$ onto a smooth frame and make a matrix of E-valued complex anti-linear 1-forms. As $\overline{\partial}$ is itself a $\mathbb{C}$-linear C-R type operator we can take $D' = \overline{\partial}$ and recover a Cristoffel symbols type theorem.

Take some connection $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(T\Sigma, E))$, then

$$\nabla + J \circ \nabla \circ j : \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E))$$

is a $\mathbb{C}$-linear C-R type operator.

Proof. Firstly, $(\nabla + J \circ \nabla \circ j)$ is $\mathbb{C}$-linear from the fact that ∇ is $\mathbb{C}$-linear. The complex anti-linearity is by construction as the addition of $J \circ \nabla \circ j$ projects off the complex linear term.

Now for the Leibniz rule, let $f \in C^\infty(\Sigma)$ and $v \in \Gamma(E)$ be arbitrary

$$\begin{aligned}(\nabla + J \circ \nabla \circ j)(fv) &= \nabla_\bullet(fv) + J(\nabla_{j\bullet}(fv)) \\ &= df \otimes v + f\nabla_\bullet(v) + J(df(j\bullet) \otimes v + f\nabla_{j\bullet}(v)) \\ &= df \otimes v + J(df(j\bullet)) \otimes v + f(\nabla_\bullet(v) + \nabla_{j\bullet}(v)) \\ &= (\overline{\partial} f) \otimes v + f(\nabla + J \circ \nabla \circ j)(v).\end{aligned}$$

□

We will take the following to be fact.

Proposition 5 (Proposition 2.44, [10])**.** *If there exists a Hermitian vector bundle $(E, J) \to (\Sigma, j)$ with $\mathbb{C}$-linear C-R type operator $D : \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E))$, there exists a unique Hermitian connection ∇ such that $D = \nabla + J \circ \nabla \circ j$.*

We will need real-linear Cauchy-Riemann type operators in the sequel which necessitates a definition.

Definition 30 (Definition 2.48, [10])**.** A *Real-Linear Cauchy-Riemann type operator* on a complex vector bundle $(E, J) \to (\Sigma, j)$ is a real-lienar map $D : \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E))$ such that the Leibniz rule is satisfied

$$D(fv) = (\overline{\partial} f)v + f(Dv), \quad \text{for all } f \in C^\infty(\Sigma, \mathbb{R}), v \in \Gamma(E).$$

We will write, simply $\mathbb{R}$-linear C-R type operator for Real-Linear Cauchy-Riemann type operator.
Returning for a moment to the state of affairs, first of all (Σ, j) a Riemann surface and (M, J) an almost complex manifold. Then $\mathcal{B} = C^\infty(\Sigma, M)$ is an infinite-dimensional smooth manifold (Banach manifold) and symbolically $\mathcal{E} \to \mathcal{B}$ is a Banach space bundle over $\mathcal{B}$ with fibers $\mathcal{E}_u = \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$. By pulling back J on TM define a complex bundle structure on $u^*TM \to \Sigma$. The tangent space over the manifold $\mathcal{B}$, $T_u\mathcal{B} = \Gamma(u^*TM)$ so that tangent vectors at a point $u \in \mathcal{B}$ are just vector fields along u.

We define, as discussed before, the section of the Banach space bundle by

$$\begin{aligned}\overline{\partial}_J &: \mathcal{B} \to \mathcal{E}, \\ \overline{\partial}_J u &= du + J \circ du \circ j.\end{aligned}$$

Of course, we are interested in the set $\overline{\partial}_J^{-1}(\{0\})$. In an analogy to the finite dimensional case we use the inverse function theorem to create a differentiable structure. Of course, we have a problem the inverse function theorem is for finite dimensions but this technicality will not effect us too much. In any case there does exist an inverse function theorem for Banach manifolds and there is a much more pressing problem that our operator is not linear. We need our operator to be approximated by a linear operator so that we may apply Fredholm theory.

For now we will only concern ourselves with the finite dimensional case. Let $E \to B$ be a smooth vector bundle of real rank k, over an n-dimensional manifold B. Let $s : B \to E$ be a smooth section transverse everywhere to the zero section, meaning $s^{-1}(\{0\}) \subset B$ defines a smooth submanifold of dimension $(n - k)$ (Theorem 2.5.1, [10]). Now choose a (Koszul) connection ∇ on $E \to B$. The map $(\nabla s)_p : T_pB \to E_p$ is independent of this choice for $p \in s^{-1}(\{0\})$, due to the fact that TE splits canonically into horizontal and vertical subspaces along the zero section. Keep in mind this idea going forward as it will appear again in this appendix as we are aiming to have a connection independent result.

At once we linearize s at $p \in s^{-1}(\{0\})$ by writing

$$Ds(p) : T_pB \to E_p.$$

Intersections of s with the zero section happen only at $s^{-1}(\{0\})$ and we have a transverse intersection if and only if $Ds(p)$ is surjective for each $p \in s^{-1}(\{0\})$ (this is why the Fredholm operator needs to be surjective).

