Riemannian Manifolds

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1 Introduction

In this bachelor's thesis, I give a summary of the material I have covered in a reading course on differential geometry, during a three month period of study at the University of California, Berkeley, in the summer of 2004. The topics include smooth manifolds, vector fields, tensors, differential forms, integration on manifolds (the "differential" part), and Riemannian metrics, connections, geodesics, curvature and the Gauss-Bonnet theorem (the "geometry" part).

A minimum of background theory is given and only the absolutely necessary theorems are stated, always without a proof. Books that I have used can be found in the bibliography. This work consists mainly of worked examples, solutions to problems presented in the books and proofs of some theorems that will hopefully illuminate some of the abstract concepts of differential geometry.

I want to express my gratitude to UC Berkeley professor Charles C. Pugh for taking the time of supervising my study in Berkeley, in particular for always giving me useful suggestions on how to solve the various problems. Furthermore, I would like to thank my advisor in Groningen, professor Marius van der Put.

Contents

2 Smooth Manifolds and Smooth Maps

Suppose M is a topological space. We say that M is a *topological manifold of dimension n* or a *topological n-manifold* if it has the following properties:

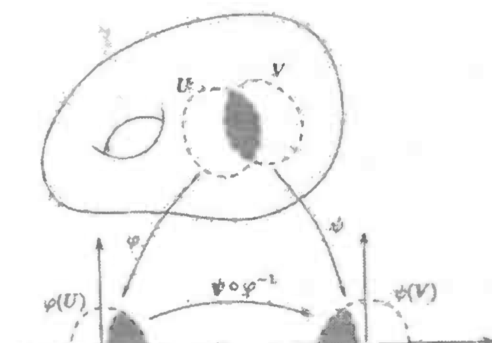
- M is a *Hausdorff* space: for every pair of points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.

- M is *second countable*: there exists a countable basis for the topology of M.

- M is locally *Euclidean of dimension n*: every point of M had a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$

Let M be a topological n-manifold. A *coordinate chart* on M is a pair (U, ϕ), where U is an open subset of M and $\phi : U \to \tilde{U}$ is a homeomorphism from U to an open subset of $\tilde{U} = \phi(U) \subset \mathbb{R}^n$. By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, ϕ). Given a chart (U, ϕ), we call the set U a *coordinate domain*, or a *coordinate neighborhood* of each of its points. The map ϕ is called a (local) *coordinate map*, and the component funcions $(x^1, ..., x^n)$ of ϕ , defined by $\phi(p) = (x^1(p), ..., \phi^n(p))$, are called *local coordinates* on U.

If (U, ϕ) and (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is called the *transition map* from ϕ to ψ. Two charts are said to be *smoothly compatible* if either $U \cap V \neq \emptyset$ or the transitions map $\phi \circ \psi^{-1}$ is a diffeomorphism. A *smooth atlas* for M is a collection of charts whose domains cover M and any two charts are compatible with each other.



Let M and N be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, ϕ)

containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U)$ to $\psi(V)$.

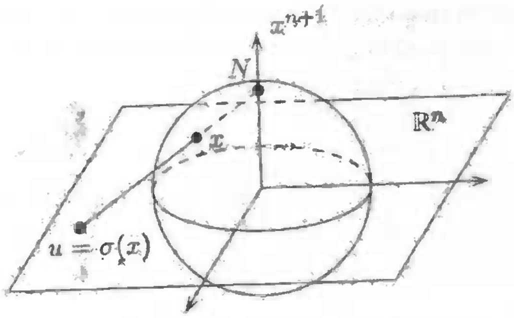
Example The line with two origins
Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Thus $M = X/\sim$.

M is locally Euclidean, i.e. for each $p \in M$, we can find an open set $U \in M$ and $\tilde{M} \subset \mathbb{R}$ such that there exists a homeomorphism $\phi : U \to \tilde{U}$, for in the case of this manifold we can take $\phi = Id$ and U and $\tilde{U}$ open sets in $\mathbb{R}$. So M is locally Euclidean. Also, M is second countable, for we can take as countable basis the open sets $B_x(\epsilon)$, where $x, \epsilon \in \mathbb{Q}$. However, M is not Hausdorff. To see this take the two points $x = (0, 1)$ and $y = (0, -1)$ in M, then $x \neq y$ by the definition of M. However $B_x(\epsilon) \cap B_y(\epsilon') = (-\lambda, 1) \cup (1, \lambda) \neq \emptyset$, for arbitrary small λ, where $\lambda = \min(\epsilon, \epsilon')$ We conclude that M is not a topological manifold. The space M is called *line with two origins.* □

2.1 Stereographic Projection

Let $N = (0, ..., 0, 1)$ be the "north pole" in $\mathbb{S}^n \subset \mathbb{R}^{n=1}$, and let $S = -N$ be the "south pole". Define *stereographic projection* $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ by

$$\sigma(x_1, ..., x^{n+1}) = \frac{(x_1, ..., x_n)}{1 - x^{n+1}}. \tag{1}$$



Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash \{S\}$.

For any $x \in \mathbb{S}^n$, the point $\sigma(x)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$, identified with $\mathbb{R}^n$ in the obvious way. By congruence of triangles,

$$\tan \alpha = \frac{\sigma(x_1, ..., x_{n+1})}{1} = \frac{(x^1, ..., x^n)}{1 - x^{n+1}},$$

where α denotes the angle between the line and the hyperplane. Hence

$$\sigma(x_1, ..., x_{n+1}) = \frac{(x^1, ..., x^n)}{1 - x^{n+1}}.$$

Similarly, $\tilde{\sigma}$ is the point where the line through S and x intersect the same subspace. (For this reason, $\tilde{\sigma}$ is called *stereographic projection from the south pole.*)

The inverse is given by

$$\sigma^{-1}(u^1, ..., u^n) = \frac{(2u^1, ..., 2u^n, |u|^2 - 1)}{u^2 + 1}.$$

This is seen by the next computation

$$\begin{aligned}\sigma(\sigma^{-1})(u^1, ..., u^n) &= \frac{(\frac{2u^1}{|u|^2+1}, ..., \frac{2u^n}{|u|^2+1})}{1 - \frac{|u|^2-1}{|u|^2+1}} \\ &= \frac{(\frac{2u^1}{|u|^2+1}, ..., \frac{2u^n}{|u|^2+1})}{\frac{2}{|u|^2+1}} = (u^1, ..., u^n),\end{aligned}$$

hence $\sigma(\sigma^{-1}) = Id_{\mathbb{R}^n}$. Furthermore, σ is bijective. To show that σ is surjective, take $(u^1, ..., u^n) \in \mathbb{R}^n$, then

$$\sigma(u^1(1-\lambda), ..., u^n(1-\lambda), \lambda) = (u^1, ..., u^n).$$

Injectivity is clear from the geometry. Furthermore,

$$\begin{aligned}\tilde{\sigma}(x^1, ..., x^n) &= -\sigma(-x^1, ..., -x^n) = -\frac{(-x^1, ..., -x^n)}{1 + x^{n+1}} \\ &= \sigma(x^1, ..., x^n, -x^{n+1}).\end{aligned}$$

Then for the transition map $\tilde{\sigma} \circ \sigma^{-1}$ we compute

$$\begin{aligned}\tilde{\sigma} \circ \sigma^{-1}(x^1, ..., x^n) &= \tilde{\sigma}\left(\frac{(2x^1, ..., 2x^n, |x|^2 - 1)}{|x|^2 + 1}\right) \\ &= \sigma\left(\frac{(2x^1, ..., 2x^n, 1 - |x|^2)}{|x|^2 + 1}\right) = \frac{(2x^1, ..., 2x^n)}{|x|^2 \cdot |x|^2 + 1}\end{aligned}$$

and thus the transition map is smooth, except when $|x| = 0$. Since $x \in \mathbb{S}^n$, we have that if $x = 0$, then $x^1 = x^2 = ... = x^n = 0$ and thus $x^{n+1} \pm 1$ which are exactly the north and south pole. This makes the atlas consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ a smooth structure on $\mathbb{S}^n$. The coordinates defined by σ and $\tilde{\sigma}$ are called *stereographic* coordinates. □

2.2 Smooth structure on a topological manifold

For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. If $F : M \to N$ is a continuous map, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.
The map F^* is linear,

$$F^*(f+g) = (f+g) \circ F = f \circ F + g \circ F = F^*(f) + F^*(g),$$

because in general $(f+g)(x) = f(x) + g(x)$ and $f+g$ is continuous because f and g are. If $F : M \to N$ is a map from the smooth manifold M to the smooth manifold N, then F is defined to be smooth if for every $p \in M$, there exists smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U)$ to $\psi(V)$.
Now, F is smooth if and only if $F^*(C^\infty(N)) \subset (C^\infty(M))$. For suppose that F is smooth, then for every $f \in C^\infty(N)$, $F^*(f)$ is a composition of smooth maps and hence a smooth map on M. So indeed, $F^*(C^\infty(N)) \subset (C^\infty(M))$. Conversely, suppose that $F^*(C^\infty(N)) \subset (C^\infty(M))$. This means that for every $f \in C^\infty(N)$, $F^*(f)$ is smooth. In particular, for smooth coordinate maps of the smooth charts (U, ϕ) and (V, ψ), $\psi : N \to \mathbb{R}^k$ and $\phi : M \to \mathbb{R}^n$, the mapping $\psi \circ F \circ \phi^{-1}$ is smooth because ϕ^{-1} and ψ are smooth by definition the smooth structures on M and N, therefore $\psi \circ F$ is smooth by hypothesis. Thus the composition of $\psi \circ F$ and ϕ^{-1} is also smooth. Hence by definition of a smooth map from the manifold M to N, F is a smooth map.

Furthermore, if F is a homeomorphism between smooth manifolds, then F is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$. First, suppose that F is a diffeomorphism, then F is smooth, bijective and has a smooth inverse. Hence, $F^{*-1} : C^\infty(M) \to C^\infty(N)$, $F^{*-1}(g) = g \circ F^{-1}$ is smooth and thus $F^*(C^\infty(N) = C^\infty(M)$ is onto. Also F^* is linear and because $\ker(F^*) = \{f \equiv 0\}$ gives that F is also injective. We conclude that F^* is an isomorphism.
Conversely, if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(N)$, then $F^*(C^\infty(N)) = C^\infty(M)$. Obviously, $F^{*-1}(C^\infty(M)) = C^\infty(N)$. So by the preceding discussion, F^{-1} is also smooth. Hence we conclude, because F is a homeomorphism with a smooth inverse, that F is a diffeomorphism.

Remark: This result shows that in a certain sense, the entire smooth structure of M is encoded in the space $C^\infty(M)$. In fact, sometimes a smooth structure is *defined* on a topological manifold to be a subalgebra of $C(M)$ with certain properties.

3 Tangent Vectors and Vector Fields

Let M be a smooth manifold and let p be a point of M. A linear map $X : C^\infty(M) \to \mathbb{R}$ is called a *derivation* if it satisfies

$$X(fg) = f(p)X(g) + g(p)X(f) \tag{2}$$

for all $f, g \in C^\infty(M)$. The set of all derivations of $C^\infty(M)$ at p is a vector space called the *tangent space* to M at p. and is denoted by T_pM. An element of T_pM is called a *tangent vector* at p.

Example The gradient Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function. The *gradient of* f in the standard basis in $\mathbb{R}^2$ is given by

$$X = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y}.$$

Define $\psi(r, \phi) = (r\cos\phi, r\sin\phi) = (x, y)$. We will express the gradient in polar coordinates, more precisely we will pushforward the vector field X under the mapping ψ^{-1}.

Let D denote the *Jacobian*, then

$$\left(D_{(r,\phi)}\psi\right)^{-1} = D_{(x,y)}\psi^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\phi & r\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},$$

where we denote $D_{(x,y)}\psi^{-1} = \frac{\partial \psi^{-1j}}{\partial x^i}$. Now, unfolding the definitions

$$\begin{aligned} \left(\psi^{-1}\right)_*(X) &= \frac{\partial \psi^{-1j}}{\partial x^i}(\psi(r,\phi))\frac{\partial f}{\partial x^i}(\psi(r,\phi))\frac{\partial}{\partial y^j} \\ &= \frac{\partial g}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial g}{\partial \phi}\frac{\partial}{\partial \phi}, \end{aligned}$$

where $g = f \circ \psi$. $\square$

4 Vector Bundles and (Co)tangent Bundles

Let M be a topological space. A *(real) vector bundle of rank k over M* is a topological space E together with a surjective continuous map $\pi : E \to M$ satisfying:

1. For each $p \in M$, the set $E_p = \pi^{-1}(p) \subset E$ (called the fibre over E) is endowed with the structure of a k-dimensional real vector space.

2. For each $p \in M$, there exists a neighbourhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a *local trivialization of E over U)*, such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

(where π_1 is the projection onto the first factor); and such that for each $q \in U$, the restriction of Φ to E_q is a linear isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

Sections of Vector Bundles

Let $\pi : E \to M$ be a vector bundle over a manifold M. A *section of* E is a section of the map π, i.e. a continuous map $\sigma : M \to E$ satisfying $\pi \circ \sigma = \mathrm{Id}_M$. Specifically, this means that $\sigma(p)$ is an element of the fiber E_p for each $p \in U$. A *local section of* E is a section $\sigma : U \to E$ defined only on some open subset $U \subset M$. For a smooth manifold M, sections of TM are vector field on M.

Dual spaces

Suppose V and W are finite-dimensional vector spaces and $A : V \to W$ is any linear map, then the following diagram commutes. The ξ_V and ξ_W denote the (canonical) isomorphism defined as $\xi(X)(\omega) = \omega(X)$.

$$\begin{tikzcd}
V \arrow[r, "A"] \arrow[d, "\xi_V"'] & W \arrow[d, "\xi_W"] \\
V^{**} \arrow[r, "(A^*)^*"'] & W^{**}
\end{tikzcd}$$

Take $X \in V$, then $\xi_V(X)$ defines a linear functional on V^*, the dual space of V and take an $\omega \in W^*$. Also $A^* : W^* \to V^*$ denotes the canonical mapping $(A^*\omega)(X) = \omega(AX)$. Furthermore, define the mapping $(A^*)^* : V^{**} \to W^{**}$ by $(A^*)^*\xi(X) = \xi(X)A^*$. This gives

$$(A^*)^*\xi_V(X)(\omega) = \xi_V(X)(A^*\omega) = (A^*\omega)(X) = \omega(AX) = \xi_W(AX)(\omega),$$

which yields,

$$(A^*)^*\xi_V(X) = \xi_W(AX),$$

hence, the diagam given above commutes.□

5 Tensors and Riemannian Metrics

Much of the machinery of smooth manifold theory is designed to allow the concepts of linear algebra to be applied to smooth manifolds. Calculus tells us how to approximate smooth objects by linear ones, and the abstract definitions of manifold theory give a way to interpret these linear approximations in a coordinate independent way. Generalizing this idea further, i.e. from linear objects to multilinear ones, this leads to the concepts of tensors and tensor fields on smooth manifolds.

5.1 Tensors

Suppose $V_1, ..., V_k$ and W are vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is said to be *multilinear* if it is linear as function of each variable separately. Examples are the dot product in $\mathbb{R}^n$, the cross product in $\mathbb{R}^n$ and the determinant function.

Let V be a finite-dimensional real vector space, and let k be a natural number. A *covariant k-tensor* in V is a real-valued multilinear function of k elements of V

$$T : V \times \cdots \times V \to \mathbb{R}.$$

The number k is called the rank of T. The set of all covariant k-tensors on V, denoted $T^k(V)$, is a vector space. Suppose $\omega, \eta \in V^*$. Define a map $c : V \times V \to \mathbb{R}$ by

$$\omega \otimes \eta(X, Y) = \omega(X)\eta(Y), \tag{3}$$

where the product on the right is just ordinary multiplication of real numbers. This definition can be generalized to tensors of any rank. Let $S \in T^k(V), T \in S^l(V)$. Define a map $S \otimes T : V \times \cdots \times V \to \mathbb{R}$ by

$$S \otimes T(X_1, ..., X_{k+l}) = S(X_1, ..., X_k)T(X_{k+1}, ..., X_{k+l}). \tag{4}$$

This map is called the *tensor product*.

5.2 The Alternating and Symmetric Product

Alternating product

A most important algebraic construction is a product called the operation called the *wedge product*, which takes alternating tensors to alternating tensors. We define the projection Alt:$T^k(V) \to \Lambda^k(V)$, called the *alternating projection*, as follows

$$Alt(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn}\sigma)\,(^{\sigma}T).$$

For every tensor T, Alt$T$ is alternating and T is alternating if and only if Alt$T = T$.

If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$, we define the *wedge product* or *interior product* of ω and η to be the alternating $(k+l)$-tensor

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta). \tag{5}$$

This product is bilinear, associative but not commutative.

Symmetric product
Symmetric tensors play an extremely inportant role in differential geometry. A covariant k-tensor T is said to be *symmetric* if its value is unchanged by interchanging any pair of arguments:

$$T(X_1, ..., X_i, ..., X_j, ..., X_k) = T(X_1, ..., X_j, ..., X_i, ..., X_k),$$

whenever $1 \leq i < j \leq k$. The set of symmetric covariant k-tensors is often denoted as $\Sigma^k(V)$ and is a subspace of $T^k(V)$. There is a natural projection Sym: $T^k(V) \to \Sigma^k(V)$ called *symmetrization*, defined by

$$\mathrm{Sym} T = \frac{1}{k!} \sum_{\sigma \in S_k} {}^{\sigma}T.$$

For instance, when ω and η are covectors, then

$$\omega\eta = \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega).$$

A *symmetric tensor field* on a manifold is simply a covariant tensor field whose value at any point is a symmetric tensor.

