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On generic pole assignment of linear time-invariant systems by memoryless periodically time-varying output feedback

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Abstract

It was shown by Wonham that the closed-loop poles of a linear time-invariant system can be assigned arbitrarily by real memoryless state feedback if and only the system is controllable. The use of output feedback can be a good alternative in case the state is not available for feedback. In 1992 it was proven by Wang that if $n < mp$ (where n is the number of states, m the number of inputs and p the number of outputs), then generically the system is arbitrary pole assignable by real memoryless time-invariant output feedback. One of the proofs of this theorem is based on a behavioral approach.

Another possibility for achieving pole assignment is applying periodic output feedback. Using the technique of lifting, exact conditions were derived in the literature for SISO systems, showing that, under these conditions, we have (almost) arbitrary pole placement by periodic output feedback with period $T = n + 1$. In this report we use this lifting technique for MIMO systems, and establish conditions on n, m, p and the period of the controller for generic arbitrary pole placement. This is achieved by formulating the pole placement problem in a behavioral context and extending the behavioral proof to periodically time-varying controllers.

Keywords: genericity, eigenvalue assignment, periodic output feedback, lifting, behavioral approach

Contents

Chapter 1

Preliminaries and introduction

In this first chapter we discuss some preliminaries and give an introduction. Readers who would like to refresh their memory or readers who are not familiar with some of the mathematical aspects might find it useful to start with the preliminaries. The first section contains some basic definitions used in mathematical systems theory. In section 1.2 we give a short introduction to behaviors. An important aspect in this report is the notion of genericity. Its definition, together with a few examples, can be found in section 1.3. Readers who skip the first sections of preliminaries can start with the introduction in section 1.4.

1.1 Linear Systems Theory

Suppose we have a linear time-invariant (LTI) system with n states, m inputs and p outputs, described by constant matrices A, B and C,

$$\sigma x = Ax + Bu, \qquad y = Cx, \tag{1.1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.
In (1.1) σ can stand for either the differentiation operator or the shift operator. The differentiation operator is used to denote *continuous-time* systems. Substituting the differentiation operator $\frac{d}{dt}$ in (1.1) leads to the continuous-time LTI system

$$\frac{d}{dt}x = Ax + Bu, \qquad y = Cx, \tag{1.2}$$

where the state x, the input u and the output y are functions of time. In this report, the time axis is taken to be $\mathbb{R}$. If σ stands for the shift operator, (1.1) describes a *discrete-time* system

$$x_{i+1} = Ax_i + Bu_i, \qquad y_i = Cx_i. \tag{1.3}$$

In (1.3) x, y and u are functions of time and $i \in \mathbb{Z}$, i.e. the time axis is taken to be $\mathbb{Z}$.
The time between two successive points is constant and is referred to as the *time interval.*

A representation of the form (1.1) is called a state space representation. System (1.1) also allows for an input-output representation, which involves the transfer matrix $C(\xi I - A)^{-1}B$. The transfer matrix is always strictly proper, i.e. the elements of the matrix are strictly proper rational functions. Recall that a proper rational function has the property that the degree of the numerator does not exceed the degree of the denominator. If the degree of the numerator is strictly smaller than the degree of the denominator, the rational function is called strictly proper.

1.1.1 Realizations

Let $H(\xi)$ be a strictly proper $p \times m$ rational matrix. A triple (A, B, C) describing a system (1.1) is called a *realization* of $H(\xi)$ if the input-output structure of (1.1) is identical to the input-output structure of $y = H(\xi)u$, i.e. if $H(\xi) = C(\xi I - A)^{-1}B$. A triple (A, B, C) is called a minimal realization if A is a $n \times n$ -matrix and no realization with A of size smaller than n exists. The integer n is known as the *McMillan degree.*

1.1.2 Control

System (1.1) can be a model of some physical system which we would like to display a certain behavior. We may achieve desired behavior by choosing a suitable input function u. The input function can be of the form

$$u = Fx, \tag{1.4}$$

with F a constant matrix of suitable dimensions with elements in $\mathbb{F}$. Here, $\mathbb{F}$ is a field, which can for instance be $\mathbb{R}$ or $\mathbb{C}$. Recall that a field $\mathbb{F}$ is algebraically closed if every polynomial of positive degree with coefficients in $\mathbb{F}$ has all its roots in $\mathbb{F}$.
The control law (1.4) is called a *state feedback* law. Clearly, state feedback can only be applied if the state is available, which may not be the case in practical problems.
Another way to control the system is to feed back the output, by using an *output feedback* law

$$u = Ky, \tag{1.5}$$

with K a constant matrix of suitable dimensions with elements in $\mathbb{F}$. The control law (1.4) (respectively (1.5)) is called *memoryless*, i.e. it only depends on the present value of the state (respectively the output). Notice that in the special case that $C = I$, output feedback is equivalent to state feedback.

Recall that system (1.1) is controllable if and only if the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{1.6}$$

has rank equal to n. System (1.1) is observable if and only if the rank of the observability matrix $\mathcal{O}$ equals n, where

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{1.7}$$

From the observability matrix we can derive p *observability indices* that are computed as follows. Select from the observability matrix as many ($\bar{n}$) linearly independent rows as possible, starting with the first row and going downwards. Let $\bar{\mathcal{O}}$ denote the $\bar{n} \times n$ matrix which consists of those selected rows. Observe that $\bar{\mathcal{O}}$ is a square matrix if the considered system is observable. Then, re-order the rows, such that the first (μ_1) rows involve c_1, the first row of C, the next (μ_2) rows involve c_2, etc.

Thus

$$\bar{O} = \begin{bmatrix} c_1 \\ c_1 A \\ \vdots \\ c_1 A^{\mu_1 - 1} \\ c_2 \\ \vdots \\ c_2 A^{\mu_2 - 1} \\ \vdots \\ c_p \\ \vdots \\ c_p A^{\mu_p - 1} \end{bmatrix}. \tag{1.8}$$

The integers $\mu_1, \ldots, \mu_p$ are called the observability indices. Notice that for an observable system $\mu_1 + \mu_2 + \ldots + \mu_p = n$.

1.1.3 Pole placement

If we connect system (1.1) with the output feedback (1.5) we obtain the *closed-loop system*

$$\sigma x = (A + BKC)x. \tag{1.9}$$

The closed-loop system is an *autonomous system*, i.e. the past completely determines the future of the trajectory of the state. The behavior of the trajectory is described by the eigenvalues of the closed-loop matrix $A + BKC$, which are exactly the zeros of $det(sI - (A + BKC))$, the *characteristic polynomial* of $A + BKC$. To achieve desired behavior, e.g. asymptotically stable, we should be allowed to choose the eigenvalues and find a feedback law $u = Ky$ such that the eigenvalues of the closed-loop matrix are the same as the ones we chose. If K is a matrix such that the set of zeros of the characteristic polynomial of $(A + BKC)$ is equal to a prescribed set of n eigenvalues $\Lambda = \{ \lambda_1, \ldots, \lambda_n \}$, we say that K *assigns the set of eigenvalues* Λ.

