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Lie groupoids, Lie algebroids and Noether's Theorem

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Student: B.A. Mintjes

Supervisor for Mathematics: K. Efstathiou, PhD

Supervisor for Physics: prof. dr. E. Pallante

Abstract

In this thesis, we provide the reader with an introduction to the concepts of groupoids, Lie groupoids, algebroids and Lie algebroids, using various examples. We then employ this framework for the formulation of Noether's Theorem involving groupoids in the case of a real scalar field.

Contents

Introduction

Symmetry became a fundamental idea in mathematics and physics over the course of the nineteenth century. The main architects of the mathematical framework used today to discuss symmetries were Felix Klein and Sophus Lie. Their work in defining symmetries by groups of automorphisms cemented group theory as an important branch of mathematics, and one would be hard-pressed to find any mathematician that does not associate the study of symmetry with the field of group theory.

The notion of a groupoid is a less widespread one. While, after their introduction by H. Brandt in 1927 [5], [1] groupoids have since seen applications in various areas of mathematics [6], most undergraduate students (and perhaps, indeed, most graduate students as well) will not come across them in any form during their education. For physicists, the groupoid is an even more obscure concept. And that while the groupoid can be a very nice tool when discussing symmetries. In the words of Weinstein [19]:

> "Mathematicians tend to think of the notion of symmetry as being virtually synonymous with the theory of groups and their actions. (...) In fact, though groups are indeed sufficient to characterize homogeneous structures, there are plenty of objects which exhibit what we clearly recognize as symmetry, but which admit few or no nontrivial automorphisms. It turns out that the symmetry, and hence much of the structure, of such objects can be characterized if we use groupoids and not just groups."

The first chapter of this paper introduces readers to groupoids, algebroids, Lie groupoids, [2] and Lie algebroids in an accessible way. It introduces the concept of a groupoid in an algebraic sense (mostly following Weinstein's 1996 paper [19]), and makes a brief detour through category theory before introducing Lie groupoids and Lie algebroids.

In the second chapter, we mainly follow Costa et al.'s 2018 paper [8] in applying this framework of Lie groupoids and Lie algebroids to come to a formulation of Noether's theorem for the real scalar field. This chapter is intended

[1] Brandt actually introduced a more restrictive definition than the one that is now commonly used, defining what is known today as a connected groupoid.

[2] Introduced by Ehresmann [10].

as a review of the theory, its core concepts, and the application to Noether's theorem. Extra care is taken to ensure that all steps towards the result are explained in a way that can be understood by those that, like the author, have undergraduate knowledge of mathematics and physics.

Chapter 1

Lie groupoids and Lie algebroids

1.1 Groupoids

In this section, we introduce the concept of groupoids using various examples. We introduce the pair groupoid, the main groupoid relevant for our approach to Noether's theorem.

Definition 1.1.1. A **groupoid** G with base B is a set G with mappings σ_G and τ_G, called the *source* and *target* mappings, respectively, from G onto B and a binary operation

$$\mu_G : G \times G \to G, \qquad (g, h) \mapsto gh$$

with the following properties:

O1. It is defined only for certain pairs of elements: gh is defined only when $\sigma(g) = \tau(h)$.

O2. It is associative: if either of the products $(gh)k$ or $g(hk)$ is defined, then so is the other, and they are equal.

O3. For each g in G, there are left and right identity elements λ_g and ρ_g such that $\lambda_g g = g = g\rho_g$.

O4. Each g in G has an inverse, and hence G has an inverse map $\iota_G : G \to G, g \mapsto g^{-1}$ such that $gg^{-1} = \lambda_g$ and $g^{-1}g = \rho_g$.

⌟

For the maps defining the groupoid, we will often leave out the subscript G if no confusion can arise.

Lemma 1.1.2. *For any groupoid G over a base B, there exists a bijective map $1_G : B \to G$ such that,*

$$1_G(\tau(g)) = \lambda_g, \quad 1_G(\sigma(g)) = \rho_g.$$

Example 1.1.3. Consider a tiling of $\mathbb{R}^2$ by the set $X = H \cup V$, where $H = \mathbb{R} \times \mathbb{Z}$ and $V = 2\mathbb{Z} \times \mathbb{R}$. Call each connected component of $\mathbb{R}^2 \setminus X$ a *tile*. The symmetry of this tiling is described by the group Γ, consisting of the normal subgroup of translations by elements of $\Lambda = H \cap V = 2\mathbb{Z} \times \mathbb{Z}$, and reflections through each of the points in $\frac{1}{2}\Lambda = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ and across the horizontal and vertical lines through the points.

We define the **transformation groupoid** of the action of Γ on $\mathbb{R}^2$ to be the set

$$G(\Gamma, \mathbb{R}^2) = \{(x, \gamma, y) | x \in \mathbb{R}^2, \gamma \in G, x = \gamma y\},$$

with the binary operation

$$(x, \gamma, y)(y, \nu, z) = (x, \gamma\nu, z).$$

We will now show that all properties above hold for this operation.

O1. gh is defined only if $\sigma(g) = \alpha(h)$ for $\alpha : \Gamma \to \mathbb{R}^2, \quad (x, \gamma, y) \mapsto x$ and $\beta : \Gamma \to \mathbb{R}^2, \quad (x, \gamma, y) \mapsto y$.

O2. If we have $g = (x, \gamma, y)$, $h = (y, \nu, z)$ and $k = (z, \mu, u) \in G$, then clearly

$$gh = (x, \gamma, y)(y, \nu, z) = (x, \gamma\nu, z)$$

and hence $\beta(gh) = \alpha(k) = y$, so $(gh)k$ exists. Similarly

$$hk = (y, \nu, z)(z, \mu, u) = (y, \nu\mu, u)$$

so $\sigma(g) = \alpha(hk) = y$ so $g(hk)$ exists. Finally

$$(gh)k = (x, \gamma\nu, z)(z, \mu, u) = (x, \gamma\nu\mu, u) = (x, \gamma, y)(y, \nu\mu, u) = g(hk).$$

O3. For $g = (x, \gamma, y)$, let $\lambda_g = (x, \mathbb{1}, x)$, where $\mathbb{1}$ is the identity element of Γ, then

$$\lambda_g g = (x, \mathbb{1}, x)(x, \gamma, y) = (x, \gamma, y) = g.$$

Similarly, let $\rho_g = (y, \mathbb{1}, y)$.

O4. For $g = (x, \gamma, y)$, $g^{-1} = (y, \gamma^{-1}, x)$, since

$$gg^{-1} = (x, \gamma, y)(y, \gamma^{-1}, x) = (x, \gamma\gamma^{-1}, x) = (x, \mathbb{1}, x) = \lambda_g$$

and

$$g^{-1}g = (y, \gamma^{-1}, x)(x, \gamma, y) = (y, \mathbb{1}, y) = \rho_g.$$

Example 1.1.4. If B is any set, the product $B \times B$ is a groupoid over B with $\tau(y,x) = y$, $\sigma(y,x) = x$ and $(y,x)(z,y) = (z,x)$. The left and right-identities are given by $\lambda_{(y,x)} = (x,x)$ and $\rho_{(y,x)} = (y,y)$ (and the function 1_G is given by $x \mapsto (x,x)$). The inverse is given by $(y,x)^{-1} = (x,y)$. This is called the **pair groupoid** of B.

We can think of each element g of G as an arrow pointing from $\sigma(g)$ to $\tau(g)$. The notion of arrows is also prevalent in category theory, and we will show in the next section that *a groupoid is a (small) category in which every morphism has an inverse.*

1.2 Categories

In this section, we make a brief detour to category theory. We show that a groupoid is a small category in which every morphism has an inverse. While this will ultimately not come into play in our approach to Noether's theorem, it gives appropriate mathematical context to the concept of groupoids and aims to further the reader's understanding of this concept. The main reference for this section is Aluffi's *Algebra: Chapter 0* [1].

Definition 1.2.1. A **category** C consists of a class $\mathrm{Obj}(\mathsf{C})$ of *objects* of the category, and, for every two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, a set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ of *morphisms* with the following properties:

M1. For every object A of C, there exists (at least) one morphism $\mathbb{1}_A \in \mathrm{Hom}(A, A)$, the identity on A.

M2. For every triple of objects A, B, C of C there is a function

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \times \mathrm{Hom}_{\mathsf{C}}(B, C) \to \mathrm{Hom}_{\mathsf{C}}(A, C)$$

and the image of the pair (f, g) is denoted gf.