5.3 Wedge Product

Let V be a finite dimensional real vector space. We have two ways to think of the tensor space $T^k(V)$: concretely, as the space of k-multilinear functionals on V; and abstractly, as the tensor product space $V^* \otimes \cdots \otimes V^*$. Often, the alternating (and symmetric) products are defined only in terms of the discrete defintion. Now we will outline an abstact approach to alternating tensors. Let Σ denote the subspace of $V^* \otimes \cdots \otimes V^*$ spanned by all elements of the form $\alpha \otimes \phi \otimes \phi \otimes \beta$ for a covector ϕ and arbitrary tensors α and β, and let $A^k(V^*)$ denote the quotient vector space $(V^* \otimes \cdots \otimes V^*)/\Sigma$. Then there is an isomorphism $F : A^k(V) \to \Lambda^k(V)$, such that the following diagram commutes:

$$\begin{tikzcd}
V^* \otimes \cdots \otimes V^* \arrow[r, "\cong"] \arrow[d, "\pi"'] & T^k(V) \arrow[d, "Alt"] \\
A^k(V^*) \arrow[r, "F"'] & \Lambda^k(V)
\end{tikzcd}$$

Suppose $x \in A^k(V^*)$ and take the sum of two covectors $\phi = \phi^1 + \phi^2$, then

$$\begin{aligned}
\alpha \otimes (\phi^1 + \phi^2) \otimes (\phi^1 + \phi^2) \otimes \beta &= \alpha \otimes \phi^1 \otimes \phi^1 \otimes \beta + \alpha \otimes \phi^1 \otimes \phi^2 \otimes \beta + \\
&\quad \alpha \otimes \phi^2 \otimes \phi^1 \otimes \beta + \alpha \otimes \phi^2 \otimes \phi^2 \otimes \beta \\
&= \alpha \otimes \phi^1 \otimes \phi^2 \otimes \beta + \alpha \otimes \phi^2 \otimes \phi^1 \otimes \beta \\
&= 0,
\end{aligned}$$

in $A^k(V^*)$. From this it is seen that the elements in $A^k(V^*)$ are alternating, i.e.

$$\alpha \otimes \phi^1 \otimes \phi^2 \otimes \beta = -\alpha \otimes \phi^2 \otimes \phi^1 \otimes \beta.$$

We will use this to show that there exists an isomorphism between $A^k(V^*)$ and $\Lambda^k(V)$. We know that the dimension of $\dim \Lambda^k(V) = \begin{pmatrix} n \\ k \end{pmatrix}$. Now, the elements $A^k(V^*)$ are alternating and the basis of $A^k(V^*)$ consists of elements of the form $\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$, where the i_j are all different. So $\dim A^k(V^*)$ is equal to the number of ways in which we can choose a sequence of k different elements out of in total n elements. This can be done in $\begin{pmatrix} n \\ k \end{pmatrix}$ ways, so $\dim A^k(V^*) = \begin{pmatrix} n \\ k \end{pmatrix}$. This also implies that $\dim \ker(\pi) = \dim \ker(Alt)$, because $V^* \otimes \cdots \otimes V^*$ and $T^k(V)$ both have dimension of n^k. If we can show that $\ker(\pi)$ is mapped onto $\ker(\mathrm{Alt})$, then this implies that, because the dimensions of the kernels are equal, that $\ker(\pi) \cong \ker(\mathrm{Alt})$, which in turn implies that $A^k(V^*) = (V^* \otimes \cdots \otimes V^*)/\ker(\pi) \cong T^k(V)/\ker(\mathrm{Alt}) = \Lambda^k(V)$.
First take an element $x \in V^* \otimes \cdots \otimes V^*$, such that $x \in \ker(\pi)$, i.e. $x \in \Sigma$. Then x is of the form $\alpha \otimes \phi \otimes \phi \otimes \beta$ and is identified with an element y of this same form in $T^k(V)$ and $\mathrm{Alt}(y) = 0$. Hence $\pi(x) = 0$ implies $\mathrm{Alt}(y) = 0$. Conversely, suppose $x' \in V^* \otimes \cdots \otimes V^*$ such that $\pi(x') \neq 0$, then x must be alternating and the element $y' \neq 0 \in T^k(V)$ identified with x' is alternating and since the Alt-projection is the identity with respect to alternating elements, $\mathrm{Alt}(y') = y' \neq 0$, which we needed to show and we conclude that the diagram given above commutes.

Define a wedge product on $A^k(V^*)$ by $\omega \wedge \eta = \pi(\tilde{\omega} \otimes \tilde{\eta})$, where $\pi : V^* \otimes \cdots \otimes V^* \to A^k(V^*)$ is the projection, and $\tilde{\omega}, \tilde{\eta}$ are arbitrary tensors such that $\pi(\tilde{\omega}) = \omega, \pi(\tilde{\eta}) = \eta$. We will show that this wedge product is well defined. Suppose $\pi(\tilde{\omega}) = \pi(\tilde{\omega}') = \omega$ and $\pi(\tilde{\eta}) = \pi(\tilde{\eta}') = \eta$. Then we must have $\pi(\tilde{\omega}' \otimes \tilde{\eta}') = \pi(\tilde{\omega} \otimes \tilde{\eta})$, or equivalently, $\pi(\tilde{\omega}' \otimes \tilde{\eta}' - \tilde{\omega} \otimes \tilde{\eta}) = 0$. Well

$$\begin{aligned}
\pi(\tilde{\omega}' \otimes \tilde{\eta}' - \tilde{\omega} \otimes \tilde{\eta}) &= \pi((\tilde{\omega}' - \tilde{\omega}) \otimes \tilde{\eta}' + \tilde{\omega} \otimes (\tilde{\eta}' - \tilde{\eta})) \\
&= \pi((\tilde{\omega}' - \tilde{\omega}) \otimes \tilde{\eta}') + \pi(\tilde{\omega} \otimes (\tilde{\eta}' - \tilde{\eta}))) \\
&= 0
\end{aligned}$$

in $A^k(V^*)$. And thus the wedge product in well defined.

Because F is an isomorphism, F takes the wedge product defined on $A^k(V^*)$ to a wedge product on $\Lambda^k(V)$. Since the Alt wedge product is the unique wedge product on $\Lambda^k(V)$, it follows that F takes the wedge product as defined above to the Alt wedge product.

5.4 Riemannian Metrics

A *Riemannian metric* on a smooth manifold M is a smooth symmetric 2-tensor field that is positive definite at each point. A *Riemannian manifold* is a pair (M, g), where M is a smooth manifold a g a Riemannian metric. In any smooth local coordinates (x_i), a Riemannian metric can be written

$$g = g_{ij} dx^i \otimes dx^j, \tag{6}$$

where g_{ij} is a symmetric positive definite matrix of smooth functions. Because of the symmetry, g can also be written as

$$\begin{aligned} g = g_{ij} dx^i \otimes dx^j &= \frac{1}{2}(g_{ij} dx^i \otimes dx^j + g_{ji} dx^i \otimes dx^j) \\ &= \frac{1}{2}(g_{ij} dx^i \otimes dx^j + g_{ij} dx^j \otimes dx^i) = g_{ij} dx^i dx^j, \end{aligned}$$

by definition of the symmetric product

5.4.1 Euclidean Metric

The simplest example of a Riemannian metric is the *Euclidean metric* $\bar{g}$ on $\mathbb{R}^n$, defined in standard coordinates by

$$\bar{g} = \delta_{ij} dx^i dx^j, \tag{7}$$

where δ_{ij} is the Kronecker delta. Thus

$$\bar{g} = (dx^1)^2 + ... + (dx^n)^2.$$

Applied to vectors, $v, w \in T_p\mathbb{R}$, this yields

$$\bar{g}_p(v, w) = \delta_{ij} v^i w^j = \sum_{i=1}^{n} v^i w^j = v \cdot w.$$

We can express the Euclidean metric in polar coordinates on $\mathbb{R}^2$. The Euclidean metric in this case is $g = dx^2 + dy^2$. Substituting $x = r\cos\phi$ and $y = r\sin\phi$ and expanding, we obtain

$$\begin{aligned} \bar{g} = dx^2 + dy^2 &= d(r\cos\phi)^2 + d(r\sin\phi)^2 \\ &= (\cos\phi dr - r\sin\phi d\phi)^2 + (\sin\phi dr + r\cos\phi d\phi)^2 \\ &= dr^2 + r^2 d\phi^2. \end{aligned}$$

Often, the Riemannian metric g is written as $ds^2 = g_{ij}dx^i dx^j$, where ds is exactly the length of an infinitesimal tangent vector, and is called the element of *arc length*. Suppose $C : u^i = u^i(t), t_0 \leq t \leq t_1$, is a continuous and piecewise smooth parametrized curve on M. Then the arc length of C is defined to be

$$s = \int_{t_0}^{t_1} \sqrt{g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}} dt.$$

In the simple case of a graph of a function $(t, f(t))$, embedded in $\mathbb{R}^2$, we have $g_{ij} = \delta_{ij}$ and

$$\frac{du^1(t)}{dt} = 1, \quad \frac{du^2(t)}{dt} = f'(t),$$

so the arc length is given by

$$s = \int_{t_0}^{t_1} \sqrt{1 + (f'(t))^2} dt,$$

a formula well known in classical analysis.

5.4.2 The Gradient

The gradient of f is defined by

$$\text{grad} f = \tilde{g}^{-1}(df),$$

where $\tilde{g}(X)(Y) = g(X, Y)$. In smooth coordinates, grad$f$ has the expression

$$\text{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j},$$

where $(g_{ij})^{-1} = g^{ij}$ In particular, on $\mathbb{R}^n$ with the Euclidean metric, this reduces to

$$\text{grad} f = \delta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

In the previous example, we saw that the matrix of $\tilde{g}$ in polar coordinates is $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, so its inverse is $\begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$. Inserting this into the formula for the gradient, we obtain

$$\text{grad} f = \frac{\partial g}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial g}{\partial \phi} \frac{\partial}{\partial \phi},$$

where $g(r, \phi) = f(r \cos \phi, r \sin \phi)$.

5.4.3 Shortest Path in $\mathbb{R}^n$

Let us consider all the paths between two points in Euclidean space and determine the shortest one. More precisely, for $x, y \in \mathbb{R}^n$, let $\gamma : [0,1] \to \mathbb{R}^n$ be the curve segment

$$\gamma(t) = (1-t)x + ty.$$

and we will show that any other piecewise smooth curve segment $\tilde{\gamma}$ from x to y satisfies $L_{\bar{g}}(\tilde{\gamma}) \geq L_{\bar{g}}(\gamma)$, where $\bar{g}$ denotes the Euclidean metric on $\mathbb{R}^n$.

First we remark that, without loss of generality, we can choose the points x and y to be on the x_1-axis. Then if $\gamma(t)$ is given as above, that is $\gamma(t) = ((1-t)x_1^1 + tx_2^1, 0, ..., 0)$, then $L_{\bar{g}}(\gamma) = |x_2^1 - x_1^1|$. Let us define $\lambda \equiv |x_2^1 - x_1^1|$. Now suppose

$$\tilde{\gamma}(t) = ((1-t)x_1^1 + tx_2^1 + \alpha^1(t), \alpha^2(t), ..., \alpha^n(t))$$

, then

$$L_{\bar{g}}(\tilde{\gamma}) = \int_0^1 \sqrt{\dot{\tilde{\gamma}}^1(t)^2 + ... + \dot{\tilde{\gamma}}^n(t)^2 dt}.$$

But

$$\sqrt{\dot{\tilde{\gamma}}^1(t)^2 + ... + \dot{\tilde{\gamma}}^n(t)^2} \geq \sqrt{\dot{\tilde{\gamma}}^1(t)^2} = |\lambda + \dot{\alpha}(t)|.$$

By definition of $\tilde{\gamma}$, we see that $\alpha(0) = \alpha(1) = 0$. And thus

$$L_{\bar{g}}(\tilde{\gamma}) \geq \int_0^1 |\lambda + \dot{\alpha}(t)| dt = |\lambda t + \alpha(t)||_0^1 = |\lambda| = L_{\bar{g}}(\gamma).$$

We conclude that the straight line is the shortest path between two points in Euclidean space. □

6 Differential Forms and Integration

6.1 Integration on Manifolds

Differential forms are the objects that can be integrated on a manifold in a coordinate-independent way. A *domain of integration* is a bounded subset of $\mathbb{R}^n$ whose boundary has n-dimensional measure zero. Let $D \subset \mathbb{R}^n$ be such a compact domain of integration, and let ω be an n-form on D. Any such form can be written as

$$\omega = f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \tag{8}$$

for a continuous real-valued function f on D. We define the integral of ω *over* D to be

$$\int_D \omega = \int_D f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \int_D f dx^1 dx^2 \cdots dx^n. \tag{9}$$

Let M be a smooth, oriented n-manifold , and let ω be an n-form on M. Suppose that ω is compactly supported in the domain of a single smooth coordinate chart (U, ϕ). We define the integral of ω over M to be

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega. \tag{10}$$

With ω as above, $\int_M \omega$ does not depend on the choice of oriented smooth chart whose domain contains supp ω.

Example
Let $\omega = x^2 dx \wedge dy$ and $\mathbb{D}^2$ be the unit disc. Furtermore, $\psi : D \subset \mathbb{R}^2 \to \mathbb{D}^2$, $\psi(r, \theta) = (r\cos\theta, r\sin\theta) = (x, y)$. Let $D = [0, 1] \times [0, 2\pi)$. We compute

$$dx \wedge dy = d(r\cos\theta) \wedge d(r\sin\theta) = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) = r dr \wedge d\theta.$$

So we obtain

$$\begin{aligned} \int_{\mathbb{D}^2} \omega &= \int_D \psi^* \omega = \int_D r^3 \cos^2\theta dr \wedge d\theta \\ &= \int_0^1 \int_0^{2\pi} r^3 \cos^2\theta dr d\theta = \frac{\pi}{4}. \end{aligned}$$

□
Example
The *length* of a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ is defined to be the value of the integral

$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

There is, however, no smooth covector field $\omega \in \chi^*(M)$ with the property that $\int_\gamma \omega = L(\gamma)$. for every smooth curve γ. To see this, suppose $\omega = |\gamma'(t)|$ is such a covector. By linearity (of every covector), we must have $\omega(\alpha'(t) + \beta'(t)) = \omega(\alpha'(t)) + \omega(\beta'(t))$, instead

$$\begin{aligned} \omega(\alpha'(t) + \beta'(t)) &= |\alpha'(t) + \beta'(t)| = \langle \alpha'(t) + \beta'(t), \alpha'(t) + \beta'(t) \rangle^{1/2} \\ &= \left(\langle \alpha'(t), \alpha'(t) \rangle + \langle \beta'(t), \beta'(t) \rangle + 2\langle \alpha'(t), \beta'(t) \rangle \right)^{1/2}, \end{aligned}$$

and, in general, $\langle \alpha'(t), \beta'(t) \rangle \neq \langle \alpha'(t), \alpha'(t) \rangle^{1/2} \langle \beta'(t), \beta'(t) \rangle^{1/2}$, and thus $|\alpha'(t) + \beta'(t)| \neq |\alpha'(t)| + |\beta'(t)|$. We conclude that $\omega = |\gamma'(t)|$ is no covector. □

6.2 Stokes' formula

The central result in the theory of integration on manifolds is *Stokes' formula*. This is a far reaching generalisation of the fundamental theorem of calculus and of the classical theorems of vector analysis.

Theorem (Stokes' formula) *Let M be a smooth, oriented n-dimensional manifold with boundary, and let ω be a compactly supported smooth (n-1)-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega \tag{11}$$

It has to be mentioned that ∂M is understood to have the induced orientation, and the ω on the right-hand side is to be interpreted as $\omega|_{\partial M}$. If $\partial M = \emptyset$, then the right-hand side is to be interpreted as zero. When M is 1-dimensional, the right-hand integral is really just a finite sum.

Example
Let N be a smooth manifold and suppose $\gamma : [a,b] \mapsto N$ is a smooth embedding, so that $M = \gamma[a,b]$ is an embedded 1-dimensional submanifold without boundary in N. Now Stokes' formula says that

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_M df = \int_{\partial M} f = f(\gamma(b)) - f(\gamma(a)).$$

Thus Stokes' formula reduces to the fundamental theorem for line integrals in this case. In particular, when $\gamma : [a,b] \mapsto \mathbb{R}$ is the inclusion map, then Stokes' formula is just the ordinary fundamental theorem of calculus. □

These two corollaries follow immediately from Stokes' theorem.

Corollary
Suppose M is a compact smooth manifold without boundary. Then the integral of every exact form over M is zero:

$$\int_M d\omega = 0 \quad \text{if } \partial M = \emptyset.$$

Corollary
Suppose M is a compact smooth manifold without boundary. If ω is a closed form on M, then the integral of ω over ∂M is zero:

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0 \text{ on } M.$$

Example Green's Theorem
Suppose D is a domain in $\mathbb{R}^2$, and P, Q are smooth real-valued functions on D. Take $\omega = Pdx + Qdy$ and apply Stokes' formula to obtain

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_{\partial D} Pdx + Qdy.$$

□

6.3 Integration on Riemannian Manifolds

Let V be a finite-dimensional vector space, and let $X \in V$. We define a linear map $\iota_X : \Lambda^k(V) \to \Lambda^{k-1}(V)$, called the *interior multiplication* with X by,

$$\iota_X\omega(Y_1, ..., Y_{k-1}) = \omega(X, Y_1, ..., Y_{k-1}). \tag{12}$$

Let (M, g) be an oriented Riemannian manifold. Then in any oriented smooth coordinates (x^i), the *Riemannian volume form* has the expression

$$dV_g = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n, \tag{13}$$

where g_{ij} are the components of g in these coordinates. Now let (M, g) be an oriented Riemannian manifold, let $S \subset M$ be an immersed hypersurface, and let $\tilde{g}$ denote the induced metric on S. Suppose furthermore that N is a smooth unit vector field along S. Then the volume form is given by

$$dV_{\tilde{g}} = \iota_N dV_g|_S.$$

If X is any vector field along S, then we have

$$\iota_X dV_g|_S = \langle X, N\rangle_g dV_{\tilde{g}}. \tag{14}$$

Let (M, g) be a Riemannian manifold. Multiplication by the Riemannian volume form defines a linear map $\star : C^\infty \to A^n(M)$:

$$\star f = f dV_g,$$

where $A^n(M)$ denotes the space of smooth n-forms. Define the *divergence operator* div: $\chi(M) \to C^\infty$ by

$$\text{div}X = \star^{-1}d(\iota_X dV_g), \tag{15}$$

or equivalently

$$d(\iota_X dV_g) = (\text{div}X)dV_g$$

Example The Divergence Theorem
If M is a oriented manifold with boundary, and X any compactly supported vector field on M. Using Stokes' formula on the form $\omega = \iota_X dV_g$ we obtain

$$\int_M \omega = \int_M d(\iota_X dV_g) = \int_M (\text{div}X)dV_g = \int_{\partial M} d(\iota_X dV_g) = \int_{\partial M} \langle X, N\rangle_g dV_{\tilde{g}},$$

where N is outward-pointing unit normal vector field along ∂M and $\tilde{g}$ is the induced Riemannian metric on ∂M.□

Let (M, g) be an oriented Riemannian 3-manifold. Then the *curl* is defined to be

$$\iota_{\text{curl}X} dV_g = d(X^\flat) \tag{16}$$

Example
To see that this definition coincides with our classical definition on $\mathbb{R}^3$, recall that $X^\flat = \tilde{g}(X) = g_{ij}X^i dy^j$, where $g_{ij} = \delta_{ij}$. Notice that $X^\flat = X^1dx+X^2dy+X^3dz$, so

$$d(X^\flat) = \left(\frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y}\right) dx\wedge dy + \left(\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z}\right) dy\wedge dz + \left(\frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x}\right) dz\wedge dx.$$

On the other hand we have, denoting $\mathrm{curl}X = Y$,

$$\iota_Y dV_g = dx \wedge dy \wedge dz(Y) = Y^3 dx \wedge dy + Y^1 dy \wedge dz + Y^2 dz \wedge dx,$$

and so we obtain

$$\mathrm{curl}X = \left(\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z}, \frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x}, \frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y}\right),$$

and this is a familiar result.□

Example Stokes' formula for Surface Integrals
Suppose S is a compact, oriented, embedded, 2-dimensional submanifold with boundary in an oriented Riemannian 3-manifold M. For any smooth vector field X on M,

$$\int_S \langle \mathrm{curl}X, N\rangle_g dA = \int_{\partial S} \langle X, T\rangle_g ds, \tag{17}$$

where N is the smooth unit normal vector field along S that determines its orientation, ds is the Riemannian volume form for ∂S and T is the positively oriented unit tangent vector field on ∂S. By equation (14), we have that $dX^\flat = \iota_{\mathrm{curl}X} dV_g = \langle \mathrm{curl}X, N\rangle dV_{\tilde{g}}$. Furthermore, $X^\flat$ is a smooth 1-form on a 1-manifold, and thus must equal fds for some smooth function f. Note that $ds(T) = 1$, and so

$$f = fds(T) = X^\flat(T) = \langle X, T\rangle_g ds.$$

Now using Stokes' formula on the form $X^\flat$ we obtain,

$$\int_S d(X^\flat) = \int_S \iota_{\mathrm{curl}X} dV_g = \int_{\partial S} X^\flat = \int_{\partial S} \langle X, T\rangle_g ds \tag{18}$$

and this gives the result. □

6.4 Integration by parts

Let (M, g) be a compact Riemannian manifold with boundary, let $\tilde{g}$ denote the induced Riemannian metric on ∂M, and let N be the outward unit normal vector field along ∂M. Equation (20) gives the definition of the divergence. The divergence operator satisfies a certain product rule for $f \in C^\infty$, $X \in \chi(M)$:

$$\mathrm{div}(fX) = f\mathrm{div}X + \langle \mathrm{grad}f, X\rangle_g. \tag{19}$$