Observe that each choice of K defines a particular closed-loop system and hence the characteristic polynomial of $A + BKC$, and its set of zeros, can be regarded as a function of K. Define a monic polynomial

$$\alpha(s) = \alpha_0 + \alpha_1 s + \ldots + \alpha_{n-1} s^{n-1} + s^n, \tag{1.10}$$

with $\alpha_i \in \mathbb{F}$, $i = 0, \ldots, n-1$. We identify the polynomial $\alpha(s)$ with the n-dimensional vector space by associating its coefficients with a vector in $\mathbb{F}^n$. Hence, we identify (1.10) with the vector $\bar{\alpha} := (\alpha_0, \ldots, \alpha_{n-1})$. Furthermore, we define a map φ: $\mathbb{F}^{m \times p} \rightarrow \mathbb{F}^n$, with $\varphi(K) = (\alpha_0, \ldots, \alpha_{n-1})$. The map φ is called the *pole assignment map* or *pole placement map*.

Definition 1.1.1 The system (1.1) is said to be arbitrary pole assignable by memoryless output feedback (1.5), if for every polynomial (1.10) there exist a matrix K such that $\varphi(K) = \bar{\alpha}$.

The problem of determining whether there exists a control law $u = Ky$ such that the characteristic polynomial of the closed-loop system coincides with a prescribed polynomial, i.e. such that $\varphi(K) = \bar{\alpha}$, is known as the *pole placement problem*.
The pole placement problem thus deals with the surjectivity of the placement map. It would be useful to have necessary and sufficient conditions for the surjectivity of the pole placement map,

since this property of φ would guarantee arbitrary eigenvalue assignment. Of particular interest are sufficient conditions for which, for almost all systems, the pole placement map is surjective. The notion of "almost all" is captured by the notion of genericity and is explained in section 1.3. The search for sufficient conditions on the parameters n, m and p for generic arbitrary pole assignment has lead to a number of results. An overview of these results can be found in Chapter 2.

1.2 Introduction to behaviors

While in the classical state-space representation a mathematical system is described by its input-output structure, this structure does not readily appear when modeling from a behavioral point of view. In the behavioral approach we consider a mathematical system, loosely speaking, more generally as a set of trajectories. More precisely:

Definition 1.2.1 A pair $(\mathbb{U}, \mathcal{B})$ defines a mathematical model, where $\mathbb{U}$ is the universum and $\mathcal{B} \subset \mathbb{U}$ is referred to as the behavior. The universum is the set of all outcomes.

The variables that we set out to describe are called the *manifest variables*. When modeling a certain phenomenon, it is very likely that we need to introduce variables other than the manifest variables to describe the phenomenon. Those auxiliary variables are called *latent variables*.

Definition 1.2.2 A mathematical model with latent variables is a triple $(\mathbb{U}, \mathbb{U}_\ell, \mathcal{B}_f)$. The universum of manifest variables is denoted by $\mathbb{U}$, the universum of latent variables is referred to by $\mathbb{U}_\ell$, and $\mathcal{B}_f$ is a subset of $\mathbb{U} \times \mathbb{U}_\ell$, denoting the full behavior.

A triple $(\mathbb{U}, \mathbb{U}_\ell, \mathcal{B}_f)$ defines the mathematical model described by $(\mathbb{U}, \mathcal{B})$. $\mathcal{B}$ is called the *manifest behavior* or external behavior, and equals $\{u \in \mathbb{U} \mid \exists \ell \in \mathbb{U}_\ell \text{ such that } (u, \ell) \in \mathcal{B}_f\}$.
A special type of a mathematical model is a dynamical system. A dynamical system aims to describe variables that are functions of time.

Definition 1.2.3 A dynamical system is a triple $(\mathbb{T}, \mathbb{W}, \mathcal{B})$ where $\mathbb{T} \subseteq \mathbb{R}$ represents the time axis, $\mathbb{W}$ is the signal space and $\mathcal{B} \subset \mathbb{W}^{\mathbb{T}}$ is the behavior.

The behavior is the set of all trajectories $w : \mathbb{T} \to \mathbb{W}$ that can occur according to the model. In the present report we let $\mathbb{T} = \mathbb{R}$ represent the time-axis if the considered system is a continuous-time system, while for discrete-time systems we take $\mathbb{T} = \mathbb{Z}$.
The behavior of a mathematical model is usually described by a set of equations. In this report we restrict ourselves to linear systems. Then the behavior is described by linear difference equations (if $\mathbb{T} = \mathbb{Z}$) or linear differential equations (if $\mathbb{T} = \mathbb{R}$). Such behavioral equations are represented by polynomial matrices, i.e. matrices with polynomial entries. The set $\mathbb{R}[\xi]$ denotes the set of real polynomials in the indeterminate ξ. An element of the set $\mathbb{R}^{p \times q}[\xi]$ is a $p \times q$ -matrix with real polynomial entries.
A mathematical model allows several types of representations. Here, we discuss the *kernel representation* and the *input-output representation*. A kernel representation is of the form

$$R(\sigma)w = 0, \tag{1.11}$$

where σ can stand for the differentiation operator or the shift operator. By making a suitable input-output partition of the vector w, $w = \begin{bmatrix} y \\ u \end{bmatrix}$, a kernel representation can be written as an input-output representation, which is of the form

$$P(\sigma)y = Q(\sigma)u, \tag{1.12}$$

with $P^{-1}(\sigma)Q(\sigma)$ proper, i.e. a matrix of proper rational functions. The input vector is given by u, while y is the output vector.

Applying elementary matrix operations, such as adding a multiple of one row to another, can be done by pre-multiplication by a unimodular matrix.

Definition 1.2.4 A polynomial matrix $U(\xi) \in \mathbb{R}^{q\times q}[\xi]$ is called unimodular if there exists a polynomial matrix $V(\xi) \in \mathbb{R}^{q\times q}[\xi]$ such that $V(\xi)U(\xi) = I$. An equivalent condition is that the determinant of $U(\xi)$ is a non-zero constant.

Two representations $R_1(\sigma)w = 0$ and $R_2(\sigma)w = 0$ are called equivalent if they represent the same behavior.

Theorem 1.2.5 [14] Two representations $R_1(\sigma)w = 0$ and $R_2(\sigma)w = 0$, with $R_1(\xi)$ and $R_2(\xi)$ having the same number of rows describe the same behavior if and only if there exists a unimodular matrix $U(\xi)$ such that $R_1(\xi) = U(\xi)R_2(\xi)$.

Definition 1.2.6 Let $R(\xi) \in \mathbb{R}^{p\times q}[\xi]$. $R(\xi)$ is called row reduced if the coefficient matrix of the highest order terms in each row of $R(\xi)$ has full row rank.

