On a variational principle for Newton-Cartan Gravity

Eric Spreen

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Primary supervisor: Prof. Dr. E.A. Bergshoeff
Secondary supervisor: Prof. Dr. D. Roest
Student number: s1958208

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Introduction

For centuries Newton's theory of motion and gravitation were regarded as the pinnacle of physics. Together with the theory of optics, culminating in Maxwell's equations, it was thought that most of physics had been discovered. Albert A. Michelson even remarked in 1894 that "it seems probable that most of the grand underlying principles have been firmly established and that further advances are to be sought chiefly in the rigorous application of these principles to all the phenomena which come under our notice."[**17**]

This view was however proven to be utterly misguided when at the beginning of the twentieth century two startling new theories emerged. Quantum Mechanics completely redefined the microscopic realm, whereas General Relativity shook up the way we thought about gravity. Together these two theories showed that our familiar Newtonian description of the world only turned out to be valid in day-to-day situations, whereas a whole world was to be explored in the microscopic and the high-energy realm. Especially the elegance of Special and General Relativity, being derived from only a handful of key principles was, was inspirational for formulating new theories. The idea that gravity works the way it does purely due to geometrical reasons turned out to be a guiding principle for the coming century.

In 1923 Elie Cartan reformulated Newton's theory of gravitation as a geometrical theory as well [**8**], showing that this classical theory could also be seen as the consequences of matter deforming space. This "Newton-Cartan Gravity" allowed for a more complete parallelism between Newtonian gravity and General Relativity.

The rising interest in quantum mechanics also sparked new research into Hamiltonian and Lagrangian mechanics. Especially the connection between the invariance of a theory under some fundamental group of transformations and the resulting emergence of a set of conserved currents became the cornerstone for modern theories. This idea was utilized in the formulation of Quantum Electrodynamics (with the gauge group $U(1)$ and the Yang-Mills theory for the weak force (with $U(2)$). A short time time later it was shown [**13, 18**] that General Relativity can, like almost any modern physical theory, be derived as a gauge theory for a group, the symmetry group being the Poincaré group of Lorentz transformations and translations.

Recently, Andringa et al. showed that Newton-Cartan Gravity can be obtained by a gauging procedure as well, the relevant group being an extension of the Galilean group with the Bargmann algebra as its Lie algebra[**3**]. It was also shown that by performing a procedure akin to the Inönü-Wigner group contraction on an extension of the Poincaré group, one obtains the centrally extended Galilean group with the Bargmann Algebra[**7**]. It is intuitive to view this procedure as a kind of non-relativistic limit. After all, Newtonian gravity reigns in the non-relativistic realm. This limit has however classically always been performed by considering bodies at speeds very much lower than the speed of light and then taking the limit in the metric, whereas this contraction shows that by considering the speed of light as infinite, one in fact obtains such a "limit" at the level of the underlying symmetries.

This contraction procedure may be extended so far as to the field equations. In quantum theories, however, there is always an underlying action from which these field equations may be derived. This action, together with the symmetries under which it should be invariant (as given by a specific Lie group), then form the foundations of the physical theory under consideration. It is here that Newton-Cartan Gravity falls short as of yet, as there is no appropriate action known.

The question of course arises why we are even interested in Newton-Cartan Gravity, when we have General Relativity as a more accurate theory. The first reason is that although General Relativity is mathematically very elegant, it is also much more difficult to find solutions for the field equations when the system under consideration even slightly departs from being trivial. We can often make a non-relativistic approximation of the system and make our calculations there, allowing us to solve a wide range of everyday problems. The independence of special coordinate systems that Newton-Cartan Gravity provides then allows for an ever wider range of methods to calculate solutions to these problems.

Secondly, Newton-Cartan geometry has in recent years been utilized in the study of various quantum mechanical systems. In particular it was used by Son et al. as a natural background upon which to study the Fractional Quantum Hall effect [**15**]. The requirement of general coordinate invariance (for which the coordinate independent formulation of Newton-Cartan geometry lends itself perfectly) is used to construct an effective action from which universal features of the Quantum Hall effect can be derived.

Thirdly, the theory is very useful when combined with the AdS/CFT correspondence to describe non-relativistic field theories in condensed matter physics. The holographic principle is used here to convert problems of a strongly coupled field theory into equivalent theories in the language of Newton-Cartan Gravity, in particular stringy versions of the theory (see e.g. [**12**]). This renewed interest in Newton-Cartan Gravity motivates us to examine the possibility of construcing an action for the theory.

Our main question in this thesis is then whether it is possible to produce an action for Newton-Cartan Gravity by considering procedures that produce Newton-Cartan Gravity as a non-relativistic limit of General Relativity, and if such a procedure turns out to be fruitless we wish to investigate what the underlying for failing to produce an action is. The hope is that this will provide us insight into why such an action eludes us, or might not even be possible to construct at all. Of course a full proof that an action does not exist is much more difficult to formulate and is thus beyond the scope of this thesis, but the investigation of the procedures in this thesis can provide some guidance as to where one must seek such a proof.

As our first procedure will consider the Inönü-Wigner contraction and investigate whether also produces an action. We will see that due to a divergence that emerges when we finally take the limit as $c \to \infty$, this is not the case. This is of course very disappointing, especially since it turns out that a similar procedure that produces the Galilei algebra, and so-called "Galilei Gravity" (which is significantly different from the day-to-day gravity we see around us) *does* produce an action that correctly describes the dynamics of the theory.[**5**]

Another procedure that one might consider is an *expansion* of the Poincaré algebra [**9**]. This expansion produces an infinite set of new algebras, that one might – in some sense – consider to be closer and closer to General Relativity as the order at which the expansion is truncated increases. This procedure has as a benifit that it is able to automatically produce actions that are invariant under one of the truncated algebras. The second-order truncation of the expansion of the Poincaré algebra turns out to be an extension of the Bargmann algebra, which produces a theory similar to Newton-Cartan Gravity. However, in order to produce an invariant action one needs a larger algebra: the third-order truncation. The resulting theory does produce normal Newtonian Gravity when coupled to a point particle (as shown in [**11**]), but clearly this theory is much larger than needed to produce the geometrical structure of Newtonian Gravity.

Structure

The structure of this thesis is divided in three parts. In the first part we will review the three types of theories for gravity that we will consider. In chapter 1 we will review General Relativity as a gauge theory of the Poincaré Group. Here, we will also introduce the notions and formalisms we will need in the rest of the thesis. chapter 2 and chapter 3 are reviews of Galilei Gravity and Newton-Cartan Gravity respectively.

The second part is dedicated to Lie algebra contractions. In chapter 4 we show how one obtains Galilei by performing a (naive) contraction procedure on the Poincaré algebra, and how one automatically obtains an action for this theory. After that, we show in chapter 5 the contraction procedure for Newton-Cartan Gravity, and show that this does not produce an action even though the procedure is able to reproduce the correct equations of motion.

The third part treats Lie algebra expansions. In chapter 6 we will develop the framework that is needed for the expansions, which we put to use in chapter 7 where we discuss the expansion of the Poincaré algebra and the corresponding invariant actions. The first and second order truncations are considered and identified as Galilei Gravity and Extended Newton-Cartan Gravity respectively.

Conventions

In this thesis, we will work in a $(d+1)$-dimensional spacetime, where d denotes the number of spatial dimensions. The Minkowski metric will always have the signature $(-,+,\dots,+)$: a negative sign for the temporal part and positive sign for the spatial part. We also adopt the Einstein summation convention, always summing over upper and lower indices appearing in pairs within a term, unless indicated otherwise.

We will frequently make use of vielbeins. It is therefore necessary to carefully distinguish the indices used in general frames and in the vielbein. We will use greek indices for general spacetime coordinates, small latin indices starting at $i, j, \dots$ for spatial spacetime coordinates, capital latin indices for coordinates in a vielbein, and small latin indices starting at $a, b, \dots$ for spatial coordinates in a vielbein.

The vielbein fields will be used to convert greek indices to latin indices and vice versa. The Minkowskian metric η will be used to raise and lower vielbein indices, while the normal metric g can be used to raise and lower greek indices.

We will be considering quite a few algebras with very similar sets of generators. In order to distinguish them, we therefore strictly place hats on the generators belonging to the Poincaré algebra, tildes on redefined generators before taking a limit during a contraction, and no signs on top of the generators belonging to the Bargmann or Galilei algebras. A similar distinction is made for the fields. The fields corresponding to the Poincaré algebra will be capitalized, those corresponding to the intermediate algebras will have tildes, and the fields corresponding to the Bargmann or Galilei algebras will have nothing on top.

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Part 1

Gravity and Lie Algebras

CHAPTER 1

General Relativity

When Einstein first wrote down his field equations, he did so in the language of (pseudo-)Riemannian geometry. This framework of geometry allows us to view spacetime as an abstract space and to describe it using arbitrary coordinate systems. The geometry of the spacetime is determined by the metric and its corresponding covariant derivative (the Levi-Civita connection). This formulation is generally the way students first come in contact with General Relativity.

This formulation does however fail to express one particularly nice fact about General Relativity, namely that it is locally equivalent to Special Relativity where the spacetime is flat. This is very important, because Special Relativity is completely defined by the postulate that the laws of physics are independent of your choice of *inertial* coordinate systems. That is, the laws of physics should be invariant under translations and Lorentz transformations.

The local equivalence to Special Relativity is expressed by the fact that on a small enough neighbourhood of any point in spacetime, one can choose some basis ${E_\mu}^A$ for the tangent spaces such that the metric reduces to the Minkowski metric in that basis: $g_{\mu\nu} = {E_\mu}^A {E_\nu}^B \eta_{AB}$. Such a basis that diagonalizes the metric is called a vielbein. A second set of fields that is very useful in this formulation are the spin-connection fields. Just like the covariant derivative can be expressed by the Christoffel symbols ${\Gamma_\mu}^\rho{}_\nu$ with respect to a coordinate basis, we can represent the covariant derivate with respect to the vielbein basis. The corresponding symbols are called the spin-connection. For General Relativity we will denote them by ${\Omega_\mu}^A{}_B$. It is possible to reformulate the entire theory of General Relativity in terms of these vielbeins. Instead of needing to calculate the metric one now needs to calculate the vielbeins in order to solve the Einstein equations form some particular system.

In modern theories, such an invariance under transformations is formulated in the language of group and gauge theory. The transformations under which the theory should be invariant is represented by a group, and the invariance of the theory is then expressed by requiring that the action of the theory should remain constant when the group acts on it. The group that represents the Lorentz transformations and translations is called the Poincaré Group.

In Special Relativity this invariance under the Poincaré Group is global. That is, we only require the action to be invariant when we perform the same transformation at each point in spacetime. This gives us precisely the independence of choice of inertial coordinate system. However, in General Relativity, we are not restricted to inertial coordinate systems. Instead, the theory is invariant under the choice of *any* coordinate system. This corresponds to the fact that General Relativity is locally invariant under the Poincaré Group, i.e. the action should be invariant even when we apply a (smoothly) changing transformation at each point in spacetime. The locality of this invariance introduces a set of extra fields that are known as gauge fields. It turns out that these gauge fields are the vielbein fields and their corresponding spin-connection fields. This then allows us to construct General Relativity from it's underlying local symmetry alone!

Such a gauging procedure also allows one to make contact with quantum field theories, so that one may start to explore quantum gravitional theories. With the rise of these gauge theories such as quantum electrodynamics and the Yang-Mills theory of the weak force, interest rose in expressing General Relativity as a gauge theory as well. Utiyama [**18**] made a first attempt at this in 1956, which was expanded upon by Kibble [**13**] in 1961. We will discuss in this chapter how General Relativity may be viewed as a gauge theory, following [**7**]. This discussion will be our guiding light throughout the rest of our journey, where we will explore how we can do the same thing for non-relativistic gravity and how we can even use this framework to relate General Relativity and non-relativistic gravity. Unless stated otherwise, we will be working on a $(d+1)$-dimensional spacetime, where d is the number of spatial dimensions and we will denote the total number of dimensions by $D = d+1$.

1. The gauge fields

As stated above, the symmetries of General Relativity are represnted by the Poincaré Group $\mathrm{Iso}(d,1)$. It is a standard result of Lie group theory that the structure of this group is determined by its Lie algebra $\mathfrak{iso}(d,1)$. Being the semidirect product of the ($\frac{D(D-1)}{2}$-dimensional) Lorentz group and the (D-dimensional) group of translations, it is a $\frac{D(D+1)}{2}$-dimensional Lie group. The Lie algebra therefore has $\frac{D(D+1)}{2}$ independent generators: D associated to translations ($\hat{P}_A$ with parameters η_A) and $\frac{D(D-1)}{2}$ associated to Lorentz transformations ($\hat{J}_{AB} = -\hat{J}_{BA}$ with parameters Λ_{AB}), where $A, B = 0, \dots, d$. The structure of the Poincaré algebra is represented by the commutators of these generators. These are given by:

$$[\hat{P}_A, \hat{P}_B] = 0, \tag{1.1}$$

$$[\hat{P}_A, \hat{J}_{BC}] = 2\eta_{A[B}\hat{P}_{C]}, \tag{1.2}$$

$$[\hat{J}_{AB}, \hat{J}_{CD}] = 4\eta_{[A[C}\hat{J}_{D]B]}. \tag{1.3}$$

We can introduce the gauge fields by introducing an object called a Cartan connection. This is a 1-form α_μ on the spacetime with values in the

Lie algebra $\mathfrak{iso}(d,1)$.[1] Since this form has values in the Lie algebra, we can express it in terms of the generators:[2]:

$$\alpha_\mu = E_\mu{}^A \hat{P}_A + \frac{1}{2}\Omega_\mu{}^{AB}\hat{J}_{AB}. \tag{1.4}$$

These $E_\mu{}^A$ and $\Omega_\mu{}^{AB}$ are now regular 1-forms, and they are known as the gauge fields. These fields transform under translations and Lorentz transformations as [**5**]:

$$\delta E_\mu{}^A = \partial_\mu \zeta^A + \Lambda^A{}_B E_\mu{}^B - \Omega_\mu{}^{AB}\zeta_B, \tag{1.5a}$$

$$\delta \Omega_\mu{}^{AB} = \partial_\mu \Lambda^{AB} + \Omega_\mu{}^A{}_C \Lambda^{BC} - \Omega_\mu{}^{BC}\Lambda^A{}_C. \tag{1.5b}$$

All the components of the theory are summarized in Table 1.

Symmetry	Generators	Fields	Parameters	Curvatures
Translations	$\hat{P}_A$	$E_\mu{}^A$	ζ^A	$R_{\mu\nu}{}^A(\hat{P})$
Lorentz tranformations	$\hat{J}_{AB}$	$\Omega_\mu{}^{AB}$	Λ^{AB}	$R_{\mu\nu}{}^{AB}(\hat{J})$

TABLE 1. All components that arise in the gauging procedure of the Poincaré algebra to obtain General Relativity[**5**].

As discussed above, we wish to interpret the $E_\mu{}^A$ as a vielbein, and $\Omega_\mu{}^{AB}$ as the corresponding spin-connection. This then introduces a metric on the spacetime in the usual manner:

$$g_{\mu\nu} := E_\mu{}^A E_\nu{}^B \eta_{AB}. \tag{1.6}$$

The possibility of having different metrics on M corresponds to the freedom in choosing the Cartan connection. However, as it stands we cannot yet call this General Relativity. Although we may have found a metric, we cannot yet interpret $\Omega_\mu{}^{AB}$ as the spin-connection, since we are yet unable to solve for the spin-connection in terms of the vielbein. We will address this problem in the next section.

Since the vielbein $E_\mu{}^A$ is to be a complete set of orthonormal vector fields, we are able to invert this basis and thereby obtain the dual vierbein $E^\mu{}_A$. These fields can be defined by their inversion properties:

$$E^\mu{}_A E_\nu{}^A = \delta^\mu_\nu \quad E_\mu{}^A E^\mu{}_B = \delta^A_B, \tag{1.7}$$

[1]Mathematically, we introduce a principal Iso$(d,1)$-bundle with our spacetime as its base and define the Cartan connection on that bundle. The form α_μ that we use is then the Lie-algebra-valued 1-form that is induced on the underlying spacetime by the solder form the Cartan connection generates. The principal bundle ensures that we can actually perform a group action on the gauge fields by introducing a local copy of the group itself at each point in spacetime. For a comprehensive discussion of the mathematical concepts the reader is referred to [**16**]. For a more extensive treatment, see [**14**].

[2]The factor of $\frac{1}{2}$ in the second term is to compensate for the fact that the $\hat{J}_{AB}$ are not independent. In fact, $\hat{J}_{AB} = -\hat{J}_{BA}$, and hence $\Omega_\mu{}^{AB} = -\Omega_\mu{}^{BA}$.

and we will use the notation:

(1.8a) $$E^{\mu A} := \eta^{AB} E^{\mu}{}_{B}, \qquad E_{\mu A} := \eta_{AB} E_{\mu}{}^{B},$$

(1.8b) $$\Omega_{\mu}{}^{A}{}_{B} = -\Omega_{\mu B}{}^{A} := \eta_{BC} \Omega_{\mu}{}^{AC}.$$

That is, we can use the Minkowski metric to raise and lower the vielbein indices. Using this notation, it will be convenient for further calculations to record the following rules for the vierbein:

(1.9a) $$E_{\mu A} E^{\mu}{}_{B} = \eta_{AB},$$

(1.9b) $$E_{\mu}{}^{A} E^{\mu B} = \eta^{AB},$$

(1.9c) $$E_{\mu A} E^{\mu B} = \delta_{A}{}^{B}.$$

2. Curvature and Conventional Contraints

We now have all the necessary fields to our disposal. However, there is one more ingredient we are missing to complete our theory and make contact with General Relativity: the curvatures of the fields. (These curvatures are also often called the field strengths. The origin of this is that when the curvature of a particular field is zero, one can eliminate the field entirely by performing a gauge transformation, indicating that the field is "not there".)