Let $V_i, i = 0, ..., k-1$ denote smooth vector fields, then this formula can be derived as follows,

$$\begin{aligned}
\mathrm{div}(fX)dV_g(V_1,...,V_{k-1}) &= d(\iota_{fX}dV_g)(V_1,...,V_{k-1}) = d(dV_g(fX,V_1...,V_{k-1})) \\
&= df \wedge dV_g(X,V_1...,V_{k-1}) + fd(dV(X,V_1...,V_{k-1}) \\
&= df(X)dV(V_1...,V_{k-1}) + fd(\iota_X dV_g)(V_1,...,V_{k-1}) \\
&= \langle \mathrm{grad} f, X\rangle_g dV(V_1...,V_{k-1}) + f\mathrm{div}X dV_g(V_1,...,V_{k-1}).
\end{aligned}$$

And so we obtain equation (19). Now define the k-form $\omega = \mathrm{div}(fX)dV$. Then, using Stokes' formula and that $\iota_{fX}dV_g = f\langle X, N\rangle_g dV_{\tilde{g}}$ by linearity , we see that

$$\begin{aligned}
\int_M \mathrm{div}(fX)dV &= \int_M f\mathrm{div}(X)dV + \int_M \langle \mathrm{grad} f, X\rangle dV \\
&= \int_{\partial M} d(\mathrm{div}(fX)dV) \\
&= \int_{\partial M} f\langle X, N\rangle_g dV_{\tilde{g}}.
\end{aligned}$$

Hence we obtain the important formula

$$\int_M \langle \mathrm{grad} f, X\rangle dV = \int_{\partial M} f\langle X, N\rangle_g dV_{\tilde{g}} - \int_M f\mathrm{div}(X)dV. \tag{20}$$

But what has this to do with integration by parts? For this to see, consider the simple 1-dimensional case, i.e. where $f : \mathbb{R} \to \mathbb{R}$ and the manifold M is 1-dimensional. Take $M = [a, b]$, an interval in $\mathbb{R}$. Then, using that the positive orientation on an interval is the direction to the right and that the unit normal is pointing outward, we have that $\partial M = \{b\} - \{a\}$. Now $dV_g = dx$, $X = g(x)$ a ordinary function, $\langle \mathrm{grad} f, X\rangle dV = f'(x)g(x)dx$ and $f\mathrm{div}(X)dV = f(x)g'(x)dx$. Then equation (20) is easily seen to reduce to

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx,$$

since integration over a 0-dimensional manifold is defined to be summation.

6.5 Green's Identities

Let (M, g) be a oriented Riemannian manifold with or without boundary. The linear operator $\Delta : C^\infty(M) \to C^\infty(M)$ defined by $\Delta u = -\mathrm{div}(\mathrm{grad})u$ is called the *Laplace operator* or *Laplacian.* A function $u \in C^\infty(M)$ is said to be *harmonic* if $\Delta u = 0$. Now define $f = u$ and $X = \mathrm{grad}v$. Then, using equation (19), we obtain

$$\begin{aligned}
\mathrm{div}(fX) = \mathrm{div}(u \cdot \mathrm{grad}v) &= u\mathrm{div}(\mathrm{grad}v) + \langle \mathrm{grad}u, \mathrm{grad}v\rangle_g \\
&= -u\Delta v + \langle \mathrm{grad}u, \mathrm{grad}v\rangle_g
\end{aligned}$$

But also

$$\operatorname{div}(u\operatorname{grad}v) = d(\iota_{u\operatorname{grad}v}dV_g) = d(u \cdot \iota_{\operatorname{grad}v}dV_g)$$

and because $\langle \operatorname{grad}v, X\rangle_g = Xv$ we have,

$$\iota_{\operatorname{grad}v}dV_g = \langle \operatorname{grad}v, N\rangle_g dV_{\tilde{g}} = NvdV_{\tilde{g}}.$$

Applying this to Stokes' formula in a fashion similar to the case of equation (20) we obtain *Green's First Identity*,

$$\int_M u\Delta v dV_g = \int_M \langle \operatorname{grad}u, \operatorname{grad}v\rangle_g dV_g - \int_{\partial M} uNvdV_{\tilde{g}}. \tag{21}$$

Switching the u and the v in equation (21) and subtracting both equations we obtain *Green's Second Identity*

$$\int_M (u\Delta v - v\Delta u)dV_g = \int_{\partial M} (vNu - uNv)dV_{\tilde{g}} \tag{22}$$

Suppose that the function v is constant and that M is compact and connected and $\partial M = \emptyset$. Then $\Delta v = 0$. Conversely, suppose $\Delta v = 0$ and suppose that $u \equiv 1$. Then by equation (22) we must have $Nv \equiv 0$. This is the case only if v is constant. Hence, the only harmonic functions on M are the constants.

Suppose M is compact and connected, $\partial M \neq \emptyset$, and u,v are harmonic functions on M whose restrictions to ∂M agree. Then, by equation (22) we have that

$$\int_M (u\Delta v - v\Delta u)dV_g \equiv 0,$$

that is, $u\Delta v \equiv v\Delta u$. This can only hold when $u \equiv v$. We conclude that u and v agree anywhere on the manifold M whenever they agree on the boundary ∂M.

Example
Let M be a compact, connected, oriented Riemannian n-manifold with nonempty boundary. A number $\lambda \in \mathbb{R}$ is called a *Dirichlet eigenvalue* for M if there exists a smooth real-valued function u on M, not identically zero, such that $\Delta u = \lambda u$ and $u|_{\partial M} = 0$.

When we choose u and v equal in equation (22), say u and note that $u|_{\partial M} = 0$, we see

$$\int_M u\Delta u dV_g = \lambda \int_M u^2 dV_g = \int_M |\operatorname{grad}u|^2 dV_g$$

and so $\lambda \geq 0$. However, suppose $\lambda = 0$, then $\Delta u = 0$, so $u =$ constant. But the fact that M is connected and $u|_{\partial M} = 0$ gives exactly that $u \equiv 0$, a case that we excluded by definition. Hence every Dirichlet eigenvalue is strictly positive.

Similarly, λ is called a *Neumann eigenvalue* if there exists such a u satisfying $\Delta u = \lambda u$ and $Nu|_{\partial M} = 0$, where N is the outward unit normal.

When we choose $\lambda = 0$, we get $\Delta u = 0$, which implies $u =$ constant. But if $u =$ constant, then $Nu = 0$, and so is indeed an Neumann eigenvalue. Noting that, by hypothesis, $Nu|_{\partial M} = 0$ we obtain, again by (22), that $\lambda \geq 0$. □

6.6 Hodge Star operator

In this section we will generalize the mapping used to define the divergence in section 6.3. Suppose (M, g) is a Riemannian manifold. We start by defining an inner product for k-forms.

For each $k = 0, ..., n$, g determines a unique inner product on $\Lambda^k(T_pM)$, denoted $\langle \cdot, \cdot \rangle_g$, defined locally by

$$\langle \omega^1 \wedge \cdots \wedge \omega^k, \eta^1 \wedge \cdots \wedge \eta^k \rangle_g = \det \left(\langle (\omega^i)^\sharp, (\eta^j)^\sharp \rangle_g \right), \tag{23}$$

where $\omega^1, ..., \omega^k, \eta^1, ..., \eta^k$ are 1-forms and $\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$ with $(i_1, ..., i_k)$ a strictly increasing index an orthonormal basis for $\Lambda^k(T_pM)$ whenever (ϵ^j) is the coframe dual to the orthonormal frame. First we will show that this definition satisfies the properties of an inner product. For this, we will use the following lemma.

Lemma *On a finite-dimensional vector space, the covectors $\omega^1, ..., \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.*

Proof If the ω^j are linearly dependent then, without loss of generality, we may assume that $\omega^1 = \sum_{i=2}^k a_i \omega^i$ and we see

$$\omega^1 \wedge \cdots \wedge \omega^k = \sum_{i=2}^k a_i \omega^i \wedge \omega^2 \wedge \cdots \wedge \omega^k = 0,$$

because in every term we have $\omega^j \wedge \omega^j$ for a certain j and is zero by anticommutativity. Conversely, suppose that the ω^i are linearly independent. Then these ω^i can be extended to a basis of $\Lambda^n(TM)$. Then $\omega^1 \wedge \cdots \wedge \omega^k \wedge \cdots \wedge \omega^n \neq 0$ and hence $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$. □

Next we verify the properties pointwise.
Symmetry follows immediately from symmetry of the inner product defined on vector fields, i.e. $\langle (\omega^i)^\sharp, (\eta^j)^\sharp \rangle_g = \langle (\eta^j)^\sharp, (\omega^i)^\sharp \rangle_g$.
Bilinearity follows from the fact the inner product on vector fields is bilinear, together with the fact that the determinant is multilinear and hence bilinear which follows immediately from the definition.
To show that $\langle \omega, \omega \rangle_g = 0$ only if $\omega = 0$ we note that $\det \left(\langle (\omega^i)^\sharp, (\omega^j)^\sharp \rangle_g \right) = \det \left(\langle \omega^i, \omega^j \rangle_g \right) = 0$ if and only if ω^i are linearly dependent, where we denote $\omega = \omega^1 \wedge \cdots \wedge \omega^k$. By the lemma above, this holds if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

Hence we conclude that this indeed defines an inner product. Since this definition is given in a local form, we need to show that this definition is independent of the choice of local frame. Denote the orthonormal frame defined above (E_j). For another (not necessarily orthonormal) frame $\tilde{E}_i$ we can write

$$\tilde{E}_i = A_i^j E_j,$$

for some matrix (A_i^j) of smooth functions. Because $\omega = \omega_j \epsilon^j$ and $\omega = \tilde{\omega}_i \tilde{\epsilon}^i$ we have

$$\omega(\tilde{E}_j) = \tilde{\omega}_j = \omega(A_j^k E_k) = A_j^k \omega(E_k) = A_j^k \omega_k,$$

and thus $A_j^k \omega_k = \tilde{\omega}_j$. Furthermore

$$\omega = \tilde{\omega}_i \tilde{\epsilon}^i = A_i^k \omega_k \tilde{\epsilon}^i = \omega_k \epsilon^k,$$

hence $\epsilon^k = A_i^k \tilde{\epsilon}^i$ or equivalently $(A^{-1})_k^i \epsilon^k = \tilde{\epsilon}^i$. Then

$$\begin{aligned}
\langle (\tilde{\omega}^i)^\sharp, (\tilde{\eta}^j)^\sharp \rangle_g &= \tilde{g}^{st} (\tilde{\omega}^i)_t (\tilde{\eta}^j)_s = \langle \tilde{\epsilon}^s, \tilde{\epsilon}^t \rangle (\tilde{\omega}^i)_t (\tilde{\eta}^j)_s \\
&= \langle (A^{-1})_\lambda^s \epsilon^\lambda, (A^{-1})_\mu^t \epsilon^\mu \rangle A_t^p A_s^q (\omega^i)_p (\eta^j)_q \\
&= (A^{-1})_\lambda^s (A^{-1})_\mu^t A_t^p A_s^q \langle \epsilon^\lambda, \epsilon^\mu \rangle (\omega^i)_p (\eta^j)_q \\
&= (A^{-1})_\lambda^s \delta_\mu^p A_s^q g^{\lambda\mu} (\omega^i)_p (\eta^j)_q = (A^{-1})_\lambda^s A_s^q g^{\lambda\mu} (\omega^i)_\mu (\eta^j)_q \\
&= g^{\lambda\mu} (\omega^i)_\mu (\eta^j)_\lambda = \langle (\omega^i)^\sharp, (\eta^j)^\sharp \rangle_g.
\end{aligned}$$

Hence, $\langle (\omega^i)^\sharp, (\eta^j)^\sharp \rangle_g$ is independent of the choice of local frame and therefore $\langle \omega^1 \wedge \cdots \wedge \omega^k, \eta^1 \wedge \cdots \wedge \eta^k \rangle_g$ is also independent of the choice of local frame which proves that the inner product is well defined.

For each $k = 0, ..., n$, there is a unique smooth bundle map $\star : \Lambda^k M \to \Lambda^{n-k} M$ satisfying

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle_g dV_g, \tag{24}$$

this map is called the *Hodge star operator*. For $k = 0$ the inner product can be interpreted as ordinary multiplication.Because the inner product is linear in its second entry, the hodge star operator is linear, i.e.

$$\omega \wedge \star(\eta + \eta') = \langle \omega, \eta + \eta' \rangle_g dV_g = \langle \omega, \eta \rangle_g dV_g + \langle \omega, \eta' \rangle_g dV_g = \omega \wedge \star\eta + \omega \wedge \star\eta'$$

First we will prove uniqueness of this mapping. Suppose there exists another mapping $\sharp$, such that

$$\omega \wedge \sharp\mu = \langle \omega, \mu \rangle_g dV_g,$$

where $\omega, \mu \in \Lambda^k(T_pM)$. Then for every $\eta \in \Lambda^k(T_pM)$,

$$\omega \wedge \sharp\mu - \omega \wedge \star\eta = \langle \omega, \mu \rangle_g dV_g - \langle \omega, \eta \rangle_g dV_g = \langle \omega, \mu - \eta \rangle_g dV_g.$$

However $\langle \omega, \mu - \eta \rangle_g dV_g = 0$ if and only if $\langle \omega, \mu - \eta \rangle_g = 0$ and if and only if $\mu = \eta$. Hence for every $\omega, \eta \in \Lambda^k(T_pM)$ we have

$$\omega \wedge \sharp\eta = \omega \wedge \star\eta,$$

which proves uniqueness.
We define the mapping $\star : \Lambda^k M \to \Lambda^{n-k} M$ locally by setting

$$\star\left(\epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}\right) = \operatorname{sgn}(\sigma)\epsilon^{t_1} \wedge \cdots \wedge \epsilon^{t_{n-k}}, \tag{25}$$

for a certain sequence of indices $(t_1, ..., t_{n-k})$ and permutation σ. To justify this definition, take (E_i) a local orthonormal frame and (ϵ^i) the dual coframe. The Riemannian volume form in this frame is given by $dV_g = \epsilon^1 \wedge \cdots \wedge \epsilon^n$. From this it is immediately seen that $dV_g(E_1, ..., E_n) = 1$. Because, in general, $\langle \omega^\sharp, \eta^\sharp \rangle_g = \langle \omega, \eta \rangle_g$ and orthonormality, it is seen that

$$\langle \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}, \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k} \rangle_g = \det\left(\langle \epsilon^{i_p}, \epsilon^{j_q} \rangle\right) = \det\left(\delta^{i_p j_q}\right).$$

We define $\det\left(\delta^{i_p j_q}\right) = \delta^{IJ}$, where $I = (i_1, ..., i_k)$ and $J = (j_1, ..., j_k)$ and $\delta^{IJ} = 0$ if I is not a permutation of J and $\delta^{IJ} = \operatorname{sgn}(\tau)$, whenever $\tau(I) = J$ for $\tau \in S_k$. Suppose there exists an index i_r such that $i_r \neq j_q$ for all j_q, then $\langle \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}, \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k} \rangle_g = 0$. Hence, for the inner product to be unequal to zero, we must have $\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} = \pm \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}$, or otherwise stated, the sequence of indices $(i_1, ..., i_k)$ is a permutation of $(j_1, ..., j_k)$, which justifies the definition of δ^{IJ}. The increasing sequence $(t_1, ..., t_{n-k})$ is then to be chosen such that $\sigma(j_1, ..., j_k, t_1, ..., t_{n-k}) = (1, ..., n)$.
First $\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} \wedge \star\left(\epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}\right)$ equals

$$\begin{cases} 0 & \text{if I is no permutation of J} \\ \operatorname{sgn}(\tau)\epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k} \wedge \star\left(\epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}\right) & \text{if } \tau(I) = J \end{cases}$$

which by definition of δ^{IJ} equals

$$\delta^{IJ} \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k} \wedge \star\left(\epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}\right).$$

Furthermore, define $\omega = \omega_{i_1 \cdots i_k} \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$ and $\eta = \eta_{j_1 \cdots j_k} \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}$. Then

$$\begin{aligned} \langle \omega, \eta \rangle_g &= \langle \omega_{i_1 \cdots i_k} \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}, \eta_{j_1 \cdots j_k} \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k} \rangle_g \\ &= \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_k} \langle \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}, \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k} \rangle_g \\ &= \delta^{IJ} \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_k}, \end{aligned}$$

By linearity of the Hodge star operator, we now obtain,

$$\begin{aligned} \omega \wedge \star\eta &= \omega_{i_1 \cdots i_k} \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} \wedge \star\left(\eta_{j_1 \cdots j_k} \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}\right) \\ &= \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_k} \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} \wedge \star\left(\epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k}\right) \\ &= \delta^{IJ} \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_k} \operatorname{sgn}(\sigma) \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_k} \wedge \epsilon^{t_1} \wedge \cdots \wedge \epsilon^{t_{n-k}} \\ &= \langle \omega, \eta \rangle_g \epsilon^1 \wedge \cdots \wedge \epsilon^n = \langle \omega, \eta \rangle_g dV_g. \end{aligned}$$

Finally, because the mapping is defined locally, we need to show that the mapping is independent of the choice of the local orthonormal frame. It has been shown already that the inner product is independent of the choice of the frame, so it will suffice to show the independence of dV_g. If $(\tilde{E}_i)$ is another orthonormal frame, with dual coframe $(\tilde{\epsilon}^i)$, let $dV'_g = \tilde{\epsilon}^1 \wedge \cdots \wedge \tilde{\epsilon}^n$. As before, $\tilde{E}_i = A_i^j E_j$ for some matrix. However, since both frames are orthonormal, $\det A = \pm 1$, and the fact that the frames are oriented positively forces the sign to be 1. But then

$$dV_g(\tilde{E}_1, ..., \tilde{E}_n) = \det\left(\epsilon^j(\tilde{E}_i)\right) = \det(A_i^j) = 1 = dV'_g(\tilde{E}_1, ..., \tilde{E}_n),$$

which proves the independence.

As a special case of the Hodge star mapping, take $\Lambda^k M = \Lambda^0 M$, $\star : \Lambda^0 M \to \Lambda^n M$. Then $g \wedge \star f = g \cdot \star f = \langle g, f \rangle_g dV_g = g \cdot f dV_g$. Hence $\star f = f dV_g$, and this is exactly the mapping defined in section 6.3.

