Rigidity of Brillouin Zones

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RuG

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CHAPTER 1

Introduction and Definitions

In general, geometric properties of a manifold are not determined by topological invariants of this manifold. Starting in the 1960's, however, a number of fascinating results have been proved that show that, under certain conditions, the topology of a manifold *can* determine its geometry. In this case, one often speaks of *rigidity*.

The prototype rigidity theorem is due to Mostow [**1**]. This result, also known as the *strong rigidity theorem*, can be stated as

THEOREM 1.1 (Mostow's Rigidity Theorem, 1968). *Suppose M and N are closed manifolds of constant sectional curvature -1 with the dimension of M is at least 3. If $\pi_1(M) \cong \pi_1(N)$, then M and N are isometric.*

In this thesis we study the rigidity of the *focal decomposition* of the flat 2-torus, as introduced and studied in [**2**]. We show that the focal decomposition determines the torus up to conformal equivalence.

First we need a number of definitions. A lattice Λ is a discrete subgroup of $\mathbb{R}^2$ generated by two linearly independent vectors $\omega_1, \omega_2 \in \mathbb{R}^2$, i.e.

$$\Lambda = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}. \tag{1.1}$$

We define two elements $x, y \in \mathbb{R}^2$ to be equivalent, $x \sim y$, if and only if $x - y \in \Lambda$. The flat 2-torus $\mathbb{T}_\Lambda = \mathbb{R}^2/\Lambda$ is the quotient space of $\mathbb{R}^2$ under the equivalence $\sim$. We identify $\mathbb{T}_\Lambda$ with the fundamental parallellogram of the lattice centered at $0 \in \mathbb{R}^2$. Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{T}_\Lambda$ be the canonical projection $\pi : x \mapsto [x]$ that sends x to its equivalence class $[x]$ and let $d(x, y) = |x - y|$ be the standard Euclidean metric on $\mathbb{R}^2$. Locally π is an isometry and induces the covering metric $\tilde{d}$ defined by $\pi^*\tilde{d} = d$ on the torus.

Suppose we are given a flat two-dimensional torus $\mathbb{T}_\Lambda$ with its geodesic flow. We will consider the focal decomposition on the tangent plane T_0, which we identify with $\mathbb{R}^2$, at the base point 0. It describes arithmetic properties of the geodesic flow. Namely, the number of geodesics of the same length starting at the base point with the same endpoint.

According to the original study in [**2**], the focal decomposition is characterized by *Brillouin lines*.

DEFINITION 1.2 (Brillouin line). *A* Brillouin line $L_g \subset \mathbb{R}^2$ *is defined as the perpendicular bisector of the line connecting the origin* 0 *and* $g \in \Lambda$, *i.e.*

$$L_g = \{x \mid g \in \Lambda \text{ and } |x| = |x - g|\}$$

For $0 \in \Lambda$, *we define* $L_0 = \{0\}$.

DEFINITION 1.3. *Let $M_\Lambda \subset \mathbb{R}^2$ be the set of all Brillouin lines relative to the lattice Λ, i.e.*

$$M_\Lambda = \bigcup_{g \in \Lambda_*} L_g, \tag{1.2}$$

where $\Lambda_* = \Lambda - \{0\}$.

In the case the flat torus, the focal decomposition can be identified with the set M_Λ, because the exponential map $\exp : T_0 \to \mathbb{T}$ can be identified with the projection map π. Therefore we call M_Λ the focal decomposition of the torus. These Brillouin lines were used by Brillouin in the quantum study of wave propagation in crystals and give rise to the Brillouin zones, as follows.

Let ℓ_x be the open line segment connecting the origin 0 and x and let $\bar{\ell}_x$ be the closure of ℓ_x also containing 0 and x.

Definition 1.4. *Let* $\iota, \chi, \mu : \mathbb{R}^2 \to \mathbb{N}$ *be the* indices *defined by*

$$\iota(x) = \#\{g \in \Lambda \mid L_g \cap \ell_x \neq \emptyset\} \tag{1.3}$$

$$\chi(x) = \#\left\{g \in \Lambda \mid L_g \cap \bar{\ell}_x \neq \emptyset\right\} \tag{1.4}$$

$$\mu(x) = \#\{g \in \Lambda \mid L_g \ni x\} \tag{1.5}$$

where # *means the cardinality of the set. The index* $\mu(x)$ *is referred to as the* multiplicity *of* x.

It follows that $\chi(x) = \iota(x) + \mu(x) + 1$.

Definition 1.5 (Brillouin zone). *The* n*-th Brillouin zone relative to a lattice* Λ *is the set*

$$B_n = \{x \in \mathbb{R}^2 \mid \iota(x) \leq n \text{ and } \chi(x) \geq n+1\}. \tag{1.6}$$

Notation 1. *Although the Brillouin zones* B *and the torus* $\mathbb{T}$ *are defined relative to a lattice* Λ*, we omit the subscripts referring to the lattice. The results will hold for any (but fixed) lattice* Λ*, unless explicitly stated otherwise.*

Example 1.6. *In figure 1.1, we see the first 9 Brillouin zones,* 0 *through* 8*, relative to* $\mathbb{Z}^2$*. The consecutive Brillouin zones are alternately shaded and unshaded.*

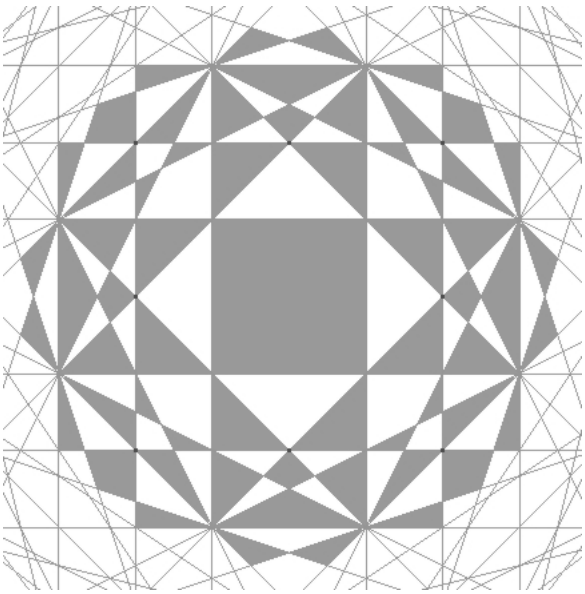


Figure 1.1. The first 9 Brillouin zones relative to $\mathbb{Z}^2$ of Example 1.6.

The topology of the decomposition M_Λ contains information about the geometry of the underlying torus. The main content of the rigidity theorem we prove here is that it actually uniquely determines the geometry of the torus.

We say that the focal decompositions associated to two flat tori are *equivalent* if there exists a homeomorphism between the corresponding tangent planes that maps the decomposition associated to the one torus homeomorphically onto the decomposition of the other.

THEOREM 1.7. *The focal decomposition of two tori are equivalent if and only if the tori are conformally equivalent.*

The idea to study the rigidity of 2-tori through focal decompositions is inspired by Mostow's rigidity theorem. An important ingredient in the proof of theorem 1.7 is the asymptotic shape of Brillouin zones which, independent of the lattice, is a circle. This was shown by Jones in [**4**]. Using a result from analytic number theory [**5**], we give bounds on the distance of B_n from the origin and show that B_n is contained in an annulus with decreasing modulus.

A classical result by Bieberbach [**3**], states that each zone is a fundamental domain for the covering transformation. That is, each Brillouin zone gives rise to a tiling of the torus, which we call a *torus puzzle*. To every torus, we can associate a sequence of these torus puzzles. We show that, arbitrarily close to a given lattice, there exists a lattice such that the sequence of torus puzzles of the associated tori will be distinct, in the sense that there exists at least one pair of torus puzzles that are not homeomorphic.

We define an equivalence relation between torus puzzles, which in addition to requiring the puzzles to be homeomorphic, involves a fixed-point condition. We show that under this equivalence, generically, the torus puzzles relative to two tori are pairwise equivalent if and only if the tori are conformally equivalent. We use theorem 1.7 to prove this result.

CHAPTER 2

Torus Puzzles

In this chapter, we study the topological properties of B_n and show that the projection of every B_n tiles the torus. Such a tiling of the torus is called a torus puzzle. Our special interest lies in determining what combinatorial information about the set M_Λ is encoded in these torus puzzles.

LEMMA 2.1. *Let $x \in \mathbb{R}^2$, then*

$$\iota(x) = \#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\}. \tag{2.1}$$

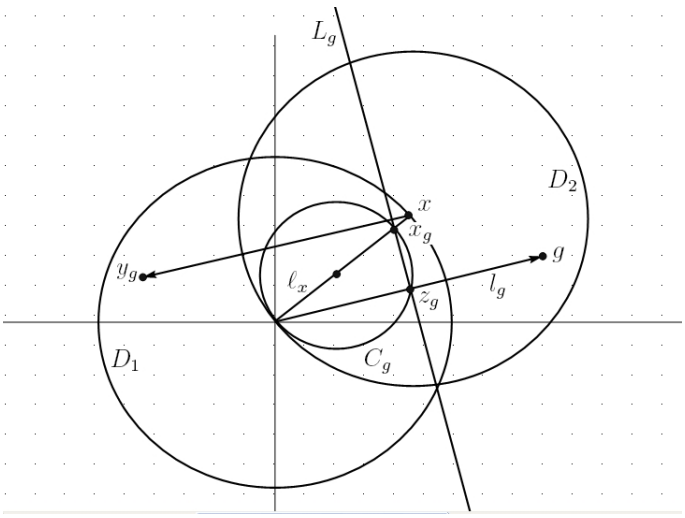


FIGURE 2.1. Proof of Lemma 2.1

PROOF. Let $x \in \mathbb{R}^2$ with index $\iota(x)$. We let $D_1 = D(0, |x|)$, be the open disc with center 0 and radius $|x|$ and similarly $D_2 = D(x, |x|)$ see figure 2.1.
We show that for $g \in \Lambda$ the following are equivalent:

1) $\ell_x \cap L_g \neq \emptyset$,
2) $g \in \Lambda \cap D_2$,
3) $y_g = x - g \in D_1$.

