- Bachelor Thesis -

On Dupin cyclides

University of Groningen
Department of mathematics

Author:
Martijn van der Valk

Supervisor:
Prof.Dr. Gert Vegter

Abstract

Dupin cyclides are surfaces all lines of curvature of which are circular. We study, from an idiosyncratic approach of inversive geometry, this and more geometric properties, e.g., symmetry, of these surfaces to obtain a clear geometric description of Dupin cyclides. Furthermore, we investigate in terms of inversive geometry the application of Dupin cyclides in Computer Aided Geometric Design (CAGD) in blending between intersecting natural quadrics. Nowhere else in the literature, we found such a method and results of blending intersecting natural quadrics.

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Chapter 1

Introduction

Dupin cyclides

In the 19th century the French mathematician Charles Pierre Dupin discovered surfaces all lines of curvature of which are circular. He called these surfaces cyclides in his book *Applications de Géometrie* (1822).
Dupin cyclides have been studied by several mathematicians like Cayley [Cayley] and Maxwell [Max]. These mathematicians studied Dupin cyclides in the late 19th century which resulted in an enormous list of properties of Dupin cyclides.
Interest in cyclides revived in the 1980's. It was motivated by research in Computer Aided Geometric Design (CAGD) and from the viewpoint of Lie Sphere Geometry by Pinkall, Cecil and Ryan. In CAGD Dupin cyclides are used among others in blending intersecting natural quadrics (cones, cylinders, planes and spheres).

Problem statement and main results

Studying the paper of Chandru, Dutta and Hoffmann [ChDuHo] it becomes clear that the paper is quite intuitive and contains faults. Chandru, Dutta and Hoffmann state six definitions of a Dupin cyclide in [ChDuHo] and try to show the equivalence between these definitions. We state four of these six definitions of a Dupin cyclide and provide, in an idiosyncratic approach of inversive geometry, a rigorous mathematical proof for the equivalence between these four definitions. From this we obtain geometric properties of Dupin cyclides concerning (i) existence, (ii) uniqueness, (iii) lines of curvature, (iv) symmetry and (v) classes, i.e., different types of Dupin cyclides. Then we prove in terms of inversive geometry the existence of cyclide blends in the case of (i) two intersecting spheres, (ii) an intersecting sphere and a plane, (iii) an intersecting cylinder and a plane and (iv) an intersecting cone and a plane. It follows from these proofs that these cases are equivalent blending problems. This is a result which hitherto was not known.

Related work

The main article for our study is the paper by Hoffmann, Dutta and Chandru [ChDuHo]. Their work is based on Maxwell [Max], Cayley [Cayley] and Boehm [Boehm I]. In Pratt [Pratt], Pottmann [Pott], Krasauskas [KrasMä], Boehm [Boehm II] and Cecil [Cecil], (Dupin) cyclides are studied from the viewpoint of Laguerre Geometry.
In a series of two papers [AllDut I] and [AllDut II] Allen and Dutta give an extensive overview of Dupin cyclides in blending intersecting natural quadrics. In [AllDut I] theoretical aspects of cyclides in blending intersecting natural quadrics are studied. In [AllDut II] Allen and Dutta consider cyclide blends in all possible cases of intersecting natural quadrics.

Overview

In chapter 2 we study the geometry of Dupin cyclides. First, we give in section 2.1 the definition of a cyclide according to Dupin. Then we give in section 2.2 an introduction to inversive geometry in $\mathbb{R}^3$. In section 2.3 we study the image of a torus, defined as in the appendix, under inversion in $\mathbb{R}^3$. With this knowledge we prove in section 2.4 the equivalence between four different definitions of a Dupin cyclide in $\mathbb{R}^3$. To conclude chapter 2, we study from the viewpoint of inversive geometry in section 2.5 the main geometric properties of Dupin cyclides concerning (i) existence, (ii) uniqueness, (iii) lines of curvature, (iv) symmetry and (v) classes.
In chapter 3 we consider the application of Dupin cyclides in blending intersecting natural quadrics. In section 3.1 we start with an informal discussion of cyclide blends. Then we give a formal definition of a cyclide blend and we give a definition that reduce the problem of blending intersecting natural quadrics to the problem of finding two so-called extreme circles. In section 3.2 we prove from the viewpoint of inversive geometry the existence of a cyclide blend in the case of (i) two intersecting spheres, (ii) an intersecting sphere and a plane, (iii) an intersecting cylinder and a plane and (iv) an intersecting cone and a sphere. From this proof it follows that all these cases are equivalent blending problems.
In chapter 4 we give suggestions for further research and our conclusions.

Chapter 2

On Dupin cyclides

According to Dupin a cyclide is the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres. In this chapter we state four definitions of a Dupin cyclide and provide a mathematical proof for the equivalence between these definitions from an idiosyncratic approach of inversive geometry. Furthermore, we give the main geometric properties of Dupin cyclides. This is not only for theoretical interest, practically this means that Dupin cyclides can be constructed in several ways.

2.1 Dupin cyclides according to Dupin

First we introduce some definitions. In the appendix we give some deviations from standard terminology which we use frequently throughout the text.

Definition 1. *A 1-parameter family of spheres is a collection of spheres* $S(t) \subset \mathbb{R}^3$ *defined by*

$$\|x - c(t)\| - r(t) = 0,$$

where $c : [a, b] \subset \mathbb{R} \to \mathbb{R}^3,\ \ t \mapsto (c_1(t), c_2(t), c_3(t))$ *is the center of* $S(t)$*,* $r : I \subseteq \mathbb{R} \to \mathbb{R},\ \ t \mapsto r(t)$ *is the radius of* $S(t)$*, where* I *is an interval in* $\mathbb{R}$ *and* t *is a parameter.*

Unless otherwise stated d is the standard Euclidean distance, $||.||$ is the standard Euclidean norm and $\langle ., . \rangle$ is the standard inner product.

Definition 2. *A sphere* $S \subset \mathbb{R}^3$ *with radius* r *is proper if* $0 < r < \infty$ *and* S *is non-proper if* $r = 0$ *or* $r = \infty$*.*

Therefore, a point in $\mathbb{R}^3$ and a plane in $\mathbb{R}^3$ are non-proper spheres.

Definition 3. *The envelope surface of a 1-parameter family of spheres is the boundary of the union of the spheres in the 1-parameter family.*

Consider two surfaces S_1 and S_2 from a 1-parameter family of spheres $\mathcal{F}$ defined by an implicit equation $F(x,y,z;t)=0$ and an implicit equation $F(x,y,z;t+\epsilon)$=0, respectively. The set of points that belongs to both S_1 and S_2 satisfies $\frac{F(x,y,z;t+\epsilon)-F(x,y,z;t)}{\epsilon}=0$. When we take the limit $\epsilon \to 0$ we obtain $\frac{\partial F}{\partial t}=0$. Assume that $\frac{\partial^2 F}{\partial t^2} \neq 0$, then solving $\frac{\partial F}{\partial t}=0$ for t gives $t(x,y,z)$.
The envelope surface of $\mathcal{F}$ is defined by $F(x,y,z;t(x,y,z))=0$. Therefore, the envelope surface consists of those points which belong to each pair of infinitely near surfaces in the 1-parameter family.

Example (envelope surfaces) The 1-parameter family of spheres $\mathcal{F}$ defined by the implicit equation $F(x,y,z;t)=(x-t)^2+(y-t)^2+(z-t)^2-t^2=0$ consists of spheres with center on a line l parameterized by $l(t)=(t,t,t)$ and radius $r(t)=t$. Let $t \in [-a,a]$ where $a \in \mathbb{R}$, then the envelope surface of the 1-parameter family is a right circular cone (figure 2.1)and the envelope surface of the 1-parameter family according to definition 3 is a cone with spheres on the ends (figure 2.2).

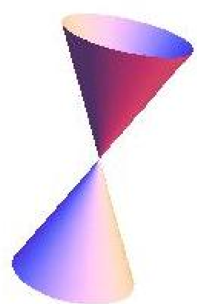

Figure 2.1: *The envelope surface according to the formal definition.*

Figure 2.2: *The envelope surface according to definition 3.*

The difference between these envelope surfaces arises because $t \in [-a,a] \subset \mathbb{R}$. If we let $t \in \mathbb{R}$ then there is no difference between the envelope surface and the envelope surface according to definition 3.

Definition 4 (Dupin)**.** *A cyclide is the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres.*

Examples (Dupin cyclides) Let P_1, P_2 and P_3 be three fixed planes in $\mathbb{R}^3$ such that P_1 is parallel to P_2 and P_3 is perpendicular to P_1. All 1-parameter spheres in $\mathbb{R}^3$ tangent to P_1, P_2 and P_3 have their center on a line $l \subset \mathbb{R}^3$ parallel to P_1, P_2 and P_3. Furthermore, $d(l,P_1)=d(l,P_2)=d(l,P_3)=r$, where r is the radius of all spheres tangent to P_1, P_2 and P_3 (figure 2.3). The set of these spheres is 1-parameter family of spheres in $\mathbb{R}^3$ and the envelope surface of the 1-parameter family is a cylinder in (figure 2.4).

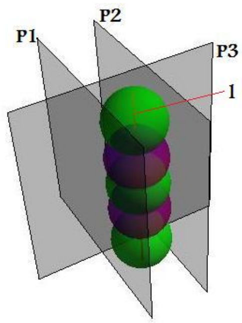

Figure 2.3: *Spheres tangent to the three fixed planes P_1, P_2 and P_3.*

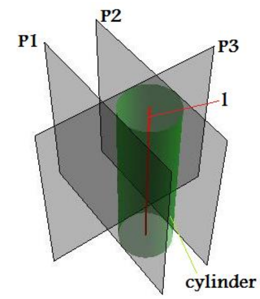

Figure 2.4: *The envelope surface of a 1-parameter family of spheres (a cylinder) tangent to P_1, P_2 and P_3.*

Let $P_1' \subset \mathbb{R}^3$ be the plane defined by $x + y = 0$, let $P_2' \subset \mathbb{R}^3$ be the plane defined by $x - y = 0$ and let $P_3' \subset \mathbb{R}^3$ be the plane defined by $z - x = 0$. A 1-parameter family of spheres in $\mathbb{R}^3$ tangent to P_1', P_2' and P_3' consists of spheres with centers on a line $l' \subset \mathbb{R}^3$ parameterized by $l'(t) = (t, 0, 0)$ and radius $r(t) = t$ for $t \in \mathbb{R}$ (figure 2.5). Hence, the envelope surface of the 1-parameter family is a cone (figure 2.6).