M3. If $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ and $h \in \mathrm{Hom}_{\mathsf{C}}(C, D)$, then

$$(hg)f = h(gf).$$

M4. For all $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ we have

$$f\mathbb{1}_A = f, \quad \mathbb{1}_B f = f$$

Moreover, $\mathrm{Hom}_{\mathsf{C}}(A, B)$ and $\mathrm{Hom}_{\mathsf{C}}(C, D)$ are disjoint unless $A = C$, $B = D$. ⌟

Definition 1.2.2. A category C is called a **small category** if $\mathrm{Obj}(\mathsf{C})$ is a set. ⌟

Example 1.2.3. Let S be a set, and let $\sim$ be a relation on S that is reflexive (i.e. for all $a \in S$, $a \sim a$), and transitive (i.e. for all $a, b, c \in S$, $a \sim b$ and $b \sim c$ implies $a \sim c$). Then we can create a category with as objects the elements of S, and with sets of morphisms defined as, for $a, b \in S$

$$\mathrm{Hom}(a, b) = \begin{cases} (a, b) & \text{if } a \sim b, \\ \emptyset & \text{otherwise.} \end{cases}$$

We check the morphism properties

M1. Let $a \in S$. Since $\sim$ is reflexive, we have $(a, a) \in \mathrm{Hom}(a, a)$, and we thus set $\mathbb{1}_a = (a, a)$.

M2. Let $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$. Then both these sets are nonempty, and hence $a \sim b$ and $b \sim c$. By transitivity, then, $a \sim c$. Thus, $\mathrm{Hom}(a, c) = \{(a, c)\}$, and we set $hg = (a, c)$.

M3. Let $f \in \mathrm{Hom}(a,b)$, $g \in \mathrm{Hom}(b,c)$ and $h \in \mathrm{Hom}(c,d)$. Then $f = (a,b)$, $g = (b,c)$ and $h = (c,d)$. Then

$$h(gf) = (c,d)(a,c) = (a,d) = (d,b)(a,b) = (hg)f.$$

M4. Let $f \in \mathrm{Hom}(a,b)$. Then $f = (a,b)$, and

$$f\mathbb{1}_a = (a,b)(a,a) = (a,b) = f$$

and

$$\mathbb{1}_b f = (b,b)(a,b) = (a,b) = f.$$

Definition 1.2.4. Let C be a category. A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ is an **isomorphism** if it has a (two-sided) inverse under composition, i.e. if there exists $g \in \mathrm{Hom}_{\mathsf{C}}(B,A)$ such that

$$gf = \mathbb{1}_A, \qquad fg = \mathbb{1}_B.$$

⌟

Proposition 1.2.5. *The inverse of an isomorphism is unique.*

Proof. Let $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ and let $g_1, g_2 \in \mathrm{Hom}_{\mathsf{C}}(B,A)$ be inverses of f. Then

$$g_1 = g_1\mathbb{1}_B = g_1(fg_2) = (g_1f)g_2 = \mathbb{1}_A g_2 = g_2.$$

□

Proposition 1.2.6. *We have that*

1. *each identity $\mathbb{1}_A$ is an isomorphism, and it is its own inverse;*
2. *if f is an isomorphism, then f^{-1} is an isomorphism, and $(f^{-1})^{-1} = f$;*
3. *if $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B,A)$ are isomorphisms, then gf is an isomorphism and $(gf)^{-1} = f^{-1}g^{-1}$*

Definition 1.2.7. Objects A, B of a category are **isomorphic** if there is an isomorphism $f : A \to B$. We write $A \cong B$. ⌟

Proposition 1.2.8. *A groupoid is a small category in which every morphism is an isomorphism.*

Proof. Let G be a groupoid over a set B with mappings τ, σ from G onto B and binary operation $(g,h) \mapsto gh$.

Define, for all $a, b \in B$, the sets

$$\mathrm{Hom}(a,b) = \{g \in G \mid \tau(g) = b, \sigma(g) = a\}.$$

M1. Let $a \in B$. Then, since σ is onto, there exists $g \in G$ s.t. $\sigma(g) = a$. Moreover, there exist ρ_g and g^{-1} such that $g^{-1}g = \rho_g$. Then

$$\tau(\rho_g) = \tau(g^{-1}g) = \tau(g^{-1}) = \sigma(g) = a$$

and

$$\sigma(\rho_g) = \sigma(g^{-1}g) = \sigma(g) = a.$$

Hence we let $\mathbb{1}_a = \rho_g \in \mathrm{Hom}(a, a)$.

M2. Let $a, b, c \in B$. Let $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$. Then $\sigma(g) = \tau(f)$, and hence gf is defined. Moreover, $\tau(gf) = \tau(g)$ and $\sigma(gf) = \sigma(f)$ so $gf \in \mathrm{Hom}(a, c)$.

M3. Let $f \in \mathrm{Hom}(a, b)$, $g \in \mathrm{Hom}(b, c)$, $h \in \mathrm{Hom}(c, d)$. Then $\sigma(hg) = \sigma(g) = b$ and $\tau(f) = b$, so $(hg)f$ is defined. Then, $h(gf)$ is defined, and

$$(hg)f = h(gf)$$

M4. Let $f \in \mathrm{Hom}(a, b)$. Then $f \in G$, so there exist ρ_f, λ_f such that

$$\lambda_f f = f = f\rho_f.$$

By similar reasoning as **M1**, we have $\rho_f = \mathbb{1}_a$, and $\lambda_f = \mathbb{1}_b$. Then

$$f\mathbb{1}_a = f\rho_f = f = \lambda_f = \mathbb{1}_b f.$$

We now prove every morphism is an isomorphism. Let $f \in \mathrm{Hom}(a, b)$. Then $f \in G$, and hence there exists f^{-1} such that

$$f^{-1}f = \rho_f = \mathbb{1}_a, \qquad ff^{-1} = \lambda_f = \mathbb{1}_b.$$

□

Example 1.2.9. The category defined in Example 1.2.3 is a groupoid if the relation is an **equivalence relation**, i.e. next to being reflexive and transitive it is also *symmetric* ($a \sim b$ implies $b \sim a$). This is easily proven.

Let $f \in \mathrm{Hom}(a, b)$. Then $f = (a, b)$, and $a \sim b$. By symmetry, then, $b \sim a$, so there exists $g = (b, a) \in \mathrm{Hom}(b, a)$, and

$$gf = (b, a)(a, b) = (a, a) = \mathbb{1}_a, \qquad fg = (a, b)(b, a) = (b, b) = \mathbb{1}_b.$$

Therefore, f is an isomorphism. Since f was arbitrary, this holds for all morphisms, and the category is thus a groupoid.

Proposition 1.2.10. *A group is the set of isomorphisms of a groupoid with a single object, with the operation of composition of morphisms.*

Proof. Let Γ be a groupoid, and let A be its single object. Define

$$G = \mathrm{Hom}_\Gamma(A, A) = \mathrm{Aut}_\Gamma(A).$$

Then by definition of a category,

1. There exists $\mathbb{1}_G \in G$.
2. There exists $\circ : G \times G \to G$.
3. For all $g, h, k \in G$,
$$(g \circ h) \circ k = g(h \circ k).$$

And, since Γ is a groupoid:

4. For all $g \in G$ there exists $g^{-1} \in G$ such that
$$g \circ g^{-1} = g^{-1} \circ g = \mathbb{1}_G.$$

$\square$

1.3 Lie groups

As the concept of a Lie groupoid is key to this thesis, we give a short introduction of Lie groups, assuming the reader has a familiarity with group theory.

We also introduce Lie algebras, so that we can more easily introduce Lie algebroids later on in this paper.

In this section we also introduce the Lie algebra of Killing vector fields, which we will apply in our treatment of Noether's theorem.

Definition 1.3.1. A topological space M is **locally Euclidean of dimension** n if for all $p \in M$ there is a neighbourhood U of p such that there exists a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$. We call $(U, \varphi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighbourhood** and φ a **coordinate map** on U. We say that a chart (U, φ) is **centered** at $p \in U$ if $\varphi(p) = 0$. ⌟

Definition 1.3.2. A **topological manifold** is a Haussdorff, second countable, locally Euclidean space. It is said to be **of dimension** n if it is locally Euclidean of dimension n. ⌟

Definition 1.3.3. Two charts $(U, \varphi : U \to \mathbb{R}^n), (V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are (C^∞-)**compatible** if the **transition functions**

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V), \quad \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

are C^∞. ⌟

Definition 1.3.4. A ($C^\infty-$)**atlas** on a locally Euclidean space M is a collection $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ of pairwise C^∞ compatible charts that cover M (i.e. $\cup_\alpha U_\alpha = M$). ⌟

Definition 1.3.5. A **smooth** or C^∞ **manifold** is a topological manifold together with a maximal atlas (i.e. an atlas that is not contained in another atlas). This atlas is also called a **differentiable structure** on M. ⌟

Definition 1.3.6. A **Lie group** is a C^∞ manifold G that is also a group, such that the two group operations, multiplication

$$\mu : G \times G \to G, \quad \mu(a, b) = ab$$

and inverse

$$\iota : G \to G, \quad \iota(a) = a^{-1}$$

are C^∞. ⌟

Example 1.3.7. The *general linear group*

$$GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} | \det A \neq 0\} = \det^{-1}(\mathbb{R} \setminus 0)$$

is a smooth manifold, since the determinant is a continuous function, and hence $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, and $\mathbb{R}^{n \times n} \simeq \mathbb{R}^{n^2}$.