Finally, an important fact about the Hodge star operator is the following. For arbitrary $\omega \in \Lambda^k(T_pM)$ and $\eta \in \Lambda^l(T_pM)$, we have $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$. Define the sequence of indices of length n, $(j_1, ..., j_t, i_1, ..., i_k, j_{t+1}, ..., j_{n-k})$ and take the following elementary k-forms $\mu = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$ and $\eta = \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_{n-k}}$. Then it is easily seen that,

$$\begin{aligned}
\star\star\mu &= \star\left(\star\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}\right) = \star\left((-1)^{kt} \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_{n-k}}\right) \\
&= (-1)^{kt}(-1)^{(n-k-t)k} \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} = (-1)^{(n-k)k} \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} \\
&= (-1)^{(n-k)k} \mu.
\end{aligned}$$

Since this holds for every elemetary k-form and because every k-form ω can be written as a sum of elementary k-forms, we conclude, by linearity of the Hodge star operator, that $\star\star\omega = (-1)^{k(n-k)}\omega$.

Example
Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric and the standard orientation. We will calculate $\star dx^i$. We see

$$dx^i \wedge \star dx^i = (-1)^i dx^i \wedge dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n = dx^1 \wedge \cdots \wedge dx^n,$$

hence $\star dx^i = (-1)^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$.

Next we will calculate $\star dx^i \wedge dx^j$ in the case $n = 4$. Some calculations give

$$\begin{aligned}
\star(dx^1 \wedge dx^2) &= dx^3 \wedge dx^4, & \star(dx^2 \wedge dx^3) &= dx^1 \wedge dx^4, \\
\star(dx^3 \wedge dx^4) &= dx^1 \wedge dx^2, & \star(dx^2 \wedge dx^4) &= -dx^1 \wedge dx^3, \\
\star(dx^1 \wedge dx^4) &= dx^2 \wedge dx^3, & \star(dx^1 \wedge dx^3) &= -dx^2 \wedge dx^4.
\end{aligned}$$

A closer look reveals that $\star(dx^i \wedge dx^j) = (-1)^{i+j+1} dx^p \wedge dx^q$, where (p, q) is the (increasing) complement of (i, j) in $(1, 2, 3, 4)$.□

Hodge star decomposition
Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be *self-dual* if $\star\omega = \omega$, and *anti-self-dual* if $\star\omega = -\omega$. Every 2-form ω on M can

be written uniquely as a sum of a self-dual form and an anti-self-dual form as follows. Take any 2-form ω. Denote the self-dual form λ and the anti-self-dual form μ. Define $\lambda = \frac{1}{2}(\omega + \star\omega)$ and $\mu = \frac{1}{2}(\omega - \star\omega)$. Then

$$\omega = \frac{1}{2}(\omega + \star\omega) + \frac{1}{2}(\omega - \star\omega) = \lambda + \mu.$$

Since $\star\star\,\omega = (-1)^{k(n-k)}\omega$, in this case, with $n = 4$ and $k = 2$, we see that $\star\star\,\omega = \omega$. From this it follows,

$$\star\lambda = \frac{1}{2}(\star\omega + \star\star\,\omega) = \frac{1}{2}(\star\omega + \omega) = \lambda,$$

and

$$\star\mu = \frac{1}{2}(\star\omega - \star\star\,\omega) = \frac{1}{2}(\star\omega - \omega) = -\mu.$$

Hence, λ is self-dual and μ is anti-self-dual and from this we see that every 2-form can be decomposed into a self-dual form and an anti-self-dual form. To prove uniqueness, suppose that ω has another decomposition, $\omega = \lambda' + \mu'$, where $\star\lambda' = \lambda'$ and $\star\mu' = -\mu'$. Then

$$\lambda \equiv \frac{1}{2}(\omega + \star\omega) = \frac{1}{2}\left((\lambda' + \mu') + (\lambda' - \mu')\right) = \lambda'.$$

Similarly, $\mu = \mu'$ and hence the decomposition is unique.

Example Consider the case of $M = \mathbb{R}^4$ with the Euclidean metic, as in the previous example. We will determine the self-dual and anti-self-dual forms in standard coordinates. In the previous example, we already calculated the Hodge star map of 2-forms. By the theory (and notation) in this section

$$\lambda^{ij} = \frac{1}{2}\left(dx^i \wedge dx^j + \star\left(dx^i \wedge dx^j\right)\right) = \frac{1}{2}\left(dx^i \wedge dx^j + (-1)^{i+j+1} dx^p \wedge dx^q\right),$$

and, similarly,

$$\mu^{ij} = \frac{1}{2}\left(dx^i \wedge dx^j - \star\left(dx^i \wedge dx^j\right)\right) = \frac{1}{2}\left(dx^i \wedge dx^j + (-1)^{i+j} dx^p \wedge dx^q\right).$$

Let us verify that λ^{ij} and μ^{ij} are indeed self-dual and anti-self-dual respectively.

$$\begin{aligned}
\star\lambda^{ij} &= \frac{1}{2}\left(\star\left((-1)^{i+j+1} dx^p \wedge dx^q\right) + (-1)^{i+j+1} dx^p \wedge dx^q\right) \\
&= \frac{1}{2}\left((-1)^{i+j+1+p+q+1} dx^i \wedge dx^j + (-1)^{i+j+1} dx^p \wedge dx^q\right) \\
&= \frac{1}{2}\left(dx^i \wedge dx^j + (-1)^{i+j+1} dx^p \wedge dx^q\right) = \lambda^{ij},
\end{aligned}$$

where we used that $(-1)^{i+j+1+p+q+1} = (-1)^{12} = 1$ always. A similar calculation gives $\star\mu^{ij} = -\mu^{ij}$. $\square$

6.7 The Hairy Ball Theorem

In this section we will prove that there exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd. This theorem (by the Dutch mathematician Brouwer) is often stated more popularly by saying: "You cannot comb the hair on a ball", referring to the case $n = 2$. First we will state without proof a theorem on homotopy and then we will prove two lemmas to give the necessary tools to proof the actual hairy ball theorem.

Theorem (Smooth Homotopy Theorem)
If $F, G : M \to N$ are homotopic smooth maps, then they are smoothly homotopic, i.e. there is a smooth map $H : M \times I \to N$ that is a homotopy between F and G.

Lemma 1
Let M and N be compact, connected, oriented, smooth manifolds, and suppose $F, G : M \to N$ are diffeomorphisms. If F and G are homotopic, then they are either both orientation-preserving or both orientation-reversing.

Proof
Suppose F and G are diffeomorphisms and are homotopic then, by the smooth homotopy theorem, there exists a smooth map $H : M \times I \to N$ that is a homotopy between F and G, i.e. $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$. Assume that M has no boundary, then we have $\partial(M \times I) = M \times \partial I$. Because N is an oriented manifold there exists an orientation n-form on N, Ω, and then $d\Omega = 0$. Define a n-form on $M \times I$, $\Omega' = H^*\Omega$. This form is well defined since H is smooth. Then, because in general $H^* d\Omega = d(H^*\Omega)$, we have that the $(n+1)$-form $d\Omega' = 0$. By Stokes' formula

$$0 = \int_{M \times I} d\Omega' = \int_{M \times \partial I} \Omega' = \int_{M \times \partial I} H^*\Omega = \int_M (G^*\Omega - F^*\Omega),$$

because integration over $\partial I = \{1\} - \{0\}$ is just (signed) summation. And thus

$$\int_M G^*\Omega = \int_M F^*\Omega = \pm \int_N \Omega,$$

where we used that F and G are diffeomorphisms and where the plus or minus sign depends on the orientation of F and G. And as we see, the orientation of F and G is equal; either both F and G are orientation-preserving or both orientation-reversing and this is exactly what we needed to show.□

Lemma 2
The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving if and only if n is odd.

Proof The antipodal map is defined as $\alpha : \mathbb{S}^n \to \mathbb{S}^n$, $\alpha(x) = -x$. Now in general, a map $F : M \to N$ is orientation-preserving if and only if the Jacobian matrix has positive determinant. In this case it is seen that $\det D_x\alpha(x) = (-1)^{n+1}$

and so we see immediately that $\det D_x\alpha(x)$ is positive if and only if n is odd. □

Theorem (The Hairy Ball Theorem)
There exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.

Proof We will prove this by showing that the following are equivalent:

1. There exists a nowhere-vanishing vector field on $\mathbb{S}^n$
2. There exists a continuous map $V : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$
3. The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.
4. The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.
5. n is odd.

(1) $\Rightarrow$ (2) A vector field V on $\mathbb{S}^n$ by definition defines a continuous map $\mathbb{S}^n \to \mathbb{R}^{n+1}$. Now, because the vector field is non-vanishing, we can divide by the norm to obtain a continuous map $\mathbb{S}^n \to \mathbb{S}^n$. Since the vector field is everywhere tangent to $\mathbb{S}^n$, we have that $V(x) \perp x$ for all $x \in \mathbb{S}^n$.
(2) $\Rightarrow$ (3) Suppose we have a continuous map $V : \mathbb{S}^n \to \mathbb{S}^n$ where $V(x) \perp x$ for all $x \in \mathbb{S}^n$. Then define $H(x,t) := \cos(\pi t)x + \sin(\pi t)V(x)$. Then

$$\begin{aligned} |H(x,t)| &= \langle \cos(\pi t)x + \sin(\pi t)V(x), \cos(\pi t)x + \sin(\pi t)V(x)\rangle^{1/2} \\ &= \left(\cos^2(\pi t)\langle x,x\rangle + \sin^2(\pi t)\langle V(x),V(x)\rangle\right)^{1/2} = 1, \end{aligned}$$

since $\langle V(x), x\rangle = 0$. Furthermore, $H(x,0) = x$ and $H(x,1) = -x$. So $H(x,t) : \mathbb{S}^n \to \mathbb{S}^n$ and this defines a homotopy between $\mathrm{Id}_{\mathbb{S}^n}$ and $\alpha(x)$, the antipodal map.
(3) $\Rightarrow$ (4) Both $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ and $\mathrm{Id}_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{S}^n$ are diffeomorhisms. By hypothesis, they are homotopic and so, by **lemma 1**, we conclude that, since $\mathrm{Id}_{\mathbb{S}^n}$ is always orientation-preserving, α also must be orientation-preserving.
(4) $\Rightarrow$ (5) If the antipodal map is orientation-preserving, by **lemma 2**, we must have that n is odd.
(5) $\Rightarrow$ (1) Suppose n is odd, then $\mathbb{S}^n = \mathbb{S}^{2k+1} \subset \mathbb{C}^{k+1} \cong \mathbb{R}^{k+1} \times \mathbb{R}^{k+1}$, where we identify $(x,y) = x + iy$. Define the curve $\gamma_z : \mathbb{R} \to \mathbb{C}^{k+1}$, $\gamma_z(t) = e^{it}z$, $z \in \mathbb{S}^{2k+1}$. Then $\gamma_z'(t)$ is tangent to the sphere and $\gamma_z'(0) = iz$. This vector field can be expressed as

$$\gamma_z'(0) = x^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial x^j}.$$

Since the component functions of this vector field are (smooth) coordinate functions, the vector field is smooth. Since $|z| = 1$, not all x^j, y^j are zero, and thus the vector field is non-vanishing. Hence for every n odd there exists a nowhere-vanishing vector field on $\mathbb{S}^n$ and this concludes the proof.□

7 Connections

Before we can define curvature on Riemannian manifolds, we need to study geodesics, the Riemannian generalizations of straight lines. A curve in Euclidian space is a staight line if and only if its accelaration is identically zero. This is the property that we choose to take as a defining property of geodesics on a Riemannian manifold. In order to define geodesics on a manifold we need the concept of a connection; essentially a set of rules for taking directional derivatives of vector fields.

To see why we need a new kind of differentiation operator, consider a submanifold $M \subset \mathbb{R}^n$ with the induced Riemannian metric, and a smooth curve γ lying entirely in M. We want to think of a geodesic as a curve in M that is "as straight as possible". An intuitively plausible way to measure straightness is to compute the Euclidean acceleration $\ddot{\gamma}(t)$ as usual. From this we can derive the *tangential acceleration* of γ. We could then define a geodesic as a curve in M whose tangential aceleration is zero. But the problem with this is: If we wanted to make sense of $\ddot{\gamma}(t)$ by differentiating $\dot{\gamma}(t)$ with respect to t, we would have to write a difference quotient involving vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(t+h)$. However, these tangent vectors live in different tangent spaces and consequently, it doesn't make much sense to just subtract them. To interpret the acceleration of a curve in a manifold in a coordinate invariant way, we need a way to "connect" nearby tangent spaces.

Let $\pi : E \to M$ be a vector bundle over a manifold M, and let $\Sigma(M)$ denote the space of smooth sections of E. A *connection* in E is a map

$$\nabla : \chi(M) \times \Sigma(M) \to \Sigma(M),$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X:

   $$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y,$$

   for $f, g \in C^\infty(M)$.

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

   $$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X aY_2,$$

   for $a, b \in \mathbb{R}$.

3. ∇ satisfies the following product rule:

   $$\nabla_X fY = f\nabla_X Y + X(f)Y,$$

   for $f \in C^\infty(M)$.

$\nabla_X Y$ is called the *covariant derivative of Y in the direction of X*. We can specialize to linear connections in the tangent bundle of a manifold. A *linear connection* on M is a connection in TM, i.e. a map

$$\nabla : \chi(M) \times \chi(M) \to \chi(M),$$

satisfying the properties of a connection. For any choices of indices, we can express the connection in terms of a (local) frame

$$\nabla_{E_i} E_j = \Gamma_{ij}^k.$$

These n^3 functions Γ_{ij}^k are called the *Christoffel symbols* of ∇ with respect to this frame. A connection can be expressed in terms of its Christoffel symbols as follows

$$\nabla_X Y = \left(XY^k + X^i Y^j \Gamma_{ij}^k\right) E_k, \tag{26}$$

for a general frame $\{E_i\}$. In coordinates, this reads

$$\nabla_X Y = \left(X^i \partial_i Y^j + X^i Y^j \Gamma_{ij}^k\right) \partial_k. \tag{27}$$

Example Let ∇ be a linear connection. If ω is a 1-form and X is a vector field, we will show that the coordinate expression for $\nabla_X \omega$ is

$$\nabla_X \omega = \left(X^i \partial_i \omega_k - X^i \omega_j \Gamma_{ij}^k\right) dx^k,$$

where Γ_{ij}^k are the Christoffel symbols of the given connection ∇ on TM.

It can be shown that,

$$\begin{aligned}(\nabla_X F)(\omega^1, ..., \omega^l, Y_1, ..., Y_k) &= X(F(\omega^1, ..., \omega^l, Y_1, ..., Y_k)) \\ &\quad - \sum_{j=1}^{l} F(\omega^1, ..., \nabla_X \omega^j, ..., \omega^l, Y_1, ..., Y_k) \\ &\quad - \sum_{i=1}^{k} F(\omega^1, ..., \omega^l, Y_1, \nabla_X Y_i, ..., Y_k).\end{aligned} \tag{28}$$

In this case we see

$$\begin{aligned}(\nabla_X \omega)(Y) &= X(\omega(Y)) - \omega(\nabla_X Y) \\ &= X(w_k dx^k(Y)) - \omega((X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k)\partial_k) \\ &= \omega_k X(Y^k) + Y^k X(\omega_k) - \omega_k dx^k(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k)\partial_k \\ &= (X^i \partial_i \omega_k - X^i \omega_k \Gamma_{ij}^k) Y^k \\ &= (X^i \partial_i \omega_k - X^i \omega_k \Gamma_{ij}^k) dx^k(Y),\end{aligned}$$

and thus,

$$\nabla_X \omega = (X^i \partial_i \omega_k - X^i \omega_k \Gamma^k_{ij}) dx^k. \tag{29}$$

□

Because the covariant derivative $\nabla_X Y$ of a vector field (or tensor field) Y is linear over $C^\infty(M)$ in X, it can be used to construct another tensor field called the *total covariant derivative*, as follows. If ∇ is a linear connection on M, and $F \in \chi^k_l(M)$, the following map defines a $(k+1, l)$ tensor field.

$$\nabla F(\omega^1, ..., \omega^l, Y^1, ..., Y^k, X) = \nabla_X F(\omega^1, ..., \omega^l, Y^1, ..., Y^k). \tag{30}$$

By linearity, multiplication by a real number leaves the result invariant, and thus this map is unique up to multiplication by a diffeomorphism $f \in C^\infty(M)$.
Example We will show that for any $u \in C^\infty(M)$ and $X, Y \in \chi(M)$,

$$\nabla^2 u(X, Y) = Y(Xu) - (\nabla_Y X)u,$$

where the tensor field ∇F denotes the total covariant derivative of F, where $F \in \chi^k_l(M)$.
The action of ∇u is the same as the 1-form du: $\langle \nabla u, X \rangle = \nabla_X u = Xu = \langle du, X \rangle$. The 2-tensor $\nabla^2 u = \nabla(\nabla u)$ is called the *covariant Hessian of u*. In this case

$$\nabla^2 u(X, Y) = \nabla_Y(\nabla u(X)) = Y(Xu) - \nabla_X u(\nabla Y) = Y(Xu) - (\nabla_Y X)u,$$

where we have used equation (28). □

A *curve* in a manifold means a smooth, parametrized curve; a smooth map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is some interval. A *vector field along a curve* $\gamma : I \to M$ is a smooth map $V : I \to TM$ such that $V(t) = T_{\gamma(t)}M$ for every $t \in I$. Let $\chi(M)$ denote the space of vector fiels along γ. This operator has the (local) expression

$$D_t V(t_0) = \left(\dot{V}^j(t_0)\partial_j + V^j(t_0)\dot{\gamma}^i \Gamma^k_{ij}(\gamma(t_0)) \right) \partial_k. \tag{31}$$

where $V(t) = V^j(t)\partial_j$ is the vector field near $\gamma(t_0)$.

7.1 Geodesics

Let M be a manifold with a linear connection ∇, and let γ be a curve in M. The *acceleration* of γ is the vector field $D_t \dot{\gamma}$ along γ. A curve is said to be a *geodesic* with respect to ∇ if its acceleration is zero: $D_t \dot{\gamma} \equiv 0$. When we write $\gamma(t) = (x^1(t), ..., x^n(t))$, then, by equation (31), a geodesic solves the equation

$$\ddot{x}^k(t) + \dot{x}^i \dot{x}^j \Gamma^k_{ij}(x(t)) = 0 \tag{32}$$

Example
We will show that the geodesics on $\mathbb{R}^n$ with respect to the Euclidian connection

are exactly straight lines with constant speed parametrizations.

