UNIVERSITY OF GRONINGEN

MASTER THESIS

The a-theorem in gauge field theories

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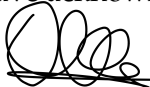Van Swinderen Institute for Particle Physics and Gravity

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Declaration of Authorship

I, ALFREDO PROFUMO, declare that this thesis titled, "The a-theorem in gauge field theories" and the work presented in it are my own. I confirm that:

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Abstract

Faculty Name
Van Swinderen Institute for Particle Physics and Gravity

Master in Physics

The a-theorem in gauge field theories

by ALFREDO PROFUMO

In this thesis we explore the constraints posed by the a-theorem to the renormalization group flows between fixed points in supersymmetric gauge theories. We reproduce some known results. We argue that the analogies between the supersymmetric gauge theory and the massless Veneziano limit of large-N QCD (and in particular the recently proposed exact beta function) point to the possibility of the extension of similar constraints to QCD.

Acknowledgements

I want to thank my people, friends and teachers who made this thesis possible.

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Contents

Chapter 1

Introduction

1.1   RG-Flow & Irreversibility

The renormalization group is a mathematical apparatus that allows to systematize the investigation of the effects of scale transformations on the observables of a physical system.

Our main focus in this thesis will be its application to the study of quantum field theories at the scale of particle physics and smaller, but applications are plentiful in every physical system in which these concepts are relevant (e.g. condensed matter systems, chaos theory) and also outside of physics (e.g. neural networks).

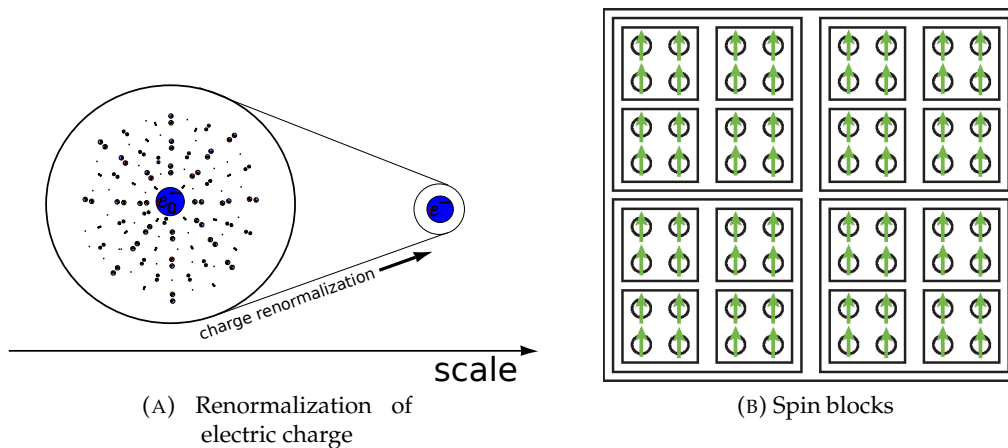(A)   Renormalization   of
electric charge

(B) Spin blocks

FIGURE 1.1: Two physical systems whose dynamics can be described at different scales: (A) when we look at the electron from afar the degrees of freedom of the virtual particles get absorbed in the redefinition of the electric charge [1] (B) the *block spin* picture devised by Leo P. Kadanoff in which we describe a system of spins through block variables of increasing sizes [2]; the block variables describing a block of a particular size are obtained by averaging the behaviour of the blocks it contains.

The main idea behind the renormalization group is that a system described by many degrees of freedom at a certain microscopic scale can be described in general by less degrees of freedom at a larger scale.

We will see that we can describe the theories at different scales with an effective action S_μ appropriate for the particular scale. The action is composed by the fields of interest for the physical theory we want to describe and a set of operators coupled through a set of couplings $\{g_i\}$ which can be thought of as coordinates in *theory space*. The renormalization group flow then connects theories at different scales (see Fig.

1.2). This process can be formalized, as we will see in section 2.1.2, as an integration over the high-energy modes.
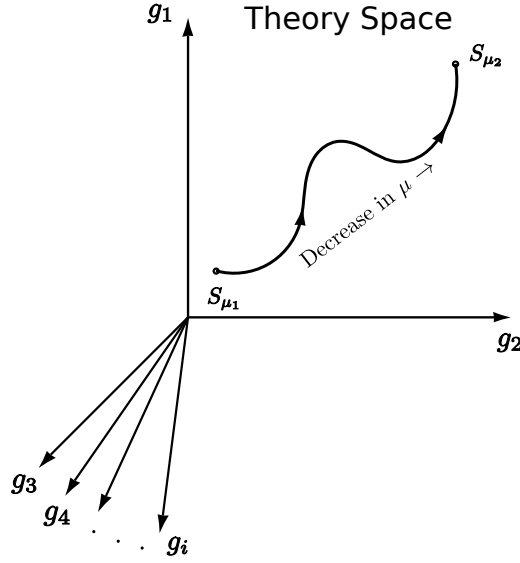

FIGURE 1.2: Flow between theories

1.2   a-theorem

The process of integration over high-energy modes can be intuitively understood to be irreversible. We will see in chapter 3 that this idea can be formalized thanks to the c- and a-theorems, respectively valid in 2 and 4 dimensions. These theorems show that there exists a quantity specific to a particular renormalization group flow that is monotonic during the flow and stationary only at the fixed points where it counts the true degrees of freedom of the system.

1.3   Fixed Points of SQCD & QCD

The study of the fixed points of the renormalization group flow of a theory is very important because it gives the information on the high and low energy behaviour of said theory (see chapter 2.1.4) and can help us understanding which flows may or may not be realized. Unfortunately, in most practical cases we are limited to computations within perturbation theory, i.e. near the free limit of the theory, being it in the ultraviolet or the infrared.

   In this thesis, we will show how the a-theorem can be used to infer non-perturbative constraints on the fixed point structure of a theory in a case in which the symmetries are calculable enough to allow the exact calculation of the *a*-function. In particular we will explore the case of SQCD for which we will show some non perturbative results valid in the conformal window and above it.

   In QCD a recently proposed beta function for the massless Veneziano limit of large-N gives hope that similar constraints may be extended giving new non-perturbative information on the theory (see chapter 4.1.1).

Chapter 2

RG-Flow

2.1 Renormalization Group Flow[1]

2.1.1 Introduction to RG-flows

In theoretical physics the *renormalization group flow* (RG-flow) is a mathematical apparatus that allows to connect the behaviour of a theory at different energy scales. The necessity of a renormalization group picture emerges naturally after the problem of infinities has been taken care with renormalization.

In particle physics, whenever ultraviolet divergences occurs, their cancellation is necessary to yield physical predictions. In renormalization this is taken care by absorbing the infinite quantities in the coupling constants and masses and introducing a cutoff scale Λ—which can eventually taken to be infinite. The dependence of the physical quantities on the scale Λ is hidden, traded for the large scales at which the quantities are to be measured and as a result this quantities end up being finite even for an infinite Λ.

One of the fundamental features of QFT is locality, that, in this framework, constraints different space-time points to have independent quantum fluctuations and degrees of freedom. High-momentum quanta appear in the calculations as virtual particles arising from quantum fluctuations at arbitrary short distances. In every renormalizable theory this short distances (high energy) quantum fluctuation are dominated in the loop integrals by the contributions due to the finite external particle momenta. However, at an intuitive level is perhaps not clear why high-momentum quanta can have so little physical effect on a theory.

In the first part of this chapter we will introduce a physical picture, due to Kenneth Wilson that will shed some light on this phenomenon. This picture, of difficult practical implementation, is however effective in giving a physical understanding of the underlying phenomena in the emerging of scale dependent physical quantities.

We will subsequently introduce a different, and more practical, description of the scaling of renormalized quantities through the use differential equations called Callan-Symanzik equations; we will also introduce the *beta function* an important instrument that encloses the information on the scaling properties of the couplings of the theory.

We will then discuss the important topic of the fixed points of the renormalization group flow and see how they can be used in model building. We will end the chapter by discussing some of the scaling properties of the couplings and the Green's functions and how they relate to anomalous dimensions.

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[1]This section is partially adapted from [3] and [4]

2.1.2   Wilson's Picture

Wilson's approach to renormalization is based on the path integral approach to field theory. With this method, ultraviolet divergences can be studied isolating the contributions of the high-frequency degrees of freedom of the field.

The modern view of renormalization is that our quantum field theories should be regarded only as effective field theories, valid up to some energy scale Λ[2]. We will then insist on a hard cutoff Λ in our path integral. We will analyze how the integration of the high-energy modes close to Λ affects the generating functional and see how this operation can be interpreted as a flow in the space of the possible Lagrangians.

Our analysis wil be specialized to a scalar field theory and subsequently to a ϕ^4 theory but the idea is similar, modulo technical nuances, for any field theory.

Let's then consider the generating functional $Z[J]$ in which the integration variables are the Fourier components of the field $\phi(k)$

$$Z[J] = \int \mathcal{D}\phi e^{i \int [\mathcal{L} + J\phi]} = \left(\prod_k \int d\phi(k) \right) e^{i \int [\mathcal{L} + J\phi]} \tag{2.1}$$

To impose the sharp cutoff we integrate only over $\phi(k)$ with $|k| \leq \Lambda$, and set $\phi(k) = 0$ for $|k| > \Lambda$. However, in Minkowski space this kind of cutoff is not completely effective in controlling large momenta, as in lightlike directions the components of k can be vary large while k^2 remains small. We will therefore impose the cutoff to the Wick rotated (Euclidean) version of the functional integral: k from now on will be Euclidean and the cutoff will be imposed to in the same way as before.

We will specialize furthermore to a ϕ^4 theory, with $J = 0$ for simplicity. The path integral with the cutoff is written as

$$Z = \int [\mathcal{D}\phi]_\Lambda \exp\left(-\int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \right) \tag{2.2}$$

with

$$[\mathcal{D}\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k) \tag{2.3}$$

Note that we are carrying this analysis in d spacetime dimensions and both m and λ are still the bare parameters of the theory.

One way to perform the integration on the high momentum shell is to divide the integration variables in two groups. The one pertaining to the shell and the rest. Choose a fraction $b < 1$. The variables $\phi(k)$ in the high-momentum shell $b\Lambda \leq |k| < \Lambda$ are the ones to be integrated over. We can label these variables $\hat\phi$, defined as

$$\hat{\phi}(k) = \begin{cases} \phi(k) & \text{for } b\Lambda \leq |k| < \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

We can replace the old ϕ in the Lagrangian by $\phi + \hat\phi$ where this new ϕ is identical to the old one for $|k| < b\Lambda$ and zero otherwise. With this substitution eq. (2.2)

---

[2]An alternative approach, called *dimensional regularization*, is to analytically continue the spacetime dimensions. The idea is that, just like the cutoff Λ, the regularizator should disappear from all physical quantities.

becomes

$$
\begin{aligned}
Z &= \int \mathcal{D}\phi \int \mathcal{D}\hat{\phi} \exp\left(-\int d^d x \left[\frac{1}{2}\left(\partial_\mu \phi + \partial_\mu \hat{\phi}\right)^2 + \frac{1}{2}m^2(\phi + \hat{\phi})^2 + \frac{\lambda}{4!}(\phi + \hat{\phi})^4\right]\right) \\
&= \int \mathcal{D}\phi\, e^{-\int \mathcal{L}(\phi)} \int \mathcal{D}\hat{\phi} \exp\left(-\int d^d x \left[\frac{1}{2}\left(\partial_\mu \hat{\phi}\right)^2 + \frac{1}{2}m^2\hat{\phi}^2\right.\right. \\
&\quad \left.\left. + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right]\right)
\end{aligned}
$$

(2.5)

(mettere a posto parentesi e allineamento)

We have isolated all the terms independent from $\hat{\phi}$ in $\mathcal{L}(\phi)$. Note also that terms of the form $\phi\hat{\phi}$ automatically vanish, since Fourier components of different wavelengths are orthogonal.