1) $\Leftrightarrow$ 2). Suppose $\ell_x \cap L_g \neq \emptyset$ for some $g \in \Lambda$. Let $\ell_x \cap L_g = \{x_g\}$, then $|x_g| < |x|$. Let C_g be the circle centered at $\frac{1}{2}x_g$ and radius $\rho_g = \frac{1}{2}|x_g|$. Let l_g be the line segment connecting the origin O and g and let $z_g = \frac{1}{2}g$. Since L_g is perpendicular to l_g, $z_g \in l_g \cap C_g$. Since $|x_g| < |x|$ and $\rho_g < \frac{1}{2}|x|$, by congruence, $g \in D_2 \cap \Lambda$. Reading the previous arguments backwards yields the other direction.
2) $\Leftrightarrow$ 3). By symmetry, $\alpha \in D_2$ if and only if $x - \alpha \in D_1$.

Hence, there is a one-to-one correspondence between the set of points

$$\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\}$$

and the set of Brillouin lines L_g such that $\ell_x \cap L_g \neq \emptyset$ and this proves the lemma. □

Definition 2.2. *For $x \in \mathbb{R}^2$, let*

$$\mathcal{O}(x) = \left\{y \in \mathbb{R}^2 \mid \pi(x) = \pi(y),\ |x| = |y|\right\}. \tag{2.2}$$

and $\sigma : \mathbb{R}^2 \to \mathbb{N}$, $\sigma(x) = \#\mathcal{O}(x)$.

Lemma 2.3. *Let $x \in B_n$ and $v = \pi(x)$. Then $\sigma(x) = \mu(x) + 1$ and ι, σ, χ and μ are constant on $\mathcal{O}(x)$. Moreover,*

$$\pi^{-1}(v) \cap B_n = \mathcal{O}(x). \tag{2.3}$$

Proof. In the notation of the proof of lemma 2.1, let $C_i = \partial D_i$ for $i = 1, 2$. If $x \in L_g$, then $g \in C_2$ and $y_g = x - g \in C_1$. Hence $y_g \in \mathcal{O}(x)$. Moreover, if $L_g \neq L_{g'}$, i.e. $g \neq g'$, then $y_g \neq y_{g'}$. Conversely, every $y \in \mathcal{O}(x)$ gives rise to a L_g such that $x \in L_g$; because $\pi(x) = \pi(y)$, $x - y = g$ for some $g \in \Lambda$ and it is easily seen that $x \in L_g$. So $\sigma(x)$ equals the number of points y_g plus x itself, hence $\sigma(x) = \mu(x) + 1$.

Since $\sigma(x) = \sigma(y)$ for all $y \in \mathcal{O}(x)$, μ is constant on $\mathcal{O}(x)$. From lemma 2.1 it is easy to see that $\iota(x)$ (and hence $\chi(x)$) is constant on $\mathcal{O}(x)$.

To prove (2.3), first note that $|x'| = |x|$ for all $x' \in \pi^{-1}(v) \cap B_n$. For suppose that $x, x' \in \pi^{-1}(v) \cap B_n$ but $|x| \neq |x'|$, say $|x| > |x'|$. Then, by lemma 2.1, $\iota(x) \geq \chi(x') > n$, a contradiction. And since the indices $\iota(x)$ and $\chi(x)$ are constant on $\mathcal{O}(x)$, it follows that $y \in B_n$ for all $y \in \mathcal{O}(x)$. □

Example 2.4. *Let $\Lambda = \mathbb{Z}^2$ and $N \in \mathbb{N}$ with prime factorisation*

$$N = 2^\alpha \prod_{i=1}^{k} p_i^{\beta_i} \prod_{j=1}^{l} q_j^{\gamma_j},$$

where $p_i \equiv 1 \bmod 4$ and $q_j \equiv 3 \bmod 4$. Denote $R(N)$ be the number of solutions in $\mathbb{Z}^2$ of $n^2 + m^2 = N$. If all γ_j are even, which is the case for $N = |g|^2$ for $g \in \mathbb{Z}^2$, then $R(N) = 4\prod_{i=1}^{k}(1 + \beta_i)$. See for instance [**6**] *for this result. Thus we have $\sigma(g) = R(|g|^2)$.*

Lemma 2.5. *B_n is closed.*

Proof. Let $x \in B_n^c$, the complement of B_n. Then either $\iota(x) \geq n + 1$ or $\chi(x) \leq n$. The latter is equivalent to $\iota(x) + \sigma(x) \leq n$. In both cases, because Λ and hence $\pi^{-1}(v)$ with $v = \pi(x)$ is discrete, there exists an open neighbourhood around x for which $\iota(x) \geq n + 1$ or $\iota(x) + \sigma(x) \leq n$ respectively, which shows that the complement of B_n is open and hence B_n is closed. □

Lemma 2.6. *Let $x \in B_n$, then $x \in \mathrm{Int}(B_n)$ if and only if $\sigma(x) = 1$. Consequently,*

$$M_\Lambda = \bigcup_{g \in \Lambda} L_g = \bigcup_{n \in \mathbb{N}} \partial B_n. \tag{2.4}$$

Proof. If $\sigma(x) = 1$, then $\iota(x) = n$ and $\mu(x) = 0$. Therefore, there exists a small neighbourhood around x such that $\iota(y) = n$ and $\mu(y) = 0$. Thus $y \in B_n$ for all y in this neighbourhood, so $x \in \mathrm{Int}(B_n)$. Conversely, if $\sigma(x) \geq 2$, then $\mu(x) \geq 1$. Let $y = tx$ with $t = 1 + \epsilon$. Then $\iota(y) \geq \chi(x) \geq n + 1$ for all $\epsilon > 0$, hence $y \in B_n^c$ and thus $x \in \partial B_n$.

Since $\sigma(x) = \mu(x) + 1$, $x \in \partial B_n$ if and only if $\mu(x) \geq 1$, hence (2.4) follows. □

If $x \in \mathrm{Int}(B_n)$, then $\iota(x) = n$ and $\chi(x) = n+1$. This yields that the zones tile $\mathbb{R}^2$ in the sense that

$$\bigcup_{n\in\mathbb{N}} B_n = \mathbb{R}^2 \quad \text{and} \quad \mathrm{Int}(B_n) \cap \mathrm{Int}(B_m) = \emptyset \quad \text{if} \quad n \neq m. \tag{2.5}$$

Definition 2.7. *Define*

$$\partial_n^- = B_n \cap B_{n-1} \quad \text{and} \quad \partial_n^+ = B_n \cap B_{n+1}. \tag{2.6}$$

If $x \in \partial B_n$, then since $\sigma(x) \geq 2$ (or equivalently $\mu(x) \geq 1$), either $\iota(x) \leq n-1$ and $\chi(x) \geq n+1$ or $\iota(x) = n$ and $\chi(x) \geq n+2$, corresponding to $x \in B_n \cap B_{n-1}$ and $x \in B_n \cap B_{n+1}$ respectively. It follows that

$$\partial B_n = \partial_n^- \cup \partial_n^+ \tag{2.7}$$

We denote $\partial_n = \partial_n^- \cup \partial_n^+$ and (2.4) rewrites as

$$M_\Lambda = \bigcup_{n\in\mathbb{N}} \partial_n = \bigcup_{n\in\mathbb{N}} \partial_n^+, \tag{2.8}$$

since $\partial_n^+ = \partial_{n+1}^-$ and $\partial_0^- = \emptyset$.

Topological properties of Brillouin zones are given in the next proposition, see for instance [**4**].

Proposition 2.8 (Topology of B_n). *For every Brillouin zone B_n, the following holds:*

(i) *B_n is compact and,*
(ii) *$\partial_n^\pm \simeq \mathbb{S}^1$ and B_n is path-connected.*

Although B_n is connected, the interior of B_n is in general not connected. Let $\{b_n^j\}_{j\in J_n}$ be the set of connected components of $\mathrm{Int}(B_n)$, then

$$\bigcup_{j\in J_n} b_n^j = \mathrm{Int}(B_n). \tag{2.9}$$

The set $B_n^j = b_n^j \cup \partial b_n^j$ is called a *subzone*[1] and we have

$$B_n = \bigcup_{j\in J_n} B_n^j. \tag{2.10}$$

Lemma 2.9. *B_n is a finite union of convex polygons.*

Proof. Because Λ is discrete only finitely many Brillouin lines can meet B_n because B_n is bounded. This yields that every B_n consists of finitely many subzones and that the boundary of a subzone is comprised of finitely many edges, so every subzone is a polygon.

To prove convexity, notice that every Brillouin line L_g divides $\mathbb{R}^2$ into two half planes H_g^i, $i = 1, 2$. Since $M_\Lambda = \bigcup_{g\in\Lambda_*} L_g = \bigcup_{n\in\mathbb{N}} \partial_n$, every subzone is the intersection of finitely many convex half planes and thus is convex. □

A point $x \in M_\Lambda$ is called a *vertex* if $\mu(x) \geq 2$. The connected components of $\{x \in M_\Lambda \mid \mu(x) = 1\}$ are the *edges* of M_Λ.

Let $\mathcal{P}_n = \bigcup_{j\in J_n} \mathcal{P}_n^j$ with $\mathcal{P}_n^j = \pi(B_n^j)$. Moreover, let $\partial^\pm \mathcal{P}_n = \pi(\partial_n^\pm)$ and $\partial \mathcal{P}_n = \pi(\partial_n)$.

We define $\tilde{e} \subset \mathbb{T}$ to be an *edge* if $e \subset M_\Lambda$ is an edge and $\tilde{e} = \pi(e)$. Similarly, we say a region $\tilde{P} \subset \mathbb{T}$ is a *convex polygon* on the torus, if $P \subset \mathbb{R}^2$ is a convex polygon and $\pi(P) = \tilde{P}$ and π injective on $\mathrm{Int}(P)$.