We can compute the curvature of the fields by calculating the curvature of the entire Cartan connection α_μ and inspecting its components. The curvature is $R_{\mu\nu} := 2\partial_{[\mu}\alpha_{\nu]} + [\alpha_{[\mu}, \alpha_{\nu]}]$. When we write this out in components, we obtain:

$$R_{\mu\nu} = R_{\mu\nu}{}^{A}(\hat{P})\hat{P}_A + \frac{1}{2} R_{\mu\nu}{}^{AB}(\hat{J})\hat{J}_{AB},$$

(1.10a) $$R_{\mu\nu}{}^{A}(\hat{P}) = 2\partial_{[\mu} E_{\nu]}{}^{A} + 2\Omega_{[\mu}{}^{AB} E_{\nu]B}$$

(1.10b) $$R_{\mu\nu}{}^{AB}(\hat{J}) = 2\partial_{[\mu} \Omega_{\nu]}{}^{AB} + 2\Omega_{[\mu}{}^{AC} \Omega_{\nu]C}{}^{B}$$

These curvatures transform as:

(1.11a) $$\delta R_{\mu\nu}{}^{AB}(\hat{P}) = \Lambda^{A}{}_{B} R_{\mu\nu}{}^{B}(\hat{P}) - \zeta^{B} R_{\mu\nu}{}^{A}{}_{B}(\hat{J})$$

(1.11b) $$\delta R_{\mu\nu}{}^{AB}(\hat{J}) = \Lambda^{A}{}_{C} R_{\mu\nu}{}^{CB}(\hat{J}) - \Lambda^{B}{}_{C} R_{\mu\nu}{}^{CA}(\hat{J})$$

In the last section we raised the problem that we cannot yet interpret $\Omega_{\mu}{}^{AB}$ as the spin-connection corresponding to the vielbein. At this point the $\Omega_{\mu}{}^{AB}$ are still independent fields, but we want to be able to solve for them. The only expressions that relate the vielbein and the spin-connection are the curvatures (1.10), in particular (1.10a). We therefore set this curvature to zero:

(1.12) $$R_{\mu\nu}{}^{A}(\hat{P}) = 0.$$

We can now make contact with General Relativity by defining the covariant derivative (through its Christoffel symbols). This is done by the following vielbein postulate:

(1.13) $$\partial_{\mu} E_{\mu}{}^{A} - \Gamma_{\nu}{}^{\rho}{}_{\mu} E_{\rho}{}^{A} - \Omega_{\mu}{}^{A}{}_{B} E_{\nu}{}^{B} = 0.$$

We can now calculate the torsion of this covariant derivative by taking the antisymmetric part of this equation. Together with (1.12) this implies that the torsion vanishes. This further justifies the constraint, since we need the torsion of the covariant derivative to vanish in General Relativity. A further requirement for this covariant derivative is that the covariant derivative of the metric needs to vanish (i.e. that it is metric-compatible). This is automatically satisfied through (1.13) and the fact that ${\Omega_\mu}^{AB}$ is antisymmetric in its upper two indices.

As stated before, the constraint (1.12) allows us to solve for ${\Omega_\mu}^{AB}$ in terms of the vielbein ${E_\mu}^A$ and the inverse vielbein ${E^\mu}_A$. The solution is given by:

$$ {\Omega_\mu}^{AB} = 2E^{\rho[A}\partial_{[\mu}{E_{\rho]}}^{B]} - E_{\mu C}E^{\rho A}E^{\sigma B}\partial_{[\rho}{E_{\sigma]}}^{C}. \tag{1.14} $$

This is in fact exactly the equation for the spin-connection in terms of the vierbein when derived from metric compatibility and zero torsion.

Note that the curvature constraint (1.12) is not absolutely necessary to impose. In fact, if one ommits this constraint, one obtains general relativity with a connection that has non-zero torsion. This is known as Einstein-Cartan theory, and is useful for introducing particles with spin. This does not allow you to solve for the spin-connection in terms of the vielbein, so the spin-connection becomes a completely independent set of fields. This extra freedom corresponds to a choice in how much torsion the underlying geometry has.

3. Field equations

At this point we have a complete picture of the kinematics of General Relativity. If we want to put the theory on-shell, we must impose the usual equations of motion: the Einstein field equations.

In order to connect to the classic field equations of General Relativity, we observe that the Riemann curvature tensor corresponding to the introduced covariant derivative is given by:

$$ {R_{\mu\nu\rho}}^A(\Gamma) := {R_{\mu\nu}}^{AB}(\hat{J})E_{\rho B}. \tag{1.15} $$

The constraint (1.12) ensures that the first Bianchi identity is satisfied:

$$ {R_{[\mu\nu\rho]}}^A(\Gamma) = 0. \tag{1.16} $$

Similarly, the Ricci curvature and scalar become:

$$ R_{\mu\nu}(\Gamma) = {R_{\rho\mu}}^{AB}(\hat{J})E_{\nu B}{E^\rho}_A, \tag{1.17} $$

$$ R := R(\Gamma) = {R_{\mu\nu}}^{AB}(\hat{J}){E^\mu}_A{E^\nu}_B. \tag{1.18} $$

The Einstein field equations can now be expressed by:

$$ R_{AB} - \frac{1}{2}\eta_{AB}R = \kappa T_{AB}, \tag{1.19} $$

where we defined $R_{AB} := E^{\mu}{}_{A}E^{\nu}{}_{B}R_{\mu\nu}(\Gamma)$ and $T_{AB} = E^{\mu}{}_{A}E^{\nu}{}_{B}T_{\mu\nu}$. We can write this out explicitly in terms of the vielbein fields and curvatures to obtain:

$$E^{\rho}{}_{A}E^{\sigma}{}_{C}R_{\rho\sigma}{}^{C}{}_{B}(\hat{J}) + \frac{1}{2}\eta_{AB}E^{\rho}{}_{C}E^{\sigma}{}_{D}R_{\rho\sigma}{}^{CD}(\hat{J}) = -\frac{8\pi G}{c^4}T_{AB}\,. \tag{1.20}$$

4. Action

Shortly after Einstein published his field equations, Hilbert proposed a variational principle for it, proposing the now-called Einstein-Hilbert action, given by [**10**, p. 172]:

$$S = \frac{1}{2\kappa}\int \sqrt{-g}\,\mathrm{d}^D x. \tag{1.21}$$

Here, we have $g = \det(g_{\mu\nu})$. One can obtain the (vacuum) Einstein field equations from this action by varying with respect to only the metric and considering the curvatures to be dependent on the metric (the second-order formulation), or one can assume the metric and curvatures to be independent, varying with respect to both. In order to obtain the field equations with matter present, one can add a matter Lagrangian density to the action:

$$S = \int \sqrt{-g}\left(\frac{R}{2\kappa} + \mathcal{L}_{\text{matter}}\right)\mathrm{d}^D x \tag{1.22}$$

Let us rewrite this action in terms of the vielbein fields. Since $g_{\mu\nu} = E_{\mu}{}^{A}E_{\mu}{}^{B}\eta_{AB}$, we can immediately calculate:

$$\det(g_{\mu\nu}) = \det(E_{\mu}{}^{A}E_{\mu}{}^{B}\eta_{AB}) = \det(E_{\mu}{}^{A})^2\det(\eta_{AB}) = -\det(E_{\mu}{}^{A})^2. \tag{1.23}$$

So, if we define $E := \det(E_{\mu}{}^{A})$ (which must indeed be positive), we obtain:

$$S = \int E\left(\frac{R}{2\kappa} + \mathcal{L}_{\text{matter}}\right)\mathrm{d}^4 x. \tag{1.24}$$

It is now straightforward to show that the Einstein field equations (1.19) can be derived by varying this action with respect to the vielbein fields $E_{\mu}{}^{A}$ and $\Omega_{\mu}{}^{AB}$ independently. This will also produce the constraint (1.12), showing that the first order formulation automatically allows us to solve for the spin-connection in terms of the vielbein. Similarly to the metric formulation, one can also start by imposing (1.12) and then only varying with respect to $E_{\mu}{}^{A}$. This will indeed also produce the correct equations, but in a second order formalism.

It is thus clear that General Relativity can be expressed as a full fledged gauge theory based on the Einstein-Hilbert action. We will go on in the next chapter to investigate the form of such a gauge theory for non-relativistic gravity, and whether it is possible to obtain an action for this theory.

CHAPTER 2

Galilei Gravity

In the last chapter we saw that General Relativity can be obtained by gauging the Poincaré algebra and imposing a constraint on the curvature ${R_{\mu\nu}}^{A}(\hat{P})$. This all feels very natural, as the group that represents the fundamental spacetime symmetries in GR under which the equations of motion should be invariant is, in fact, the Poincaré group. It should then come naturally to ask ourselves whether we can obtain "classic" Newtonian Gravity via a similar gauging procedure. If so, what group should we use as our starting point?

As is the case for Special Relativity, in Newtonian physics the notion of an inertial frame is central to the theory. In General and Special Relativity, all these special frames are related by transformations of the Poincaré group: Lorentz boosts, rotations, and spatial translations. Similarly, in Newtonian physics the inertial frames are related by non-relativistic boosts (frames moving at constant velocity with respect to another), spatial translations, temporal translations, and spatial rotations. The group that is formed by these transformations is known as the Galilean Group. It is this group that transforms the non-relativistic inertial frames into each other.

However, in regular Newtonian physics we are completely restricted to these inertial frames. So when we switch to an arbitrary frame, artifacts start to appear in the expression for the forces, such as the Coriolis force in a rotating frame. If we are to formulate an inherently covariant theory, we have to promote this global invariance under the Galilean Group to a local invariance, just like we did when moving from Special Relativity to General Relativity. That is, intuitively, we must gauge the Galilean Group.

I will spoil the end of this chapter, as this intuition is misguided. The reasons for this are however profound, so it will be fruitful to make an attempt anyway. An overview of this gauge theory, which we will dub Galilei Gravity in line with other literature, will moreover prove useful in the discussion of the limiting procedure that we encounter in the next part.

1. The gauge fields

As in the last chapter, we begin by studying the group in question. The Galilean group is again a $\frac{D(D+1)}{2}$-dimensional Lie Group. The generators of its Lie algebra correspond to the different symmetries involved: temporal translation (H with parameter ζ), spatial translations (P_a with parameter

ζ^a), boosts (G_a with parameter λ^a) and rotations ($J_{ab} = -J_{ba}$ with parameter λ^{ab}). The commutators of this group are given by:

$$[P_a, J_{bc}] = 2\delta_{a[b}P_{c]}, \qquad [G_a, J_{bc}] = 2\delta_{a[b}G_{c]},$$
$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]}, \qquad [H, G_a] = P_a. \tag{2.1}$$

These will be the starting point of our derivation.

Again we introduce a Cartan connection α_μ for the Galilei group, defining the gauge fields as the components of the connection corresponding to their respective generators:[1]

$$\alpha_\mu = \tau_\mu H + e_\mu{}^a P_a + \omega_\mu{}^a G_a + \frac{1}{2}\omega_\mu{}^{ab} J_{ab}. \tag{2.2}$$

These fields then transform as:

$$\delta\tau_\mu = \partial_\mu\zeta, \tag{2.3a}$$
$$\delta e_\mu{}^a = \partial_\mu\zeta^a + \zeta\omega_\mu{}^a - \lambda^a\tau_\mu + \lambda^{ab}e_{\mu b} - \omega_\mu{}^a{}_b\zeta^b, \tag{2.3b}$$
$$\delta\omega_\mu{}^a = \partial_\mu\lambda^a + \lambda^{ab}\omega_{\mu b} - \omega_\mu{}^a{}_b\lambda^b, \tag{2.3c}$$
$$\delta\omega_{\mu\nu}{}^{ab} = \partial_\mu\lambda^{ab} + 2\lambda^{c[a}\omega_\mu{}^{b]}{}_c. \tag{2.3d}$$

Again, all the components are summarized in Table 1.

Symmetry	Generators	Fields	Parameters	Curvatures
Temporal translation	H	τ_μ	ζ	$R_{\mu\nu}(H)$
Spatial translations	P_a	$e_\mu{}^a$	ζ^a	$R_{\mu\nu}{}^a(P)$
Boosts	G_a	$\omega_\mu{}^a$	λ^a	$R_{\mu\nu}{}^a(G)$
Rotations	J_{ab}	$\omega_\mu{}^{ab}$	λ^{ab}	$R_{\mu\nu}{}^{ab}(J)$

TABLE 1. All components that arise in the gauging procedure of the Galilei Algebra to obtain Galilei Gravity[**5**].

The interpretation of these fields will be that τ_μ is a sort of *temporal metric*, that is able to measure time intervals, and $e_\mu{}^a$ as a sptial frame in which we are able to measure distances. Hence, we can again view the set of fields $(\tau_\mu, e_\mu{}^a)$ as an orthonormal basis. We can then define the inverse fields as follows:

$$e_\mu{}^a e^\mu{}_b = \delta^a_b, \qquad \tau_\mu\tau^\mu = 1,$$
$$\tau_\mu e^\mu{}_a = 0, \qquad \tau^\mu e_\mu{}^a = 0, \tag{2.4}$$
$$\tau_\mu\tau^\nu + e_\mu{}^a e^\nu{}_a = \delta^\mu_\nu.$$

[1]Again, the factor of $\frac{1}{2}$ in the term for J_{ab} is introduced for the fact that the J_{ab} are not all independent.

Note that in the third equations, we need to add $\tau_\mu \tau^\nu$, since τ_μ acts as the fourth basis field. From this definition we can find the following transformation rules for these fields:

$$\delta\tau^\mu = -\tau^\mu\tau^\nu\delta\tau_\nu - e^\mu{}_b\tau^\nu\delta e_\nu{}^b, \tag{2.5a}$$

$$\delta e^\mu{}_a = -e^\mu{}_b e^\nu{}_a\delta e_\nu{}^b - \tau^\mu e^\nu{}_a\delta\tau_\nu\,. \tag{2.5b}$$

2. Curvature and Constraints

Again, the curvature of the connection α_μ is given by $R_{\mu\nu} = 2\partial_{[\mu}\alpha_{\nu]} + [\alpha_{[\mu}, \alpha_{\nu]}]$ and define the curvatures as the fields as the corresponding components of this total curvature:

$$R_{\mu\nu} := R_{\mu\nu}(H)H + R_{\mu\nu}{}^a(P)P_a + R_{\mu\nu}{}^a(G)G_a + \frac{1}{2}R_{\mu\nu}{}^{ab}(J)J_{ab}\,,$$

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]}\,, \tag{2.6a}$$

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu}e_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}e_{\nu]b} + 2\tau_{[\mu}\omega_{\nu]}{}^a, \tag{2.6b}$$

$$R_{\mu\nu}{}^a(G) = 2\partial_{[\mu}\omega_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}\omega_{\nu]b}, \tag{2.6c}$$

$$R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b. \tag{2.6d}$$

These curvatures transform as:

$$\delta R_{\mu\nu}(H) = 0, \tag{2.7a}$$

$$\delta R_{\mu\nu}{}^a(P) = \zeta R_{\mu\nu}{}^a(G) - \lambda^a R_{\mu\nu}(H) + \lambda^a{}_b R_{\mu\nu}{}^b(P) - \zeta^b R_{\mu\nu}{}^a{}_b(J), \tag{2.7b}$$

$$\delta R_{\mu\nu}{}^a(G) = \lambda^a{}_b R_{\mu\nu}{}^b(P) - \lambda^b R_{\mu\nu}{}^a{}_b(J), \tag{2.7c}$$

$$\delta R_{\mu\nu}{}^{ab}(J) = 2\lambda^{c[a}R_{\mu\nu}{}^{b]}{}_c(J). \tag{2.7d}$$

Again we wish to introduce some kind of covariant derivative, where $\omega_\mu{}^a$ and $\omega_\mu{}^{ab}$ represent the spin-connection corresponding to that covariant derivative. We can define that connection by imposing vielbein postulates, just as in the case of General Relativity. However, this time we need to impose two sets of postulates for τ_μ and $e_\mu{}^a$:

$$\begin{aligned} \partial_\mu\tau_\nu - \Gamma_\mu{}^\rho{}_\nu\tau_\rho - \omega_\mu{}^a e_{\nu a} = 0 \\ \partial_\mu e_\nu{}^a - \Gamma_\mu{}^\rho{}_\nu e_\rho{}^a - \omega_\mu{}^{ab}e_{\nu b} \end{aligned} \tag{2.8}$$

Since τ_μ and $e_\mu{}^a$ form a complete basis of covector fields, this completely determines the covariant derivative.

Of course we now want to solve $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ in terms of $e_\mu{}^a$ and τ_μ (and their inverses). This is partly possible using the Galilei equations of motion, but we will delay this discussion after we have gotten familiar with the Galilei action and the resulting field equations.

3. Action and Field equations

In the case of General Relativity we were able to write down a very elegant action that only contained the determinant of the vielbein and the trace of the $\hat{J}$-curvatures. We can readily write down a similarly natural action for Galilei gravity using corresponding objects. This Galilean action is given by [**5**]:

$$S = -\frac{1}{2\kappa}\int e e^{\rho}{}_{a} e^{\sigma}{}_{b} R_{\rho\sigma}{}^{ab}(J), \tag{2.9}$$

where we defined $e = \det(\tau_{\mu}, e_{\mu}{}^{a})$. By using the earlier given transformation rules is straightforward to check that this action is indeed invariant under Galilean transformations.

We can now derive the field equations for Galilei Gravity by working out the Euler-Lagrange equations for the action. The results for the various fields are:

(2.10a) $\boxed{\tau_{\mu}}$ $\quad 2e^{\mu}{}_{a}\tau^{\rho}e^{\sigma}{}_{b}R_{\rho\sigma}{}^{ab}(J) = \tau^{\mu}R(J)$

(2.10b) $\boxed{e_{\mu}}$ $\quad 2e^{\rho}{}_{a}e^{\mu}{}_{c}e^{\sigma}{}_{d}R_{\rho\sigma}{}^{cd}(J) = e^{\mu}{}_{a}R(J)$

(2.10c) $\boxed{\omega_{\mu}{}^{a}}$ $\quad$ No new information,

(2.10d) $\boxed{\omega_{\mu}{}^{ab}}$
$$\begin{aligned} 4\omega_{\nu}{}^{c}{}_{[a}e^{[\mu}{}_{b]}e^{\nu]}{}_{c} =& -4e^{\rho}{}_{[a}e^{[\mu}{}_{b]}\tau^{\nu]}\partial_{\nu]}\tau_{\rho} - 4e^{[\mu}{}_{c}e^{\nu]}{}_{[a}e^{\rho}{}_{b]}\partial_{\nu}e_{\rho}{}^{c} \\ & - 2e^{[\mu}{}_{a}e^{\nu]}{}_{b}\left(\tau^{\rho}\partial_{\nu}\tau_{\rho} + e^{\rho}{}_{c}\partial_{\nu}e_{\rho}{}^{c}\right) \end{aligned}$$

where $R(J) := e^{\rho}{}_{a}e^{\sigma}{}_{b}R_{\rho\sigma}{}^{ab}(J)$. Note that the Euler-Lagrange equations give no field equations when varying for $\omega_{\mu}{}^{a}$, because those fields don't occur in the action.