Example 1.3.8. The *orthogonal group*

$$O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | A^T A = AA^T = I\}$$

is a Lie group.

Definition 1.3.9. Let $\mathbb{K}$ be a field. A **Lie algebra** over $\mathbb{K}$ is a vector space V over $\mathbb{K}$ together with a product $[\ ,\ ] : V \times V \to V$ called the **bracket** satisfying the following properties: for all $a, b \in \mathbb{K}$ and $X, Y, Z \in V$,

- $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$;
- $[Y, X] = -[X, Y]$;
- $\sum_{\text{cyclic}} [X, [Y, Z]] = 0$.

⌟

Example 1.3.10. Let $\mathbb{K}^{n\times n}$ be the vector space of all $n \times n$ matrices over a field $\mathbb{K}$. Define, for $X, Y \in \mathbb{K}^{n\times n}$

$$[X, Y] = XY - YX$$

where XY is the matrix product of X and Y. Then $\mathbb{K}^{n\times n}$ with this bracket is a Lie algebra.

Example 1.3.11. If M is a manifold, then the vector space $\mathcal{X}(M)$ of C^∞ vector fields on M is a real Lie algebra with the Lie bracket as the bracket.

Example 1.3.12. Given a manifold M with metric g, the set of Killing vector fields on (M, g), i.e. the set of vector fields $X \in \mathcal{X}(M)$ such that

$$\mathcal{L}_X \mathsf{g} = 0,$$

where $\mathcal{L}$ is the *Lie derivative*, is a subset of $\mathcal{X}(M)$ closed under the Lie bracket (a **Lie subalgebra**). This is shown as follows. Let X, Y be Killing vector fields. Then

$$\mathcal{L}_{[X,Y]}\mathsf{g} = (\mathcal{L}_X \circ \mathcal{L}_Y)\mathsf{g} - (\mathcal{L}_Y \circ \mathcal{L}_X)\mathsf{g} = \mathcal{L}_X 0 - \mathcal{L}_Y 0 = 0$$

so $[X, Y]$ is a Killing vector field.

1.4 Lie groupoids and Lie algebroids

This section introduces the Lie groupoids as a specific subset of the groupoids. It gives various examples of Lie groupoids, some of them analogous to examples introduced in Section 1.1.

The section introduces various groupoids important to our treatment of Noether's theorem, i.e. the pair groupoid, the linear and orthonormal frame groupoids. The notion of a bisection of a Lie groupoid is also treated.

Definition 1.4.1. A **Lie groupoid** G over a base M is a groupoid for which G and M are differentiable manifolds, and σ, τ and multiplication are differentiable maps. ⌟

Example 1.4.2. Any Lie group is a Lie groupoid over a single point.

Example 1.4.3. Any manifold M is a Lie groupoid with $\sigma(p) = \tau(p) = p$ for all $p \in M$, and $pp = p$.

Example 1.4.4. For any manifold M, the pair groupoid $G = M \times M$ is a Lie groupoid over M.

Example 1.4.5. The *action groupoid* $G \times M$ defined by a smooth left action $\cdot$ of a Lie group G on a manifold M is a Lie groupoid over M. Multiplication is given by

$$(g, m)(h, n) = (gh, g \cdot n)$$

and we have

$$\sigma(g, m) = g^{-1} \cdot m, \qquad \tau(g, m) = m.$$

Then,

O1. We have

$$m = \tau(g, m) = \tau((g, m)(h, n)) = \tau(gh, g \cdot n) = g \cdot n$$

so

$$\tau(h, n) = n = g^{-1} \cdot m = \sigma(g, m).$$

O2. Similar.

O3. For all (g, m) in $G \times M$ we have

$$\lambda_{(g,m)} = (\mathbb{1}_G, g^{-1} \cdot m), \quad \rho_{(g,m)} = (\mathbb{1}_G, m).$$

O4. The inverse is given by

$$(g, m)^{-1} = (g^{-1}, g^{-1} \cdot m)$$

Example 1.4.6. Given a manifold M, consider its tangent bundle TM and set

$$GL(TM) = \bigsqcup_{x,y\in M} {}_yGL(TM)_x \quad \text{where} \quad {}_yGL(TM)_x = GL(T_xM, T_yM)$$

is the set of invertible linear transformations from T_xM to T_yM. Then, $GL(TM)$ is a Lie groupoid, with, for $g \in {}_yGL(TM)_x \subset GL(TM)$,

$$\sigma(g) = x, \qquad \tau(g) = y.$$

We can prove this. Since $GL(TM)$ is a disjoint union, the source and target maps are onto. Take composition as our operation, then

O1. The element hg is defined only when $\sigma(g) = \tau(h)$, as we can only compose maps to a specific tangent space with maps from that same tangent space.

O2. If $(hg)k$ is defined, then so is $h(gk)$, and they are equal, as linear maps between tangent spaces are associative.

O3. For $g \in {}_yGL(TM)_x \subset GL(TM)$, there exist

$$\lambda_g = \mathbb{1}_{T_xM} \qquad \rho_g = \mathbb{1}_{T_yM}$$

such that $\lambda_g g = g = g\rho_g$.

O4. Obviously, since we are considering invertible linear transformations, we have that for each $g \in {}_yGL(TM)_x$ there exists $g^{-1} \in {}_xGL(TM)_y$ s.t. $gg^{-1} = \mathbb{1}_{T_xM}$ and $g^{-1}g = \mathbb{1}_{T_yM}$.

Example 1.4.7. Similarly, for a manifold equipped with a pseudoriemannian metric g, the set

$$O(TM, \mathsf{g}) = \bigsqcup_{x,y\in M} {}_yO(TM, \mathsf{g})_x, \text{ with } {}_yO(TM, \mathsf{g})_x = O\left((T_xM, \mathsf{g}_x), (T_yM, \mathsf{g}_y)\right)$$

the set of orthogonal linear transformations from (T_xM, g_x) to (T_yM, g_y), is a Lie groupoid over M called the **orthonormal frame groupoid**. This is in fact a *subgroupoid* of $GL(TM)$.

Definition 1.4.8. A (smooth) **bisection** of a Lie groupoid G over a manifold M is a (smooth) map $\beta : M \to G$ such that $\sigma \circ \beta = \mathbb{1}_M$ and $\tau \circ \beta \in \mathrm{Diff}(M)$. We denote the set of all bisections of G by $\mathrm{Bis}\,(G)$. ⌟

Proposition 1.4.9. *For any Lie groupoid G, the set* $\mathrm{Bis}\,(G)$ *is a group, with product defined by*

$$(\beta_2\beta_1)(x) = \beta_2(\tau(\beta_1(x)))\beta_1(x), \quad \textit{for } x \in M.$$

Proof. We first check that the product is well-defined. Let $x \in M$, and $\beta_1, \beta_2 \in \mathrm{Bis}\,(G)$. Then, since $\sigma(\beta_1(x)) = x$, and, since $\tau \circ \beta_1 \in \mathrm{Diff}(G)$ there exists unique $y \in M$ such that $\tau(\beta_1(x)) = y$. We get

$$(\beta_2\beta_1)(x) = \beta_2(y)\beta_1(x).$$

Now, since $\beta_2 \in \mathrm{Bis}\,(G)$, we have

$$\sigma(\beta_2(y)) = y = \tau(\beta_1(x))$$

so this product is defined. It is immediately evident that we have closure by elementary groupoid properties, as

$$\sigma\left((\beta_2\beta_1)(x)\right) = \sigma(\beta_1(x)) = x$$

and

$$\tau \circ (\beta_2\beta_1) = (\tau \circ \beta_2) \circ (\tau \circ \beta_1).$$

Associativity follows from associativity in the groupoid, the identity element of the group is the map 1_G, and the inverse of a bisection $\beta : x \mapsto g$ is given by the bisection $\beta^{-1} : \tau(g) \mapsto g^{-1}$. □

Example 1.4.10. The set of bisections for the pair groupoid is given by the diffeomorphism group of M, i.e. for $G = M \times M$,

$$\mathrm{Bis}\,(G) = \mathrm{Diff}(M).$$

This is easily seen, as any bisection β of G can be uniquely identified with a diffeomorphism $\tau \circ \beta$.

