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Modeling the conditional correlation between United States stock and bond returns

A multivariate GARCH approach

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Abstract

This paper investigates the intertemporal interaction between returns on the S&P 500 index and the Lehman US Aggregate Bond index using daily data from 6 May 1988 until 8 May 2008. We allow the conditional covariance matrix to vary over time, according to the bivariate *BEKK* GARCH(1,1) model. The results indicate that our estimated conditional correlation varies considerably over time. Based on several test statistics, we can conclude that it is a statistically adequate representation of the dataseries. Finally, we investigate how the interest rate is related to our estimated conditional correlation and find that they are positively related.

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1 Introduction

The advantage of knowing about risks is that we can properly decide if the risk is worth to take or not. The central optimization problem for investors is to maximize rewards and minimize risks. A key issue in this problem is to estimate risk. Since risk is usually measured by the variance of the asset return, an estimate of the variance is required. Typically the square root of the variance, called volatility, is used.

Volatility plays a central role in asset pricing and many other areas of finance. Therefore, the volatility of asset returns has been the topic of study for decades. In most financial data the volatility varies over time, so the volatility may reasonably be expected to be larger for some points in time. These dataserries are said to suffer from heteroskedasticity. The first model which treats heteroskedasticity as a variance to be modeled was proposed by Engle (1982) and is called the ARCH model. A generalization of this model is the GARCH model introduced by Bollerslev (1986).

When investing in many assets at the same time, not only the volatilities of the individual asset returns, but also the covariances or the correlations between these asset returns are important. If correlations between asset returns are high, a loss in one asset is likely to be accompanied by a loss in other assets as well. This implies that diversification benefits are greater when correlations between the asset returns are low. Volatilities and correlations of asset returns are not only relevant for portfolio management, but are also important for option pricing, hedging strategies and risk measures. Because of these applications, accurate estimates of volatilities and correlations are very important to banks and pension funds. Also on a bigger scale comovements are noticed, namely comovements of world equity markets.

In this paper we investigate the conditional covariance between the returns on the S&P 500 index and the Lehman US Aggregate Bond index using the bivariate *BEKK* GARCH(1,1) model, proposed by Engle and Kroner (1995). The *BEKK* model is named after Baba, Engle, Kraft and Kroner (1990) who wrote the preliminary version of Engle and Kroner. We find that our estimated conditional correlation varies considerably over time. Even though several factors may cause this time variation, we will only investigate the relation between the interest rate and our estimated conditional correlation. We find that they are positively related. This could be caused by the common discount rate effect.

The remainder of this paper is organized as follows. In section 2 we describe several methods which could be used to model a relationship between two financial dataserries. Section 3 describes and analyzes the data used, followed by Section 4 in which we estimate the *BEKK* GARCH(1,1) model and explain why we use this model. In the next section we investigate the adequacy of our estimated model and find that it is consistent. The relation between the interest rate and our estimated conditional correlation is

investigated in Section 6. Finally, Section 7 concludes this paper.

2 Outline of different Methods

In this section we will discuss several methods which could be used modeling the dependence between variables. After discussing OLS briefly we will discuss ARCH and GARCH models and Copulas. OLS will only be discussed briefly because of its important disadvantages. In the last subsection of this paragraph we will compare the other methods by looking at their advantages and disadvantages.

2.1 OLS

One of the first methods to model dependence between two variables is the ordinary least squares method (OLS). The basic version of OLS assumes that the expected value of all error terms, when squared, is the same at any given point in time; assuming that $E(\epsilon_t^2) = c$ is called homoskedasticity.

One of the disadvantages of this method is that it is very sensitive to the presence of unusual outcomes of the variables. Just a few outliers can sometimes seriously skew the results of a least squares analysis.

An even more important disadvantage is the assumption of homoskedasticity. In most datasets and particularly in applications in finance this assumption is rejected.

2.2 ARCH and GARCH

Models which do not assume homoskedasticity are ARCH and GARCH, which stand for autoregressive conditional heteroskedasticity and generalized autoregressive conditional heteroskedasticity. They treat heteroskedasticity as a conditional variance to be modeled. Typically they model the square root of the variance, the standard deviation, also referred to as volatility. This implies that a prediction of high volatility is really just a prediction of high variance, a prediction that the potential size of a price move is great. In financial applications where the dependent variable is the return on an asset or portfolio, the variance of the return represents the risk level of those returns. Even a cursory look at returns on financial data suggests that some time periods are riskier than others. That is, the expected value of the magnitude of error terms differs over time. Moreover, these risky times are usually not scattered randomly across quarterly or annual data. Financial analysts describe this as volatility clustering. ARCH and GARCH models are designed to deal with this. Returns also have surprisingly large numbers of extreme values, which is often described as unpredictability. ARCH and GARCH can deal with this feature also.

2.2.1 Why do most financial time series display volatility clustering?

To find an answer to this question we first have to understand why asset prices change over time. Financial assets are purchased and owned because of the future payments that can be expected. Because these payments are uncertain and depend upon unknown future developments, the fair price of the asset requires forecasts of the distribution of these payments based on the information available today. As time goes by, we get more information on these future events and revalue the asset. So at a basic level, financial price volatility is due to the arrival of new information. Volatility clustering is simply clustering of information arrivals, and since information is typically clustered in time, this implies that volatility is clustered.

One other phenomenon that causes volatility clustering is that investors tend to copy each other's behavior. A subgroup of investors imitating one another's actions, is called herding. There are many different definitions for herding, but we will keep it simple and will use the one just explained. It simply means that when one investor starts to buy a certain stock, more and more investors start buying. The result of this is that the price rises and that high returns are followed by high returns. The reverse also holds; when investors start to sell their stocks, more and more investors start selling, so that low returns are followed by low returns. Herding is closely related to the arrival of new information. If for example some financial specialists reveal their expectations about a certain stock index, this could be seen as the arrival of new information. As a consequence of herding behavior, investors tend to adapt their expectations towards those of the 'specialists'.

2.2.2 Univariate ARCH and GARCH models

Since this paper focuses on financial series, we will use financial notation. Let r_t be the dependent variable, which is the return on an asset at time t . Returns on assets will usually be measured in terms of asset prices. Differences of asset prices are random variables, since it is not possible to determine exactly what they will be in the future. Differences of financial asset prices are often assumed to be lognormal random variables. Therefore returns on financial assets, the relative price changes, are usually measured by the difference in log prices, which will be normally distributed. If we now say that P_t is the price of an asset on time t , it follows that:

$$r_t = \text{Log} \left(\frac{P_t}{P_{t-1}} \right)$$

The dependent variable r_t can be modeled by the regression $r_t = \mu + h_t \cdot \epsilon_t$, with μ being the unconditional expectation of r_t , ϵ_t being the error term of

r_t and h_t being the standard deviation of the return defined relative to the past information set, which will be explained below. h_t^2 is also referred to as the conditional variance. Since μ often appears to be zero in high frequency data, the regression equation becomes more simple. The return r in the present will be equal to the conditional standard deviation of r times the error term for the present period:

$$r_t = h_t \cdot \epsilon_t$$

Now the conditional variance and conditional expectation of r_t can only be specified by assuming a specific conditional distribution for ϵ_t . Most of the time the error terms are assumed to be Student's t or normally distributed. Since the latter is most used, we will assume that based on the past informationset: $\epsilon_t \sim N(0, 1)$. It now follows that:

$$E_{t-1}[r_t] = E_{t-1}[h_t \cdot \epsilon_t] = 0$$

$$Var_{t-1}[r_t] = Var_{t-1}[h_t \cdot \epsilon_t] = h_t^2 \cdot Var_{t-1}[\epsilon_t] = h_t^2 \cdot Var[\epsilon_t] = h_t^2$$

In the rest of this paper we will often refer to these equations as the conditional mean equation and the conditional variance equation.

Now the econometric challenge is to specify how the past information is used to forecast the variance of the return, conditional on the past information. Virtually no methods were available for the variance before Engle (1982) introduced the ARCH model. In the ARCH(p) model the variance is calculated using the last p observations. Instead of using short or long sample standard deviations, the ARCH(p) model takes weighted averages of past squared returns. These weights could give more influence to recent information and less to the distant past. In mathematical terms the conditional variance defined by the ARCH model of order p is equal to:

$$h_t^2 = \omega + \sum_{i=1}^p \alpha_i \cdot r_{t-i}^2$$

where ω is defined as the long-term average variance, $\omega > 0$, $\alpha_1, \dots, \alpha_p \geq 0$ and $\epsilon_t \sim N(0, 1)$.

Bollerslev (1986) generalized the ARCH process by allowing past conditional variances in the above equation:

$$h_t^2 = \omega + \sum_{i=1}^p \alpha_i \cdot r_{t-i}^2 + \sum_{j=1}^q \beta_j \cdot h_{t-j}^2$$

with the additional assumption that all $\beta_i \geq 0$. This process is called generalized ARCH of order (p, q) , or GARCH(p,q). The (p, q) in the parentheses

is a standard notation in which the first number refers to how many autoregressive lags, or ARCH terms appear in the equation, and the second number refers to how many moving average lags are specified, which is often called the number of GARCH terms.