The equations in this form are slightly difficult to work with. It is luckily straightforward to show that (2.10a) and (2.10b) together are completely equivalent to:

$$R_{\mu b}{}^{ab}(J) := e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(J) = 0. \tag{2.11a}$$

With this in hand, we can also show that for $D > 2$ the field equation (2.10d) is then equivalent to the contraint on the geometry:

$$R_{ab}(H) := 2e^{\rho}{}_{a}e^{\sigma}{}_{b}\partial_{[\rho}\tau_{\sigma]} = 0, \tag{2.11b}$$

and the field equations

$$R_{0a}(H) = \frac{D-3}{D-2}R_{ab}{}^{b}(P), \tag{2.11c}$$

$$R_{ab}{}^{c} = -\frac{2}{D-2}\delta^{c}_{[a}R_{b]d}{}^{d}(P). \tag{2.11d}$$

Note that the dependence on the dimensionality originates in traces that are taken of (2.10d), introducing factors of $\delta^{a}_{a} = d = D-1$.

The constraint (2.11b) implies that the Galilean geometry has *twistless torsion.* This is in fact an integrability condition that implies the existence of a foliation of spacetime, thereby separating the spacetime globally (i.e. independent of choice of coordinate system) in one temporal and d spatial directions. The (temporal) torsion is not completely elimated by this constraint, so there may occur effects such as time dilation, but there is always global agreement on the ordering of events, and in particular on the simultaneity

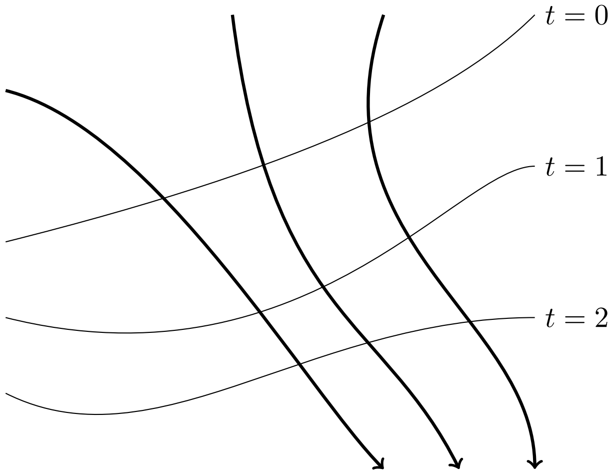

FIGURE 1. The Galilean spacetime might have some torsion that can cause time dilation and other similar effects, but the existence of a foliation shows that there is a concept of global ordering of events.

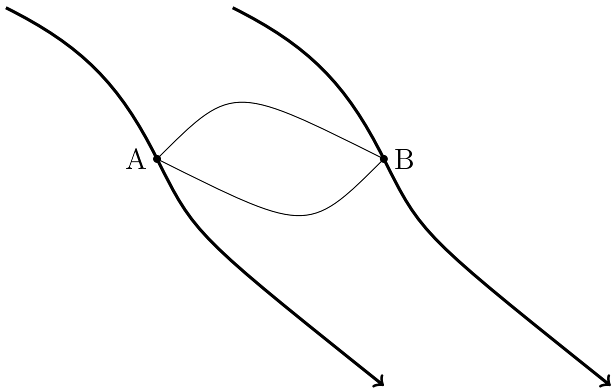

FIGURE 2. The integrability condition (2.11b) implies that the amount of time measured between any two points is independent of the path taken.

of them (whereas in General Relativity you can only define an ordering of events inside the lightcone of your coordinate system).

For $D > 3$ we can use (2.11c) and (2.11d) to partially solve for ${\omega_\mu}^{ab}$ in terms of τ_μ, ${e_\mu}^a$ and their inverses. This "solution" is given by:

$$\begin{aligned}{\omega_\mu}^{ab} = \tau_\mu A^{ab} + e_{\mu c}\left(e^{\rho[a}e^{b]\nu}\partial_\rho {e_\nu}^c + e^{\rho[a}e^{c]\nu}\partial_\rho {e_\nu}^b - e^{\rho[b}e^{c]\nu}\partial_\rho {e_\nu}^a\right) \\ + \frac{4}{D-3} e^{\rho[a}{e_\mu}^{b]}\tau^\nu \partial_{[\rho}\tau_{\nu]}\,. \end{aligned} \tag{2.12}$$

The field A^{ab} that occurs in the first term is an undetermined anti-symmetric tensor field. Also note that we cannot solve for ${\omega_\mu}^a$ using the field equations. This prevents us from interpreting ${\omega_\mu}^{ab}$ and ${\omega_\mu}^a$ as the spin- and boost-connections of a covariant derivative. As we will see in the next chapter, such a covariant derivative is instrumental in connecting to Newtonian Gravity. Of course, one could define such a covariant derivative by introducing an appropriate vielbein postulate and then fixing values for the auxiliary field A^{ab} and the gauge field ${\omega_\mu}^a$, but we then end up with a covariant derivative whose geodesics do not necessarily correspond to the worldlines of free-falling particles. That is, we do not produce Newtonian Gravity.

One might of course now ask whether (2.9) really is the correct action for Galilei Gravity. We claim that it is, and this will be further justified in chapter 4, where we produce Galilei Gravity as the non-relativistic limit of General Relativity. This limit can even be taken in the action, precisely producing (2.9). So in that sense, this action is not only the simplest invariant action we can write down, but it is also really the non-relativistic version of the Einstein-Hilbert action.

So, while we started out with the underlying symmetries of Newtonian physics, the Galilean Group, and proceeded in a very similar fashion to the gauging procedure for General Relativity, we end up with a theory that does not correspond to Newtonian Gravity. Instead, the theory is underdetermined and may produce wildly varying geometries for different choices of A^{ab} and ${\omega_\mu}^a$. This deficiency will be remedied in the next chapter where we consider Newton-Cartan Gravity. This theory works with a central extension of the Galilei Group, thereby introducing another gauge field. It turns out that (together with appropriate curvature constraints) this gauge field allows us to now uniquely solve for the spin- and boost-connections and thereby fixes the geometry. This then finally allows us to interpret the geodesics of the corresponding covariant derivative as the worldlines of free-falling particles in a Newtonian gravitational field.

CHAPTER 3

Newton-Cartan Gravity

In the previous chapter we attempted to produce a theory of Newtonian gravity from a gauging procedure starting with the Galilei Algebra. The rationale behind this was that the Galilean Group exactly consists of the transformations under which the laws of Newtonian physics are invariant. However, we ended up with an underdetermined system of field equations. In particular, the solution of ${\omega_\mu}^{ab}$ in terms of the vielbeins was only determined up to a term $\tau_\mu A^{ab}$ and we were completely unable to solve for ${\omega_\mu}^a$. This makes the resulting theory very different from "regular" Newtonian gravity.

Luckily, this deficiency can be remedied by the introduction of another gauge field. This of course requires that we start out with a larger algebra. This will be the so-called Bargmann Algebra, which is a central extension of the Galilei Algebra. When we perform a gauging procedure on the Bargmann Algebra (as we will do in this chapter), we will end up with the well-known Newton-Cartan Gravity.

Newton-Cartan Gravity is a geometrical representation of the classical Newtonian Gravity, but contrary to its classical counterpart its formulation is independent of the choice of coordinate systems. It has a similar structure as General Relativity. However, in this case it is not possible to treat time and space at the same footing, i.e. in one metric, since time has a very special postion in Newtonian physics in the sense that there is a concept of global time. Hence, we shall need to consider two "metrics": one for space, and one for time. Just like in General Relativity, the paths of free-falling particles (in a gravitational field) will then correspond to the geodesics of a covariant derivative that is compatible with these two metrics.

In this chapter we review how Newton-Cartan Gravity can be obtained as a gauge theory, similar to General Relativity and Galilei Gravity. We will be closely following [**3**], performing a gauging procedure similar to the previous two chapters. We shall begin by writing down the Bargmann Algebra, and follow a similar procedure as in the last chapter. The equivalence of the resulting theory with Newtonian Gravity is demonstrated in Appendix A. The downside of Newton-Cartan Gravity is that as of the moment of writing there is no known action that produces all field equations. At the end of this chapter we will discuss

1. The Gauge Fields

We wish to obtain Newton-Cartan theory from a gauging procedure in the same way as we obtained General Relativity and Galilei Gravity. As we saw in the last chapter, gauging the Galilei Algebra does not suffice. The shortcoming of gauging the Galilei Algebra lies in the fact that it leaves a degree of freedom in determining the covariant derivative from the gauge fields. This indeterminacy is a motivation for the introduction of an extra gauge field.

We are therefore led to extend the Galilei algebra, and instead gauge the so-called Bargmann algebra $\mathfrak{b}(d,1)$. This is essentially the Galilean algebra, but extended by an extra generator that commutes with all other generators (a central extension). So, we have the generators for temporal translations (H with parameter ζ), spatial translations (P_a with parameter ζ^a), boosts (G_a with parameter λ^a), rotations (J_{ab} with parameter λ^{ab}) and the central generator (M with parameter σ). The non-zero commutators of this algebra are:

$$\begin{aligned} [P_a, J_{bc}] &= 2\delta_{a[b}P_{c]}, \qquad [G_a, J_{bc}] = 2\delta_{a[b}G_{c]}, \\ [J_{ab}, J_{cd}] &= 4\delta_{[a[c}J_{d]b]}, \qquad [H, G_a] = P_a, \\ [P_a, G_b] &= \delta_{ab}M. \end{aligned} \tag{3.1}$$

Notice that while M commutes with all other generators, it does appear as the right-hand side of the commutator of spatial translations and boost. This can be interpreted by saying that a translation and a boost differ by some transformation that commutes with all other transformations. We might consider this as the manifestation of inertia. The (conserved) charge that corresponds to the central generator M can then be viewed as a measure of the number of particles minus antiparticles of the system.

Let us introduce the gauge fields that correspond to these generators. Again, we do this by introducing a Cartan connection α_μ with values in $\mathfrak{b}(d,1)$[1]:

$$\alpha_\mu = m_\mu M + \tau_\mu H + e_\mu{}^a P_a + g_\mu{}^a G_a + \frac{1}{2}\omega^{ab} J_{ab}. \tag{3.2}$$

These fields then transform as:

$$\delta\tau_\mu = \partial_\mu \zeta, \tag{3.3a}$$

$$\delta e_\mu{}^a = \partial_\mu \zeta^a - \omega_\mu{}^a{}_b \zeta^b + \lambda^a{}_b e_\mu{}^b + \zeta\omega_\mu{}^a - \lambda^a \tau_\mu, \tag{3.3b}$$

$$\delta\omega_\mu{}^a = \partial_\mu \lambda^a - \omega_\mu{}^a{}_b \lambda^b + \lambda^a{}_b \omega_\mu{}^b, \tag{3.3c}$$

$$\delta\omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} - \omega_\mu{}^a{}_c \lambda^{cb} - \omega_\mu{}^b{}_c \lambda^{ac}, \tag{3.3d}$$

$$\delta m_\mu = \partial_\mu \sigma - \lambda_a e_\mu{}^a + \zeta_a \omega_\mu{}^a. \tag{3.3e}$$

All components of our gauging procedure are listed in Table 1.

We now interpret $e_\mu{}^a$ as a "spatial" frame, and τ_μ as a "temporal" frame, able to measure temporal intervals, whereas the fields $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ will be interpreted as the spin- and boost-connection. Again we will require that together they form a complete vielbein for the manifold as a whole. We

[1] Again, the factor of $\frac{1}{2}$ in front of the J_{ij} is to compensate for the antisymmetricity of J_{ij}.

Symmetry	Generators	Fields	Parameters	Curvatures
Temporal translation	H	τ_μ	ζ	$R_{\mu\nu}(H)$
Spatial translations	P_a	${e_\mu}^a$	ζ^a	${R_{\mu\nu}}^a(P)$
Boosts	G_a	${\omega_\mu}^a$	λ^a	${R_{\mu\nu}}^a(G)$
Rotations	J_{ab}	${\omega_\mu}^{ab}$	λ^{ab}	${R_{\mu\nu}}^{ab}(J)$
Central generator	M	m_μ	σ	$R_{\mu\nu}(M)$

TABLE 1. All components that arise in the gauging procedure of the Bargmann Algebra to obtain Newton-Cartan Gravity.

therefore introduce the dual fields by their inversion properties:

$$
\begin{aligned}
&{e_\mu}^a {e^\mu}_b = {\delta^a}_b && \tau^\mu \tau_\mu = 1 \\
&\tau^\mu {e_\mu}^a = 0 && \tau_\mu {e^\mu}_a = 0 \\
&{e_\mu}^a {e^\nu}_a = {\delta_\mu}^\nu - \tau_\mu \tau^\nu .
\end{aligned} \tag{3.4}
$$

These inverse fields again transform as:

$$\delta\tau^\mu = -\tau^\mu\tau^\nu\delta\tau_\nu - {e^\mu}_b\tau^\nu\delta{e_\nu}^b, \tag{3.5a}$$

$$\delta{e^\mu}_a = -{e^\mu}_b{e^\nu}_a\delta{e_\nu}^b - \tau^\mu{e^\nu}_a\delta\tau_\nu . \tag{3.5b}$$

2. Curvature and Constraints

In complete parallel with the previous chapters, we calculate the curvature of the connection $R_{\mu\nu} = 2\partial_{[\mu}\alpha_{\nu]} + [\alpha_{[\mu}, \alpha_{\nu]}]$ to obtain the curvatures of the gauge fields:

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]} , \tag{3.6a}$$

$${R_{\mu\nu}}^a(P) = 2\partial_{[\mu}{e_{\nu]}}^a + 2{\omega_{[\mu}}^{ab}e_{\nu]b} - 2{\omega_{[\mu}}^a\tau_{\nu]} , \tag{3.6b}$$

$${R_{\mu\nu}}^a(G) = 2\partial_{[\mu}{\omega_{\nu]}}^a + 2{\omega_{[\mu}}^{ab}\omega_{\nu]b}, \tag{3.6c}$$

$${R_{\mu\nu}}^{ab}(J) = 2\partial_{[\mu}{\omega_{\nu]}}^{ab} + 2{\omega_{[\mu}}^{bc}{\omega_{\nu]}}^a{}_c, \tag{3.6d}$$

$$R_{\mu\nu}(M) = 2\partial_{[\mu}m_{\nu]} + 2{e_{[\mu}}^a\omega_{\nu]a}. \tag{3.6e}$$

These curvatures transform as:

$$\delta R_{\mu\nu}(H) = 0, \tag{3.7a}$$

$$\delta{R_{\mu\nu}}^a(P) = \zeta{R_{\mu\nu}}^a(G) - \lambda^a R_{\mu\nu}(H) + {\lambda^a}_b{R_{\mu\nu}}^b(P) - \zeta^b{R_{\mu\nu}}^a{}_b(J), \tag{3.7b}$$

$$\delta{R_{\mu\nu}}^a(G) = {\lambda^a}_b{R_{\mu\nu}}^b(P) - \lambda^b{R_{\mu\nu}}^a{}_b(J), \tag{3.7c}$$

$$\delta{R_{\mu\nu}}^{ab}(J) = 2\lambda^{c[a}{R_{\mu\nu}}^{b]}{}_c(J), \tag{3.7d}$$

$$\delta R_{\mu\nu}(M) = \zeta_a{R_{\mu\nu}}^a(G) - \lambda_a{R_{\mu\nu}}^a(P). \tag{3.7e}$$

Wishing to make contact with classical gravity (see Appendix A), we introduce the spatial metric h by setting $h_{\mu\nu} = \delta_{ij} e_\mu{}^i e_\nu{}^j$. h allows us to measure distances, whereas τ lets us measure temporal intervals. We then wish to introduce a torsion-free covariant derivative that is compatible with h and τ and interpret $\omega_\mu{}^i$ and $\omega_\mu{}^{ij}$ as the components of the spin-connection.[2] We therefore introduce the following vielbein postulates:

$$\begin{cases} \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\mu{}^a \tau_\nu - \Gamma_\mu{}^\rho{}_\nu e_\rho{}^a = 0, \\ \partial_\mu \tau_\nu - \Gamma_\mu{}^\rho{}_\nu \tau_\rho = 0. \end{cases} \tag{3.8}$$

If we are to be able to solve for $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ in terms of the vielbeins, we see from (3.7) that we must impose the conventional constraints:

$$\begin{cases} R_{\mu\nu}{}^a(P) = 0, \\ R_{\mu\nu}(M) = 0. \end{cases} \tag{3.9}$$

This indeed allows us to obtain the solutions:

$$\omega_\mu{}^{ab} = 2e^{\nu[a}\partial_{[\mu} e_{\nu]}{}^{b]} - e_{\mu c} e^{\rho a} e^{\sigma b} \partial_{[\rho} e_{\sigma]}{}^c + \tau_\mu e^{\rho[a} \omega_\rho{}^{b]}, \tag{3.10}$$

$$\omega_\mu{}^a = e^{\nu a}\partial_{[\mu} m_{\nu]} + \tau^\nu \partial_{[\mu} e_{\nu]}{}^a + e^{\rho a} \tau^\sigma e_{\mu b} \partial_{[\rho} e_{\sigma]}{}^b + \tau_\mu \tau^\rho e^{\sigma a} \partial_{[\rho} m_{\sigma]}. \tag{3.11}$$

By antisymmetrizing (3.8) in μ and ν, we can also see that these constraints are equivalent to the statement that the resulting covariant derivative has zero torsion.