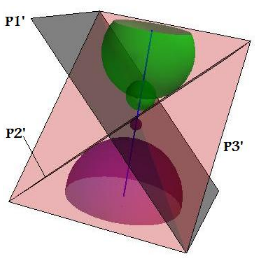

Figure 2.5: *Spheres tangent to the three fixed planes P_1', P_2' and P_3'.*

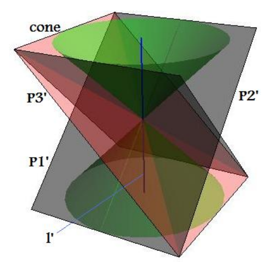

Figure 2.6: *The envelope surface of a 1-parameter family of spheres (a cone) tangent to P_1', P_2' and P_3'.*

Let S_1, S_2 and S_3 be three fixed spheres in $\mathbb{R}^3$ such that S_1 and S_2 (not necessarily of the same radius) are contained in S_3 and let the center of S_1, S_2 and S_3 be on a line $l'' \subset \mathbb{R}^3$. Then there exists a 1-parameter family of spheres in $\mathbb{R}^3$ tangent to S_1, S_2 and S_3 (figure 2.7). Moreover, the centers of spheres from the 1-parameter family lie on a circle and the radius of these spheres is constant (figure 2.8). Hence, the envelope surface of the 1-parameter family is a torus in $\mathbb{R}^3$.
Therefore, a cylinder, a cone and a torus are the envelope surfaces of a 1-parameter family of spheres tangent to three fixed spheres. Hence, a cylinder, a cone and a torus are Dupin cyclides.

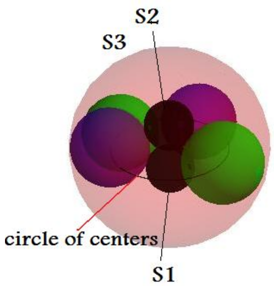

Figure 2.7: *Spheres tangent to the three fixed spheres S_1, S_2 and S_3.*

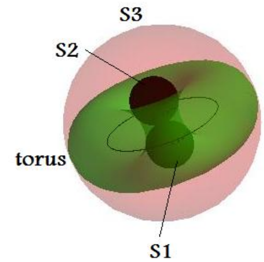

Figure 2.8: *The envelope surface of a 1-parameter family of spheres (a torus) tangent to S_1, S_2 and S_3.*

In particular, natural quadrics (a plane, a sphere, a cylinder and a cone) are Dupin cyclides. Moreover, it is not always possible to find a Dupin cyclide as the envelope surface of a 1-parameter family of spheres in $\mathbb{R}^3$ tangent to three fixed spheres in $\mathbb{R}^3$. Consider for example three concentric fixed spheres S_1, S_2 and S_3 in $\mathbb{R}^3$. Assume without loss of generality that S_1 and S_2 are both contained in S_3. It follows that a 1-parameter family of spheres in $\mathbb{R}^3$ tangent to S_1 and tangent to S_2 cannot be tangent to S_3. Hence, not every triple of fixed spheres in $\mathbb{R}^3$ defines a Dupin cyclide. In the following section we give a brief introduction to inversive geometry in $\mathbb{R}^3$. From this we obtain four equivalent definitions of a Dupin cyclide and their main geometric properties, e.g., curvature and symmetry.

2.2 Inversion

In this section we define inversion in $\mathbb{R}^3$ and give properties of these inversions. These properties are used to provide a rigorous mathematical proof for the equivalence between four definitions of a Dupin cyclide and their main geometric properties.

Definition 5. *Let $S \subset \mathbb{R}^3$ be a sphere with center O and radius k. Inversion i with respect to S is the map $i : \mathbb{R}^3 \setminus \{O\} \to \mathbb{R}^3 \setminus \{O\}$ given by*

$$p' = i(p) = O + k^2 \cdot \frac{p - O}{||p - O||^2}, \quad \forall p \in \mathbb{R}^3 \setminus \{O\}.$$

We call p' the inverse of p with respect to the sphere S. The point O is called the center of inversion, k is called the radius of inversion and S is called the sphere of inversion.

Let $S \subset \mathbb{R}^3$ be a sphere of inversion with center O and radius k and let $p = (p_1, p_2, p_3) \in \mathbb{R}^3 \setminus \{O\}$. Note that p' is the point on the line Op that lies on the same side of p such that $d(O,p) \cdot d(O,p') = k^2$. In other words, inversion in $\mathbb{R}^3$ with respect to a sphere $S \subset \mathbb{R}^3$ maps the inside of S onto the outside of S and vice versa, i.e., inversion is a kind of reflection in a sphere.

In the following proposition we list those properties of inversion in addition to other properties of inversion that enable use to obtain four equivalent definitions of Dupin cyclides and which reveals their main geometric properties.

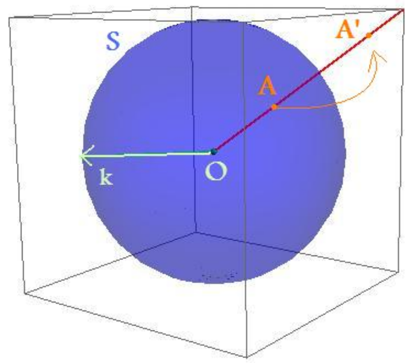

Figure 2.9: *Inversion of a point A with respect to a sphere of inversion S with center O and radius k.*

Proposition 1. *Let $S \subset \mathbb{R}^3$ be a sphere of inversion with center O, then inversion with respect to S maps*

1. *a plane through O onto itself,*
2. *a plane not through O onto a sphere through O,*
3. *a sphere through O onto a plane not through O,*
4. *a sphere not through O onto a sphere not through O, and*
5. *a sphere which intersect S orthogonally onto itself.*

Furthermore, inversion is a conformal bijective map.

Proof. We prove statement 4. Let $S \subset \mathbb{R}^3$ be a sphere of inversion with center O and radius k and let $S'(c,r) \subset \mathbb{R}^3$ be a sphere with center c and radius r, i.e., $S'(c,r) = \{p \in \mathbb{R}^3 | \, ||p-c||^2 = r^2\}$. Let $p \in S'(c,r)$ and let i be the inversion with respect to S. If $||i(p) - e \cdot c||^2 = \tilde{r}^2$ with e and $\tilde{r}$ constants to be determined, then i maps $S'(c,r)$ onto a sphere. From the definition of inversion it follows that

$$\begin{aligned}
||i(p)-e\cdot c||^2 &= \left|\left|\left(O+k^2\frac{p-O}{||p-O||^2}\right)-e\cdot c\right|\right|^2\\
&= \left\langle\left(O+k^2\frac{p-O}{||p-O||^2}\right)-e\cdot c,\left(O+k^2\frac{p-O}{||p-O||^2}\right)-e\cdot c\right\rangle\\
&= \left\langle O+k^2\frac{p-O}{||p-O||^2},O+k^2\frac{p-O}{||p-O||^2}\right\rangle\\
&\quad -2e\left\langle c,O+k^2\frac{p-O}{||p-O||^2}\right\rangle+e^2\,||c||^2\\
&= ||O||^2+\frac{2k^2}{||p-O||^2}\left(\langle O,p\rangle-||O||^2\right)+\frac{2ek^4}{||p-O||^2}\langle c,p\rangle\\
&\quad +e^2\,||c||^2\\
&= \frac{k^2}{||p-O||^2}\left(-2\,||O||^2+2\,\langle O,p\rangle+k^2+e\left(2\,\langle c,O\rangle-2\,\langle c,p\rangle\right)\right)\\
&\quad +||O||^2-2e\,\langle c,O\rangle+e^2\,||c||^2\,. \qquad (2.1)
\end{aligned}$$

Assume without loss of generality that O is the origin. Then equation (2.1) becomes

$$||i\,(p)-e\cdot c||^2=\frac{k^2}{||p||^2}\left(k^2-2e\,\langle c,p\rangle\right)+e^2\,||c||^2\,. \qquad (2.2)$$

From $p\in S'\,(c,r)$ it follows that

$$r^2=||p-c||^2=\langle p-c\rangle=||p||^2-2\,\langle p,c\rangle+||c||^2 \qquad (2.3)$$

which implies that

$$-2\,\langle p,c\rangle=r^2-||p||^2-||c||^2\,. \qquad (2.4)$$

Substitution of equation (2.4) into equation (2.2) gives

$$\begin{aligned}
||i\,(p)-e\cdot c||^2 &= \frac{k^2}{||p||^2}\left(k^2+e\left(||p||^2+||c||^2-r^2\right)\right)+e^2\,||c||^2\\
&= \frac{k^2}{||p||^2}\left(k^2+e\,||c||^2-er^2\right)+k^2e-e^2\,||c||^2\,. \qquad (2.5)
\end{aligned}$$

From $O\notin S'\,(c,r)$ it follows that $||c||^2\neq r^2$. Therefore, set $e=\frac{k^2}{||c||^2-r^2}$. Then it follows that equation (2.5) is the equation of a sphere. □

Remark Consider a 1-parameter family of spheres in $\mathbb{R}^3$ tangent to three fixed spheres in $\mathbb{R}^3$. Let the center of inversion be the origin of $\mathbb{R}^3$ and let the origin be contained in one of the three fixed spheres. Then it follows that none of the spheres from the 1-parameter family passes through the origin. Hence, we can always compute the image of the center of a sphere from the 1-parameter family after inversion with the method of the proof of proposition 1.

Another property of inversion in $\mathbb{R}^3$ that will be used to obtain four equivalent definitions of a Dupin cyclide and their main geometric properties is that it maps two disjoint spheres in $\mathbb{R}^3$ onto two concentric spheres in $\mathbb{R}^3$.