[1]Subzones are also referred to as *Landsberg subzones*.

Let $\{P_i\}_{i\in I}$ be a finite family of polygons on $\mathbb{R}^2$ and $v \in \mathbb{T}$ such that $v \in \tilde{P}_i = \pi(P_i)$ for all $i \in I$. Then v is called a *vertex* if $\pi^{-1}(v) \cap P_i$ is a vertex of P_i for every $i \in I$. We call an edge $\tilde{e} \subset \partial\mathcal{P}_n$ a *plus edge* if $\tilde{e} \subset \partial^+\mathcal{P}_n$ and a *minus edge* if $\tilde{e} \subset \partial^-\mathcal{P}_n$.

Definition 2.10 (Torus Puzzle). *A* torus puzzle *is a finite family of convex polygons, $\{P^j\}_{j\in J_n}$ with $P^j \subset \mathbb{T}$, such that*

(i) *the union of the polygons covers the torus,*
(ii) *if $i \neq j$, then the intersection $P^i \cap P^j$ is either empty, or a single vertex of both P^i and P^j or a single edge of both.*

When the polygons are all triangles, the notion of a torus puzzle coincides with that of a triangulation.

Theorem 2.11. *Every $\mathcal{P}_n$ is a torus puzzle.*

Proof. By lemma 2.1, $\{B_n^j\}_{j\in J_n}$ is a finite family of convex polygons on $\mathbb{R}^2$, hence $\{\mathcal{P}_n^j\}_{j\in J_n}$ is a finite family of convex polygons on $\mathbb{T}$. To show that $\pi : B_n \to \mathbb{T}$ is surjective, let $v \in \mathbb{T}$ and consider $\pi^{-1}(v)$. Because Λ is discrete, $\pi^{-1}(v)$ is discrete. Hence, there exists an $x \in \pi^{-1}(v)$ such that

$$\#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\} \leq n$$

and

$$\#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| \leq |x|\right\} \geq n+1.$$

We have shown this to be equivalent to $\iota(x) \leq n$ and $\chi(x) \geq n+1$, hence $x \in B_n$. This shows that $\{\mathcal{P}_n^j\}_{j\in J_n}$ satisfies property (i) of definition 2.10.

To prove $\mathcal{P}_n$ satisfies part (ii), $\pi : \mathrm{Int}(B_n) \to \mathbb{T}$ is injective since $\pi^{-1}(v) \cap B_n = \mathcal{O}(x)$ and $\sigma(x) = 1$ if and only if $x \in \mathrm{Int}(B_n)$. This shows that $\mathrm{Int}(\mathcal{P}_n^i) \cap \mathrm{Int}(\mathcal{P}_n^j) = \emptyset$ if $i \neq j$ for $i, j \in J_n$. Let $x \in \partial_n$ and $\pi(x) = v$. A point $x \in \partial_n$ is a vertex if and only if $\mu(x) \geq 2$. By lemma 2.3, for all points $y \in \mathcal{O}(x)$, $\mu(y) = \mu(x)$ and these are exactly all the points in ∂_n that are mapped to v, hence v is a vertex. Conversely, if x is not a vertex then $\mu(x) = 1$ and $\mu(y) = 1$ for the other $y \in \mathcal{O}(x)$ so y is not a vertex and this proves $\{\mathcal{P}_n^j\}_{j\in J_n}$ satisfies part (ii) of definition 2.10. □

It particular, this shows that B_n is a *fundamental domain* for Λ. That is, B_n is closed by proposition 2.8 (i) and, moreover,

$$\bigcup_{g\in\Lambda} gB_n = \pi^{-1}(\pi(B_n)) = \pi^{-1}(\mathcal{P}_n) = \mathbb{R}^2$$

and

$$\mathrm{Int}(gB_n) \cap \mathrm{Int}(g'B_n) = \emptyset \qquad \text{if } g \neq g'.$$

The first equality follows from surjectivity of $\pi : B_n \to \mathbb{T}$ and the second by injectivity of $\pi : \mathrm{Int}(B_n) \to \mathbb{T}$. This result was first shown by Bieberbach in [**3**] and later by Jones in [**4**]. It also follows that

Corollary 2.12. *The measure of B_n is equal for all $n \in \mathbb{N}$.*

Proof. Since π is an isometry on every element of $\{\mathrm{Int}(B_n^j)\}_{j\in J_n}$ and since the measure of ∂_n and hence $\partial\mathcal{P}_n$ is zero, the measure of B_n equals the measure of $\mathcal{P}_n$ which in turn is equal to the measure of $\mathbb{T}$. □

Example 2.13. *Figure 2.2 shows the puzzle $\mathcal{P}_4$ (right figure) relative to $\mathbb{Z}^2$. The left and middle figure shows the decomposition of $\partial\mathcal{P}_4$ into $\partial^-\mathcal{P}_4$ and $\partial^+\mathcal{P}_4$ respectively.*

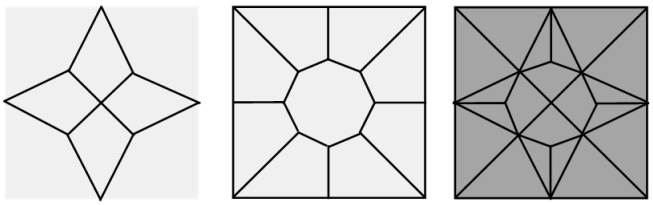


FIGURE 2.2. The puzzle $\mathcal{P}_4$ of Example 2.13.

See Appendix A for the puzzles $\mathcal{P}_n$, $n = 1, ..., 8$ relative to $\mathbb{Z}^2$, cf. example 1.6.

Let $x \in \partial_n^- \cap \partial_n^+$, or equivalently, $x \in B_{n-1} \cap B_n \cap B_{n+1}$. Then $\iota(x) \leq n-1$ and $\chi(x) \geq n+2$. Hence, $\mu(x) \geq 2$. Every $x \in \partial_n^- \cap \partial_n^+$ is a vertex.

DEFINITION 2.14. *Let*

$$\mathcal{I}_n = \{x \in \mathbb{R}^2 \mid x \in \partial_n^- \cap \partial_n^+\}, \tag{2.11}$$

the set of intermediate *vertices of* ∂_n *and let* $\gamma_n^\pm = \partial_n^\pm - \mathcal{I}_n$. *The vertices of* $\gamma_n^\pm$ *are called* plus *and* minus *vertices respectively, see figure 2.3.*

Since the union of $\mathcal{I}_n$ and $\gamma_n^\pm$ is ∂_n, every vertex of ∂_n is either a plus, minus or intermediate vertex.

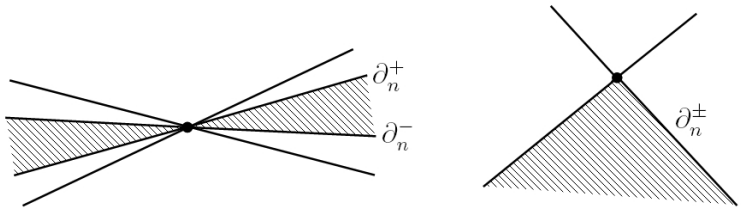


FIGURE 2.3. An intermediate vertex (left) and a plus/minus vertex (right) of ∂_n.

LEMMA 2.15. *Let* $x \in \partial_n$ *a vertex. If* x *is a plus, minus or intermediate vertex, then* y *is plus, minus or intermediate vertex respectively for all* $y \in \mathcal{O}(x)$.

PROOF. For $x \in \mathcal{I}_n$ we have $\iota(x) \leq n-1$ and $\chi(x) \geq n+2$. If $x \in \gamma_n^+$, then $x \in B_n \cap B_{n+1}$ but $x \notin B_{n-1}$, hence $\iota(x) = n$ and $\chi(x) \geq n+2$. For a vertex we must have $\mu(x) \geq 2$ hence a vertex in γ_n^+ satisfies $\iota(x) = n$ and $\chi(x) \geq n+3$. Similarly, if $x \in \gamma_n^-$, then $\iota(x) \leq n-1$ and $\chi(x) = n+1$. So a vertex in γ_n^- satisfies $\iota(x) \leq n-2$ and $\chi(x) = n+1$.

Since these conditions are mutually exclusive, and, by lemma 2.3, $\iota(x) = \iota(y)$ and $\chi(x) = \chi(y)$ for all $y \in \mathcal{O}(x)$, the result follows. □

DEFINITION 2.16. *Let* v *be a vertex of* $\mathcal{P}_n$, *then* v *is a* vertex of type I *if all edges incident to* v *are plus edges and* v *is a* vertex of type II *if all edges incident to* v *are minus edges. Finally, a vertex* v *is a* vertex of type III, *if the edges incident to* v *are alternately plus and minus edges.*

Definition 2.17. *Let $x \in \partial_n$ and $v = \pi(x) \in \partial\mathcal{P}_n$. We define $\tilde{\mu}(v)$ to be the number of edges that are* locally *incident to v. If x lies on the interior of an edge, we define $\tilde{\mu}(v) = 1$.*

By *locally* in definition 2.17 we mean the number of edges incident to a vertex in a small neighbourhood, since an edge can have its vertices identified on the torus, see for instance the puzzles $\mathcal{P}_1, \mathcal{P}_6$ and $\mathcal{P}_7$ relative to $\mathbb{Z}^2$ in Appendix A.

If we set $\tilde{\mathcal{I}}_n = \pi(\mathcal{I}_n)$, then

Lemma 2.18.

$$\partial^-\mathcal{P}_n \cap \partial^+\mathcal{P}_n = \tilde{\mathcal{I}}_n \tag{2.12}$$

Proof. We need to show that $\pi(\gamma_n^-) \cap \pi(\gamma_n^+) = \emptyset$. Let $v \in \mathbb{T}$ and $x \in \gamma_n^+$ such that $\pi(x) = v$. Since $\pi^{-1}(v) \cap B_n = \mathcal{O}(x)$ by lemma 2.3 and $y \in \gamma_n^\pm$ for all $y \in \mathcal{O}(x)$ if $x \in \gamma_n^\pm$ by the proof of lemma 2.15, we have $\pi(\gamma_n^-) \cap \pi(\gamma_n^+) = \emptyset$ and hence (2.12). □

The following proposition relates the combinatorial properties of B_n to that of the torus puzzles $\mathcal{P}_n$ on the torus $\mathbb{T}$.