Definition 1.4.11. The **action** of a Lie groupoid on a fibre bundle E is a map

$$\Phi_E : G \times_M E \to E, \qquad (g, e) \mapsto g \cdot e$$

defined on the submanifold

$$G \times_M E = \{(g, e) \in G \times E \mid \sigma_G(g) = \pi_E(e)\},$$

such that

$$\pi_E \circ \Phi_E = \tau_G \circ \mathrm{pr}_1\,.$$

and, similar to group action, we have

1. $\rho_g \cdot e = e$ with $\sigma_G(g) = \pi_E$;
2. $h \cdot (g \cdot e) = (hg) \cdot e$ for $h, g \in G$, $e \in E$ such that $\sigma_G(h) = \tau_G(g)$ and $\sigma_G(g) = \pi_E(e)$.

⌟

Proposition 1.4.12. *Any Lie groupoid action Φ_E induces a homomorphism*

$$\Pi_E : \mathrm{Bis}\,(G) \to \mathrm{Aut}(E), \qquad \beta \to \Pi_E(\beta)$$

defined by

$$\Pi_E(\beta) = \Phi_E \circ (\beta \circ \pi_E, \mathbb{1}_E).$$

Explicitly,

$$\Pi_E(\beta) : E \to E, \qquad e \mapsto \beta(\pi(e)) \cdot e.$$

Proof. First, we prove that, for $\beta \in \mathrm{Bis}\,(G)$, $\Pi_E(\beta)$ is indeed an automorphism on E, i.e. an isomorphism to itself.

We first check surjectivity, i.e. for all $f \in E$, there exists $e \in E$ such that $\Pi_E(\beta)(e) = f$.

Let $f \in E$. Then, since π is onto, there exists $y \in M$ such that $\pi(f) = y$. Since τ is onto, there exists $g \in G$ such that $\tau(g) = y$. Then, as $\beta \circ \tau$ is onto, there exists $x \in M$ such that $(\beta \circ \tau)(x) = y$, and then, for this x, $\beta(x) = g$. Now, π is onto, so there then exists $e \in E$ such that $\pi(e) = x$. Hence,

$$\Pi_E(\beta)(e) = \beta(\pi_E(e)) \cdot e = \beta(x) \cdot e = g \cdot e = f.$$

Now, we prove injectivity. Let $e_1, e_2 \in E$, with $\pi(e_1) = x_1$, $\pi(e_2) = x_2$. Let $\Pi_E(\beta)(e_1) = \Pi_E(\beta)(e_2) = f$. Then $\beta(x_1) \cdot e_1 = \beta(x_2) \cdot e_2 = f$. Hence, $(\tau \circ \beta)(x_1) = (\tau \circ \beta)(x_2)$. Since $\tau \circ \beta$ is injective, $x_1 = x_2$. Then, for $g = \beta(x_1) = \beta(x_2)$, $g \cdot e_1 = g \cdot e_2 = f$. Hence $e_1 = e_2$.

The fact that Π_E is a homomorphism follows directly from the properties of Φ_E as a group action. Let $\pi(e) = x$, $\beta_1(x) = h$, $\beta_2(\tau(h)) = g$, then

$$\Pi_E(\beta_1)(\Pi_E(\beta_2)(e)) = \Pi_E(\beta_1)(h \cdot e) = g \cdot (h \cdot e) = (gh) \cdot e = \Pi_E(\beta_2\beta_1)(e).$$

$\square$

1.5 Lie algebroids

Definition 1.5.1. A **Lie algebroid** over a manifold M is defined to be a vector bundle $\mathfrak{g}$ over M with a Lie bracket $[\cdot,\cdot]$ on its space of smooth sections, together with a bundle map $\rho : E \to TM$ called the **anchor** of the Lie algebroid satisfying

1. $[\rho(X), \rho(Y)] = \rho([X,Y])$;
2. $[X, \varphi Y] = \varphi\,[X,Y] + (\rho(X)\cdot\varphi)Y$.

Here, X and Y are smooth sections of E, φ is a smooth function on M, and the bracket is the Lie bracket. ⌟

Example 1.5.2. The tangent bundle TM with the Lie bracket, and the identity map as anchor, is a Lie algebroid over M. Any section of TM is a smooth vector field on M, and with the identity map as anchor, the first condition holds immediately, and the second reduces to an elementary property of the Lie bracket.

From any Lie groupoid G over M, we can construct a corresponding Lie algebroid $\mathfrak{g}$. We use the unit map 1_G to consider M as an embedded submanifold of G, that is, we define

$$G|_M = \{1_G(x) \mid x \in M\} \subset G.$$

We restrict the tangent maps to the source and target projections to this submanifold. We then define

$$\mathfrak{g} = \ker T\sigma_G|_M.$$

Example 1.5.3. We consider the pair groupoid $G = M \times M$. Then

$$G|_M = \{(x,x) \mid x \in M\}$$

and

$$TG|_M = T_{(x,x)}(M \times M) \cong T_xM \times T_xM.$$

Then, any map in $\ker T\sigma|_M$ is a map $T_xM \times T_xM \to T_xM \times \{0\} \cong T_xM$ for some $x \in M$, such that for the pair groupoid we have the Lie algebroid

$$\mathfrak{g} = TM.$$

Chapter 2

Noether's Theorem

2.1 The classical Noether's Theorem

In this section, we give the classical formulation of Noether's Theorem for the real scalar field. We will derive an analogous version using the groupoid framwork in the rest of the chapter.

Consider a field φ. Its transformation in infinitesimal form reads

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \alpha\Delta\varphi(x)$$

where α is an infinitesimal parameter and $\alpha\Delta\varphi$ is some field deformation. This transformation is called a **symmetry** of the classical theory if the action is invariant under this transformation. The action is given by the time integral of the Lagrangian L, which we can write in terms of the *Lagrangian density* $\mathcal{L}$, a function of the fields and their derivatives, that is,

$$S = \int L\,dt = \int \mathcal{L}(\varphi, \partial_\mu\varphi)d^n x.$$

From the *principle of least action*, which says that a system evolves between two given configurations between times t_1 and t_2 along a path for which the action S is a minimum or a maximum, we get

$$\begin{aligned} 0 &= \delta S \\ &= \int d^n x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)\right] \\ &= \int d^n x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)\right]. \end{aligned}$$

If we restrict ourselves to deformations that vanish on the spatial boundary of the region of integration, the last term drops out. Noting that the total integral

must be 0 for arbitrary $\delta\varphi$, we get the **Euler-Lagrange equation**

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0.$$

This is the equation of motion for our system. We note, then, that a change by a surface term leaves the Euler-Lagrange equation invariant, so that the Lagrangian must be invariant under this equation up to a divergence

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x)$$

for some $\mathcal{J}^\mu$. If we vary the fields

$$\begin{aligned} \alpha \Delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \varphi}(\alpha \Delta \varphi) + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \partial_\mu(\alpha \Delta \varphi) \\ &= \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \Delta \varphi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \Delta \varphi. \end{aligned}$$

Here, the term between brackets drops out by the Euler-Lagrange equation, and we get

$$\begin{aligned} \alpha \Delta \mathcal{L} &= \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta \varphi \right) + 0 \\ &= \alpha \partial_\mu \mathcal{J}^\mu. \end{aligned}$$

so that

$$\partial_\mu j^\mu(x) = 0, \qquad \text{for } j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta \varphi - \mathcal{J}^\mu.$$

and hence the current $j^\mu(x)$ is conserved. This leads us to the general statement of Noether's theorem:

Theorem 2.1.1 (Noether's Theorem)**.** *For every continuous symmetry of the action there exists a conserved current.*

For a spacetime transformation

$$x^\mu \to x^\mu - a^\mu$$

or, alternatively

$$\varphi(x) \to \varphi(x + a) = \varphi(x) + a^\mu \partial_\mu \varphi(x)$$

we get that the Lagrangian transforms as

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

which gives us four separately conserved currents

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \mathcal{L} \delta^\mu_\nu.$$

This is the **energy-momentum tensor** of the field φ. The Hamiltonian is then

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x.$$

2.2 Fibre bundles

In this section, we provide the mathematical framework for treating classical field theory. We introduce the notion of fibre bundles and jet bundles, and describe the Lagrangian and Hamiltonian in this context. We use the real scalar field and with that, the trivial bundle, as a guideline.