The Euclidian connection is given by

$$\bar{\nabla}_X(Y^j\partial_j) = (XY^j)\partial_j,$$

in other words, the Euclidian connection is just the vector field whose components are the ordinary directional derivatives of the components of Y in the direction X. A curve $\gamma(t)$ being a geodesic means $D_t\dot{\gamma}(t) \equiv 0$. By definition

$$D_t\dot{x}(t) = \left(\ddot{x}^k(t) + \dot{x}^j(t)\dot{x}^i(t)\Gamma^k_{ij}(\gamma(t))\right)\partial_k \equiv 0.$$

However, the Christoffel symbols of the Euclidian connection are all zero, and thus $\ddot{x}^k(t) \equiv 0$. From this we see that $\dot{x}^k(t)$ =const and indeed, $\gamma(t)$ represents a straight line with constant speed parametrization. □

7.2 Connecting tangent spaces

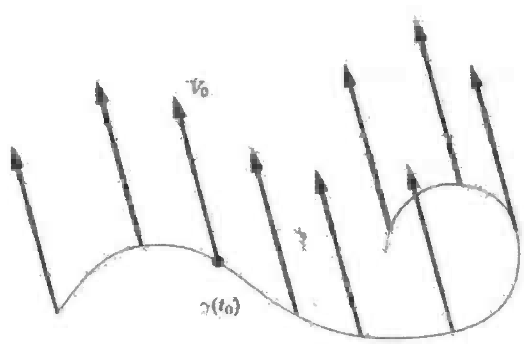
We say that a vector field V is *parallel along* γ (with respect to ∇ if $D_tV = 0$ along γ. In general, given a curve $\gamma : [a, b] \to M$, and a vector $V_0 \in T_pM$, there is a unique vector field V along γ which is parallel along γ. This is because the equations

$$\left(\dot{V}^j(t_0)\partial_j + V^j(t_0)\dot{\gamma}^i\Gamma^k_{ij}(\gamma(t_0))\right)\partial_k$$

are linear differential equations with unique solutions $V^i(t)$, defined on all of $[a, b]$, for given initial conditions. The vector $V(t) \in T_{\gamma(t_1)}M$ is said to be obtained from $V(t) \in T_{\gamma(t_0)}M$ by *parallel translation along* γ. We define a linear transformation

$$\tau_t : T_{\gamma(t_0)}M \to T_{\gamma(t)}M, \quad V_0 \to V_t.$$

Clearly, τ_t is one-to-one, for its inverse is just parallel translation along the reversed portion of the curve γ from t to 0. Thus, parallel translation gives an isomorphism between tangent spaces at different points on M.



Parallel translation τ_t is defined in terms of ∇, but we can also reverse the process.

Suppose γ is a given curve with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. Then

$$\nabla_{X_p} Y = D_t Y = \lim_{h\to 0} \frac{1}{h}\left(\tau_h^{-1} Y_{\gamma(h)} - Y_p\right). \tag{33}$$

We will demonstrate this as follows. Let $V_1, ..., V_1$ be parallel vector fields along γ which are linearly independent at $\gamma(0)$, and hence all points of γ. Set

$$Y(\gamma(t)) = \sum_{i=1}^{n} \lambda^i(t) \cdot V_i(t).$$

Then

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\left(\tau_h^{-1}Y_{\gamma(h)} - Y_p\right) &= \lim_{h\to 0}\frac{1}{h}\left(\sum_{i=1}^{n}\lambda^i(t)\tau_h^{-1}V_i(h) - \lambda^i(0)V_i(0)\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\sum_{i=1}^{n}\lambda^i(t)V_i(0) - \lambda^i(0)V_i(0)\right) \\
&= \sum_{i=1}^{n}\lim_{h\to 0}\frac{\lambda^i(h)-\lambda^i(0)}{h}\cdot V_i(0) \\
&= \sum_{i=1}^{n}\lim_{h\to 0}\frac{d\lambda^i}{dt}(0)\cdot V_i(0) = D_t|_{t=0}\frac{d\lambda^i}{dt}(t)\cdot V_i(t) \\
&= \nabla_{X_p}Y.
\end{aligned}$$

This possibility of comparing, or *connecting*, tangent spaces at different points gives rise to the term "connection". It was invented by Levi-Civita.

7.3 The Torsion Tensor

Let ∇ be a linear connection on M, and define a map $\tau : \chi(M)\times\chi(M) \mapsto \chi(M)$ by

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{34}$$

First we will show that this defines a $(2,1)$ tensor field, called the *torsion tensor* of ∇. For this, we have to verify that $\tau(X,Y)$ is bilinear. Well,

$$\tau(fX_1gX_2, Y) = \nabla_{fX_1gX_2}Y - \nabla_Y(fX_1gX_2) - [fX_1gX_2, Y]$$

$$f\nabla_{X_1}Y + g\nabla_{X_2}Y - f\nabla_Y X_1 - g\nabla_Y X_2 - (Yf)X_1 - (Yg)X_2 - [fX_1, Y] - [gX_2, y],$$

but because the Lie-bracket can be written as $[fX, Y] = f[X,Y] - (Yf)X$, this is equal to

$$= f(\nabla_{X_1}Y - \nabla_Y X_1 - [X_1, Y]) + g(\nabla_{X_2}Y - \nabla_Y X_2 - [X_2, Y]).$$

Indeed, τ is linear in X and by symmetry it is clear that it is also linear in Y. Hence, τ defines a tensor field.

We say ∇ is *symmetric* if its torsion vanishes identically. This condition however is equivalent to the statement that its Christoffel symbols with respect to any coordinate frame are symmetric, $\Gamma^k_{ij} = \Gamma^k_{ji}$, as we will show.
In coordinates, a linear connection has the form

$$\nabla_X Y = (X^i \partial_i Y^k + X^i Y^j \Gamma^k_{ij})\partial_k.$$

Now, first assume that the torsion of τ vanishes identically. In coordinates, the Lie bracket has the form

$$[X,Y] = (X^i \partial_i Y^k - Y^i \partial_i X^k)\partial_k.$$

Hence

$$0 \equiv \nabla_X Y - \nabla_Y X - [X,Y] = [X,Y] - [X,Y] + X^i Y^j \Gamma^k_{ij}\partial_k - X^j Y^i \Gamma^k_{ij}\partial_k.$$

From this we obtain

$$X^i Y^j \Gamma^k_{ij} = X^j Y^i \Gamma^k_{ij},$$

which implies $\Gamma^k_{ij} = \Gamma^k_{ji}$. On the other hand, if the Christoffel symbols are symmetric, by the computation above, it is seen that

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = [X,Y] - [X,Y] \equiv 0,$$

so indeed, τ is symmetric.

Now we will show that ∇ being symmetric is also equivalent to the covariant Hessian being a symmetric 2-tensor field.
Recall that the covariant Hessian $\nabla^2 u$ can be written as

$$\nabla^2 u = Y(Xu) - (\nabla_Y X)u.$$

Now, ∇ symmetric means $\nabla_X Y - \nabla_Y X - [X,Y] \equiv 0$
First assume that the Hessian is a symmetric 2-tensor field. Then

$$\nabla^2 u(X,Y) - \nabla^2 u(Y,X) = 0 = Y(Xu) - X(Yu) - (\nabla_Y X)u + (\nabla_Y X)u = \tau(X,Y).$$

On the other hand, if ∇ is symmetric, then

$$\tau(X,Y)(u) = \nabla_X Y u - \nabla_Y X u - [X,Y]u \equiv 0,$$

hence

$$Y(Xu) - (\nabla_Y X)u = \nabla^2 u(X,Y) = X(Yu) - (\nabla_X Y)u = \nabla^2 u(Y,X),$$

so indeed $\nabla^2 u$ is a symmetric 2-tensor. $\square$

7.4 The Difference Tensor

If ∇^0 and ∇^1 are any two linear connections on M, then

$$A(X,Y) = \nabla^1_X Y - \nabla^0_X Y \tag{35}$$

defines a $(2,1)$ tensor field, called the *difference tensor.* For if $f, g \in C^\infty(M)$ and $X, Y \in \chi(M)$, then

$$A(fX_1+gX_2,Y) = f\nabla^1_{X_1}Y+g\nabla^1_{X_2}-f\nabla^0_{X_1}Y-g\nabla^0_{X_2}Y = fA(X_1,Y)+gA(X_2,Y),$$

by definition of a linear connection. Also

$$A(X, fY_1 + gY_2) = \nabla^1_X fY_1 - \nabla^0_X gY_2,$$

which, by the product rule of connections, equals

$$f\nabla^1_X Y_1 + (Xf)Y_1 + g\nabla^1_X Y_2 + (Xg)Y_2 - f\nabla^0_X Y_1 - (Xf)Y_1 - g\nabla^1_X Y_2 - (Xg)Y_2$$
$$= f(\nabla^1_X Y_1 - \nabla^0_X Y_1) + g(\nabla^1_X Y_2 - \nabla^0_X Y_2) = A(X,Y_1) + A(X,Y_2).$$

Hence, $A(X,Y)$ defines a tensor field.

Next we show that ∇^0 and ∇^1 determine the same geodesics if and only if their difference tensor is antisymmetric, that is $A(X,Y) = -A(Y,X)$.
Assume first that the difference tensor is antisymmetric, it then follows

$$\nabla^1_X Y - \nabla^0_X Y = -\nabla^1_Y X + \nabla^0_Y X,$$

so

$$\nabla^1_X Y + \nabla^1_Y X = +\nabla^0_Y X + \nabla^0_X Y.$$

Now put $X = Y = \dot\gamma(t)$, so then, because $\nabla_{\dot\gamma(t)}\dot\gamma(t) = D_t\dot\gamma(t)$, we see from the equation above that $2D^1_t\dot\gamma(t) = 2D^0_t\dot\gamma(t)$ and we conclude that $D^1_t\dot\gamma(t) \equiv 0$ if and only if $D^0_t\dot\gamma(t) \equiv 0$ and thus ∇^0 and ∇^1 determine the same geodesics.
Now suppose that ∇^0 and ∇^1 do not define the same geodesics, then there exists a curve $\alpha(t)$ such that $D^0_t\dot\alpha(t) \equiv 0$, but $D^1_t\dot\alpha(t) \neq 0$. If ∇^0 and ∇^1 were to be antisymmetric then, in particular $\nabla^1_X X - \nabla^0_X X = -\nabla^1_X X + \nabla^0_X X$ so $\nabla^1_X X = \nabla^0_X X$. But $\nabla_{\dot\alpha(t)}\dot\alpha(t) = D_t\dot\alpha(t)$ and by hypothesis, $D^0_t\dot\alpha(t) \neq D^1_t\dot\alpha(t)$, a contradiction. Hence, if the two connections do not define the same geodesics, then their difference tensor can't be antisymmetric.

Now we will show that ∇^1 and ∇^0 have the same torsion tensor if and only if their difference tensor is symmetric, i.e. $A(X,Y) = A(Y,X)$.
First suppose that ∇^1 and ∇^0 have the same tension tensor, i.e. $\tau^1(X,Y) = \tau^0(X,Y)$. From this we see,

$$\nabla^1_X Y - \nabla^1_Y X - [X,Y] = \nabla^0_X Y - \nabla^0_Y X - [X,Y],$$

hence

$$\nabla^1_X Y - \nabla^0_X Y = \nabla^1_Y X - \nabla^0_Y X,$$

and thus $A(X,Y) = A(Y,X)$.
Now suppose $A(X,Y) = A(Y,X)$, from this it follows that, going in the opposite direction as above we arrive at $\tau^1 = \tau^0$. □

7.5 Cartan's First Structure Equation

Let ∇ be a linear connection on M, let E_i be a local frame on some open subset $U \subset M$, and let ϕ^i be the dual coframe.
We can write $\nabla_X E_i = \nabla_{X^k E_k} E_i = X^k \nabla_{E_k} E_i = X^k \Gamma^j_{ik} E_j$, so we can define $\omega_i^j = \Gamma^j_{ik}\phi^k$, so that $\nabla_X E_i = \omega_i^j(X) E_j$.

For every smooth 1-form α and smooth vector fields X and Y, we have the following identity

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]). \tag{36}$$

Now let E_i be a local frame and (ϕ^i) the dual coframe, as above. Let c^j_{ik} be the component functions of the Lie bracket $[E_i, E_k]$ in this frame:

$$[E_i, E_k] = c^j_{ik} E_j.$$

Then applying ϕ^j to (36) we get:

$$d\phi^j(E_i, E_k) = E_i(\phi^j(E_k)) - E_k(\phi^j(E_i)) - \phi^j([E_i, E_k]) = 0 + 0 - c^j_{ik}.$$

On the other hand

$$c^j_{ik}\phi^i \wedge \phi^k(E_i, E_k) = c^j_{ik},$$

and thus

$$d\phi^j = -c^j_{ik}\phi^i \wedge \phi^k. \tag{37}$$

Now,

$$(\phi^i \wedge \omega_i^j)(X,Y) = X^i\omega_i^j(Y) - Y^i\omega_i^j(X).$$

Furthermore

$$\nabla_X Y = \left(Y^i\omega_i^j(X) + XY^j\right)E_j,$$

$$(\nabla_X Y)^j = Y^i\omega_i^j(X) + XY^j.$$

The torsion tensor was defined to be $\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.
Hence we get

$$(\phi^i \wedge \omega_i^j)(X,Y) + \tau^j(X,Y) = XY^j - YX^j - [X,Y]^j.$$

Now take $X = E_j$ and $Y = E_k$, then

$$\phi^i \wedge \omega_i^j(E_i, E_k) + \tau^j(E_i, E_k) = -[E_i, E_k]^j = -c^j_{ik}.$$

But also, because $d\phi^j = -c^j_{ik}\phi^j \wedge \phi^k$, we see that

$$d\phi^j(E_i, E_k) = -c^j_{ik}\phi^j \wedge \phi^k(E_i, E_k) = -c^j_{ik} = \phi^i \wedge \omega_i^j(E_i, E_k) + \tau^j(E_i, E_k),$$

from which we obtain the equation

$$d\phi^j = \phi^i \wedge \omega_i^j + \tau^j. \tag{38}$$

This formula is known as *Cartan's First Structure Equation*, due to E. Cartan.
□

8 Riemannian Geodesics

On each Riemannian manifold there is a natural connection that is particularly suited to computations in Riemannian geometry. The Euclidean connection has one very nice property with respect to the Euclidean metric: it satisfies the product rule

$$\bar{\nabla}_X \langle Y, Z\rangle = \langle \bar{\nabla}_X Y, Z\rangle + \langle Y, \bar{\nabla}_X Z\rangle.$$

This can be generalized. Let g be a Riemannian metric on an manifold M. A linear connection ∇ is said to be *compatible with* g if it satisfies the following product rule for all vector fields X, Y, Z, i.e.

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Proposition *The following conditions are equivalent for a linear connection ∇ on a Riemannian manifold.*

1. *∇ is compatible with g*

2. $\nabla g \equiv 0$

3. *If $V, W)$ are vector fields along any curve γ,*

$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle.$$

4. *If V, W are parallel vector fields along a curve γ, then $\langle V, W\rangle$ is constant*

5. *Parallel translation $\tau_{t_0 t_1} : T_{\gamma(t_1)}M$ is an isometry for each t_0, t_1.*

Proof (1) $\Rightarrow$ (2) Suppose ∇ is compatible with g, then

$$\nabla g(Y, Z, X) = \nabla_X g(Y, Z) = X\langle Y, Z\rangle - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle \equiv 0,$$

hence $\nabla g \equiv 0$.
(2) $\Rightarrow$ (3) Suppose $\nabla g \equiv 0$. Noting that, in general, $\nabla_X f = X(f)$ we obtain,

$$D_t\langle V, W\rangle = \nabla_{\dot{\gamma}(t)}\langle V, W\rangle = \dot{\gamma}(t)(\langle V, W\rangle) = \frac{d}{dt}\langle V, W\rangle,$$

because V, W are taken along $\gamma(t)$, i.e. $V = V(\gamma(t))$ and $W = W(\gamma(t))$. Plugging this into the equation given in (2) immedatiately gives

$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle$$

(3) $\Rightarrow$ (4) If V, W are parallel vector fields along a curve $\gamma(t)$ then, by definition $D_t V \equiv 0$ and $D_t W \equiv 0$. Then we see that $\frac{d}{dt}\langle V, W\rangle \equiv 0$, hence $\langle V, W\rangle$ =const.
(4) $\Rightarrow$ (5) Parallel translating two vectors in space leaves the angle between them unchanged. Hence $\cos\theta = \frac{\langle V,W\rangle}{|X||Y|}$ =const. But, we saw that if V, W parallel vector fields, then $\langle V, W\rangle$ =const, so $|X||Y|$ =const . This means that the norms of the vectors are left unchanged under the parallel translation mapping and thus is an isometry.
(5) $\Rightarrow$ (1) Suppose parallel translation $\tau_{t_0 t_1} : T_{\gamma(t_1)}M$ is an isometry for each t_0, t_1. Then, because τ is norm-preserving, parallel vector fields being orthonormal at one point, are orthonormal at every point along γ. Choose parallel vector fields $P_1, ..., P_n$ along γ which are orthonormal. Let

$$V(t) = \sum_{i=1}^{n} v^i(t) P_i(t), \qquad W(t) = \sum_{i=1}^{n} w^j(t) P_j(t).$$

Then

$$\langle V, W\rangle = \sum_{i=1}^{n} v^i \cdot w^i,$$

and remembering that $D_t P_i = 0$ we have

$$D_t V = \sum_{i=1}^{n} \frac{dv^i}{dt} P_i, \quad D_t W = \sum_{i=1}^{n} \frac{dw^j}{dt} P_j.$$

So

$$\langle D_t V, W\rangle + \langle V, D_t W\rangle = \sum_{i=1}^{n} \left(\dot{v}^i w^j + v^j \dot{w}^i\right) = \frac{d}{dt}\langle V, W\rangle$$

When we apply this to a curve γ with $\dot\gamma(0) = X_p$, and note that $D_t V = \nabla_{\dot\gamma(t)} Y(\gamma(t))$, we obtain

$$X_p\langle Y, Z\rangle = \langle \nabla_{X_p} Y, Z_p\rangle + \langle \nabla_{X_p}, Z_p\rangle. \tag{39}$$

Since (39) holds for every $p \in M$ and every $X_p \in T_p M$ we conclude that ∇ is compatible with g. $\square$

Without a proof we state the following important theorem.

Theorem (Fundamental Theorem of Riemannian Geometry)
Let (M, g) be a Riemannian manifold. There exists a unique linear connection ∇ or M that is compatible with g and symmetric.

This connection is called the *Riemannian connection* of *Levi-Civita connection* of g. There is an explicit formula for computing the Christoffel symbols of the Riemannian connection in any coordinate chart

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{40}$$

Geodesics with respect to this connection are called *Riemannian geodesics*, or simply *geodesics*. If γ is a curve in a Riemannian manifold, the *speed* of γ at any time t is the length of its velocity vector $|\dot{\gamma}(t)|$. We say γ is constant speed if $|\dot{\gamma}(t)|$ is independent of t, and unit speed if the speed is identically equal to 1.

Example
Consider the manifold $M = \mathbb{R}^2$. Suppose that $|\text{grad} f| \equiv 1$. Then $\dot{x}^2(t)+\dot{y}^2(t) \equiv 1$, where $\dot{x} = \frac{\partial f}{\partial x}$ and $\dot{y} = \frac{\partial f}{\partial y}$. We will show that the integral curves are geodesics, that is straight lines. Differentiating $\dot{x}^2(t) + \dot{y}^2(t) \equiv 1$ with respect to time we obtain $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$. Furthermore

$$\partial_x \dot{x}^2 = \partial_x (\partial_x f)^2 = 2\partial_x f \cdot \partial^2_{xx} f$$

$$\partial_x \dot{y}^2 = \partial_x (\partial_y f)^2 = 2\partial_y f \cdot \partial^2_{xy} f.$$

Hence

$$0 = \frac{1}{2}\partial_x\{\dot{x}^2(t) + \dot{y}^2(t)\} = \partial_x f \cdot \partial^2_{xx} f + \partial_y f \cdot \partial^2_{xy} f \tag{41}$$

and similarly

$$0 = \partial_y f \cdot \partial^2_{yy} f + \partial_x f \cdot \partial^2_{yx} f. \tag{42}$$

Multiplying (41) by $\partial_x f$ and (42) by $\partial_y f$ and using that $\partial^2_{xy} f = \partial^2_{yx} f$, we obtain, upon subtracting the two equations

$$(\partial_x f)^2 \partial^2_{xx} f = (\partial_y f)^2 \partial^2_{yy} f. \tag{43}$$

Also, $\ddot{x}(t) = \partial^2_{xx} f \partial_x f + \partial^2_{xy} f \partial_y f$ and $\ddot{y}(t) = \partial^2_{yy} f \partial_y f + \partial^2_{xy} f \partial_x f$. After multiplying $\ddot{x}(t)$ with $\dot{x}(t) = \partial_x f$ and $\ddot{y}(t)$ with $\dot{y}(t) = \partial_y f$, using equation (43) and subtracting this yields $\dot{x}\ddot{x} - \dot{y}\ddot{y} = 0$. This together with our first equation, $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$, gives that the integral curves are straight lines, which we know to be geodesics on $\mathbb{R}^2$.□

This result can be generalized as the next theorem shows.