Proposition 2.19. *Let $x \in \partial_n$ be a vertex and $v = \pi(x)$. If x is a plus, minus or intermediate vertex, then v is of type I, II or III respectively and*

(i) $\tilde{\mu}(v) = \mu(x) + 1$ if v is of type I or II,

(ii) $\tilde{\mu}(v) = 2\mu(x) + 2$ if v is of type III.

Proof. By lemma 2.15, if x is a plus or minus or intermediate vertex, then all vertices in $\mathcal{O}(x)$ are plus or minus vertices respectively. If x is a plus or minus vertex, it is clear that the corresponding vertex v is of type I or II respectively. So consider the case where $\mathcal{O}(x)$ consists of all intermediate vertices. For every subzone B_n^j sharing the intermediate vertex $y \in \mathcal{O}(x)$, the two edges contained in ∂B_n^j incident to y consists of one edge contained in ∂_n^- and one edge contained in ∂_n^+, cf. figure 2.3. Hence, for every $\mathcal{P}_n^j$ that shares the common vertex v, there is one minus edge and one plus edge incident to v. By lemma 2.18, $\partial^-\mathcal{P}_n \cap \partial^+\mathcal{P}_n = \tilde{\mathcal{I}}_n$, so the minus edge incident to v of one subzone $\mathcal{P}_n^i$ is identified to the minus edge incident to v of the neighbouring subzone $\mathcal{P}_n^j$ for certain $i, j \in J_n$. Similarly, plus edges are mapped to plus edges, thus the edges incident to v are alternately plus and minus edges, so v is of type III.

To prove the second statement, note that if v is of type I or II, then to every vertex $y \in \mathcal{O}(x)$ there are exactly two plus or minus edges of ∂B_n^j for of some $j \in J$ incident to y. Exactly $2\sigma(x) = 2(\mu(x) + 1)$ plus or minus edges are mapped to $\mathbb{T}$ and are incident to v. For any edge $\tilde{e} \subset \partial^-\mathcal{P}_n$, $\pi^{-1}(\tilde{e}) \cap \partial_n^- = e \cup e'$ for certain edges $e, e' \subset \partial_n^-$ by theorem 2.11 and this yields that $\tilde{\mu}(v) = \sigma(x) = \mu(x) + 1$ which proves (i). If v is of type III, then incident to every vertex $y \in \mathcal{O}(x)$ are exactly two plus edges and two minus edges of ∂_n. By similar reasoning, we have that $\tilde{\mu}(v) = 2\sigma(x) = 2\mu(x) + 2$ and proves (ii). □

Definition 2.20. *A lattice Λ is in* general position *if the Brillouin lines of M_Λ intersect at most pairwise.*

Almost all lattices are in general position, in the sense that the set of lattices in general position has full measure in the set of all lattices. However, lattices not in general position are also dense in this set, see [**4**].

EXAMPLE 2.21. *Consider the following family of lattices*

$$\Lambda(\theta) = (1,0)\mathbb{Z} \oplus (\cos\theta, \sin\theta)\mathbb{Z},$$

with $\theta \in (0,\pi)$. *It is clear that* $L_{(2,0)}$ *intersects* $(1,0)$. *Consider the points* $g_1, g_2 \in \Lambda(\theta)$,

$$g_1 = (1+\cos\theta, \sin\theta) \qquad \text{and} \qquad g_2 = (1-\cos\theta, -\sin\theta). \tag{2.13}$$

An easy computation shows that both lines L_{g_1} *en* L_{g_2} *intersect* $(1,0)$. *Hence, this family of lattices is not in general position. In particular, the set of lattices not in general position is uncountable.*

We can write $\Lambda = B\mathbb{Z}^2$ with $B \in \mathrm{GL}(2,\mathbb{R})$. This matrix B gives rise to a (positive definite) quadratic form induced by the positive definite matrix B^tB. The Brillouin lines relative to the Euclidean metric and the lattice $\Lambda = B\mathbb{Z}^2$ are identical to the Brillouin lines relative to the lattice $\mathbb{Z}^2$ with the metric induced by the matrix B^tB.

An interesting result, proved by Kupka, Peixoto and Pugh in [**9**], is the following relation between the coefficients of a quadratic form and the notion of general position.

THEOREM 2.22. *If the coefficients* a, b, c *of the positive definite quadratic form* Q *are rationally independent, then no three of its Brillouin lines meet at a common point.*

It is understood that the Brillouin lines in theorem 2.22 are the Brillouin lines relative to the metric induced by the quadratic form Q.

So if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$, then M_Λ with $\Lambda = B\mathbb{Z}^2$ is in general position if the coefficients $a^2+c^2, ab+cd$ and b^2+d^2 are rationally independent. It is not known whether the converse of theorem 2.22 holds.

DEFINITION 2.23. *Two puzzles* $\mathcal{P}_n$ *and* $\mathcal{P}'_n$ *are homeomorphic if there exists a homeomorphism* $h_n : \mathbb{T} \to \mathbb{T}'$ *such that* $h_n(\partial\mathcal{P}_n) = \partial\mathcal{P}'_n$.

PROPOSITION 2.24. *Let* Λ *be in general position and* Λ' *not in general position. Then there exists an* $n \in \mathbb{N}$ *such that* $\mathcal{P}_n$ *and* $\mathcal{P}'_n$ *are not homeomorphic.*

PROOF. By assumption, $\mu(x) = 2$ for every vertex $x \in M_\Lambda$, hence $\tilde{\mu}(u) = 2+1 = 3$ or $2(2+1) = 6$ for $u = \pi(x)$ of type I/II or III respectively, for every vertex u of every $\mathcal{P}_n$ by proposition 2.19. On the other hand, there exists at least two (antipodal) vertices $y \in M_{\Lambda'}$ for which $\mu(y) \geq 3$. For a certain n, $y \in \partial'_n$ is an intermediate vertex. Thus $\tilde{\mu}(v) \geq 2(3+1) = 8$ for $v = \pi'(y) \in \mathcal{P}'_n$. Hence, these puzzles can't be homeomorphic. $\square$

Hence, arbitrarily close to a given lattice, there exists a lattice such that the torus puzzles of the associated tori are not pairwise homeomorphic.

CHAPTER 3

Asymptotic Behaviour of B_n

In this chapter we study the behaviour of B_n for $n \to \infty$. More precisely, we derive bounds on the distance of B_n from the origin and show that B_n is contained in a circular annulus with decreasing modulus. Consequently, B_n always becomes circular shaped, independent of the underlying lattice.

If we let G be the set of all lattices in $\mathbb{R}^2$, then we define two lattices $\Lambda, \Lambda' \in G$ to be *conformally equivalent*, $\Lambda \sim \Lambda'$, if there exists a conformal matrix A,

$$A = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{3.1}$$

where $\lambda > 0$ and $\theta \in [0, \pi)$, such that $\Lambda' = A(\Lambda)$. We denote $\mathcal{G} = G/\sim$.

Remark 3.1. *Note that A is orientation preserving and that $A(L_g) = L_{A(g)}$, hence*

$$A\left(M_\Lambda\right) = M_{A(\Lambda)}. \tag{3.2}$$

Every lattice $\Lambda \in \mathcal{G}$ can be written as $\Lambda = B(\mathbb{Z}^2)$ where

$$B = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \tag{3.3}$$

with $(\alpha, \beta) \in \mathcal{H} = (-\infty, \infty) \times (0, \infty) \subset \mathbb{R}^2$, the upper half plane. In other words, a lattice in $\mathcal{G}$ has the form

$$\Lambda = (1, 0)\mathbb{Z} \oplus (\alpha, \beta)\mathbb{Z}.$$

By *modular symmetry*, this representation is not unique. That is, if two lattices Λ, Λ' are generated by the vectors (ω_1, ω_2) and (ω_1', ω_2') then $\Lambda = \Lambda'$ if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix},$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. For $\Lambda, \Lambda' \in \mathcal{G}$, we have $\Lambda = \Lambda'$ if (and only if) the associated matrix has the form $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ with $n \in \mathbb{Z}$. Hence, the points $(\alpha + n, \beta) \in \mathcal{H}$ for $n \in \mathbb{Z}$ all represent the same lattice.

If $x \in \mathrm{Int}(B_n)$, then $\iota(x) = n$ and by lemma 2.1,

$$\iota(x) = \#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\} = n,$$

which we proved to be equivalent to

$$\#\left\{g \mid g \in \Lambda \cap D(x, |x|)\right\} = n. \tag{3.4}$$

The following (classical) result is essential in this respect, the proof of which can be found in [**5**].

THEOREM 3.1 (Van der Corput, 1920). *Let D be a region bounded by a convex simple closed curve, piecewise twice differentiable, with radius of curvature bounded above by R. The discrepancy Δ of D, the difference between the number of integer points in D and the area of D, satisfies*

$$\Delta = O(R^{2/3}) \tag{3.5}$$

Theorem 3.1 gives rise to the following bounds on the distance of a point $x \in B_n$ from the origin.