Definition 2.2.1. Let M be a smooth manifold. Given any map $\pi : E \to M$, we call the inverse image

$$\pi^{-1}(p) = \pi^{-1}\{(p)\}$$

of a point p the **fibre at** p, denoted E_p. ⌟

Definition 2.2.2. For any two maps $\pi : E \to M$, $\pi' : E' \to M$, a map $\varphi : E \to E'$ is said to be **fibre-preserving** if $\varphi(E_p) \subset E'_p$ for all $p \in M$ ⌟

Definition 2.2.3. A surjective smooth map $\pi : E \to M$ of manifolds is said to be **locally trivial of rank** r if:

1. Each fibre $\pi^{-1}(p)$ has the structure of a vector space of rank r.
2. For all $p \in M$ there exists an open neighbourhood U of p and a fibre-preserving diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for all $q \in U$ the restriction

   $$\varphi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$$

   is a vector space isomorphism. Such an open set U is called a **trivialising open set** for E, and φ is called a **trivialisation** of E over U.

⌟

Definition 2.2.4. A **fibre bundle** is a structure (E, B, π, F) where E, B and F are topological spaces and $\pi : E \to B$ is a continuous surjection such that for all $e \in E$ there is an open neighbourhood $U \subset B$ of $\pi(e)$ such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

commutes. ⌟

Example 2.2.5. Let $E = B \times F$, and let $\pi : E \to B$ be the projection onto the first factor. Then E is a fibre bundle (of F) over B, called the **trivial bundle**.

Example 2.2.6. The tangent bundle

$$TM = \{(x, X) \mid x \in M,\ X \in T_xM\}$$

with projection $\pi(x, X) = x$, is a fibre bundle for which the fibres $\pi^{-1}(x) = T_xM$ are vector spaces.

Definition 2.2.7. A **section** of a fibre bundle is a map $\sigma : B \to E$ is a smooth right-inverse of π, i.e.

$$\pi(\sigma(x)) = x \quad \text{for all } x \in B.$$

We denote the set of sections of E as $\Gamma(E)$. ⌟

Proposition 2.2.8. *For the trivial bundle $E = M \times \mathbb{R}$, we have $\Gamma(E) = C^\infty(M, \mathbb{R})$.*

Proof. Denote $f : M \to \mathbb{R} \in C^\infty(M, \mathbb{R})$ by its graph $(x, f(x))$. Then

$$\mathrm{pr}_1((x, f(x)) = x$$

so $f \in \Gamma(E)$. Conversely, $\sigma : M \to M \times \mathbb{R} \in \Gamma(E)$ is a smooth function $x \mapsto (x, y)$ and can therefore be identified with $(x, f(x)) \in C^\infty(M, \mathbb{R})$ with $f(x) = y$. □

Given a manifold M on which we fix a metric tensor g, the real scalar fields are sections of the trivial bundle for M.

Example 2.2.9. The sections of the tangent bundle TM of a manifold M are the vector fields on M (i.e. $\Gamma(TM) = \mathcal{X}(M)$), since for all $x \in M$, and $X \in \mathcal{X}(M)$ we have

$$\pi(X(x)) = \pi(x, X) = x.$$

2.3 Jet bundles

In this section, we introduce a specific kind of fibre bundle, namely the *jet bundle*. For any fibre bundle E over a manifold M, these jet bundles are defined in such a way that they are both a bundle over the original fibre bundle E, and a bundle over its base space M.

The importance of jet bundles stems from the fact that any section φ of E induces a section of its fibre bundle which we can see, in some sense, as the derivative of φ. Moreover, the Lagrangian and Hamiltonian of a physical system can be defined as functions on specific jet bundles.

Definition 2.3.1. For any point $e \in E$ with base point $x = \pi_E(e)$ in M, let $L(T_xM, T_eE)$ denote the space of linear maps from T_xM to T_eE. Consider the affine subspace

$$J_eE = \{u_e \in L(T_xM, T_eE \mid T_e\pi_E \circ u_e = \mathbb{1}_{T_xM}\}.$$

and its difference vector space

$$\vec{J}_eE = \{\vec{u}_e \in L(T_xM, T_eE \mid T_e\pi_E \circ \vec{u}_e = 0\},$$

i.e.

$$\vec{J}_eE = L(T_xM, V_eE) = T_x^*M \otimes V_eE$$

where $V_eE = \ker T_e\pi_E$ is the vertical space of E at e. Taking the disjoint union as e varies over E, this defines JE and $\vec{J}E$ as **jet bundles**. ⌟

These jet bundles are bundles over E with respect to the projections $\pi_{JE} : JE \to E$, and $\pi_{\vec{J}E} : \vec{J}E \to E$. They are also bundles over M, where the corresponding projections are $\pi_E \circ \pi_{JE}$ and $\pi_E \circ \pi_{\vec{J}E}$, respectively.

Proposition 2.3.2. *For $E = M \times \mathbb{R}$, we have*

$$JE \cong \vec{J}E \cong \mathrm{pr}_1^*(T^*M)$$

Proof. First, note that, for $e = (x, y) \in E$ we have

$$T_eE = T_{(x,y)}(M \times \mathbb{R}) = T_xM \times T_y\mathbb{R} = T_xM \times \mathbb{R}.$$

Let $u \in JE$. Then $u \in J_eE$ for some $e \in E$. Then $u \in L(T_xM, T_eE)$ with $\mathrm{pr}_1(e) = x$, and $T_e \,\mathrm{pr}_1 \circ u = \mathbb{1}_{T_xM}$. Then

$$u : T_xM \to T_xM \times \mathbb{R}, \quad X_x \mapsto (X_x, \xi), \quad \text{for } X_x \in T_xM,\ \xi \in \mathbb{R}$$

which means we can identify u with a linear map $\vec{u} \in \vec{J}E$, given by

$$\vec{u} : T_xM \to T_xM \times \mathbb{R}, \quad X_x \mapsto (0, \xi).$$

which must exist since $u \in L(T_xM, T_eE)$. Hence, these two sets are isomorphic.

We have,

$$\mathrm{pr}_1^*(T^*M) = \bigsqcup_{e\in E} \{(e,(x,\omega)) \mid (x,\omega) \in T_x^*M\} \quad \text{where } x = \mathrm{pr}_1(e)$$

For $u \in JE$ there exists unique corresponding $e \in E$. We can identify $u \in JE : X_x \mapsto (X_x, \xi)$ with the element

$$(e,(x,\omega)) \in \mathrm{pr}_1^*(T^*M) \mid (x,\omega) : X_x \mapsto \xi.$$

Hence, these two spaces are also isomorphic. □

We then see that any section φ of a fibre bundle E induces a section $j\varphi$ of JE, which we call its **jet prolongation**. This is nothing more than a reinterpretation of the tangent map $T\varphi$, since

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\mathbb{1}_M"'] & E \arrow[dl, "\pi_E"] \\ M & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} T_xM \arrow[r, "T_x\varphi"] \arrow[d, "\mathbb{1}_{T_xM}"'] & T_eE \arrow[dl, "T_e\pi_E"] \\ T_xM & \end{tikzcd}$$

For every $u_e \in J_eE$, we can find $\varphi \in \Gamma(E)$ such that $u_e = T_x\varphi$. Now, we look at those sections of JE that can be written as the jet prolongation of some section of E.

Definition 2.3.3. A section $\tilde{\varphi}$ of JE is called **holonomous** if it can be written as the jet prolongation of some section of E. ⌟

Proposition 2.3.4. *A section $\tilde{\varphi}$ of JE is holonomous if and only if*

$$\tilde{\varphi} = j\varphi \quad \textit{where } \varphi = \pi_{JE} \circ \tilde{\varphi}$$

Proof. Let $\tilde{\varphi}$ be a section of JE such that $\tilde{\varphi} = j\varphi$. Then

$$\mathbb{1} = (\pi_E \circ \pi_{JE}) \circ \tilde{\varphi} = \pi_E \circ (\pi_{JE} \circ \tilde{\varphi})$$

so that φ is a section of E if and only if $\varphi = \pi_{JE} \circ \tilde{\varphi}$. □

Definition 2.3.5. The **Lagrangian** is a homomorphism

$$\mathcal{L} : JE \to \bigwedge^n T^*M$$

of fiber bundles over M. ⌟

When composed with the jet prolongation of a section φ of E, this provides an n-form (volume form) on M, such that we can define the **action functional** S as, for a compact subset of K of M

$$S_K[\varphi] = \int_K \mathcal{L}(\varphi, \partial\varphi).$$

Definition 2.3.6. The **affine dual** $J^\star E$ of JE and the **linear dual** $\vec{J}^* E$ of $\vec{J}E$ are defined by setting, for any point $e \in E$ with $\pi(e) = x$,

$$J_e^\star E = \{z_e : J_e E \to \mathbb{R} \mid z_e \text{ is affine}\},$$

and

$$\vec{J}_e^* E = \{\vec{z}_e : \vec{J}_e E \to \mathbb{R} \mid \vec{z}_e \text{ is linear}\}.$$