We will also impose the following geometrical constraint:

$$R_{\mu\nu}(H) = 0. \tag{3.12}$$

This constraint implies that $\mathrm{d}\tau = 0$, so that we may write $\tau_\mu = \partial_\mu f$ for some global function f. This function will be designated as *global time*. This also induces a foliation of the spacetime, seperating it in one purely temporal and d purely spatial directions. The existence of such a global time function is defining of Newtonian mechanics. However, it is not strictly necessary to impose this condition. Indeed, one may still obtain this foliation by imposing the less restrictive constraint $e^\mu{}_a e^\nu{}_b R_{\mu\nu}(H) = 0$. This is known as Newton-Cartan with twistless torsion, and it still has the notion of a global ordering of events and global agreement of simultaneity. For a broader discussion, see [**4**].

Another motivation for introducing these three curvature constraints is that they allow us to replace the H-, P- and M-transformations by general coordinate transformations, inherently making the theory invariant under a general change of coordinates. For a full discussion see [**1**, ch. 4].

We now want to solve for the Christoffel symbols in terms of the vielbein fields. These equations can be solved uniquely to give:

$$\Gamma^\rho_{\mu\nu} = \tau^\rho \partial_{(\mu}\tau_{\nu)} + e^\rho{}_a \left(\partial_{(\mu} e_{\nu)}{}^a + \omega_{(\mu}{}^a{}_b e_{\nu)}{}^b + \omega_{(\mu}{}^a \tau_{\nu)} \right). \tag{3.13}$$

---

[2] This is conceptually similar to the Levi-Civita connection, but instead of being compatible with a single global metric, we now need the connection to be compatible with two degenerate metrics that together foliate the entire space into a temporal and spatial part.

This covariant derivative is derived from the explicit compatibility with the vielbeins. However, we actually only require compatibility with the spatial metric $h^{\mu\nu} = e^{\mu}{}_{a}e^{\nu}{}_{b}\delta^{ab}$. That compatibility allows a shift in the Christoffel symbols:

$$\Gamma_{\mu}{}^{\rho}{}_{\nu} \to \Gamma_{\mu}{}^{\rho}{}_{\nu} + h^{\rho\sigma}K_{\sigma(\mu}\tau_{\nu)} \tag{3.14}$$

We then find that we must have:

$$K_{\mu\nu} = 2\omega_{[\mu}{}^{a}e_{\nu]a} = 2\partial_{[\mu}m_{\nu]}, \tag{3.15}$$

(where the second equality follows from $R_{\mu\nu}(M) = 0$). So, we see that the gauge field m_{μ} makes this extra degree of freedom (which also caused the indeterminacy in Galilei Gravity) explicit. By promoting this to a gauge field, we automatically have a mechanism that fixes this degree of freedom when a particular gauge is chosen.

3. Field Equations and (Lack of an) Action

We are now ready to put this theory on shell and make full contact with Newtonian Gravity. First we need to impose one more curvature constraint. In Appendix A we show that in order to make contact with Newtonian Gravity, we need the Ehler's condition (A.15). This condition is easily seen to be equivalent to the fourth curvature constraint:

$$R_{\mu\nu}{}^{ab}(J) = 0. \tag{3.16}$$

This constraint implies (through the Bianchi identities of the curvature) that the only non-zero projection of $R_{\mu\nu}{}^{a}(G)$ is [**1**]:

$$\tau^{\mu}e^{\nu(a}R_{\mu\nu}{}^{a)}(G) = \delta^{c(a}R^{b)}{}_{0c0}(\Gamma), \tag{3.17}$$

where we set the right-hand side to make the connection with (A.13). We can again make use of the Bianchi identity to transform this into the full set of equations of motion:

$$\tau^{\mu}e^{\nu}{}_{a}R_{\mu\nu}{}^{a}(G) = 4\pi G\rho, \qquad \tau^{\mu}e^{\nu}{}_{a}R_{\mu\nu}{}^{ab}(J) = 0. \tag{3.18}$$

Of course, we already set $R_{\mu\nu}{}^{ab}(J)$ to be completely zero as a constraint, but we must make a careful distinction between the constraints on the curvatures and equations of motion.

This then finally is a complete description of Newton-Cartan Gravity. At this point we wish to ask whether there is an action that produces these equations of motion and all the curvature constraints. Unfortunately, as of yet we are unable to provide such an action. One might think that an action similar to the Galilei action (2.9) could suffice, but this is clearly not the case, since this action does not contain m_{μ} at all and thus will not produce any equations of motion for this field.

However, we know that Newtonian Gravity (of which Newton-Cartan Gravity is just a geometric reformulation) is just the non-relativistic limit of General Relativity for which we do have an action available. This motivates us to consider procedures that produce Newton-Cartan Gravity as a non-relativistic approximation to General Relativity, and have the potential to also produce an appropriate action. This attempt will ultimately prove to be fruitless, but the reasons why these procedures fail to produce an action are profound and give us a hint at the underlying reason why there might not be an action for Newton-Cartan Gravity at all.

Part 2

Algebra Contractions as Limits

CHAPTER 4

General Relativity to Galilei

Having discussed General Relativity and the two "classical" theories of gravity as gauge theories, we are now ready to make the connection between them. In particular, we will show that one can view the classical theories as a certain limit of General Relativity, where the speed of light is very large compared to the typical velocities involved.

The limit procedure that we will use will be the Inönü-Wigner Contraction. Such a contraction consists of an invertible linear transformation on a Lie algebra, dependent om some parameter ω. We then calculate the resulting structure constants in the new basis, and send the parameter to infinity. In this last step, the structure constants (still dependent on ω) should converge, effectively yielding a new algebra. The parameter ω will represent the speed of light (relative to typical speeds in the system). One can then view this as taking the low-speed limit.

The advantage of this procedure is that it automatically induces a limit for the fields, and thus has the potential to produce a convergent limit for the field equations and even for the action. This procedure is thus a serious candidate for producing an action for Newton-Cartan Gravity. We will however start out with a simpler contraction that produces Galilei Gravity, as this also illustrates that it is indeed possible to obtain an action this way.

1. The Galilei Algebra as a Contraction

Since we want to view classical gravity as a limit of General Relativity, we will make a contraction of the Poincaré Algebra. We will start with the most simple contraction that we can imagine, which will produce the Galilei Algebra.

One of the defining features of Newtonian Gravity is the concept of global time. Motivated by this, we want to single out a timelike direction. Noting that in general relavity distance and time are related by the speed of light as $x^0 = ct$, we make a corresponding transformation in the generator for temporal translations P_0 and the generators for the boosts (which are in a sense timelike Lorentz transformations). Concretely we have:

$$\hat{P}_0 = \omega^{-1}\tilde{H}, \quad \hat{P}_a = \tilde{P}_a, \quad \hat{J}_{a0} = -\hat{J}_{0a} = \omega\tilde{G}_a, \quad \hat{J}_{ab} = \tilde{J}_{ab} \tag{4.1}$$

The inverse of this transformation is given by:

$$\tilde{H} := \omega\hat{P}_0, \quad \tilde{P}_a := \hat{P}_a, \quad \tilde{G}_a := \frac{1}{\omega}\hat{J}_{a0}, \quad \tilde{J}_{ab} := \hat{J}_{ab}. \tag{4.2}$$

It is a simple exercise to now calculate the commutators of these new generators by just writing out the above transformations. The non-zero commutators are:

$$\begin{aligned} [\tilde{P}_a, \tilde{J}_{bc}] &= 2\delta_{a[b}\tilde{P}_{c]}, & [\tilde{G}_a, \tilde{J}_{bc}] &= 2\delta_{a[b}\tilde{G}_{c]}, \\ [\tilde{P}_a, \tilde{G}_b] &= \omega^{-2}\delta_{ab}\tilde{H}, & [\tilde{G}_a, \tilde{G}_b] &= \omega^{-2}\tilde{J}_{ab}, \\ [\tilde{J}_{ab}, \tilde{J}_{cd}] &= 4\delta_{[a[c}\tilde{J}_{d]b]}, & [\tilde{H}, \tilde{G}_a] &= \tilde{P}_a. \end{aligned} \tag{4.3}$$

For any finite ω this is of course still the same Lie algebra, just written out in a different basis. The big change occurs when we now take the limit $\omega \to \infty$. As the structure constants that occur in the commutators all converge, this gives us a new algebra. As a notational convention we will use tildes to denote redefined generators before taking the limit, and we will drop these tildes when we have taken the limit and are thus working in the new algebra. The same convention will be followed in the corresponding fields.

If we take the limit $\omega \to \infty$ and drop the tildes, we obtain:

$$\begin{aligned} [P_a, J_{bc}] &= 2\delta_{a[b}P_{c]}, & [G_a, J_{bc}] &= 2\delta_{a[b}G_{c]}, \\ [J_{ab}, J_{cd}] &= 4\delta_{[a[c}J_{d]b]}, & [H, G_a] &= P_a. \end{aligned} \tag{4.4}$$

As we anticipated, this is indeed the Galilei Algebra!

2. Limit of the Fields and Curvatures

When gauging the Poincaré Algebra, our prime instrument was the Cartan connection α_μ through which we defined the gauge fields $E_\mu{}^A$ and $\Omega_\mu{}^{AB}$. Let us see what this connection looks like in terms of our new generators:

$$\begin{aligned} \alpha_\mu &= E_\mu{}^A \hat{P}_A + \frac{1}{2}\Omega_\mu{}^{AB}\hat{J}_{AB} \\ &= \underbrace{\omega^{-1}E_\mu{}^0}_{\tilde{\tau}_\mu}\tilde{H} + \underbrace{E_\mu{}^a}_{\tilde{e}_\mu{}^a}\tilde{P}_a + \underbrace{\omega\Omega_\mu{}^{a0}}_{\tilde{\omega}_\mu{}^a}\tilde{G}_a + \frac{1}{2}\underbrace{\Omega_\mu{}^{ab}}_{\tilde{\omega}_\mu{}^{ab}}\tilde{J}_{ab}. \end{aligned} \tag{4.5}$$

The fields of our theory are nothing more than the components of this connection corresponding to the different generators of the algebra. We are therefore led to define a redefinition of the fields that is dual to the redefinition of the generators. This transformation is given by:

$$E_\mu{}^0 = \omega\tilde{\tau}_\mu, \quad E_\mu{}^a = \tilde{e}_\mu{}^a, \quad \Omega_\mu{}^{a0} = -\Omega_\mu{}^{0a} = \omega^{-1}\tilde{\omega}_\mu{}^a, \quad \Omega_\mu{}^{ab} = \tilde{\omega}_\mu{}^{ab} \tag{4.6}$$

This redefinition exactly produces the fields that keep the connection form invariant. The inverse transformation is given by:

$$\tilde{\tau}_\mu = \omega^{-1}E_\mu{}^0, \quad \tilde{e}_\mu{}^a = E_\mu{}^a, \quad \tilde{\omega}_\mu{}^a = \Omega_\mu{}^{a0}, \quad \tilde{\omega}_\mu{}^{ab} = \Omega_\mu{}^{ab}. \tag{4.7}$$

This redefinition will serve us as the definition of the new fields before we have taken the limit. Again, we carefully place tildes on the fields before taking the limit, when ω is still finite. After we take the limit $\omega \to \infty$, we drop the tildes on these fields, indicating that they belong to the Galilei algebra. Our strategy for taking limits in objects that are build up from the fields (such as the the curvatures, the field equations, and the action) will be to just begin with the expression in terms of the GR fields and fill in the redefinition. This will give us a new expression in terms of the fields with the tildes and powers of ω. The limit $\omega \to \infty$ then converges only when we have no positive powers of ω. If this is the case, we can then finally take the limit by setting terms containing negative powers of ω to zero, and dropping the tildes on the fields.

We will calculate the first limit in the curvatures of the fields. If the fields without the tildes are really to belong to the Galilei algebra, they must of course have the same curvatures. We will now show that we can in fact take the limit of the curvatures, and that these indeed converge to the curvatures of the Galilei algebra.

The curvatures undergo a similar transformation as the fields by requiring that the total curvature of the Cartan connection remains invariant under the redefinition:

(4.8)
$$\begin{aligned} R_{\mu\nu} &= R_{\mu\nu}{}^{A}(\hat{P})\hat{P}_A + \frac{1}{2}R_{\mu\nu}{}^{AB}(\hat{J})\hat{J}_{AB} \\ &= \underbrace{\omega^{-1}R_{\mu\nu}{}^{0}(\hat{P})}_{R_{\mu\nu}(\tilde{H})}\,\tilde{H} + \underbrace{R_{\mu\nu}{}^{a}(\hat{P})}_{R_{\mu\nu}{}^{a}(\tilde{P})}\,\tilde{P}_a + \underbrace{\omega R_{\mu\nu}{}^{a0}(\hat{J})}_{R_{\mu\nu}{}^{a}(\tilde{G})}\,\tilde{G}_a + \frac{1}{2}\underbrace{R_{\mu\nu}{}^{ab}(\hat{J})}_{R_{\mu\nu}{}^{ab}(\tilde{J})}\,\tilde{J}_{ab}. \end{aligned}$$

Again, this leads us to redefine the curvatures as follows:

$$\begin{aligned} R_{\mu\nu}{}^{0}(\hat{P}) = \omega R_{\mu\nu}(\tilde{H}), \qquad R_{\mu\nu}{}^{a}(\hat{P}) = R_{\mu\nu}{}^{a}(\tilde{P}), \\ R_{\mu\nu}{}^{a0}(\hat{J}) = -R_{\mu\nu}{}^{0a}(\hat{J}) = \omega^{-1}R_{\mu\nu}{}^{a}(\tilde{G}), \qquad R_{\mu\nu}{}^{ab}(\hat{J}) = R_{\mu\nu}{}^{ab}(\tilde{J}). \end{aligned} \tag{4.9}$$

The inverse of these redefinitions allows us to calculate the curvatures of the redefined fields. We can do that by simply filling in the curvatures of GR, and replacing the fields of GR by the redefined fields (4.6). We can then take the limit as described above. For example, for $R_{\mu\nu}(\tilde{H})$ we can calculate:

$$\begin{aligned} R_{\mu\nu}(\tilde{H}) &= \omega^{-1}R_{\mu\nu}{}^{0}(\hat{P}) = \omega^{-1}\left(2\partial_{[\mu}E_{\nu]}{}^{0} + 2\Omega_{[\mu}{}^{0a}E_{\nu]a}\right) \\ &= 2\partial_{[\mu}\tilde{\tau}_{\nu]} + 2\omega^{-2}\tilde{e}_{[\mu}{}^{a}\tilde{\omega}_{\nu]b} \\ &\to 2\partial_{[\mu}\tau_{\nu]}\,. \end{aligned} \tag{4.10}$$

Note that in the first term, the ω^{-1} gets absorbed by the $\tilde{\tau}_\nu$, whereas in the second term another factor of ω^{-1} is introduced by writing out $\Omega_\mu{}^{0a}$ using (4.6). So this indeed produces the correct curvature $R_{\mu\nu}(H)$ of the Galilei algebra.

Similarly, we can calculate:

$$\begin{aligned} R_{\mu\nu}{}^{a}(\tilde{P}) = R_{\mu\nu}{}^{a}(\hat{P}) &= 2\partial_{[\mu}E_{\nu]}{}^{a} + 2\Omega_{[\mu}{}^{aB}E_{\nu]B} \\ &= 2\partial_{[\mu}\tilde{e}_{\nu]}{}^{a} + 2\tilde{\omega}_{[\mu}{}^{ab}\tilde{e}_{\nu]b} + 2\tilde{\tau}_{[\mu}\tilde{\omega}_{\nu]}{}^{a} \\ &\to 2\partial_{[\mu}e_{\nu]}{}^{a} + 2\omega_{[\mu}{}^{ab}e_{\nu]b} + 2\tau_{[\mu}\omega_{\nu]}{}^{a}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} R_{\mu\nu}{}^{a}(\tilde{G}) = \omega R_{\mu\nu}{}^{a0}(\tilde{J}) &= \omega\left(2\partial_{[\mu}\Omega_{\nu]}{}^{a0} + 2\Omega_{[\mu}{}^{aC}\Omega_{\nu]C}{}^{0}\right) \\ &= 2\partial_{[\mu}\tilde{\omega}_{\nu]}{}^{a} + 2\tilde{\omega}_{[\mu}{}^{ac}\tilde{\omega}_{\nu]c} \\ &\to 2\partial_{[\mu}\omega_{\nu]}{}^{a} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} R_{\mu\nu}{}^{ab}(\tilde{J}) &= R_{\mu\nu}{}^{ab}(\tilde{J}) = 2\partial_{[\mu}\Omega_{\nu]}{}^{ab} + 2\Omega_{[\mu}{}^{aC}\Omega_{\nu]C}{}^{b} \\ &= 2\partial_{[\mu}\tilde{\omega}_{\nu]}{}^{ab} + 2\Omega_{[\mu}{}^{ac}\Omega_{\nu]c}{}^{b} - 2\Omega_{[\mu}{}^{a0}\Omega_{\nu]}{}^{0b} \\ &= 2\partial_{[\mu}\tilde{\omega}_{\nu]}{}^{ab} + 2\tilde{\omega}_{[\mu}{}^{ac}\tilde{\omega}_{\nu]c}{}^{b} + 2\omega^{-2}\tilde{\omega}_{[\mu}{}^{a}\tilde{\omega}_{\nu]}{}^{b} \\ &\to 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b}. \end{aligned} \tag{4.13}$$

This shows that the curvatures really converge to those of the Galilei algebra.