Now let us apply the ideas from the finite dimensional case to the infinite. First, consider the Banach space bundle $\hat{\mathcal{E}}$ that contains $\mathcal{E}$ as subbundle, with fibers $\hat{\mathcal{E}}_u = \Gamma(\mathrm{Hom}_{\mathbb{R}}(T\Sigma, u^*TM))$. Choose a connection ∇ on M and $\hat{\mathcal{E}}$, meaning it has as first slot a vector field on M and second slot a smooth section of $\mathrm{Hom}(T\Sigma, \bullet^*TM)$ where the $\bullet$ will be replaced by $u \in \mathcal{B}$.

Take a smooth parametrization of a path $\tau \mapsto u_\tau \in \mathcal{B}$ (a path of smooth functions) and a section $\ell_\tau \in \hat{\mathcal{E}}_{u_\tau}$, along the path. We will assume that the covariant derivative $\nabla_\tau \ell_\tau \in \hat{\mathcal{E}}_{u_\tau}$ has the form

$$(\nabla_\tau \ell_\tau)(X) = \nabla_\tau(\ell_\tau(X)) \in (u_\tau^*TM)_z = T_{u_\tau(z)}M, \quad z \in \Sigma, X \in T_z\Sigma$$

A simplification in symbols has been made by $\nabla_\tau = \nabla_{\partial/\partial\tau}$. We will also use $\partial_\tau = \partial/\partial\tau$, both out of convenience. Notice also, as was the case for the finite dimensions, $\nabla_\tau \ell_\tau$ is independent of the connection ∇ at any τ such that $\ell_\tau = 0$.

Now to linearize $\overline{\partial}_J$. Let $u \in \overline{\partial}_J^{-1}(\{0\})$ further let $\{u_\tau\}_{\tau\in(-1,1)}$ be a smooth family of maps such that $u_0 = u$ with $\partial_\tau u_\tau|_{\tau=0} =: \eta \in \Gamma(u^*TM)$. The linearization is

$$D\overline{\partial}_J(u) : \Gamma(u^*TM) \to \Gamma(\mathrm{Hom}(T\Sigma, u^*TM)),$$

the unique linear map such that

$$\begin{aligned} D\overline{\partial}_J(u)\eta &= \nabla_\tau(\overline{\partial}_J u_\tau)|_{\tau=0} \\ &= \nabla_\tau[du_\tau + J(u_\tau) \circ du \circ j]|_{\tau=0}. \end{aligned} \tag{B.1}$$

The $J(u_\tau)$ appears to emphasize that the almost complex structure on TM has been pulled back to u_τ^*TM.

To obtain a tangible expression pick holomorphic coordinates $s + it$ near $z \in \Sigma$ and evaluate on ∂_s by (B.1)

$$\nabla_\tau[du_\tau + J(u_\tau) \circ du \circ j]|_{\tau=0}\partial_s = \nabla_\tau[\partial_s u_\tau + [J(u_\tau)\partial_t]u_\tau]|_{\tau=0}. \tag{B.2}$$

A symmetric connection ∇ on M satisfies the condition $\nabla_X Y - \nabla_Y X = [X, Y]$. Where $[X, Y]$ is the commutator bracket of vector fields, is zero for commuting vector fields which are all we will encounter. So for us a symmetric connection amounts to the property $\nabla_X Y = \nabla_Y X$. We are allowed to enforce this condition without penalty due to the result being connection independent. Therefore,

$$\begin{aligned} \nabla_\tau \partial_s u_\tau|_{\tau=0} &= \nabla_s \partial_\tau u_\tau|_{\tau=0} \\ &= \nabla_s \eta \qquad\qquad \text{and} \\ \nabla_\tau \partial_t u_\tau|_{\tau=0} &= \nabla_t \partial_\tau u_\tau|_{\tau=0} \\ &= \nabla_t \eta. \end{aligned}$$

Notice the swap via the symmetry of the connection.

Then (B.2) can be re-written after the following computation,

$$\begin{aligned} \nabla_s\eta + \nabla_\tau([J(u_\tau)\partial_t]u_\tau)|_{\tau=0} &= \nabla_s\eta + (J(u_\tau)|_{\tau=0})\nabla_t\eta + [\nabla_\tau(J(u_\tau))\partial_t]u_\tau|_{\tau=0} \\ &= \nabla_s\eta + J(u)\nabla_t\eta + [\nabla_\tau(u_\tau^*J)\partial_t]u_\tau|_{\tau=0} \qquad \text{connection acting on the pullback} \\ &= \nabla_s\eta + J(u)\nabla_t\eta + [(\nabla_{\partial_\tau u_\tau|_{\tau=0}}J)\partial_t]u \\ &= \nabla_s\eta + J(u)\nabla_t\eta + [(\nabla_\eta J)\partial_t]u. \end{aligned}$$

The first equality is from applying the Leibnitz rule. Removing the coordinates we result in

$$D\overline{\partial}_J(u) = \nabla\eta + J(u) \circ \nabla\eta \circ j + (\nabla_\eta J)du \circ j. \tag{B.3}$$

What is surprising is that the above expression is not only linear but also complex anti-linear thus $D\overline{\partial}_J$ is not only a section of $\hat{\mathcal{E}}$ but of $\mathcal{E}$ as well (pages 44-45, [10]). Now to prove this.