Lemma 1. *There exist infinitely many inversions which map two disjoint spheres in $\mathbb{R}^3$ onto two concentric spheres in $\mathbb{R}^3$.*

Proof. Let S_1 and S_2 in $\mathbb{R}^3$ be two disjoint spheres. Then there exist two spheres S_3 and S_4 in $\mathbb{R}^3$ such that the angle between S_l and S_j for $l = 1, 2$ and $j = 3, 4$ is right, i.e., the center of S_j for $j = 3, 4$ lies on the power line of S_1 and S_2. Let the center of inversion O be a point on the intersection curve $S_3 \cap S_4$. Then it follows from proposition 1 that the image of S_3 and the image of S_4 under inversion i with respect to O are two planes which intersect both spheres $i(S_1)$ and $i(S_2)$ at a right angle. Hence, $i(S_1)$ and $i(S_2)$ are concentric spheres.
Because the center of S_j for $j = 3, 4$ lies on the power line of S_l for $l = 1, 2$ and because O is a point on the intersection curve $S_3 \cap S_4$ it follows that there exist infinitely many inversions in $\mathbb{R}^3$ which map S_1 and S_2 onto two concentric spheres. □

From the proof that inversion in $\mathbb{R}^3$ maps a sphere in $\mathbb{R}^3$ onto a sphere in $\mathbb{R}^3$ we obtain a formula for the center and a formula for the radius of a sphere after inversion. From these formulas and the properties of inversion in $\mathbb{R}^3$ given in this section we obtain four equivalent definitions of a Dupin cyclide and their main geometric properties.
Consider a 1-parameter family of spheres F with the centers of spheres on a plane curve $c(t) \subset \mathbb{R}^3$, where t is a parameter. Let $S \subset \mathbb{R}^3$ be a sphere of inversion with the center at the origin O and radius k. Then inversion with respect to S maps a sphere $S(t)$ from F onto a sphere with center $c'(t) = \frac{k^2}{c_1(t)^2 + c_2(t)^2 - r(t)^2} c(t)$. Therefore, $c'(t) \subset \mathbb{R}^3$ is a constant multiple of $c(t)$ for all $t \in \mathbb{R}$, i.e., the centers of the spheres from F under inversion are contained in a plane.
Moreover, from the proof of statement 4 of proposition 1 we obtain the radius of a sphere $S(t)$ from F under inversion. Substitution of $e = \frac{k^2}{||c||^2 - r^2}$ into equation (2.5) gives

$$\left|\left|i(p) - \frac{k^2 c}{||c||^2 - r^2}\right|\right|^2 = k^2 \frac{k^2}{||c||^2 - r^2} - (\frac{k^2}{||c||^2 - r^2})^2 \, ||c||^2 = \left(\frac{kr}{||c||^2 - r^2}\right)^2. \qquad (2.6)$$

Hence, the radius of a sphere $S(t)$ from F under inversion is given by $\frac{k^2 r}{||c||^2 - r^2}$, where k is the radius of inversion, c the center of the original sphere and r the radius of the original sphere. We conclude that if the center of inversion is the origin of $\mathbb{R}^3$, then for a sphere S from a 1-parameter family of spheres in $\mathbb{R}^3$ it follows that

1. the image of the center of S under inversion is

$$c'(t) = \frac{k^2 c(t)}{||c(t)||^2 - r(t)^2}, \;\; \textit{and} \qquad (2.7)$$

2. the image of the radius of S under inversion is

$$r'(t) = \frac{k^2 r(t)}{||c(t)||^2 - r(t)^2}, \tag{2.8}$$

where k is the radius of inversion, $c(t)$ the locus of the centers of the original spheres from the 1-parameter family and $r(t)$ the radius of the original spheres from the 1-parameter family.
In the following section we study with formula (2.7) what kind of locus of the centers of spheres from a 1-parameter family of spheres such that the envelope surface of the 1-parameter family is a Dupin cyclide define.

2.3 The image of a torus in $\mathbb{R}^3$ under inversion

In this section we study the image of a torus in $\mathbb{R}^3$ under inversion with respect to the origin of $\mathbb{R}^3$. First, we prove that the image of a torus given as the envelope surface of two 1-parameter families of spheres tangent to three fixed spheres is a cyclide according to the definition of Dupin. Finally, we consider the image of the locus of the centers of spheres from the two 1-parameter families of spheres which define the torus.

Lemma 2. *Inversion maps a torus in $\mathbb{R}^3$ onto a Dupin cyclide in $\mathbb{R}^3$.*

Proof. Let $T \subset \mathbb{R}^3$ be a ring torus, a horn torus or a spindle torus given as the envelope surface of a 1-parameter family of spheres F tangent to three fixed spheres S_1, S_2 and S_3. Let without loss of generality i be an inversion with respect to the origin O of $\mathbb{R}^3$ and let O be contained in one of the three fixed spheres S_1, S_2 or S_3. Then i maps any sphere S from F onto a sphere tangent to the spheres $i(S_1), i(S_2)$ and $i(S_3)$. Hence, the envelope surface of the image of F under inversion i is a cyclide according to the definition of Dupin. □

From lemma 2 and formula (2.7) we obtain the locus of the centers of spheres from a 1-parameter family of spheres tangent to three fixed spheres. Moreover, we prove that this locus is a conic, i.e., we prove that the centers of spheres from a 1-parameter family of spheres such that the envelope surface of the 1-parameter family is a Dupin cyclide lie on a conic.

Corollary 1. *Let a torus T in $\mathbb{R}^3$ be the envelope surface of two 1-parameter families of spheres F_1 and F_2 in $\mathbb{R}^3$ with the centers of spheres on a circle and a line, respectively. Then inversion with respect to the origin of $\mathbb{R}^3$ maps F_1 and F_2 onto a 1-parameter family of spheres in $\mathbb{R}^3$ with the centers of spheres on a conic. Moreover, this conic is independent of the radius of inversion.*

Proof. Consider a torus T in $\mathbb{R}^3$ (the standard axis system) and let T be the envelope surface of two 1-parameter families of spheres F_1 and F_2 in $\mathbb{R}^3$ such that the locus of the centers of spheres from F_1 is the circle c and the locus of the center of spheres from F_2 is the line l. Assume without loss of generality that c is contained in the $z = 0$ plane and that the center of c lies on the x-axis (figure 2.10).
It follows that c can be parameterized by $c(t) = (R\cos t + a, R\sin t, 0)$ and that l can be parameterized by $l(t) = (a, 0, t)$. From formula (2.7) it follows that the image of $c(t)$ and the image of $l(t)$ under inversion with respect to the origin of $\mathbb{R}^3$ is given by $c'(t) = \frac{k^2}{||c(t)||^2 - r(t)^2} \cdot c(t)$ and $l'(t) = \frac{k^2}{||l(t)||^2 - r(t^2)} \cdot l(t)$, respectively. Furthermore, the spheres with center on c which define T

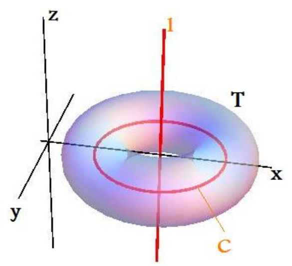

Figure 2.10: *The torus T, the circle c and the line l.*

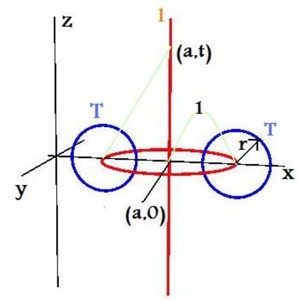

Figure 2.11: *The intersection of the torus T with the $x = 0$ plane.*

have equal radius. Assuming that $R = 1$, i.e., if $0 < r < 1$ then T is a ring torus, if $r = 1$ then T is a horn torus and if $r > 1$ then T is a spindle torus, it follows that

$$c'(t) = k^2 \left(\frac{\cos t + a}{2a \sin t + 1 + a^2 - r^2}, \frac{\sin t}{2a \sin t + 1 + a^2 - r^2}, 0 \right), \tag{2.9}$$

where r is the radius of spheres with center on c.
Now consider the intersection of T with the $x = 0$ plane (figure 2.11). It follows that the radius of spheres with center on l which define T is given by $r(t) = d\left(T, (a, t)\right) = \sqrt{1 + t^2} - r^2$. Hence,

$$l'(t) = k^2 \left(\frac{a}{2r\sqrt{1 + t^2} + a^2 - 1 - r^4}, 0, \frac{t}{2r\sqrt{1 + t^2} + a^2 - 1 - r^4} \right). \tag{2.10}$$

From equation (2.9) it follows that in $\mathbb{R}^3$ (the standard axis system)

$$(x, y, z) = \left(\frac{k^2(\cos t + a)}{2a \sin t + 1 + a^2 - r^2}, \frac{k^2 \cdot \sin t}{2a \sin t + 1 + a^2 - r^2}, 0 \right). \tag{2.11}$$

Solving $\cos t$ and $\sin t$ in terms of x and y with equation (2.11) it follows that

$$\begin{aligned} \cos t &= \frac{-ak^2 - x - a^2 x + r^2 x - 2ay^2}{k^2 - 2ay}, \\ \sin t &= \frac{-y - a^2 + r^2 y}{k^2 - 2ay} \end{aligned}$$

From these expressions of $\cos t$ and $\sin t$ and from the equality $\cos^2 t + \sin^2 t = 1$ we obtain a rational equation in x and y

$$\tfrac{1}{(k^2 - 2ay)^2}(-k^2 + a^2 k^4 - 2ak^2 x - 2a^3 k^2 x + 2ak^2 r^2 x + x^2 + 2a^2 x^2 + a^4 x^2 - 2r^2 x^2 - 2a^2 r^2 x^2 + r^4 x^2 + 4ak^2 y - 4a^3 k^2 y + 4a^2 xy + 4a^4 xy - 4a^2 r^2 xy + y^2 - 2a^2 y^2 + 5a^4 y^2 - 2r^2 y^2 - 2a^2 r^2 y^2 + r^4 y^2 = 1.$$

The numerator of this expression is a quadratic equation in x and y, so its zero set is a conic

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0,$$

where

$$\begin{aligned}
A &= \left(1+a^2-r^2\right)^2, \\
B &= 2a^2(1+a^2-r^2), \\
C &= 5a^4+\left(r^2-1\right)^2-2a(1+r^2), \\
D &= -2ak^2(1+a^2-r^2), \\
E &= -4ak^2\left(a^2-1\right), \textit{and} \\
F &= k^2(a^2-1).
\end{aligned} \tag{2.12}$$

Furthermore, $c'(t)$ is

1. an ellipse if $A^2-4BC<0$,
2. a circle if $A=C$ and $B=0$,
3. a parabola if $A^2-4BC=0$, and
4. a hyperbola if $A^2-4BC>0$.