Today, the most widely used univariate GARCH model is the GARCH(1,1) model, also called the 'vanilla' GARCH model. It is specified by:

$$r_t = h_t \cdot \epsilon_t$$

$$h_t^2 = \omega + \alpha_1 \cdot r_{t-1}^2 + \beta_1 \cdot h_{t-1}^2$$

where $\omega > 0, \alpha_1 \geq 0, \beta_1 \geq 0$ and $\epsilon_t \sim N(0, 1)$

This GARCH specification asserts that the predictor of the variance in the next period is a weighted average of the long-term average variance, the variance predicted for this period, and the new information in this period that is captured by the most recent squared return. Although this model is directly set up to forecast the variance for just one period, it turns out that based on the one-period forecast, a two-period forecast can be made. By repeating this step, long-term forecasts can be constructed. Eventually, the variance will converge to ω , the long-term average variance.

ARCH models are not often used in financial markets because the simple GARCH models perform much better. In fact the ARCH model with exponentially declining lag coefficients is equivalent to a GARCH(1,1) model. In other words, the GARCH(1,1) model actually models an infinite ARCH process with using only very few parameters. A proof for this is given in Appendix A.

We will end this section by answering the question of how it is possible to estimate ARCH or GARCH models while there are only data available on r_t . The simple answer is to use maximum likelihood estimation by substituting h_t^2 for σ^2 in the likelihood equation, where σ^2 is the variance of the normal distribution, and then maximizing with respect to the parameters. An even more simple answer is to use software such as for example Eviews. Since our focus will not be on how this estimation is carried out exactly, we will not discuss this further and use Eviews for estimation.

2.2.3 Multivariate GARCH models

As mentioned before in the introduction, not only volatilities of individual asset returns matter, but correlations between asset returns are also important. Comovements between different markets are noticed a lot. A multivariate GARCH model is a model that describes correlation between different assets or markets. There are several models available, which can result in different estimates of the same conditional correlations. We will

start this subsection by explaining the multivariate *vec* GARCH(p,q) model followed by the *diagonal* model and *BEKK* model. We will also specify what the bivariate GARCH(1,1) specification of these models looks like. Our focus is on defining the conditional variance and covariance equations and therefore pay no further attention to the conditional expectation equations, since they are defined in exact the same way as in section 2.2.2.

In a bivariate model, the correlation between two asset returns is modeled. If the return of asset 1 at time t is named $r_{1,t}$ and the return of asset 2 at time t is named $r_{2,t}$, the return equations for both assets are given by $r_{1,t} = h_{1,t} \cdot \epsilon_{1,t}$ and $r_{2,t} = h_{2,t} \cdot \epsilon_{2,t}$. The difference between univariate and bivariate modeling is that in the latter is assumed that $\text{cov}(r_{1,t}, r_{2,t}) = h_{12,t}^2$ with $h_{12,t}^2 \neq 0$, while in the previous $\text{cov}(r_{1,t}, r_{2,t}) = 0$.

Let $\mathbf{r}_t = \begin{pmatrix} r_{1,t} \\ r_{2,t} \end{pmatrix}$ and let \mathbf{H}_t be the 2×2 conditional covariance matrix of \mathbf{r}_t . \mathbf{H}_t is given by:

$$\mathbf{H}_t = \text{cov}(\mathbf{r}_t) = \begin{pmatrix} \text{var}(r_{1,t}) & \text{cov}(r_{1,t}, r_{2,t}) \\ \text{cov}(r_{2,t}, r_{1,t}) & \text{var}(r_{2,t}) \end{pmatrix} = \begin{pmatrix} h_{11,t}^2 & h_{12,t}^2 \\ h_{21,t}^2 & h_{22,t}^2 \end{pmatrix}$$

To define the *vec* parameterization of the two-dimensional GARCH(1,1) model, we have to use the vector operator $\text{vec}()$. This vector operator stacks the columns of a matrix into a column vector. We now define $\mathbf{h}_t^2 = \text{vec}(\mathbf{H}_t)$ and $\boldsymbol{\eta}_t = \text{vec}(\mathbf{r}_t \mathbf{r}_t')$. The *vec* parameterization is given by:

$$\mathbf{h}_t^2 = \boldsymbol{\omega} + \mathbf{A}_1 \boldsymbol{\eta}_{t-1} + \mathbf{B}_1 \mathbf{h}_{t-1}^2 \quad (2.1)$$

where $\boldsymbol{\omega}$ is a 4×1 parameter vector and \mathbf{A}_1 and \mathbf{B}_1 are 4×4 parameter matrices.

If we write out (2.1), the bivariate *vec* GARCH(1,1) model is given by:

$$\begin{aligned} \mathbf{h}_t^2 = \begin{bmatrix} h_{11,t}^2 \\ h_{12,t}^2 \\ h_{22,t}^2 \end{bmatrix} &= \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} r_{1,t-1}^2 \\ r_{1,t-1} \cdot r_{2,t-1} \\ r_{2,t-1}^2 \end{bmatrix} \\ &+ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} h_{11,t-1}^2 \\ h_{12,t-1}^2 \\ h_{22,t-1}^2 \end{bmatrix} \end{aligned} \quad (2.2)$$

Notice that we have omitted the equation for $h_{21,t}^2$ since it is equal to $h_{12,t}^2$. Thereby the number of parameters to be estimated are nine for \mathbf{A}_1 , nine for \mathbf{B}_1 and three for $\boldsymbol{\omega}$ which gives a total 21 parameters.

The bivariate *vec* GARCH(1,1) model can easily be extended to a multivariate *vec* GARCH(p,q) model. The return equations for n assets can be defined in exact the same way as we have done before. \mathbf{r}_t now is the $n \times 1$ column vector of the returns and \mathbf{H}_t is the conditional variance covariance matrix of the returns, where each element of \mathbf{H}_t depends on q lagged values of the squares and cross-products of \mathbf{r}_t and on the p lagged values of the elements of \mathbf{H}_t . Again we define $\mathbf{h}_t = \text{vec}(\mathbf{H}_t)$ and $\boldsymbol{\eta}_t = \text{vec}(\mathbf{r}_t \mathbf{r}_t')$, so that the *vec* parameterization can be written as:

$$\begin{aligned} \mathbf{h}_t^2 &= \boldsymbol{\omega} + \mathbf{A}_1 \boldsymbol{\eta}_{t-1} + \dots + \mathbf{A}_p \boldsymbol{\eta}_{t-p} + \mathbf{B}_1 \mathbf{h}_{t-1}^2 + \dots + \mathbf{B}_q \mathbf{h}_{t-q}^2 \\ &= \boldsymbol{\omega} + \sum_{i=1}^p \mathbf{A}_i \text{vec}(\mathbf{r}_{t-i} \mathbf{r}_{t-i}') + \sum_{j=1}^q \mathbf{B}_j \text{vec}(\mathbf{H}_{t-j}) \end{aligned} \quad (2.3)$$

where $\boldsymbol{\omega}$ is a $n^2 \times 1$ parameter vector and $\mathbf{A}_1, \dots, \mathbf{A}_p$ and $\mathbf{B}_1, \dots, \mathbf{B}_q$ are $n^2 \times n^2$ parameter matrices. Notice that some of the covariance equations appear twice, since there is an equation for $h_{ij,t}$ as well as for $h_{ji,t}$. The redundant terms can be eliminated without affecting the model. The number of elements of \mathbf{H}_t to be estimated is reduced to $n(n+1)/2$, so that there are $((n(n+1))/2)^2$ instead of n^4 parameters left in each of the \mathbf{A}_i and \mathbf{B}_j parameter matrices. The total number of parameters to be estimated is $n(n+1)/2 + (p+q) \cdot (n(n+1)/2)^2$.

Even though the number of parameters is reduced when deleting redundant covariance equations, it is desirable to restrict the parameterization further. Bollerslev, Engle and Wooldridge (1988) produced the *diagonal representation*, in which each element of the variance covariance matrix, $h_{km,t}$, depends only on past values of itself and past values of $\mathbf{r}_{k,t} \mathbf{r}_{m,t}$. In other words, variances depend only on their own past values and own past squared returns. Covariances depend on their own past values and own cross-products of returns. The bivariate *diagonal* GARCH(1,1) model can be obtained from the bivariate *vec* GARCH(1,1) model by assuming that \mathbf{A}_1 and \mathbf{B}_1 are a diagonal matrices:

$$\begin{aligned} \mathbf{h}_t^2 = \begin{bmatrix} h_{11,t}^2 \\ h_{12,t}^2 \\ h_{22,t}^2 \end{bmatrix} &= \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} r_{1,t-1}^2 \\ r_{1,t-1} \cdot r_{2,t-1} \\ r_{2,t-1}^2 \end{bmatrix} \\ &+ \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \begin{bmatrix} h_{11,t-1}^2 \\ h_{12,t-1}^2 \\ h_{22,t-1}^2 \end{bmatrix} \end{aligned} \quad (2.4)$$

If we write out (2.4), we get:

$$\begin{cases} h_{1,t}^2 = \omega_1 + a_{11} \cdot r_{1,t-1}^2 + b_{11} \cdot h_{1,t-1}^2 \\ h_{2,t}^2 = \omega_2 + a_{22} \cdot r_{2,t-1}^2 + b_{22} \cdot h_{2,t-1}^2 \\ h_{12,t}^2 = \omega_3 + a_{33} \cdot r_{1,t-1} \cdot r_{2,t-1} + b_{33} \cdot h_{12,t-1}^2 \end{cases} \quad (2.5)$$

In the bivariate model illustrated here there are nine parameters to be estimated; three parameters of \mathbf{A}_1 , three parameters of \mathbf{B}_1 and three of $\boldsymbol{\omega}$. Compared to the bivariate *vec* model the number of parameters is reduced with 12. In general, in the n -variate *diagonal* model there are $((n(n+1))/2)$ parameters in each matrix to be estimated, so that the total number of parameters to be estimated is equal to $(p+q+1) \cdot (n(n+1)/2)$. The n -variate *diagonal* model is given by (2.3), with the restriction that all \mathbf{A}_i and \mathbf{B}_j are diagonal matrices.