Proof. We will begin by showing that for any vector field X, $\nabla_X J$ and J anticommute. Firstly, take some vector field X, remembering that J maps TM to TM and thus is a $(1, 1)$-tensor, the covariant derivative then acts like so

$$(\nabla_X J)(Y) = \nabla_X(J(Y)) - J(\nabla_X Y), \text{ for all vector fields } Y.$$

We need to show the expression $(\nabla_X J)(JY) + J(\nabla_X J)(Y)$ vanishes for all vector fields X, Y. Indeed, let X, Y be any two vector fields then

$$\begin{aligned} (\nabla_X J)(JY) + J(\nabla_X J)(Y) &= \nabla_X(J(J(Y))) - J(\nabla_X(J(Y))) + J(\nabla_X(J(Y))) - J^2(\nabla_X Y) \\ &= -\nabla_X Y + \nabla_X Y - J(\nabla_X(J(Y))) + J(\nabla_X(J(Y))) = 0. \end{aligned}$$

Therefore, $J(\nabla_X J)J = \nabla_X J$ and in particular for $X = \partial_\eta$, $J(\nabla_\eta J)J = \nabla_\eta J$. The first two terms of (B.3) form a complex anti-linear map so that we need only concern ourselves with the third term. By projecting off the complex linear term we result in

$$(\nabla_\eta J)du \circ j - J \circ ((\nabla_\eta J)du \circ j) \circ J = 0$$

which follows precisely from the anti-commutativitiy property we have shown already. ☐

From now on we use (B.3) as a definition

$$\mathcal{D}_u := D\overline{\partial}_J(u),$$

this is a real-linear map taking vector fields along u to sections of $\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TM)$. To prove that $\mathcal{D}_u$ defines a $\mathbb{R}$-linear C-R type operator it is left to show that the Leibnitz rule is satisfied

Proof.

$$\begin{aligned}\mathcal{D}_u(f\eta) &= \nabla(f\eta) + J(u)\circ\nabla(f\eta)\circ j + (\nabla_{f\eta}J)\circ du\circ j\\ &= df\otimes\eta + f\nabla\eta + J(u)\circ(df\otimes\eta + f\nabla\eta)\circ j + f(\nabla_\eta J)du\circ j\\ &= [df + J(u)\circ df\circ j]\otimes\eta + f\nabla\eta + fJ(u)\circ\nabla\eta\circ j + f[(\nabla_\eta J)du\circ j]\\ &= [df + J(u)\circ df\circ j]\otimes\eta + f[\nabla\eta + J(u)\circ\nabla\eta\circ j + (\nabla_\eta J)du\circ j]\\ &= (\overline{\partial}f)\otimes\eta + f\mathcal{D}_u\eta.\end{aligned}$$

□

There is a theorem we have missed that we will be state now as it pertains to creating a holomorphic structure on a vector bundle. The vector bundle in question will be $\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TM)$.

Theorem 9 (Theorem 2.45, [10]). *For any complex-linear Cauchy-Riemann type operator D on a complex vector bundle (E, J) over a Riemann surface (Σ, j), there is a unique holomorphic structure on (E, J) such that the naturally induced $\overline{\partial}$-operator is D.*

The idea is to use the linearized operator $\mathcal{D}_u$ to define a holomorphic structure on $\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TM)$ so that du is a holomorphic section. As we have covered (Σ, j) is in fact a complex manifold so that $T\Sigma\to\Sigma$ has a natural holomorphic structure, it is thus possible to talk about holomorphic vector fields on Σ. To this end we now supply two more lemmas without proof.

Lemma 5 (Lemma 2.55, [10]). *Suppose X is a holomorphic vector field on some open subset $U\subset\Sigma$, $U'\subset U$ is another open subset and $\varepsilon > 0$ a number such that the flow $\varphi_X^t : U'\to\Sigma$ is well-defined for $t\in(-\varepsilon,\varepsilon)$. Then the maps φ_X^t are holomorphic.*

This is a characteristic situation of flows and the idea of the proof is to work in local holomorphic coordinates. Then apply the local $\overline{\partial}$ operator to the flow and use uniqueness results of ODEs to surmise the flow, for any time, vanishes under the operator.

Lemma 6 (Lemma 2.56, [10]). *For any holomorphic vector field X defined on an open subset $U\subset\Sigma$, $\mathcal{D}_u[du(X)] = 0$ on U.*

In a sufficiently small neighbourhood it is possible to define the flow φ_X^t which is holomorphic by the above lemma so that maps along the flow are J-holomorphic.