In chapter 2 we introduced an inverse set of fields $(e^{\mu}{}_{a}, \tau^{\mu})$. We can also do that for the fields before we take the limit. These fields are instrumental when taking the limit in expressions that contain the inverse GR vielbein $E^{\mu}{}_{A}$ such as the action and field equations. We can introduce these pre-limit inverses again by defining them by their invertion properties:

$$\begin{gathered} \tilde{e}^{\mu}{}_{a}\tilde{e}_{\mu}{}^{b} = \delta_{a}^{b}, \qquad \tilde{\tau}^{\mu}\tilde{\tau}_{\mu} = 1, \qquad \tilde{e}^{\mu}{}_{a}\tilde{e}_{\nu}{}^{a} + \tilde{\tau}^{\mu}\tilde{\tau}_{\nu} = \delta_{\nu}^{\mu}, \\ \tilde{\tau}^{\mu}\tilde{e}_{\mu}{}^{a} = 0, \qquad \tilde{\tau}_{\mu}\tilde{e}^{\mu}{}_{a} = 0. \end{gathered} \tag{4.14}$$

We can calculate how these fields relate to the relativistic inverse vielbein $E^{\mu}{}_{a}$ by:

$$\begin{aligned} E^{\mu}{}_{A} &= E^{\nu}{}_{A}\delta_{\nu}^{\mu} = E^{\nu}{}_{A}\left(\tilde{e}^{\mu}{}_{b}\tilde{e}_{\nu}{}^{b} + \tilde{\tau}^{\mu}\tilde{\tau}_{\nu}\right) \\ &= \tilde{e}^{\mu}{}_{b}E^{\nu}{}_{A}E_{\nu}{}^{b} + \tilde{\tau}^{\mu}E^{\nu}{}_{A}\omega^{-1}E_{\nu}{}^{0} \\ &= \delta_{A}^{b}\tilde{e}^{\mu}{}_{b} + \delta_{A}^{0}\omega^{-1}\tilde{\tau}^{\mu}. \end{aligned} \tag{4.15}$$

Note that this is precisely dual to the redefinition (4.6), without any terms lower than ω^{-1}. That allows us to write down a lot of exact expressions without considering lower orders of ω.. This simplicity turns out to be a feature particular to the contraction to the Galilei algebra, which we will see in the next chapter, where the more intrigate redefinitions we will have to make to arrive at the Bargmann algebra lead to a mixing of fields in these inverse fields that make the expression a lot more complicated.

3. Limit of the action

Having discussed how the fields transform under our redefinitions (4.1), and how we can take limits in expressions with these new fields, we will immediately aim at the big prize: taking the limit of the Einstein-Hilbert action (1.24).

The prime components of the Einstein-Hilbert action are the determinant $E = \det(E_{\mu}{}^{A})$ and the trace $R = E^{\mu}{}_{A}E^{\nu}{}_{B}R_{\mu\nu}{}^{AB}(\hat{J})$. By filling in the redefined fields and the expansion of the inverse GR fields in the inverse Galilei fields, we obtain:

$$E = \det(E_{\mu}{}^{A}) = \det(\omega\tilde{\tau}^{\mu}, \tilde{e}^{\mu}{}_{a}) = \omega\det(\tilde{\tau}^{\mu}, \tilde{e}^{\mu}{}_{a}) = \omega\tilde{e}, \tag{4.16}$$

where we have defined $\tilde{e} := \det(\tilde{\tau}^{\mu}, \tilde{e}^{\mu}{}_{a})$, and:

$$\begin{aligned} R &= E^{\mu}{}_{A}E^{\nu}{}_{B}R_{\mu\nu}{}^{AB}(\hat{J}) \\ &= \tilde{e}^{\mu}{}_{a}\tilde{e}^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(\tilde{J}) + 2\omega^{-2}\tilde{e}^{\mu}{}_{a}\tilde{\tau}^{\nu}R_{\mu\nu}{}^{a}(\tilde{G}) \end{aligned} \tag{4.17}$$

So when we put this together in the Einstein-Hilbert action, we get:

$$\begin{aligned} S &= -\frac{1}{2\kappa}\int E R \, \mathrm{d}^D x \\ &= -\frac{1}{2\kappa}\int \omega e \left(\tilde{e}^\mu{}_a \tilde{e}^\nu{}_b R_{\mu\nu}{}^{ab}(\tilde{J}) + 2\omega^{-2}\tilde{e}^\mu{}_a \tilde{\tau}^\nu R_{\mu\nu}{}^a(\tilde{G})\right) \mathrm{d}^D x. \end{aligned} \tag{4.18}$$

In order to obtain convergence, we need to rescale the gravitational constant: $G_G := G_N/\omega$, or equivalently $\kappa_G := \kappa/\omega$. Then finally taking the limit (while treating κ_G as a constant), we get:

$$S = \frac{1}{2\kappa_G}\int e e^\mu{}_a e^\nu{}_b . R_{\mu\nu}{}^{ab}(J) \tag{4.19}$$

This is indeed the action (2.9) that we wrote down for Galilei Gravity and from which we were able to derive its equations of motion.

The results of this procedure seem very promising, since they produced an entire non-relativistic version of gravity by starting the limit at a geometrical level, and allowing us to proceed up to the level of the action. However, this does clearly not produce gravity as we see it in our day to day lives, i.e. Newtonian Gravity. For that we really need Newton-Cartan Gravity with it's extra fields. The natural question than arises whether we can produce Newton-Cartan Gravity through a similar procedure, and whether this also produces an action. We will address these questions in the next chapter.

4. Limit of the field equations

Now that we have produced Galilei Gravity by taking the limit $\omega \to \infty$ at the level of the action, we might ask whether taking the same limit at the level of the Einstein field equations together with the constraint $R_{\mu\nu}{}^A(\hat{P}) = 0$ produces the same equations of motion as those that we derive from the Galilei action. To answer this question, let us begin by considering the limit of the curvature constraint. Through (4.11) it is clear that the curvature constraint readily translates to constraints on these new fields. If we write this out for $A = a$, we get:

$$R_{\mu\nu}{}^a(\tilde{P}) = R_{\mu\nu}{}^a(\hat{P}) = 0. \tag{4.20}$$

When we drop the tilde, this gives us the constraint $R_{\mu\nu}{}^a(P) = 0$. Doing the same for $A = 0$, we obtain:

$$R_{\mu\nu}(\tilde{H}) = \omega^{-1}R_{\mu\nu}{}^0(\hat{P}) = \omega^{-1}0 = 0. \tag{4.21}$$

Again dropping the tilde, this is a stronger version of the geometrical constraint (2.11b) that provides us with absolute time, instead of just a foliation. Note that it is very important that the identities (4.20) and (4.21) hold for any finite ω, so that we may indeed conclude that in the limit both curvatures become zero.

We now move on to the limit of the (vacuum) Einstein field equations:

$$E^\rho{}_A E^\sigma{}_C R_{\rho\sigma}{}^C{}_B(\hat{J}) + \frac{1}{2}\eta_{AB} E^\rho{}_C E^\sigma{}_D R_{\rho\sigma}{}^{CD}(\hat{J}) = 0. \tag{4.22}$$

We can simply fill in (4.15) and then look at the various values for the indices A and B. When we take $A = 0, B = 0$, we get:

$$\tilde{e}^{\rho}{}_{c}\tilde{e}^{\sigma}{}_{d}R_{\rho\sigma}{}^{cd}(\tilde{J}) + 4\omega^{-2}\tilde{\tau}^{\rho}\tilde{e}^{\sigma}{}_{c}R_{\rho\sigma}{}^{c}(\tilde{G}) = 0. \tag{4.23}$$

Taking the limit $\omega \to \infty$ causes the second term to vanish. Then dropping the tildes, we obtain:

$$e^{\mu}{}_{a}e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(J) = 0. \tag{4.24}$$

The equation for $A = a, B = 0$ gives:

$$\omega^{-1}\tilde{e}^{\mu}{}_{a}\tilde{e}^{\nu}{}_{b}R_{\mu\nu}{}^{b}(\tilde{G}) = 0. \tag{4.25}$$

Note that since this must hold for any finite ω, we can multiply by a factor of ω and then take the limit. This gives us:

$$\boxed{e^{\mu}{}_{a}e^{\nu}{}_{b}R_{\mu\nu}{}^{b}(G) = 0.} \tag{4.26}$$

Taking $A = 0, B = b$, we then obtain:

$$\omega^{-1}\tilde{\tau}^{\mu}\tilde{e}^{\nu}{}_{a}R_{\mu\nu}{}^{ab}(\tilde{J}) - \omega^{-3}\tilde{\tau}^{\mu}\tilde{\tau}^{\nu}R_{\mu\nu}{}^{b}(\tilde{G}) = 0. \tag{4.27}$$

Again multiplying by ω and then taking the limit, we obtain:

$$\tau^{\mu}e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(J) = 0, \tag{4.28}$$

The final set of equations is given by $A = a, B = b$, from which we obtain:

(4.29)

$$\tilde{e}^{\mu}{}_{a}\tilde{e}^{\nu}{}_{c}R_{\mu\nu}{}^{cb}(\tilde{J})+\frac{1}{2}\delta^{b}_{a}\tilde{e}^{\mu}{}_{c}\tilde{e}^{\nu}{}_{d}R_{\mu\nu}{}^{cd}(\tilde{J})-\omega^{-2}\left(\tilde{e}^{\mu}{}_{a}\tilde{\tau}^{\nu}R_{\mu\nu}{}^{b}(\tilde{G}) - \delta^{b}_{a}\tilde{e}^{\mu}{}_{c}\tilde{\tau}^{\nu}R_{\mu\nu}{}^{c}(\tilde{G})\right) = 0.$$

Taking the limit now gives us:

$$e^{\mu}{}_{a}e^{\nu}{}_{c}R_{\mu\nu}{}^{bc}(J) + \frac{1}{2}\delta^{b}_{a}e^{\mu}{}_{c}e^{\nu}{}_{d}R_{\mu\nu}{}^{cd}(J) = 0. \tag{4.30}$$

This equation can be combined with (4.24) and (4.28) to get:

$$\boxed{R_{\mu b}{}^{ab}(J) := e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(J) = 0,} \tag{4.31}$$

This is indeed the field equation (2.11a) of Galilei Gravity. However, the equation (4.26) did not occur as a field equation that can be derived from the Galilei action! Furthermore, the limits of the curvature constraint of General Relativity (4.20) and (4.21) – while definitely compatible with the field equations (2.11c) and (2.11d) – impose a much stricter stricter constraints on these curvatures than may be derived from the Galilei action alone. This is particularly unsettling since the original constraint (1.12) can be derived from the Einstein-Hilbert action. This then shows that deriving field equations by varying the action and taking the limit $\omega \to \infty$ do not commute.

CHAPTER 5

General Relativity to Newton-Cartan

In the previous chapter we performed a (naive) Inönü-Wigner contraction on the Poincaré algebra, which produced the Galilei algebra. We were able to take this limit at the top-level of theory: the action, producing an entire theory of non-relativistic gravity, known as Galilei Gravity. We were also able to take the limit at the level of the field equations, but this turned out to give a more restricted set of field equations than can be derived from the Galilei Action.

Our following question is whether we can perform a similar procedure to obtain the Bargmann algebra, and – by taking the limit in the action or field equations – Newton-Cartan Gravity. As we were able to obtain Galilei Gravity together with its action in this way, we might be hopeful that this procedure eventually produces an action for Newton-Cartan Gravity. Unfortunately this hope would be misplaced as the limit on the level of the action turns out to be divergent. We will investigate how far we can come, and why the action diverges as $\omega \to \infty$. The limit can however be taken at the level of the field equations, indeed producing the correct field equations. This then provides us with a clue as to why we are as of yet unable to write down an action.

1. The Bargmann algebra as a contraction

Just as the Galilei algebra could be seen as a limit of the Poincaré algebra, we wish to view the Bargmann algebra as a limit of the Poincaré algebra. We could attempt to begin our contraction procedure simply with the normal Poincaré algebra, but this would be futile. Since the Bargmann algebra has 11 generators and the Poincaré algebra only has 10 generators, we need to extend the Poincaré algebra with a trivial generator that commutes with all other generators, i.e. we work with a direct product of the Poincaré Group and $U(1)$. We will denote this generator by $\hat{M}$, and the accompanying field by M_μ. All other generators and commutators of the original Poincaré Algebra remain the same.

We wish to see the Bargmann Algebra as a non-relativistic limit of the Poincaré Algebra. We can of course keep the same redefinition for $\tilde{G}_a$, but in order to obtain a proper definition for $\tilde{M}$, we need to mix some generators together, since if we perform a simple scaling this $\tilde{M}$ would not show up in the commutator of P_a and G_b. Let us recall the remark we made in chapter 3 that conceptually the generator M has to do with the conservation of matter. We can also consider the non-relativistic approximation for the timelike component of the 4-momentum[1]:

$$\hat{P}_0 = \sqrt{\hat{P}_a \hat{P}^a c^2 + M^2 c^4} \approx Mc^2 + \frac{\hat{P}_a \hat{P}^a}{2M}. \tag{5.1}$$

We also then take into account that in General Relativity $x^0 = ct$. Motivated by this we make the following redefinitions:

$$\hat{P}_0 = \frac{1}{2\omega}\tilde{H} + \omega\tilde{M}, \quad \hat{M} = \frac{1}{2\omega}\tilde{H} - \omega\tilde{M}, \quad \hat{J}_{a0} = \omega\tilde{G}_a. \tag{5.2}$$

Equivalently, we have the inverse transformations:

$$\begin{gathered} \tilde{H} := \omega(\hat{P}_0 + \hat{M}), \quad \tilde{M} := \frac{1}{2\omega}(\hat{P}_0 - \hat{M}), \quad \tilde{G}_a := \frac{1}{\omega}\hat{J}_{a0}. \\ \tilde{P}_a := \hat{P}_a, \quad \tilde{J}_{ab} := \hat{J}_{ab} \end{gathered} \tag{5.3}$$

Again, it is now a simple exercise to show that as $\omega \to \infty$, all the structure constants in the redefined basis converge, and correspond exactly to the structure constants of the Bargmann algebra (dropping the tildes). In fact, the non-zero commutators are:

$$\begin{gathered} [\tilde{P}_a, \tilde{J}_{bc}] = 2\delta_{a[b}\tilde{P}_{c]}, \qquad [\tilde{G}_a, \tilde{J}_{bc}] = 2\delta_{a[b}\tilde{G}_{c]}, \\ [\tilde{J}_{ab}, \tilde{J}_{cd}] = 4\delta_{[a[c}\tilde{J}_{d]b]}. \qquad [\tilde{H}, \tilde{G}_a] = \tilde{P}_a, \\ [\tilde{M}, \tilde{G}_a] = \frac{1}{2\omega^2}\tilde{P}_a \to 0, \quad [\tilde{G}_a, \tilde{G}_b] = \frac{1}{\omega^2}\tilde{J}_{ab} \to 0, \\ [\tilde{P}_a, \tilde{G}_b] = \delta_{ab}\left(\tilde{M} + \frac{1}{2\omega^2}\tilde{H}\right) \to \delta_{ab}\tilde{M}, \end{gathered} \tag{5.4}$$

This provides the connection between the Poincaré algebra and the Bargmann algebra we sought.

2. Limit of the gauge fields

Let us again investigate how the Cartan connection α_μ of General Relativity transforms, only now for the extended Poincaré algebra. This connection can be written in the standard and redefined bases as:

(5.5)

$$\begin{aligned} \alpha_\mu &= E_\mu{}^A \hat{P}_A + \frac{1}{2}\Omega_\mu{}^{AB}\hat{J}_{AB} + M_\mu \hat{M} \\ &= \underbrace{\frac{1}{2\omega}(E_\mu{}^0 + M_\mu)}_{\tilde{\tau}_\mu}\tilde{H} + \underbrace{E_\mu{}^a}_{\tilde{e}_\mu{}^a}\tilde{P}_a + \underbrace{\omega\Omega_\mu{}^{a0}}_{\tilde{\omega}_\mu{}^a}\tilde{G}_a + \frac{1}{2}\underbrace{\Omega_\mu{}^{ab}}_{\tilde{\omega}_\mu{}^{ab}}\tilde{J}_{ab}, + \underbrace{\omega(E_\mu{}^0 - M_\mu)}_{\tilde{m}_\mu}\tilde{M} \end{aligned}$$

We are therefore led to introduce the gauge fields as indicated with the brackets. These new fields can be expressed in terms of the Poincaré fields

[1]The generators $\hat{P}_A$ generate the 4-momentum as their conserved current through Noether's theorem. Also recall that these generators correspond to the momentum operators in quantum mechanics.

as:
(5.6)

$$E_\mu{}^0 = \omega\tilde{\tau}_\mu + \frac{1}{2\omega}\tilde{m}_\mu, \quad E_\mu{}^a = \tilde{e}_\mu{}^a,$$
$$\Omega_\mu{}^{a0} = \omega^{-1}\tilde{\omega}_\mu{}^a, = -\Omega_\mu{}^{0a}, \qquad \Omega_\mu{}^{ab} = \tilde{\omega}_\mu{}^{ab}, \qquad M_\mu = \omega\tilde{\tau}_\mu - \frac{1}{2\omega}\tilde{m}_\mu,$$

with the inverse transformations:

$$\begin{gathered}\tilde{\tau}_\mu = \frac{1}{2\omega}(E_\mu{}^0 + M_\mu), \qquad \tilde{e}_\mu{}^a = E_\mu{}^a \\ \tilde{\omega}_\mu{}^a = \omega\Omega^{a0}, \qquad \tilde{\omega}_\mu{}^{ab} = \Omega_\mu{}^{ab}, \qquad \tilde{m}_\mu = \omega(E_\mu{}^0 - M_\mu).\end{gathered} \tag{5.7}$$

We will interpret the fields as the corresponding fields in Newton-Cartan theory as $\omega \to \infty$. Since in Newton-Cartan we also could introduce an inverse vielbein, we are motivated to do the same for the corresponding fields before the limit. We define these inverses by:

$$\begin{gathered}\tilde{e}^\mu{}_a\tilde{e}_\mu{}^b = \delta_a^b, \qquad \tilde{\tau}^\mu\tilde{\tau}_\mu = 1, \qquad \tilde{e}^\mu{}_a\tilde{e}_\nu{}^a + \tilde{\tau}^\mu\tilde{\tau}_\nu = \delta_\nu^\mu, \\ \tilde{\tau}^\mu\tilde{e}_\mu{}^a = 0, \qquad \tilde{\tau}_\mu\tilde{e}^\mu{}_a = 0.\end{gathered} \tag{5.8}$$

Just as in the case of Galilei Gravity we can relate these inverse fields to the relativistic inverse vielbein by manipulating these equations. However, this time the equations are not so simple:

$$\begin{aligned}E^\mu{}_A &= E^\nu{}_A\delta_\nu^\mu = E^\nu{}_A\left(\tilde{e}^\mu{}_a\tilde{e}_\nu{}^a + \tilde{\tau}^\mu\tilde{\tau}_\nu\right) \\ &= \tilde{e}^\mu{}_a E^\nu{}_A E_\nu{}^a + \tilde{\tau}^\mu E^\nu{}_A \cdot \frac{1}{2\omega}\left(E_\nu{}^0 + M_\nu\right) \\ &= \delta_A^a\tilde{e}^\mu{}_a + \delta_A^0\frac{1}{2\omega}\tilde{\tau}^\mu + \frac{1}{2\omega}\tilde{\tau}^\mu E^\nu{}_A M_\nu.\end{aligned} \tag{5.9}$$