In both the *vec* representation and the *diagonal* representation it can be difficult to check if \mathbf{H}_t is positive definite for all values of \mathbf{r}_t . The restriction that \mathbf{H}_t should be a positive definite matrix is required for any parameterization to be sensible. Engle and Kroner (1995) proposed a new parameterization that easily imposes this restriction and that eliminates very few interesting models allowed by the *vec* representation. It is called the *BEKK* model. In the introduction we already mentioned that it is named after Baba, Engle, Kraft and Kroner (1990) who wrote the preliminary version of Engle and Kroner. It is characterized by the following equation:

$$\mathbf{H}_t = \boldsymbol{\omega}^* \boldsymbol{\omega}^{*'} + \sum_{i=1}^p \mathbf{A}_i^* \mathbf{r}_{t-i} \mathbf{r}_{t-i}' \mathbf{A}_i^{*'} + \sum_{j=1}^q \mathbf{B}_j^* \mathbf{H}_{t-j} \mathbf{B}_j^{*'} \quad (2.6)$$

where $\boldsymbol{\omega}^*$, all \mathbf{A}_i^* and all \mathbf{B}_j^* are $n \times n$ matrices and $\boldsymbol{\omega}^*$ is an upper triangular.

The bivariate GARCH(1,1) *BEKK* model is now easily deduced from (2.6):

$$\mathbf{H}_t = \boldsymbol{\omega}^* \boldsymbol{\omega}^{*'} + \mathbf{A}_1^* \mathbf{r}_{t-1} \mathbf{r}_{t-1}' \mathbf{A}_1^{*'} + \mathbf{B}_1^* \mathbf{H}_{t-1} \mathbf{B}_1^{*'} \quad (2.7)$$

and written out, the model is equal to:

$$\begin{aligned} \mathbf{H}_t &= \begin{bmatrix} \omega_{11}^* & \omega_{12}^* \\ 0 & \omega_{22}^* \end{bmatrix}' \begin{bmatrix} \omega_{11}^* & \omega_{12}^* \\ 0 & \omega_{22}^* \end{bmatrix} \\ &+ \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{bmatrix}' \begin{bmatrix} r_{1,t-1}^2 & r_{1,t-1} \cdot r_{2,t-1} \\ r_{2,t-1} \cdot r_{1,t-1} & r_{2,t-1}^2 \end{bmatrix} \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{bmatrix} \\ &+ \begin{bmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{bmatrix}' \mathbf{H}_{t-1} \begin{bmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{bmatrix} \end{aligned} \quad (2.8)$$

The *BEKK* parameterization for a bivariate model involves 11 parameters, only two more than the *vec parameterization*, but for higher-dimensional systems the extra number of parameters in the *BEKK* model increases. In general, there are $n(n + 1)/2$ elements of \mathbf{H}_t to be estimated, so that the total number of parameters to be estimated is $n(n + 1)/2 + (p + q) \cdot n^2$.

Next to the models discussed so far there are many more multivariate models proposed by many researchers, such as the AARCH, APARCH, TARCH, MARCH, EGARCH, STARCH, Component ARCH, Asymmetric Component ARCH, Student-t-ARCH and many others. Many of these models recognize that there could be important nonlinearity and asymmetry properties and that returns can be nonnormal distributed. We will not discuss these models, since we prefer to work with the more simple models discussed earlier.

2.3 Copula

Dependency between random variables can also be modeled by copulas. A copula enables the calculation of the joint probability of events from the marginal probabilities of each event. This makes copulas attractive, as the univariate marginal behavior of random variables can be modeled separately from their dependence.

For the random variables X_1, \dots, X_n , the copula with cumulative distribution function $C()$ gives the cumulative probability for the events x_1, \dots, x_n :

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))$$

Since Sklar's (1959) proved that any multivariate continuous distribution function can be uniquely factored into its margins and a copula, the applicability of copulas is wide.

2.4 Copula versus multivariate GARCH

When using a Multivariate GARCH model, you have to specify distributions for the error terms. Usually they are assumed to be normally distributed with mean 0 and variance 1. By assuming this dependency between series, series can be completely modeled by means of the correlation, which is the concept of multivariate GARCH modeling. In situations where multivariate normality does not hold, Copula-based GARCH is useful. So when univariate distributions are complicated and cannot be easily extended to a multivariate setup, Copula is a good way of modeling the dependency between variables. On the other hand, even though Copula seems to have more advantages than Multivariate GARCH modeling, the last one is very attractive because of its simplicity.

3 The dataseries

In this paper we use daily returns on the S&P 500 index¹ and the Lehman US Aggregate Bond index². The data cover the period May 6, 1988 to May 8, 2008 and are obtained from Datastream. Because of holidays and other market closures, some days are missing in both datasets (in both sets the same days are missing), resulting in 5178 observations in both datasets. Graphs showing the price changes of the S&P 500 index and Lehman US Aggregate Bond index over the last twenty years can be found in Figure 1. The graph of the Lehman US Aggregate Bond index does not really show a trend; in some periods the price goes up and in other periods it goes down. The reasons for these upgoing and downgoing periods are unfortunately hard to find out.

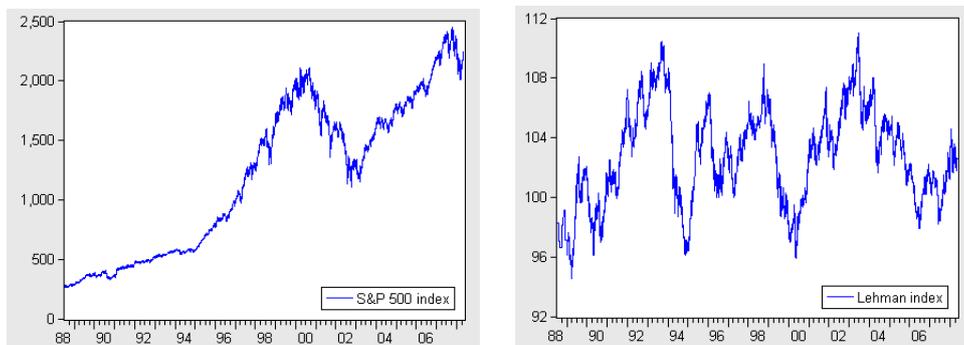


Figure 1: The S&P 500 index and the Lehman Aggregate Bond index

We can say more about the graph of the S&P 500 index. Up to 2000 the prices kept increasing, which can probably be explained by the dot-com bubble³. In the beginning of September 2000 the prices declined quickly until

¹The S&P 500 is one of the biggest stock market indices of the United States. It is owned by Standard & Poor's, a department of McGraw-Hill and contains stocks of the 500 largest corporations, measured through their market capitalization. Market capitalization is a measurement of corporate size and calculated by multiplying the share price times the number of outstanding shares.

²The Lehman US Aggregate Bond index, is a broad index often used to measure the relative performance of bond funds traded in the United States. The index is constructed by Lehman Brothers and is used by more than 90% of the investors in the United States. It includes many different securities, which all have a maturity of more than one year. The Lehman US Aggregate Bond index is calculated by weighting the securities according to the market size of each bond type.

³The dot-com bubble was a speculative bubble which took from about 1995 up to the beginning of 2001. During this period values of stock markets in Western nations increased rapidly as a consequence of the growth in the internet sector. The bubble officially burst at March 10, 2000 when the first market scandals were announced.

somewhere in 2003. Around that time the United States housing bubble⁴ started. The S&P 500 increased until October 9, 2007 when it reached its highest value of \$2447.03. After that day prices declined which can be ascribed to the burst of the housing bubble.

Even though we have examined both price levels shortly, we will not use them directly to examine the intertemporal interaction between the S&P 500 index and Lehman US Aggregate Bond index, because both price levels appear to be non-stationary⁵. For the S&P 500 this can be seen from Figure 1, since it appears to have a global trend in the mean. In order to get a more convincing proof we apply the 'Augmented Dicky-Fuller test' (ADF) in Eviews. When carrying out this test you have to choose your exogenous regressors. You can choose to include a constant, a constant and a linear trend, or neither. We have applied the test for all three options (see Appendix B.1) and it appears that on a 1% confidence level the null hypothesis of stationarity is rejected for all options. Based on these test results we can conclude that the price levels of the S&P 500 and the Lehman US Aggregate Bond are not stationary.