⌟

Definition 2.3.7. The **twisted affine dual** $J^\circledast E$ and the **twisted linear dual** $\vec{J}^\circledast E$ are defined as

$$J_e^\circledast E = \{z_e : J_e E \to \bigwedge^n T^*M \mid z_e \text{ is affine}\},$$

and

$$\vec{J}_e^\circledast E = \{z_e : J_e E \to \bigwedge^n T^*M \mid z_e \text{ is linear}\}.$$

⌟

Proposition 2.3.8. *For the trivial bundle $E = M \times \mathbb{R}$, we have*

$$\vec{J}^\circledast E = \mathrm{pr}_1^* \left(\bigwedge^{n-1} T^*M \right), \qquad \vec{J}^\circledast E = \mathrm{pr}_1^* \left(\bigwedge^{n-1} T^*M \oplus \bigwedge^n T^*M \right)$$

Proof. We have shown earlier that, for $E = M \times \mathbb{R}$,

$$\vec{J}E \cong JE \cong \mathrm{pr}_1^*(T^*M).$$

Hence, to prove that

$$\vec{J}^\circledast E = \mathrm{pr}_1^* \left(\bigwedge^{n-1} T^*M \right),$$

we need only show that

$$\{L : \bigwedge^1 T^*M \to \bigwedge^n T^*M \mid L \text{ is linear}\} = \bigwedge^{n-1} T^*M.$$

The $(n-1)$-form $\Omega \in \bigwedge^{n-1} T^*M$, given by

$$\Omega = \sum_{i=1}^n c_i e_1^* \wedge \cdots \wedge \widehat{e_i^*} \wedge \cdots e_n^*$$

can be uniquely identified with the linear map $L : \bigwedge^1 T^*M \to \bigwedge^n T^*M$ given by

$$L(\omega) = \omega \wedge \Omega = f e_1^* \wedge \cdots \wedge e_n^*.$$

For the dual basis vectors we have

$$L(e_j^*) = e_j^* \wedge \sum_{i=1}^n c_i e_1^* \wedge \cdots \wedge \widehat{e_i^*} \wedge \cdots \wedge e_n^* = (-1)^{j-1} c_j e_1^* \wedge \cdots \wedge e_n^*$$

so that for a general 1-form $\omega = \sum_{k=1}^{n} b_k e_k^*$, we have

$$L(\omega) = \sum_{k=1}^{n} b_k L(e_k^*) = \left(\sum_{k=1}^{n} (-1)^{k-1} b_k c_k \right) e_1^* \wedge \cdots \wedge e_n^* = f e_1^* \wedge \cdots \wedge e_n^*.$$

Here, f and $\{c_1, \ldots, c_k\}$, and hence $L(\omega)$ and Ω, uniquely determine one another.

This quite immediately shows that

$$J^{\circledast} E = \mathrm{pr}_1^* \left(\bigwedge^{n-1} T^* M \oplus \bigwedge^{n} T^* M \right)$$

as for any affine map $A : T^* M \to \bigwedge^n T^* M$ we can write

$$A(\omega) = L(\omega) + \Xi$$

with L as above and Ξ some n-form. $\square$

We can view $J^{\circledast} E$ as an affine line bundle over $J^{\circledast} E$, with projection

$$\eta : J^{\circledast} E \to \vec{J}^{\circledast} E$$

defined by taking the linear part of an affine map. The *hamiltonian* of the classical field theory is a section of this bundle.

$$\mathcal{H} : \vec{J}^{\circledast} E \to J^{\circledast} E.$$

2.4 The jet groupoid

Analogous to the notion of jet bundles of a bundle over a manifold, we now introduce the concept of the *jet groupoid* of a groupoid over a manifold.

Proposition 2.4.1. *Let G be a Lie groupoid over a manifold M. Set*

$$JG = \bigsqcup_{g \in G} J_g G$$

where, for $g \in G$ with with $\sigma_G(g) = x$ and $\tau_G(g) = y$,

$$J_g G = \{u_g \in L(T_x M, T_g G) \mid T_g \sigma_G \circ \ u_g = \mathbb{1}_{T_x M} \text{ and } T_g \tau_G \in GL(T_x M, T_y M)\}.$$

Furthermore, let

$$\sigma_{JG}(u_g) = \sigma_G(g), \qquad \tau_{JG}(u_g) = \tau_G(g)$$

and

$$\mu_{JG}(v_h, u_g) = T_{(h,g)}\mu_G \circ (T_g \tau_G \circ u_g), u_g).$$

Then, JG is a Lie groupoid over M.

Proof. We check all properties:

O1. We have the linear maps

$$T_g \tau_G \circ u_g : T_{\sigma_G(g)} M \to T_{\tau_G(g)} M$$

and

$$v_h : T_{\sigma_G(h)} M \to T_h G.$$

For these to be composable (and hence for our operation to be defined) we must have that $\sigma_{JG}(v_h) = \sigma_G(h) = \tau_G(g) = \tau_{JG}(u_g)$.

O2. Since composition of linear maps is associative, this property follows quite immediately from the one above.

O3. For $u_g \in J_g G$, we have

$$\lambda_g = T_{\tau(g)} 1_G, \qquad \rho_g = T_{\sigma(g)} 1_G.$$

O4. For any $u_g \in J_g G$, we have

$$u_g^{-1} = T_g \iota_G \circ u_g \circ (T_g \tau_G \circ u_g)^{-1}.$$

□

Next to being a Lie groupoid over M, JG is also a fibre bundle over G, under the projection

$$\pi_{JG} : JG \to G, \qquad u_g \mapsto g.$$

Example 2.4.2. In the case of the pair groupoid $G \times G$ we have,

$$J(M \times M) = \bigsqcup_{g \in M \times M} J_g(M \times M)$$

where, for $g = (y, x) \in M \times M$,

$$\begin{aligned} J_g(M \times M) &= \{u_g \in L(T_xM, T_yM \times T_xM) \mid T_g\sigma \circ u_g = \mathbb{1}_{T_xM} \text{ and} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_g\tau \circ u_g \in GL(T_xM, T_yM)\} \\ &= \{u_g \in L(T_xM, T_yM \times T_xM) \mid u_g((x, X)) = (v(x, X), (x, X)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } v \in GL(T_xM, T_yM)\} \\ &\cong GL(T_xM, T_yM) = {}_yGL(TM)_x \end{aligned}$$

so that

$$J(M \times M) \cong GL(TM).$$

Similar to sections of a fibre bundle inducing bisections of its jet bundle, any bisection β of G induces a bisection $j\beta$ of JG, its **jet prolongation**, which is a reinterpretation of the tangent map $T\beta$ to β.

Definition 2.4.3. The **group of holonomous bisections** of a Lie subgroupoid $\tilde{G}$ of JG is given by

$$HB(\tilde{G}, G) = \{\tilde{\beta} \in \mathrm{Bis}\,(\tilde{G}) \mid \tilde{\beta} = j\beta \text{ for some } \beta \in \mathrm{Bis}\,(G)\}.$$

or, equivalently, by

$$HB(\tilde{G}, G) = \{\beta \in \mathrm{Bis}\,(G) \mid j\beta \in \mathrm{Bis}\,(\tilde{G})\}.$$

⌟

Proposition 2.4.4. *For the pair groupoid $G = M \times M$, and $\tilde{G} = O(TM, \mathsf{g})$, we have*

$$HB(G, \tilde{G}) = \mathrm{Isom}(M, \mathsf{g}).$$

Proof. For $G = M \times M$, $\mathrm{Bis}\,(G) = \mathrm{Diff}(M)$. The set of holonomous bisections is then the set

$$HB(G, \tilde{G}) = \{\beta \in \mathrm{Diff}(M) \mid j\beta \in \mathrm{Bis}\,(\tilde{G})\}$$

This is the set of diffeomorphisms of M such that their derivative $T\beta$ preserves the metric g. These are precisely the isometries on M with respect to the metric g. □

Analogously, we can define the **jet algebroid** of a Lie algebroid.