The equation becomes more complicated than its Galilei counterpart through the factors $\Xi_A := E^\mu{}_A M_\mu$. We can solve these to give:

$$\Xi_0 = 1 - \frac{\tilde{\tau}^\mu\tilde{m}_\mu}{\omega^2 + \frac{1}{2}\tilde{\tau}^\mu\tilde{m}_\mu}, \tag{5.10}$$

and

$$\Xi_a = -\frac{\omega\tilde{e}^\mu{}_a\tilde{m}_\mu}{\omega^2 + \frac{1}{2}\tilde{\tau}^\mu\tilde{m}_\mu.} \tag{5.11}$$

This gives us the expression:
(5.12)

$$E^\mu{}_A = \delta_A^a\left(\tilde{e}^\mu{}_a - \frac{\tilde{\tau}^\mu}{2}\frac{\tilde{e}^\rho{}_a\tilde{m}_\rho}{\omega^2 + \frac{1}{2}\tilde{\tau}^\rho\tilde{m}_\rho.}\right) + \delta_A^0\frac{1}{\omega}\left(\tilde{\tau}^\mu - \frac{\tilde{\tau}^\mu}{2}\frac{\tilde{\tau}^\mu\tilde{m}_\mu}{\omega^2 + \frac{1}{2}\tilde{\tau}^\mu\tilde{m}_\mu}\right).$$

Both fractions in this expression can be expanded as a series in ω^{-2}. As we will need only the terms up to ω^{-1}, we will write:

$$E^\mu{}_A = \omega^{-1}\delta_A^0(\tilde{\tau}^\mu + \mathcal{O}(\omega^{-2})) + \delta_A^a(\tilde{e}^\mu{}_a + \mathcal{O}(\omega^{-2})). \tag{5.13}$$

The first limits we will now consider are those of the curvatures. Again, we require the total curvature to be invariant, so that the curvatures of the redefined fields are given by:

$$\begin{aligned}
R_{\mu\nu}{}^{0}(\hat{P}) &= \omega R_{\mu\nu}(\tilde{H}) + \frac{1}{2\omega}R_{\mu\nu}(\tilde{M}), \qquad R_{\mu\nu}{}^{a}(\hat{P}) = R_{\mu\nu}{}^{a}(\tilde{P}),\\
R_{\mu\nu}{}^{a0}(\hat{J}) &= \omega^{-1}R_{\mu\nu}{}^{a}(\tilde{G}) = -R_{\mu\nu}{}^{0a}(\hat{J}), \qquad R_{\mu\nu}{}^{ab}(\hat{J}) = R_{\mu\nu}{}^{ab}(\tilde{J}),\\
R_{\mu\nu}(\hat{M}) &= \omega R_{\mu\nu}(\tilde{H}) - \frac{1}{2\omega}R_{\mu\nu}(\tilde{M}),
\end{aligned} \tag{5.14}$$

with the inverse transformation:

$$\begin{aligned}
R_{\mu\nu}(\tilde{H}) &= \frac{1}{2\omega}\left(R_{\mu\nu}{}^{0}(\hat{P}) + R_{\mu\nu}(\hat{M})\right), \qquad R_{\mu\nu}{}^{a}(\tilde{P}) = R_{\mu\nu}{}^{a}(\hat{P}),\\
R_{\mu\nu}{}^{a}(\tilde{G}) &= \omega R_{\mu\nu}{}^{a0}(\hat{J}), \qquad R_{\mu\nu}{}^{ab}(\tilde{J}) = R_{\mu\nu}{}^{ab}(\hat{J}),\\
R_{\mu\nu}(\tilde{M}) &= \omega\left(R_{\mu\nu}{}^{0}(\hat{P}) - R_{\mu\nu}(\hat{M})\right),
\end{aligned} \tag{5.15}$$

Just as in the Galilei case, it is straightforward to show by filling in the definitions of the Poincaré curvatures that these redefined curvatures converge exactly to the corresponding curvatures of the Bargmann Algebra. This will allow us to simply drop the tildes in these curvatures when we take the limit.

3. Limit of the Action?

Seeing how we were able to derive an action for Galilei Gravity we have hope that we might derive an action for Newton-Cartan Gravity in the same way. Indeed, filling in the redefined fields and then take the limit $\omega \to \infty$ will give us a convergent limit. In fact, this will give us the Galilei action (2.9) again. Of course, this is not the right action for Newton-Cartan Gravity. First of all, the action does not contain m_μ in any way, so we will be unable to derive equations of motion for this field using the action. Second of all, the Galilei action is not invariant under all transformations of the Bargmann algebra.

We of course arrive at this defective action because the Einstein-Hilbert Action we start off with does not contain the extra field M_μ that we introduced in order to obtain the Bargmann Algebra through the contraction. When redefining the fields, we do introduce m_μ in the action, but these all occur with a factor of ω^{-2} or lower orders, so they vanish in the limit. This could be remedied when M_μ occurs in the original action, so we might try to add terms to the Einstein-Hilbert action containing this field. In particular, this term should give us the constraint $R_{\mu\nu}(\hat{M}) = 0$, indicating that the field has no physical meaning. Sadly this is impossible with just M_μ and the other fields from General Relativity. We could make invariant combinations with another trivial field (so that we have a double trivial extension of the Poincaré Algebra), but this obviously leads to a larger algebra than the Bargmann Algebra (in fact, this is known as the Extended Bargmann Algebra and it can lead to an appropriate action, see [**11**]). While having similar features to Newton-Cartan Gravity, this theory is much larger and one can encounter phenomena that are not present in Newton-Cartan Gravity.

Hence, we conclude that since we have no appropriate action for the trivially extended Poincaré Algebra to begin with, we are unable to take the limit at the level of the action.

4. Limit of the Field Equations

Having failed to produce an appropriate action for Newton-Cartan Gravity, we recall the lessons we learned from constructing Galilei Gravity. We saw then that taking the limit of the Einstein field equations and the corresponding curvature constraint gave us a more restricted set of equations of motion than we could derive from the Galilei action. Hence, we might expect that the requirements for the limit of the field equations to converge are less restrictive. We therefore now make the attempt to construct the field equations for Newton-Cartan Gravity.

Let us again begin with the curvature constraints. First of all, since we introduced M_μ as a pure gauge field (i.e. it does not correspond to physical objects and therefore should have zero field strength), we are lead to impose the constraint:

$$R_{\mu\nu}(\hat{M}) = 0. \tag{5.16}$$

We also recall the normal constraint $R_{\mu\nu}{}^A(\hat{P}) = 0$, that allowed us to solve for $\Omega_{\mu\nu}{}^A$ in terms of the $E_\mu{}^A$ and its inverses. Starting out with this solution (1.14) and inserting the expressions for the inverse vielbein (5.13), we obtain:

$$\Omega_\mu{}^{ab} = \tilde{\omega}'_\mu{}^{ab} + \mathcal{O}(\omega^{-2}), \tag{5.17}$$

$$\Omega_\mu{}^{0a} = \omega^{-1}\tilde{\omega}'_\mu{}^{a} + \mathcal{O}(\omega^{-3}), \tag{5.18}$$

where we have defined the lowest-order parts of the spin- and boost-connections as:

$$\tilde{\omega}'_\mu{}^{ab} = 2\tilde{e}^{\nu[a}\partial_{[\mu}\tilde{e}_{\nu]}{}^{b} - \tilde{e}_\mu{}^c\tilde{e}^{\rho a}\tilde{e}^{\sigma b}\partial_{[\rho}\tilde{e}_{\sigma]}{}^c + \tilde{\tau}_\mu\,\tilde{e}^{\rho[a}\tilde{\omega}_\rho{}^{b]}, \tag{5.19}$$

$$\tilde{\omega}'_\mu{}^{a} = \tilde{\tau}^\nu\partial_{[\mu}\tilde{e}_{\nu]}{}^a + \tilde{e}^{\nu a}\partial_{[\mu}\tilde{m}_{\nu]} + \tilde{e}_{\mu b}\tilde{e}^{\rho a}\tilde{\tau}^\sigma\partial_{[\rho}\tilde{e}_{\sigma]}{}^b + \tilde{\tau}_\mu\tilde{\tau}^\rho\tilde{e}^{\sigma a}\partial_{[\rho}\tilde{m}_{\sigma]}. \tag{5.20}$$

We now take the limit $\omega \to \infty$, dropping the tildes on the fields. We then see that (5.19) and (5.20) have the same form as we derived in (3.10) and (3.11). From the fact that curvatures converge to curvatures, we can of course also immediately assertain that that:

$$\begin{cases} R_{\mu\nu}(H) = 0, \\ R_{\mu\nu}{}^a(P) = 0, \\ R_{\mu\nu}(M) = 0. \end{cases} \tag{5.21}$$

This concludes the so-called kinematic limit. This defines the correspondence of the two theories on a geometrical level. We will now consider the dynamics of the theory by taking the limit of the vacuum Einstein field equations:

$$E^\rho{}_A E^\sigma{}_C R_{\rho\sigma}{}^C{}_B(\hat{J}) + \frac{1}{2}\eta_{AB}E^\rho{}_C E^\sigma{}_D R_{\rho\sigma}{}^{CD}(\hat{J}) = 0. \tag{5.22}$$

Taking $A = 0, B = 0$ gives us:

$$\tilde{e}^{\rho}{}_{a}\tilde{e}^{\sigma}{}_{b}R_{\rho\sigma}{}^{ab}(\tilde{J}) + \mathcal{O}(\omega^{-2}) = 0. \tag{5.23}$$

From this we can conclude that in the limit:

$$e^{\rho}{}_{a}e^{\sigma}{}_{b}R_{\rho\sigma}{}^{ab}(J) = 0. \tag{5.24}$$

For $A = a, B = 0$ we obtain:

$$\omega^{-1}e^{\rho}{}_{a}e^{\sigma}{}_{b}R_{\rho\sigma}{}^{b}(\tilde{G}) + \mathcal{O}\omega^{-3} = 0. \tag{5.25}$$

Multiplying through by ω (again, since this is true for any finite ω we are allowed to do this before taking the limit) and then taking the limit, we get:

$$e^{\rho}{}_{a}e^{\sigma}{}_{b}R_{\rho\sigma}{}^{b}(G) = 0. \tag{5.26}$$

This can be manipulated to obtain:

$$\boxed{\tau^{\mu}e^{\nu}{}_{a}R_{\mu\nu}{}^{a}(G) = 0,} \tag{5.27}$$

which is precisely the first field equation of Newton-Cartan Gravity!

Now taking $A = 0, B = b$ we get:

$$\tilde{\tau}^{\rho}\tilde{e}^{\sigma}{}_{b}R_{\rho\sigma}{}^{ab}(\tilde{J}) + \mathcal{O}(\omega^{-2}) = 0, \tag{5.28}$$

so that:

$$\boxed{\tau^{\rho}e^{\sigma}{}_{b}R_{\rho\sigma}{}^{ab}(J) = 0,} \tag{5.29}$$

which is indeed the second field equation!

Finally we take $A = a, B = b$, which gives us:

$$\tilde{e}^{\rho}{}_{a}\tilde{e}^{\sigma}{}_{c}R_{\rho\sigma}{}^{bc}(\tilde{J}) + \frac{1}{2}\delta^{b}_{a}\tilde{e}^{\rho}{}_{c}\tilde{e}^{\sigma}{}_{d}R_{\rho\sigma}{}^{cd}(\tilde{J}) + \mathcal{O}(\omega^{-2}) = 0, \tag{5.30}$$

which gives us:

$$e^{\rho}{}_{a}e^{\sigma}{}_{c}R_{\rho\sigma}{}^{bc}(\tilde{J}) + \frac{1}{2}\delta^{b}_{a}e^{\rho}{}_{c}e^{\sigma}{}_{d}R_{\rho\sigma}{}^{cd}(J) = 0. \tag{5.31}$$

This can be combined with (5.24) and (5.29) to obtain:

$$\boxed{e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(J) = 0.} \tag{5.32}$$

This then is indeed the full set of field equations for Newton-Cartan Gravity.

So even though we failed to produce an action for Newton-Cartan Gravity, we did manage to produce its equations of motion as a non-relativistic limit of General Relativity. The failure to produce an action is strongly related to the order of ω at which the fields $\tilde{m}_{\mu}$ occur in expressions through their mixing with $\tilde{\tau}_{\mu}$ in the redefinitions. If we are to keep m_{μ} in such expressions, we are to cancel the $\tilde{\tau}_{\mu}$ so that we may raise the orders of ω. But this would in turn introduce invariant terms of $e^{\mu}{}_{a}m_{\mu}$ into the expression, making it unsuitable for an action. In the next part we will also see that it is this inability to produce invariant combinations containing m_{μ} that hinders the production of an action, giving us a hint at where we might look for a proof for the inexistence of an action for Newton-Cartan Gravity.

Part 3

Actions: From Limits to Algebra Expansions

CHAPTER 6

Lie algebra expansions

In the previous part we considered specific contractions that enabled us to obtain Galilei Gravity and Newton-Cartan Gravity from General Relativity as low-speed limits. It turned out that in the Galilei case we were able to push this limit as far as the action, but in the Newton-Cartan case the action diverged. In this part we will consider a different procedure to obtain approximations from the Lie algebra, this time to arbitrary order.

This method will be akin to Post-Newtonian approximation: we will "expand the algebra" (we will elaborate below what this means precisely) in terms of a parameter ω (representing the speed of light) and then truncate this expansion to the level of accuracy we desire. This induces an expansion of the fields and the field equations, and even allows us to construct actions for the approximated theories. The framework closely follows [**6**].

1. Maurer-Cartan forms

Generally the structure of a Lie algebra is expressed in terms of a set of generators and their commutators (or structure constants). There is however a dual formulation in terms of the Maurer-Cartan form. Let us be given a Lie group G with Lie algebra $\mathfrak{g}$. Let us choose a basis for $\mathfrak{g}$ (i.e. a set of generators): $\{X_i\}, i = 1, \dots, \dim(\mathfrak{g})$. The structure constants are defined by $[X_i, X_j] = C^k_{ij} X_k$.

We now introduce a set of left-invariant 1-forms $\{\theta^i\}$ on G that are defined by $\theta^i(\mathrm{id})(X_j) = \delta^i_j$. [1] We call these forms the *Maurer-Cartan forms*[**14**, p. 322]. They satisfy a set of equations known as the *Maurer-Cartan equations*:

$$\mathrm{d}\theta^k = -\frac{1}{2} C^k_{ij} \theta^i \wedge \theta^j. \tag{6.1}$$

These equations also uniquely determine the structure of the Lie algebra. We will work with the Maurer-Cartan forms as the carriers of this information during the expansion of the algebra. This dual formulation has the particularly nice feature that it provides us with natural objects to expand in terms of ω, namely these Maurer-Cartan forms.

[1] Note that we identify $\mathfrak{g}$ with the tangent space of G at the identity. If we define the 1-forms θ^i at the identify, we can use the left-invariancy to define them at the rest of G.

2. Expansions

In order to keep the expansion framework simple, we will only consider algebras that can be split into two subspaces $\mathfrak{g} = V_0 \oplus V_1$ with the particular restrictions:

$$[V_0, V_0] \subseteq V_0, \quad [V_0, V_1] \subseteq V_1, \quad [V_1, V_1] \subseteq V_0. \tag{6.2}$$

We assume that we are given a set of generators that can be split over these subspaces: $\{X_i\} = \{X_{i_0}\} \cap \{X_{i_1}\}$ ($i_0 = 1, \ldots, \dim(V_0)$; $i_1 = 1, \ldots, \dim(V_1)$).[2]

Let us now expand the Maurer-Cartan forms of the original algebras in terms of some parameter ω:

$$\theta^{i_0} = \sum_{\substack{\alpha_0=0 \\ \text{even}}}^{\infty} \overset{(\alpha_0)}{\theta^{i_0}} \omega^{\alpha_0}, \qquad \theta^{i_1} = \sum_{\substack{\alpha_1=1 \\ \text{odd}}}^{\infty} \overset{(\alpha_1)}{\theta^{i_1}} \omega^{\alpha_1}. \tag{6.3}$$

If we insert this into the Maurer-Cartan equations, we obtain:

$$\begin{aligned} (\mathrm{d}\theta)^{k_0} &= \sum_{\substack{\alpha_0=0 \\ \text{even}}}^{\infty} \overset{(\alpha_0)}{\mathrm{d}\theta^{k_0}} \omega^{\alpha_0} \\ &= -\frac{1}{2} \left(\begin{array}{c} \displaystyle\sum_{\substack{\beta_0=0 \\ \text{even}}}^{\infty} \sum_{\substack{\gamma_0=0 \\ \text{even}}}^{\infty} \omega^{\beta_0+\gamma_0} C^{k_0}_{i_0 j_0} \overset{(\beta_0)}{\theta^{i_0}} \wedge \overset{(\gamma_0)}{\theta^{j_0}} \\ \displaystyle + \sum_{\substack{\beta_1=1 \\ \text{odd}}}^{\infty} \sum_{\substack{\gamma_1=1 \\ \text{odd}}}^{\infty} \omega^{\beta_1+\gamma_1} C^{k_0}_{i_1 j_1} \overset{(\beta_1)}{\theta^{i_1}} \wedge \overset{(\gamma_1)}{\theta^{j_1}} \end{array} \right), \end{aligned} \tag{6.4}$$

and:

$$(\mathrm{d}\theta)^{k_1} = \sum_{\substack{\alpha_1=1 \\ \text{odd}}}^{\infty} \overset{(\alpha_1)}{\mathrm{d}\theta^{k_1}} \omega^{\alpha_1} = - \sum_{\substack{\beta_0=0 \\ \text{even}}}^{\infty} \sum_{\substack{\gamma_1=1 \\ \text{odd}}}^{\infty} \omega^{\beta_0+\gamma_1} C^{k_0}_{i_0 j_1} \overset{(\beta_0)}{\theta^{i_0}} \wedge \overset{(\gamma_1)}{\theta^{j_1}}. \tag{6.5}$$

If we now collect like powers of ω, we obtain the following equations:

$$\overset{(\alpha_s)}{\mathrm{d}\theta^{k_s}} = -\frac{1}{2} C^{k_s, \alpha_s}_{i_p, \beta_p,\ j_q, \gamma_q} \overset{(\beta_p)}{\theta^{i_p}} \wedge \overset{(\gamma_q)}{\theta^{j_q}}, \tag{6.6}$$

where

$$C^{k_s, \alpha_s}_{i_p, \beta_p,\ j_q, \gamma_q} := \begin{cases} C^{k_s}_{i_p j_q} & \text{if } \alpha_s = \beta_p + \gamma_q, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

In these equations, we have $p, q, s = 0, 1$.