Theorem 1.2.7 For every polynomial matrix $R(\xi) \in \mathbb{R}^{p\times q}[\xi]$ there exists a unimodular matrix $U(\xi) \in \mathbb{R}^{p\times p}[\xi]$ such that $U(\xi)R(\xi)$ is row reduced.

In particular, every behavior described by $R(\sigma)w = 0, R(\xi) \in \mathbb{R}^{p\times q}[\xi]$ can also be described by a full row rank representation $R'(\sigma)w = 0, R'(\xi) \in \mathbb{R}^{p'\times q}[\xi]$, with R' of full row rank. If $R(\xi) \in \mathbb{R}^{p\times q}[\xi]$ has full row rank p, then the McMillan degree is the maximal degree of all the minors of $R(\xi)$ of size p. Moreover, if R is row reduced then the McMillan degree is equal to the sum of the row degrees.

For more details on systems theory in a behavioral setting, we refer to [14]. The notion of feedback control in a behavioral framework is dealt with in [20].

1.3 Genericity

To explain the notion of genericity, we need to introduce the notion of an *algebraic variety*. Recall that $\mathbb{F}$ denotes a field which can for instance be $\mathbb{R}$ or $\mathbb{C}$. A set $S \subseteq \mathbb{F}^n$ is called an algebraic variety, if there exists a polynomial in n variables and coefficients in $\mathbb{F}$ such that its set of zeros is equal to the set S. If S is a strict subset of $\mathbb{F}^n$, then S is called a proper algebraic variety. A proper algebraic variety has Lebesgue measure zero. Its complement is open and dense.

Definition 1.3.1 The complement of a proper algebraic variety is called a generic set.

Example 1.3.2 Consider the set $S := \{(x, y) \in \mathbb{R}^2 \mid x \neq y\}$. Its complement S^c, $S^c = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ is obviously an example of a proper algebraic variety, for S^c is a strict subset of $\mathbb{R}^2$ and the points that belong to S^c are defined as zeros of a polynomial. S^c being a proper algebraic variety implies that its complement S is a generic set. Intuitively, a pair at random chosen numbers $(x, y) \in \mathbb{R}^2$ will most likely belong to the set S. If the numbers happen to belong to S^c, then a small perturbation on one of the numbers drives them out of the set S^c. □

Lemma 1.3.3 Let M be a square matrix. Then $det(M) \neq 0$ is generically satisfied.

Intuitively, this is clear: if the elements of M are chosen at random, the rows (or columns) are very likely to be linear independent. And if one row (or column) is actually a linear combination of the other rows (columns) a small perturbation on one of the elements of that row (column) makes the rows (columns) linearly independent. Here is the proof of Lemma 1.3.3:

Proof Write $M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}$ and define the set $S = \{M \in \mathbb{R}^{n \times n} \mid det(M) \neq 0\}$.
To prove that S is a generic set, we prove that its complement $S^c = \{M \in \mathbb{R}^{n \times n} \mid det(M) = 0\}$ is a proper algebraic variety. The determinant of the matrix M is a polynomial expression in n^2 variables which can be written as $p(m_{11}, \dots, m_{1n}, \dots, m_{n1}, \dots, m_{nn})$. Therefore, $S^c = \{M \in \mathbb{R}^{n \times n} \mid p(m_{11}, \dots, m_{nn}) = 0\}$ is an algebraic variety, since the entries of M satisfy the algebraic equation $p(m_{11}, \dots, m_{nn}) = 0$. Moreover, because S is not the empty set, e.g. the identity matrix belongs to S, S^c is a strict subset of $\mathbb{R}^{n \times n}$. So, S^c is a proper algebraic variety. □

We now formulate two more lemma's. The first lemma is used in the proof of the second one.

Lemma 1.3.4 Let $N \in \mathbb{R}^{n \times m}$,with $n \leq m$, be a matrix. Then $rank(N) < n$ if and only if $det(X) = 0$ for all matrices $X \in \mathbb{R}^{n \times n}$ consisting of n columns of N.

Proof Let $N \in \mathbb{R}^{n \times m}$, with $n \leq m$. (Only if part) Assume there is a matrix X with non-zero determinant and containing n columns of N. Then N contains, among others, the columns of X, which are linear independent. But then N has full row rank, which is in contradiction with the assumption that $rank(N) < n$. (If part) Assume that (i) $det(X) = 0$ for all matrices X consisting of n columns of N. Furthermore, assume that (ii) $rank(N) = n$. We prove that this is a contradiction. If $rank(N) = n$ then N contains n linear independent columns. The matrix X consisting of those n columns that are linear independent has a non-zero determinant. This is in contradiction with the assumption (i). □

Lemma 1.3.4 also holds if $n \geq m$. In that case, exchange the roles of n and m and exchange the words "rows" and "columns". This remark also applies for the next lemma, which is a generalization of Lemma 1.3.3.

Lemma 1.3.5 Let $N \in \mathbb{R}^{n \times m}$, with $n \leq m$. The set of matrices N of full row rank is a generic set.

Proof To be proven is that the set

$$\{N \in \mathbb{R}^{n \times m}, n \leq m \mid rank(N) < n\} \tag{1.13}$$

is a proper algebraic variety. By Lemma 1.3.4 $rank(N) < n$ if and only if every $n \times n$ submatrix of N has a zero determinant. So the elements of N with $rank(N) < n$ satisfy a polynomial expression. Hence (1.13) is an algebraic variety. It is a proper one, for the matrix $[I_{n \times n} \quad O_{n \times (m-n)}]$, with $I_{n \times n}$ the identity matrix of size n, and $O_{n \times (m-n)}$, the $n \times (m-n)$ zero matrix, has rank n. □

With this result, it is easily seen that for generic (A, B, C), the linear time-invariant system

$$\sigma x = Ax + Bu \qquad y = Cx, \tag{1.14}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, is in fact controllable and observable.
Indeed, it is well known that system (1.14) is controllable if and only if the controllability matrix $\mathcal{C}$ (1.6) has rank n. Since $\mathcal{C}$ generically has full rank, it follows that for a generic set of matrices (A, B, C) system (1.14) is controllable. Observability of system (1.14) is guaranteed if and only if the observability matrix $\mathcal{O}$ (1.7) has full rank. The matrix $\mathcal{O}$ has full rank for generic (A, B, C). Hence system (1.14) is for generic (A, B, C) observable.

Lemma 1.3.6 The set of systems that can be arbitrary pole assigned by real memoryless state feedback is a generic set.