Therefore, if we plot the expression A^2-4BC for a and r we get a surface from which we obtain what type of conic $c'(t)$ is for given a and r (figure 2.12). Moreover, the expression A^2-4BC is independent of the radius of inversion k. Hence, the type of conic which we obtain from $c(t)$ after inversion is independent of the radius of inversion.

Figure 2.12: *Some plots of the expression A^2-4BC.*

Similarly, it follows that $l'(t)$ is a conic. Hence, inversion with respect to the origin of $\mathbb{R}^3$ maps the 1-parameter families F_1 and F_2 onto 1-parameter families of spheres such that the centers of these spheres lie on a conic which is independent of the radius of inversion. □

From the results of this section it follows that we can compute what the locus of the centers of spheres from a 1-parameter family of spheres tangent to three fixed spheres is, i.e., we can compute the locus of the centers of spheres which define a dupin cyclide defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres. Among others, we use these results to obtain four equivalent definitions of a Dupin cyclide and to obtain the main geometric properties of these surfaces.

2.4 Equivalent definitions of Dupin cyclides

In this section we state four definitions of Dupin cyclides and provide a proof in terms of inversive geometry for the equivalence between them. First we introduce the concept of a pair of anti-conics.

Definition 6. *Two conics c_1 and c_2 form a pair of anti-conics if*

1. *c_1 and c_2 lie in mutually perpendicular planes and*
2. *the vertices (on the major axes) of c_1 are the foci of c_2, and vice versa.*

There exist four types of pairs of anti-conics. These pairs consist of (i) an ellipse and a hyperbola, (ii) two parabolas, (iii) a (straight) line and a circle or (iv) a pair of (straight) lines.

Theorem 1. *A Dupin cyclide is*

1. *the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres,*
2. *the envelope surface of two 1-parameter families of spheres tangent to three fixed spheres,*
3. *the envelope surface of a 1-parameter family of spheres with center in a plane and tangent to two fixed spheres, and*
4. *the envelope surface of a 1-parameter family of spheres with center on a conic and tangent to a fixed sphere.*

Remark From the proof of theorem 1 it follows that a Dupin cyclide is the envelope surface of a 1-parameter family of spheres with center on a conic and tangent to a sphere with the center on another conic such that these conics forms a pair of ant-conics.

In this section we only consider Dupin cyclides obtained from three proper, disjoint fixed spheres. Moreover, in figure 2.13 and in figure 2.14 we give Dupin cyclides such that their pair of anti-conics consists of two parabolas. Furthermore, the lines on the left Dupin cyclides in figure 2.13 and in figure 2.14 are (circular) lines of curvature on the cyclides.

Figure 2.13: *Dupin cyclides defined by two parabolas.*

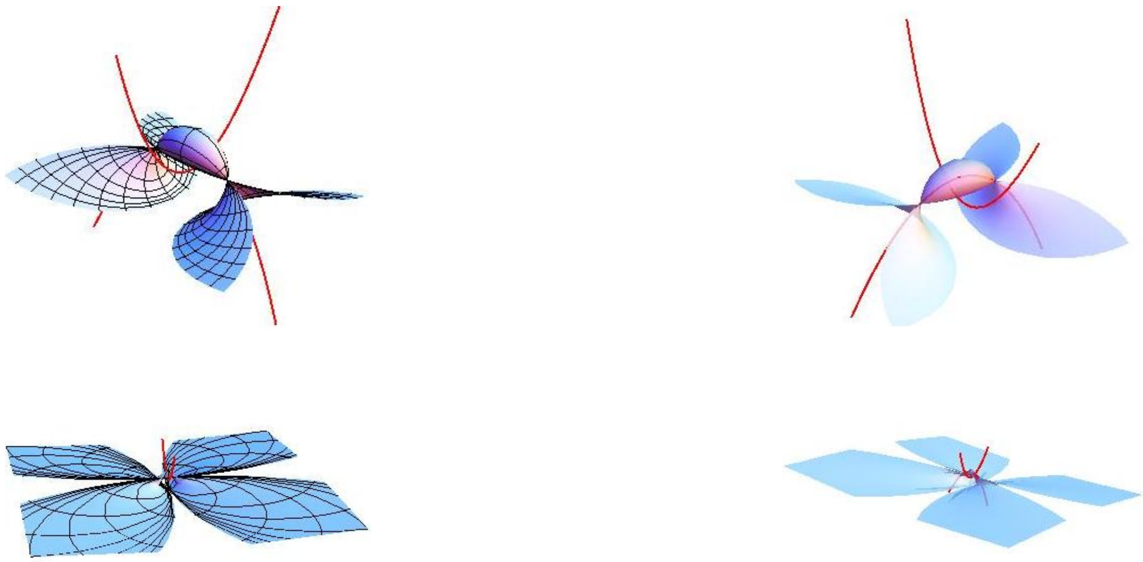

Figure 2.14: *Dupin cyclides defined by two parabolas.*

Proof. Definition (1) is the definition of a cyclide according to Dupin and it follows directly that (2) $\Rightarrow$ (1). We now prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Therefore, let S_1, S_2 and S_3 be three proper, disjoint fixed spheres in $\mathbb{R}^3$, let the origin $O \in \mathbb{R}^3$ be the center of inversion and let i be an inversion with respect to O. From lemma 1 it follows that there exists an inversion i with respect to O such that without loss of generality $i(S_1)$ and $i(S_2)$ are concentric spheres. Let (x_j, y_j, z_j) be the center of S_j and let r_j the radius of S_j for $j = 1, 2, 3$. Let S be a sphere with center (x, y, z), radius r and tangent to the spheres S_1, S_2 and S_3, then

$$(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 = (r \pm r_j)^2 \quad for \;\; j = 1, 2, 3. \tag{2.13}$$

Given that S_1, S_2 and S_3 are disjoint it follows that the system of equations (2.13) have at least one solution, i.e., there exists a sphere S tangent to S_1, S_2 and S_3. Hence, $i(S_3)$ is contained in the annulus of $i(S_1)$ and $i(S_2)$ because $i(S)$ is tangent to $i(S_j)$ for $j = 1, 2, 3$.
It follows that the centers of the 1-parameter spheres tangent to $i(S_j)$ for $j = 1, 2, 3$ lie on a circle c and have constant radius (figure 2.15). Hence, there exists a 1-parameter family of spheres F_1 with center on a circle, constant radius and tangent to $i(S_j)$ for $j = 1, 2, 3$. Therefore, the envelope surface of F_1 is a torus T (figure 2.16). From lemma 2 it follows that the envelope surface of $i(F_1)$ is a cyclide according to the definition of Dupin. Furthermore, it follows from corollary 1 that the centers of spheres from $i(F_1)$ lie on a conic c'. Moreover, the centers of the spheres $i(S_1), i(S_2)$ and $i(S_3)$ lie on a straight line l since $i(S_1)$ and $i(S_2)$ are concentric spheres. Furthermore, the spheres $i(S_j)$ for $j = 1, 2, 3$ are tangent to all spheres from F_1. Hence, l is the locus of the center of spheres from a 1-parameter family of spheres F_2 tangent to the envelope of F_1. Hence, the envelope surface of F_1 and the envelope surface of F_2 are equal. Therefore, the envelope surface of $i(F_1)$ and the envelope surface of $i(F_2)$ are equal. Hence, $i(F_1)$ and $i(F_2)$ determine the same Dupin cyclide. It follows from corollary 1 that the centers of spheres from $i(F_2)$ lie on a conic l'. Hence, a Dupin

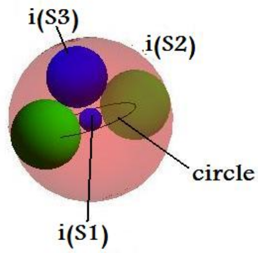

Figure 2.15: *Two spheres with center on a circle and equal radius tangent to $i(S_j)$ for $j = 1, 2, 3$.*

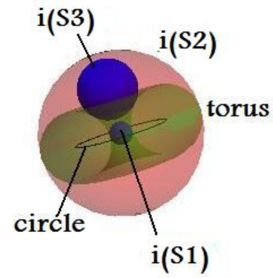

Figure 2.16: *The envelope surface of a 1-parameter family of spheres (a ring torus) tangent to $i(S_j)$ for $j = 1, 2, 3$.*

cyclide is the envelope of (1) two 1-parameter families of spheres tangent to three fixed spheres, (2) a 1-parameter family of spheres tangent to two fixed spheres with center in a plane (figure 2.17) and (3) a 1-parameter family of spheres with center on c' and tangent to a sphere with center l' or a 1-parameter family of spheres with center on l' and tangent to a sphere with center on a c'(figure 2.18). Hence, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, it follows directly from this proof that (4) $\Rightarrow$ (3) and (4) $\rightarrow$ (3).