The objective is to perform the integral over $\hat{\phi}$. What we want to obtain from this integration is an expression of the form

$$
Z = \int [\mathcal{D}\phi]_{b\Lambda} \exp\left(-\int d^d x \mathcal{L}_{\text{eff}}\right),
$$

(2.6)

where $\mathcal{L}_{\text{eff}}(\phi)$ is an effective Lagrangian containing only the low momentum Fourier components $\phi(k)$. It can be shown [3] that this Lagrangian is of the form

$$
\mathcal{L}_{\text{eff}} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + (\text{corrections proportional to } \lambda)
$$

(2.7)

The corrections can be shown to be a sum of connected diagrams that contains corrections to m^2 and λ, as well as all possible higher-dimension operators.

Since we only included renormalizable interaction in our starting Lagrangian one could be worried by the appearance of higher-dimensional nonrenormalizable interactions when we integrate out $\hat{\phi}$. However, we will now show that this procedure keeps this contributions under control. In fact we will see that the presence of a very large cutoff in the original Lagrangian already implies that the presence of nonrenormalizable interactions has negligible effect at scales far below Λ.

Let us now rewrite (2.6) in a form closer to the one we started with (2.2). To do this we can rescale the distances and momenta in (2.6) in such a way that the cutoff can be written as $|k'| < \Lambda$, meaning

$$
k' = k/b, \quad x' = xb.
$$

(2.8)

If we write (2.7) with the correction to λ and m and the higher-dimensional operators as

$$
\begin{aligned}
\int d^d x \mathcal{L}_{\text{eff}} = \int d^d x \left[\frac{1}{2}(1 + \Delta Z)\left(\partial_\mu \phi\right)^2 + \frac{1}{2}\left(m^2 + \Delta m^2\right)\phi^2 \right. \\
\left. + \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta C \left(\partial_\mu \phi\right)^4 + \Delta D\phi^6 + \cdots\right]
\end{aligned}
$$

(2.9)

the substitution leads to

$$
\begin{aligned}
\int d^d x \mathcal{L}_{\text{eff}} = \int d^d x' b^{-d} \left[\frac{1}{2}(1 + \Delta Z)b^2 \left(\partial'_\mu \phi\right)^2 + \frac{1}{2}\left(m^2 + \Delta m^2\right)\phi^2 \right. \\
\left. + \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta C b^4 \left(\partial'_\mu \phi\right)^4 + \Delta D\phi^6 + \cdots\right]
\end{aligned}
$$

(2.10)

To complete the rewriting lets define

$$\phi' = \left[b^{2-d}(1 + \Delta Z) \right]^{1/2} \phi, \tag{2.11}$$

and the new parameters of the Lagrangian

$$
\begin{aligned}
m'^2 &= \left(m^2 + \Delta m^2 \right) (1 + \Delta Z)^{-1} b^{-2}, \\
\lambda' &= (\lambda + \Delta\lambda)(1 + \Delta Z)^{-2} b^{d-4}, \\
C' &= (C + \Delta C)(1 + \Delta Z)^{-2} b^{d}, \\
D' &= (D + \Delta D)(1 + \Delta Z)^{-3} b^{2d-6}
\end{aligned}
\tag{2.12}
$$

which brings the Lagrangian in its initial form, with the added appearance of the higher-dimensional operators:

$$
\begin{aligned}
\int d^d x \mathcal{L}_{\text{eff}} = \int d^d x' &\left[\frac{1}{2} \left(\partial'_\mu \phi' \right)^2 + \frac{1}{2} m'^2 \phi'^2 \right. \\
&\left. + \frac{1}{4} \lambda' \phi'^4 + C' \left(\partial'_\mu \phi' \right)^4 + D' \phi'^6 + \cdots \right].
\end{aligned}
\tag{2.13}
$$

Note that, in the original Lagrangian, the coefficient C and D where set to be equal to 0, but we could very well have had them to be nonzero and the same equation would apply. It is also important to notice that throughout the previous derivations terms such as Δm^2 and $\Delta\lambda$ are to be considered as small with respect to the leading terms as they arise from perturbation theory.

Observing Eq. (2.13) we can see that the combined effect of the integration on the high-momentum shell and rescaling has resulted in a transformation of the Lagrangian. Iterating this procedure [3] we can obtain entire families of Lagrangians. If we set the b parameter close to 1, so that the shells are infinitesimally thin, the transformation becomes a continuous one. We can then describe it as a trajectory, or flow, in the space of all possible Lagrangians.

For historical reasons, this operation takes the name of *renormalization group flow* or only *renormalization group*. However, we must remember that the flow through the space of Lagrangians is not a group operation in the mathematical sense, since the operation of integrating out degrees of freedom is not invertible.

Now imagine we want to calculate some observable at some momentum scale such that all external momenta p_i are much smaller than Λ. We could compute the value of this observable using the original Lagrangian $\mathcal{L}$ and perturbation theory, or the effective Lagrangian obtained after a renormalization group flow down to the the scale of the external momenta p_i. Both procedures must yield the same results. In the first one however, the effects of high-momentum fluctuations do no show up until we calculate loop diagrams. In the second case, this effects are absorbed in the redefinition of the coupling constants (λ', m') arising from the renormalization group flow. In the first procedure, divergences appear suddenly in one-loop diagrams, and seem to invalidate the use of perturbation theory. In the second case the effect of divergences is slowly absorbed in the corrections to the coupling constants and provided these effective couplings remain small the perturbative treatment is valid every step of the way.

---

[3]We can do this by changing b or by integrating on a different momentum shell, in practice the two operations are equivalent. Furthermore, the operation is transitive: if we call $b\Lambda \equiv \Lambda'$ and we repeat the procedure by integrating on a shell $\Lambda'' \leq |k'| < \Lambda'$ we would obtain the same Lagrangian we would have obtained by integrating directly on the shell $\Lambda'' \leq |k| < \Lambda$.

In general, there is no guarantee that from the values of the bare couplings of a Lagrangian the renormalization group flow will bring us to useful or even finite values of the new renormalized parameters. Let shed light on this point by continuing our analysis of the ϕ^4 theory with particular focus on how the Lagrangian varies when subjected to a renormalization group transformation.

A point to start is the free Lagrangian where the values of all the bare parameters are set to 0, which is simply the kinetic term

$$\mathcal{L}_0 = \frac{1}{2} \left(\partial_\mu \phi \right)^2. \tag{2.14}$$

Looking at the iterations equations (2.12) that define the transformation we see that this Lagrangian is left unchanged; we say that it is a *fixed point* of the renormalization group flow.

Continuing in the vicinity of the free-field Lagrangian $\mathcal{L}_0$, we can keep only the linear terms in the transformations and obtain:

$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad C' = C b^d, \quad D' = D b^{2d-6}, \quad \text{etc.} \tag{2.15}$$

We can observe that, since $b < 1$ during the flow–which is an iteration of this transformation–the terms that are multiplied by positive powers of b decay, while those which are multiplied by negative powers grow. Eventually the growing coefficients, if present in the Lagrangian, will carry it away from $\mathcal{L}_0$.

In *renormalized perturbation theory* [3] is conventional to call the operators whose coefficients grow during the flow *relevant*, those whose coefficients die *irrelevant* and those whose coefficient is multiplied by b^0 in the transformation are called *marginal*; to find out if these last ones grow or die during the flow we must include higher-order corrections.

More generally an operator with N powers of the scalar field, M derivatives in d dimensions transforms as

$$C'_{N,M} = b^{N(d/2-1)+M-d} C_{N,M}. \text{[4]} \tag{2.16}$$

Thus we have shown that, at least in the vicinity of the free-field fixed point, a Lagrangian with an arbitrary number of interactions at the scale of the cutoff reduces to a Lagrangian containing only a finite number of renormalizable terms. We can compare this to the interpretation of renormalized perturbation theory and see how this way of seeing things is much more satisfying. In renormalized perturbation theory we see the cutoff Λ as an artifice to be disposed of by taking the limit $\Lambda \to \infty$ as quickly as possible. The theory then gives sensible predictions only if the Lagrangian contains no non-renormalizable parameters. In this interpretation, it can seem just a fortuitous circumstance that the theories which make up the Standard Model, such as QED or QCD, contain no such parameters.

The renormalization group offers a different point of view on this issue. That is that every quantum field theory is fundamentally defined with a UV cutoff Λ that has some physical significance perhaps not already discovered [5]. If we consider field theories associated with solid state systems this is quite obvious: the cutoff is the inverse atomic spacing. However, even if the precise nature of the cutoff in field

---

[4]The reader already knowledgeable in renormalized perturbation theory will recognize $N(d/2 - 1) + M$ as the mass dimension operator, which means that the definitions of relevant, marginal and irrelevant operators correspond precisely to the definitions of super-renormalizable, renormalizable and non-renormalizable interactions.

[5]Obviously, already finite or scale invariant theories are an exception.

theories regarding fundamental interactions still eludes us—be it a consequence of some fundamental graininess of spacetime or else—it makes a lot of sense to consider our theories as effective low energies Lagrangians of some more fundamental theory defined at energies not yet accessible.

However, when we move far from the free fixed point strong field interactions can alter this simple picture beyond the validity of perturbation theory. The corrections containing higher powers of the coupling constants arising from (2.12) can change drastically the flow, generate new fixed points which in turn can create new types of asymptotic behaviours as $\Lambda \to \infty$.

Let us look at how interactions change the renormalization group flow by specializing to the $d = 4$ case. This will provide a reason to introduce the concept of beta function which will simplify the discussion of fixed points and their charaterization.

For the scalar theory two operator are of interest in 4 dimensions: the mass operator–which is relevant in any number of dimensions–and the ϕ^4 interaction–which is marginal.