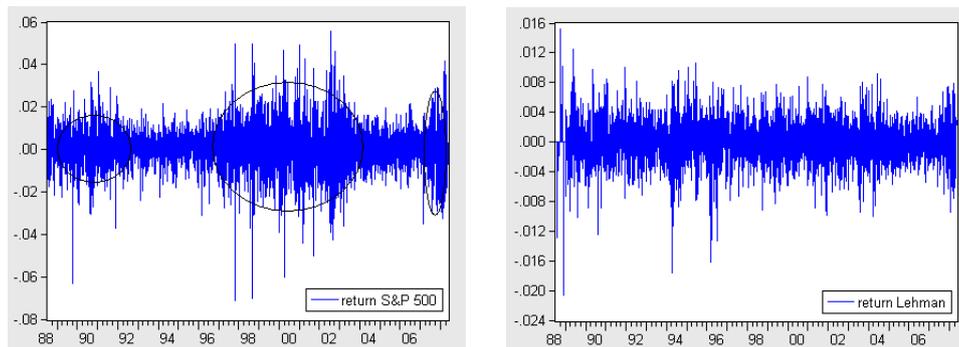


Figure 2: Returns on the S&P 500 and the Lehman US Aggregate Bond Index

By examining the logreturns of both series, computed as the logarithm of the price today divided by the price yesterday, we find that they are stationary. Results of the ADF test can be found in Appendix B.2 and the graphs of the returns can be found in Figure 2. Both graphs show a changing amplitude of the returns. Since the magnitude of the changes is sometimes large

⁴The United States housing bubble began roughly in 2001 when the valuations of real property started increasing rapidly. This period was followed by decreases in home prices. This resulted in a crisis in August 2007, because many homeowners were unable to pay their mortgages.

⁵A time series is stationary when the whole joint distribution is independent of the date at which it is measured. When this does not hold a series is non-stationary.

and sometimes small, both series show signs of GARCH effects. Note that the S&P 500 returns exhibit volatility clustering, since in the three marked periods large returns tend to be followed by large returns. On first sight, we can not give clear reasons for these periods. The last two do not exactly coincide with the dot-com bubble and the housing bubble, but what we do see is that the second circle in Figure 2 roughly coincides with the period before and after the first big peak in Figure 1 (September 2000) and the third circle coincides with the period before and after the second big peak in Figure 1 (October 2007). As mentioned earlier these peaks in Figure 1 can be explained by the burst of the dot-com bubble and the housing-bubble.

If volatility is present this can also be shown by looking at autocorrelations. Autocorrelations are correlations calculated between the value of a random variable today and its value some days in the past. That means that if you have a random variable x , then the autocorrelation coefficient of lag s is given by the following formula:

$$\rho_s = \frac{cov(x_t x_{t+s})}{\sqrt{var(x_t)} \sqrt{var(x_{t+s})}}$$

In Eviews we now make graphs of the squared returns on both series (see Appendix B.3) and a correlogram of the squared returns on both indices, which displays the autocorrelation values (see Appendix B.4). The autocorrelations of the squared returns on Lehman US Aggregate Bond are not large, but they are significant. In the last column of the correlograms p-values are given. When Eviews makes a correlogram, it automatically tests the hypothesis that there is no autocorrelation. This implicitly means that you test if there are no GARCH effects. We see that after 5 lags the p-values become zero, so that the no autocorrelation hypothesis is rejected. This means that there are signs for the presence of volatility clustering and hence for GARCH effects. The correlogram of the S&P 500 gives stronger evidence for the clustering of volatility, since the autocorrelations are larger and from the first lag on the p-value is zero. The fact that all autocorrelations of both series are all positive also indicates the presence of GARCH effects.

4 Modeling the data

Now we turn to the problem of estimating volatility. In the previous section we have found clear evidence for both series displaying volatility clustering. Since GARCH models are designed to deal with volatility clustering, and these models are more simple than copulas, we choose to model the volatilities with the first. The specific model we will use is the *BEKK* model since the conditional covariance matrix is positive semi-definite and we will

include one ARCH and one GARCH term because we do not want the number of parameters to be estimated to be too large.

We start defining the bivariate *BEKK* GARCH(1,1) model, by specifying the return equations. From now on we will use the notation r_s for the return on the S&P 500 index and r_b for Lehman US Aggregate Bond index. The returns are given by the following equations:

$$r_{s,t} = h_{s,t} \cdot \epsilon_{s,t}$$

$$r_{b,t} = h_{b,t} \cdot \epsilon_{b,t}$$

where $\epsilon_{s,t}$ ($\epsilon_{b,t}$) is the error term of $r_{s,t}$ ($r_{b,t}$), with $\epsilon_{s,t} \sim N(0, 1)$ ($\epsilon_{b,t} \sim N(0, 1)$) and $h_{s,t}$ ($h_{b,t}$) is the standard deviation of $r_{s,t}$ ($r_{b,t}$) defined relative to the past information set.

The conditional covariance matrix is given by:

$$\mathbf{H}_t = \begin{pmatrix} \text{var}(r_{s,t}) & \text{cov}(r_{s,t}, r_{b,t}) \\ \text{cov}(r_{b,t}, r_{s,t}) & \text{var}(r_{b,t}) \end{pmatrix} = \begin{pmatrix} h_{s,t}^2 & h_{sb,t}^2 \\ h_{bs,t}^2 & h_{b,t}^2 \end{pmatrix}.$$

Note that $h_{sb,t}^2$ and $h_{bs,t}^2$ are equal and that the estimation of \mathbf{H}_t is already defined in equation (2.7) and (2.8). Eviews 6 includes a program, named **BV_GARCH** (see Appendix B.8), which can estimate the bivariate *BEKK* GARCH(1,1) model, with the restriction that the off-diagonal terms of matrix \mathbf{A}_1 and \mathbf{B}_1 are zero. This means that **BV_GARCH** estimates the parameters of:

$$\begin{aligned} \mathbf{H}_t &= \begin{bmatrix} \omega_{11}^* & \omega_{12}^* \\ 0 & \omega_{22}^* \end{bmatrix}' \begin{bmatrix} \omega_{11}^* & \omega_{12}^* \\ 0 & \omega_{22}^* \end{bmatrix} \\ &+ \begin{bmatrix} a_{11}^* & 0 \\ 0 & a_{22}^* \end{bmatrix}' \begin{bmatrix} r_{s,t-1}^2 & r_{s,t-1} \cdot r_{b,t-1} \\ r_{b,t-1} \cdot r_{s,t-1} & r_{b,t-1}^2 \end{bmatrix} \begin{bmatrix} a_{11}^* & 0 \\ 0 & a_{22}^* \end{bmatrix} \\ &+ \begin{bmatrix} b_{11}^* & 0 \\ 0 & b_{22}^* \end{bmatrix}' \mathbf{H}_{t-1} \begin{bmatrix} b_{11}^* & 0 \\ 0 & b_{22}^* \end{bmatrix} \end{aligned} \quad (4.1)$$

which is equivalent to the following set of equations:

$$\begin{cases} h_{s,t}^2 = \omega_{11}^2 + a_{11}^2 \cdot r_{s,t-1}^2 + b_{11}^2 \cdot h_{s,t-1}^2 \\ h_{b,t}^2 = \omega_{22}^2 + \omega_{12}^2 + a_{22}^2 \cdot r_{b,t-1}^2 + b_{22}^2 \cdot h_{b,t-1}^2 \\ h_{sb,t}^2 = \omega_{12} \cdot \omega_{11} + a_{11} \cdot a_{22} \cdot r_{s,t-1} \cdot r_{b,t-1} + b_{11} \cdot b_{22} \cdot h_{sb,t-1}^2 \end{cases} \quad (4.2)$$

We now apply the program **BV_GARCH** to our two datasets. Note that **BV_GARCH** does not assume that the unconditional means of r_s and r_b

are zero, so it estimates those values too. For reasons explained in Section 2.2.2 we assume that they are zero and therefore we will not include them in our model. The result of the estimation can be found in Appendix B.5, in which the estimates of the parameters, their standard errors, Z-statistics and p-values are given. The estimates for the parameters are given by:

$$\begin{aligned}
\hat{\omega}_{11} &= 0.00062 \\
\hat{\omega}_{22} &= 0.00005 \\
\hat{\omega}_{12} &= 0.00023 \\
\hat{a}_{11} &= 0.20104 \\
\hat{a}_{22} &= 0.15018 \\
\hat{b}_{11} &= 0.97776 \\
\hat{b}_{22} &= 0.98420
\end{aligned}$$

The estimation of our bivariate GARCH(1,1) model therefore is given by the following set of equations:

$$\begin{cases}
h_{s,t}^2 &= 0.0000004 + 0.0404175 \cdot r_{s,t-1}^2 + 0.9560185 \cdot h_{s,t-1}^2 \\
h_{b,t}^2 &= 0.0000001 + 0.0225549 \cdot r_{b,t-1}^2 + 0.9686437 \cdot h_{b,t-1}^2 \\
h_{sb,t}^2 &= 0.0000001 + 0.0301929 \cdot r_{s,t-1} \cdot r_{b,t-1} + 0.9623104 \cdot h_{sb,t-1}^2
\end{cases} \quad (4.3)$$

From the p-values given in the table in Appendix B.5, it follows that all estimated parameters are statistically significant. Hence we may conclude that the covariances are not constant.