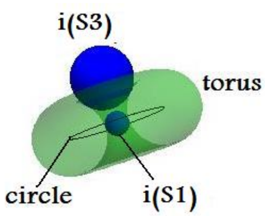

Figure 2.17: *The torus defined by the circle c and two fixed spheres.*

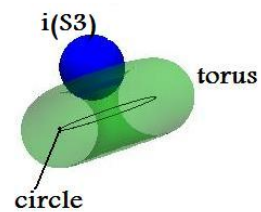

Figure 2.18: *The torus defined by the circle c and one fixed sphere.*

Hence, it remains to prove that the conics c' and l' form a pair of anti-conics. Therefore, let c' be parameterized by $c'(t)$ and let l' be parameterized by $l'(t)$. Furthermore, let the circle c be parameterized by $c(t)$ and let c be contained in a plane P_1. Similarly, let the line l be parameterized by $l(t)$ and let l be contained in a plane P_2. Because T is a torus it follows that P_1 and P_2 are perpendicular to each other. Moreover, from formula (2.7) it follows that $c'(t)$ is a multiple of $c(t)$ and $l'(t)$ is a multiple of $l(t)$ for given t. Hence, $c'(t)$ is contained in P_1 and $l'(t)$ is contained in P_2. Therefore, the conics c' and l' are contained in planes which are perpendicular to each other. An explicit calculation of the length of the major axis and minor axis of these conics shows that the vertices on the major axis of c' are the foci of l' and it shows that the foci of c' are the vertices on the major axis of l'. Therefore, the conics c' and l' form a pair of anti-conics.
Hence, it follows that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). $\square$

2.5 Miscellaneous about Dupin cyclides

In this section we study first, from the viewpoint of inversive geometry some configurations of three fixed spheres. For example, three disjoint fixed spheres or three mutually intersecting fixed spheres, which define Dupin cyclides as the envelope surface of a 1-parameter family of spheres tangent to these fixed spheres. Indeed, we study the existence and uniqueness of Dupin cyclides given as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres. Then we prove that there exist three classes of Dupin cyclides given that their pair of anti-conics consists of an ellipse and a hyperbola. After that we prove that all lines of curvature on a Dupin cyclide are circular and we conclude this section by proving that Dupin cyclides have at least two planes of symmetry.
Nowhere else in the (studied) literature, we found a similar approach to obtain these geometric properties of Dupin cyclides.

Existence and uniqueness of Dupin cyclides

Let three proper, disjoint fixed spheres be given. It follows from the proof of theorem 1 that there exists a Dupin cyclide such that the cyclide is the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres. Let three proper fixed spheres which are mutually tangent be given and let three fixed spheres which are mutually intersecting be given. In this paragraph we examine the existence and uniqueness of a Dupin cyclide given as the envelope surface of a 1-parameter family of spheres tangent to these configurations of three fixed spheres.

Proposition 2. *Let S_1, S_2 and S_3 be three fixed spheres. Let, without loss of generality, S_1 be tangent to S_2, S_2 be tangent to S_3 and let S_3 be disjoint with respect to S_1 and S_2, then there exist two Dupin cyclides such that the cyclides are the envelope surface of a 1-parameter family of spheres tangent to these fixed spheres.*

Proof. Let O be the point of tangency of S_1 and S_2 and let i be the inversion with respect to O. Then $S_1' = i(S_1)$ is a plane parallel to the common tangent plane of S_1 and S_2 at O and $S_2' = i(S_2)$ is a plane parallel to the common tangent plane of S_1 and S_2 at O. Furthermore, $S_3' = i(S_3)$ is a disjoint sphere with respect to S_1' and S_2' (figure 2.19).

Figure 2.19: *Left: the three fixed spheres S_1, S_2 and S_3. Right: the image of S_1, S_2 and S_3 under inversion i.*

Hence, the centers of spheres tangent to S_1' and S_2' lie in a plane P_1 parallel to S_1' and parallel to S_2' such that $d(P, S_1') = d(P, S_2') = \frac{1}{2}d(S_1', S_2')$. Therefore, all spheres tangent to S_1' and S_2' have radius $R = \frac{1}{2}d(S_1', S_2')$. Let c be the center of S_3', let r be the radius of S_3 and let P_2 be a plane which contains c. Consider the intersection of P_2 with P_1, S_1', S_2' and S_3' (figure 2.20).

Figure 2.20: *The intersection of P_2 with P_1, S_1', S_2' and S_3'.*

Let $C_1(m_1, R)$ and $C_2(m_2, R)$ be two circles contained in P_2 and tangent to S_1', S_2' and S_3'. It follows that the centers of these circles lie on a circle $c_1(m, \tilde{r_1})$ contained in P_1, where m is the center of c_1 and the orthogonal projection of c onto P_1 and $\tilde{r_1} = \sqrt{(R+r)^2 - d\,(c,m)^2}$ is the radius of c_1. Similarly, there exist two circles $C_3(m_3, R)$ and $C_4(m_4, R)$ with center on a circle $c_2(m, \tilde{r_2})$ contained in P_1, where m is the center of c_2 and $\tilde{r_2} = \sqrt{(R-r)^2 - d\,(c,m)^2}$ the radius of c_2. Hence, there exist two 1-parameter families of spheres F_1 and F_2 tangent to S_1', S_2' and S_3'. Therefore, $i(F_1)$ and $i(F_2)$ are two 1-parameter families of spheres tangent to S_1, S_2 and S_3. Hence, the envelope surface of $i(F_1)$ and the envelope surface of $i(F_2)$ are Dupin cyclides D_1 and D_2, respectively. Moreover, because $\tilde{r_1} \neq \tilde{r_2}$ it follows that the envelope surface of F_1 is not the same as the envelope surface of F_2. Hence, D_1 and D_2 are different Dupin cyclides. □

Proceeding as in the proof of proposition 2 one can prove the following proposition.

Proposition 3. *Let S_1, S_2 and S_3 be three proper, fixed spheres.*

1. *If S_1, S_2 and S_3 are pairwise tangent to each other, then there exists one Dupin cyclide such that the cyclide is the envelope surface of a 1-parameter family of spheres tangent to S_1, S_2 and S_3.*
2. *If, without loss of generality, S_1 is tangent to S_2, S_2 is tangent to S_3 and S_3 is not tangent to S_1 nor intersects S_1, then there exists two Dupin cyclides such that these cyclides are the envelope surface of a 1-parameter family of spheres tangent to S_1, S_2 and S_3.*

Remark If we count also the degenerate Dupin cyclides (spheres, cylinders and cones), then we find instead of one Dupin cyclide in the case of (1) of proposition 3 two Dupin cyclides.

Instead of a combination of tangent and disjoint fixed spheres we now study the existence and uniqueness of Dupin cyclides defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres such that these fixed spheres intersect each other.

Proposition 4. *Let S_1, S_2 and S_3 be three fixed spheres. Assume, without loss of generality, that S_1 intersects S_2, S_1 intersects S_2 and assume that S_3 is disjoint with respect to S_1 and S_2, then there exist two Dupin cyclides such that the cyclides are the envelope surface of a 1-parameter family of spheres tangent to these fixed spheres.*

Proof. Let c be the intersection curve of $S_1 \cap S_2$ and take a point O on c as the center of inversion and let i be an inversion with respect to O. Then $S_1' = i(S_1)$ is a plane parallel to the tangent plane of S_1 at O and intersects the plane $S_2' = i(S_2)$ which is a plane parallel to the tangent plane of S_2 at O. Furthermore, i maps the sphere S_3 onto the sphere S_3' between S_1' and S_2' (figure 2.21).

Figure 2.21: *Left: the three fixed spheres S_1, S_2 and S_3. Right: the image of S_1, S_2 and S_3 under inversion i.*

Hence, the center of spheres tangent to S_1' and S_2' lie in a plane B_1 or in a plane B_2 such that $\angle(B_k, S_1') = \angle(B_k, S_2')$ for $k = 1, 2$. Let P be the equatorial plane of S_3' perpendicular to B_k for $k = 1, 2$. Consider the intersection of P with S_1', S_2' and S_3' (figure 2.22).

Figure 2.22: *The intersection of P with B_1, B_2, S_1', S_2' and S_3'.*

Let $Cr_1(m_1, R_1)$ and $Cr_2(m_2, R_2)$ be two circles contained in P and tangent to $S_1^{'}, S_2^{'}$ and $S_3^{'}$ (figure 2.22). Furthermore, let $Cr_3(m_3, R_3)$ be a circle contained in P with center m_3 in B_1 tangent to $Cr_1(m_1, R_1)$ and $Cr_2(m_2, R_2)$. Then the ellipse E_1 contained in B_1 such that m_1 and m_2 are the vertices and m_3 is a focal point of E_1 is the locus of the center of spheres tangent to $S_1^{'}, S_2{'}$ and $S_3^{'}$, i.e., the length of the major axis of E_1 is $R_1 + R_2 + 2R_3$ and the length of the minor axis is $\sqrt{(\frac{1}{2}(R_1 + R_2 + 2R_3))^2 - (R_1 + R_3)^2}$. Similarly, there exists two circles $C_4(m_4, R_4)$ and $C_5(m_5, R_5)$ which define an ellipse E_2 which is the locus of the center of spheres tangent to $S_1^{'}, S_2^{'}$

and $S_3^{'}$. Hence, there exist two 1-parameter families of spheres F_1 and F_2 with the center of spheres on E_1 and E_2, respectively, and tangent to $S_1^{'}, S_2^{'}$ and $S_3^{'}$. Therefore, the 1-parameter families $i(F_1)$ and $i(F_2)$ are tangent to S_1, S_2 and S_3. Hence, the envelope surface of $i(F_1)$ and the envelope surface of $i(F_2)$ are two different cyclides D_1 and D_2, respectively, according to the definition of Dupin. □

Proceeding as in the proof of proposition 4 one can prove the following proposition.

Proposition 5. *Let S_1, S_2 and S_3 be three proper, fixed spheres.*

1. *If S_1, S_2 and S_3 intersect each other pairwise, then there exist four Dupin cyclides such that these cyclides are the envelope surface of a 1-parameter family of spheres tangent to S_1, S_2 and S_3.*
2. *If, without loss of generality, S_1 intersects S_2, S_2 intersects S_3 and if S_3 does not intersects S_1, then there exist four Dupin cyclides such that these cyclides are the envelope surface of a 1-parameter family of spheres tangent to S_1, S_2 and S_3.*

Remark In the case of statement (2) of lemma 5 one can argue that in a certain case there exist two Dupin cyclides, i.e., in the case that the planes obtained from inversion both passes through the origin of the fixed sphere after inversion and if the angle between these planes is right. Indeed, if the three fixed spheres intersect each other at a right angle, then there exist two Dupin cyclides according to the definition of Dupin.

Classes of Dupin cyclides

From the proof of lemma 2 it follows that a Dupin cyclide defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres is the image of a torus under inversion. In the appendix we have defined three classes of tori. From these three classes of torii and formula (2.8) we define three classes of Dupin cyclides. Nowhere else in the literature, we found such an approach to define different types (classes) of Dupin cyclides.