Starting with the mass operator we can see that close to the fixed point, after n iterations of the linear transformation, the mass parameter becomes $m'^2 = m^2 b^{-2n}$. As $b < 1$ this means that the mass operator gets larger and larger during the flow and eventually becomes comparable to the cutoff. However, we must remember that until now we have discussed the ϕ^4 theory in the limit in which the mass parameter is small compared to the cutoff. To mantain this criterion intact we must impose that the mass parameter $m'^2 \sim \Lambda$ only after a large number of iterations of the transformation. This condition is met every time the initial conditions for the renormalization group flow are adjusted in such a way that the trajectory eventually passes near a fixed point. One can even imagine to construct a complicated nonlinear Lagrangian in $d = 4$ and, as long as the initial value of m^2 is adjusted in such a way that the trajectory comes close to a fixed point, the effective theory at low energy compared to the cutoff would be extremely simple: basically a free field theory with negligible nonlinear interactions. But what if the coupling associated to ϕ^4 does not die down, you may ask. We will show in a moment that it does–but in a more general theory–it may not. It can happen in fact, and will become important further on, that for some theories the renormalization group flow does lead to interacting fixed points. Let's for now conclude our example, we will pause for a moment at the end of it to discuss this point a little further.

In $d = 4$ the operator associated to the ϕ^4 interaction is marginal. For marginal operators the linear transformation doesn't give enough information on whether the interaction gets larger or dies out at large distances; one has to go back to the complete transformations (2.12). The leading contribution to $\Delta\lambda$ is $-\frac{3\lambda^2}{16\pi^2}\log(1/b)$ [3]. The leading contribution to ΔZ is of order λ^2 and gets grouped with the higher orders. We find the transformation

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\log(1/b) + \mathcal{O}(\lambda^2). \tag{2.17}$$

This, as promised, shows that λ slowly dies down when we integrate out high-momentum degrees of freedom. Near the fixed point $\mathcal{L}_0$ the renormalization group flow has the structure shown in Fig. 2.1 with one slowly decaying direction.

With a bit of work we can put the informations of eq.(2.17) in differential form. Let us define $\Lambda' \equiv b\Lambda$. We can think of λ' and λ as functions respectively of Λ' and Λ in the specific sense that λ is the coupling for the effective Lagrangian defined with cutoff Λ whereas λ' is the coupling for the effective Lagrangian defined with
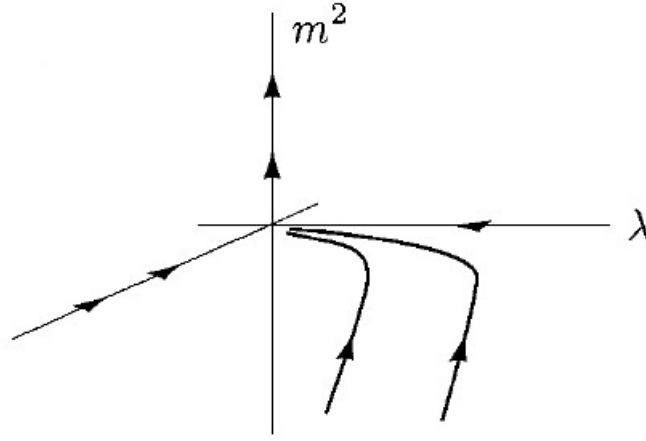
FIGURE 2.1: Renormalization group flow near the free-fixed point in
d=4. The arrows denote the direction of decreasing momentum (IR).

cutoff Λ' obtained after having integrated out high momentum degrees of freedom
through the transformations (2.12). Now (2.17) can be rewritten as follows

$$\lambda'(\Lambda') = \lambda(\Lambda) + \frac{3\lambda(\Lambda)^2}{16\pi^2} \log(\frac{\Lambda'}{\Lambda}) + \mathcal{O}(\lambda(\Lambda)^2), \qquad (2.18)$$

it is then a simple exercises to show that this satisfies the differential flow equation

$$\Lambda\frac{d\lambda}{d\Lambda} = \frac{3\lambda(\Lambda)^2}{16\pi^2} + \mathcal{O}(\lambda(\Lambda)^2). \qquad (2.19)$$

This is an example of a *beta function*. The beta function $\beta(g)$ describes the dependence of a coupling parameter g on an energy scale, Λ, through a relationship of the type

$$\Lambda\frac{\partial g}{\partial \Lambda} = \beta(g). \qquad (2.20)$$

Knowing the full beta function for a particular coupling would provide in principle complete information on the flow of the particular coupling hence give access to the full scale dependence of the theory. In practice we are almost never that lucky and, unless strong symmetries are present in the theory, beta functions are usually calculated order by order through perturbation theory.

Unfortunately the problem of using perturbation theory to compute the orders of a beta function is that this procedure is strictly accurate only close to the free Lagrangian. In principle, fixed points of the beta function which are strongly coupled can exist, but obviously this type of renormalization group flow cannot be understood through the traditional use of Feynman diagrams within perturbation theory. Luckily a lot of quantum field theory that are known to be important for physical applications have been found to contain only free-field fixed points or interacting fixed points that can however be controlled by some limit and be brought to be arbitrarily close to a free-field fixed point. This will be in fact the case for the theories we will work with.

2.1.3   The Callan-Symanzik equation

Wilson's picture, although intuitive in it's approach, is in most practical cases of difficult implementation. Performing the integrations required to obtain Wilson's effective function is, if not impossible, almost always awkward in that the integrals are on finite domains and they involve a parameter that has to cancel in the final results.

We will introduce here an approach to the Renormalization Group that, although more abstract and formal, will prove to be much more systematic and useful in its implementation.

Let's consider a renormalizable theory with a massless scalar field ϕ and one self-coupling g . Let's call the bare field ϕ_0 and the renormalized one ϕ defined at some renormalization scale μ with:

$$\phi = Z^{-1/2}\phi_0. \tag{2.21}$$

where Z is the field strength renormalization. Obviously the renormalization scale μ is arbitrary. It makes no appearance in the bare Green's functions,

$$G_0^{(n)}(x_1, ..., x_n) = \langle \Omega \,|T\phi_0\,(x_1)\,\phi_0\,(x_2)\cdots\phi_0\,(x_n)|\,\Omega \rangle\,. \tag{2.22}$$

The μ dependence enters only when we rescale the fields with Z so that the relation between the bare and renormalized n-point Green's function is:

$$\langle \Omega \,|T\phi\,(x_1)\,\phi\,(x_2)\cdots\phi\,(x_n)|\,\Omega \rangle = Z^{-n/2}\,\langle \Omega \,|T\phi_0\,(x_1)\,\phi_0\,(x_2)\cdots\phi_0\,(x_n)|\,\Omega \rangle \tag{2.23}$$

This n-point function depends on the scale μ and the renormalized coupling g. But we could very well have chosen a different scale μ'; in that case we would have a different rescaling factor Z' and coupling g'.
Which means that

$$Z(\mu)^{-n/2}G^{(n)}(x; \mu, g(\mu)) = Z(\mu')^{-n/2}G^{(n)}(x; \mu', g(\mu')) \tag{2.24}$$

Let's analyze the effect of this shift more explicitly. Suppose we apply a shift of $\delta\mu$. We will have a corresponding coupling constant and field renormalization shift such that the bare Green's function stays the same:

$$\begin{aligned} \mu &\to \mu + \delta\mu \\ g &\to g + \delta g \qquad . \\ \phi &\to (1 + \delta\eta)\phi \end{aligned} \tag{2.25}$$

The shift in the renormalized Green's functions is the product of the ones induced by the the field rescaling,

$$G^{(n)} \to (1 + n\delta\eta)G^{(n)} \tag{2.26}$$

Then writing the differential change in G^n thought as a function of μ and g and setting it equal to the one we found out in the previous equation

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial \mu}\delta\mu + \frac{\partial G^{(n)}}{\partial g}\delta g = n\delta\eta G^{(n)} \tag{2.27}$$

It is conventional to rewrite this differential equation in terms of the dimensionless parameters

$$\beta \equiv \frac{\mu}{\delta\mu}\delta g \quad \& \quad \gamma_\phi \equiv -\frac{\mu}{\delta\mu}\delta\eta, \tag{2.28}$$

This β can be intuitively understood to be equivalent to the beta function previously introduced but we will expand on this point in a little bit. The second parameter γ_ϕ is called *anomalous dimension* of the field ϕ.

After having substituted these definitions in Eq. (2.28) and multiplying by $\mu/\delta\mu$ we can be rewrite it as

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + n\gamma_\phi\right] G^{(n)}(x_1, \cdots, x_n; \mu, g) = 0. \tag{2.29}$$

Looking at their definition we see that the parameters β and μ are the same for every n and independent of the x^i. Moreover, since the G^n are renormalized, β and μ do not depend on the cut-off, and hence, by dimensional analysis, they do not depend on μ. They therefore depend only on g. The relation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + n\gamma_\phi(g)\right] G^{(n)}(\{x_i\}; \mu, g) = 0 \tag{2.30}$$

valid for every massless scalar theory is called Callan-Symanzik equation.

These arguments generalize nicely to more complicated massless theories. In theories with multiple fields and couplings, there is a γ term for each field and a *beta* function for each coupling. For example, the a version of QED with massless electron, provided with adequate renormalization conditions, satisfies the Callan-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + n\gamma_2(e) + m\gamma_3(e)\right] G^{(n,m)}(\{x_i\}; \mu, e) = 0 \tag{2.31}$$

where n and m are the number of electron and photon fields in the Green's function $G^{(n,m)}$ and γ_2 and γ_3 are the rescaling functions of the electron and photon fields.

Let us spend some words on the meaning of γ and β. Always referring to the massless scalar theory, we can find a more useful way of expressing them in terms of the parameters of bare perturbation theory: Z, g_0 and Λ.

Recalling that Z is a function of μ and using the relationship between bare and renormalized fields expressed in Eq. (2.21), we can express the shift in the renormalized field when μ is shifted by $\delta\mu$ as

$$\delta\eta = \frac{Z(\mu + \delta\mu)^{-1/2}}{Z(\mu)^{-1/2}} - 1. \tag{2.32}$$

Hence, using our original definition of γ, Eq. (2.27), we can immediately write

$$\gamma_\phi(g) = -\frac{1}{2}\frac{\mu}{Z}\frac{\partial}{\partial\mu}Z. \tag{2.33}$$

This expression clarifies the relation between γ and the field strength rescaling.

We can find a similarly useful expression for β again using the parameters of bare perturbation theory. In our definition of *beta* we used the quantity δg which is the shift in the renormalized coupling g necessary to keep the bare Green's functions constant after the renormalization scale μ is shifted infinitesimally. So this definition can be rewritten as

$$\beta(g) = \mu\frac{\partial}{\partial\mu}\,g\big|_{g_0,\Lambda} \tag{2.34}$$

calculated at g_0, Λ since the bare Green's functions depend on the bare variables. As

before, if there is more than one coupling there will be a β_{g_i} for each coupling and each of this can in general depend on each of the other couplings.

In both these two formulas we just proved, the independence from the cut-off Λ which was apparent in their original form, is somewhat hidden. To understand this fact we have to go back to their definitions in terms of the renormalized Green's functions, whose cut-off independence follows from the renormalizability of the theory.