Estimates for the coefficients on the product of the return shocks (i.e. $r_{s,t-1}^2$, $r_{b,t-1}^2$ and $r_{s,t-1}r_{b,t-1}$) appear to be larger than zero. For the estimated coefficient on $r_{s,t-1}r_{b,t-1}$, this implies that two shocks of the same sign affect the conditional covariance between the return on the S&P 500 and the return on the Lehman US Aggregate Bond index positively, while two shocks of opposite signs have a negative effect on the forecast covariance. Apparently a negative (positive) return on the S&P 500 and a negative (positive) return on the Lehman US Aggregate Bond index leads to an increase in next period's covariance. If we now examine the estimates for the coefficients on the lagged volatilities (i.e. $h_{s,t-1}^2$, $h_{b,t-1}^2$ and $h_{sb,t-1}^2$), we see that they are close to 1. Clearly, the bulk of the information comes from the previous day forecast volatility. The new information gathered from the returns and the long-run variance has a very small effect. Even though the estimated long-run variances, $\hat{\omega}_{11}$, $\hat{\omega}_{22}$ and $\hat{\omega}_{12}$, are very small, they are still important. When forecasting volatility in the far future, the long-run variance eventually dominates as the importance of news and other recent information fades away. The reason for the long-run variances to be close to zero is that we have used daily data.

To get better knowledge of what our estimated model means, we will start by looking at the graph of $h_{s,t}^2$ and $h_{b,t}^2$ given in Figure 3. Consistent with our expectations, the estimated squared volatilities of the returns on the S&P 500 are higher than the estimated volatilities of the returns on the Lehman US Aggregate Bond index. In general, the stock market is more risky than the bond market, which should result in a higher volatility. We also notice that the three marked periods (by means of circles) which display relatively higher variances, coincide with the periods marked in Figure 2.

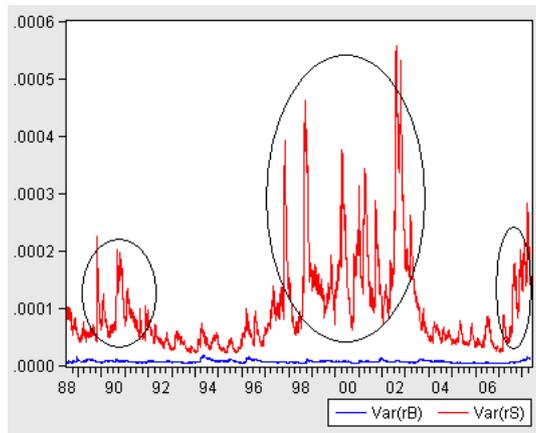


Figure 3: Estimated conditional variances

Figure 4 shows the conditional correlation⁶ between the two returns. We see that the conditional correlation varies over time. This implies that the unconditional correlation between r_s and r_b of 0.25, certainly is a too simple measure for the correlation between r_s and r_b . We notice a turning point in the graph, which corresponds roughly with October 2000 and is marked by the red point (the intersection of the vertical red line and the null axis). Up to the turning point the conditional correlation is mostly positive and after the turning point it is alternating negative and positive. This is confirmed by the mean value of the conditional correlation before and after the turning point. Before the turning point it is equal to 0.27 and after it is -0.10. The turning point could be related to the burst of the dot-com bubble. The S&P 500 experienced the burst in the beginning of September 2000 when it declined quickly (see Figure 1). This decline of the S&P 500 roughly coincides with the turning point. Even though we notice that the turning point coincides with the dot-com crash, we still can not explain why the correlation was mostly positive before the turning point and mostly negative afterwards. To find an answer to this problem, more investigation is needed.

⁶The conditional correlation, $\rho_{sb,t}$, is calculated using the formula: $\rho_{sb,t} = \frac{h_{sb,t}^2}{h_{s,t}h_{b,t}}$.

As mentioned before, Figure 4 indicates that the correlations change substantially in very short periods of time. For instance, on October 17, 1997 the conditional correlation estimate was 0.51 and only 18 days later it had dropped to -0.19. Usually it is difficult to find one particular reason for such sudden changes, since they depend on many different factors. However, the decline in correlation just mentioned could be assigned to the Asian Financial Crisis⁷. During periods of financial market turbulence investors tend to become more risk averse and thereby prompting shifts of funds out of the stock market into safer asset classes, such as bonds are noticed. Since Figure 4 plots the conditional correlation between the stock index S&P 500 and the bond index Lehman, this could be the reason for the sudden drop in October 1997. Notice that the sudden drop in correlation in September 2000, described in the paragraph above, can be explained in the same way.

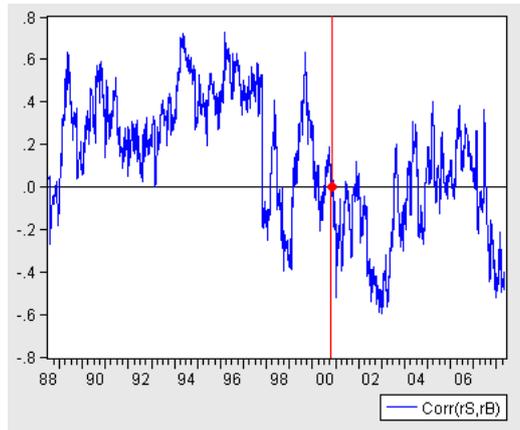


Figure 4: Estimated conditional correlation

⁷The Asian Financial Crisis was a period of financial crisis that gripped much of Asia beginning in the summer of 1997. The United States market was also affected by this crisis. It did not collapse, but was severely hit.

5 Diagnostic Tests

When modeling the variances and covariances, it is important that the model specification is a statistically adequate representation of the dataseries. In particular, it must be the case that the standardized returns show no significant signs of autocorrelation. This means that we want to test the null hypothesis that:

$$\text{cov}(\bar{r}_{s,t}^2, \bar{r}_{s,t+k}^2) = 0, \text{ where } \bar{r}_{s,t}^2 = \frac{r_{s,t}^2}{h_{s,t}^2} \text{ and } k = 1, 2, \dots$$

and

$$\text{cov}(\bar{r}_{b,t}^2, \bar{r}_{b,t+k}^2) = 0, \text{ where } \bar{r}_{b,t}^2 = \frac{r_{b,t}^2}{h_{b,t}^2} \text{ and } k = 1, 2, \dots$$

Plots of the standardized returns are shown in Figure 5 and correlograms of the standardized returns can be found in Appendix B.6. Compared to the squared returns on the S&P 500 (see Appendix B.4), the autocorrelations are dramatically reduced. The corresponding p-values are over 0.5, indicating that we can not reject the hypothesis of no autocorrelation. If we now examine the correlogram of the standardized returns on the Lehman US Aggregate Bond index, we also find that the autocorrelations are almost equal to zero. The p-values on the first twenty lags are all over 0.5, indicating that we can not reject that $\text{cov}(\bar{r}_{b,t}^2, \bar{r}_{b,t+k}^2) = 0$. Based on these results we can conclude that the specification of our model is a statistically adequate representation of our dataseries.

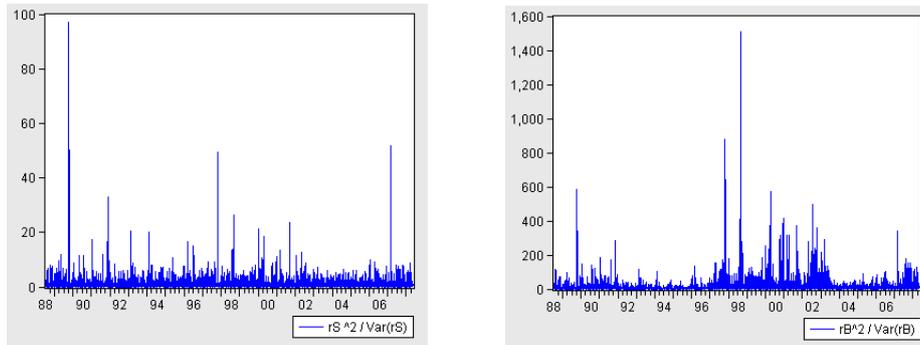


Figure 5: Standardized returns on the S&P 500 and Lehman US Aggregate Bond index

6 Relation with interest rate

Our analysis in the preceding sections evidently demonstrated that the relation between the returns on the S&P 500 index and the Lehman US Aggre-

gate Bond index varies considerably over time. Against this background, it is interesting to examine if certain factors may cause this time variation in the conditional correlation. We will do this for the interest rate as measured by the federal funds rate⁸. We have obtained the timeseries of this rate over the last twenty years. The plot of the interest rate is given in Figure 6. The Fed chose to let the rate vary from 0.860% to 10.7%.

Interest rates determine the propensity of people to either save or consume. If interest rates are low, people prefer to spent their money over saving their money and if interest rates are high the reverse holds. To support this, we will give an example of how the interest rate affected the housing market. After the dot-com crash and the subsequent 2001-2002 recession the Fed cut the interest rates to historically low levels, from about 6.5% in 2001 to just 1% in 2004. One of the results of the decrease in interest rates was an increase in the number of houses sold in the United States. But when the Fed raised the interest rates between 2004 and 2006 several times, homeowners were unable to pay their mortgages anymore, which resulted in the housing market crash.