Corollary 2. *There exist three classes of Dupin cyclides such that the cyclides are the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres.*

Proof. Let $O = (0, 0, 0) \in \mathbb{R}^3$ be the center of inversion, let F_1 and F_2 be two 1-parameter families of spheres and let E be the envelope surface of F_1 and the envelope surface of F_2. From equation (2.8) it follows that the radius of spheres after inversion with respect to O is a constant multiple of the original spheres. From the proof of lemma 2 it follows that a Dupin cyclide given as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres is the image of a torus under inversion.

First, let the envelope surface E be a ring torus. Then all spheres from F_1 and F_2 which contribute to E have radius different from zero. Hence, none of the spheres which define the Dupin cyclide as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres, which is the image of E under inversion with respect to O, is equal to a point.

Secondly, let the envelope surface E be a horn torus. Then there exists a sphere equal to a point which defines E as the envelope surface of a 1-parameter family of spheres. Therefore, a Dupin cyclide defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed

spheres, which is the image of E under inversion with respect to O, consists of a point.
Finally, let the envelope surface E be a spindle torus. Similarly, there exist two spheres equal to two different points contained in a 1-parameter family of spheres such that the envelope surface of the 1-parameter family is a Dupin cyclide. □

From corollary 2 it follows that there exist three classes of Dupin cyclides. We will refer to these three classes of Dupin cyclides as

1. a *ring cyclide* if none of the spheres from the two 1-parameter families of spheres which define the Dupin cyclide consist of a sphere which is equal to a point,
2. a *horn cyclide* if there exists a sphere such that the sphere is equal to a point and contained in one of the two 1-parameter families of spheres which define the Dupin cyclide, and
3. a *spindle cyclide* if there exists two spheres such that these spheres are equal to two different points and are contained in one of the two 1-parameter families of spheres which define the Dupin cyclide.

A Dupin cyclide defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres is the image under a conformal map of a ring torus, a horn torus and a spindle torus. Hence, it follows from corollary 2 that a Dupin cyclide is a ring cyclide, a horn cyclide or a spindle cyclide if it is the image of a conformal map of a ring torus, a horn torus or a spindle torus, respectively.

Lines of curvature on a Dupin cyclide

A cylinder, a cone and a torus are in particular Dupin cyclides. All lines of curvature on these surfaces are circular. We extend this observation for any Dupin cyclide.

Lemma 3. *All lines of curvature on a Dupin cyclide such that the cyclide is the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres are circular.*

Proof. It follows from the theory of smooth surfaces in $\mathbb{R}^3$ [Gray] that a conformal transformation of $\mathbb{R}^3$ maps lines of curvature on any smooth surface to lines of curvature on the image surface. In lemma 2 we proved that any Dupin cyclide defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres is the image a torus T given as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres under inversion. An inversion of $\mathbb{R}^3$ is in particular a conformal transformation of $\mathbb{R}^3$ and all lines of curvature on the smooth surface T are circular, i.e., the intersection curve of infinitely near intersecting spheres from the 1-parameter family which defines T. Because inversion maps intersecting spheres onto intersecting spheres and lines of curvature on a smooth surface onto lines of curvature on the image surface it follows that all lines of curvature on the Dupin cyclide are circular. □

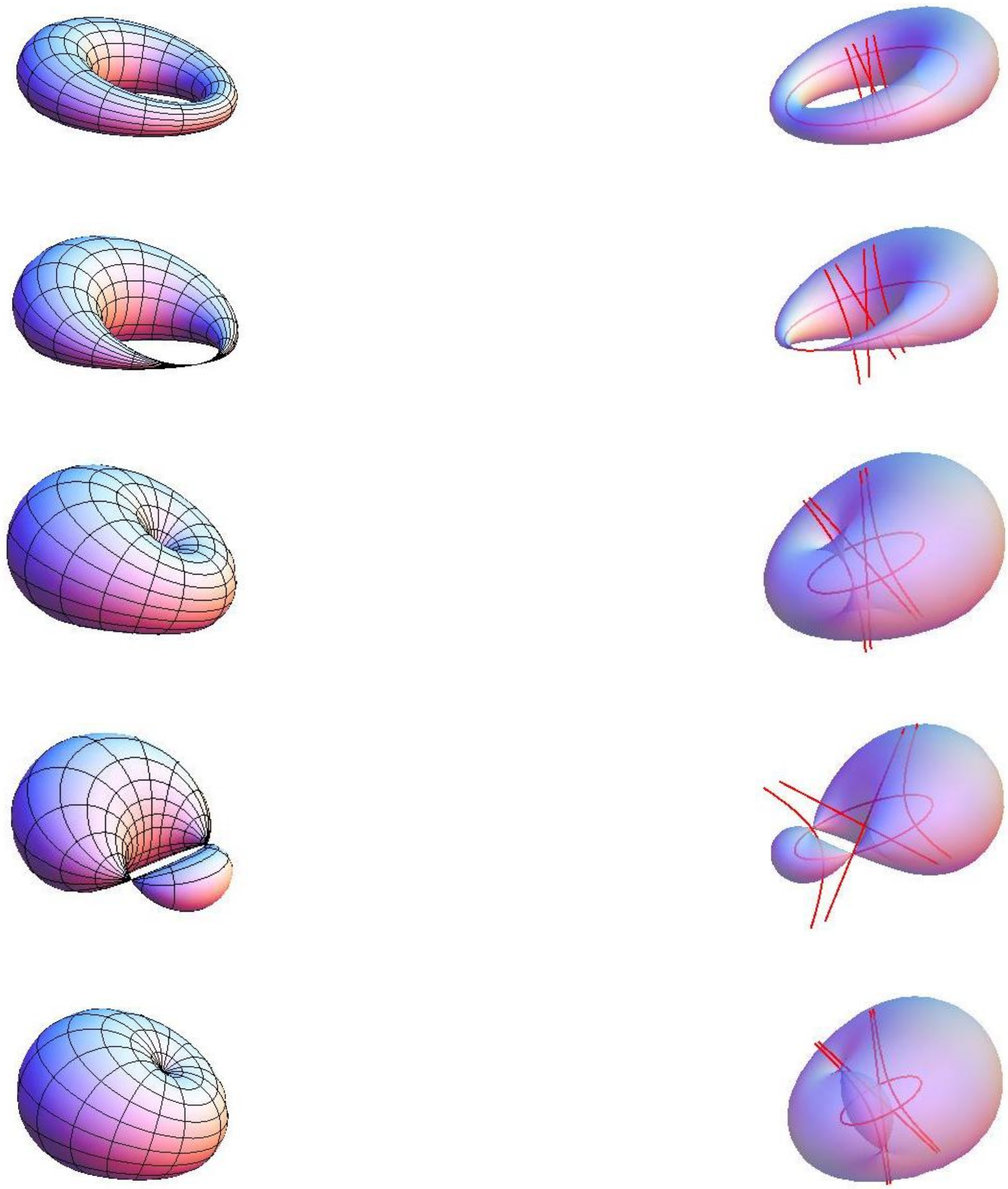

Figure 2.23: *From top to bottom: a ring cyclide, two horn cyclides and two spindle cyclides. Left: lines of curvature on the Dupin cyclides. Right: the pair of anti-conics of the Dupin cyclides.*

Symmetry of a Dupin cyclide

A cylinder, a cone and a torus have infinitely many planes of symmetry. We prove that in general a Dupin cyclide has two planes of symmetry.

Lemma 4. *A Dupin cyclide such that the cyclide is the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres has at least two planes of symmetry.*

Proof. Let the envelope surface of two 1-parameter families of spheres F_1 and F_2 be a Dupin cyclide. From theorem 1 it follows that the centers of spheres from F_1 lie on a conic in a plane P_1 and the centers of spheres from F_2 lie on a conic in a plane P_2. From the symmetry of these conics and from the symmetry of the spheres from the 1-parameter families which define the cyclide it follows that P_1 and P_2 are planes of symmetry of the cyclide. □

Figure 2.24: *The planes of symmetry of a ring cyclide.*

It is easily seen that a torus, a cylinder, a cone and a sphere are examples of Dupin cyclides which has infinitely many planes of symmetry such that these cyclides are defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres.

Chapter 3

Dupin cyclides in blending

In this chapter we consider an application of Dupin cyclides such that their pair of anti-conics consists of an ellipse and a hyperbola in blending intersecting natural quadrics (cones, cylinders, spheres and planes). Cyclide blends are important in design both for functional reasons, i.e., simplifying manufacture, and for cosmetic reasons. For other applications of Dupin cyclides we refer to DePont [DePont] and Martin [Mart].

3.1 Dupin cyclides in blending between intersecting natural quadrics

For a detailed introduction to cyclide blends we refer to Allen and Dutta [AllDut I] and [AllDut II]. We only state the formal definition of a cyclide blend according to Allen and Dutta [AllDut I] and give an intuitive description of Dupin cyclides in blending intersecting natural quadrics. Furthermore, we state the so-called extreme circle condition of blending according to Allen and Dutta [AllDut I]. This condition reduces the problem of blending intersecting natural quadrics with Dupin cyclides to the problem of finding appropriate extreme circles of a Dupin cyclide (definition 9).
Intuitively, a cyclide blend can be considered as a transition surface between intersecting natural quadrics such that the connection between them becomes smooth, i.e., tangent continuous. We require that the cyclide blend follows approximately the intersection curve of the surfaces being blended, i.e., the blend surface should roughly have the same shape as the intersection curve. Moreover, by the tangent continuously property and the requirement that the blend surface follows the intersection curve of the surfaces being blended it follows that the blend surface stays on one side of each of the surfaces being blended (see Allen and Dutta [AllDut I]). Furthermore, we assume that the intersection curve of the intersecting surfaces being blended is a non-empty closed curve.

Definition 7. *Latitudinal lines of curvature on a Dupin cyclide such that its pair of anti-conics consists of an ellipse and a hyperbola are lines of curvature obtained from spheres with their center on the hyperbola of the cyclide. Longitudinal lines of curvature on a Dupin cyclide are lines of curvature obtained from spheres with their center on the ellipse of the cyclide.*

The intuitive description of cyclide blends between intersecting natural quadrics in the beginning of this section and definition 7 leads Allen and Dutta to the following formal definition of a cyclide blend between intersecting natural quadrics.