THEOREM 3.2. *Let $\Lambda = B(\mathbb{Z}^2)$ where $B = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$, $(\alpha, \beta) \in \mathcal{H}$. Then there exists a constant $K_\Lambda > 0$ depending only on the lattice Λ such that for $x \in B_n$ and $n \geq 1$,*

$$|x| \in \left[\left(\frac{\beta n}{\pi}\right)^{1/2} - \frac{K_\Lambda}{n^{1/6}}, \left(\frac{\beta n}{\pi}\right)^{1/2} + \frac{K_\Lambda}{n^{1/6}} \right]. \tag{3.6}$$

PROOF. First let $x \in \mathrm{Int}(B_n)$. Since $\det(B) = \beta \neq 0$, B is invertible. Let $C_x = \partial D(x, |x|)$, then $\mathcal{E}_x := B^{-1}(C_x)$ is an ellipse and the region bounded by this ellipse satisfies the requirements of theorem 3.1. The radius of curvature of an ellipse with major and minor axes given by a and b respectively is bounded from above by $R = \frac{a^2}{b}$. Let R_x denote the upper bound on the radius of curvature of $\mathcal{E}_x$ and let $t_n(x) = (\frac{\pi}{\beta})^{1/2}|x|$. The (semi)axes of $\mathcal{E}_x$ are proportional to $|x|$ and hence to $t_n(x)$, thus R_x is proportional to $t_n(x)$ where the constant of proportionality depends only the lattice Λ and

$$|B^{-1}(D(x, |x|))| = \det(B^{-1})\pi|x|^2 = \frac{\pi}{\beta}|x|^2 = t_n(x)^2. \tag{3.7}$$

From equation (3.4), it follows that

$$\#\left\{g \mid g \in \mathbb{Z}^2 \cap B^{-1}(D(x, |x|))\right\} = n, \tag{3.8}$$

so by theorem 3.1 and (3.7)

$$n = |B^{-1}(D(x, |x|))| + O(t_n(x)^{2/3}) = t_n(x)^2 + O(t_n(x)^{2/3}). \tag{3.9}$$

Put $t_n(x) = \sqrt{n}\,(1 + z_n(x))$, with $z_n(x)$ the error term. Since $t_n(x) > 0$, $1 + z_n(x) > 0$ and by (3.9),

$$|n - n\,(1 + z_n(x))^2\,| \leq C_\Lambda(\sqrt{n}\,(1 + z_n(x)))^{2/3}, \tag{3.10}$$

for some constant $C_\Lambda > 0$ depending only on the lattice Λ. Then (3.10) for $n \geq 1$, after some manipulation, reads

$$|z_n(x)| \leq \frac{C_\Lambda}{n^{2/3}} \frac{(1 + z_n(x))^{2/3}}{z_n(x) + 2}. \tag{3.11}$$

For $z_n \in (-1, \infty)$, $0 < \frac{(1+z_n)^{2/3}}{z_n+2} \leq \frac{2^{2/3}}{3}$, so (3.11) reduces to $|z_n(x)| \leq \frac{C'_\Lambda}{n^{2/3}}$, where $C'_\Lambda = \frac{2^{2/3}}{3} C_\Lambda$ yielding

$$|z_n(x)|\sqrt{n} \leq \frac{C'_\Lambda}{n^{2/3}}\sqrt{n} = \frac{C'_\Lambda}{n^{1/6}}. \tag{3.12}$$

Since $t_n(x) = (\frac{\pi}{\beta})^{1/2}|x|$ the result follows for all $x \in \mathrm{Int}(B_n)$ with $K_\Lambda = (\frac{\beta}{\pi})^{1/2} C'_\Lambda$. Letting x approach ∂_n, we see that these bounds are in fact valid for all $x \in B_n$. □

REMARK 3.2. *Note that $\det(B) = \beta$ is independent of the representation of the lattice, so the statement of the theorem 3.2 is well-defined.*

CHAPTER 4

Rigidity of M_Λ

In this chapter we prove our main result that the focal decomposition M_Λ is *rigid* in the sense that M_Λ and $M_{\Lambda'}$ are homeomorphic if and only if Λ and Λ' are conformally equivalent.

DEFINITION 4.1. *We define $M_\Lambda \simeq M_{\Lambda'}$, if there exists an orientation preserving homeomorphism*

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad \text{such that} \quad \varphi(M_\Lambda) = M_{\Lambda'}. \tag{4.1}$$

NOTATION 2. *In order to distinguish between the Brillouin zones relative to Λ and Λ', we denote these B_n and B'_n respectively.*

THEOREM 4.2 (Rigidity Theorem). *$M_\Lambda \simeq M_{\Lambda'}$ if and only if Λ and Λ' are conformally equivalent.*

For the proof of the theorem, we need the following lemmas.

LEMMA 4.3. *Let φ be as in definition 4.1. Then φ induces a bijection $\psi : \Lambda_* \to \Lambda'_*$, defined by*

$$\varphi(L_g) = L_{\psi(g)} = L_{g'}. \tag{4.2}$$

PROOF. Because φ is a homeomorphism, $\mu(x) = \mu(x')$ where $x, x' \in \mathbb{R}^2$, $\varphi(x) = x'$. In particular, φ maps vertices to vertices. Consider a vertex x that is the intersection point of m Brillouin lines L_{g_i}, $g_i \in \Lambda$ for $i = 1, ..., m$, so $\mu(x) = m$.

If $\varphi(x) = x'$, then x' is the intersection point of n Brillouin lines $L_{g'_j}$, $g'_j \in \Lambda'$, $j = 1, ..., m$. Let $g = g_i$ for some $i = 1, ..., m$. The plane minus L_g divides $\mathbb{R}^2$ into two connected half-planes H^1_g and H^2_g, i.e. $\mathbb{R}^2 \backslash L_g = H^1_g \cup H^2_g$. Locally, there are exactly $m-1$ edges e^1_k incident to x such that $e^1_k \subset H^1_g$ and $m-1$ edges e^2_k incident to x with $e^2_k \subset H^2_g$. Hence, $\varphi(H^1_g)$ contains $m-1$ edges $\tilde{e}^1_k = \varphi(e^1_k)$ incident to x' and $\varphi(H^2_g)$ contains $m-1$ edges $\tilde{e}^2_k = \varphi(e^2_k)$ incident to x'. So locally the image $\varphi(L_g)$ goes across x' as a straight line segment. Since this holds for every vertex, $\varphi(L_g) \subseteq L_{g'}$ for some $g' = g'_j \in \Lambda'_*$. The same arguments show that $\varphi^{-1}(L_{g'}) \subseteq L_g$, thus $\varphi(L_g) = L_{g'}$.

Since φ is a homeomorphism, it is seen that the map $\psi : \Lambda_* \to \Lambda'_*$ defined by (4.2) is a bijection and this concludes the proof. □

LEMMA 4.4. *Given φ as in definition 4.1. There exists a uniform $N \in \mathbb{N}$ such that, if $x \in \mathrm{Int}(B_n)$, i.e. $\iota(x) = n$, then $n - N \leq \iota(x') \leq n + N$*

PROOF. Let $x \in \mathrm{Int}(B_n)$, then $\iota(x) = n$, i.e. there are n lines L_g such that $L_g \cap \ell_x \neq \emptyset$. Let $x' = \varphi(x)$. By lemma 4.3, $\varphi(L_g) = L_{g'}$, with $g \in \Lambda_*$ and $g' \in \Lambda'_*$. Let $\gamma = \varphi(\ell_x)$, then γ is a continuous curve between $\varphi(0)$ and x'. Every L_{g_i}, $g_i \in \Lambda_*$, $i = 1, ..., n$ has exactly one point of intersection with ℓ_x and is transversal to ℓ_x. Hence there are exactly n Brillouin lines $L_{g'_j}$, $g'_j \in \Lambda'_*$, $j = 1, ..., n$ and every such line has exactly one point of

intersection with γ. Moreover, because L_{g_i} are transversal to ℓ_x for all $i = 1, ..., n$, $L_{g'_j}$ are transversal to γ for all $j = 1, ..., n$.

Let D be the disc $D(O, R)$ with $R = |\varphi(0)| < \infty$. The curve γ can have (multiple) intersection points with $\ell_{x'}$ and γ meets $\ell_{x'}$ at the point x', see figure 4.1.

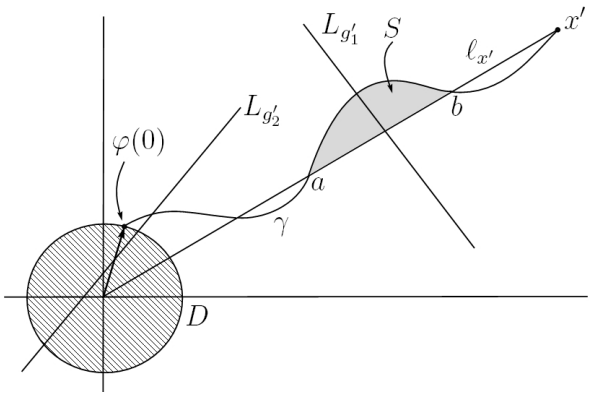


FIGURE 4.1. Proof of Lemma 4.4.

Suppose γ has intersection points with $\ell_{x'}$ and let a, b be two consecutive intersection points. Let S be the Jordan domain enclosed by $\ell_{x'}$ and γ between a and b. If a Brillouin line $L_{g'_1}$, $g'_1 \in \Lambda'_*$, enters S by crossing γ, then it has to leave S through $\ell_{x'}$, since $L_{g'_1}$ has only one point of intersection with γ. Suppose however that a line L_{g_2} intersects ℓ_x, but that the image $L_{g'_2}$ does not intersect $\ell_{x'}$. In this case, the line $L_{g'_2}$ has to escape through the disc D. But because at most finitely many Brillouin lines can meet any bounded subset of $\mathbb{R}^2$, the number of Brillouin lines that can escape through the disc D is uniformly bounded by a certain $N \in \mathbb{N}$. Conversely, there are lines that could intersect with $\ell_{x'}$, but not with γ. Again, since $\iota(\varphi(0)) \leq N$, this number of lines is uniformly bounded by N. Hence $\iota(x') = n + \iota(\varphi(0)) \leq n + N$. Hence, if $\iota(x) = n$ and $\varphi(x) = x'$, then $n - N \leq \iota(x') \leq n + N$. □

Let $\mathcal{B}_g$ be the *bundle of Brillouin lines* consisting of all Brillouin lines parallel to L_g. The Brillouin lines in a bundle are parallel, so by lemma 4.3 and injectivity of φ, we see that bundles are mapped to bundles,

$$\varphi(\mathcal{B}_g) = \mathcal{B}_{g'} \quad \text{where } g' = \psi(g). \tag{4.3}$$

We call an element g that is the generator of the subgroup formed by all lattice points on the line through 0 and g the *generator of the bundle* $\mathcal{B}_g$.