Proof Consider the state feedback law $u = Fx$. Applied to system (1.14), the state x then satisfies $\sigma x = (A+BF)x$. It is well known (Wonham, [23]) that the closed-loop poles can be chosen arbitrarily if and only if system (1.14) is controllable. Since for generic (A, B, C) system (1.14) is controllable, Lemma 1.3.6 holds. □

1.4 Introduction to this report

It is a well known result in control theory that the closed-loop poles of a linear controllable time-invariant system can be assigned arbitrarily by real memoryless state feedback. This was shown in 1967 by Wonham [23]. In case the state is not available, one could try to feed back the output to assign the poles. Over the past few decades, a reasonable amount of attention has been paid to the problem of determining sufficient conditions for eigenvalue assignability by memoryless output feedback.
The emphasis has been on finding sufficient conditions on the parameters n, m and p, where n is the number of states of the system, m the number of inputs and p the number of outputs. This search has lead to a number of results concerning sufficient conditions on the parameters n, m and p for generic arbitrary pole assignment by memoryless output feedback. An overview of the main results can be found in section 2.2. The best known sufficient condition was claimed by Wang [18], proving that $n < mp$ implies generic arbitrary pole assignment by memoryless time-invariant output feedback. One of the proofs is based on the behavioral approach, [19], which provides a useful framework for studying the pole placement problem.

Pole assignment may also be achieved by the use of periodically time-varying controllers. In some problems, performance of periodic controllers may be better than performance achieved by time-invariant controllers. For example, periodic controllers were shown to be useful for decentralized systems, [22], where the use of time-invariant controllers involved some major restrictions. Another advantage of periodically time-varying controllers was established in [4], where it is argued that systems with unstable decentralized fixed modes cannot be stabilized by decentralized time-invariant controllers but can be stabilized by linear time-varying controllers.
Greschak and Verghese studied the problem of pole assignment by means of periodic output feedback. In [8] they proved that, under certain conditions, for second order SISO systems, arbitrary pole assignment can be achieved by using a 3-periodic output feedback law, see also [1]. In 1992, Aeyels and Willems [3] extended their result, showing that, under mild conditions, a SISO system with n states is (almost) arbitrarily pole assignable by periodic output feedback with period $n + 1$.

In this report we study the problem of pole assignment of MIMO discrete-time systems by T - periodic output feedback. Our goal is to find sufficient conditions on the parameters n, m, p and T for generic arbitrary pole placement. This is established by extending the behavioral proof in [19] to periodically time-varying output feedback.

Chapter 2

Memoryless output feedback

In this chapter we give an overview of results concerning generic arbitrary eigenvalue assignment by memoryless time-invariant output feedback. The pole placement problem was originally formulated in a state-space framework and therefore, we start with considering state-space models. In the first section we give a preliminary result on generic pole assignment for single-input systems. Then, in section 2.2, we give an overview of the main results that can be found in the literature. The last section contains a behavioral proof of one of the results summarized in section 2.2.

2.1 A preliminary result

Consider the LTI system

$$\sigma x = Ax + Bu, \qquad y = Cx, \tag{2.1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and consider the memoryless time-invariant controller described by $u = Ky$ with $K \in \mathbb{F}^{m\times p}$. The closed-loop system then becomes

$$\sigma x = (A + BKC)x. \tag{2.2}$$

The pole placement problem was formulated in section 1.1.3 as follows: Does there exist, for every polynomial $\alpha(s) = \alpha_0 + \alpha_1 s + \ldots + \alpha_{n-1}s^{n-1} + s^n$, with $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{F}$, a $K \in \mathbb{F}^{m\times p}$ such that

$$det(sI - (A + BKC)) = \alpha(s)? \tag{2.3}$$

Denoting the pole placement map by φ, as introduced in section 1.1.2, (2.3) is equivalent to $\varphi(K) = \bar{\alpha} = (\alpha_0, \cdots, \alpha_{n-1})$, with $\alpha_0, \cdots, \alpha_{n-1}$ the coefficients of $\alpha(s)$. The pole placement problem is equivalent to the question whether φ is onto or not. Now the question arises under what conditions this map φ is onto. It is easily seen by an argument on the dimensions that $n \leq mp$ is a necessary condition on φ to be onto. Could this be a sufficient condition as well?
For the single-input case we have the following lemma.

Lemma 2.1.1 Let $n = mp$ and $m = 1$. The set of systems that can be arbitrarily pole assigned by memoryless output feedback is a generic set.

Proof Consider the LTI system with scalar input $\sigma x = Ax+bu$, $y = Cx$ with $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}^p$ and consider the feedback law $u = Ky$, with $K \in \mathbb{R}^{1\times p}$. The feedback law can be written as

$$u = Ky = KCx. \tag{2.4}$$

If $n = mp$ and $m = 1$, then C is a $n \times n$ matrix which, by Lemma 1.3.3, is generically invertible. This implies that x and y are related in a special way: For every n-vector K, there exists a n-vector $\hat{K}$, such that $K = \hat{K}C^{-1}$. Then, generically, (2.4) can be written

$$u = \hat{K}x. \tag{2.5}$$

So in the special case that $n = mp$ and $m = 1$, output feedback can be regarded as state feedback, that is, it is no restriction to take $C = I$. Lemma 1.3.6 now implies Lemma 2.1.1. □

2.2 Overview of results on memoryless output feedback

The following results are on generic eigenvalue assignability, where conditions are given on the parameters n, m and p.

Theorem 2.2.1 (Kimura, 1975 [11]) If $n \leq m + p - 1$, then for generic A, B, C the poles of the closed-loop system (2.2) can be chosen arbitrarily.

Hermann and Martin (1977) [9] showed that $n \leq mp$ is in fact a sufficient condition for generic eigenvalue assignment if we allow complex feedback gains, i.e. if $\mathbb{F} = \mathbb{C}$. The extension to the case $\mathbb{F} = \mathbb{R}$ failed. A year later, Willems and Hesselink showed that for the first non-trivial case where $n = mp$, i.e. $n=4, m=p=2$, generic pole placement does not hold if $\mathbb{F} = \mathbb{R}$.

Theorem 2.2.2 (Willems and Hesselink, 1978 [21]) Let $\mathbb{F} = \mathbb{R}$. If $n=4, m=2$ and $p=2$ then the pole placement map φ is not generically onto.

Theorem 2.2.3 (Brockett and Byrnes, 1981 [6]) Let the field $\mathbb{F}$ be algebraically closed. If $n \leq mp$ then generically φ is onto. If $n = mp$, then there exist (counting multiplicity)

$$d(m,p) = \frac{1!2!\cdots(p-1)!(mp)!}{m!(m+1)!\cdots(m+p-1)!} \tag{2.6}$$

different feedback compensators assigning a prescribed set of poles $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$. This number $d(m,p)$ is independent of Λ.