Focusing again on Eq. (2.34) we see that, as promised, this is just the beta function we introduced in chapter 2.1.2. The difference is only in the substitution μ in Eq. (2.34) with Λ. In one formula the derivative is taken with respect to the cutoff, in the other with respect to the renormalization scale at which we want to calculate the physical quantities. But this is little more than a matter of interpretation. There is no reason why we cannot take the physical cutoff $\Lambda = \mu$. In fact this is an optimal choice since it involves in the effective theory only modes with energies $\leq \mu$ which are the ones involved in the physical process. Furthermore, as it should be Eq. (2.19) can be reproduced using the Callan Symanzik equation for the ϕ^4 theory [3].

2.1.4   Running of couplings and asymptotic safety

Let us discuss now in some generality the type of behaviours and fixed points that can occur in beta functions of the type we will encounter. We will then give a brief introduction to the phenomenon of asymptotic safety and to its relevance in BSM (Beyond Standard Model) models.

One of the chief interests when looking at the renormalization group flow of a theory is its asymptotic behaviour. As we emphasized before, a theory can have an ill-defined behaviour when looking at high momenta particles in loop calculations—this very fact motivated the introduction of a UV cut off in the first place—or could have IR divergences too, which are usually a less serious problem since experiments take place in limited regions of spacetime. Now that we have built a systematic way to approach this problems–the renormalization group flow–we are still interested in taking the UV and IR limits of our theory. Roughly speaking, we would like the particular trajectory our theory lives on to have finite limits in both directions.

Having this issue in mind, we can well understand how important the fixed points of the renormalization group flow are. Having an UV (IR) fixed point i.e. a collection of g_i^* for which, in the limit of $\mu \to \infty$ ($\mu \to 0$), all of the beta function vanish, ensures us that the theory is complete in that direction.

Looking at the beta functions at the fixed point

$$\mu \left. \frac{dg_i}{d\mu} \right|_{g_j} = 0 \tag{2.35}$$

we can see that a theory arbitrarily close to it becomes scale invariant. It happens usually that to scale invariance corresponds also conformal-invariance so that the theories at fixed points theories are also *conformal field theories* (CFTs)[6]. We will discuss the scaling close to fixed points more thoroughly in chapter 2.1.5.

Let's go back for now to fixed points. The first, and historically more important, type of fixed point we want to introduce is the one that gives rise to the phenomenon of *asymptotic freedom*. A complete treatment of asymptotic freedom is beyond the

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[6]It is possible, although rare, for a theory at the fixed point to be scale invariant without being conformally-invariant [5][6]. Usually in QFT the two terms are used almost interchangeably.

scope of the present text but, as we will use often the term in the following, it seems only right that we introduce it.

We have already seen how free fixed points can arise when not only $\beta(g) = 0$ but g itself is equal to zero e.g the ϕ^4 theory in d=4. In the case of the scalar theory the first coefficient in the perturbative expansion of the beta function was positive (see Eq. (2.19)). This means that the beta function close to $g = 0$ is positive and approaches 0 when $\mu \to 0$ (see Fig. 2.2). This type of fixed point is called an IR free fixed point.
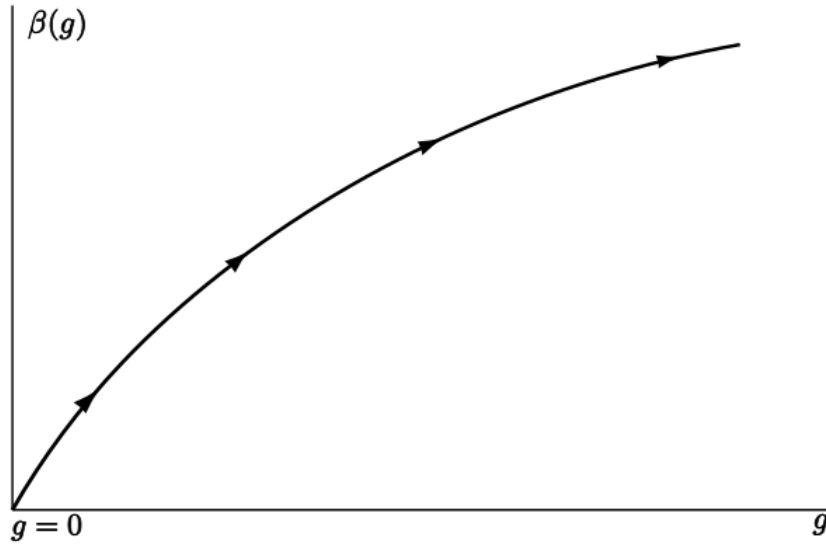

FIGURE 2.2: IR free fixed point. The arrows denote the direction of increasing momentum (UV).

It reproduces the normal physical intuition that a force should vanish asymptotically when the interaction distance gets infinite. This is for example the low energy behaviour of QED in $d = 4$ since to one-loop its beta function is

$$\beta(e) = \frac{e^3}{12\pi^2} \tag{2.36}$$

or often, in terms of the fine structure constant $\alpha = e^2/4\pi$

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi}. \tag{2.37}$$

However, in the late 60s experimentalists studying the deep inelastic scattering of electrons on protons were puzzled in finding out that building blocks of nucleons followed a different behaviour. Their data indicated that the quarks inside the protons, when hit by an highly energetic electron, were propagating freely without interacting with the other quarks. This behaviour, as the three quarks composing the protons are otherwise strongly bound, implies that the force holding them together, instead of getting stronger with a decreasing distance of interaction as in QED, someway gets instead asymptotically weak at smaller distances. This implies a negative beta function to lowest non trivial order, however, no theory known at the time exhibited such feature.

This puzzle was solved in 1973 by David Gross and Frank Wilczek, and independently by David Politzer in the same year, when they discovered that a large

family of gauge theories were capable of reproduce this asymptotic freedom behaviour[7]. The main feature of this gauge theories is that, in contrast to QED, their gauge group is not abelian. The discovery of asymptotic freedom in QCD i.e. a non-abelian gauge theory with gauge group $SU(3)$ garnered the three theorists the Nobel Prize in Physics in 2004.

To lowest nontrivial order in perturbation theory, the beta function of a theory with gauge group $SU(N)$ and n_f number of interacting fermions in the fundamental (or anti-fundamental) representation is

$$\beta(g)_{1\text{loop}} = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right).$$ (2.38)

If this function is negative than the theory is asymptotically free i.e. when $\mu \to \infty$ the coupling $g \to 0$, as in Fig. 2.3.
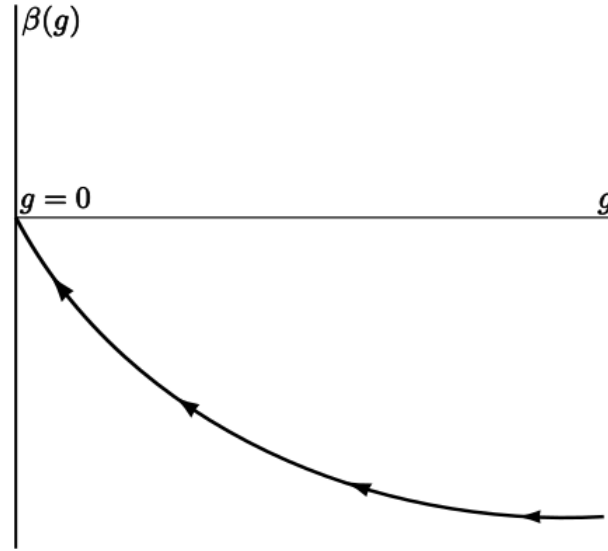

FIGURE 2.3: Beta function of an asymptotically free theory with one coupling near $g = 0$.

For the case of $SU(3)$ (QCD), to have this function negative, the number of flavours of quarks must respect $n_f \leq \frac{33}{2}$, so that one could accommodate up to 16 triplets in this theory.[8]

A part from the experimental evidence of deep inelastic scattering of electrons on protons there is a more obvious reason why asymptotic freedom is a nice feature to have in a QFT. In both QED and the ϕ^4 theory the renormalized coupling grows with the energy. The formula for the renormalized (in this case is useful to think of it as observable) electric charge is [3]

$$e_R^2 = \frac{e_0^2}{1 + (e_0^2/12\pi^2)\ln\Lambda^2/m^2}.$$ (2.39)

---

[7]A big stepping stone in this direction was the formulation in 1954 of the first non-abelian gauge theory by Chen Ning Yang and Robert Mills to try and explain strong interactions.

[8]For $n_f = 0$ we obtain a pure Yang-Mills theory and asymptotic freedom is still present. We can see from this that asymptotic freedom is really a property of the Yang-Mills part of the theory.

If we invert it for e_0^2 we obtain

$$e_0^2 = \frac{e_R^2}{1 - (e_R^2/12\pi^2)\ln \Lambda^2/m^2} \tag{2.40}$$

from which we can see that if we keep the observable charge fixed and increase the energy scale Λ we eventually hit a pole at the finite energy of

$$\Lambda^2 = m^2 \exp\left(12\pi/e_R^2\right) \tag{2.41}$$

and e_0 diverges. This type of divergence at finite energy is called a Landau pole. This particular evidence of a Landau pole is easily disproved since Eq. (2.39) is obtained under the hypothesis of the validity of perturbation theory and the growth of g_0 invalidates perturbation theory altogether. Furthermore, if one computes the scale at which the Landau should appear we find out that it is approximately $\Lambda = 10^{286}$eV so much larger than the Plank scale 10^{28}eV—the scale at which quantum gravity should become important—and makes questionable the use of a quantum field theory altogether.[9]

One could anyway wonder if there are ways for theories that are free in the IR to obtain a UV completion that avoids the issue of Landau poles altogether. One way to do this is the approach originally suggested by Weinberg [8] in 1976 to avoid the UV divergences arising when we try to quantize in the standard perturbative way the Einstein-Hilbert action for classic general relativity.[10] The idea of asymptotic safety is that a non-trivial UV fixed point can be used to limit unphysical divergences as in the UV all the couplings flow to their asymptotic values, see 2.4.
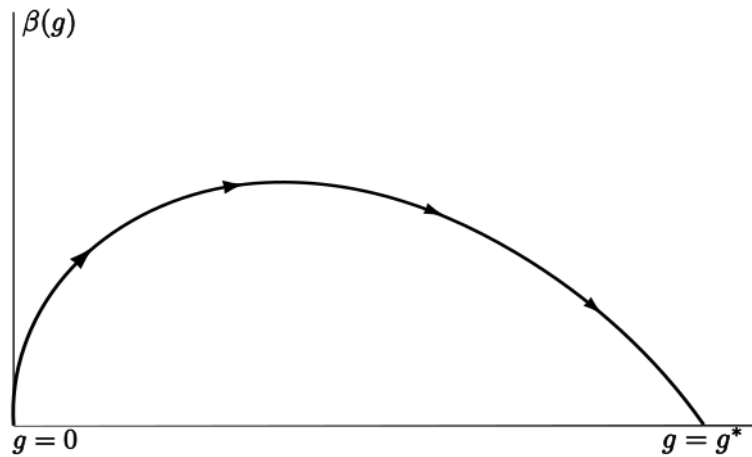

FIGURE 2.4: An example of a theory with one coupling which is free in the IR and asymptotically safe in the UV. The coupling flows to g^* for $\mu \to \infty$.