Definition 8 (Cyclide blend)**.** *A ring cyclide blends two intersecting natural quadrics when*

1. *the intersection curve of the two natural quadrics being blended is a non-empty closed curve,*
2. *the cyclide is tangent to each natural quadric along a latitudinal line of curvature and*
3. *the intersection curve of the natural quadrics being blended wrap around the axis of each axial natural quadric (a cylinder of a cone).*

Allen and Dutta proved that in any case of intersecting natural quadrics their exits a cyclide blend between these intersecting natural quadrics such that the Dupin cyclide is a ring cyclide. Therefore, it is sufficient to consider ring cyclides as cyclide blends between intersecting natural quadrics.
As noted before, Allen and Dutta simplify the problem of blending intersecting natural quadrics to the problem of finding two so-called extreme circles of a Dupin cyclide.

Definition 9. *The extreme circles of a Dupin cyclide are all circles obtained from intersecting the cyclide with its planes of symmetry.*

In particular, the extreme circles of a Dupin cyclide such that its pair of anti-conics consists of an ellipse and a hyperbola are the circles obtained from intersecting the cyclide with its planes of symmetry. Moreover, any Dupin cyclide consists of at least four extreme circles.

Definition 10 (Extreme circles conditions)**.** *Let Q be a natural quadric and let P be a plane. Two circles c_1 and c_2 contained in P satisfy the extreme circles conditions when*

1. *c_1 and c_2 are disjoint,*
2. *both c_1 and c_2 are tangent to the intersection $Q \cap P$, and*
3. *both c_1 and c_2 lie either entirely inside or entirely outside Q.*

For axial natural quadrics Allen and Dutta give the following extreme circles conditions.

Definition 11 (Axial extreme circles conditions)**.** *Let Q be an axial natural quadric and let P be a plane such that the axis of Q is contained in P. Two circles c_1 and c_2 contained in P satisfy the extreme circles conditions when*

1. *c_1 and c_2 satisfy the extreme circles conditions with respect to Q, and*
2. *if c_i is tangent to Q for $i = 1, 2$ at a point t_i, then the line t_1t_2 is perpendicular to the axis of Q.*

Theorem 2. *There exist two circles which satisfy the extreme circles conditions with respect to intersecting natural quadrics iff there exist a cyclide blend between the intersecting natural quadrics.*

For a proof of theorem 2 we refer to Allen and Dutta [AllDut I]. Moreover, our aim is to study cyclide blends from the viewpoint of inversive geometry which have never previously be done. Therefore, we not give a rigorous mathematical proof of theorem 2, but we refer to the intuitive proof of Allen and Dutta.
From theorem 2 if follows that the problem of finding a Dupin cyclide which blends two intersecting natural quadrics is reduced to the problem of finding two circles which satisfy the extreme circles conditions. Therefore, we use in the following section the definition of (axial) extreme circles conditions and theorem 2 to prove the existence of cyclide blends between intersecting natural quadrics.

3.2 Existence of cyclide blends

In this section we prove the existence of cyclide blends in the case of (i) two intersecting proper spheres, (ii) an intersecting proper sphere and a plane, (iii) an intersecting cylinder and a plane and (iv) an intersecting cone and a plane. In the case of two proper spheres we prove the existence of a cyclide blend using of inversive geometry. From this it follows that all four cases are equivalent blending problems, this is an approach which hitherto was not known. For other cyclide blends between intersecting natural quadrics we refer to Allen and Dutta [AllDut II]. For a detailed description of a cyclide blend between two intersecting cones we refer to Pratt [Pratt] and Srinivias and Dutta [Srin].

Theorem 3. *Let S_1 and S_2 be two intersecting, proper spheres, then there exists a cyclide blend between S_1 and S_2. Furthermore, the cyclide blend between S_1 and S_2 is not unique.*

Proof. Let S_1 and S_2 be intersecting, proper spheres. Consider the intersection of S_1 and S_2 with a plane P which contain the center of S_j for $j = 1, 2$. Let q_j for $j = 1, 2$ be the points of intersection of S_1 and S_2 in P and let t_j be the tangent to S_j at q_j (figure 3.1). Let C_j for $j = 1, 2$ be a circle

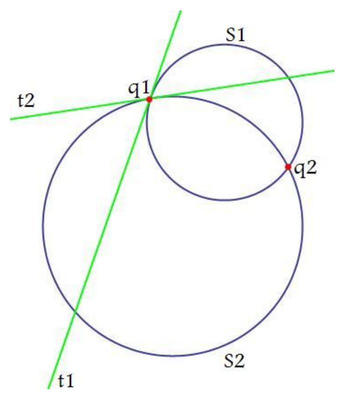

Figure 3.1: *The intersection of two intersecting proper spheres with a plane which contain the center of these proper spheres.*

of inversion with center q_j and let i_j be an inversion with respect to C_j for $j = 1, 2$. Then i_1 maps S_j for $j = 1, 2$ onto two straight lines l_1 and l_2, respectively, and i_2 maps S_j for $j = 1, 2$ onto two straight lines l_3 and l_4, respectively. Let b_1 be a bisector of the angle between l_1 and l_2 and let b_2 be ta bisector of the angle between l_3 and l_4. Any point p on b_1 is the center of a circle C_p tangent to l_1 and l_2 and any point q on b_2 is the center of a circle C_q tangent to l_3 and l_4. Hence, there exist infinitely many pairs of circles $(i_j(C_p), i_j(C_q))$ for $j = 1, 2$ which satisfy the extreme circles conditions with respect to S_1 and S_2. Therefore, if follows from theorem 2 that there exist infinitely many Dupin cyclides such that these cyclides are blending surfaces between the intersecting, proper spheres S_1 and S_2. □

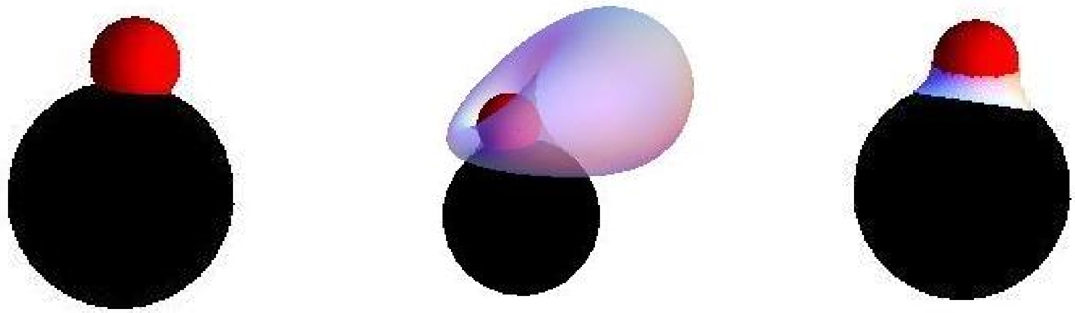

Figure 3.2: *A cyclide blend between two intersecting, proper spheres.*

The intersection of a proper sphere with a plane is circular. From theorem 3 it follows that there exist infinitely many cyclide blends between a plane and a proper sphere which intersects each other. Furthermore, two planes cannot be blended because their intersection curve is empty or their intersection curve is not closed, i.e., the planes are parallel or the planes intersect each other.

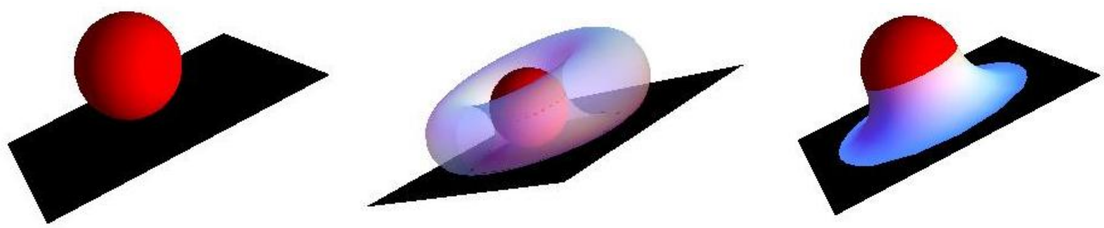

Figure 3.3: *A cyclide blend between a intersecting proper sphere and a plane such that the sphere and the plane are tangent to each other, i.e., the intersection curve of the sphere and the plane is a point.*

Theorem 4. *Let C be a cylinder and let P be a plane such that C and P intersect each other. The axis of C is not parallel to P iff there exists a cyclide blend. Furthermore, the cyclide blend between C and P is not unique.*

Proof. Let C be a cylinder and let P be a plane such that C and P intersect each other. First, assume that the axis a of C is parallel to P, i.e., a is contained in P or there exists a line in P parallel to a. If a is contained in P then the intersection of C with P is a pair of straight lines. Moreover, if there exists a line l in P parallel to a, then the intersection of C with P is a (pair of) straight line(s). Hence, there does not exists a cyclide blend if P is parallel to the axis of C.
Secondly, assume that P is not parallel to the axis a of C and let P' be a plane that contains the axis a, i.e., $P \neq P'$. Consider the intersection $C \cap P'$ (figure 3.4).
Assume without loss of generality that b_1 is a bisector of the angle between $C_1 \cap P'$ and $P \cap P'$ and assume that b_2 is a bisector of the angle between $C_2 \cap P'$ and $P \cap P'$. Any point p_1 on b_1 is the center of a circle c_{1i} for $i \in \mathbb{N}$ tangent to C_1 and tangent to P and any point on b_2 is the center of a circle c_{2j} for $j \in \mathbb{N}$ tangent to C_2 and tangent to P. Moreover, for any p_1 on b_1 there exists a p'_1 on b_2 such that $p_1p'_1$ is perpendicular to a. Hence, there exist pairs of circles consisting of a circle

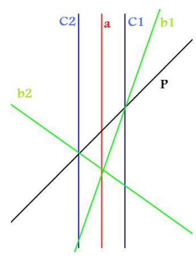

Figure 3.4: *The intersection of the cylinder C and the plane P with the plane P' which contains the axis C.*

c_{1i} and a circle c_{2j} for $i, j \in \mathbb{N}$ which satisfy the extreme circles conditions with respect to C and P. Therefore, it follows from theorem 2 that there exist infinitely many Dupin cyclides which are the blending surfaces between the intersecting cylinder C and the plane P. □

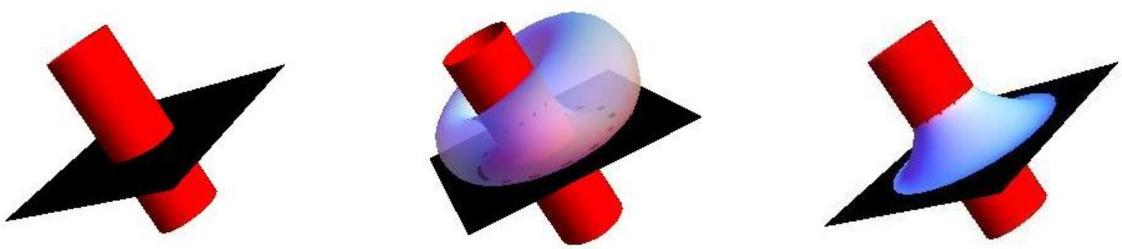

Figure 3.5: *A cyclide blend between a cylinder and a plane which intersect each other.*

Similarly as in the proof of theorem 4, it follows that there exists a cyclide blend between a cone and a plane if the plane is not parallel to the axis of the cone.