Given a Lie algebroid $\mathfrak{g}$ over a manifold M with anchor $\rho_{\mathfrak{g}}$ amd bracket $[\cdot, \cdot]_{\mathfrak{g}}$, its jet bundle $J\mathfrak{g}$ is a Lie algebroid over M with anchor $\rho_{J\mathfrak{g}}$ and bracket $[\cdot, \cdot]_{J\mathfrak{g}}$

defined so that jet prolongation of sections (under $j : \Gamma(E) \to \Gamma(JE)$ preserves the anchor and the bracket

$$\rho_{J\mathfrak{g}}(j\xi) = \rho_{\mathfrak{g}}(\xi), \quad [j\xi, j\eta]_{J\mathfrak{g}} = j([\xi, \eta]_{\mathfrak{g}}] \quad \text{for } \xi, \eta \in \Gamma(\mathfrak{g}).$$

If $\mathfrak{g}$ is the Lie algebroid associated to a Lie groupoid G, then $J\mathfrak{g}$ is the Lie algebroid associated to the Lie groupoid JG. We also provide another definition similar to the holonomous bisections.

Definition 2.4.5. Let $\mathfrak{g}$ be a Lie algebroid over a manifold M and $\tilde{g}$ a Lie subalgebroid of its jet algebroid $J\mathfrak{g}$. Then the **Lie algebra of holonomous sections** of $\tilde{g}$, denoted by $H\Gamma(\mathfrak{g}, \tilde{\mathfrak{g}})$ to be the Lie subalgebra of $\Gamma(\tilde{g})$ given by

$$H\Gamma(\mathfrak{g}, \tilde{\mathfrak{g}}) = \{\tilde{X} = jX \text{ for some } X \in \Gamma(\mathfrak{g})\}.$$

⌟

2.5 The real scalar field

In this section, we show how a formulation of Noether's theorem involving groupoids can be used to arrive at conserved currents in the case of the real scalar field.

Noether's theorem for groupoids

This section gives an illustration of the version of Noether's theorem for groupoids, as given in [8]. Its full statement is as follows:

Theorem 2.5.1. *Let E be a fibre bundle over the manifold M, endowed with the action of a Lie groupoid G over the same manifold M, and consider the induced action of JG on the ordinary multiphase space $\vec{J}^{\circledast}E$ as well as the corresponding infinitesimal actions of the Lie algebroids $\mathfrak{g}$ (by fundamental vector fields on E) and $J\mathfrak{g}$ (by fundamental vector fields on $\vec{J}^{\circledast}E$). Given a full Lie subgroupoid $\tilde{G}$ of JG, with corresponding full Lie subalgebroid $\tilde{g}$ of $J\mathfrak{g}$, and a $\tilde{G}$-invariant Hamiltonian $\mathcal{H} : \vec{J}^{\circledast}E \to J^{\circledast}E$, the **Noether current** associated with a "generator" $X \in H\Gamma(\mathfrak{g}, \tilde{\mathfrak{g}})$ and a section Φ of $\vec{J}^{\circledast}E$ is the pull-back $\Phi^*\mathcal{J}(X) \in \Omega^{n-1}(M)$. Then, if Φ satisfies the equations of motion, i.e., the De Donder-Weyl equations, this current is conserved, i.e. a closed form:*

$$d\left[\Phi^*\mathcal{J}(X)\right] = 0.$$

A full proof of the theorem is outside the scope of this text, and for this we refer the reader to the paper by Costa et al.. In this section, we limit ourselves to the real scalar field, and show the following

Proposition 2.5.2. *Let $E = M \times \mathbb{R}$ be the trivial bundle over the manifold M, endowed with the action of the pair groupoid $G = M \times M$ over the same manifold M. Given the Lie subgroupoid $\tilde{G} = O(TM, \mathfrak{g})$ of $JG = GL(TM)$, with corresponding Lie subalgebroid $\tilde{g}$ of $J\mathfrak{g}$, and a $\tilde{G}$-invariant Hamiltonian $\mathcal{H} : \vec{J}^{\circledast}E \to J^{\circledast}E$, the **Noether current** associated with a "generator" $K \in H\Gamma(\mathfrak{g}, \tilde{\mathfrak{g}}) = \mathrm{Kill}(M, \mathfrak{g})$ and a section Φ of $\vec{J}^{\circledast}E$ is the pull-back*

$$\Phi^*\mathcal{J}(K) = \iota_{K_{\vec{J}^{\circledast}E}}\theta_{\mathcal{H}} \in \Omega^{n-1}(M).$$

This current is conserved.

The real scalar field

We consider a pseudoriemannian manifold M with metric tensor $\mathfrak{g}$. In our geometrical approach to field theory, the fields under consideration are sections of a fibre bundle (the configuration bundle) over our (spacetime) manifold M. We take the trivial bundle

$$E = M \times \mathbb{R}$$

as our configuration bundle, since, as we showed in Proposition 2.2.8, its sections are the smooth functions on M, i.e.

$$\Gamma(E) = C^\infty(M, \mathbb{R})$$

which are the real scalar fields, and hence the fields we consider.

Symmetry groupoid and algebroid

The Lie groupoid for symmetry considerations is the pair groupoid. Its bisections are, as shown earlier in Example 1.4.10, the smooth diffeomorphisms of M. These correspond to coordinate transformations $x^\mu \to y^\mu(x)$ that induce a change in the metric, so

$$G = M \times M, \qquad \mathrm{Bis}\,(G) = \mathrm{Diff}(M).$$

The corresponding Lie algebroid and its sections are given by (see Example 1.5.3 and Example 2.2.9, respectively)

$$\mathfrak{g} = TM, \qquad \Gamma(\mathfrak{g}) = \mathcal{X}(M).$$

We have shown in Proposition 2.3.2 that the jet bundles are given by

$$JE = \vec{J}E = \mathrm{pr}_1^*\,(T^*M)$$

and in Proposition 2.3.8 that the ordinary and extended multiphase spaces are given by

$$\vec{J}^{\circledast}E = \mathrm{pr}_1^*\left(\bigwedge^{n-1} T^*M\right), \qquad J^{\circledast}E = \mathrm{pr}_1^*\left(\bigwedge^{n-1} T^*M \oplus \bigwedge^{n} T^*M\right).$$

The jet groupoid of the pair groupoid is given by (Example 2.4.2)

$$JG = GL(TM)$$

and hence, we have

$$J\mathfrak{g} = \mathfrak{gl}(TM) = TM \oplus L(TM).$$

In the case of the real scalar field, the Lagrangian is then the homomorphism

$$\mathcal{L} : JE = \mathrm{pr}_1^*\,(T^*M) \to \bigwedge^{n} T^*M,$$

which, when composed with a section $(\varphi, d\varphi)$ of JE, where $d\varphi = \partial_\mu\varphi dx^\mu$, gives the n-form $L(\varphi, d\varphi)d^nx$, where

$$L(\varphi, d\varphi) = \tfrac{1}{2}\mathsf{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi).$$

The Hamiltonian composed with a section $\Phi = (\varphi, \pi)$ of $\vec{J}^{\circledast}E$, is given by $\mathcal{H} = Hd^nx$, with

$$H(\varphi, \pi) = \tfrac{1}{2}\mathsf{g}_{\mu\nu}\pi^\mu\pi^\nu + V(\varphi),$$

where $\pi = \pi^\mu d^n x_\mu$, and V is some potential. The symmetry groupoid for this theory is the groupoid under which $\mathcal{H}$ is invariant, so that we have, in this case,

$$\tilde{G} = O(TM, \mathsf{g}).$$

The group of holonomous bisections of $\tilde{G}$ is then

$$HB(G, \tilde{G}) = \mathrm{Isom}(M, \mathsf{g}),$$

as shown in Proposition 2.4.4, so that the Lie algebra of holonomous sections of $\tilde{\mathfrak{g}}$ is the Lie algebra of Killing vector fields on (M, g).

From Killing field to Noether current

Now that we have shown that in this case, our Noether currents are generated by Killing vector fields on M, we compute the Noether current associated to such a vector field, given by.