An asymptotically safe theory can be well defined at all scales even while being perturbatively non-renormalizable.

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[9]Lattice QED calculations suggest that a Landau pole is indeed present but in phase space region not accessible because of spontaneous chiral symmetry breaking [7].

[10]In 2 dimensions Newton's constant G is dimensionless and gravity can be perturbatively renormalized. In 4 dimensions the mass dimension of G is -2 and thus is perturbatively non-renormalizable.

Actually, fixed points like the asymptotic safe one can tell us much more. From the Wilson point of view one can consider the basic input of their model i.e. the quantum fields of the theory and the symmetries they respect. These two inputs determine the theory space where the renormalization group flow occurs. Each point in this space is one possible action composed by linear combinations of the monomials of fields selected and respecting the symmetry principles. The coefficients of these linear combinations are the coupling constants g_i.

The set of points (theories) which are pulled towards the fixed point under the renormalization group flow in the $\mu \to \infty$ direction is referred to as UV critical surface. The hypothesis underling the asymptotic safety program is that a trajectory can be realized in nature only if it's contained in the UV critical surface and thus has a well-behaved high-energy limit (black trajectories in Fig. 2.5). Trajectories outside this surface are unacceptable since they develop divergences in the UV (red trajectory in fig. 2.5).
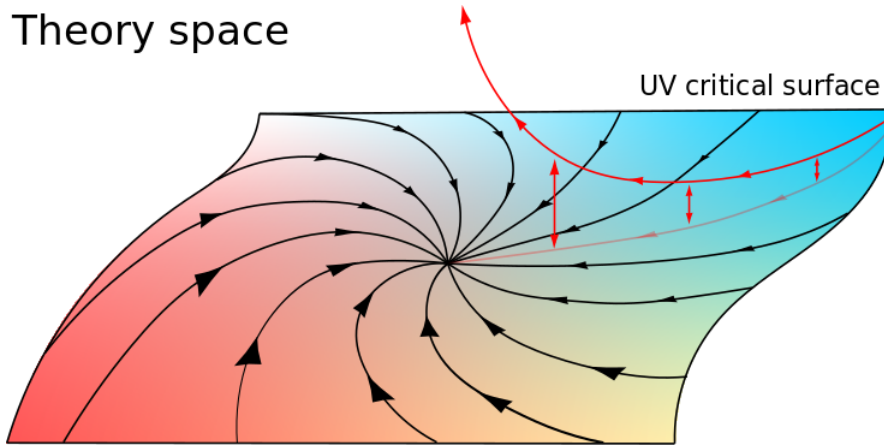

FIGURE 2.5

Let's assume that a given UV fixed point has n attractive directions, then the UV critical surface will be of dimension n. One way to go and analyze it can be to linearize the couplings near the fixed point and studying the directions where the couplings flow towards the UV fixed point (*relevant directions*) and the ones where the couplings flow away from the fixed point (*irrelevant directions*). The irrelevant directions will be infinite in number corresponding to all the irrelevant couplings that can be added to the Lagrangian. The relevant directions will pin down the couplings of the theory univocally.

Or at least this is true at high energies. Unfortunately when one moves towards lower energies the critical surface deviates from its tangent space, nonlinear effects kick in and everything becomes much more complicated.

Obviously one has to somehow bypass the problem of not having always access to perturbation theory for this kind of calculations. Even if the tools at our disposal outside perturbation theory—such as Functional Renormalization and lattice gauge theory—are limited and have their own drawbacks, the research in these fields is still flourishing.

The program for an asymptotic safe gravity is unfortunately still far from being completely successful in its original aim but there have been many advancement in asymptotic safety since the idea was originally proposed. For example it has been shown that the non-linear sigma model [9] and a variant of the Gross-Neveu model [10] exhibit non-trivial UV fixed points and are renormalizable even if they

are perturbatively non-renormalizable. This confirms that asymptotic safety as was originally proposed is indeed possible.

Applications of functional renormalization methods suggest that the existence of a highly predictive asymptotically safe gravity-Standard-Model-fixed point is indeed possible. Lastly, another approach used has been to try and analyze non-trivial fixed points in the limit of small coupling so that perturbation theory is still accessible. With this approach, based on limits such as the large-N Veneziano limit, it has been shown that 4 dimensional QFTs involving gauge fields, fermions and scalars exhibiting asymptotic safety could be built [11].

2.1.5 The scaling close to critical points

Let us now study the behaviour of the beta function close to fixed points.

To this aim it is useful first to introduce the dimensionless couplings defined as

$$\alpha_i = \mu^{-d_i} g_i \tag{2.42}$$

where d_i is the canonical energy dimension of the coupling g_i.

From dimensional analysis, $\beta(g_j, k) = k^{d_i} \beta_i^Q(\alpha_j)$ where $\beta_i^Q(\alpha_j) = \beta_i(\alpha_j, 1)$. The beta functions of the dimensionless variables will then be given by

$$\tilde{\beta}_i(\alpha_j) = \mu \frac{d\alpha_i}{d\mu} = -d_i \alpha_i + \beta_i^Q(\alpha_j). \tag{2.43}$$

They depend on k only implicitly through α_j. The Q in β_i^Q stands for quantum. We can in fact think about the first term as the classical scaling and the second as the non-trivial quantum fluctuation integrated along the RG flow. Indeed in the classical limit the second term would vanish.

Now, suppose α_i^* is a critical point of the flow. We can find the tangent space to the UV critical surface at the fixed point by linearizing the flow. Let's write the coupling near the fixed point as $\alpha_i = \alpha_i^* + \delta\alpha_i$, then the linearized RG equations are:

$$\mu \left. \frac{d\alpha_i}{d\mu} \right|_{\alpha_j^* + \delta\alpha_j} = A_{ij}\delta\alpha_j + \mathcal{O}\left(\delta\alpha_j^2\right) \tag{2.44}$$

Finding a diagonal basis $\{y_i\}$ for $\{\alpha_i\}$ we can write

$$\mu \frac{dy_i}{d\mu} = (\Delta_i - d) y_i + \mathcal{O}\left(y^2\right) \tag{2.45}$$

The quantity Δ_i is called the *scaling* (or *conformal*) *dimension* of the operator associated to y_i. In a general interacting QFT, it will not be given by the classical scaling dimension of the operator and the difference $\gamma_i = \Delta_i - d_i$ is known as the *anomalous dimension* of the operator. Then to linear order the RG flow is

$$y_i(\mu) = \left(\frac{\mu}{\mu'}\right)^{\Delta_i - d} y_i(\mu'). \tag{2.46}$$

Let's now see how the correlation functions scale in the vicinity of the fixed point. As an example, let's consider the 2-point correlation function $G^{(2)}(x) = \langle \phi(x)\phi(0)\rangle$.

This satisfies the RG equation (2.24)

$$Z(\mu)^{-1} G^{(2)}\left(x; \mu, g_i(\mu)\right) = Z\left(\mu'\right)^{-1} G^{(2)}\left(x; \mu', g_i\left(\mu'\right)\right) \tag{2.47}$$

At the fixed point $g_i(\mu) = g_i\left(\mu'\right) = g_i^*$. Moreover, since $\gamma_\phi(g_i^*)$ is a constant we can solve Eq. (2.33) to obtain

$$Z(\mu) = \left(\mu'/\mu\right)^{2\gamma_\phi^*} Z\left(\mu'\right) \tag{2.48}$$

. Then using dimensional analysis we can isolate a dimensional part which takes dimension from the field ϕ and an adimensional part $\mathcal{G}$ likewise

$$G^{(2)}\left(x; \mu, g_i^*\right) = \mu^{2d_\phi} \mathcal{G}(x\mu). \tag{2.49}$$

Where $\mathcal{G}$ can depend only on the adimensional combination $x\mu$ and d_ϕ is the classical dimension of ϕ. Substituting back into Eq.(2.47) allows us to write

$$\frac{\mathcal{G}(x\mu')}{\mathcal{G}(x\mu)} = (\frac{\mu}{\mu'})^{2d_\phi + 2\gamma_\phi^*} \tag{2.50}$$

which means that up to a constant c

$$\mathcal{G}(x\mu) = c(x\mu)^{d_\phi + 2\gamma_\phi^*}. \tag{2.51}$$

We can then write

$$G^{(2)}\left(x; \mu, g_i^*\right) = \frac{c}{\mu^{2\gamma_\phi^*} x^{2d_\phi + 2\gamma_\phi^*}} \propto \frac{1}{x^{2\Delta_\phi}} \tag{2.52}$$

which displays the typical power-law behaviour of correlation functions in a CFT.

2.2   QCD and SQCD

Let's discuss now in some more details the properties of the functions of QCD and SQCD.

2.2.1   The beta function of QCD

In chapter 2.1.4 we have written the perturbative beta function of a non-abelian gauge theory with gauge group $SU(N)$, and N_f fermions in the fundamental N-dimensional representation of $SU(N)$ at first loop order as

$$\mu \frac{dg}{d\mu} = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right). \tag{2.53}$$

Let's write it as $\mu dg/d\mu = -\beta_1 g^3/(4\pi)^2$ with

$$\beta_1 = \frac{11}{3}N - \frac{2}{3}N_f. \tag{2.54}$$

We had previously noted the remarkable fact that, since the contribution in the one loop coefficient from the gauge part and from the matter field have opposite signs, we could change the sign of β_1—hence turn on/off asymptotic freedom by changing the matter content and/or the dimension of the gauge group of the theory.

Before discussing the theory beyond one-loop let's shortly expand on the previous point.

If we solve (2.53) we obtain

$$g^2(\mu) = \frac{8\pi^2}{C + \beta_1 \log \mu} = \frac{8\pi^2}{\beta_1 \log(\mu/\Lambda)} \tag{2.55}$$

where the integration constant Λ is a parameter with unit mass dimension. When the matter content is sufficient so that $\beta_1 < 0$ we will have the same behaviour of QED. There will be an IR fixed point at $g = 0$, while for $\mu \to \Lambda$ there is a Landau pole and the coupling diverges.

Instead, if $\beta_1 > 0$ (as in QCD with $N_f = 3$) we have a UV fixed point for $g = 0$ and asymptotic freedom is realized. In this case, Λ signals the scale at which the coupling becomes large and perturbation theory breaks down. This is the scale at which, in QCD, the phenomenon of confinement sets in.[11] In a theory like QCD this scale can be actually measured.[12]

Let's go now beyond the one-loop order. We can write the general form for the QCD beta function at 2 loop as

$$\mu \frac{dg}{d\mu} = -\frac{\beta_1}{(4\pi)^2} g^3 - \frac{\beta_2}{(4\pi)^4} g^5 \tag{2.56}$$

Then the second coefficient for an SU(N) gauge theory with N_f flavours of quarks in the N representation can be written as

$$\beta_2 = \frac{34N^2}{3} - \frac{10NN_f}{3} - \frac{N_f \left(N^2 - 1\right)}{N}. \tag{2.57}$$

and β_1 is the same as before.