If the field $\mathbb{F}$ is closed, Theorem 2.2.3 provides the best possible bound you can hope for, since $n \leq mp$ was shown to be necessary. Even though the real numbers are not algebraically closed, Theorem 2.2.3 can be used to derive a result for real output feedback by using the following argument which can also be found in [6]. If the set Λ is symmetric, so if $\lambda \in \Lambda$ then its complex conjugate $\bar{\lambda}$ is also in the set Λ, then a feedback gain matrix K will assign the same poles as $\bar{K}$. Consequently, in the case that $\mathbb{F} = \mathbb{R}$ and $d(m,p)$ is odd, there will be at least one real K that assigns the set of poles.

Corollary 2.2.4 If $\mathbb{F} = \mathbb{R}, n=mp$ and $d(m,p)$ is odd, then generically φ is onto.

Berstein concluded that $d(m,p)$ is odd only in a few cases.

Theorem 2.2.5 (Berstein [5]) $d(m,p)$ is odd if and only if $\min(m,p)=1$ or if $\min(m,p)=2$ and $\max(m,p)=2^{\ell}-1$, with ℓ a positive integer.

Observe that Theorem 2.2.5 is consistent with Lemma 2.1.1 and Theorem 2.2.2. In the latter case $d(m,p) = d(2,2) = 2$.
Rosenthal and Sottile [17] showed that for $n=mp$ in fact $n=4, m=2$ and $p=2$ is not the only case when arbitrary pole assignment by real output feedback is not generically possible. For the case $\mathbb{F} = \mathbb{R}, n = mp$ and $d(m,p)$ even, they provide a necessary and sufficient condition which guarantees that the pole placement map is not generically surjective.
The best known sufficient condition was claimed by Wang in 1992.

Theorem 2.2.6 (Wang [18]) If $\mathbb{F} = \mathbb{R}$ and $n < mp$, then generically φ is onto.

A few other proofs of Wang's theorem can be found in the literature, [13, 15, 16, 19]. An insightful proof was provided by Willems, [19], who looked at the problem from a behavioral point of view. In the next section we recall this behavioral proof of Theorem 2.2.6.

2.3 Behavioral proof of Wang's theorem

In this section we recall the proof of Theorem 2.2.6 in [19], though we made a few simplifications. Throughout the proof we indicate what the differences are. Although Willems merely considered continuous-time systems, Theorem 2.2.6 also holds for discrete-time systems, since the pole placement problem is of an algebraic nature. Throughout the proof, we therefore use representations involving the σ -notation, where σ can stand for either the differentiation operator or the shift operator.

Preliminaries

In the behavioral approach control is viewed as the interconnection of a plant and a controller. Suppose we are given a system in kernel representation

$$R(\sigma)w = 0, \tag{2.7}$$

where $R \in \mathbb{R}^{p\times(m+p)}[\xi]$, and a real memoryless controller

$$Kw = 0, \tag{2.8}$$

with $K \in \mathbb{R}^{m\times(m+p)}$. Then the *controlled system* is given by

$$\begin{bmatrix} R(\sigma) \\ K \end{bmatrix} w = 0. \tag{2.9}$$

The trajectory w thus has to satisfy both the equations of the plant and the controller. The characteristic polynomial of the controlled system is defined as the monic polynomial which has the same set of zeros as

$$det \begin{bmatrix} R(\xi) \\ K \end{bmatrix}. \tag{2.10}$$

Therefore, the degree of the characteristic polynomial is $\leq n$. When considering systems in kernel representation we adopt a more general definition of arbitrary pole assignability than the one given in Definition 1.1.1.

Definition 2.3.1 The system $R(\sigma)w = 0$ is said to be arbitrary pole assignable if for any given monic polynomial $\alpha(\xi)$ of degree $\leq n$ there exists a control law $Kw = 0$ such that the characteristic polynomial of the controlled system is equal to $\alpha(\xi)$.

We distinguish three types of controllers, depending on the value of (2.10). A controller K is called a

- dependent controller if (2.10) $= 0$
- singular controller if (2.10) has degree $\leq n$
- regular controller if (2.10) has degree n

If a controller is dependent, some non-trivial behavioral equations of the controller are linearly dependent of those of the plant.

In the next chapters we need a convenient expression for determinants. The next lemma provides such an expression.

Lemma 2.3.2 The determinant of a $n \times n$ matrix can be written as a sum of $n!$ products. Each of those products contains n elements of the matrix, such that exactly one element is chosen from each row and exactly one element from each column.

Outline of the proof

The proof is organized as follows. First, we focus on behavioral systems. We consider systems in kernel representation (2.7) with $n < mp$ and real memoryless controllers (2.8). By examining the Taylor expansion of the pole placement map we can prove that generically the pole placement map is surjective. To prove this we construct a controller having special properties, such that the pole placement map is onto. Then we prove that for a generic set of systems (2.7) arbitrary pole assignability (in the sense of Definition 2.3.1) holds.

Then in section 2.3.2 we show that for a generic set of systems

$$\sigma x = Ax + Bu, \qquad y = Cx, \tag{2.11}$$

$n < mp$ is a sufficient condition for arbitrary pole assignment (in the sense of Definition 1.1.1), where n is the number of states, m the number of inputs and p the number of outputs. This result can be regarded as a corollary of the previously established result.

2.3.1 Behavioral part

Consider the linear time-invariant system

$$R(\sigma)w = 0, \tag{2.12}$$

with $R \in {}^{p\times(m+p)}[\xi]$. Assume $n < mp$, where n is the McMillan degree. Furthermore, we assume, without loss of generality that R is row reduced (see Theorem 1.2.7) and that R has full row rank. The observability indices of R are exactly the row degrees of R, hence $\sum_{i=1}^{p} \mu_i = n$. Let the row degrees be denoted by $\mu_1, \mu_2, \ldots, \mu_p$ and suppose they are ordered: $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$.

In this section we prove the following theorem.

Theorem 2.3.3 The set of systems (2.12) with $n < mp$ for which arbitrary pole assignability holds, is a generic set.

Proof Interconnection of (2.12) and a memoryless time-invariant controller

$$Kw = 0, \tag{2.13}$$

with $K \in \mathbb{R}^{m\times(m+p)}$, yields the controlled system

$$\begin{bmatrix} R(\sigma) \\ K \end{bmatrix} w = 0. \tag{2.14}$$

Define the pole assignment map by $\varphi : \mathbb{R}^{m\times(m+p)} \to \mathbb{R}^{n+1}$. It maps a controller $\in \mathbb{R}^{m\times(m+p)}$ into the $n+1$ -dimensional vector space by associating the coefficients of the characteristic polynomial of (2.14) with a vector in $\mathbb{R}^{n+1}$. Let $K^\star$ be a point in $\mathbb{R}^{m\times(m+p)}$. Then the Taylor expansion of φ around $K^\star$ can be written as (2.15).