This procedure leaves us with an infinite set of Maurer-Cartan equations (6.6) and an infinite set of structure constants (6.7). The idea is to now cut the expansions (6.3) off at some order ω^{N_s} and define a new Lie algebra

[2]Indices over V_0 will be denoted by i_0, j_0, etc., whereas indices over V_1 will be denoted by i_1, j_1, etc.

with the resulting Maurer-Cartan forms and structure constants. In order to obtain a consistent Lie algebra, the orders have to satisfy:[9]

$$N_0 = N_1 + 1, \quad \text{or } N_1 = N_0 + 1. \tag{6.8}$$

We will denote the resulting algebras by $\mathfrak{g}(N, N+1)$ and $\mathfrak{g}(N+1, N)$ respectively (with N being even in the first case, and odd in the second).

We have now efectively constructed a new Lie Algebra, with new generators (each one corresponding to the expansion terms $\overset{(\alpha_s)}{\theta^{k_s}}$ of the Maurer-Cartan forms). So if we are given a connection on this expanded algebra, we can introduce the gauge fields as the components of this connection with respect to the expanded generators: $\overset{(\alpha_s)}{\theta^{k_s}} \leftrightarrow \overset{(\alpha_s)}{A^{k_s}}$. The corresponding curvatures can be calculated as follows:

$$\overset{(\alpha_s)}{R^{k_s}} = \mathrm{d}\overset{(\alpha_s)}{A^{k_s}} + \frac{1}{2} C^{k_s,\alpha_s}_{i_p,\beta_p\ j_q,\gamma_q} \overset{(\beta_p)}{A^{i_p}} \wedge \overset{(\gamma_q)}{A^{j_q}}. \tag{6.9}$$

Note that the expansion of the Maurer-Cartan forms also induces an expansion on the fields:

$$A^{k_s} = \sum_{\alpha_s} \overset{(\alpha_s)}{A^{k_s}} \omega^{\alpha_s} \tag{6.10}$$

In the next chapter we will apply this expansion procedure to the Poincaré Algebra. The algebra will be split into a "spatial" and a "temporal" part, and we will see that this splitting obeys (6.2).

3. Expanded Actions

Having constructed a new algebra by expanding the Maurer-Cartan forms, we can now ask what happens if we apply this expansion to a given action. To this end, assume that we are given an action:

$$S = \int B, \tag{6.11}$$

where B is an n-form, dependent on the fields A^{k_s} and their curvatures R^{k_s} only. Plugging in the expanded fields (up to arbitrary order), we get:

$$S = \int \sum_{k=0}^{\infty} \overset{(k)}{B} \omega^k = \sum_{k=0}^{\infty} \omega^k \int \overset{(k)}{B} = \sum_{k=0}^{\infty} \omega^k \overset{(k)}{S}. \tag{6.12}$$

Our wish is now to also truncate this expansion of the action. However, the action must always be invariant under the symmetries of the new Lie algebra and this imposes requirements on the order at which we truncate our expansion. In fact, each term $\overset{(k)}{S}$ is only invariant under the new Lie algebra if and only if this algebra is the expansion up to the smallest order that contains all fields (and corresponding curvatures) appearing in $\overset{(k)}{S}$. This will give us a way to determine for which order of expansion we may construct corresponding invariant actions for the new Lie algebra.

In the next chapter we will apply this framework to the Poincaré Algebra, in order to produce non-relativistic theories of gravity. We can use the expansion of the actions to construct invariant actions for these theories, and can immediately determine for which order of the truncation we can obtain such an action. The hope is of course that this provides us with an action for Newton-Cartan Gravity, or at least with insight into why this action keeps eluding us.

CHAPTER 7

Expansion of the Poincaré Algebra

In the last chapter we discussed a framework for expanding a Lie algebra in terms of a parameter ω. We now apply this procedure to the Poincaré Algebra. The parameter ω will again represent the ratio of the typical speed of the system under consideration to the speed of light, so that low-order expansions indeed correspond to a non-relativistic approximation of General Relativity.

We consider the first two orders at which we can truncate the expansion, and investigate which theories these truncated algebras correspond to. It will become clear that the first-order expansion gives us Galilei Gravity, complete with the appropriate action. Hopeful that the second-order expansion might then produce Newton-Cartan Gravity with an appropriate action, we consider this truncation. This however gives us an algebra that is similar to, but larger than the Bargmann Algebra. Moreover, this method does not produce an action for this theory! The reason for this does however gives us a hint at why we are hindered in constructing an action for Newton-Cartan Gravity (even though this is a larger algebra).

1. Splitting of the algebra

The first step of the procedure is to appropriately split the Poincaré Algebra into two subspaces, so that we can identify which fields correspond to odd and even powers of ω in the rest of the procedure. To refresh our memory, the commutators of the Poincaré Algebra are:

$$[\hat{P}_A, \hat{P}_B] = 0, \quad [\hat{P}_A, \hat{J}_{BC}] = 2\eta_{A[B}\hat{P}_{C]}, \quad [\hat{J}_{AB}, \hat{J}_{CD}] = 4\eta_{[A[C}\hat{J}_{D]B]}. \tag{7.1}$$

We will split the algebra in two subspaces, where one subspace is spanned by the generators with a 0 as index (temporal generators), and the other subspace is spanned by the remaining generators (spatial generators). To this end, let us define:

$$\tilde{H} := \hat{P}_0, \quad \tilde{P}_a := \hat{P}_a, \quad \tilde{G}_a := \hat{J}_{a0}, \quad \tilde{J}_{ab} := \hat{J}_{ab}. \tag{7.2}$$

The non-zero commutators in this representation are given by:

$$\begin{aligned} [\tilde{P}_a, \tilde{J}_{bc}] &= 2\delta_{a[b}\tilde{P}_{c]}, \quad & [\tilde{P}_a, \tilde{G}_b] &= \delta_{ab}\tilde{H}, \\ [\tilde{G}_a, \tilde{J}_{bc}] &= 2\delta_{a[b}\tilde{G}_{c]}, \quad & [\tilde{G}_a, \tilde{G}_b] &= \tilde{J}_{ab} \\ [\tilde{J}_{ab}, \tilde{J}_{cd}] &= 4\delta_{[a[c}\tilde{J}_{d]b]}, \quad & [\tilde{H}, \tilde{G}_a] &= \tilde{P}_a. \end{aligned} \tag{7.3}$$

The relevant subspaces are $V_0 := \text{span}(\tilde{H}, \tilde{J}_{ab})$ and $V_1 := \text{span}(\tilde{P}_a, \tilde{G}_a)$. These subspaces then satisfy (6.2), so we may indeed readily apply our previously developped framework.

For our new algebra, we now introduce the following fields:

$$\overset{(\alpha_0)}{H} \leftrightarrow \overset{(\alpha_0)}{\tau}, \quad \overset{(\alpha_0)}{J_{ab}} \leftrightarrow \overset{(\alpha_0)}{\omega^{ab}}, \quad \overset{(\alpha_1)}{P_a} \leftrightarrow \overset{(\alpha_1)}{e^a}, \quad \overset{(\alpha_1)}{G_a} \leftrightarrow \overset{(\alpha_1)}{\omega^a}. \tag{7.4}$$

It will become very clear why we choose this notation later, when we consider a first order expansion.

2. Expansion of the action

The next step is to expand the Einstein-Hilbert action:

$$S = \int E E^{\mu}{}_{A} E^{\nu}{}_{B} R_{\mu\nu}{}^{AB}(\hat{J}). \tag{7.5}$$

The procedure that we outlined in the last chapter assumes that this action is invariant under the transformations represented by the Poincaré Algebra. However, this is not the case! In fact, this action is invariant under the $\hat{J}$-transformations, and under general coordinate tranformations ($\delta x^\mu = \xi^\mu$). It is *not* invariant under the $\hat{P}$-transformations. For a detailled treatment of the invariance of the action (7.5) see [**13**]. Kibble considers there two transformations: δ (representing $\hat{J}$-transformations and general coordinate transformations) and δ_0 (representing $\hat{J}$-transformations and $\hat{P}$-transformations). The Einstein-Hilbert action that is constructed there from symmetry principles is only invariant under δ, but not under δ_0! This stems froms the fact that the $\hat{P}$-transformations not only act on the fiels, but also on the coordinates of the underlying manifold.

It turns out that one can in fact write the $\hat{P}$-transformations as a linear combination of a general coordinate transformation, a Lorentz ($\hat{J}$-)transformation and a certain trivial symmetry that is proportional to the equations of motion[**6**]. This implies that we may just expand this action and that we can just take the expanded algebra that contains the highest orders that occur in the term of the expanded action that we are interested in.

For simplicity we only consider the 4-dimensional case. Using the standard formula for the determinant and collecting some like terms of the vielbein and inverse vielbein, we can then write:

$$S = \int \mathcal{L} = \frac{1}{4!} \int \epsilon^{\mu\nu\rho\sigma} \epsilon_{ABCD} E_{\mu}{}^{A} E_{\nu}{}^{B} R_{\rho\sigma}{}^{CD}. \tag{7.6}$$

The gauge fields can be expanded as:

$$E_\mu{}^0 = \sum_{\alpha_0 \text{ even}} \omega^{\alpha_0} \overset{(\alpha_0)}{\tau}_\mu \tag{7.7a}$$

$$\Omega_\mu{}^{ab}, = \sum_{\alpha_0 \text{ even}} \omega^{\alpha_0} \overset{(\alpha_0)}{\omega}_\mu{}^{ab} \tag{7.7b}$$

$$\Omega_\mu{}^{a0}, = \sum_{\alpha_1 \text{ odd}} \omega^{\alpha_1} \overset{(\alpha_1)}{\omega}_\mu{}^{a}, \tag{7.7c}$$

$$E_\mu{}^a = \sum_{\alpha_1 \text{ odd}} \omega^{\alpha_1} \overset{(\alpha_1)}{e}_\mu{}^{a}, \tag{7.7d}$$

where we understand that the summation over α_0 only runs over even terms, and α_1 only runs over odd terms. We can also write[1]:

$$4!\mathcal{L} = \underbrace{2\epsilon^{\mu\nu\rho\sigma}\epsilon_{a0cd}E_\mu{}^a E_\nu{}^0 R_{\rho\sigma}{}^{cd}(\hat{J})}_{\text{odd terms}\geq 1} + \underbrace{2\epsilon^{\mu\nu\rho\sigma}\epsilon_{abc0}E_\mu{}^a E_\nu{}^b R_{\rho\sigma}{}^{c0}(\hat{J})}_{\text{odd terms}\geq 3}\,. \tag{7.8}$$

Using the explicit formula for $R_{\mu\nu}{}^{AB}(\hat{J})$, we can obtain:

$$\begin{aligned} 4!\mathcal{L} = {} & \underbrace{4\epsilon^{\mu\nu\rho\sigma}\epsilon_{a0cd}E_\mu{}^a E_\nu{}^0 \partial_{[\rho}\Omega_{\sigma]}{}^{cd}}_{\text{odd terms}\geq 1} + \underbrace{4\epsilon^{\mu\nu\rho\sigma}\epsilon_{a0cd}E_\mu{}^a E_\nu{}^0 \Omega_{[\rho}{}^{ce}\Omega_{\sigma]e}{}^{d}}_{\text{odd terms}\geq 1} \\ & + \underbrace{4\epsilon^{\mu\nu\rho\sigma}\epsilon_{a0cd}E_\mu{}^a E_\nu{}^0 \Omega_{[\rho}{}^{c0}\Omega_{\sigma]}{}^{d0}}_{\text{odd terms}\geq 3} \\ & + \underbrace{4\epsilon^{\mu\nu\rho\sigma}\epsilon_{abc0}E_\mu{}^a E_\nu{}^b \partial_{[\rho}\Omega_{\sigma]}{}^{c0}}_{\text{odd terms}\geq 3} + \underbrace{4\epsilon^{\mu\nu\rho\sigma}\epsilon_{abc0}E_\mu{}^a E_\nu{}^b \Omega_{[\rho}{}^{cd}\Omega_{\sigma]d}{}^{0}}_{\text{odd terms}\geq 3}\,. \end{aligned} \tag{7.9}$$

This form in particular is very useful when constructing the first few order terms of the expanded action. We will see that all these terms in different orders of ω can be gathered to form curvatures of the expanded algebras.

3. First-order expansion

Let us consider a minimal expansion of the Poincaré Algebra, i.e. to first order. We obtain the fields $\overset{(0)}{\tau}$, $\overset{(0)}{\omega}{}^{ab}$, $\overset{(1)}{e}{}^{a}$ and $\overset{(1)}{\omega}{}^{a}$. Their curvatures are given by:

$$\overset{(0)}{R}(H) = \mathrm{d}\overset{(0)}{\tau}, \tag{7.10a}$$

$$\overset{(0)}{R}{}^{ab}(J) = \mathrm{d}\overset{(0)}{\omega}{}^{ab} + \overset{(0)}{\omega}{}^{ac} \wedge \overset{(0)}{\omega}{}^{db}\delta_{cd}, \tag{7.10b}$$

$$\overset{(1)}{R}{}^{a}(P) = \mathrm{d}\overset{(1)}{e}{}^{a} + \overset{(0)}{\omega}{}^{ab} \wedge \overset{(1)}{e}{}^{c}\delta_{bc} + \overset{(0)}{\tau} \wedge \overset{(1)}{\omega}{}^{a}, \tag{7.10c}$$

$$\overset{(1)}{R}{}^{a}(G) = \mathrm{d}\overset{(1)}{\omega}{}^{a} + \overset{(0)}{\omega}{}^{ab} \wedge \overset{(1)}{\omega}{}^{c}\delta_{bc}. \tag{7.10d}$$

[1]Note that the term with ϵ_{abcd} becomes zero because of overantisymmetrization.

These are of course indeed the curvatures of the Galilei fields!

Let us now consider the first order expansion of the Einstein Hilbert Lagrangian density. This expansion only contains the first two terms of (7.9):

(7.11)

$$\overset{(1)}{\mathcal{L}} = \omega^1 \underbrace{\frac{1}{4!}\left(4\epsilon^{\mu\nu\rho\sigma}\epsilon_{a0cd}\overset{(1)}{e}_{\mu}{}^{a}\overset{(0)}{\tau}_{\nu}\partial_{[\rho}\overset{(0)}{\omega}_{\sigma]}{}^{cd} + 4\epsilon^{\mu\nu\rho\sigma}\epsilon_{a0cd}\overset{(1)}{e}_{\mu}{}^{a}\overset{(0)}{\tau}_{\nu}\overset{(0)}{\omega}_{[\rho}{}^{ce}\overset{(0)}{\omega}_{\sigma]e}{}^{d}\right)}_{\overset{(1)}{B}}$$

$$= \omega^1 \underbrace{\frac{1}{4!}\left(2\epsilon^{\mu\nu\rho\sigma}\epsilon_{a0cd}\overset{(1)}{e}_{\mu}{}^{a}\overset{(0)}{\tau}_{\nu}\overset{(0)}{R}_{\rho\sigma}{}^{cd}(J)\right)}_{\overset{(1)}{B}}$$

Let us now use the identity

$$\overset{(3)}{e} := \det(\overset{(0)}{\tau}_{\mu}, \overset{(1)}{e}_{\mu}{}^{a}) = \frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abc}\overset{(0)}{\tau}_{\mu}\overset{(1)}{e}_{\nu}{}^{a}\overset{(1)}{e}_{\rho}{}^{b}\overset{(1)}{e}_{\sigma}{}^{c} \tag{7.12}$$

We can easily show that $\overset{(1)}{B}$ is just:

$$\overset{(1)}{B} = \overset{(3)}{e}\,\overset{(-1)}{e^{\mu}}_{a}\,\overset{(-1)}{e^{\nu}}_{b}\,\overset{(0)}{R}_{\mu\nu}{}^{ab}(J). \tag{7.13}$$

This is indeed the Lagrangian Density for the action (2.9) of Galilei Gravity! This action is by construction invariant under the transformations of the Galilei Algebra. The key to this invariancy is that we have enough fields to combine into invariant combinations.

4. Second-order expansion

We now consider a second-order expansion. One might conjecture that, since this is the next term in our expansion, this should give us the Bargmann algebra. This turns out not to be the case, however. Instead, we obtain a larger algebra, with the fields $\overset{(0)}{\tau}$, $\overset{(0)}{\omega}{}^{ab}$, $\overset{(1)}{e}{}^{a}$, $\overset{(2)}{\tau}$, and $\overset{(2)}{\omega}{}^{ab}$.

$$\overset{(0)}{R}(H) = \mathrm{d}\overset{(0)}{\tau}, \tag{7.14a}$$

$$\overset{(0)}{R}{}^{ab}(J) = \mathrm{d}\overset{(0)}{\omega}{}^{ab} + \overset{(0)}{\omega}{}^{ac} \wedge \overset{(0)}{\omega}{}^{db}\delta_{cd}, \tag{7.14b}$$

$$\overset{(1)}{R}{}^{a}(P) = \mathrm{d}\overset{(1)}{e}{}^{a} + \overset{(0)}{\omega}{}^{ab} \wedge \overset{(1)}{e}{}^{c}\delta_{bc} + \overset{(0)}{\tau} \wedge \overset{(1)}{\omega}{}^{a}, \tag{7.14c}$$

$$\overset{(1)}{R}{}^{a}(G) = \mathrm{d}\overset{(1)}{\omega}{}^{a} + \overset{(0)}{\omega}{}^{ab} \wedge \overset{(1)}{\omega}{}^{c}\delta_{bc}, \tag{7.14d}$$

$$\overset{(2)}{R}(H) = \mathrm{d}\overset{(2)}{\tau} + \overset{(1)}{e}{}^{a} \wedge \overset{(1)}{\omega}{}^{b}\delta_{ab}, \tag{7.14e}$$

$$\overset{(2)}{R}{}^{ab}(J) = \mathrm{d}\overset{(2)}{\omega}{}^{ab} + \overset{(1)}{\omega}{}^{a} \wedge \overset{(1)}{\omega}{}^{b} + \overset{(0)}{\omega}{}^{ac} \wedge \overset{(2)}{\omega}{}^{db}\delta_{cd} + \overset{(2)}{\omega}{}^{ac} \wedge \overset{(0)}{\omega}{}^{db}\delta_{cd}. \tag{7.14f}$$

The astute reader may now recognize the first five of these curvatures as the curvatures of Newton-Cartan gravity, identifying:

$$\overset{(0)}{\tau} \leftrightarrow \tau, \quad \overset{(0)}{\omega}{}^{ab} \leftrightarrow \omega^{ab}, \quad \overset{(1)}{e}{}^{a} \leftrightarrow e^{a}, \quad \overset{(1)}{\omega}{}^{a} \leftrightarrow \omega^{a}, \quad \overset{(2)}{\tau} \leftrightarrow m. \tag{7.15}$$

However, we also obtain a sixth set of fields: $\overset{(2)}{\omega}{}^{ab}$. The curvature of these fields indicate that they correspond to a spin-like field, tranforming under boosts and spatial rotations. Luckily we could simply dispose of these fields by noting that the Bargmann Algebra is a proper subalgebra of this second-order algebra. So we might take some lessons from this expansion after all.