We have that the C-R type operator $\mathcal{D}_u$ is $\mathbb{R}$-linear, what we require instead is $\mathbb{C}$-linear. So we project off the complex anti-linear part

$$\mathcal{D}_u^{\mathbb{C}} := \frac{1}{2}(\mathcal{D}_u - J\circ\mathcal{D}_u\circ J),$$

with the innocuous introduction of a half. This defines a $\mathbb{C}$-linear map

$$\Gamma(u^*TM)\to\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)),$$

and, as has nearly always been the case, we prove the defined map satisfies the Leibnitz rule

$$\begin{aligned}\mathcal{D}_u^{\mathbb{C}}(f\eta) &= \frac{1}{2}[(\overline{\partial}f)\otimes\eta + (\mathcal{D}_u\eta)f - J\circ((\overline{\partial}f)\otimes\eta + (\mathcal{D}_u\eta)f)\circ J]\\ &= \frac{1}{2}[(\overline{\partial}f)\otimes\eta - J^2((\overline{\partial}f)\otimes\eta)] + [\mathcal{D}_u^{\mathbb{C}}\eta]f\\ &= \frac{1}{2}[(\overline{\partial}f)\otimes\eta + (\overline{\partial}f)\otimes\eta] + f\mathcal{D}_u^{\mathbb{C}}\eta\\ &= (\overline{\partial}f)\otimes\eta + f\mathcal{D}_u^{\mathbb{C}}\eta.\end{aligned}$$

This makes $\mathcal{D}_u^{\mathbb{C}}$ into a $\mathbb{C}$-linear C-R type operator.

The induced bundle $u^*TM\to\Sigma$ for any smooth J-holomorphic curve $u:\Sigma\to M$ admits a holomorphic structure for which holomorphic vector fields along u satisfy $\mathcal{D}_u^{\mathbb{C}}\eta = 0$. In addition, for any local holomorphic vector field X on Σ,

$$\mathcal{D}_u^{\mathbb{C}}[du(X)] = \frac{1}{2}\mathcal{D}_u[du(X)] - \frac{1}{2}J\mathcal{D}_u[J\circ du(X)] = -\frac{1}{2}J\mathcal{D}_u[du(jX)] = 0.$$

The last equality follows from (jX) also being a holomorphic vector field, and $\overline{\partial}_J(u) = 0$. Now $du(X)$ is a holomorphic section on u^*TM whenever X is holomorphic on $T\Sigma$. Therefore, $du\in\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ is holomorphic. Now we are able to enforce conditions on the space of critical points of u.

Corollary 3 (Corollary 2.59, [10])**.** *If $u : \Sigma \to M$ is smooth and J-holomorphic and not constant then*

$$\mathrm{Crit}(u) := \{z \in \Sigma \mid du(z) = 0\}$$

contains at most a countable number of points.

This corollary is important for making the jump from somewhere-injective to almost everywhere injective J-holomorphic curves, see Section 2.

B.4 Sobolev Completions of Vector Bundles

To begin talking about Sobolev Completions we must understand the motivation. In the theory of PDEs some solutions may not carry the same amount of derivatives that would be expected from being a solution. In these cases a type differentiability is associated, weak differentiability. We are supported in this discussion by the book of Evans [46].

We begin with the most well-behaved functions we could conceivably come up with, smooth functions with compact support on an open domain $U \subset \mathbb{R}^n$. We denote the space of these functions by $C_c^\infty(U)$ and ϕ belongs to this space if it vanishes outside U and we can take an infinite number of its derivatives. Sometimes ϕ is called a test function. On this space the following is well-defined, for $\alpha = (i_1, ..., i_n)$ a multi-index we write $D^\alpha \phi = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \phi$. This operator on ϕ takes each partial derivative in the coordinate x_j of ϕ, i_j-times for $j = 1, ..., n$. We let $|\alpha| = i_1 + \cdots + i_n$ be the degree of the operator D^α. We will need a further definition to apply this operator to functions not of this well-behaved type.

In the definition below we will employ $L^1_{\mathrm{loc}}(U)$ which is the space of locally L^1-functions on U i.e. if $u \in L^1_{\mathrm{loc}}(U)$ then for each open relatively compact V such that $\overline{V} \subset U$, u is L^1-integrable on V. We assume in this subsection that the reader is familiar with basic integration theory, if not review here [47].

Definition 31 (Section 5.2.1, [46])**.** Suppose $u, v \in L^1_{\mathrm{loc}}(U)$ and α a multi-index. We say v has a weak α-th partial derivative of u, which we write $v = D^\alpha u$. This is only possible if

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi$$

for every test function $\phi \in C_c^\infty(U)$.

The integral identity can be shown to be satisfied by a test function ϕ and a "normally" degree $|\alpha|$ differentiable function u. This is via $|\alpha|$ applications of integration by parts, which picks up the sign $(-1)^{|\alpha|}$. It is then natural to ask for such an identity to hold for the weak-case.

Now to define Sobolev spaces, which are essentially function spaces that have weak derivatives lying in L_p spaces.

Definition 32 (Section 5.2.2, [46])**.** The Sobolev space $W^{k,p}(U)$ consists of all locally L^1 functions $u : U \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$ we have that u is weak α differentiable and its weak derivative $D^\alpha u$ is in $L^p(U)$.

From now on we identify functions in $W^{k,p}(U)$ that are equal almost everywhere. It is possible to define a norm that makes $W^{k,p}(U)$ into a Banach space (Theorem 2, [46]) which is precisely the reason we took this detour in the first place. For applications, we care about taking $n = 2$ and identifying $\mathbb{R}^2$ with $\mathbb{C}$. This is because the domain of our functions will be a Riemann surface Σ. Define the locally Sobolev functions $W^{k,p}_{\mathrm{loc}}(U)$ as those functions on $U \subset \mathbb{C}$ that belong to $W^{k,p}(V)$ for each relatively compact V such that $\overline{V} \subset U$.