LEMMA 4.5. *Let φ be as in definition 4.1, then*

$$\varphi(x) = Q(x) + \delta(x),$$

with Q linear and $|\delta(x)| \leq \mathcal{K}$ for all $x \in \mathbb{R}^2$ for some $\mathcal{K} > 0$.

PROOF. Label the Brillouin lines in a bundle $\mathcal{B}_g$ $1, 2, 3, ...$ where the two (on either side of the origin) Brillouin lines closest to the origin are labeled 1, the lines second closest to the origin 2 and so forth and then let $\tilde{\mathcal{B}}_g$ denote the bundle with all even-labeled lines deleted.

The union of two such bundles $\tilde{\mathcal{B}}_{g_1} \cup \tilde{\mathcal{B}}_{g_2}$, with $g_1, g_2 \in \Lambda_*$ independent, tile $\mathbb{R}^2$ into identical parallellograms. Now, φ maps these bundles $\tilde{\mathcal{B}}_{g_i}, i = 1, 2$ bijectively onto two

bundles $\tilde{\mathcal{B}}_{g_1'}$ and $\tilde{\mathcal{B}}_{g_2'}$ with $g_i' = \psi(g_i) \in \Lambda_*'$, $i = 1, 2$ independent. These two bundles tile $\mathbb{R}^2$ into parallellograms with certain diameter $\mathcal{D}$. Without loss of generality, we may assume that the the lattice points g_1 and g_1' lie on the $x-$axis and that if g_2 lies in the upper half plane, then g_2' lies in the upper half plane. Since φ is a homeomorphism, it sends these parallellograms to the corresponding parallellograms. Moreover, because φ is orientation preserving, it sends the left and right (vertical) edges of a parallellogram to the left and right (vertical) edges of the image parallellogram. Similarly, it sends the upper and lower edges to the corresponding upper and lower image edges. There exists a non-singular linear map Q such that $Q(\tilde{\mathcal{B}}_{g_1} \cup \tilde{\mathcal{B}}_{g_2}) = \tilde{\mathcal{B}}_{g_1'} \cup \tilde{\mathcal{B}}_{g_2'}$.

If we write $\varphi(x) = Q(x) + (\varphi(x) - Q(x)) = Q(x) + \delta(x)$, then, up to the fixed translation $\varphi(0)$, $\delta(x)$ is uniformly bounded by $\mathcal{D}$. Setting $\mathcal{K} = \mathcal{D} + |\varphi(0)|$, we have that $|\delta(x)| \leq \mathcal{K}$. □

Proof of theorem 4.2. The *if* part easily follows, because if $\Lambda' = A(\Lambda)$ with A conformal, then $A(M_\Lambda) = M_{\Lambda'}$, hence $M_\Lambda \simeq M_{\Lambda'}$ with $\varphi = A$.

To prove the *only if* part, by lemma 4.5, the linear part $|Q(x)| \to \infty$ for $|x| \to \infty$. Since $\delta(x)$ is bounded, $\frac{|\delta(x)|}{|Q(x)|} \to 0$ for $|x| \to \infty$. Thus for $|x| \to \infty$, the behaviour of φ is completely determined by Q. By theorem 3.2, B_n converges to a large circle, for $n \to \infty$. This implies in turn, by lemma 4.4, that $\varphi(B_n)$ converges to a large circle for $n \to \infty$. Hence Q maps circles to circles. The only possible non-singular linear map that maps circles is to circles is a rotation or reflection combined with dilatation. A reflection reverses orientation, and φ is orientation preserving if and only if Q is orientation preserving. So Q cannot be a reflection. Hence, Q is a combination of a rotation and dilatation, that is, Q is conformal.

We show that $\varphi(M_\Lambda) = Q(M_\Lambda)$ by showing that $\varphi(\mathcal{B}_g) = Q(\mathcal{B}_g)$ for every bundle. There exists a conformal map $A = \lambda R(\theta)$ with $\lambda \neq 0$ and $R(\theta)$ a rotation, such that $\varphi(\mathcal{B}_g) = A(Q(\mathcal{B}_g))$. We claim that $\theta = 0 \pmod{2\pi}$ and $\lambda = 1$, i.e. $A = \mathrm{Id}$. First suppose that $\theta \neq 0$. Then $Q(L_g)$ and $\varphi(L_g)$ with $L_g \in \mathcal{B}_g$ are non-parallel lines in $\mathbb{R}^2$ and hence $|\varphi(x) - Q(x)| > \mathcal{K}$ for $x \in L_g$ and $|x|$ sufficiently large, contradicting the uniform bound on $\delta(x)$ we found in lemma 4.5. To show that $\lambda = 1$, consider the points $x_k = \frac{1}{2}gk \in L_{gk}$, $k \in \mathbb{Z}^*$. By what we just showed, $Q(g)$ a multiple of the vector g' and since Q is linear, $Q(x_k) = \frac{1}{2}kQ(g)$. The Brillouin lines in a bundle are mapped by φ in a bijective and order preserving manner to the image bundle $\mathcal{B}_{g'}$, hence $\varphi(x_k) \in L_{g'(k+m)}$ for some fixed $m \in \mathbb{Z}$. Now, if $\lambda \neq 1$, we may assume that $\lambda = 1 + \epsilon$ with $\epsilon > 0$. Let $K = |\phi(x_1) - Q(x_1)| \leq \mathcal{K}$, then $|\phi(x_k) - Q(x_k)| \geq |K + k\epsilon| > \mathcal{K}$ for $|k|$, or equivalently $|x_k|$, large enough again contradicting the uniform bound on $\delta(x)$.

Hence $A = \mathrm{Id}$ and it follows that

$$\varphi(M_\Lambda) = Q(M_\Lambda) = M_{Q(\Lambda)} = M_{\Lambda'}.$$

Thus $\Lambda' = Q(\Lambda)$, so $\Lambda \sim \Lambda'$ and this proves the theorem. □

CHAPTER 5

Rigidity of Torus Puzzles

Next we study the rigidity of torus puzzles. We define an equivalence relation on torus puzzles and show that, for almost all lattices, the torus puzzles relative to two lattices are pairwise equivalent if and only if the lattices are conformally equivalent. We use the rigidity of M_Λ to prove this result.

Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$, $\tau(x) = -x$ be the *antipodal map*. By symmetry, $\tau(M_\Lambda) = M_\Lambda$. Let $\tilde{\tau} : \mathbb{T} \to \mathbb{T}$ be the map that satisfies $\pi \circ \tau = \tilde{\tau} \circ \pi$. Let $\Lambda = (1,0)\mathbb{Z} \oplus (\alpha,\beta)\mathbb{Z} \in \mathcal{G}$. Denote symbolically the points $0,1,2,3 \in \mathbb{T}$ defined by $i = \pi(x_i)$, $i = 0,1,2,3$ with

$$x_0 = (1,0),\ x_1 = \frac{1}{2}(1,0),\ x_2 = \frac{1}{2}(\alpha+1,\beta),\ x_3 = \frac{1}{2}(\alpha,\beta). \tag{5.1}$$

A straightforward computation shows that the points $0,1,2,3$ are the only fixed points of $\tilde{\tau}$.

Example 5.1. *Figure 5.1 depicts $\mathcal{P}_1$ for $\Lambda = (1,0)\mathbb{Z} \oplus (\frac{1}{4},\frac{3}{4})\mathbb{Z}$. The associated fixed points $0,1,2,3$ discussed above are indicated with dots.*

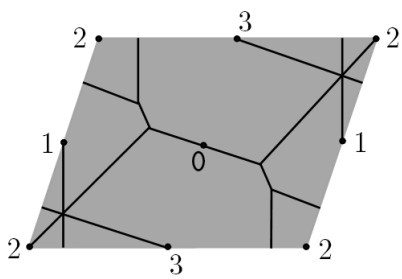


Figure 5.1. Puzzle $\mathcal{P}_1$ of Example 5.1.

Remark 5.1. *The points 0 and 1 are independent of the representation of the lattice, but the the points 2 and 3 are not. If (α,β) represents Λ, then so does $(\alpha+n,\beta)$ with $n \in \mathbb{Z}$. The points 2 and 3 flip according to n being even or odd.*

Lemma 5.2. *Let Λ be in general position, then $0,1 \in \partial\mathcal{P}_n$ but the points are not vertices, for all $n \geq 1$. In addition*

(i) $\quad 0 \in \partial^-\mathcal{P}_n$ and $1 \in \partial^+\mathcal{P}_n \qquad$ if n is even,

(ii) $\quad 0 \in \partial^+\mathcal{P}_n$ and $1 \in \partial^-\mathcal{P}_n \qquad$ if n is odd.

Proof. Let $x \in \pi^{-1}(0)$ or $\pi^{-1}(1)$. Because 0 and 1 are the fixed points of $\tilde{\tau}$, $\sigma(x)$ is always even. This yields that $\sigma(x) = 2$. Because if $\sigma(x) > 2$, i.e. $\sigma(x) \geq 4$, then $\mu(x) \geq 3$, contradicting the assumption that Λ is in general position. Hence, $\mu(x) = 1$, hence these points always lie on the interior of an edge of M_Λ.