$$\begin{aligned} \varphi(K^\star + \Delta) &= det \begin{bmatrix} R(\xi) \\ K^\star + \Delta \end{bmatrix} \\ &= det \begin{bmatrix} R(\xi) \\ K^\star \end{bmatrix} + \sum_{i=1}^{m} \sum_{j=1}^{m+p} \Delta_{ij} \bar{R}_{ij}(\xi) + \text{ higher order terms in } \Delta \end{aligned} \tag{2.15}$$

In (2.15) we have used the following: Let $a_1, a_2, \cdots, a_n$ be n columns of a square matrix. Suppose one of the columns, say a_1, can be written as the sum of two vectors, e.g. $a_1 = b_1 + b_2, b_1, b_2 \in \mathbb{R}^n$. Then

$$det \begin{bmatrix} b_1 + b_2 & a_2 & \cdots & a_n \end{bmatrix} = det \begin{bmatrix} b_1 & a_2 & \cdots & a_n \end{bmatrix} + det \begin{bmatrix} b_2 & a_2 & \cdots & a_n \end{bmatrix}. \tag{2.16}$$

The disturbance matrix Δ has the same dimensions as $K^\star$. The element in the i^{th} row and j^{th} column is denoted by Δ_{ij}. $\bar{R}_{ij}$ is equal to the product of $(-1)^{i+j+p}$ and the minor of the $(p+i, j)^{th}$ element of the matrix $\begin{bmatrix} R(\xi) \\ K^\star \end{bmatrix}$.

We show that if $K^\star$ has the properties

P.1 $det \begin{bmatrix} R(\xi) \\ K^\star \end{bmatrix} = 0,$

P.2 The set of polynomials $\bar{R}_{ij}(\xi)$ for $i = 1 \ldots m$ and $j = 1 \ldots m+p$ spans the vector space $\mathbb{R}^{n+1}$ of all polynomials of degree $\leq n$,

then the pole placement map is generically surjective. This is shown as follows.

(i) First, we prove that if $K^\star$ has the properties P.1, P.2, the pole assignment map is surjective.

(ii) Next, for every plant R (2.12), we construct a controller with the property P.1.

(iii) Finally, we establish that for a generic set of plants (2.12), arbitrary pole assignability holds.

(i) By using the Implicit Function Theorem it can be shown that the image of φ contains an open neighborhood of the origin. Details can be found in Appendix B. If the image of φ contains an open neighborhood of the origin, then the image contains all the polynomials of degree $\leq n$ with very small coefficients. This is all we need, since we now can obtain any polynomial of degree $\leq n$, by 'blowing up the coefficients'. Indeed, if

$$\varphi(K) = det \begin{bmatrix} R(\xi) \\ K \end{bmatrix} := \pi(\xi), \tag{2.17}$$

with $\pi(\xi)$ some polynomial of degree $\leq n$ that is in the image of φ, then pre-multiplication of K by the matrix

$$D := \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{2.18}$$

with $d \in \mathbb{R}$, yields

$$\varphi(DK) = d\pi(\xi). \tag{2.19}$$

Thus, if $\pi(\xi)$ is in the image of φ, then so will $d\pi(\xi)$. This means that we can reach every polynomial. Notice that it is crucial that the image of the pole placement map contains an open ball with the origin as center. □

(ii) In this step we construct a (non-trivial) dependent controller. We show that for every plant R (2.12) such a controller can be constructed. The last (p^{th}) row of $R(\xi)$ can be written as below:

$$r_p^0 + r_p^1\xi + \cdots + r_p^{\mu_p-1}\xi^{\mu_p-1} + r_p^{\mu_p}\xi^{\mu_p}. \tag{2.20}$$

We can use the vector coefficients $r_p^0, r_p^1, \ldots, r_p^{\mu_p-1}, r_p^{\mu_p}$ of (2.20) to construct a dependent controller. Define

$$K^* = \begin{bmatrix} r_p^0 \\ r_p^1 \\ \vdots \\ r_p^{\mu_p-1} \\ r_p^{\mu_p} \\ e_{p+\mu_p+1} \\ \vdots \\ e_{m+p-1} \end{bmatrix} \in \mathbb{R}^{m\times(m+p)}. \tag{2.21}$$

In (2.21) e_j is used to denote the j^{th} unit-vector. Indeed, because $\sum_{i=1}^{p} \mu_i = n$ and $\mu_p \leq \ldots \leq \mu_1$ by assumption, the condition $n < mp$ implies that $\mu_p < m$, i.e. the smallest observability index is always smaller than m. Therefore, we can always define such a controller. By construction, the controller is a dependent one, i.e. $det \begin{bmatrix} R(\xi) \\ K^* \end{bmatrix} = 0$. □

Remark 2.3.4 This part differs from [19]. There it was first proven that generically, all observability indices are equal or their value differs by one (see Appendix A), to conclude that in any case $\mu_p < m$ holds. □

(iii) Finally, we show that generically, the minors $\bar{R}_{ij}(\xi), i = 1\ldots m, j = 1\ldots m+p$ span the space of real polynomials of degree $\leq n$ for the constructed dependent controller, implying that generically, arbitrary pole assignability holds. Consider the set S

$$S := \{R \mid \text{ P.2 holds for } K^* \text{ described in (2.21)}\}. \tag{2.22}$$

It can be shown that $R \in \mathbb{R}^{n(m+p)}$, see Appendix A, and hence $S \subset \mathbb{R}^{n(m+p)}$. We first prove that S^c is an algebraic variety. Since K^* is a linear function of the coefficients of R, the coefficients of the polynomials $\bar{R}_{ij}(\xi)$ are polynomial expressions in the coefficients of R. So, if the polynomials $\bar{R}_{ij}(\xi)$ do not span the space of real polynomials of degree $\leq n$, then some polynomial in the coefficients of R must be zero. Therefore there exists a polynomial in $n(m+p)$ variables, such that its set of zeros is equal to the complement of S. Hence S^c is an algebraic variety. Next we prove that S^c is a proper algebraic variety, by providing an element that belongs to S, where we choose the values of the row degrees as follows. Let μ denote the smallest integer that is greater than or equal to $\frac{n}{p}$. Define $p_2 := \mu p - n$ and $p_1 := p - p_2$, so $p_1 + p_2 = p$. If μ is equal to $\frac{n}{p}$ then $p_2 = 0$ and consequently $p_1 = p$. Generically, if n divides p then the observability indices are equal: $\mu_1 = \ldots = \mu_p = \mu$, or else their value differs by one: $\mu_1 = \ldots = \mu_{p_1} = \mu, \mu_{p_1+1} = \ldots = \mu_p = \mu - 1$. Consider the plant given by

$$\begin{bmatrix} \xi^{\mu_1} & 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \xi^{\mu_2} & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \xi^{\mu_{p-1}} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \xi^{\mu_p} & \xi^{\mu_p - 1} & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.23}$$