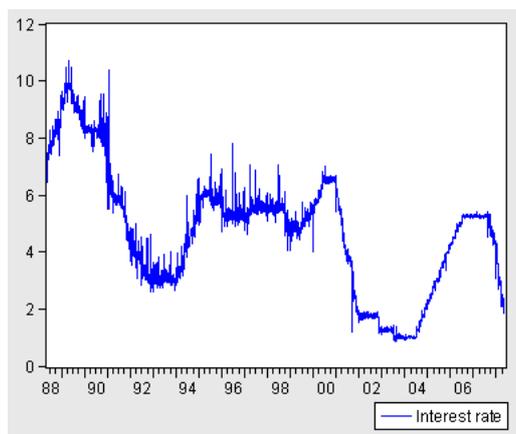


Figure 6: Interest rate

At first sight Figure 6 seems to show a similar pattern as the graph of the estimated conditional correlation. To investigate this further, we plot the conditional correlation with the interest rate divided by 10 in Figure 7. Figure 7 demonstrates that the stock-bond return correlations and the interest rate indeed exhibit rather similar patterns over time. This is confirmed by the correlation between both being 0.40. This implies that the interest rate and the correlation between the stock and bond returns is positively related.

⁸The federal funds rate is the interest rate in the United States at which private depository institutions (mostly banks) lend federal funds at the central bank of the United States called the Federal Reserve (the Fed).

To further examine the impact of the interest rate on the conditional correlation between stock and bond returns, we regress the stock-bond return correlation estimates on the interest rate. A potential difficulty in this regression is that the stock-bond return correlations are restricted to the range $[-1, 1]$, whereas the predicted values of the regression are not restricted to produce values within this range. In order to transform the range of the correlation estimates to $[-\infty, \infty]$ we apply the generalized logit transformation. Consequently, we estimate the following regression model:

$$\text{Log} \left(\frac{1 + \rho_{sb,t}}{1 - \rho_{sb,t}} \right) = C_1 + C_2 \cdot I_{t-1} + \epsilon_t$$

where I_{t-1} is the interest rate at time $t - 1$. From the test results (see Appendix B.7) it follows that the estimation of this regression is given by the following formula:

$$\widehat{\text{Log}} \left(\frac{1 + \rho_{sb,t}}{1 - \rho_{sb,t}} \right) = -0.28 + 0.12 \cdot I_{t-1}$$

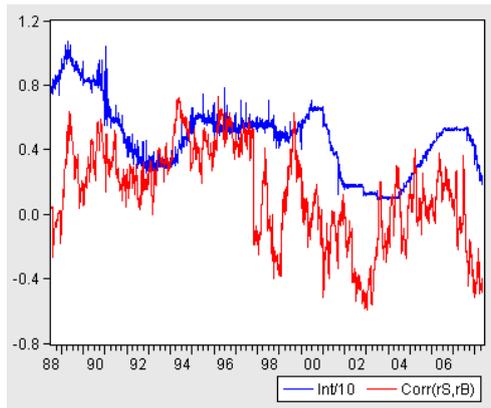


Figure 7: $\rho_{sb,t}$ and adjusted interest rate

It can be seen that the interest rate is positively related to $\rho_{sb,t}$. This implies that if the interest rate increases, $\rho_{sb,t}$ increases also. By looking at the way stocks and bonds are priced we explain this positive relation. There are several methods used to value stocks and bonds, but in most methods stock and bond are priced by discounting the sums of their future cash flows. The rate at which the cash flows are discounted is the interest rate. Therefore, interest rate shocks are likely to move stock and bond prices in the same direction. If for example the interest rate increases, both bond and stock returns are likely to fall. Also notice that if the interest rate increases, the discount factor becomes more important. This implies that if the interest rate increases, $\rho_{sb,t}$ increases also.

Even though it follows from the estimation results that the estimated coefficients are statistically significant, we can not say that the stock-bond return correlations are completely described by the interest rate. This can be seen from the graph in Appendix B.7, in which the residuals⁹ are displayed. To better explain the stock-bond correlations, more variables such as for example the inflation rate or exchange rates should be included. Further research is needed to investigate these options.

7 Conclusion

In this paper we analyzed the intertemporal interaction between returns on the S&P 500 and the Lehman US Aggregate Bond index. When analyzing both datasets we found clear evidence for both series displaying volatility clustering. Therefore, we estimated the time-varying covariances using the bivariate *BEKK* GARCH(1,1) model. We have chosen this specific GARCH model because of its simplicity and the fact that its conditional covariance matrix is positive semi-definite.

The model appears to fit the data well. The standardized returns show no signs of autocorrelation and there are no signs of remaining GARCH effects anymore. The estimated conditional correlation appears to vary considerably over time and in its graph we note a remarkable turning point. Up to the turning point the conditional correlation is mostly positive and after the turning point it is alternating negative and positive. To find a reason for this, further investigation is needed. There might be several factors which caused the time variation in the conditional correlation. One of these factors appears to be the interest rate as measured by the federal funds rate. The graph of the interest rate and the graph of the stock-bond return correlations exhibit rather similar patterns over time and this is confirmed by the correlation between both being 0.40. By regressing the generalized logit transformation of the stock-bond return correlation estimates on the interest rate, again we see that the interest rate is positively related to the stock-bond return correlation. This is probably caused by the fact that they are discounted in the same way.

But even though the estimated regression coefficients are statistically significant, we certainly can not say that the stock-bond return correlations are completely described by the interest rate. To better explain the conditional correlation between the returns on the S&P 500 index and the Lehman US Aggregate Bond index, the impact of other factors, such as for example the inflation rate, should be investigated.

⁹The residual at time t is the difference between $\log\left(\frac{1+\rho_{sb,t}}{1-\rho_{sb,t}}\right)$ and $\widehat{\log}\left(\frac{1+\rho_{sb,t}}{1-\rho_{sb,t}}\right)$

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Appendix A. GARCH(1,1)=ARCH(∞)

By writing out the definition for the univariate GARCH(1,1) model, defined in Section 2.2.2 we get:

$$\begin{aligned}
 h_t^2 &= \omega + \alpha_1 \cdot r_{t-1}^2 + \beta_1 \cdot h_{t-1}^2 \\
 &= \omega + \alpha_1 \cdot r_{t-1}^2 + \beta_1(\omega + \alpha_1 \cdot r_{t-2}^2 + \beta_1 \cdot h_{t-2}^2) \\
 &= \omega + \alpha_1 \cdot r_{t-1}^2 + \beta_1(\omega + \alpha_1 \cdot r_{t-2}^2 + \beta_1(\omega + \alpha_1 \cdot r_{t-3}^2 + \beta_1(\dots))) \\
 &= \omega(1 + \beta_1 + \beta_1^2 + \dots) + \alpha_1(r_{t-1}^2 + \beta_1 \cdot r_{t-2}^2 + \beta_1^2 \cdot r_{t-3}^2) \\
 &= \omega/(1 - \beta_1) + \alpha_1(r_{t-1}^2 + \beta_1 \cdot r_{t-2}^2 + \beta_1^2 \cdot r_{t-3}^2)
 \end{aligned}$$

Herefrom it follows that the GARCH(1,1) model is equivalent to an infinite ARCH model with exponentially declining weights.

Appendix B. Results from Eviews

Appendix B.1 ADF test for price levels

ADF test for S&P 500 index

Exogenous regressors	t-statistic	p-value
constant	-0.634974	0.8605
trend and intercept	-2.100216	0.5450
none	1.414153	0.9612

ADF test for Lehman US Aggregate Bond index

Exogenous regressors	t-statistic	p-value
constant	-3.024005	0.0328
trend and intercept	-2.995249	0.0328
none	0.139966	0.7266

Appendix B.2 ADF test for returns

ADF test for returns on the S&P 500

Exogenous regressors	t-statistic	p-value
constant	-73.49284	0.0001
trend and intercept	-73.50596	0.0001
none	-73.37341	0.0001

ADF test for the returns on the Lehman US Aggregate Bond

Exogenous regressors	t-statistic	p-value
constant	-69.43340	0.0001
trend and intercept	-69.42899	0.0001
none	-69.43936	0.0001