One very important property of these two coefficients is that differently from the subsequent ones they are universal i.e. they do not depend on the renormalization scheme used. Because of this universality the properties we are going discuss do not depend on the renormalization scheme employed and are physical features of the theory.

In particular, we observe that there are three behaviours when we vary N_f for a fixed N.

The first two are completely analogous to the one discussed before. When

$$N_f < \frac{34N^3}{13N^2 - 3} \tag{2.58}$$

the theory is asymptotically free in the UV and the coupling runs to large values in the IR. This theory confines. If instead

$$N_f > \frac{11N}{2} \tag{2.59}$$

---

[11]Confinement is the phenomenon for which two color charges cannot be observed isolated. It doesn't exist yet a proof of color confinement from first principles even in the simplest cases but it can be intuitively understood as follows. When two color charges get separated the interaction between them can be depicted as a narrow flux tube of gluons. The energy density of the flux tube and its radius are constant regardless of separation, so the force associated will be weak at low distances and strong at large distances. At some point as the two charges get separated it becomes energetically favorable for a pair of quark anti-quark to appear rather than extend the tube even further. The scale at which this happens is called Λ_{QCD} [3], which is the only scale present in the theory. It is important to notice also that this scale is generated in the UV, as it is clear from the derivation, so it is present in the theory at every energy.

[12]Or more precisely, since it is renormalization scheme dependent, its value in a particular renormalization scheme can be deduced from experiments.

the theory is free in the IR and in the UV there will be a perturbative Landau pole.

The third case is more interesting. Whenever

$$\frac{34N^3}{13N^2 - 3} < N_f < \frac{11N}{2} \tag{2.60}$$

the theory is still asymptotically free in the UV but now there seems to appear a new IR interacting fixed point. This is exactly what we had in the 1-loop case as this bound depends only by the behaviour of β_1. This fixed point is called Banks-Zaks fixed point [12][13]. The region of N_f where this new fixed point actually exists is called the *conformal window*.

Now the question is, given the limit of applicability of perturbation theory, is the new fixed point really present? And if so, in what region? The fixed point value obtained solving $\beta(g) = 0$, with $\beta(g)$ given by Eq. (2.56) is

$$g_*^2 = (4\pi)^2 \frac{11N^2 - 2NN_f}{34N^3 - 13NN_f + 3N_f} \tag{2.61}$$

If we look at the top of the conformal window for QCD i.e. $N = 3$ and $N_f = 16.5$, then close to the upper edge, e.g. $N_f = 16$ one has $g_*^2 \sim 0.52$. For larger N and still close to $N_f = 11N/2$, $g_*^2(N)$ scales like N^{-1} and the Banks Saks fixed point can be brought arbitrarily close to the free fixed point in the Veneziano limit ($N \to \infty$, $N_f \to \infty$ $N_f/N = const$). On the contrary, near the lower edge of the conformal window, let's say again for QCD (which perturbatively at the second order has the lower edge located at $N_f = 8.05$) then $g_*^2 \sim 11.96$ and perturbation theory is clearly not applicable. The exact location of the bottom of the conformal window even in the large-N limit is still an open question but new lattice QCD results have determined it to be bound between $6 < N_f^c < 8$ [14].

2.2.2   The beta function of SQCD

The presence of the additional symmetry between fermions and bosons makes QFTs with supersymmetry particularly interesting to analyze from the point of view of the renormalization group.

An example particularly pertinent for our interests is the one of supersymmetric Yang-Mills theories in $d = 4$. For $\mathcal{N} = 1, 2, 4$[13] it has been known for a while that a beta function, called NSVZ (Novikov-Shifman-Vainshtein-Zakharov) beta function, which links the coupling to the anomalous dimension of the matter fields, can be derived exactly to all orders [15] [16]. For $\mathcal{N} = 1$ this result can be used to show the presence of a conformal window in the framework of Seiberg duality [17].

The NSVZ beta function for an SQCD theory with dimension of the gauge group N_c and N_f flavour of fermions is

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{3N_c - N_f + N_f \gamma_m(g)}{1 - N_c g^2/(8\pi^2)} \tag{2.62}$$

with

$$\gamma_m(g) = -\frac{g^2}{8\pi^2} \frac{N_c^2 - 1}{N_c} + O\left(g^4\right) \tag{2.63}$$

---

[13]$\mathcal{N}$ refers to supersymmetric theories where the generators of the supersymmetry Q_i^α carry not only a spinor index α but also an integer index $i = 1, 2 \ldots \mathcal{N}$. It is also called extended supersymmetry.

being the anomalous mass dimension in perturbation theory.

The observation by Seiberg about the existence a dual description in terms of magnetic variables of this theory (vs. the fundamental theory which is described in terms of electric variables), can be used to determine the edges of the conformal window[14] yielding:

$$3N_c/2 < N_f < 3N_c. \tag{2.64}$$

In this range there exist both the electric and the magnetic description. The theory is asymptotically free in the UV and an additional interacting fixed point is present in the IR.

An additional useful property is that in SQCD the R symmetry[15] and the exact beta function can be used to determine exactly the mass anomalous dimension $\gamma_m^*(N_f)$ along the IR fixed points line $g_*(N_f)$ in the conformal window. The relationship between the scale dimension $D_{\tilde{Q}Q}$ and the R-charge $R_{\tilde{Q}Q}$ for the meson operator $M = \tilde{Q}Q$, is [17]

$$D_{\tilde{Q}Q} = \frac{3}{2}R_{\tilde{Q}Q} = 3R = 3\frac{N_f - N_c}{N_f} \tag{2.65}$$

then using $D_{\tilde{Q}Q} = 2 + \gamma_m^*$ we can obtain the following expression for γ_m^*

$$\gamma_m^*\left(N_f\right) = 1 - \frac{3N_c}{N_f}. \tag{2.66}$$

By substituting this into the NSVZ beta function we can indeed check that it vanishes, provided the pole in the denominator of Eq. (2.62) is not hit, that is for $Ng_*^2/\left(8\pi^2\right) < 1$.

---

[14]In Seiberg's analysis the lower edge is determined with a physical condition i.e. the saturation of the unitarity bound. Such condition is independent of the beta function.

[15]The R symmetry is a symmetry transforming different supercharges into each others in a theory with supersymmetry [18].

Chapter 3

a-theorem

3.1 Introduction

In our explanation of the renormalization group we had briefly touched upon the fact that this type of fluxes do not form a group in the mathematical sense because they are generally not invertible. This is a commonly accepted phenomenon because from an intuitive point of view, an high-energy theory contains more degrees of freedom than a low energy theory and so whenever one flows from the UV to the IR information is lost; this is one way to interpret the lesson of Wilson's approach.

A proof of this however was, till not so long ago, available only for QFTs in 2 dimensions. This theorem, known as the c-theorem, was proved by Alexander Zamolodchikov in 1986 [19]. The usefulness of the theorem however goes far beyond the intuition. It establishes the existence of a monotonically decreasing function interpolating between the central charges of the CFTs at the UV and the IR, and that this function is stationary only at the fixed points. This can be used to prove constraints on RG-flows in a fundamentally non-perturbative way. It also provides an effective measure of the degrees of freedom of the theory and it shows that they indeed decrease when we integrate out high-energy modes.

The naive generalization of Zamolodchikov's theorem to four dimensions however doesn't work and even if a proposal for an equivalent quantity to the c-function was know as far back as 1988 [20], only recently (2011) a non-perturbative proof that this quantity has in fact the required properties has garnered acceptance [21][22]. This proof, by Zohar Komargodski and Adam Schwimmer, establishes what is known as the a-theorem.

In the next chapters we will discuss Zamolodchikov's c-theorem. After, we will introduce the a-theorem and discuss the a-function of SQCD.

3.2 The c-theorem

The c-theorem is a theorem valid for 2 spacetime dimensions which establishes the existence of a function $c(g_i, \mu) \geqslant 0$ such that

- $\mu \frac{d}{d\mu} c = \beta^i(g) \frac{\partial}{\partial g^i} c(g) \geqslant 0$ and it is stationary only at the fixed points where $\beta^i(g*) = 0$.

- At the fixed points the theory is conformal [23] and the value of the c-function equals the value of the central charge $\tilde{c}$ associated to the CFT at the fixed point $c(g^*) = \tilde{c}(g^*)$.

Let's analyze in more detail the structure and properties of the c-function. Let's rewrite the local symmetric energy momentum tensor in complex coordinates $(z, z) = (x^1 + ix^2, x^1 - ix^2)$ and define $T = T_{zz}$. Let's also define $\Phi_i(g, x) = \frac{\partial}{\partial g^i} \mathcal{L}(g, x, \mu)$.

Than the c-function is defined as the combination

$$c(g) = C(g) + 4\beta^i H_i(g) - 6\beta^i \beta^j G_{ij}(g) \tag{3.1}$$

where

$$
\begin{aligned}
C(g) &= 2z^4 \left\langle T(x)T(0)\right\rangle\big|_{x^2=x_0^2} \\
H_i(g) &= z^2 x^2 \left\langle T(x)\Phi_i(0)\right\rangle\big|_{x^2=x_0^2} \\
G_{ij}(g) &= x^4 \left\langle \Phi_i(x)\Phi_j(0)\right\rangle\big|_{x^2=x_0^2}
\end{aligned}
\tag{3.2}
$$

and x_0^2 is an arbitrary scale above such that $x_0^2 \gg \mu^{-1}$.

The central charge is an important characteristic of a conformal field theory. In the case of 2 dimensions the generators of the conformal symmetry L_n, $n = 0, +1, \pm 2, \dots$ form a Visaroro algebra

$$[L_{n'}, L_m] = (n - m)L_{n+m} + \frac{\tilde{c}}{12}\left(n^3 - n\right)\delta_{n+m,0}. \tag{3.3}$$

It can be shown that the central charge equals the number of degrees of freedom of the theory. So for example $\tilde{c} = 1$ for a single free boson and $\tilde{c} = 1/2$ for a single free fermion. This observation connects the c-theorem to the interpretation that says that along the renormalization group flow the degrees of freedom of the theory decrease.

In $d = 2$ the central charge can be also connected to the value of the trace anomaly i.e the nonvanishing trace of the energy momentum tensor; for a conformal field theory on a curved background [24]:

$$\left\langle T^\mu_\mu \right\rangle = -\frac{\tilde{c}}{12} R \tag{3.4}$$

This observation gives a direction for a possible generalization of the c-theorem in $d \neq 2$. Cardy [20] in 1988 suggested to utilize the generalization of this quantity in even d dimensions (there is no anomaly in odd dimensions) as

$$a \sim \int_{S^d} \left\langle T^\mu_\mu \right\rangle \tag{3.5}$$

The general form of the external trace anomaly contains multiple local terms formed from the metric but restricting to a space of constant curvature we can reduce it to a single number.