Properties of these spaces follow now. Let $U \subset \mathbb{C}$ with smoothly varying boundary, then there are natural continuous inclusions of $W^{k+d,p}(U)$ into $C^d(U)$ whenever $kp > 2$ (for d non-negative integer). We also have the obvious inclusion of $W^{k,p}$ into $W^{k-1,p}$.

Further related properties of $W^{k,p}(U)$ need the requirements, $k \geq 1$ and $p > 2$. First being that $W^{k,p}(U)$ forms a Banach algebra. The second is pairing that composes a continuous function f, defined on an open subset Ω with $u \in W^{k,p}(U)$ is continuous. For this to be well-defined $u(U) \subset \Omega$, the pairing explicitly is $(f, u) \mapsto f \circ u$.

The properties of the last 2 paragraphs is the content of Lemmma 2.96 of Wendl [10]). They will be needed to define a differentiable structure on the Sobolev type sections of vector bundles displayed in Section 2. The theory behind those bundles will be exposed now.

Exposition of Banach manifold and Banach space bundles rely on Lang's book ([48]). First some preliminaries.

Let E be a real vector space. We call E a topological vector space when E has a topology in which $+$ and $\cdot$ are continuous. Also assume E is Hausdorff and locally convex. By the later we mean every neighbourhood of 0 contains a neighbourhood U of 0 such that for $0 \leq t \leq 1$, $tU + (1-t)U \subset U$.

The maps between topological vector spaces E and F are continuous linear maps. Let $L(E, F)$ denote the space of these maps. If $T \in L(E, F)$ is invertible with continuous and linear inverse then T is called a toplinear isomorphism.

The set of toplinear isomorphism, Lang denotes by $\mathrm{Lis}(E, F)$ and defines $\mathrm{Laut}(E) := \mathrm{Lis}(E, E)$. The linear maps of linear maps will be denoted $L^2(E, F) = L(E, L(E, F))$. Those maps of $L^2(E, F)$ that are symmetric in their arguments will be $L^2_{\mathrm{sym}}(E, F)$.

A Banachable space is a complete topological vector space with a topology induced by a norm. With a norm that induces the topology, a Banachable space becomes a Banach space. Such a norm is not unique but there is no harm in just as well calling Banachable spaces with some norm a Banach space. Via corollary to the open mapping theorem any continuous bijective linear map between Banach spaces E and F is actually a toplinear isomorphism. The set $L(E, F)$ is also a Banach space. From now on all topological vector spaces are Banach spaces.

Now to discuss the generalization of differentiation to this situation. Let E and F be two topological vector spaces. Assuming the existence of norms on E and F, we call φ tangent to 0 if $\|\varphi(x)\|_F \leq \|x\|_E \psi(x)$. Where ψ is such that $\lim_{x\to 0} \|\psi(x)\|_F = 0$. The maps $\|\cdot\|_E$ and $\|\cdot\|_F$ are norms on E and F, respectively.

Let U be open in E and $f : U \to F$ a continuous map. Define f to be differentiable at $x_0 \in U$ if there exists λ a continuous linear map from E to F by letting

$$f(x_0 + y) = f(x_0) + \lambda y + \varphi(y)$$

for y near 0 and φ tangent to 0. The map λ is called the derivative of f at x_0 and written $Df(x_0)$ or $f'(x_0)$ or $df(x_0)$. If f is differentiable at all $x_0 \in U$ then f is just called differentiable and we denote the derivative $Df \in L(E, F)$.

If f is differentiable then f is of class C^1. Define inductively when f is of class of C^k by saying Df is of class C^{k-1}. Note that for f of class C^2, $D^2 f := D(Df) \in L^2(E, F)$. Furthermore, if f is locally homogeneous of degree 2 i.e. when $f(tx) = t^2 f(x)$ for $t > 0$ then action of f is $(1/2)D^2(f)(0))(x, x)$. If f is of two variables then D_1 and D_2 are the partial derivatives with respect to each coordinate.

By an extension of the chain rule we can say if two maps f and g are of class C^k then so is their composition $f \circ g$. From the setting of Banach spaces we have an analogous implicit function theorem which means that it is possible to write down the inverse of a function f that has locally non-singular derivative df.

Now to deal with actual manifolds. The following is a radically generalized definition which we will restrict very soon.

Definition 33 (Section 2.1, [48])**.** Let X be a set. An *atlas of class* C^k $(k \geq 1)$ on X starts as a collection of pairs (U_i, φ_i) (called charts) with i in some indexing set. This collection must satisfy the following conditions

(i) Each U_i is a subset of X and $\cup_i U_i \supset X$.

(ii) Each φ_i is a bijection from U_i onto an open subset $\varphi_i(U_i)$ of some Banach space E_i and for any i, j, $\varphi_i(U_i \cap U_j)$ is open in E_i.

(iii) For U_i, U_j with non-empty intersection, the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a C^k-isomorphism for each pair of indices i, j.