We have that $\partial^-\mathcal{P}_0 = \emptyset$ and $\partial^+\mathcal{P}_0 = \partial\mathcal{P}_0$. Since $g \in L_{2g}$, $\partial_0 = \partial_0^+$ can't contain a lattice point, so $0 \notin \partial\mathcal{P}_0$. Since $1 \neq 0$ and $1 \in \partial\mathcal{P}_0$, $1 \in \partial^+\mathcal{P}_0$. For $n = 1$, we have that $0 \in \partial^+\mathcal{P}_1$ and $1 \in \partial^-\mathcal{P}_1$. Repeating this argument shows, since these points always lie on the interior of a edge and since $\partial^+\mathcal{P}_n = \partial^-\mathcal{P}_{n+1}$, that 0 and 1 are alternately and exactly oppositely contained in $\partial^-\mathcal{P}_n$ and $\partial^+\mathcal{P}_n$. By induction one finishes the argument. □

Definition 5.3 (Equivalence of Puzzles). *Let $\Lambda, \Lambda' \in \mathcal{G}$. Two puzzles $\mathcal{P}_n, \mathcal{P}'_n$ are equivalent, $\mathcal{P}_n \sim \mathcal{P}'_n$, if there exists an orientation preserving homeomorphism $h_n : \mathbb{T} \longrightarrow \mathbb{T}'$ such that*

a) $h_n(\partial\mathcal{P}_n) = \partial\mathcal{P}'_n$ and
b) $h_n(0) = 0'$ and $h_n(1) = 1'$.

Comparing $\mathcal{P}_1$ relative to $\Lambda = (1,0)\mathbb{Z} \oplus (\frac{1}{4}, \frac{3}{4})\mathbb{Z}$ of example 5.1 and $\mathcal{P}_1$ relative to $\Lambda = \mathbb{Z}^2$ of Appendix A, it is clear that these two puzzles are not equivalent (or even homeomorphic).

Theorem 5.4. *Let $\Lambda, \Lambda' \in \mathcal{G}$ in general position, then $\Lambda = \Lambda'$ if and only if $\mathcal{P}_n \sim \mathcal{P}'_n$ for all $n \in \mathbb{N}$.*

The proof of theorem 5.4 will be preceded by the following two lemmas.

Notation 3. *In what follows, if a map on $\mathbb{R}^2$ or $\mathbb{T}$ has the property that it maps plus/minus or intermediate vertices (for a map on $\mathbb{R}^2$) or vertices of type I, II, or III (for a map on $\mathbb{T}$) to vertices of the same type, we say for short that the map* preserves the types of vertices.

Lemma 5.5. *Let Λ, Λ' in general position and $\mathcal{P}_n \sim \mathcal{P}'_n$, then*

$$h_n(\partial^\pm\mathcal{P}_n) = \partial^\pm\mathcal{P}'_n. \tag{5.2}$$

Consequently, h_n preserves the types of vertices.

Proof. Since $\tilde{\mu}(v) = 3$ if v is of type I or II and $\tilde{\mu}(v) = 6$ if v is of type III, h_n maps vertices of type III to vertices of type III, i.e. $h_n(\tilde{\mathcal{I}}_n) = \tilde{\mathcal{I}}'_n$. Hence, $h_n(\partial^+\mathcal{P}_n) \cap h_n(\partial^-\mathcal{P}_n) = \partial^+\mathcal{P}'_n \cap \partial^-\mathcal{P}'_n$.

Suppose that $e \in \partial^+\mathcal{P}'_n$ is a plus edge and $e' = h_n(e) \in \partial^+\mathcal{P}'_n$. Let $\partial e = \{u, v\}$ and $\partial e' = \{u', v'\}$ and suppose that $h_n(v) = v'$. Then either

(i) v and v' are of type I, hence every edge incident to v and v' is a plus edge. Because h_n maps edges incident to v to edges incident to v', $h_n(e) \subset \partial^+\mathcal{P}'_n$ if (and only if) $e \subset \partial^+\mathcal{P}_n$ incident to v.
(ii) v and v' are of type III, hence alternately 3 plus and 3 minus edges are incident to v and v'. Label these edges e_i, $i = 1, .., 6$ in clockwise order with $e_1 = e$. Then e_1, e_3 and e_5 are plus edges and e_2, e_4 and e_6 are minus edges. Since $h_n(e_1) = e'_1 = e'$ and h_n is an orientation preserving homeomorphism, it preserves the order of these edges, i.e. $h_n(e_i) = e'_i$, where e'_i, $i = 1, ...6$ are the edges incident to v' labeled in clockwise order. Because e_1 and e'_1 by assumption are plus edges, we see that $e'_i = h_n(e_i) \subset \partial^+\mathcal{P}'_n$ if and only if $e_i \subset \partial^+\mathcal{P}_n$.

In both cases the image under h_n of the plus edges incident to v are plus edges incident to v'. Because $\partial_n^+ \simeq \mathbb{S}^1$ is path-connected, $\partial^+\mathcal{P}_n$ is path-connected. Taking a path through $\partial^+\mathcal{P}_n$, traversing every plus edge at least once (possibly some edges more than once), the above arguments show that $h_n(\partial^+\mathcal{P}_n) \subseteq \partial^+\mathcal{P}'_n$. Similarly, $h_n^{-1}(\partial^+\mathcal{P}'_n) \subseteq \partial^+\mathcal{P}_n$. This yields we have in fact $h_n(\partial^+\mathcal{P}_n) = \partial^+\mathcal{P}'_n$ and hence also $h_n(\partial^-\mathcal{P}_n) = \partial^-\mathcal{P}'_n$.

To finish the proof, we need to show that we indeed have a pair of plus edges e and $e' = h_n(e)$ to start with. If n is odd, then the edge e through 0 on $\mathbb{T}$ and e' through $0'$ on $\mathbb{T}'$ is a plus edge by lemma 5.2. Since $h_n(0) = 0'$, $h_n(e) = e'$. Let $\partial e = \{u, v\}$ and $\partial e' = \{u', v'\}$. Since every torus puzzle is pointsymmetric with respect to 0, we may assume that $h_n(v) = v'$.

The proof for n is even is analogous; one replaces $\partial^+\mathcal{P}_n$ by $\partial^-\mathcal{P}_n$ and repeates the above proof. □

Lemma 5.6. *Let $\Lambda, \Lambda' \in \mathcal{G}$ in general position and $\mathcal{P}_n \sim \mathcal{P}'_n$. Then there exist an orientation preserving homeomorphism $\varphi_n : B_n \to B'_n$ such that*

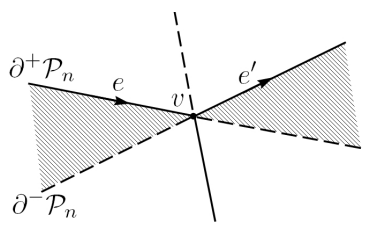
$$\begin{tikzcd} B_n \arrow[r, "\varphi_n"] \arrow[d, "\pi"'] & B'_n \arrow[d, "\pi'"] \\ \mathcal{P}_n \arrow[r, "h_n"'] & \mathcal{P}'_n \end{tikzcd}$$

Consequently, $\varphi_n(\partial_n^\pm) = \partial'^{\pm}_n$ and φ_n preserves the types of vertices.

Proof. Write $B_n = \bigcup_{j\in J_n} B_n^j$ and $B'_n = \bigcup_{j'\in J'_n} B_n^{j'}$ and let $\varphi_n : B_n \to B'_n$ a map. Since $\mathcal{P}_n \sim \mathcal{P}'_n$, $|J_n| = |J'_n|$. Let $h_n^j = h_n|\mathrm{Int}(\mathcal{P}_n^j)$ and $\varphi_n^j = \varphi_n|\mathrm{Int}(B_n^j)$ with $h_n^j(\mathrm{Int}(\mathcal{P}_n^j)) = \mathrm{Int}({\mathcal{P}'_n}^{j'})$. By relabeling the indices, we may assume that the zones are indexed in clockwise order and that $j = j'$. Since $\pi : \mathrm{Int}(B_n) \to \mathbb{T}$ is a bijection, the map

$$\varphi_n^j : \mathrm{Int}(B_n^j) \to \mathrm{Int}({B'_n}^j), \quad x \mapsto \pi'^{-1}\left(h_n^j(\pi(x))\right) \cap \mathrm{Int}({B'_n}^j)$$

is a homeomorphism. We can extend φ_n^j uniquely to a homeomorphism on the boundary ∂B_n^j, which we denote again φ_n^j. Hence $\varphi_n^j : B_n^j \to {B'}_n^j$ is a homeomorphism for every $j \in J_n$ that by construction commutes with the given diagram. To show that $\varphi : B_n \to B'_n$ is a homeomorphism, it suffices to show that, if $\varphi_n^j(B_n^j) = {B'}_n^j$, then $\varphi_n^{j+1}(B_n^{j+1}) = {B'}_n^{j+1}$. That is, that φ preserves the clockwise ordering of the subzones. Orient ∂_n^+ and ${\partial'_n}^+$ clockwise and consider a vertex $x \in \mathcal{I}_n$; since $\mu(x) = 2$ it is the intersection of two lines. Consider the plus edge e of $\partial\mathcal{P}_n^j$ incident to $v = \pi(x)$.