So, generically, either (2.23) has row degrees μ, or the first p_1 rows have row degree μ, while the last p_2 rows have row degree $\mu - 1$. In both cases we have $p_1\mu + p_2(\mu - 1) = n$. Examining the controlled system matrix $det \begin{bmatrix} R(\xi) \\ K^* \end{bmatrix}$ and using Lemma 2.3.2, it is readily seen that the set of minors $\bar{R}_{ij}(\xi)$ consists of the polynomials

$$\begin{array}{l} 1, \xi, \ldots, \xi^{\mu_p}, \\ \xi^{\mu_1}, \xi^{\mu_1+1}, \ldots, \xi^{\mu_1+\mu_p} \\ \xi^{\mu_1+\mu_2}, \xi^{\mu_1+\mu_2+1}, \ldots, \xi^{\mu_1+\mu_2+\mu_p} \\ \vdots \\ \xi^{\mu_1+\mu_2+\ldots+\mu_{p-1}}, \xi^{\mu_1+\mu_2+\ldots+\mu_{p-1}+1}, \ldots, \xi^{\mu_1+\mu_2+\ldots+\mu_{p-1}+\mu_p}. \end{array} \tag{2.24}$$

The polynomials are listed in increasing powers of ξ such that (2.24) contains every power of ξ up to $\mu_1 + \mu_2 + \ldots + \mu_p = n$, i.e. it contains all the monomials $1, \xi, \ldots, \xi^{n-1}, \xi^n$. These polynomials thus span the $n+1$ -dimensional vector space of polynomials of degree $\leq n$. Therefore, we can conclude that S^c is a proper algebraic variety, implying that S is a generic set. □

Remark 2.3.5 Though the proof in [19] does not cover the cases $n = mp, m = 1$ and $n = mp, p = 1$, we know that in these cases generically, arbitrary pole assignability holds (see Corollary 2.2.4 and Theorem 2.2.5). We cannot define a controller K^* like (2.21), because in these cases we have, generically, $\mu_p = m$. However, in the case $n = mp, m = 1$, we can find one with the properties P.1 and P.2. Take

$$\tilde{K}^* = \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{n+1}. \tag{2.25}$$

Then the set $S := \{R \mid \text{ P.2 holds for } \tilde{K}^*\}$ is a generic set in $\mathbb{R}^{n(n+1)}$. It is easily seen, by a similar argument as the one in step (iii), that the complement S^c is an algebraic variety. Moreover, S^c is a proper algebraic variety, since the following $\tilde{R}$ does belong to the set S.

$$\tilde{R}(\xi) = \begin{bmatrix} \xi & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \xi & 1 & 0 \\ 0 & \cdots & 0 & \xi & 1 \end{bmatrix} \in \mathbb{R}^{n\times(n+1)}[\xi]. \tag{2.26}$$

Clearly,

$$det \begin{bmatrix} \tilde{R}(\xi) \\ \tilde{K}^{\star} \end{bmatrix} = 0. \tag{2.27}$$

and the set of polynomials $\tilde{R}_{ij}(\xi)$ contains the monomials $1, \xi, \ldots, \xi^n$. □

2.3.2 State-space part

The pole placement problem was originally formulated for state-space models. To complete the proof of Theorem 2.2.6, we need to make the connection between kernel representations and state-space models. A state-space model can always be written in a kernel representation $R(\sigma)w = 0$. In Appendix A a procedure is described for the computation of $R \in \mathbb{R}^{p\times(m+p)}[\xi]$, applicable for a generic set of systems. We recall one aspect of this procedure. It is presented that the transformation of state-space model to kernel representation can be regarded as a mapping. Since the computed matrix R has a certain structure, we might say that $R \in \mathbb{R}^{n\times(m+p)}$ (see Appendix A). Hence, the transformation of state-space model to kernel representation can be viewed as a mapping

$$\Psi(A, B, C) = (R_0, R_1, \ldots, R_\mu), \tag{2.28}$$

with $(R_0, R_1, \ldots, R_\mu)$ as defined in Appendix A. The map Ψ is defined for a generic set of (A,B,C), so it is a mapping from a subset of $\mathbb{R}^{n^2+nm+pn}$ to $\mathbb{R}^{n(m+p)}$. Moreover, Ψ is a rational map.
In this section we prove the next corollary.

Corollary 2.3.6 If $n < mp$ then the set of systems $\sigma x = Ax + Bu, y = Cx$ that can be arbitrarily pole assigned by real memoryless output feedback $u = Ky$ is a generic set.

Proof The feedback law $u = Ky$ can be written in kernel representation and is then given by

$$[-K \mid I] \begin{bmatrix} y \\ u \end{bmatrix} = 0. \tag{2.29}$$

We argue that this controller is regular, the proof is given at the end of this section.
If a plant R can be arbitrarily pole assigned by real memoryless output feedback in the sense of Definition 2.3.1, then in particular does there exist a regular feedback law (2.29) such that the characteristic polynomial of the closed-loop system is equal to a given monic polynomial of degree n.
Let $\Upsilon : \mathbb{R}^{n(m+p)} \to \mathbb{R}$ denote a non-zero polynomial map with the property that if

$$\Upsilon(R_0, R_1, \ldots, R_\mu) \neq 0, \tag{2.30}$$

then R is arbitrary pole assignable. In section 2.3.1 we proved that such a map Υ exists. The composition $\Upsilon \circ \Psi$ is a rational mapping from a subset of $\mathbb{R}^{n^2+nm+pn}$ to $\mathbb{R}$. By multiplying $\Upsilon \circ \Psi(A, B, C)$

by a suitable power of $det\ S(A,B,C)$, we can eliminate the denominator. We write $\tilde{\Upsilon}$ for the remainder. Consider the set

$$U := \{(A,B,C) \mid [det\ S(A,B,C)]^2 + [\tilde{\Upsilon}(A,B,C)]^2 \neq 0\} \in \mathbb{R}^{n^2+nm+pn}. \tag{2.31}$$

For systems (2.7) with $(A,B,C) \in U$ arbitrary eigenvalue assignability holds. Since the elements of U^c satisfy an algebraic equation in the elements of the matrices A, B and C, U^c is an algebraic variety in $\mathbb{R}^{n^2+nm+pn}$. We have also seen that there exist matrices (A,B,C) that belong to U. These are the matrices (A,B,C) such that $\Psi(A,B,C) =$(2.23). Thus we can conclude that U is a generic set. □

Remains to prove that the controller $\begin{bmatrix}-K \mid I\end{bmatrix}\begin{bmatrix}y\\u\end{bmatrix} = 0$ is indeed a regular one. This can be concluded from the following lemma.

Lemma 2.3.7 Assume that $R(\xi)$ is derived from a state-space representation defined by matrices A, B, C. Write $R(\xi) = \begin{bmatrix}R_1(\xi) & R_2(\xi)\end{bmatrix}$, with $R_1(\xi) \in \mathbb{R}^{p\times p}, R_2(\xi) \in \mathbb{R}^{p\times m}$. Then $det\begin{bmatrix}R_1(\xi) & R_2(\xi)\\ K_1 & K_2\end{bmatrix}$ has degree $n \Leftrightarrow K_2$ is invertible.