Note that even though X is just a set, the topology on X can be induced from the atlas. It should be remarked upon that the C^k-isomorphism can only be guaranteed by the implicit function. Furthermore, in the definition X does not even need to be necessarily Hausdorff (or second countable), nor do the E_i in (i) need to be the same. However, we will not need this type of generality and require X to be Hausdorff, second countable and there to exist a Banach space E that replaces E_i for each i. In the language of Lang this would be an E-atlas.

We call a chart (U, φ), an open subset U of X is compatible with the atlas $\{(U_i, \varphi_i)\}$ if $\varphi : U \to U'$ with U' an open subset of E and $\varphi_i \circ \varphi^{-1}$ is a C^k-isomorphism for each i. Two atlases are compatible if their charts are mutually compatible. Compatibility of atlases is an equivalence relation and by defining a C^k differentiable structure on X we mean to choose a equivalence class of compatible atlases. Then by saying that X is an E-manifold, or a manifold modelled by E, we mean that X has a differentiable structure, a collection of atlases that are all E-atlases. We recover finite dimensional manifolds by letting $E = \mathbb{R}^n$ for some $n > 0$.

From manifolds we move to vector bundles immediately. This brings us one step closer to the technicalities of Theorem 2 as we will now understand the framework to define the Banach space bundles used there.

Definition 34 (Section 3.1, [48])**.** Let X be a manifold of class C^k, $k \geq 0$, $\pi : E \to X$ is a linear continuous map, where E is a Banach space. Let $\{U_i\}$ be an open covering of X such that, for each i, we have a mapping

$$\tau_i : \pi^{-1}(U_i) \to U_i \times E.$$

Together the pairs $\{(U_i, \tau_i)\}$ must satisfy

1. Each map τ_i is a C^k-isomorphism and commutes with the projection on U_i. This means, the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\tau_i} & U_i \times E \\ \downarrow & \swarrow & \\ U_i & & \end{array}$$

Specifically, there is an isomorphism on each fiber (write $\tau_i(x)$ or τ_{ix})

$$\tau_{ix} : \pi^{-1}(x) \to E.$$

2. For two U_i and U_j, the map

$$\tau_{jx} \circ \tau_{ix} : E \to E$$

is a toplinear isomorphism.

3. For U_i, U_j with non-empty intersection, the map from $U_i \cap U_j$ into $\mathrm{Laut}(E, E)$ is portrayed as

$$x \mapsto (\tau_j \circ \tau_i^{-1})_x.$$

The collection of pairs $\{(U_i, \tau_i)\}$ is a trivializing covering for π (or for E), and $\{\tau_i\}$ are the trivializing maps. Under $x \in U_i$, τ_i trivializes at x.

We call two trivializing covers $\{(U_i, \tau_i)\}$ and $\{(V_j, \mu_j)\}$ equivalent if their pairs mutually satisfy (*ii*) and (*iii*). This is an equivalence relation and an equivalence class of trivializing cover defines a vector bundle structure on π (or E). We say E is the total space and X the base space as in the finite dimensional case. By moving the Banach space structure of E to each fiber $\pi^{-1}(x)$ via the toplinear isomorphism τ_{ix} of (*ii*), these fibers become Banachable spaces. Then using (*iii*) this Banachable structure will be same. Same meaning that choosing distinct norms via two distinct toplinear isomorphisms τ_{ix} and τ_{jx} on the same fiber, these norms will turn out to be equivalent.

Before describing the Banach space bundles used in the technical details and connect this train of theory back to Sobolev completions will need a very useful function in many sub-fields of geometry, the exponential map. The exponential map for our purposes will define a differentiable structure on maps of class $W^{k,p}$ from the Riemann surface Σ and the almost complex manifold M.

First of all the first example of a Banach space bundle is the tangent bundle over a manifold. Let X be a manifold of class C^k ($k \geq 1$). Let x be a point in X, we will consider triples (U, φ, v) with (U, φ) a chart about x and v an element of $\varphi(U)$ lying in a vector space that X is modelled on. Two triples (U, φ, v) and (V, ψ, w) will be called equivalent if and only if the following identity holds

$$(\psi \circ \varphi^{-1})'(\varphi(x))v = w.$$

From the chain rule this is an equivalence relation and we call equivalence classes of triples $[(U, \varphi, v)]$ tangent vectors of X at x. The tangent space of X at x, $T_x(X)$, is the collection of these equivalence classes. The tangent bundle of X will be $T(X) = \{(x, [U, \varphi, v]) \mid [U, \varphi, v] \in T_x(X)\}$ with natural projection $\pi : T(X) \to X$ by $\pi(x, [U, \varphi, v]) = x$ is a vector bundle, in Lang's sense, of class C^{k-1}. In analogy to the differential or pushforward we will define a map between tangent spaces from a map C^k, $f : X \to Y$ between manifolds X and Y. At $x \in X$ denote $T_x(f)$ for the map between $T_x(X)$ and $T_{f(x)}(Y)$ for the pushforward of f at x. This map acts locally as simply the derivative. By simplifying an equivalence class $[U, \varphi, v]$ to just v we find $T_x(f)(v) = (f(x), f'(x)v)$.