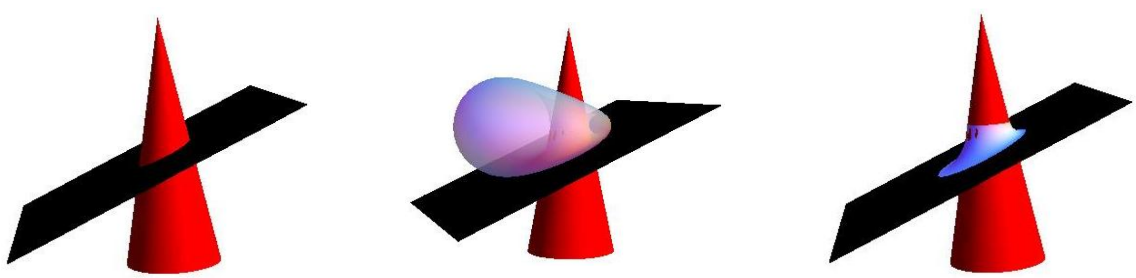

Figure 3.6: *A cyclide blend between a cone and a plane which intersect each other.*

Furthermore, from the proof of theorem 3 and from the proof of theorem 4 it follows that finding a cyclide blend in the case of (i) two intersecting proper spheres and (ii) an intersecting cylinder and a plane can be reduced to the problem of finding pairs of circles consisting of disjoint circles tangent to two straight lines, i.e., finding pairs of circles which satisfy the extreme circles conditions with respect to two straight, intersecting lines. Similarly, finding a cyclide blend in the case of (i) an intersecting proper sphere and a plane and (ii) an intersecting cone and a plane is equivalent to the problem of finding pairs of circles which satisfy the extreme circles conditions with respect to two straight, intersecting lines. Hence, finding a cyclide blend in the case of (i) two intersecting proper spheres, (ii) an intersecting proper sphere and a plane, (iii) an intersecting cylinder and a plane and (iv) a intersecting cone and a plane are equivalent blending problems.

Chapter 4

Conclusions

In this chapter we give suggestions for further research and we give our main conclusions.

4.1 Suggestions for further research

Inversion in $\mathbb{R}^3$ reveals the main geometric properties of Dupin cyclides in $\mathbb{R}^3$ such that the cyclide is the envelope surface of two 1-parameter families of spheres both tangent to three fixed spheres. Therefore, one can study all surfaces defined as the envelope surface of an m-parameter family of spheres in $\mathbb{R}^3$ for $m \in \mathbb{N}$ with inversive geometry to obtain, probably, their main geometric properties, e.g., symmetry and (lines of) curvature. More generally, one can study the envelope surface of an l-parameter family of surfaces in $\mathbb{R}^3$ for $l \in \mathbb{N}$ with inversive geometry in $\mathbb{R}^3$ to obtain, probably, their main geometric properties, e.g., symmetry and (lines of) curvature.
Similarly as we did in section 2.2, one can define inversion in $\mathbb{R}^n$ for $n > 2$. We believe that this can be used to study the main geometric properties of Dupin cyclides in $\mathbb{R}^n$ for $n > 2$, e.g., the envelope hypersurface of a 1-parameter family of spheres contained in $\mathbb{R}^4$ tangent to four fixed spheres in $\mathbb{R}^4$. Moreover, the extension to higher dimensional Dupin cyclides leads to a natural extension of higher dimensional natural quadrics.
In section 2.5 we proved the existence and uniqueness of Dupin cyclides in $\mathbb{R}^3$ given three proper fixed spheres in $\mathbb{R}^3$. Therefore, we have to examine the existence and uniqueness in the case of non-proper fixed spheres or three fixed spheres consisting of a combination of proper and non-proper fixed spheres, i.e., studying the problem of Apollonius (finding circles tangent to three fixed circles) instead of $\mathbb{R}^2$ in $\mathbb{R}^3$.
In section 3.2 we proved that the problem of finding a cyclide blend in the case of (i) two intersecting proper spheres, (ii) an intersecting proper sphere and a plane, (iii) an intersecting cylinder and a plane and (iv) an intersecting cone and a plane is equivalent to the problem of finding two extreme circles tangent to two straight, intersecting lines. Therefore, one can examine if there exists more pairs of intersecting natural quadrics which can be reduced to the problem of finding two extreme circles tangent to two straight, intersecting lines. Indeed, one can classify the ten cases of intersecting natural quadrics into equivalent blending problems.

4.2 Main conclusions

We conclude that inversion in $\mathbb{R}^3$ reveals quite simply the main geometric properties of Dupin cyclides in $\mathbb{R}^3$ defined as the envelope surface of a 1-parameter family of spheres tangent to three fixed spheres. These geometric properties concerning (i) existence and uniqueness, (ii) lines of curvature and (iii) classes, and (iv) symmetry. Moreover, inversion in $\mathbb{R}^3$ reduces the number of fixed spheres in $\mathbb{R}^3$ which define a Dupin cyclide given as the envelope surface tangent to these fixed spheres from three fixed spheres to one fixed sphere, with the additional condition that spheres from the 1-parameter family have their center on a conic and the fixed spheres have its center on another conic such that these conics form a pair of anti-conics. It is this definition which simplifies the construction of a Dupin cyclide and therefore among others simplifies the process of blending intersecting natural quadrics.
Using inversion in $\mathbb{R}^3$ in blending intersecting natural quadrics we proved that finding a cyclide blend in the case of (i) two intersecting proper spheres, (ii) an intersecting proper sphere and a plane, (iii) an intersecting cylinder and a plane and (iv) an intersecting cone and a plane are equivalent blending problems.
Beyond these results we managed to give mathematically rigorous proofs of statements made by Chandru, Dutta and Hoffmann in [ChDuHo] and to omit their flawed assumptions. Therefore, conclusions that follow from these assumptions are omitted or made mathematically rigorous. Subsequently, this is important because many articles refer to the article of Chandru, Dutta and Hoffmann [ChDuHo]. Moreover, all statements in the article by Chandru and Dutta [AllDut I] and [AllDut II] concerning blending intersecting natural quadrics are correct. But some of these arguments are far from mathematically rigorous or can be done much easier. To conclude, the process of classifying equivalent blending problems in the case of intersecting natural quadrics is mentioned nowhere else.

Chapter 5

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Martijn van der Valk, summer 2009

Chapter 6

Appendix

In this chapter we give deviations from standard terminology which will be frequently used throughout the text. We define disjoint spheres and we distinguish three types of torii.

Definition 12. *The spheres* $S_1, \dots, S_n \subset \mathbb{R}^3$ *for* $n \in \mathbb{N}$ *are disjoint when*

1. $S_i \cap S_j = \emptyset$ *for* $i, j \in \mathbb{N}$ *and*
2. *neither* S_i *is contained in* S_j *nor* S_j *is contained in* S_i *for* $i, j \in \mathbb{N}$.

In standard notation the spheres $S_1, \dots S_n$ for $n \in \mathbb{N}$ are disjoint if statement (1) of definition 12 holds.
Now we distinguish three classes of tori in $\mathbb{R}^3$. A *ring torus* in $\mathbb{R}^3$ is the envelope surface (definition 3) of two 1-parameter families of spheres in $\mathbb{R}^3$ such that none of the spheres from the two 1-parameter families which contribute to the envelope surface are a point (figure 6.1). Similarly, a *horn torus* in $\mathbb{R}^3$ is the envelope surface of two 1-parameter families of spheres $\mathbb{R}^3$ such that one of the two 1-parameter families contains one sphere that is equal to a point and contributes to the envelope surface of the two 1-parameter families (figure 6.1). Finally, a *spindle torus* in $\mathbb{R}^3$ is the envelope surface of two 1-parameter families of spheres $\mathbb{R}^3$ such that one of the two 1-parameter families contains two spheres which are equal to different points and contribute to the envelope surface of the two 1-parameter families (figure 6.1).

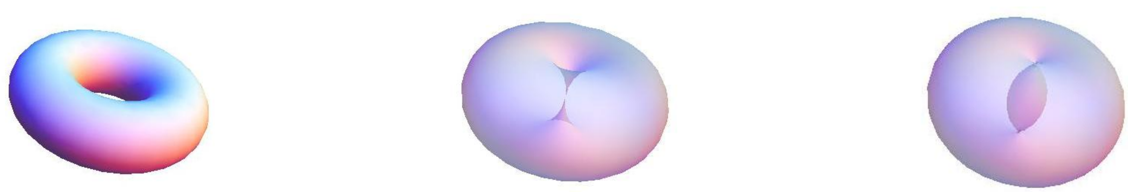

Figure 6.1: *From left to right: a ring torus, a horn torus and a spindle torus.*

In standard notation a torus in $\mathbb{R}^3$ is equal to our ring torus, i.e., a non-singular surface. Therefore, we allow a torus given as the envelope surface of two 1-parameter families of spheres in $\mathbb{R}^3$ to be a singular surface. For the purpose we refer to a torus as one of these (non-)singular surfaces.

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