Appendix B.3 Graphs of squared returns

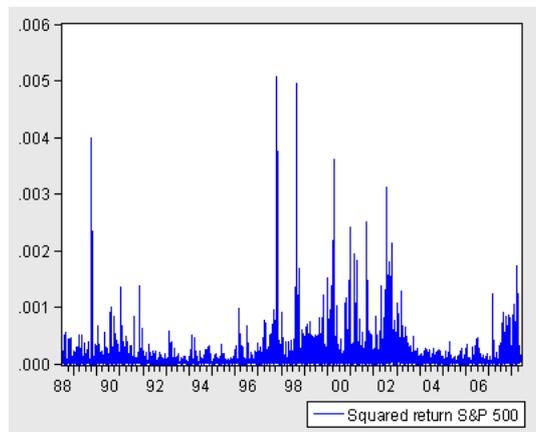


Figure 8: Squared returns on S&P 500

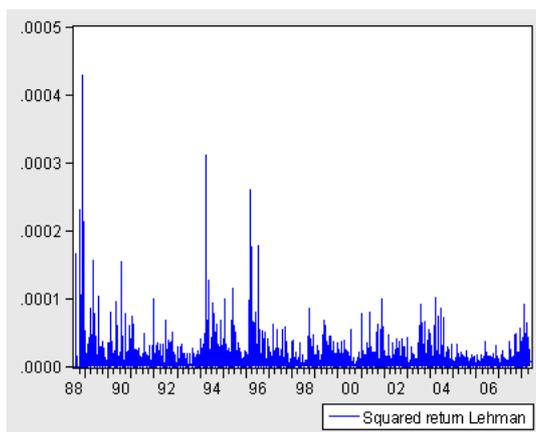


Figure 9: Squared returns on Lehman US Aggregate Bond index

Appendix B.4 Correlograms of squared returns

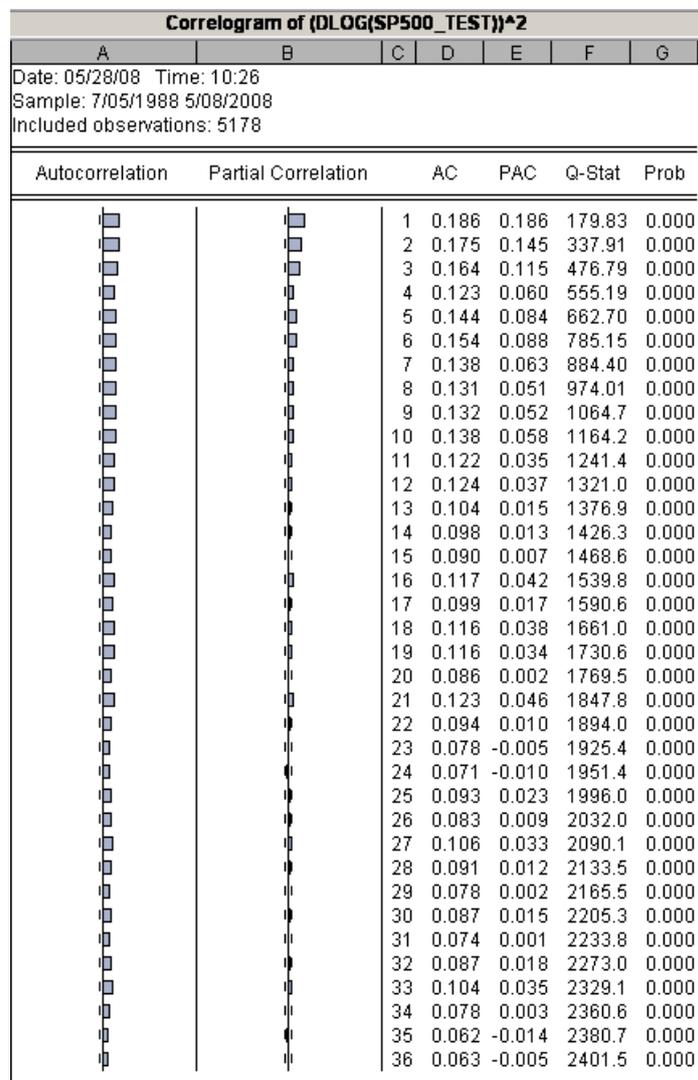


Figure 10: Correlogram of squared returns on S&P 500

Correlogram of (DLOG(LEHMAN_TEST))^2						
A	B	C	D	E	F	G
Date: 05/28/08 Time: 10:28						
Sample: 7/05/1988 5/08/2008						
Included observations: 5178						
Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob	
		1	0.023	0.023	2.6484	0.104
		2	0.024	0.024	5.7304	0.057
		3	0.011	0.010	6.3119	0.097
		4	0.028	0.027	10.438	0.034
		5	0.057	0.055	27.228	0.000
		6	0.022	0.018	29.678	0.000
		7	0.020	0.016	31.785	0.000
		8	0.029	0.026	36.064	0.000
		9	0.036	0.031	42.651	0.000
		10	0.031	0.025	47.721	0.000
		11	0.025	0.020	51.092	0.000
		12	0.008	0.002	51.458	0.000
		13	0.050	0.044	64.565	0.000
		14	0.016	0.007	65.837	0.000
		15	0.032	0.024	71.189	0.000
		16	0.026	0.019	74.778	0.000
		17	0.028	0.020	78.739	0.000
		18	0.026	0.016	82.263	0.000
		19	0.008	-0.001	82.556	0.000
		20	0.069	0.060	107.04	0.000
		21	0.064	0.055	128.62	0.000
		22	0.067	0.055	151.72	0.000
		23	0.008	-0.003	152.07	0.000
		24	0.047	0.037	163.72	0.000
		25	0.066	0.052	186.12	0.000
		26	0.024	0.006	189.09	0.000
		27	0.024	0.009	192.02	0.000
		28	0.001	-0.011	192.02	0.000
		29	0.005	-0.013	192.13	0.000
		30	0.023	0.004	194.83	0.000
		31	0.029	0.013	199.25	0.000
		32	0.006	-0.006	199.44	0.000
		33	0.002	-0.013	199.45	0.000
		34	0.030	0.018	204.28	0.000
		35	0.022	0.005	206.80	0.000
		36	-0.006	-0.019	206.96	0.000

Figure 11: Correlogram of squared returns on Lehman US Aggregate Bond index

Appendix B.5 Output of the program BV_GARCH

LogL: BVGARCH Workfile: BEIDE_APART_RESULT:Untitled\									
View	Proc	Object	Print	Name	Freeze	MergeText	Estimate	Stats	Spec
LogL: BVGARCH									
Method: Maximum Likelihood (Marquardt)									
Date: 05/20/08 Time: 13:17									
Sample: 7/05/1988 5/08/2008									
Included observations: 5178									
Evaluation order: By observation									
Estimation settings: tol= 1.0e-05, derivs=accurate numeric									
Initial Values: MU(1)=0.00055, MU(2)=3.1e-05, OMEGA(1)=0.00076,									
BETA(1)=0.97451, ALPHA(1)=0.21092, OMEGA(3)=0.00048,									
OMEGA(2)=0.00000, BETA(2)=0.96983, ALPHA(2)=0.15288									
Convergence achieved after 26 iterations									
		Coefficient	Std. Error	z-Statistic	Prob.				
	MU(1)	0.000589	0.000107	5.478410	0.0000				
	MU(2)	9.90E-06	3.13E-05	0.316640	0.7515				
	OMEGA(1)	0.000615	4.55E-05	13.51102	0.0000				
	BETA(1)	0.977762	0.001301	751.3708	0.0000				
	ALPHA(1)	0.201041	0.005880	34.19242	0.0000				
	OMEGA(3)	0.000234	1.95E-05	12.00085	0.0000				
	OMEGA(2)	4.86E-05	1.15E-05	4.236803	0.0000				
	BETA(2)	0.984197	0.001134	867.6633	0.0000				
	ALPHA(2)	0.150183	0.004082	36.79363	0.0000				
Log likelihood		41313.68	Akaike info criterion		-15.95391				
Avg. log likelihood		7.978694	Schwarz criterion		-15.94252				
Number of Coefs.		9	Hannan-Quinn criter.		-15.94993				

Figure 12: Output of the program BV_GARCH

Appendix B.6 Correlograms of standardized returns

Correlogram of (DLOG(SP500_TEST))^2VAR_Y1						
A	B	C	D	E	F	G
Date: 05/28/08 Time: 10:49						
Sample: 7/05/1988 5/08/2008						
Included observations: 5178						
Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob	
		1	0.006	0.006	0.1620	0.687
		2	0.014	0.014	1.2025	0.548
		3	-0.004	-0.004	1.2683	0.737
		4	0.003	0.003	1.3256	0.857
		5	0.005	0.005	1.4713	0.916
		6	0.003	0.002	1.5049	0.959
		7	-0.010	-0.010	2.0551	0.957
		8	-0.001	-0.001	2.0633	0.979
		9	0.006	0.007	2.2635	0.987
		10	0.014	0.014	3.3252	0.973
		11	-0.003	-0.004	3.3866	0.985
		12	-0.004	-0.004	3.4767	0.991
		13	-0.011	-0.010	4.0678	0.990
		14	0.010	0.010	4.6112	0.991
		15	-0.021	-0.021	6.8976	0.960
		16	0.003	0.003	6.9517	0.974
		17	-0.007	-0.006	7.2227	0.980
		18	-0.002	-0.002	7.2392	0.988
		19	0.010	0.010	7.7882	0.989
		20	-0.013	-0.013	8.6329	0.987
		21	0.000	0.001	8.6333	0.992
		22	-0.017	-0.017	10.202	0.984
		23	-0.009	-0.008	10.595	0.987
		24	-0.014	-0.014	11.684	0.983
		25	-0.002	-0.001	11.712	0.989
		26	-0.009	-0.008	12.125	0.990
		27	0.000	0.001	12.125	0.994
		28	-0.011	-0.011	12.727	0.994
		29	-0.013	-0.013	13.624	0.993
		30	-0.011	-0.011	14.259	0.993
		31	-0.007	-0.006	14.511	0.995
		32	-0.015	-0.014	15.667	0.993
		33	0.008	0.008	15.998	0.994
		34	-0.004	-0.002	16.066	0.996
		35	-0.006	-0.007	16.233	0.997
		36	-0.003	-0.002	16.272	0.998