In this specific example a section, a C^{k-1} toplinear map $\xi : X \to T(X)$ satisfy $\pi \circ \xi = \mathbb{1}_X$, is called a vector field as previous. In the local picture by identifying $T(U)$ with $U \times E$, U open subset of X and E a Banach space. Then ξ has two components (g_1, g_2) where the condition $\pi \circ \xi = \mathbb{1}_X$ implies that $g_1(x) = x$. So that $\xi(x) = (x, g_2(x))$, from ξ being C^{k-1} then g_2 is also C^{k-1} toplinear map. We can take this one step further by defining the double tangent bundle as $T(T(X))$ a C^{k-2} manifold with projection $T\pi : T(T(X)) \to T(X)$. Sections of this bundle will be needed in defining the exponential map through integral curves.

Firstly, a curve is a map $\alpha : J \to X$, X a manifold both of class C^k ($k \geq 2$) and J an open interval in $\mathbb{R}$. An integral curve of a vector field ξ is map of open interval J containing 0 in $\mathbb{R}$, $\alpha : J \to X$ with initial starting point $\alpha(0) = x_0$ and $\alpha'(t) = \xi(\alpha(t))$ for all $t \in J$. On points of uniqueness and existence we will rely on Section 4.1 of Lang's book and from now on assume both. For α' we mean $T\alpha \circ \iota$, with $\iota(t) = (t, 1)$. Assume from now on J is an open interval containing 0.

We call β a lifting of a curve $\alpha : J \to X$ of class C^l ($l \leq k$) into $T(X)$ such that $\beta : J \to T(X)$ and $\pi \circ \beta = \alpha$. A second-order vector field over X is a toplinear map F of class C^{k-2} such that $T\pi \circ F = \mathbb{1}_X$. For each integral curve β of F with canonical lifting $\pi \circ \beta$ then $(\pi \circ \beta)' = \beta$.

Let $\alpha : J \to X$ be a curve in X, we say α is geodesic with respect to F if the curve $\alpha' : J \to T(X)$ is an integral curve of F. In the local situation, U an open subset of X and E a Banach space, we identify $T(U)$ with $U \times E$ and $T(T(U))$ with $(U \times E) \times (E \times E)$. Here π maps from $U \times E$ to U and $T\pi$ from $(U \times E) \times (E \times E)$ to $U \times E$. By writing $(x, u, v, w) \in (U \times E) \times (E \times E)$ then $T\pi(x, v, u, w) = (x, u)$. Similar to vector fields on X there is a local representation, a C^{k-2} map $f : U \times E \to E \times E$ that has two components (f_1, f_2). Under the requirement that f is a local representation of a vector field on $T(U)$ then $f(x, v) = (v, f_2(x, v))$. Now α is a geodesic with respect to F such that $\alpha' = v$, $\alpha'' = f_2(x, v)$.

We care about those vector fields F on $T(X)$ called sprays which have local representative f_2 as a homogeneous map of degree 2 in the second variable. With f_2 as above and U an open subset of X and we require $f_2(x, tv) = t^2 f_2(x, v)$ for all $t > 0$ and $x \in U$. By the previous discussion f_2 has action as follows $f_2(x, v) = (1/2)D_2^2 f_2(x, 0)(v, v)$. It is also possible to define a spray via a symmetric bilinear map $x \mapsto B(x)$ such that $B(x) = (1/2)D_2^2 f_2(x, 0)$. Making this local situation global over the entire manifold is tackled in Proposition 3.4 of Lang and not discussed here.

Finally, by letting F be a spray on X and β_v an integral curve with respect to F with initial condition v. Then define a map $v \mapsto \beta_v(1)$ and $\exp(v) = \pi\beta_v(1)$ where v can only be mapped if β_v is at least defined on $[0, 1]$, this is achievable by Theorem 2.6 and Corollary 2.7 of [48]. Denote by $\exp_x$ the restriction of exp to the tangent space $T_x(X)$ so that $\exp_x : T_x(X) \to X$. The exponential map is of class C^{k-2} when X is a manifold of class C^k $(k \geq 2)$. The use for the exponential is a way to locally reconstruct X from its tangent bundle $T(X)$.

Now with all of this theory we can get to the applications bringing Sobolev completions together with the space of J-holomorphic curves. As a reminder the space we are trying to study is a subset of smooth functions from Riemann surface (Σ, j) and symplectic 4 dimensional almost complex manifold (M, J)

$$\{u \in C^\infty(\Sigma, M) \mid du \circ j = J \circ du\}.$$

From what we have been discussing of maps between tangent bundles an equivalent the condition could replace the above avoiding misleading notation, $Tu \circ j = J \circ Tu$. In any case we need to understand the Sobolev completions of vector bundles to apply the linearization of the Cauchy-Riemann type operator of appendix B.3 rigorously.