The next-order invariant action $(\overset{(3)}{B})$ contains terms similar to $\overset{(1)}{B}$, but with the replacements of the even terms $\overset{(0)}{\omega}{}^{ab} \to \overset{(2)}{\omega}{}^{ab}$ and $\overset{(0)}{\tau} \to \overset{(2)}{\tau}$, or the replacements of the odd terms $\overset{(1)}{e}{}^{a} \to \overset{(3)}{e}{}^{a}$.[2] That implies that to obtain an invariant action, we must go to a higher order expansion of the algebra, namely the third order! In other words, this procedure does *not* produce an invariant action for the second-order algebra. Even though we have introduced a larger set of fields, we still cannot find an action that can be viewed as the proper non-relativistic approximation of General Relativity.

We can however take away from this expansion that we need the $\overset{(3)}{e}{}^{a}$ fields to be present, since these are the only fields that occur in the algebra, but do not occur in the second-order algebra. In the third-order algebra we would of course also introduce another $\overset{(3)}{\omega}{}_{\mu}{}^{a}$, but just as with the fields $\overset{(2)}{\omega}{}_{\mu}{}^{ab}$ we might simply dispose of these. The $\overset{(2)}{\omega}{}_{\mu}{}^{ab}$ are however needed to maintain invariancy

[2]Note that the field $\overset{(3)}{\omega}{}^{a}$ only first occurs in $\overset{(5)}{B}$.

in $\overset{(3)}{B}$. We therefore conjecture that it might be possible to construct an invariant action for a larger theory of non-relativistic gravity that has the concept of spin and some kind of secondary momentum corresponding to the fields $\overset{(3)}{e}{}_{\mu}{}^{a}$.

Part 4

Conclusion

CHAPTER 8

Discussion

We began our quest in search of an invariant action for Newton-Cartan Gravity. Unfortunately this has proven unfruitful, and we still have no action that provides us with the field equations. The different methods of constructing non-relativistic theories of gravity from General Relativity have however given us a bit of insight into why this might be the case.

When performing the contraction of the centrally extended Poincaré algebra we encountered a divergence in the action that could not be avoided. This hints at the need for more fields to counteract this divergence. We might then conjecture that without those extra fields it is simply not possible to tame these divergences and that it is therefore impossible to write down an invariant action that represents the dynamics of Newton-Cartan Gravity.

The method of truncated expansions of the Poincaré algebra makes this even more explicit: it directly indicates that to construct an invariant action that represents gravity, we need either the first order – which is Galilei Gravity – or the third order truncation, which is a (much) larger algebra than the Bargmann algebra of Newton-Cartan Gravity. In **[11]** this action is in fact constructed and it is argued that when coupled to a point-particle this produces regular Newtonian Gravity (in the sense that it gives us a Poisson equation), but it can also produce artifacts such as non-relativistic time dilation. This makes it very different from "regular" Newtonian Gravity indeed. The fact that we need the third order fields to construct an invariant action again points at a fundamental need to cancel invariancies or infinities with extra fields, hinting at the impossibility of writing down an action for Newton-Cartan Gravity. In particular the field $\overset{(3)}{e}{}_{\mu}{}^{a}$, corresponding to some kind of secondary "momentum", is the prime candidate for combining with m_μ to produce invariant terms in the action.

There are several more possibilities of performing like procedures that one might consider to gain more insight in the absence of an action for Newton-Cartan Gravity. First of all, we might look at stringy versions of the theory. These are for example considered in **[2]**. Supersymmetric versions of Newton-Cartan Gravity might also provide more insight into this.

There is also the possibility of performing an alternate expansion. When considering the construction of the Bargmann Algebra through a contraction, we started out with a trivial $U(1)$-extension of the Poincaré Algebra. We might then of course also start our expansion procedure with some extension of the Poincaré Algebra. However, a simple $U(1)$-extension would prove to be fruitless, as we would be unable to write down an action for the extra field. So, we would need a larger extension of the Poincaré Algebra, but what extension this should be is unclear as of this moment.

We also wish to comment on the appearance of the fields $\overset{(2)}{\omega}{}_\mu{}^{ab}$ in the second-order expansion of the Poincaré Algebra. From the curvatures of these fields one can infer that they transform like spin. This then suggests that spin is in fact not a relativistic concept, but rather an emergent property that shows up in the non-relativistic approximation of General Relativity. This in turn vaguely hints at a deeper reason why General Relativity and the Standard Model cannot be unified as of yet, since the Standard Model treats spin as a concept that needs to be compatible with relativistic transformations.

Another point of possible further research would be a coordinate-independent formulation of the procedures used here to obtain the various non-relativistic theories of Gravity. In particular these procedures could be rigourously defined in the language of principal bundles. This might also provide more clarity into why the processes of taking limits and varying the action did not commute when constructing Galilei Gravity through a contraction.

APPENDIX A

Obtaining Newton-Cartan Gravity

In chapter 3 the outline for Newton-Cartan Gravity as a gauge theory was given. We are now going to show that this indeed produced Newtonian gravity when one fixes a frame. This chapter closely follows [**3**].

We shall begin by writing down the Newtonian equations and then we show that the equations of motion can be described as geodesics of a connection. This connection will however not be a Levi-Civita connection. Instead, it will correspond to a pair of degenerate metrics that foliate the spacetime in distinct temporal and spatial parts.

1. Newtonian gravity

Newton's equations of gravity are very simple. The gravitational force that a body of mass M, positioned at the origin, exerts on a body of mass m is:

$$\vec{F} = m\vec{a} = \frac{GMm}{r^2}\hat{r}. \tag{A.1}$$

This means that the test body feels the potential $\phi = -GM/r$. The equation of motion then becomes:

$$\ddot{x}^i(t) + \partial_i\phi(t, x(t)) = 0. \tag{A.2}$$

We can generalize the potential to that of a mass distribution ρ. The potential then satisfies the Poisson equation:

$$\partial_i\partial^i\phi = 4\pi G\rho. \tag{A.3}$$

We can interpret these equation in two ways. On one hand, this describes a curve through a flat three-dimensional manifold, parametrized by time. We could however also view this as a curve through a 4-dimensional spacetime in a special coordinate frame. In this frame time is the zeroth coordinate, and our parametrization has exactly that coordinate as its parameter. These coordinates are called *adapted coordinates*. This is the point of view we will adopt from now on.

2. Metrics and connection

We now notice that in adapted coordinates we may write:

$$\frac{\partial^2 x^\mu}{\partial t^2} + \Gamma_\rho{}^\mu{}_\sigma \frac{\partial x^\rho}{\partial t}\frac{\partial x^\sigma}{\partial t} = 0, \tag{A.4}$$

where we have set ${\Gamma_0}^i{}_0 = \delta^{ij}\partial_j\phi$ as the only non-zero components of Γ and work in adapted coordinates. This looks a lot like a geodesic equation! If one computes the Riemann curvature tensor for this "connection", we get:

$$R^i{}_{0j0} = \delta^{ik}\partial_k\partial_j\phi. \tag{A.5}$$

We therefore see, that by (A.3) we should have:

$$R = R_{00} = \partial^i\partial_i\phi = 4\pi G\rho. \tag{A.6}$$

The idea now is to really interpret this Γ as a connection. We however immediately see that this cannot possibly be a Levi-Civita connection for some nondegenerate metric, since the Riemann curvature tensor does not satisfy the proper symmetry properties. If we think about it, we know why this is: we have a foliation of spacetime into two distinguished submanifolds, one temporal and one spatial. The connection could never parallel transport a temporal vector into a spatial one. So, we are forced to conclude that we should have two degenerate metrics on each of the submanifolds that are compatible with the connection.

Having picked up intuition, we now move on to general coordinates. Let us denote the spatial metric by h. This metric has one zero eigenvalue: it is degenerate on the temporal tangent space. Similary, call the temporal metric τ. This metric has three zero eigenvalues and is precisely degenerate on the spatial tangent space. Since τ therefore is effectively a scalar operator, we can identify it with a covector such that $\tau_{\mu\nu} = \tau_\mu\tau_\nu$ [1]. The fact that the two metrics are to be degenerate on each other's spaces can be summarized as $h^{\mu\nu}\tau_\nu = 0$. This means that τ picks out a temporal component of a vector, and when the spatial metric of such a temporal projection with any other vector is taken, it should give zero. We also introduce the dual metrics:

$$\begin{aligned} h^{\mu\rho}h_{\rho\nu} &= \delta^\mu_\nu - \tau^\mu\tau_\nu, & \tau^\mu\tau_\nu &= \delta^\mu_\nu, \\ h_{\mu\nu}\tau^\nu &= 0, & h^{\mu\nu}\tau_\nu &= 0. \end{aligned} \tag{A.7}$$

To summarize: we have a symmetric and positive-definite covariant 2-tensor h and a 1-form τ such that τ spans the kernel of h. (For a rather nice discussion of this formalism, see **[19]**.) Let us introduce a covariant derivative ∇. We wish it to be compatible with our metrics. Hence, we impose:

$$\nabla_\mu h = 0 \qquad \nabla_\mu\tau = 0. \tag{A.8}$$

Using the usual definition of the Christoffel symbols, we can then write:

$$\partial_\mu h^{\rho\sigma} + \Gamma^\rho_{\mu\alpha}h^{\alpha\sigma} + \Gamma^\sigma_{\mu\alpha}h^{\alpha\rho} = 0 \tag{A.9}$$

$$\partial_\mu\tau_\nu - \Gamma^\alpha_{\mu\nu}\tau_\alpha = 0. \tag{A.10}$$

We can now see that the connection is not uniquely defined by our metrics, since we can make the transformation

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + h^{\rho\lambda}K_{\lambda(\mu}\tau_{\nu)}, \tag{A.11}$$

[1] In fact, τ_μ is the dual of the normalized eigenvectorfield corresponding to the non-zero eigenvalue of τ.

where K is an arbitrary 2-form. This leaves the compatibility conditions unchanged. The most general connection is then given by[**3**]:

$$\Gamma^{\rho}_{\mu\nu} = \tau^{\rho}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}\right) + h^{\rho\lambda}K_{\lambda(\mu}\tau_{\nu)}. \tag{A.12}$$

Since in our adapted coordinates we expect (A.6) and τ represents a temporal projection, we are led to impose the following curvature constraint as the defining equation of motion:

$$R_{\mu\nu} = 4\pi G\rho\tau_{\mu}\tau_{\nu}\,. \tag{A.13}$$

We can generalise this theory further, by introducing a matter-energy stress tensor. As we know, the matter density is normally the purely temporal component of this tensor. The matter-energy stress tensor is symmetric and should satisfy the matter equation $\nabla_{\mu}T^{\mu\nu} = 0$. Then we arrive at the final form:

$$\mathrm{Ric} = \big(4\pi G T(\tau,\tau)\big)\,\tau\otimes\tau. \tag{A.14}$$

In order to recover the usual Newtonian equations, we must impose another curvature constraint, that is known as an Ehlers condition:

$$h^{\alpha[\mu}R^{\nu]}{}_{\rho\sigma\alpha} = 0. \tag{A.15}$$

This is equivalent to requiring that the $(2,0)$-tensor $R(h(\cdot),X)Y$ is symmetric for any vector fields X and Y, where $h(\cdot)$ denotes the linear transformation on 1-forms, that when given a 1-form α produces the vector field $h(\alpha)$ that works on other 1-forms β as follows: $h(\alpha)(\beta) := h(\alpha,\beta)$. This property is the key in recovering a potential from the connection.

We summarize our discussion as:

Definition 1 (Newton-Cartan Theory). A Newton-Cartan Theory consists of a 4-dimensional smooth manifold M, a 1-form $\tau \in \Omega^1 M$, a symmetric tensor $h \in T^{(2,0)}M$, a torsion-free covariant derivative ∇, and a symmetric tensor $T \in T^{(2,0)}M$ such that:

(1) For all 1-forms α, $h(\alpha,\alpha) \geq 0$;
(2) The kernel of the linear operator $\tilde{h} : \Omega^1 M \to \mathcal{X}M$ defined by $\tilde{h}(\alpha)(\beta) := h(\alpha,\beta)$ is spanned by τ;
(3) The covariant derivative is compatible with h and τ: $\nabla h = 0$ and $\nabla\tau = 0$;
(4) The matter equation $\nabla_{\mu}T^{\mu\nu} = 0$ is satisfied;
(5) The Newtonian equations of motion $\mathrm{Ric} = \big(4\pi G T(\tau,\tau)\big)\,\tau\otimes\tau$ are satisfied;
(6) The Ehlers condition is satisfied: $R(h(\cdot),X)Y$ is a symmetric $(2,0)$-tensor for any $X, Y \in \mathcal{X}M$.

The motions of particles in such a theory are given by geodesics of the covariant derivative.

3. Recovering gravity

Let us be given a Newton-Cartan theory as defined in the previous section. We first note that since $\nabla\tau = 0$, we have a function $t : M \to \mathbb{R}$ such that $\tau_\mu = \partial_\mu t$. We call this function the *absolute time*.[2] This is the same as saying that the manifold is foliated into a spatial and temporal part. We now choose a coordinate frame with $x^0 = t$. These are the adapted coordinates we discussed earlier. This implies:

$$\begin{aligned} \tau_\mu &= \delta^0_\mu, \quad \tau^0 = 1, \\ h^{\mu 0} &= 0, \quad h_{\mu 0} = -h_{\mu i}\tau^i. \end{aligned} \tag{A.16}$$

The Christoffel symbols therefore become:

$$\begin{aligned} \Gamma^0_{\mu\nu} &= 0, \\ \Gamma^i_{00} &= h^{ij}\left(\partial_0 h_{j0} - \frac{1}{2}\partial_j h_{00} + K_{j0}\right), \\ \Gamma^i_{0j} &= h^{ik}\left(\frac{1}{2}\partial_0 h_{jk} + +\partial_{[j}h_{k]0} - \frac{1}{2}K_{jk}\right), \\ \Gamma^i_{jk} &= \frac{1}{2}h^{il}\left(\partial_j h_{lk} + \partial_k h_{lj} - \partial_l h_{jk}\right). \end{aligned} \tag{A.17}$$

We first notice that the spatial Christoffel symbols completely agrees with those of the Levi-Civita connection on a 3-dimensional manifold with Riemannian metric h. There is however a mixing term between the temporal and spatial parts. We also recall that Γ^i_{00} should represent the Laplacian of some potential. We therefore define:

$$\Phi_j := \partial_0 h_{j0} - \frac{1}{2}\partial_j h_{00} + K_{j0}, \tag{A.18}$$

$$\omega_{ij} := \partial_{[i}h_{j]0} - \frac{1}{2}K_{ij}. \tag{A.19}$$

The equation of motion becomes:

$$\mathrm{Ric} = 4\pi G T^{00}(\mathrm{d}x^0 \otimes \mathrm{d}x^0), \tag{A.20}$$

so that $R_{ij} = R_{i0} = R_{0j} = 0$. This means that the spatial hypersurfaces are flat, and we can thus choose a particular coordinate frame with:

$$h_{ij} = \delta_{ij}, \qquad h^{ij} = \delta^{ij}. \tag{A.21}$$

This simplifies the Christoffel symbols considerably:

$$\begin{aligned} \Gamma^i_{jk} &= 0, \\ \Gamma^i_{0j} &= h^{ik}\omega_{jk}, \\ \Gamma^i_{00} &= h^{ik}\Phi_k. \end{aligned} \tag{A.22}$$

We can now use the Ehlers condition to derive:

$$\partial_i \omega_{jk} = 0, \qquad \partial_0 \omega_{ij} = \partial_{[i}\Phi_{j]}. \tag{A.23}$$

In particular, ω_{jk} is a function of absolute time. We can then perform a time-dependent spatial rotation to obtain $\partial_0\omega_{ij} = 0$, while keeping $h^{ij} = \delta^{ij}$ (since

[2]In fact, the topology of the manifold must be nice enough to allow this globally. We can however always foliate the manifold locally, since $\tau \wedge \mathrm{d}\tau = 0$.

rotations preserve the Riemann metric on space). But this sets $\partial_{[i}\Phi_{j]} = 0$, so that $\Phi_i = \partial_i\phi$ for some scalar field $\phi : M \to \mathbb{R}$. Incidentally, this also sets $\sum_{i,j} \omega_{ij}^2 = 0$, so that we finally recover:

$$R_{00} = \delta^{ij}\partial_i\Phi_j = \delta^{ij}\partial_i\partial_j\phi = 4\pi G T^{00}. \tag{A.24}$$

This is the Poisson equation of Newtonian gravity. As seen in the introduction, the integral curves corresponding to the potential ϕ are precisely the geodesics for the connection in this coordinate system. Hence, we have arrived at:

THEOREM 1. *A Newton-Cartan theory is equivalent to classical Newtonian gravity in the sense that in the special coordinate system introduced above, there is an absolute time, and the geodesics are precisely the classical trajectories.*

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