Given a Killing vector field K on (M, g), we write

$$K = K^\mu \frac{\partial}{\partial x^\mu}.$$

Its lift to $\bar{J}^{\circledast} E$ is

$$K_{\bar{J}^{\circledast} E} = K^\mu \frac{\partial}{\partial x^\mu} + \left(\frac{\partial K^\mu}{\partial x^\nu} p^\nu - \frac{\partial K^\nu}{\partial x^\nu} p^\mu \right) \frac{\partial}{\partial p^\mu}$$

which we contract with the multicanonical form

$$\theta_{\mathcal{H}} = p^\mu dq \wedge d^n x_\mu - H d^n x.$$

We note that

$$dq \left(\left(\frac{\partial K^\mu}{\partial x^\nu} p^\nu - \frac{\partial K^\nu}{\partial x^\nu} p^\mu \right) \frac{\partial}{\partial p^\mu} \right) = dx^\kappa \left(\left(\frac{\partial K^\mu}{\partial x^\nu} p^\nu - \frac{\partial K^\nu}{\partial x^\nu} p^\mu \right) \frac{\partial}{\partial p^\mu} \right) = 0$$

so that we need only concern ourselves with the first part of $K_{\bar{J}^{\circledast} E}$. We get

$$dq \left(K^\mu \frac{\partial}{\partial x^\mu} \right) = 0.$$

Moreover

$$dx^\nu \left(K^\mu \frac{\partial}{\partial x^\mu} \right) = \delta^\mu_\nu K^\nu,$$

so that

$$\iota_{K_{\bar{J}^{\circledast} E}} (d^n x_\mu) = K^\nu d^n x_{\mu\nu}$$

and

$$\iota_{K_{\bar{J}^{\circledast} E}} (d^n x) = K^\mu d^n x_\mu$$

Then

$$\iota_{K_{\bar{J}^{\circledast}E}}\left(p^\mu dq \wedge d^n x_\mu\right) = -p^\mu K^\nu dq \wedge d^n x_{\mu\nu},$$

and

$$\iota_{K_{\bar{J}^{\circledast}E}}\left(-H d^n x\right) = -H K^\mu d^n x_\mu.$$

Taking all this together, we get

$$\mathcal{J}(K) = \iota_{K_{\bar{J}^{\circledast}E}}\theta_{\mathcal{H}} = -p^\mu K^\nu dq \wedge d^n x_{\mu\nu} - H K^\mu d^n x_\mu$$

Pulling back with a field configuration $\Phi = (\varphi, \pi)$ means we substitute q by φ, dq by $d\varphi = \partial_\kappa \varphi dx^\kappa$ and p^μ by $\pi^\mu = \mathsf{g}^{\mu\nu}\partial_\nu\varphi$. We then get

$$\Phi^*\mathcal{J}(K) = -\mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\kappa\varphi\, dx_\kappa \wedge d^n x_{\mu\nu} - H K^\mu d^n x_\mu.$$

Using that $dx_\kappa \wedge d^n x_{\mu\nu} = \delta^\kappa_\nu d^n x_\mu - \delta^\kappa_\mu d^n x_\nu$ and substituting H gives

$$\begin{aligned}\Phi^*\mathcal{J}(K) = -\mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\kappa\varphi\, &\left(\delta^\kappa_\nu d^n x_\mu - \delta^\kappa_\mu d^n x_\nu\right) \\ &- \left(\tfrac{1}{2}\mathsf{g}_{\alpha\beta}(\pi^\alpha)(\pi^\beta) + V(\varphi)\right) K^\mu d^n x_\mu.\end{aligned}$$

Now we substitute $\pi^\alpha = \mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi$, and similarly substitute π^β to get

$$\begin{aligned}\Phi^*\mathcal{J}(K) = -\mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu &\left(\partial_\nu\varphi d^n x_\mu - \partial_\mu\varphi d^n x_\nu\right) \\ &- \left(\tfrac{1}{2}\mathsf{g}_{\alpha\beta}(\mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi)(\mathsf{g}^{\beta\eta}\partial_\eta\varphi) + V(\varphi)\right) K^\mu d^n x_\mu.\end{aligned}$$

Expanding brackets in the first term and contracting $\mathsf{g}_{\alpha\beta}$ with $\mathsf{g}^{\beta\eta}$ in the second gives us

$$\begin{aligned}\Phi^*\mathcal{J}(K) = -\mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\nu\varphi d^n x_\mu &+ \mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\mu\varphi d^n x_\nu \\ &- \tfrac{1}{2}\delta^\eta_\alpha \mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi\partial_\eta\varphi K^\mu d^n x_\mu - V(\varphi) K^\mu d^n x_\mu.\end{aligned}$$

Now using $\delta^\eta_\alpha \partial_\eta\varphi = \partial_\alpha\varphi$ gives

$$\begin{aligned}\Phi^*\mathcal{J}(K) = -\mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\nu\varphi d^n x_\mu &+ \mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\mu\varphi d^n x_\nu \\ &- \tfrac{1}{2}\mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi K^\mu \partial_\alpha\varphi d^n x_\mu - V(\varphi) K^\mu d^n x_\mu.\end{aligned}$$

Now we relabel the second term ($\nu \to \mu, \mu \to \alpha, \rho \to \gamma$) and get

$$\begin{aligned}\Phi^*\mathcal{J}(K) = -\mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\nu\varphi d^n x_\mu &+ \mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi K^\nu \partial_\alpha\varphi d^n x_\mu \\ &- \tfrac{1}{2}\mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi K^\mu \partial_\alpha\varphi d^n x_\mu - V(\varphi) K^\mu d^n x_\mu.\end{aligned}$$

We can then subtract the third term from the second and obtain

$$\begin{aligned}\Phi^*\mathcal{J}(K) = -\mathsf{g}^{\mu\rho}\partial_\rho\varphi K^\nu \partial_\nu\varphi d^n x_\mu &+ \tfrac{1}{2}\mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi K^\mu \partial_\alpha\varphi d^n x_\mu \\ &- V(\varphi) K^\mu d^n x_\mu.\end{aligned}$$

Relabelling the first term ($\nu \to \mu, \mu \to \nu, \rho \to \kappa$), and writing $d^n x_\mu = \delta^\kappa_\mu d^n x_\kappa$ in the third term gives us

$$\begin{aligned}\Phi^* \mathcal{J}(K) = &-\mathsf{g}^{\nu\kappa}\partial_\nu\varphi K^\mu \partial_\mu \varphi d^n x_\kappa + \tfrac{1}{2}\mathsf{g}^{\alpha\gamma}\partial_\gamma\varphi K^\mu \partial_\alpha\varphi d^n x_\mu \\ &- V(\varphi) K^\mu \delta^\kappa_\mu d^n x_\kappa\end{aligned}$$

Now using $d^n x_\mu = \delta^\kappa_\mu d^n x_\kappa$ in the second term, we get

$$\begin{aligned}\Phi^* \mathcal{J}(K) = &-\mathsf{g}^{\nu\kappa}\partial_\nu\varphi K^\mu \partial_\mu \varphi d^n x_\kappa + \tfrac{1}{2}\mathsf{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi K^\mu \delta^\kappa_\mu d^n x_\kappa \\ &- V(\varphi) K^\mu \delta^\kappa_\mu d^n x_\kappa,\end{aligned}$$

so that we now can take the last two terms together to obtain

$$\Phi^* \mathcal{J}(K) = -\mathsf{g}^{\nu\kappa}\partial_\nu\varphi K^\mu \partial_\mu \varphi d^n x_\kappa + \left(\tfrac{1}{2}\mathsf{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V(\varphi)\right)\delta^\kappa_\mu K^\mu d^n x_\kappa$$

Noting that the term between brackets is precisely L as defined earlier, and writing $\delta^\kappa_\mu = \mathsf{g}_{\mu\nu}\mathsf{g}^{\nu\kappa}$, we get

$$\Phi^* \mathcal{J}(K) = -\mathsf{g}^{\nu\kappa}\partial_\nu\varphi K^\mu \partial_\mu \varphi d^n x_\kappa + \mathsf{g}_{\mu\nu} L \mathsf{g}^{\nu\kappa} K^\mu d^n x_\kappa,$$

and then taking the terms together gives

$$\Phi^* \mathcal{J}(K) = -\left(\partial_\mu\varphi\partial_\nu\varphi - \mathsf{g}_{\mu\nu}L\right) K^\mu \mathsf{g}^{\nu\kappa} d^n x_\kappa.$$

As a final result we then have

$$\Phi^* \mathcal{J}(K) = -T_{\mu\nu} K^\mu \mathsf{g}^{\nu\kappa} d^n x_\kappa.$$

where

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \mathsf{g}_{\mu\nu}L$$

is the energy-momentum tensor. Since, in the above expression, all Lorentz indices contract, we are left with a Noether current $T_{\mu\nu}K^\mu$, which is a conserved current.

Conclusion

In this paper, we have introduced our readers to groupoids, Lie groupoids, algebroids and Lie algebroids. We have introduced the reader to the necessary mathematical structure to approach classical field theory from a geometrical point of view. Within this framework, we have found a formulation of Noether's theorem using Lie groupoids and algebroids, and have shown that when applying this theorem to the case of the real scalar field, we arrive at a conserved current.

Some comments on relevance are in order, though. While the main advantage of the groupoid formulation is that it provides a unified approach to internal and external symmetries, the former are absent in the case of this example. Nevertheless, it serves as an important proof of concept of the groupoid approach. For a more general version of Noether's theorem for groupoids, we refer the reader to [8].

The groupoid approach as outlined here may have value when considering true symmetries of Lagrangians in field theory over curved spacetime. For more on this, and an application of groupoids to gauge theory, we refer the reader to [9].

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