Theorem *Let (M, g) be a Riemannian manifold and f a smooth function on M such that $|\text{grad} f| \equiv 1$. Then the integral curves of $\text{grad} f$ are geodesics.*

Proof
By definition of the gradient, $\text{grad} f = g^{ij}\partial_i f \partial_j$ and by the fact that $g_{ij}g^{ki} = \delta^k_j$, we obtain

$$1 \equiv |\text{grad } f|^2 = \langle \text{grad } f, \text{grad } f \rangle = g_{ij}(g^{ki}\partial_k f)(g^{lj}\partial_l f) = g^{lk}\partial_k f \partial_l f.$$

Hence

$$1 \equiv g^{lk}\partial_k f \partial_l f. \tag{44}$$

Suppose $\gamma(t) = (x^1(t), ..., x^n(t))$ is the integral curve of $\text{grad} f$, this means $\text{grad} f(\gamma(t)) = \dot{\gamma}(t)$, or in coordinates,

$$(\dot{x}^1(t), ..., \dot{x}^n(t)) = (g^{i1}\partial_i f, ..., g^{in}\partial_i f),$$

i.e. $\dot{x}^j(t) = g^{ij}\partial_i f$ and thus we obtain

$$1 \equiv g_{ij}\dot{x}^i(t)\dot{x}^j(t), \tag{45}$$

which reflects the fact exactly that the solution curve is unit speed, as it should be. Multiplying (44) by g_{il} we obtain

$$g_{il} = \partial_i f \partial_l f. \tag{46}$$

Equation (40) expresses the Christoffel symbols in terms of the partial derivatives of g. Taking the partial derivatives of (46) and using the fact that the second partial derivatives commute, it is seen that,

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \\
&= \frac{1}{2}g^{kl}\left(\partial_i(\partial_j f \partial_l f) + \partial_j(\partial_i f \partial_l f) - \partial_l(\partial_i f \partial_j f)\right) \\
&= \frac{1}{2}g^{kl}\left(2\partial_l f \partial^2_{ij} f\right) = g^{kl}\partial_l f \partial^2_{ij} f,
\end{aligned}$$

and thus

$$\Gamma^k_{ij} = g^{kl}\partial_l f \partial^2_{ij} f. \tag{47}$$

We compute

$$\begin{aligned}
\ddot{x}^k(t) &= \frac{d}{dt}\dot{x}^k(t) = \frac{d}{dt}g^{tk}\partial_t f \\
&= g^{t\lambda}\left\{\partial_\lambda g^{pk}\partial_t f \partial_p f + g^{pk}\partial_t f \partial^2_{\lambda p} f\right\}.
\end{aligned}$$

And furthermore

$$\dot{x}^t(t)\dot{x}^\lambda(t)\Gamma^k_{t\lambda} = g^{rt}\partial_r f g^{s\lambda}\partial_s f g^{kp}\partial_p f \partial^2_{t\lambda} f.$$

By equation (46), $\partial_r f \partial_s f = g_{rs}$ and $g_{rs}g^{rt} = \delta^t_s$, so

$$\dot{x}^t(t)\dot{x}^\lambda(t)\Gamma^k_{t\lambda} = g^{t\lambda}g^{kp}\partial_p f \partial^2_{t\lambda} f = g^{t\lambda}g^{pk}\partial_p f \partial^2_{t\lambda} f,$$

where in the last equality we used that, $g^{kp} = g^{pk}$, due to the fact that every Riemannian metric is symmetric.
These results enable us to obtain our main result, as follows,

$$\begin{aligned}
\ddot{x}^k(t) + \dot{x}^t(t)\dot{x}^\lambda(t)\Gamma^k_{t\lambda} &= g^{t\lambda}\left\{\partial_\lambda g^{pk}\partial_t f \partial_p f + g^{pk}\partial_t f \partial^2_{\lambda p}\right\} + g^{t\lambda}g^{pk}\partial_p f \partial^2_{t\lambda} f \\
&= g^{t\lambda}\left\{\partial_\lambda g^{pk}\partial_t f \partial_p f + g^{pk}\partial_t f \partial^2_{\lambda p} + g^{pk}\partial_p f \partial^2_{t\lambda} f\right\} \\
&= g^{t\lambda}\left\{\partial_\lambda\left(g^{pk}\partial_t f \partial_p f\right)\right\} \\
&= g^{t\lambda}\left\{\partial_\lambda\left(g^{pk}g_{tp}\right)\right\} = g^{t\lambda}\partial_\lambda \delta^k_t = 0.
\end{aligned}$$

The solution curves satisfy the geodesic equation and, consequently, define geodesics.□

Proposition (Naturality of the Riemannian Connection)

Suppose $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ is an isometry.

1. *ϕ takes the Riemannian connection ∇ of g to the Riemannian $\tilde{\nabla}$ of $\tilde{g}$, in the sense that*
$$\phi_*(\nabla_X Y) = \tilde{\nabla}_{\phi_* X}(\phi_* Y).$$

2. *If γ is a curve in M and V is a vector field along γ, then*
$$\phi_* D_t V = \tilde{D}_t(\phi_* V).$$

3. *ϕ takes geodesics to geodesics: if γ is the geodesic in M with initial point p and initial velocitiy V, then $\phi \circ \gamma$ is the geodesic in $\tilde{M}$ with initial point $\phi(p)$ and initial velocity $\phi_* V$.*

Proof

(1) For this, we will show that the operator $\phi^*\tilde{\nabla} : \chi(M) \times \chi(M) \to \chi(M)$, defined by
$$(\phi^*\tilde{\nabla})_X Y = \phi_*^{-1}(\tilde{\nabla}_{\phi_* X}(\phi_* Y)),$$
is a connection on M (called the *pullback connection*) and that it is symmetric and compatible with g. Then, because
$$(\phi^*\tilde{\nabla})_X Y = \phi_*^{-1}(\tilde{\nabla}_{\phi_* X}(\phi_* Y)) = \phi_*^{-1}(\phi_*(\nabla_X Y)) = \nabla_X Y,$$
we have that $\phi^*\tilde{\nabla} = \nabla$, by uniqueness of the Riemannian connection.

First let us recall the definition of a pushforward of a vector field, $F_* X(f) = X(f \circ F)$, and we see that for $h : M \to \mathbb{R}$, $F_*(hX)(f) = (hX)(f \circ F) = h \cdot X(f \circ F) = h \cdot F_* X(f)$. Using this we obtain
$$\begin{aligned}
(\phi^*\tilde{\nabla})_{fX_1+gX_2} Y &= \phi_*^{-1}(\tilde{\nabla}_{\phi_*(fX_1+gX_2)}(\phi_* Y)) \\
&= \phi_*^{-1}(\tilde{\nabla}_{f\cdot\phi_* X_1 + g\cdot\phi_* X_2}(\phi_* Y)) \\
&= \phi_*^{-1}\left\{f \cdot (\tilde{\nabla}_{\phi_* X_1}(\phi_* Y)) + g \cdot (\tilde{\nabla}_{\phi_* X_2}(\phi_* Y))\right\} \\
&= f \cdot \phi_*^{-1}(\tilde{\nabla}_{\phi_* X_1}(\phi_* Y)) + g \cdot \phi_*^{-1}(\tilde{\nabla}_{\phi_* X_2}(\phi_* Y)) \\
&= f \cdot (\phi^*\tilde{\nabla})_{X_1} Y + g \cdot (\phi^*\tilde{\nabla})_{X_2} Y,
\end{aligned}$$
so the operator satisfies the first property of a connection. The second one, linearity over $\mathbb{R}$ is immediately clear by linearity of differentiation over $\mathbb{R}$. The product rule is proven as follows:
$$\begin{aligned}
(\phi^*\tilde{\nabla})_X(fY) &= \phi_*^{-1}(\tilde{\nabla}_{\phi_* X}(\phi_* fY)) \\
&= \phi_*^{-1}(\tilde{\nabla}_{\phi_* X} f(\phi_* Y)) \\
&= \phi_*^{-1}\left\{f \cdot (\tilde{\nabla}_{\phi_* X}\phi_* Y) + \phi_* X(f) \cdot \phi_* Y\right\} \\
&= f \cdot \phi_*^{-1}(\tilde{\nabla}_{\phi_* X}(\phi_* Y)) + \phi_*^{-1}\{\phi_* X(f) \cdot \phi_* Y\} \\
&= f \cdot (\phi^*\tilde{\nabla})_X(Y) + \phi_*^{-1}\{\phi_*(X(f) \cdot Y)\} \\
&= f \cdot (\phi^*\tilde{\nabla})_X(Y) + X(f)Y.
\end{aligned}$$

Hence the operator also satisfies the product rule and we conclude that it defines a connection. Next we will show that the connection is compatible with g, or equivalently $\nabla g \equiv 0$. Suppose $\tilde{\nabla}$ is compatible with $\tilde{g}$, then $\tilde{\nabla}\tilde{g} \equiv 0$. By hypothesis, ϕ is an isometry. This implies $\phi^*\tilde{g}(X,Y) = \tilde{g}(\phi_*X, \phi_*Y) = g(X,Y)$. And if $\tilde{\nabla}\tilde{g} \equiv 0$ then $\phi_*^{-1}(\tilde{\nabla}\tilde{g}) \equiv 0$. This yields

$$0 \equiv \phi_*^{-1}\left(\tilde{\nabla}_{\phi_*X}\tilde{g}(\phi_*Y, \phi_*Z)\right) = (\phi^*\tilde{\nabla})g(X,Y,Z).$$

Hence g is compatible with $\phi^*\tilde{\nabla}$. Furthermore, if $\tilde{\nabla}$ is symmetric, then by definition $\tilde{\nabla}_V W - \tilde{\nabla}_W V = [V,W]$. We need to show that $\phi^*(\tilde{\nabla})$ is also symmetric. Note that,

$$\begin{aligned}
(\phi^*\tilde{\nabla})_X Y - (\phi^*\tilde{\nabla})_Y X &= \phi_*^{-1}\left(\tilde{\nabla}_{\phi_*X}\phi_*Y\right) - \phi_*^{-1}\left(\tilde{\nabla}_{\phi_*Y}\phi_*X\right) \\
&= \phi_*^{-1}\left(\tilde{\nabla}_{\phi_*X}\phi_*Y - \tilde{\nabla}_{\phi_*Y}\phi_*X\right) \\
&= \phi_*^{-1}\left([\phi_*X, \phi_*Y]\right)
\end{aligned}$$

So we are finished if we can show that $[\phi_*X, \phi_*Y] = \phi_*[X,Y]$. Well

$$XY(f \circ \phi) = X(\phi_*Y(f) \circ \phi) = \phi_*X\phi_*Y(f) \circ \phi$$

Interchanging X and Y and subtracting, we obtain

$$[\phi_*X, \phi_*Y]f = ([X,Y]f) \circ \phi = \phi_*[X,Y]f$$

and we can conclude that, with $\tilde{\nabla}$ symmetric, $\phi^*\tilde{\nabla}$ is also symmetric, and thus $\phi^*\tilde{\nabla}$ defines a Riemannian connection. In summary, if $\tilde{\nabla}$ is a Riemannian connection on $\tilde{M}$, then $\phi^*\tilde{\nabla}$ is also a Riemannian connection on M. Conversely, ϕ takes the Riemannian connection on M (∇) to the Riemannian connection on $\tilde{M}$ ($\tilde{\nabla}$), in the sense that $\phi^*\tilde{\nabla} = \nabla$ and the pushforward of this connection is the Riemannian connection $\tilde{\nabla}$ on $\tilde{M}$.

(2) Define the operator $\phi^*\tilde{D}_t : \chi(M) \to \chi(M)$ by

$$(\phi^*\tilde{D}_t)(V) = \phi_*^{-1}\left(\tilde{D}_t(\phi_*V)\right).$$

Then, because $D_tV = \nabla_{\dot{\gamma}(t)}V$ and (1), we see that this operator is a connection on M that is symmetric and compatible with g. Therefore, by uniqueness of the Riemannian connection, $\phi_*D_tV = \tilde{D}_t(\phi_*V)$.

(3) γ being a geodesic means that $D_t\dot{\gamma} \equiv 0$. Well, by the formula in (2), $\phi_*D_t\dot{\gamma} = \tilde{D}_t(\phi_*\dot{\gamma})$, we see that if $D_t\dot{\gamma} \equiv 0$, then $\tilde{D}_t(\phi_*\dot{\gamma}) = \tilde{D}_t(\frac{d}{dt}\phi \circ \gamma) \equiv 0$. This implies that if γ is the geodesic in M with initial point p and initial velocity $V = \dot{\gamma}(0)$, then $\phi \circ \gamma$ is the geodesic in $\tilde{M}$ with initial point $\phi(p)$ and initial velocity ϕ_*V.$\square$

Exponential Map

Any initial point $p \in M$ and any initial vector $V \in T_pM$ determine a unique maximal geodesic γ_V. This implicitly defines a map from the tangent bundle to the set of geodesics in M. More importantly, it allows us to define a map from (a subset of) the tangent bundle to M itself, by sending the vector V to the point obtained by following γ_V for time 1.

To be precise, define a subset Σ of TM, the domain of the *exponential map* by the set of curves that are defined on an interval containing $[0,1]$ and define the *exponential map* $\exp : \Sigma \to M$ by

$$\exp(V) = \gamma_V(1).$$

To see that this is a smooth map, consider the fact that γ_V is a solution to the set of differential equations

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0,$$

with initial conditions $\gamma_V(0) = p$ and $\dot{\gamma}(0) = V$. It is known from the theory of ordinary differential equations that, with γ being smooth, the set of differential equations has a solution that depends smoothly on the intial data. In particular, the solution γ depends smoothly on $\dot{\gamma}(t) = V$ Hence, the mapping $\exp : \Sigma \to M$ is a smooth mapping.

The naturality of the Riemannian connection and uniqueness of geodesics translate into the following important naturality property of the exponential map:

Proposition (Naturality of the Exponential Map)
Suppose that $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ is an isometry. Then, for any $p \in M$, the following diagram commutes:

$$\begin{tikzcd}
T_pM \arrow[r, "\phi_*"] \arrow[d, "\exp_p"'] & T_{\phi(p)}\tilde{M} \arrow[d, "\exp_{\phi(p)}"] \\
M \arrow[r, "\phi"'] & \tilde{M}
\end{tikzcd}$$

Proof
By hypothesis, ϕ is an isometry so we can use the naturality of the Riemannian connection. So on the one hand we have $\phi(\exp_p V) = \phi(\exp_p \dot{\gamma}(0)) = \phi(\gamma_{\dot{\gamma}(0)}(1)) = (\phi \circ \gamma)_{\phi_*\dot{\gamma}(0)}(1)$. By the previous proposition we know that this defines a geodesic with starting point $\phi(\gamma(0)) = \phi(p)$ and initial vector $\phi_* V$ in $\tilde{M}$. Going around the other way we have $\exp_{\phi(p)}(\phi_* V)$, which by the definition

also defines a geodesic with starting point $\phi(p)$ and initial vector $\phi_* V$ in $\tilde{M}$. By uniqueness, we conclude that these geodesics are the same and hence the diagram commutes. □.

8.1 The connection matrix

Let ∇ be a a linear connection on a Riemannian manifold (M, g). Then ∇ is compatible with g if and only if the connection 1-forms ω_i^j with respect to any local frame $\{E_j\}$ satisfy

$$g_{jk}\omega_i^k + g_{ik}\omega_j^k = dg_{ij}.$$

A linear connection ∇ is called *compatible with* g if it satisfies the following product rule for all vector fields X, Y, Z

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle .$$

Unfolding the definitions, we see

$$\begin{aligned}\nabla_X \langle Y, Z\rangle &= \nabla_X \left(g_{ij} Y^i Z^j\right) = X\left(g_{ij} Y^i Z^j\right)\\ &= Y^i Z^j X(g_{ij}) + g_{ij} X(Y^i Z^j).\end{aligned}$$

On the other hand,

$$\langle \nabla_X Y, Z\rangle = g_{ij} X Y^i Z^j + g_{ij}\omega_k^i(X) Y^k Z^j,$$

and similarly

$$\langle Y, \nabla_X Z\rangle = g_{ij} Y^i X Z^j + g_{kj} Y^k Z^i \omega_i^j(X).$$

After renaming some of the dummy indices, we get, using that the connection satisfies the product rule and cancelling of some terms,

$$Y^i Z^j X g_{ij} = g_{ik} Y^i Z^j \omega_j^k(X) + g_{kj} Y^i Z^j \omega_i^k(X),$$

from which we obtain

$$X g_{ij} = dg_{ij}(X) = g_{ik}\omega_j^k(X) + g_{kj}\omega_i^k(X),$$

hence

$$dg_{ij} = g_{ik}\omega_j^k + g_{kj}\omega_i^k.$$

Reading this proof bottom upwards yields the proof in the other direction.

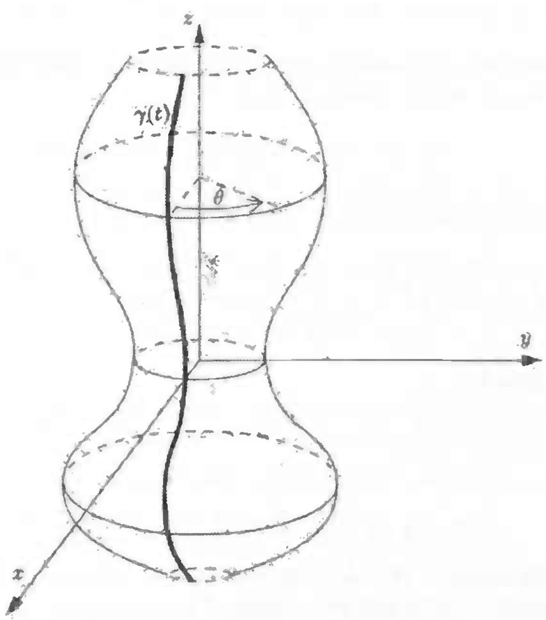
We conclude that the matrix ω_i^j of connection 1-forms for the Riemannian connection with respect to any local orthonormal frame is skewsymmetric. □

8.2 Surface of revolution

Let $M \subset \mathbb{R}^3$ be a surface of revolution, parametrized by

$$\phi(t) = (a(t)\cos\theta, a(t)\sin\theta, b(t)),$$

where $\gamma(t) = (a(t), b(t))$, $t \in I$ is a smooth injective curve in the xz-plane and suppose furthermore that $a(t) > 0$ and $\dot{\gamma}(t) \neq 0$ for all $t \in I$. Furthermore, we assume the γ to be unit speed. The *surface of revolution* is obtained by revolving the image of γ about the z-axis.