Given the induced orientation of ∂_n^+ on $\partial\mathcal{P}_n$, the plus edge e' of $\partial^+\mathcal{P}_n^{j+1}$ incident to v is the first plus edge clockwise to e, see the figure on the right[1]. Since h_n is orientation preserving and maps plus edges to plus edges, it sends e to $\tilde{e}$ and e' to $\tilde{e}'$, with $\tilde{e}$ the plus edge of $\partial^+{\mathcal{P}_n}'^j$ incident to $v' = h_n(v)$ and $\tilde{e}'$ the first plus edge clockwise to $\tilde{e}$ incident to v'. By the induced orientation on $\partial\mathcal{P}'_n$, $\tilde{e}' \subset \partial^+{\mathcal{P}'}_n^{j+1}$ incident to v'. So, $h_n(\mathcal{P}_n^{j+1}) = {\mathcal{P}'_n}^{j+1}$ if $h_n(\mathcal{P}_n^j) = {\mathcal{P}'_n}^j$. Hence, $\varphi_n^{j+1}(B_n^{j+1}) = {B'_n}^{j+1}$, if $\varphi_n^j(B_n^j) = {B'_n}^j$. Thus $\varphi_n : B_n \to B'_n$ is a homeomorphism and is orientation preserving since h_n is orientation preserving. By lemma 5.5, $h_n(\partial^\pm\mathcal{P}_n) = \partial^\pm{\mathcal{P}'}_n$, hence $\varphi_n(\partial_n^\pm) = \partial'^{\pm}_n$ and it follows that φ_n preserves the types of vertices. □

Proof of theorem 5.4. If $\mathcal{P}_n \sim \mathcal{P}'_n$ for all $n \in \mathbb{N}$, then lemma 5.6 gives us a sequence of orientation preserving homeomorphisms $\{\varphi_n\}_{n\in\mathbb{N}}$, $\varphi_n : B_n \to B'_n$, satisfying the

[1]We have indicated the plus and minus edges with solid and dashed lines respectively.

properties as stated in the lemma. Since $\partial_n^+ = \partial_{n+1}^-$,

$$\varphi_n(\partial_n^+) = \varphi_{n+1}(\partial_n^+). \tag{5.3}$$

We may assume that h_n is piecewise linear on $\partial\mathcal{P}_n$, i.e. linear on every edge of $\partial\mathcal{P}_n$, for all $n \in \mathbb{N}$. This makes the maps φ_n piecewise linear on ∂_n for all $n \in \mathbb{N}$. Assume that n is even, the case where n is odd is identical. Then $0 \in \partial^+\mathcal{P}_n$ and $\{\pm g\} = \pi^{-1}(0) \cap \partial_n$ and $\{\pm g'\} = \pi^{-1}(0') \cap \partial'_n$ for certain $g \in \Lambda_*$ and $g' \in \Lambda'_*$.

Since $\tau(B_n) = B_n$, $\tilde{\tau}(\mathcal{P}_n) = \mathcal{P}_n$. Hence, if h_n satisfies definition 5.3, then so does $\tilde{h}_n = h_n \circ \tilde{\tau}$. The map $\tilde{\varphi}_n = \varphi_n \circ \tau$ is the homeomorphism that commutes with the diagram of lemma 5.6 when one replaces h_n by $\tilde{h}_n$, so we may assume that if $\varphi_n(g) = g'$, then $\varphi_{n+1}(g) = g'$. Combining this with piecewise linearity of φ_n on ∂_n for all $n \in \mathbb{N}$ and (5.3) we have that

$$\varphi_n|\partial_n^+ = \varphi_{n+1}|\partial_n^+. \tag{5.4}$$

This holds for all $n \geq 1$. In fact, it also holds when $n = 0$, because $\partial_0^- = \emptyset$ thus $\partial_0 = \partial_0^+ = \partial_1^-$. Hence, the homeomorphisms $\{\varphi_n\}_{n\in\mathbb{N}}$ glue to a global homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, with the property that $\varphi(M_\Lambda) = M_{\Lambda'}$, since $M_\Lambda = \bigcup_{n\in\mathbb{N}} \partial_n = \bigcup_{n\in\mathbb{N}} \partial_n^+$. Hence $M_\Lambda \simeq M_{\Lambda'}$. Since $\Lambda, \Lambda' \in \mathcal{G}$, $\Lambda = \Lambda'$ by theorem 4.2. □

In effect, theorem 5.4 states that, generically, the torus is uniquely determined by the combinatorics of its puzzles $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$.

CHAPTER 6

Concluding Remarks

In this thesis, we completely focused on the focal decomposition of the tangent plane $\mathbb{R}^2$ at the base point 0 of the torus $\mathbb{T}$. The decomposition in this case is exactly described by the Brillouin zones in $\mathbb{R}^2$ relative to the lattice group Λ, which is the deck group of the torus, because the exponential map on the tangent plane can be identified with the projection map on $\mathbb{R}^2$.

Closely related to the concept of focal decomposition, is the so-called *spectrum* of a lattice. The spectrum of a lattice is a countable sequence of numbers $\{\lambda_{\nu_i}\}_{i\in\mathbb{N}}$ with $\nu_i \in \mathbb{R}^+$, $\nu_{i+1} > \nu_i$ and

$$\lambda_\nu = \#\{g \in \Lambda \mid |g| = \nu\},$$

if $\#\{\cdot\} > 0$. Two lattices are *isospectral* if $\lambda_{\nu_i} = \lambda'_{\nu_i}$ for all $i \in \mathbb{N}$. Milnor [**10**] showed in 1964, that in $\mathbb{R}^{16}$, there are non-isometric lattices that are isospectral.

The focal decomposition completely determines the spectrum of a lattice, but the converse does not hold. Namely, the spectrum does not give information on the position of the lattice points on the circles, whereas this information *is* encoded in the focal decomposition of a lattice. In fact, this information strongly influences the shape of the set M_Λ.

The definition of the set M_Λ can in a natural way be generalised for a lattice Λ of rank n in $\mathbb{R}^n$, as well as the notion of equivalence between two such sets. We make the following

Conjecture 6.1. *The rigidity theorem (theorem 4.2) generalizes to $\mathbb{R}^n$. That is, $M_\Lambda \simeq M_{\Lambda'}$ if and only if Λ and Λ' are isometric.*

Next we discuss how we can generalise the rigidity theorem to Riemannian manifolds. First, we describe how to construct Brillouin zones in Riemannian manifolds. In [**8**], it is shown that, under certain conditions, the construction of Brillouin zones can be carried out with any discrete set S in a path-connected, proper metric space N[1]. Under these conditions, the Brillouin zones are the closure of their interior and tile the space N. Moreover, if S is a discrete group acting by isometries, then the Brillouin zones form a $k-$fold cover of the fundamental domain for S. A manifold N and set S having such properties is called *Brillouin over S*. Let Γ be a group acting on N. If Γ acts discontinuously, then the orbit S_x of every $x \in N$ is a discrete set. We say that N is Brillouin over Γ is N is Brillouin over S_x for every $x \in N$.

Open Problem 6.2. *Suppose we are given a Riemannian manifold (N, g) Brillouin over Γ and Γ'. Suppose that, for every $x \in N$, the corresponding set of Brillouin "lines" relative to S_x and S'_x are homeomorphic. What does this say about the groups Γ and Γ'? And the corresponding spaces N/Γ and N/Γ'?*

[1]The word *proper* here means that the metric $d(x, \cdot)$ is a proper map for every $x \in N$.

The focal decomposition of a Riemannian manifold (N, g) is defined as follows. Let TN be the tangent bundle of N and $\exp : TN \to N$ be the exponential map. Let $v \in N$ and $x \in T_vN$. Define

$$\sigma_v(x) = \#\{y \in T_vN \mid \exp_v(y) = \exp_v(x) \text{ and } \|y\| = \|x\|\}$$

and

$$\Sigma_i = \{(v, x) \in TN \mid \sigma_v(x) = i\}.$$

The decomposition of TN into the sets $\{\Sigma_i\}_{i=0}^{\infty}$ is the focal decomposition of N. Given two Riemannian manifolds M and N, we say two decompositions are equivalent, if there exists a homeomorphism $\varphi : TN \to TM$ such that $\varphi(\Sigma_i) = \Sigma_i'$, $i = 1, 2, 3, ..., \infty$.

In the case of a flat torus, the focal decomposition is independent of the base point. Hence, the decomposition of the tangent bundle in this case is completely determined by the decomposition of the tangent plane at one base point.

Open Problem 6.3. *Given two Riemannian manifolds M and N with equivalent focal decompositions. What can we say about the manifolds M and N?*

Theorem 4.2 provides an answer to both open problems stated above in a specific case.

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APPENDIX A

Puzzles relative to $\Lambda = \mathbb{Z}^2$

In figure A.1 we see the first 8 puzzles $\mathcal{P}_n$ relative to $\mathbb{Z}^2$. The left and middle pictures are the minus and plus boundaries $\partial^-\mathcal{P}_n$ and $\partial^+\mathcal{P}_n$ respectively and the right pictures the puzzles $\mathcal{P}_n$.

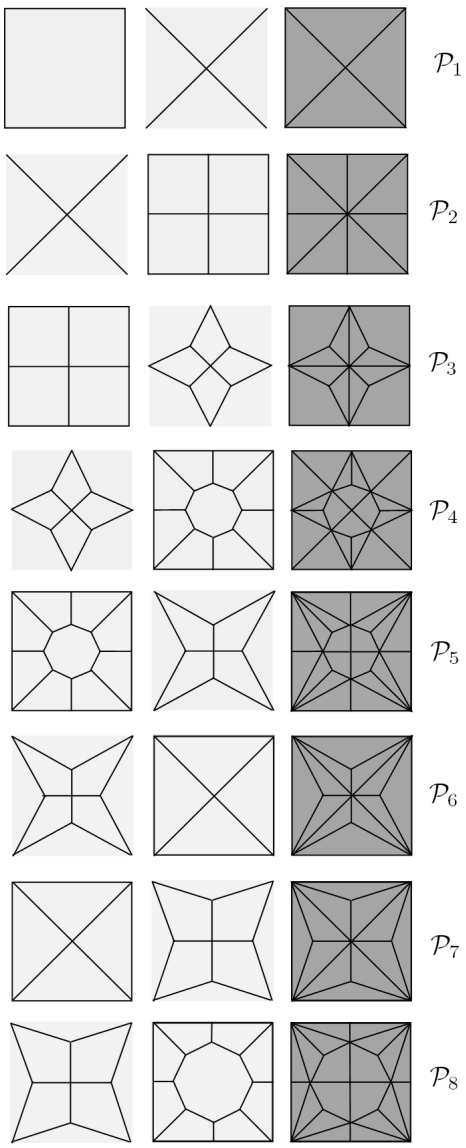

FIGURE A.1. The first 8 puzzles relative to $\mathbb{Z}^2$.