Let Σ be a Riemann surface and consider a general vector bundle of rank r, E over Σ. We write $W^{k,p}_{\text{loc}}(E)$ for those sections of $E \to \Sigma$ are of class $W^{k,p}$ on relatively compact subsets of a given open subset of Σ. Maps between manifolds can be defined similarly. When Σ is compact $W^{k,p}_{\text{loc}}(E)$ and $W^{k,p}(E)$ agree, we will assume this for Σ to avoid more complicated problems of non-canonical topology. That topology, we define now which will be put the structure of a Banach space on the fibers. From Σ being compact we can pick a finite covering $\{U_i\}_{i=1}^m, m < \infty$ and assume there exists respective chart maps φ_i such that $\Omega_i := \varphi_i(U_i) \subset \mathbb{C}$ with trivilizations $\Phi_i : E_{U_i} \to \Omega_i \times \mathbb{C}^r$. Along with, the covering, charts and trivilizations, we will need a partition of unity $\{\alpha_i : \Sigma \to [0, 1]\}_{i=1}^m$ subordinate to the covering. A partition of unity is guaranteed to exist for a topology (Theorem 36.1, [49]) and on manifolds (Prop. 13.6, [13]), they work like follows. A collection of real-valued maps with sum $\sum_{i=1}^m \alpha_i = 1$ which could be subordinate to a covering, which means $\text{supp}(\alpha_i) \subset U_i$. The exact definition is not very important, the purpose of the α_i's are to globally define the following norm, for any section $v : \Sigma \to E$

$$\|v\|_{W^{k,p}(E)} = \sum_{i=1}^m \|\text{pr}_2 \circ \Phi_i \circ (\alpha_i \cdot v) \circ \varphi^{-1}\|_{W^{k,p}(\Omega_i)}.$$

The norm inside the sum is the norm on functions of class $W^{k,p}$ from Ω_i to $\mathbb{C}^r (= \mathbb{R}^{2r})$. Let us check the function inside the sum is well-defined,

$$\Omega_i \xrightarrow{\varphi_i^{-1}} U_i \xrightarrow{\alpha_i \cdot v} E_{U_i} \xrightarrow{\Phi_i} \Omega_i \times \mathbb{C}^r \xrightarrow{\text{pr}_2} \mathbb{C}^r.$$

So that we are in the situation of the previous Sobolev theory with $n = 2r$. These fibers with this norm are actually Banachable spaces, but we are not too concerned as it is possible to make them Banach spaces. Furthermore, the norm is not canonical but, luckily enough, the topology induced is canonical.

Now we will consider maps of Sobolev type between manifolds where we will impose $kp > 2$, which implies that our maps are at least continuous. This is the beginning of making Theorem 2 rigorous as we turn to $\mathcal{B} = C^\infty(\Sigma, M)$ and making $\mathcal{B}$ into a complete space $\mathcal{B}^{k,p} = W^{k,p}(\Sigma, M)$. In that theorem we had that the tangent space at a point u was $T_u\mathcal{B} = \Gamma(u^*TM)$, so we analogize and assume $T_u\mathcal{B}^{k,p} = W^{k,p}(u^*TM)$ giving motivation to following definition.

Definition 35 (Definition 3.4, [10]). Let $k \in \mathbb{N}$ and $p \geq 1$ be such that $kp > 2$, pick any smooth connection ∇ on M and a smooth function g from Σ to M. Pick a neighbourhood U_g of the zero section in g^*TM such that for every $z \in \Sigma$, the exponential map exp restricted to $U_g \cap T_{g(z)}M$ is an embedding. Then define the space of maps of class $W^{k,p}$ between Σ and M as

$$W^{k,p}(\Sigma, M) := \{u \in C^0(\Sigma, M) \mid u = \exp_g \eta \text{ with } g \in C^\infty(\Sigma, M) \text{ and } \eta \in W^{k,p}(g^*TM)\}.$$

What do we mean then by exp as we previously defined via a spray. Well this happens via the connection ∇. We say that γ is a geodesic with respect to ∇ when $\nabla_{\dot\gamma(t)}\gamma(t) = 0$. But then

$$\nabla_{\dot\gamma(t)}\gamma(t) = 0 \iff \ddot\gamma(t) = f_2(\gamma(t), \dot\gamma(t)).$$

Noting that any γ has image in M. We now lift to $X = C^\infty(\Sigma, M)$ by defining $B(g)(\eta,\eta) = (1/2)D_2^2 f(g,0)(\eta,\eta)$ with $f(x,v) = (v, f_2(x,v))$, f_2 from above and $g \in X$, $\eta \in W^{k,p}(g^*TM)$. This defines bilinear map but also a local representation of a spray, as we know the result, in the end, is connection independent so that this choice does not matter intensely.

What does matter is that by using the Lemma 2.96 of Wendl's notes, the properties of Sobolev spaces, implies a $W^{k,p}$-differentiable structure can be put onto $W^{k,p}(\Sigma, M)$. What is entirely non-obvious is that this structure is given through maps $\eta \mapsto \exp_f \eta$ taking open subsets $W^{k,p}(g^*TM)$ into $C^0(\Sigma, M)$.

The last note to play, let $\mathcal{E}^{k-1,p}$ be a Banach space bundle with fibers $\mathcal{E}_u^{k-1,p} = W^{k,p}(\overline{\mathrm{Hom}}(T\Sigma, u^*TM))$ with Banach space structure defined via the norm as in the general case but with $E = \overline{\mathrm{Hom}}(T\Sigma, u^*TM)$. The rest of the linearization can now be carried out but using the inverse function theorem for Banach manifolds on the smooth section

$$\overline{\partial}_J : \mathcal{B}^{k,p} \to \mathcal{E}^{k-1,p} : u \mapsto du + J \circ du \circ j$$

about zero.

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