The induced metric is the Euclidean metric in $\mathbb{R}^3$, restricted to vectors tangent to the surface, i.e.

$$g = \bar{g}|_M = (dx)^2 + (dy)^2 + (dz)^2 = (d(a(t)\cos\theta))^2 + (d(a(t)\sin\theta))^2 + (d(b(t)))^2,$$

$$= (\dot{a}(t)\cos\theta dt - a(t)\sin\theta d\theta)^2 + (\dot{a}^2(t)\sin\theta dt + a^2(t)\cos\theta d\theta)^2 + (\dot{b}^2(t)dt)^2,$$

which gives

$$g = (\dot{a}^2(t) + \dot{b}^2(t))dt^2 + a^2(t)d\theta^2 = dt^2 + a^2(t)d\theta^2,$$

because γ is unit speed. So in (θ, t) coordinates, the matrix g_{ij} is given by $g_{ij} = \begin{pmatrix} a^2(t) & 0 \\ 0 & 1 \end{pmatrix}$ with inverse $g^{ij} = \begin{pmatrix} 1/a^2(t) & 0 \\ 0 & 1 \end{pmatrix}$.

Computing the Christoffel symbols gives $\Gamma^2_{11} = \dot{a}(t)a(t)$ and $\Gamma^1_{21} = \frac{\dot{a}(t)}{a(t)}$, all the others are zero.

The geodesics are the solutions to this set of ordinary differential equations,

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0.$$

Now, setting $(x^1, x^2) = (\theta_0, t)$, this curve satisfies the geodesic equation and hence is a geodesic. These curves are called "meridians" on M.

We can also determine when a "latitude circle" $\{t = t_0\}$ is a geodesic. For this, we set $\gamma(\theta) = (\theta, t_0)$. Applying this to the geodesic equation gives the condition $\dot{a}(t_0) = 0$. For instance, in the case of the unit sphere (as an example of a surface of revolution), we have $(x, y, z) = (\cos t \cos\theta, \cos t \sin\theta, \sin t)$ so $a(t) = \cos t$ and thus $\dot{a}(t) = -\sin t = 0$ gives $t = 0$, and this is seen to be the great circle in the xy-plane, consistent with the fact that great circles are geodesics on the sphere. □

An orthonormal basis $\{E_i\}$ for T_pM gives an isomorphism $E : \mathbb{R}^n \to T_pM$ by $E(x^1, ..., x^n) = x^iE_i$. If U is a normal neighborhood of p, we can combine this isomorphism with the exponential map to get a coordinate chart

$$\phi := E^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n.$$

Any such coordinates are called *(Riemannian) normal coordinates* centered at p. It can be shown, futhermore, that $(\exp_p)_* = Id$. In any normal coordinate chart centered at p, define the *radial distance function* r by

$$r(x) = \left(\sum_i (x^i)^2\right)^{1/2},$$

and the unit radial vector field by

$$\frac{\partial}{\partial r} := \frac{x^i}{r}\frac{\partial}{\partial x^i}.$$

In Euclidean space, $r(x)$ is the distance to the origin, and $\frac{\partial}{\partial r}$ is the unit vector field tangent to straight lines through the origin.
If $\epsilon > 0$ is such that $\exp_p$ is a diffeomorphism on the ball $B_\epsilon(0) \subset T_pM$ (where the radius of the norm is measured with respect to the norm defined by g), then the image set $\exp(B_\epsilon(0))$ is called a *geodesic ball* in M.

Proposition (Properties of Normal Coordinates)

Let $(U, (x^i))$ be any coordinate chart centered at p.

1. *For any $V = V^i\partial_i \in T_pM$, the geodesic γ_V starting at p with initial velocity vector V is represented in normal coordinates by the radial line segment*
$$\gamma_V(t) = (tV^1, ..., tV^n).$$
2. *The coordinates of p are $(0,...,0)$.*
3. *The components of the metric at p are $g_{ij} = \delta_{ij}$.*
4. *Any Euclidean ball $\{x|r(x) \leq \epsilon\}$ contained in U is a geodesic ball in M.*
5. *At any point $q \in U - p$, $\frac{\partial}{\partial r}$ is the velocity vector of the unit speed geodesic from p to q, and therefore has unit length with respect to g.*
6. *The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.*

Proof

(1) Because $(\exp_p)_* = Id$, we have that $\partial_i = (\exp_p)_* E_i = E_i \in T_pM$. By definition, $\exp_p(V) = \gamma_V(1)$, where $V = \dot\gamma(0)$ and $\gamma_V(t) = \exp(tV)$. It follows

$$\phi(\gamma_V(t)) = E^{-1} \circ \exp_p^{-1}\left(\exp_p(tV)\right) = E^{-1}\left(tV^i\frac{\partial}{\partial x^i}\right),$$

but also

$$E^{-1}\left(E(x^1, ..., x^n)\right) = E^{-1}(x^i E_i) = E^{-1}\left(x^i\frac{\partial}{\partial x^i}\right).$$

And thus $tV^i = x^i(t)$, and we have that $\gamma_V(t) = \left(tV^1, ..., tV^n\right)$
(2) Because $\gamma_V(0) = p$, we have that $p = (0, ..., 0)$.
(3) Because at p we have that $\partial_i = E_i$, so $g_{ij} = \langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle = \delta_{ij}$.
(4) U is a normal coordinate chart centered at p and the radial distance between p and any point in the chart is given by $r(x)$. The Euclidean ball $\{x\ |r(x) < \epsilon\}$, $(B_\epsilon(p))$, contained in U is the image set of $\exp_p(B_{\epsilon'}(0))$, where $\exp_p(B_{\epsilon'}(0)) \in T_pM$ and $|\epsilon'|_g = \epsilon$, i.e. $B_\epsilon(p)$ is a geodesic ball in M.
(5) By definition $\frac{\partial}{\partial r}$ is the unit vector field tangent to straight lines through the origin. Since the geodesics through p (the origin in normal coordinates) are straight lines, $\frac{\partial}{\partial r}$ is everywhere tangent to the unit speed geodesics through p.
(6) The components of γ_V are given by $x^i(t) = tV^i$. In general $V^i \neq 0$. Because γ_V, is a geodesic, it solves the geodesic equation, i.e.

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0.$$

But $x^i(t) = tV^i$ and thus $\ddot{x}^i(t) = 0$ for every i, and in general $\dot{x}^i(t) \neq 0$, we have that $\Gamma^k_{ij} \equiv 0$ at p. Furthermore, it can be shown that the Christoffel symbols and the g_{ij} satisfy the relation $\Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{il} = \partial_k g_{ij}$. From this we see, because the Christoffel symbols are all zero at p, that the partial derivates of g_{ij} vanish at p. $\square$

9 Geodesics and Distance

9.1 Lengths of Curves

If $\gamma : [a,b] \to M$ is a curve segment, we define the *length* of γ to be

$$L(\gamma) := \int_a^b |\dot{\gamma}| dt, \tag{48}$$

where $|\cdot|$ means the norm with respect to the metric g. The key feature of the length of a curve is that it is independent of the parametrization. A *reparametrization* of γ is of the form $\tilde{\gamma} = \gamma \circ \phi$, where $\phi : [c,d] \to [a,b]$ is a smooth map with smooth inverse. First

$$|\tilde{\gamma}'(t)| = |(\gamma \circ \phi)'(t)| = |\gamma'(s)|\dot{\phi}(t),$$

where we denote $s(t) = \phi(t)$ and so $ds = \dot{\phi}(t)dt$. Now it is seen that

$$L(\tilde{\gamma}) = \int_c^d |\tilde{\gamma}'(t)|dt = \int_c^d |(\gamma\circ\phi)'(t)|dt = \int_c^d |\gamma'(s)|\dot{\phi}(t)dt = \int_a^b |\tilde{\gamma}'(s)|ds = L(\gamma).$$

We have assumed that the reparametrization is orientation-preserving, in the case the orientation is reversed, one would have to add a minus sign on the right hand side.
A continuous map $\gamma : [a,b] \to M$ is called a *piecewise regular curve segment* if there exists a finite subdivision $a = a_0 \leq a_1 \leq ... \leq a_k = b$ such that $\gamma|_{[a_{i-1},a_i]}$ is a regular curve for $i = 1, ..., k$. A piecewise regular curve segment is also called *admissible*. If $\gamma : [a,b] \to M$ is a regular curve, and $f \in C^\infty[a,b]$, we define the *integral of f with respect to arc length* by,

$$\int_\gamma f ds = \int_a^b f(t)|\dot{\gamma}(t)|dt. \tag{49}$$

As in the previous case, this integral is independent of its parametrization. Suppose $\phi : [c,d] \to [a,b]$. Denote $\lambda = \phi(t)$, $d\lambda = \dot{\phi}(t)dt$. Then

$$\int_{\tilde{\gamma}} f ds = \int_{[c,d]} f(t)|(\gamma \circ \phi)'(t)|dt = \int_{[c,d]} f(t)|(\gamma)'(\phi)|\dot{\phi}(t)dt$$

$$= \int_{[a,b]} f(\lambda)|\gamma'(\lambda)|d\lambda = \int_\gamma f ds.$$

Again, in the case of an orientation-reversing reparametrization, we need to add an extra minus sign on the right hand side.
?

9.2 Geodesics and Minimizing Curves

An admissible curve γ in a Riemannian manifold is said to be *minimizing* if $L(\gamma) \leq L(\tilde{\gamma})$ for any other admissible curve $\tilde{\gamma}$ with the same endpoints. Hence a curve is minimizing if and only if $L(\gamma)$ is the distance between its endpionts.

An *admissible family* of curves is a continuous map $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ that is smooth on each rectangle of the form $(-\epsilon, \epsilon) \times [a_{i-1}, a_i]$ for the some subdivision $a = a_0 \leq ... \leq a_k = b$ and such that $\Gamma_s(t) := \Gamma(s, t)$ is an admissible curve for each $s \in (-\epsilon, \epsilon)$. If Γ is an admissible family, a *vector field along* Γ is a continuous map $V : (-\epsilon, \epsilon) \times [a, b] \to TM$. Any admissible family Γ defines two collections of curves: the *main* curves $\Gamma_s(t) = \Gamma(s, t)$ defined on $[a, b]$ by setting s =constant. The transverse curves $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined on $(-\epsilon, \epsilon)$ by setting t = constant. The transverse curves are smooth on $(-\epsilon, \epsilon)$ for each t, while the main curves are in general only piecewise regular.

Let us denote $\partial_t\Gamma(s,t) = \frac{\Gamma_s(t)}{dt}$; $\partial_s\Gamma(s,t) = \frac{\Gamma^{(t)}(s)}{ds}$. It can be shown that the following expression, known as the *Symmetry Lemma*, holds for every connection, in particular a Riemannian connection,

$$D_s\partial_t\Gamma = D_t\partial_s\Gamma.$$

This is a key ingredient in the proof that minimizing curves are geodesics. If $\gamma : [a, b] \to M$ is an admissible curve, a *variation* of γ is an admissible family Γ such that $\Gamma_0(t) = \gamma(t)$ for all $t \in [a, b]$. It is called a *proper* variation or *fixed-end-point* variation if in addition $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all s. If Γ is a variation of γ, the *variation field* of Γ is the vector field $V(t) = \partial_s\Gamma(0, t)$ along γ. A vector field V along γ is proper if $V(a) = V(b) = 0$.

Let γ be any admissible unit speed curve, Γ a proper variation and V its variation field, then

$$\frac{d}{ds}|_{s=0}L(\Gamma_s) = -\int_a^b \langle V, D_t\dot{\gamma}\rangle\, dt - \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i\dot{\gamma}\rangle\,,$$

where $\Delta_i\dot{\gamma} = \dot{\gamma}(a_i^+), -\dot{\gamma}(a_i^-)$ is the " jump " in the tangent vector field $\dot{\gamma}$ at a_i.

Example Let γ be a smooth, unit speed curve. We will show that $D_t(\dot{\gamma}(t))$ is orthogonal to $\dot{\gamma}(t)$ for all t. Furthermore, if Γ is a proper variation of γ such that for all s, Γ_s is a reparametrization of γ, then $L(\Gamma_s)$ vanishes.

In general, for every Riemannian conncection, the formula

$$\frac{d}{dt}\langle V, W\rangle = \langle D_tV, W\rangle + \langle V, D_tW\rangle$$

holds. Now take $V = W = \dot{\gamma}(t)$, then we obtain

$$\frac{d}{dt}\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle \equiv 0 = 2\langle D_t\dot{\gamma}(t), \dot{\gamma}(t)\rangle,$$

and thus we have that $D_t\dot\gamma(t)$ is orthogonal to $\dot\gamma(t)$.

Suppose that Γ is a proper variation of γ such that for all s, Γ_s is a reparametrization of γ. Because γ is smooth and Γ_s a (smooth) reparametrization, we have that $\Gamma_s(t)$ is smooth.

In general, length is independent of the parametrization, which in this case yields $L(\gamma) = L(\Gamma_s(t))$ and so it seen that, because $L(\gamma)$ is independent of s, $\frac{d}{ds}L(\Gamma_s(t))$, The first variation vanishes.□

Example Define a connection on $\mathbb{R}^3$ by setting

$$\Gamma^3_{12} = \Gamma^1_{23} = \Gamma^2_{31} = 1,$$

$$\Gamma^3_{21} = \Gamma^1_{32} = \Gamma^2_{13} = -1,$$

and all other Christoffel symbols to be zero. We will show that this connection is compatible with the Euclidean metric, but not symmetric.

Recall that for the Euclidean metric $g(X,Y) = \langle X,Y\rangle = \sum_i X^iY^i$ and by definition

$$\nabla_X g(Y,Z) = \nabla_X\langle Y,Z\rangle = X(\langle Y,Z\rangle) - \langle\nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle.$$

Now

$$\nabla_X\langle Y,Z\rangle = X(\langle YZ\rangle) = X(Y^1Z^1) + X(Y^2Z^2) + X(Y^3Z^3).$$

Furthermore, writing out the definition of the connection and plugging in the Christoffel symbols, we obtain

$$\nabla_X Y = XY^kE_k + X^iY^j\Gamma^k_{ij}E_k$$

$$= (XY^1+(X^2Y^3-X^3Y^2))E_1+(XY^2+(X^3Y^1-X^1Y^3))E_2+(XY^3+(X^1Y^2-X^2Y^1))E_3.$$

A similar expression can be computed for $\nabla_X Z$. These computations yield, after some manipulation and cancelling of most of the terms,

$$\nabla_X g(Y,Z) = \nabla_X\langle Y,Z\rangle \equiv 0,$$

for every X, Y, Z and thus $\nabla g \equiv 0$ what is equivalent to saying that the connection is compatible with g, the Euclidean metric.

Previously, it has been shown that a connection being symmetric is equivalent to the condition $\Gamma^k_{ij} = \Gamma^k_{ji}$. Clearly, this condition is not satisfied for our connection, as such the connection is not symmetric. □

10 Curvature

Starting with the question whether al Riemannian metric are locally isometric, one is led naturally to a definition of the Riemannian curvature tensor as a measure of the failure of second covariant derivatives to commute. A main result in this is: a manifold has zero curvature if and only if it is flat, i.e. locally isometric to Euclidean space.

If M is a Riemannian manifold, the *(Riemann) curvature endomorphism* is the map $R : \chi(M) \times \chi(M) \times \chi(M) \to \chi(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{50}$$

and this defines a $(3,1)$ tensor field. As a tensor field, the curvature endomorphism can be written in terms of any local frame with one and three lower indices. Thus, for example, the curvature endomorphism can be written in terms of local coordinates (x^i) as,

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

where the coefficients are

$$R(\partial_k, \partial_l)\partial_k = R_{ijk}{}^l \partial_l.$$

We also define the *(Riemann) curvature tensor* as the covariant 4-tensor field $Rm = R^\flat$ obtained from the $(3,1)$-tensor field R by lowering the last index. Its action on vector fields is given by

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle,$$

and in coordinates is written

$$Rm = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes dx^l.$$

10.1 Cartan's Second Structure Equation

Let ∇ be the Riemannian connection on a Riemannian manifold (M,g) and let ω_i^j be its connection 1-forms with respect to a local frame $\{E_i\}$. Define a matrix of 2-forms Ω_i^j, called the *curvature 2-forms*, by

$$\Omega_i^j = \frac{1}{2} R_{kli}{}^j \phi^k \wedge \phi^l \tag{51}$$

By equation (50), we have that

$$R(E_k, E_l)E_i = \nabla_{E_k} \nabla_{E_l} E_i - \nabla_{E_l} \nabla_{E_k} E_i - \nabla_{[E_k, E_l]} E_i. \tag{52}$$

Because the Riemannian connection is symmetric, we have $[E_k, E_l] = \nabla_{E_k} E_l - \nabla_{E_l} E_k = (\Gamma^i_{kl} - \Gamma^i_{kl})E_i = 0$ and because $[E_k, E_l] = c^i_{kl} E_i$, we thus also have $c^i_{kl} = 0$. Expanding the first term in (52),

$$\nabla_{E_k} \nabla_{E_l} E_i = \nabla_{E_k} \left(\Gamma^j_{ki} E_j\right) = E_k \left(\Gamma^\lambda_{li}\right) E_\lambda + \Gamma^j_{li} \Gamma^\lambda_{kj} E_\lambda.$$

Similarly, for the second term we obtain,

$$\nabla_{E_l}\nabla_{E_k}E_i = E_l\left(\Gamma^{\lambda}_{ki}\right)E_\lambda + \Gamma^{j}_{ki}\Gamma^{\lambda}_{lj}E_\lambda.$$

It is seen that

$$\begin{aligned}&\left(\Gamma^{j}_{li}\Gamma^{\lambda}_{kj} - \Gamma^{j}_{ki}\Gamma^{\lambda}_{lj}\right)\phi^k\wedge\phi^l = \left(\Gamma^{\lambda}_{kj}\phi^k\right)\wedge\left(\Gamma^{j}_{li}\phi^l\right) - \left(\Gamma^{j}_{ki}\phi^k\right)\wedge\left(\Gamma^{\lambda}_{lj}\phi^l\right)\\ &= -2\omega^j_i\wedge\omega^\lambda_j,\end{aligned}$$

by symmetry of the indices and the fact that $\phi^k\wedge\phi^l = -\phi^l\wedge\phi^k$. Furthermore, because $E_k\left(\Gamma^{\lambda}_{li}\right) = \frac{\partial\Gamma^{\lambda}_{li}}{\partial x^k}$ we have that because

$$d\Gamma^{\lambda}_{li}\wedge\phi^l = \frac{\partial\Gamma^{\lambda}_{li}}{\partial x^k}\phi^k\wedge\phi^l,$$

and again by antisymmetry,

$$\left\{E_k\left(\Gamma^{\lambda}_{li}\right) - E_l\left(\Gamma^{\lambda}_{ki}\right)\right\}\phi^k\ \wedge\phi^l = 2d\Gamma^{\lambda}_{li}\wedge\phi^l.$$

On the other hand, we have that

$$d\omega^\lambda_i = d\left(\Gamma^{\lambda}_{ik}\phi^k\right) = d\Gamma^{\lambda}_{ik}\wedge\phi^k + \Gamma^{\lambda}_{ik}d\phi^k.$$

But, by equation (37), $d\phi^i = -c^i_{jk}\phi^j\wedge\phi^k$, so $d\phi = 0$ because the c^i_{jk} are. Hence

$$d\omega^\lambda_i = d\Gamma^{\lambda}_{li}\wedge\phi^l.$$

Combining these equations we obtain

$$\begin{aligned}\Omega^\lambda_i = \frac{1}{2}R_{kli}{}^j\phi^k\wedge\phi^l &= \frac{1}{2}\left(\nabla_{E_k}\nabla_{E_l}E_i - \nabla_{E_l}\nabla_{E_k}E_i\right)\phi^k\wedge\phi^l\\ &= \frac{1}{2}\left(2d\Gamma^{\lambda}_{li}\wedge\phi^l - 2\omega^j_i\wedge\omega^\lambda_j\right)\\ &= d\omega^\lambda_i - \omega^j_i\wedge\omega^\lambda_j.\end{aligned}$$

The equation

$$\Omega^\lambda_i = d\omega^\lambda_i - \omega^j_i\wedge\omega^\lambda_j \tag{53}$$

is called *Cartan's Second Structure Equation.*

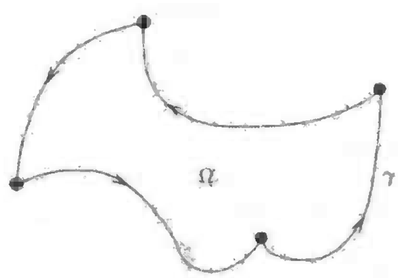
11 The Gauss-Bonnet Theorem

The *Gauss-Bonnet theorem* asserts the quality of two very differently defined quantities on a compact, orientable Riemannian 2-manifold M: the integral of the Gaussian curvature, which is determined by the local geometry of M; and 2π times the *Euler characteristic* of M, which is a global topological invariant. Although it applies only in two dimensions, it has provided a model and an inspriration for innumerable results in higher-dimensional geometry.