For example in $d = 4$

$$T^\mu_\mu = aE_4 - cW^2_{\mu\nu\rho\sigma} \tag{3.6}$$

written in terms of the Euler density E_4 and the Weyl tensor $W_{\mu\nu\rho\sigma}$[1]. The integral (3.5) performed for a theory at a fixed point isolates the a anomaly. Cardy's conjecture then was that

$$a_{IR} < a_{UV}. \tag{3.7}$$

3.3 The a-theorem

3.3.1 The statement of the theorem

Although Cardy's conjecture was tested in a number of cases a proof for the a-theorem in $d = 4$ had to wait until 2011.

---

[1]In general there are additional terms but these do not contribute in the proof as they vanish at the integration boundaries.

Komargodski and Schwimmer proved a strong version of the a-theorem in $d = 4$. They showed that Eq. (3.7) holds for all unitary RG-flows and provided also with an expression for the interpolating a function [21]. The proof of the theorem makes use of a dilaton spectator field with decay constant f in order to reinterpret every RG-flow as the result of spontaneously broken conformal symmetry. This dilaton field is weakly interacting through powers of $1/f$ with the matter field in the UV and eventually decouples completely in the IR.

The expression for the difference between the anomalies in the UV and the IR is given by

$$a_{UV} - a_{IR} = \frac{f^4}{\pi} \int_{s' > 0} ds' \frac{\sigma(s')}{s'^2} \tag{3.8}$$

where $\sigma(s)$ is the cross section for the scattering of two dilatons (always positive definite). Since $\sigma(s)$ goes as $1/f^4$ if the mass parameters respect $M_i \ll f$ the expression (3.3.1) is finite for every value of f.

A natural interpolating function $a(\mu)$ that is monotonic can then be obtained by cutting off (3.3.1) at some intermediate energy μ

$$a(\mu) \equiv a_{UV} - \frac{f^4}{\pi} \int_{s' > \mu} ds' \frac{\sigma(s')}{s'^2}. \tag{3.9}$$

This function decreases monotonically from the UV to the IR and it is stationary at the fixed points analogously to the c function for 2 dimensions.

From an operative point of view calculating these quantities in a realistic theory is often prohibitive. Just comparing the expressions for the c function and the a function we see that the first one which contains information from the 2-point correlation function of the trace of the energy momentum tensor $\langle T_\mu^\mu T_\mu^\mu \rangle$ is less complicated than the second one, which contains information from the 4-point correlator $\langle T_\mu^\mu T_\mu^\mu T_\mu^\mu T_\mu^\mu \rangle$. Usually then, exact expressions for the a function are known only in the presence of enough constraining symmetries such as in the case of SQCD.

3.3.2 The a-function of SQCD

In SQCD the additional symmetries present can be used to connect the value of the a anomaly to the $U(1)_R F^2, U(1)_R$ and $U(1)_R^3$ chiral anomalies by using the formalism of the anomaly free R current [25]. The expression for the Euler anomaly at the IR fixed point in terms of the R charge is [25]

$$a_{IR} = \frac{3}{32} \left(2 \left(N^2 - 1 \right) + 2 N_f N (1 - R) \left(1 - 3(1 - R)^2 \right) \right). \tag{3.10}$$

We can rewrite this in terms of the anomalous dimension of the matter fields through $R = \left(2 + \gamma_m^* \right) / 3$. Now similarly to Eq.(3.9) we let $\gamma_m^* \to \gamma_m(g(\mu))$ in order to obtain the natural interpolating function $a(g(\mu))$ with expression

$$a(g(\mu)) = \frac{3}{16}(N^2 - 1) + \frac{1}{16} N N_f \left(\frac{2 - 3\gamma_m^2 - \gamma_m^3}{3} \right). \tag{3.11}$$

With this information in our possession we can now proceed to infer some constraints to the RG-flow of SQCD.

Chapter 4

Applications to SQCD and the large-N Veneziano limit of QCD

4.1 Constraints on the RG-flow of SQCD

In this chapter we will show how to use the NSVZ beta function, the exact expression of the interpolating a function and the a-theorem to infer non-perturbative constraints to the RG-flow of the beta function of SQCD and the anomalous dimension of the matter fields. This part is adapted from [26].

The a-theorem says that away from the fixed points $a(g(\mu))$ satisfies

$$\frac{da}{d\log \mu} = \frac{\partial a}{\partial g}\beta(g) > 0 \tag{4.1}$$

So away from the fixed points $\frac{\partial a}{\partial g}$ has the same sign of $\beta(g)$. Furthermore, using the expression (3.11) for $a(g(\mu))$ in SQCD we can derive

$$\frac{\partial a}{\partial g} = -\frac{NN_f}{16}\gamma_m (2 - \gamma_m) \frac{\partial \gamma_m}{\partial g} \neq 0 \tag{4.2}$$

away from the fixed points. This shows that γ_m is monotonic away from fixed points. Furthermore, we can see that for $\gamma_m < 0$ and $\gamma_m > 2$ the derivative $\frac{\partial \gamma_m}{\partial g}$ has the same sign of $\frac{\partial a}{\partial g}$ (hence of $\beta(g)$), while in the interval $0 < \gamma_m < 2$ it has the opposite. Now we will see how to use this information to derive constraints to the RG-flow of the beta function.

The first result we will prove shows that if there is an interacting fixed point (UV or IR) at some coupling $g^* \neq 0$ there cannot be another interacting fixed point at some higher coupling. Let's first rewrite the NSVZ beta function from Eq. (2.62) in terms of the coupling $\alpha \equiv Ng^2$ and $\beta(\alpha) \equiv 2Ng\beta(g)$ and with the decomposition

$$
\begin{aligned}
\beta(\alpha) &= f(\alpha)h(\alpha) \\
f(\alpha) &= \frac{\alpha^2}{8\pi^2 \left(1 - \frac{\alpha}{8\pi^2}\right)} \\
h(\alpha) &= -3N + N_f - N_f\gamma_m(\alpha)
\end{aligned}
\tag{4.3}
$$

Let's say we want to rule out with the a-theorem the case of a two fixed interacting fixed points such that $0 < \alpha_{IR} < \alpha_{UV}$ as in figure 4.1.

We can see from the decomposition (4.3) that if the beta function is continuous,
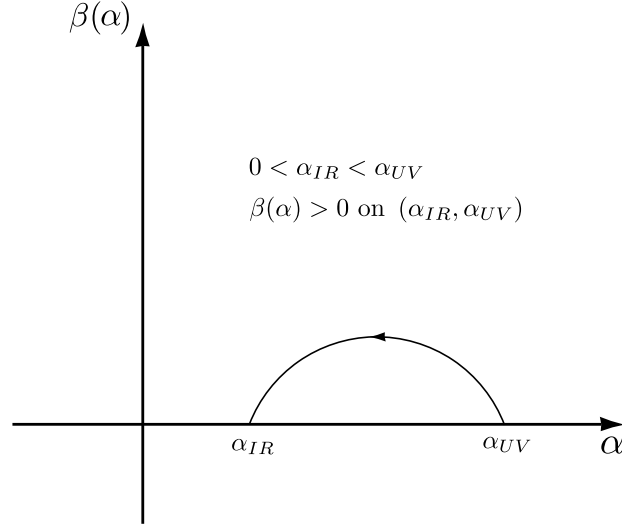
$$0 < \alpha_{IR} < \alpha_{UV}$$
$$\beta(\alpha) > 0 \text{ on } (\alpha_{IR}, \alpha_{UV})$$

FIGURE 4.1

so when [1] $1 - \alpha/\left(8\pi^2\right) > 0$, than $f(\alpha) \neq 0$ in the interval $[\alpha_{IR}, \alpha_{UV}]$ and the zeroes of $\beta(\alpha)$ are the zeroes of $h(\alpha)$. So because the $\beta(\alpha)$ is continuous and vanishes at the extremes of $[\alpha_{IR}, \alpha_{UV}]$ it has a maximum $\bar{\alpha}$ in the interval $(\alpha_{IR}, \alpha_{UV})$ which means that $h(\alpha)$ has a maximum $\bar{\bar{\alpha}}$ in the interval $(\alpha_{IR}, \alpha_{UV})$. But the γ_m would have an extremum at $\bar{\bar{\alpha}}$ away from the fixed points, thus contradicting the a-theorem.

The second case, with an inverted order for the fixed points and shown in Fig. 4.2 can be ruled out with a completely analogous reasoning.

We have thus seen how the a-theorem implies that the SQCD beta function cannot develop more than one fixed point at nonzero coupling.

$$0 < \alpha_{UV} < \alpha_{IR}$$
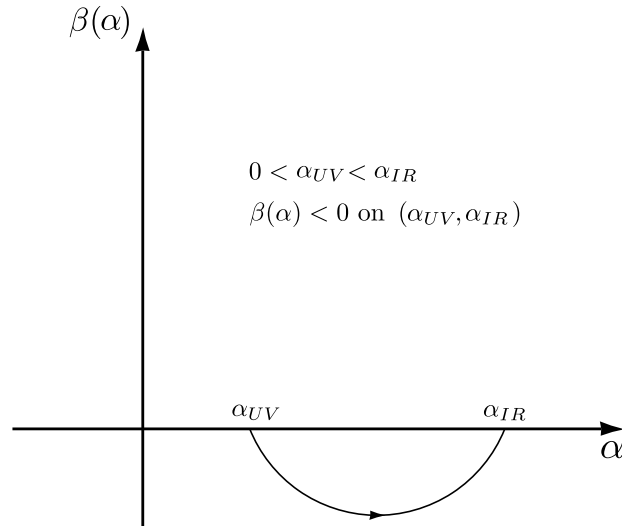$$\beta(\alpha) < 0 \text{ on } (\alpha_{UV}, \alpha_{IR})$$

FIGURE 4.2

If the one of the two fixed points occurs instead at zero coupling, we have again two cases but this time one of the two can satisfy the a-theorem whereas the other cannot. Let's first see the one that can be ruled out by the a-theorem.

---

[1]Note that the cusp singularity in Eq.(2.62) at $Ng^2/(8\pi^2) = 1$ is non physical as it is a renormalization scheme dependent condition.