Figure 13: Correlogram of standardized returns on the S&P 500

Correlogram of (DLOG(LEHMAN_TEST))^2/VAR_Y2						
A	B	C	D	E	F	G
Date: 05/28/08 Time: 10:49						
Sample: 7/05/1988 5/08/2008						
Included observations: 5178						
Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob	
⋮	⋮	1 -0.008	-0.008	0.3720	0.542	
⋮	⋮	2 -0.007	-0.007	0.6564	0.720	
⋮	⋮	3 -0.015	-0.016	1.8944	0.595	
⋮	⋮	4 -0.007	-0.008	2.1716	0.704	
⋮	⋮	5 0.013	0.012	2.9923	0.701	
⋮	⋮	6 -0.011	-0.011	3.6327	0.726	
⋮	⋮	7 -0.012	-0.012	4.3567	0.738	
⋮	⋮	8 -0.006	-0.006	4.5381	0.806	
⋮	⋮	9 0.000	-0.000	4.5386	0.873	
⋮	⋮	10 -0.005	-0.006	4.6604	0.913	
⋮	⋮	11 -0.008	-0.008	5.0083	0.931	
⋮	⋮	12 -0.018	-0.018	6.6624	0.879	
⋮	⋮	13 0.014	0.014	7.7464	0.860	
⋮	⋮	14 -0.020	-0.021	9.9174	0.768	
⋮	⋮	15 0.004	0.003	10.014	0.819	
⋮	⋮	16 -0.006	-0.006	10.197	0.856	
⋮	⋮	17 -0.007	-0.007	10.450	0.884	
⋮	⋮	18 -0.011	-0.012	11.044	0.892	
⋮	⋮	19 -0.014	-0.014	12.089	0.882	
⋮	⋮	20 0.033	0.032	17.744	0.604	
⋮	⋮	21 0.042	0.041	26.702	0.181	
⋮	⋮	22 0.058	0.058	44.170	0.003	
⋮	⋮	23 -0.009	-0.007	44.628	0.004	
⋮	⋮	24 0.002	0.004	44.648	0.006	
⋮	⋮	25 0.018	0.019	46.289	0.006	
⋮	⋮	26 -0.013	-0.014	47.193	0.007	
⋮	⋮	27 -0.002	-0.001	47.217	0.009	
⋮	⋮	28 -0.015	-0.013	48.456	0.010	
⋮	⋮	29 -0.012	-0.011	49.146	0.011	
⋮	⋮	30 -0.006	-0.007	49.347	0.014	
⋮	⋮	31 -0.001	-0.001	49.355	0.019	
⋮	⋮	32 -0.009	-0.007	49.801	0.023	
⋮	⋮	33 -0.017	-0.017	51.367	0.022	
⋮	⋮	34 0.001	0.003	51.377	0.028	
⋮	⋮	35 -0.009	-0.010	51.773	0.034	
⋮	⋮	36 -0.020	-0.019	53.898	0.028	

Figure 14: Correlogram of standardized returns on the Lehman US Aggregate Bond index

Appendix B.7 Regression analysis

Dependent Variable: LOGIT_CORR
Method: Least Squares
Date: 06/04/08 Time: 16:38
Sample: 7/05/1988 5/08/2008
Included observations: 5178
White Heteroskedasticity-Consistent Standard Errors & Covariance

	Coefficient	Std. Error	t-Statistic	Prob.
INTEREST_FRB_H15	0.118796	0.003443	34.49898	0.0000
C	-0.278537	0.018781	-14.83044	0.0000

R-squared	0.157279	Mean dependent var	0.280870
Adjusted R-squared	0.157116	S.D. dependent var	0.643120
S.E. of regression	0.590440	Akaike info criterion	1.784488
Sum squared resid	1804.453	Schwarz criterion	1.787019
Log likelihood	-4618.039	Hannan-Quinn criter.	1.785373
F-statistic	966.0094	Durbin-Watson stat	0.014303
Prob(F-statistic)	0.000000		

Figure 15: Results of regression of $\left(\frac{1+\rho_{sb,t}}{1-\rho_{sb,t}}\right)$ on the interest rate

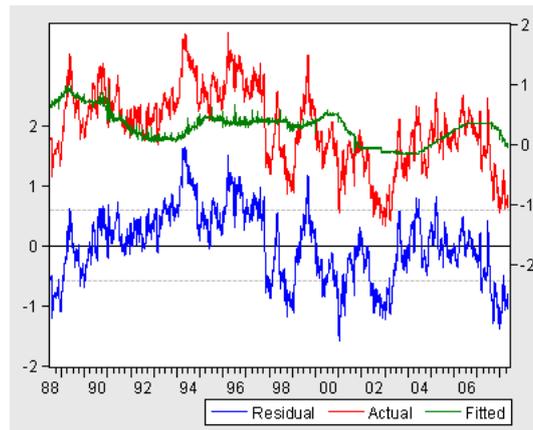


Figure 16: The actual, fitted, and residual graph of the regression of $\left(\frac{1+\rho_{sb,t}}{1-\rho_{sb,t}}\right)$ on the interest rate

Appendix B.8 GARCH program

```
' BV_GARCH.PRG (3/30/2004)
' Revised for 6.0 (3/7/2007)
' example program for EViews LogL object
,
' restricted version of
' bi-variate BEKK of Engle and Kroner (1995):
,
' y = mu + res
' res ~ N(0,H)
,
' H = omega*omega' + beta H(-1) beta' + alpha res(-1) res(-1)' alpha'
,
' where
,
'     y = 2 x 1
'     mu = 2 x 1
'     H = 2 x 2 (symmetric)
'         H(1,1) = variance of y1   (saved as var_y1)
'         H(1,2) = cov of y1 and y2 (saved as cov_y1y2)
'         H(2,2) = variance of y2   (saved as var_y2)
'     omega = 2 x 2 low triangular
'     beta = 2 x 2 diagonal
'     alpha = 2 x 2 diagonal
,

'change path to program path
%path = @runpath
cd %path

' load workfile
load beide_apart.wf1

' dependent variables of both series must be continues
smpl @all
series y1 = dlog(sp500_test)
series y2 = dlog(lehman_test)

' set sample
' first observation of s1 need to be one or two periods after
' the first observation of s0
sample s0 6/5/1988 8/5/2008
```

```
sample s1 7/5/1988 8/5/2008
```

```
' initialization of parameters and starting values  
' change below only to change the specification of model  
smpl s0
```

```
'get starting values from univariate GARCH  
equation eq1.arch(m=100,c=1e-5) y1 c  
equation eq2.arch(m=100,c=1e-5) y2 c
```

```
' declare coef vectors to use in bi-variate GARCH model  
' see above for details
```

```
coef(2) mu  
mu(1) = eq1.c(1)  
mu(2)= eq2.c(1)
```

```
coef(3) omega  
omega(1)=(eq1.c(2))^.5  
omega(2)=0  
omega(3)=eq2.c(2)^.5
```

```
coef(2) alpha  
alpha(1) = (eq1.c(3))^.5  
alpha(2) = (eq2.c(3))^.5
```

```
coef(2) beta  
beta(1)= (eq1.c(4))^.5  
beta(2)= (eq2.c(4))^.5
```

```
' constant adjustment for log likelihood  
!mlog2pi = 2*log(2*@acos(-1))
```

```
' use var-cov of sample in "s1" as starting value of variance-covariance  
' matrix
```

```
series cov_y1y2 = @cov(y1-mu(1), y2-mu(2))  
series var_y1 = @var(y1)  
series var_y2 = @var(y2)
```

```
series sqres1 = (y1-mu(1))^2  
series sqres2 = (y2-mu(2))^2  
series res1res2 = (y1-mu(1))*(y2-mu(2))
```

```

' .....
' LOG LIKELIHOOD
' set up the likelihood
' 1) open a new blank likelihood object (L.O.) name bvgarch
' 2) specify the log likelihood model by append
' .....

logl bvgarch
bvgarch.append @logl logl
bvgarch.append sqres1 = (y1-mu(1))^2
bvgarch.append sqres2 = (y2-mu(2))^2
bvgarch.append res1res2 = (y1-mu(1))*(y2-mu(2))

' calculate the variance and covariance series
bvgarch.append var_y1 = omega(1)^2 + beta(1)^2*var_y1(-1)
+ alpha(1)^2*sqres1(-1)
bvgarch.append var_y2 = omega(3)^2+omega(2)^2 + beta(2)^2*var_y2(-1)
+ alpha(2)^2*sqres2(-1)
bvgarch.append cov_y1y2 = omega(1)*omega(2) + beta(2)*beta(1)*cov_y1y2(-1)
+ alpha(2)*alpha(1)*res1res2(-1)

' determinant of the variance-covariance matrix
bvgarch.append deth = var_y1*var_y2 - cov_y1y2^2

' inverse elements of the variance-covariance matrix
bvgarch.append invh1 = var_y2/deth
bvgarch.append invh3 = var_y1/deth
bvgarch.append invh2 = -cov_y1y2/deth

' log-likelihood series
bvgarch.append logl =-0.5*(!mlog2pi + (invh1*sqres1+2*invh2*res1res2
+invh3*sqres2)+ log(deth))

' remove some of the intermediary series
' bvgarch.append @temp invh1 invh2 invh3 sqres1 sqres2 res1res2 deth

' estimate the model
smpl s1
bvgarch.ml(showopts, m=100, c=1e-5)

' change below to display different output
show bvgarch.output
graph varcov.line var_y1 var_y2 cov_y1y2

```

```
show varcov
graph corr_y1y2.line cov_y1y2/@sqrt(var_y1*var_y2)
show corr_y1y2

' LR statistic for univariate versus bivariate model
scalar lr = -2*( eq1.@logl + eq2.@logl - bvgarch.@logl )
scalar lr_pval = 1 - @cchisq(lr,1)
```