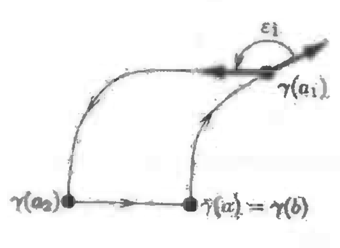
11.1 The Gauss-Bonnet formula

We begin with stating a theorem which expresses the intuitive idea that the tangent vector of a simple closed curve makes a net rotation through an angle of exactly 2π as one traverses the curve counterclockwise. This theorem is important in proving the Gauss-Bonnet formula.

A *curved polygon* in the plane is a simple, closed, piecewise smooth, unit speed curve segment, none of whose interior angles is equal to $\pm\pi$, that is the boundary of a bounded open set $\Omega \subset \mathbb{R}^2$.



If $a = a_0 < \cdots < a_k = b$ is a subdivision of $[a, b]$ such that γ is smooth on each $[a_{i-1}, a_i]$, the points $\gamma(a_i)$ are called *vertices* of γ, and the curve segments $\gamma|_{[a_{i-1}, a_i]}$ are called its *edges*. If γ is parametrized so that at points where γ is smooth, $\dot{\gamma}$ is consistent with the induced orientation on $\gamma = \partial\Omega$ in the sense of Stokes' theorem, we say γ is *positively oriented*. Furthermore, define the *exterior angle* ϵ_i at a_i to be the oriented angle from $\dot{\gamma}(a_i^-)$ to $\dot{\gamma}(a_i^+)$, chosen to be in the interval $[\pi, \pi]$.



Theorem (Rotation Angle Theorem)
If γ is a positively oriented curved polygon in the plane, the rotation angle of γ is exactly 2π.

A unit speed curve $\gamma : [a, b] \to M$ is called a curved polygon if γ is the boundary of an open set Ω with compact closure, and there is a coordinate chart containing γ and Ω under whose image γ is a curved polygon in the plane. As a straightforward consequence of the rotation angle theorem, we have the following lemma.

Lemma *If γ is a positively oriented curved polygon in M, the rotation angle of γ is 2π.*

There is a unique unit normal vector field along the smooth portions of γ such that $(\dot\gamma, N(t))$ is an oriented orthonormal basis for $T_\gamma M$ for each t. The *signed curvature* $\kappa_N(t)$ at smooth points of γ is defined to be

$$\kappa_N(t) = \langle D_t\dot\gamma(t), N(t)\rangle.$$

Now the important Gauss-Bonnet formula.

Theorem (The Gauss-Bonnet Formula)
Suppose γ is a curved polygon on an oriented Riemannian 2-manifold (M, g), and γ is positively oriented as the boundary of an open set Ω with compact closure. Then

$$\int_\Omega K dA + \int_\gamma k_N(t) ds + \sum_i \epsilon_i = 2\pi. \tag{54}$$

The following are three easy corolarries of the Gauss-Bonnet formula.

Corollary Angle-Sum Theorem
The sum of the interior angles of a Euclidean triangle is π.

Proof Because Euclidean space has zero curvature $K = 0$, and because the edges of a triangle are straight lines, we have that $\kappa_N(t) = 0$, so the Gauss-Bonnet formula reduces to

$$\sum_{i=1}^{3} \epsilon_i = 2\pi.$$

However, denoting the interior angle θ_i, it is seen that $\epsilon_i = \pi - \theta_i$ So $\sum_{i=1}^3 \epsilon_i = \sum_{i=1}^3 (\pi - \theta_i) = 3\pi - \sum_{i=1}^3 \theta_i = 2\pi$, and thus $\sum_{i=1}^3 \theta_i = \pi$. □

Corollary Circumference Theorem
The circumference of a Euclidean circle of radius R is $2\pi R$.

Proof In Euclidean space, $K = 0$, and because a circle has no vertices, the Gauss-Bonnet formula reduces to

$$\int_\gamma \kappa_N(t) ds = 2\pi,$$

where γ is the parametrization of the circle with radius R. Now because the curvature of γ is $\frac{1}{R}$, and multiplying by R we obtain

$$R\int_\gamma \kappa_N(t) ds = R\int_\gamma \frac{1}{R} ds = \int_\gamma ds = 2\pi R.$$

□

Corollary Total Curvature Theorem *If* $\gamma : [a,b] \to \mathbb{R}^2$ *is a unit speed simple closed curve such that* $\dot\gamma(a) = \dot\gamma(b)$*, and* N *is the inward pointing normal, then*

$$\int_a^b \kappa_N(t)dt = 2\pi.$$

Proof Again, $K = 0$ and because γ is unit speed we have $ds = dt$. Furthermore, γ has no vertices since it is closed and at the end points the condition $\dot\gamma(a) = \dot\gamma(b)$ is satisfied. This gives the result. □

Example Geodesic Triangle
A *geodesic triangle* on a Riemannian 2-manifold (M, g) is a three-sided geodesic polygon. Suppose M has constant curvature, say K. Because the sides of the triangle are geodesics, and since every geodesic (by definition) satisfies $D_t\dot\gamma = 0$, we have that $\kappa_N(t) = \langle D_t\dot\gamma, N(t)\rangle = 0$. As before, $\theta_i = \pi - \epsilon_i$, where θ_i denotes the interior angle. Then the Gauss-Bonnet formula yields

$$K \int_\Omega dA + \sum_i \epsilon_i = K \int_\Omega dA + \sum_i (\pi - \theta_i) = 2\pi.$$

If we write $\int_\Omega dA = A$, then we see that $\sum_i \theta_i = \pi + KA$.

In the proof of the Gauss-Bonnet formula, it is shown that $KdA = d\omega$, for a certain ω. Since in this case, K is constant, we can divide by K and obtain $dA = d\omega'$. Suppose that M is either the 2-sphere or the hyperbolic plane of radius R, in both cases the curvature K is constant. Suppose γ_1 and γ_2 are geodesic triangles with equal interior angles, then since $\sum_i \theta_i = \pi + KA$ we must have that $A_1 = A_2$. We see,

$$A_1 = \int_{\Omega_1} dA = \int_{\Omega_1} d\omega' = \int_{\gamma_1(t)} \omega',$$

by Stokes' formula. A similar expression holds for A_2, so we must have

$$\int_{\gamma_1(t)} \omega' = \int_{\gamma_2(t)} \omega'.$$

Suppose $\gamma_1 : [a,b] \to M$ and $\gamma_2 : [c,d] \to M$ and suppose $\phi : [a,b] \to [c,d]$ is an increasing diffeomorphism, then by diffeomorphism invariance of line integrals,

$$\begin{aligned}\int_{\gamma_1(t)} \omega' &= \int_{[a,b]} \gamma_1^*\omega' = \int_{\gamma_2(t)} \omega' = \int_{[c,d]} \gamma_2^*\omega' \\ &= \int_{[a,b]} \phi^*\gamma_2^*\omega' = \int_{[a,b]} (\gamma_2 \circ \phi)^*\omega' = \int_{\gamma_2\circ\phi(t)} \omega',\end{aligned}$$

so we see

$$\int_{\gamma_1(t)} \omega' = \int_{\gamma_2(t)} \omega' = \int_{\gamma_2 \circ \phi(t)} \omega'.$$

Hence ϕ is an isometry taking γ_2 to γ_1. In particular, the distances between the vertices of γ_1 are equal to the corresponding distances between the vertices of γ_2. And the distance of every point on the edge of γ_2 is equal to the distance of the corresponding point to the corresponding vertex. We conclude that, if M is a manifold with constant curvature then similar triangles, i.e. triangles with equal interior angles, are congruent triangles. □

11.2 Triangulation of smooth 2-manifolds

From the Gauss-Bonnet formula, one can relatively easily derive the Gauss-Bonnet theorem. The link between the local (an open set Ω on a manifold) and the global (the manifold) results is provided by triangulations, a construction borrowed from algebraic topology.

If M is a smooth, compact 2-manifold, a *smooth triangulation* of M is a finite collection of curved triangles (i.e. three-sided curved polygons), such that the union of the closed regions $\bar{\Omega}_i$ bounded by the triangles is M, and the intersection of any pair (if not empty) is either a single vertex of each or a single edge of each. We will prove in this section that every smooth compact surface possesses a smooth triangulation. In fact, it was proved by Tibor Radó in 1925 that every compact *topological* 2-manifold possesses a triangulation (without the assumption of smoothness at the edges). If M is a triangulated 2-manifold, the *Euler chararteristic* of M with respect to the given triangulation is defined to be

$$\chi(M) = N_v - N_e + N_f,$$

where N_v is the number of vertices in the triangulation, N_e is the number of edges, and N_f is the number of faces (the $\Omega_i s$). It is an important result of algebraic topology that the Euler characteristic is in fact a topological invariant, and is independent of the choice of the triangulation.

A subset U of a Riemannian manifold M is said to be *convex* if for each $p, q \in U$, there is a unique (in M) minimizing geodesic from p to q lying entirely in U. Without proof we state the next two theorems.

Theorem 1 *Within any geodesic ball around $p \in M$, the radial distance function $r(x)$ is equal to the Riemannian distance from p to x.*

Theorem 2 *Given $p \in M$ and any neighborhood U of p, there exists a uniformly normal neighborhood W of p contained in U.*

Lemma 3 *Every point on a Riemannian manifold has a convex neighbourhood.*

Proof Let $p \in M$ be fixed, and let W be a uniformly normal neighbourhood of p. For ϵ small enough that $B_{2\epsilon}(p) \subset W$, we define a subset $W_\epsilon \subset TM \times \mathbb{R}$ by

$$W_\epsilon = \{(q, V, t) \in TM \times \mathbb{R} : \; q \in B_\epsilon(p), V \in T_qM, |V| = 1, |t| < 2\epsilon\}.$$

Define $f : W_\epsilon \to \mathbb{R}$ by

$$f(q, V, t) = d(\exp_q(tV), p)^2.$$

Choose normal coordinates centered at p. In these coordinates $p = (0, ..., 0)$. The exponential mapping $\exp_q(tV)$ is smooth. Let us denote $\exp_q(tV) = \bar{q}_V(t)$. There is a unique (radial) geodesic starting at p with intial vector V', such that this geodesic aims at $\bar{q}$. Without loss of generality, we can assume that at $t = t_1$ $\bar{q}_V(t_1) = \exp_p(t_1 V')$, for instance by rescaling of V'. Then by theorem 1 and because the coordinates are chosen to be normal coordinates centered at p,

$$f(q, V, t) = d(\exp_q(tV), p)^2 = d(\bar{q}, p)^2 = (r(\bar{q}))^2.$$

Since the radial distance function and the exponential mapping at q are both smooth, we conclude that $f(q, V, t)$ is smooth. Suppose $V' = (v^1, ..., v^n)$. Then

$$f(q, V, t) = \sum_{i=1}^{n} (tv^i)^2 = t^2 \sum_{i=1}^{n} (v^i)^2.$$

Hence, at p, $\partial^2 f / \partial t^2 = 2 \sum_{i=1}^{n} (v^i)^2 > 0$. If ϵ is chosen small enough, then by continuity of f, $\partial^2 f / \partial t^2 > 0$ on W_ϵ. Now suppose $q_1, q_2 \in B_\epsilon(p)$ and γ is a minimizing geodesic from q_1 to q_2. Because $\partial^2 f / \partial t^2 > 0$ on W_ϵ and q_1 and V are fixed, the graph of f as a function of t is convex. Hence it attains its maximum at one of the endpoints of γ. Thus $d(\gamma(t), p)$ attains its maximum at q_1 or q_2 (or both), suppose q_1. Since $q_1 \in B_\epsilon(p)$ and $d(\gamma(t), p) < d(q_1, p)$, γ lies entirely in $B_\epsilon(p)$. So for every point p there exists an $\epsilon > 0$ such that $B_\epsilon(p)$ is convex and this concludes the proof. □

Theorem *Every smooth 2-manifold possesses a smooth triangulation.*

Proof

First we will show that it is sufficient to show there exist finitely many convex geodesic polygons whose interior cover M, and each of which lies in a uniformly convex geodesic ball.

Take an arbitrary convex geodesic polygon. Because it is convex, the geodesic connecting two vertices lies entirely within the polygon. Take one of the vertices and connect all the other vertices (except for the neighboring vertices) to this vertex. The geodesics do not intersect, because by hypothesis the convex geodesic polygon lies in a uniformly convex geodesic ball, and so the exponential mapping at every point in the ball is a diffeomorphism, in particular at the point of the *base vertex*. Hence, it is always possible to triangulate a convex geodesic polygon. And because every convex geodesic polygon can be triangulated into a finite number of triangles, and the number of geodesic polygons is finite, the total number of triangles is finite. To triangulate M, first triangulate each convex geodesic polygon separately. In general, the geodesic polygons will overlap and thus, by triangulating each polygon we will obtain a set of triangles together with a set of new (not necessarily convex) geodesic polygons. To triangulate these polygons, we use induction. Choose the left most vertex on a certain n-polygon. This vertex always exists and because it is the left most vertex, the angle between this vertex and the neighboring vertices is smaller that π. Again, let us call this left most vertex the *base vertex*. Then there are two possibilities:

1. The two neighboring vertices of this base vertex can be connected such that the entire geodesic connecting the two vertices lies inside the polygon.

2. The two neighboring vertices of this base vertex can not be connected such that the entire geodesic connecting the two vertices lies inside the polygon. But then a finite number of vertices lies inside the triangle formed by the base vertex and the two neighboring vertices. Therefore, we can choose the vertex that has the largest distance to the line connecting the two neighboring vertices. Then, the geodesic connecting the base vertex and the vertex with largest distance to the line can not intersect with any other edge of the polygon. Thus these two vertices can be connected by a geodesic.

In both cases, the original n-polygon is reduced to two polygons, each with a number of vertices less that n. By induction, this gives an algorithm to triangulate any polygon, convex or not convex. In summary, to triangulate a manifold M, the first step is to triangulate the set of convex geodesic polygons that cover M separately to obtain a set of triangles and a new set of (in general) non-convex polygons. The second step is to triangulate the given non-convex polygons, by the algorithm given above, and obtain a triangulation for the entire manifold M.

We can complete the proof by showing there exist finitely many geodesic polygons which interiors cover M, and each of which lies in a uniformly convex geodesic ball.
By lemma 3, every point p on the manifold M has a convex neighborhood. So $\forall p \in M \exists\, \epsilon_p$ such that $B_{\epsilon_p}(p)$ is convex. Furthermore, by theorem 2, $\forall p \in M \exists\, \epsilon'_p$ such that $B_{\epsilon'_p}(p)$ is uniformly normal. Because the manifold M is compact and $B_\epsilon(p)$, for arbitrary small ϵ, has positive measure, finitely many of these balls cover M. We can choose ϵ small enough and we can choose a finite number of points on the manifold $(v_1, .., v_n)$, such that the geodesic balls $B_{3\epsilon}(v_i)$ are convex, uniformly normal and the balls cover M.
For each i take the ball $B_{2\epsilon}(v_i)$. If we take points sufficiently nearby on the boundary of the ball $B_{2\epsilon}(v_i)$, then the geodesics connecting the points do not enter $B_\epsilon(v_i)$. Hence the interior of this polygon contains $B_\epsilon(v_i)$.
Furthermore, this polygon is convex. To see this, consider two points q_1, q_2 in the polygon. The vertices of this polygon lie on the circle of radius 2ϵ and $B_{2\epsilon}(v_i)$ is convex. Then the geodesic connecting q_1 and q_2 lies entirely within

$B_{2\epsilon}(v_i)$. Suppose otherwise, then the geodesic would intersect an edge of the polygon at two points. But, because the edge of the polygon is a geodesic itself, the path between the points q_1, q_2 is shorter along the edge of the polygon and hence a geodesic that does not lie entirely within the polygon violates the fact that every Riemannian geodesic is (locally) minimizing. From this we conclude that for each i, there exists a convex geodesic polygon $B_{3\epsilon}(v_i)$ whose interior contains $B_\epsilon(v_i)$.
We have shown that there exists a finite number of points, $(v_i, ..., v_k)$, such that at each point v_i there exist a convex geodesic polygon which lies in a uniformly normal convex neighborhood, and these polygons cover M. This concludes the proof. □

11.3 Gauss-Gonnet Theorem

Theorem (The Gauss-Bonnet Theorem) *If M is triangulated, compact, oriented, Riemannian 2-manifold, then*

$$\int_M K dA = 2\pi\chi(M).$$

Much of the effort in contemporary Riemannian geometry is aimed at generalizing the Gauss-Bonnet theorem and its topological consequences to higher dimensions. There is a direct generalization of the Gauss-Bonnet theorem that deserves mention: the Chern-Gauss-Bonnet theorem. This was proved by Hopf in 1925 for an n-manifold embedded in $\mathbb{R}^{n+1}$ with the induced metric, and in 1944 by Chern for abstract Riemannian manifolds. The theorem asserts that on any oriented vector space there exists a basis independent function

$$P : \{4 - \text{tensors with the symmetries of } Rm\} \to \mathbb{R},$$

called the *Pfaffian*, such that for any oriented compact even-dimensional Riemannian n-manifold M,

$$\int_M P(Rm)dV = \frac{1}{2}\mathrm{Vol}(\mathbb{S}^n)\chi(M).$$

Here $\chi(M)$ is again the Euler characteristic of M, which can be defined analogously to that of surface and is a topological invariant. The only problem with this result is that the relationship between the Pfaffian and sectional curvatures is obscure in higher dimensions, so no one seems to have any idea how to interpret the theorem geometrically! For example, it is not even known whether the assumption that M has strictly positive sectional curvatures implies $\chi(M) > 0$.

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