If we consider the SQCD above the conformal window (so for $N_f \geqslant 3N$)—where it loses asymptotic freedom and develops instead an IR free fixed point—we can consider the case of Fig. 4.3 in which we have also an additional interacting UV fixed point.

$$\beta(\alpha)$$

$\alpha_{IR} = 0 \qquad N_f \geqslant 3N$

$\beta(\alpha) > 0$ on $(\alpha_{IR}, \alpha_{UV})$

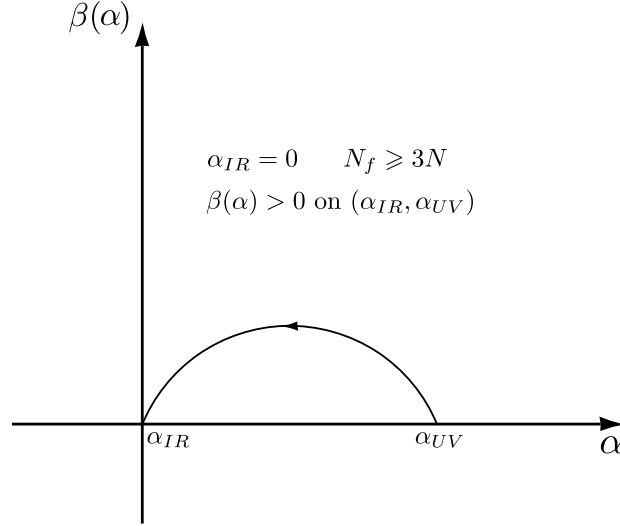α_{IR} $\qquad$ α_{UV} $\qquad$ α

FIGURE 4.3

Again, we have the condition that $\gamma_m(\alpha)$ must be strictly monotonic in $(0, \alpha_{UV})$. Furthermore, Eq. (4.1) and $\beta(\alpha) > 0$ implies $\frac{\partial a}{\partial \alpha} > 0$ in the same interval. Consequently through Eq. (4.2) we can derive two conditions in two intervals:

   i. $\frac{\partial \gamma_m}{\partial g} > 0$ for $\gamma_m < 0$ and $\gamma_m > 2$.

   ii. $\frac{\partial \gamma_m}{\partial g} < 0$ for $0 < \gamma_m < 2$.

However, if we consider γ_m in the vicinity of $\alpha = 0$ we clearly see neither of this two conditions can be satisfied. In fact, for $\alpha = 0$ we have $\gamma_m(0) = 0$ for the theory is free (see also Eq. (2.54)). Then if we move from $\alpha = 0$, $\frac{\partial \gamma_m}{\partial g}$ must be positive if γ_m becomes positive, contradicting (i).

$$\beta(\alpha)$$

$\alpha_{UV} = 0 \ ; \ \alpha_{IR} > 0$

$3N/2 \leqslant N_f < 3N$

$\beta(\alpha) < 0$ on $(\alpha_{UV}, \alpha_{IR})$

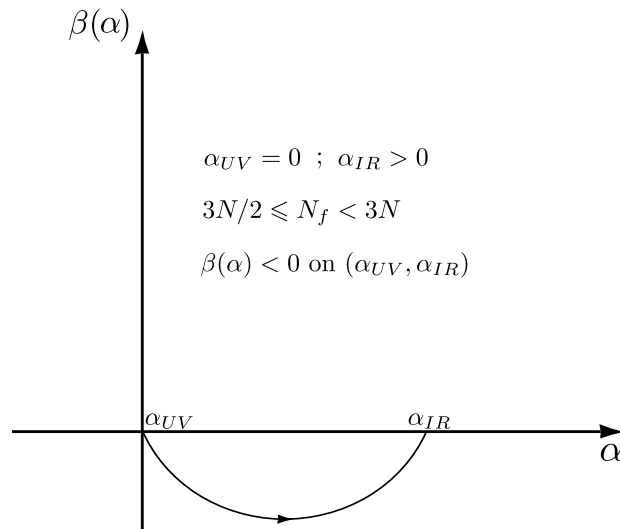α_{UV} $\qquad$ α_{IR} $\qquad$ α

FIGURE 4.4

If, on the contrary, γ_m becomes negative then $\frac{\partial \gamma_m}{\partial g}$ would be negative, contradicting (ii). Hence, the a-theorem rules out a nontrivial UV fixed point above the conformal window of SQCD.

The last case, shown in Fig.4.4 does not contradict the a-theorem and it is realized inside the conformal window. The coherence with the a-theorem can be easily seen with a very similar proof to the previous one. $\beta(\alpha) < 0$ in the interval $(\alpha_{UV}, \alpha_{IR})$ now, which implies $\frac{\partial a}{\partial \alpha} < 0$. This means that the conditions (i) and (ii) are inverted. For example (i) is $\frac{\partial \gamma_m}{\partial g} < 0$ for $\gamma_m < 0$ and $\gamma_m > 2$. Following the same reasoning as before we can see that these new conditions can be satisfied by $\gamma_m(\alpha)$ near $\alpha = 0$

4.1.1   The Large-N Veneziano Limit of QCD

An interesting question would be to investigate if this kind of constraints extend to the massless Veneziano limit of large-N QCD (defined by $N_f, N \to \infty, N_f/N =$ const). This question is motivated by a recently proposed beta function for the Veneziano limit [27]. This beta function is derived with homology methods and making use of some particular Wilson's loops. Its most striking characteristic, other than having passed a good number of consistency checks[2], is the evident analogy with the NSVZ beta function of SQCD. Its expression is given by

$$\beta(g) = \frac{\partial g}{\partial \log \mu} = -\frac{g^3}{16\pi^2} \frac{(4\pi)^2 \beta_0 - N(\partial \log Z / \partial \log \mu) + N_f \gamma_m(g)}{1 - N(g^2/4\pi^2)} \tag{4.4}$$

where β_1 is the 1-loop universal coefficient in Eq. (2.63) and

$$\frac{\partial \log Z}{\partial \log \mu} = 2\gamma_0 \left(Ng^2 + \dots\right) \quad ; \quad \gamma_0 = \frac{5}{3(4\pi)^2} \left(1 - \frac{2N_f}{5N}\right) \tag{4.5}$$

with the fermion anomalous mass dimension being

$$\gamma_m(g) = -\frac{9}{3(4\pi)^2} \frac{N^2 - 1}{N} g^2 + \dots \tag{4.6}$$

We can see from this that the only difference with the NSVZ beta function is the appearance of the anomalous dimension term $\partial \log Z / \partial \log \mu$ while the rest is completely identical to (2.62).

Another nice feature of this expression is the possibility of the determination of the lower edge of the conformal window as $N_f/N = 5/2$. For this value in fact γ_0 changes sign and as γ_0 enters the glueball kinetic term $\text{Tr}\left(G^2\right) \equiv G^a_{\mu\nu} G^{a\mu\nu}$ its change of sign signals the onset of a phase with $\left\langle \text{Tr}\left(G^2\right)\right\rangle = 0$. Because of the known relationship between the trace anomaly and the scalar glueball operator

$$T^\mu_\mu = \frac{\beta(g)}{2g} \text{Tr}\left(G^2\right) \tag{4.7}$$

the condition $\left\langle \text{Tr}\left(G^2\right)\right\rangle = 0$ implies the tracelessness of the energy momentum tensor hence the onset of a conformal phase [24]. This agrees nicely with recently obtained bound $6 < N^c_f < 8$ on the lower edge of the conformal window of $SU(3)$ QCD based on lattice calculations [14].

---

[2]For example one can verify that it reproduces the universal 2-loop perturbative beta function of QCD in the Veneziano limit.

Returning to the main question of this section, the analogies between the large-N Veneziano limit and SQCD motivate to question of whether it is possible to use the a-theorem to again infer constraints on the RG flow of the beta function (4.4) or not. The a-theorem as proved in [21] applies to all unitary theories in 4 dimensions so it covers QCD too. Unfortunately, as it was said before, an exact *a* function is known only in a very limited set of examples.

For QCD some results are known. It has been shown [20] that QCD satisfies $a_{UV} - a_{IR} > 0$ in the confined and chirally broken phase of QCD when in the IR the massless degrees of freedom consist of $N_f^2 - 1$ Goldstone bosons. Furthermore, the same relationship ($a_{UV} - a_{IR} > 0$) is shown to hold in perturbation theory in the large-N limit and with $\epsilon = 11/2 - N_f/N \ll 1$ at two loop order [28].

This is unfortunately non enough to establish the type of non-perturbative constraints we would like. The main hurdle is clearly the lack of a non-perturbative expression for the a-function of the massless large-N Veneziano limit of QCD. In comparison with the supersymmetric case, what we clearly lack is a thorough comprehension and control of the symmetries of the large-N Veneziano limit. If this would be achieved it is possible that a link between the a-function and some computable quantities could be made as it happens in the SQCD case.

Chapter 5

Conclusions

5.1 Review

In the first part of this thesis we have given an exposition of the main ideas of behind the renormalization group flow and its utility in research and model building.

In our exposition we have been guided by the connection with the a-theorem, so we have emphasized the ideas of the integration over high momentum modes and the connection between fixed points and conformal field theories. The first idea is crucial in partially understanding the mechanism that causes the irreversibility of the RG-flow and the second is central in the discourse and mathematical background behind the a-theorem. We have seen also how difficult to control is the renormalization group flow in cases of physical interest and far from the regimes where perturbation theory is applicable. In this prospective we have reviewed some know results in renormalization group theory such as the perturbative beta function of QCD and the exact NSVZ beta function of SQCD.

We have seen how the c- and a-theorem formalize the idea respectively in 2 and 4 dimensions that the renormalization group flow is irreversible and how they can be connected to physical intuition of the loss of degrees of freedom in the flow from the UV to the IR. In particular, we have seen that the c- and a-anomalies calculated at the fixed point provide a measure of the degrees of freedom for all scrutinized theories until now. We have seen how these theorems provide monotonic functions that interpolates between the anomalies at the fixed points.

In the last chapter we have seen how the a-theorem can be used in a case in which enough symmetries are known to be able to compute the a-function exactly to infer non-perturbative constraints on the RG-flow of a theory. Our testing ground has been SQCD where the symmetries present allow to compute both an exact beta function in terms of the anomalous dimension and an exact expression for the interpolating a-function in terms of the chiral anomalies (and re-express it then in terms of the anomalous dimension). With this ingredients we have shown that the a-theorem constraints the RG-flow in such a way that the beta function cannot have two interacting fixed point in sequence. We have shown that there cannot be, above the conformal window, an interacting UV fixed point after an IR free one (the so called asymptotic safety). Lastly, we have seen how the case of the Banks-Zaks fixed point, i.e. an interacting IR fixed point in an asymptotically free theory, does not contradict the a-theorem and can be realized in nature by adding a sufficient amount of matter (fermionic) degrees of freedom.

5.2   Final Remarks

The recently proposed beta function for the massless Veneziano large-N limit of QCD exhibits striking similarity to the one for SQCD. It is not clear at this time if the differences between the two are enough to justify a difference in behaviour, but the similarities provide hope that the approach to the a-theorem tested in SQCD could also be applied to this second theory. This is obviously very important as non-perturbative results are sorely lacking in QCD and non-Abelian gauge theories in general. In particular, the interests of large-N type expansions stand in the fact that they fit the experimental observations surprisingly well, even for the physical case $N = 3$ [29].

The objective now remains to find a way to write the a-function in a closed form. This is obviously much more complicated than in the SQCD case. SQCD is a theory of which we know the symmetries and conserved quantities very well. This is not the case for the Veneziano limit of QCD, for which we don't even know an expression for the action in terms of the fundamental quark and gluon degrees of freedom that produce the beta function (4.4) not to mention its symmetries or conserved quantities.

If this last difficulty was to be overcome we will be able to obtain by means of the a-theorem new breakthroughs in the understanding of the RG flows of non-Abelian gauge theories without supersymmetry and the realization of conformality, or the loss thereof.

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