Proof A matrix $R(\xi) = \begin{bmatrix}R_1(\xi) & R_2(\xi)\end{bmatrix}$ derived from a system described by matrices A, B, C has the properties that

- degree $det R_1(\xi) = n$ (see Appendix A, (A.14))
- $R_1^{-1}(\xi)R_2(\xi)$ is strictly proper (see Appendix A)

Using these properties, we have

$$det\begin{bmatrix}R_1(\xi) & R_2(\xi)\\ K_1 & K_2\end{bmatrix} = det\left(\begin{bmatrix}R_1(\xi) & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}I & R_1^{-1}(\xi)R_2(\xi)\\ K_1 & K_2\end{bmatrix}\right) \tag{2.32}$$

$$= det\, R_1(\xi)\cdot\left(det\, K_2 + \sum_i c_iY_i(\xi)\right). \tag{2.33}$$

In (2.33), $\sum_i c_iY_i(\xi)$ is a sum of products of elements of K_1, K_2 and $R_1^{-1}(\xi)R_2(\xi)$, where the c_i's are all constants and the $Y_i(\xi)$'s are strictly proper functions for all i, since $R_1^{-1}(\xi)R_2(\xi)$ is strictly proper. So the sum of c_iY_i is a sum of strictly proper functions and therefore it has degree $< n$. Now it is easily seen that (2.33) has degree n if and only if K_2 has a non-zero determinant. □

Proof that $u = Ky$ is a regular controller It follows from Lemma 2.3.7 that regular controllers $Kw = 0$ can be partitioned as $K = \begin{bmatrix}K_1 \mid K_2\end{bmatrix}$, with $K_1 \in \mathbb{R}^{m\times p}, K_2 \in \mathbb{R}^{m\times m}$ and K_2 invertible. Observe that

$$\begin{bmatrix}K_1 \mid K_2\end{bmatrix}\begin{bmatrix}y\\u\end{bmatrix} = 0 \tag{2.34}$$

is equivalent to

$$\begin{bmatrix}-(K_2)^{-1}K_1 \mid I\end{bmatrix}\begin{bmatrix}y\\u\end{bmatrix} = 0. \tag{2.35}$$

Therefore, a regular controller $\begin{bmatrix} K_1 \mid K_2 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = 0$ can be written as $\begin{bmatrix} -K \mid I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = 0$, or, equivalently, $y = Ku$. □

Chapter 3

Periodic output feedback

In the present chapter we consider the use of periodically time-varying output feedback to assign the closed-loop poles of a linear time-invariant system. Previous results on the pole assignment of SISO systems by periodic output feedback can be found in [1, 2, 3, 10]. In section 3.1 we mention one [3] that mainly covers the existing results. Then in the next section we present our problem statement, which is considered from a behavioral point of view in section 3.2.2. Our main results are obtained in section 3.2.2.

3.1 A result on periodic output feedback for SISO systems

Consider the discrete-time LTI single-input single output (SISO) system

$$x_{i+1} = Ax_i + bu_i, \qquad y_i = cx_i \tag{3.1}$$

with $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}, y_i \in \mathbb{R}$. Here the time-axis is taken to be $\mathbb{Z}$, i.e. $i \in \mathbb{Z}$. When applying a T-periodic output feedback law of the form

$$u_{\ell T+i} = k_{T-i-1} y_{\ell T+i} \tag{3.2}$$

where $i = 0, 1, \ldots, T-1$ and $\ell \in \mathbb{Z}$, we obtain the closed-loop system as a periodic system with period T. We can assign to the periodic system a time-invariant system, with time-interval equal to T. This is referred to as *lifting*, and the resulting time-invariant system is called the *lifted system*. Applying this technique to the closed-loop system of (3.1) and (3.2), the lifted system is given by

$$x_{(\ell+1)T} = (A + bk_0c)(A + bk_1c) \cdot \ldots \cdot (A + bk_{T-1}c)x_{\ell T}, \tag{3.3}$$

with $\ell \in \mathbb{Z}$. Denoting $\zeta_\ell = x_{\ell T}$, equation (3.3) is equal to

$$\zeta_{\ell+1} = (A + bk_0c)(A + bk_1c) \cdot \ldots \cdot (A + bk_{T-1}c)\zeta_\ell. \tag{3.4}$$

The technique of lifting enables us to define the notion of the characteristic polynomial of a periodically time-varying system. Indeed, the lifted system is autonomous and hence, the dynamics of system (3.4) are governed by the eigenvalues of the system matrix or, equivalently, the zeros of the characteristic polynomial $det(sI - (A + bk_0c) \cdot \ldots \cdot (A + bk_{T-1}c))$.

We would like to have explicit conditions for assignment of the closed-loop poles by means of periodic output feedback. The question is: To what extent do there exist feedback gains k_i, $i = 0, 1, \ldots, T-1$ such that the poles of the closed-loop system (3.4) can be assigned arbitrarily, i.e. such that the characteristic polynomial of the closed-loop system (3.4) is equal to a given polynomial? And how should we choose the value of T? An answer to the latter question can be found in [8]. Greschak and Verghese proved that, under certain conditions, a second order plant can be arbitrarily pole assigned by a periodic feedback law with period three, see also [1]. In 1992 Aeyels and Willems [3] extended their result to systems of arbitrary order n, and proved that for arbitrary pole assignment by periodic output feedback a period of at least $n+1$ is needed.

So let $T = n + 1$ and define

$$A_e = A(A + bk_1c) \cdot \ldots \cdot (A + bk_nc), \tag{3.5}$$

$$c_e = c(A + bk_1c) \cdot \ldots \cdot (A + bk_nc), \tag{3.6}$$

then equation (3.4) can be written as a set of equations

$$\zeta_{\ell+1} = A_e\zeta_\ell + bv_\ell, \tag{3.7}$$

$$\eta_\ell = c_e\zeta_\ell, \tag{3.8}$$

where $v_\ell = k_0\eta_\ell$.

Theorem 3.1.1 (Aeyels and Willems, 1992 [3]) Consider the system (3.1) and assume that it is controllable and observable and that A is non-singular. Consider its transfer function

$$h(s) = c(sI - A)^{-1}b = \frac{q_{n-1}s^{n-1} + \ldots + q_1s + q_0}{s^n + p_{n-1}s^{n-1} + \ldots + p_1s + p_0}. \tag{3.9}$$

Let $p_0, p_1, \ldots, p_{n-1}, q_0, q_1, \ldots, q_{n-1}$ all be unequal to zero. Moreover, let the fractions p_i/q_i be mutually different for $i = 0, 1, \ldots, n-1$, and let

$$\mathcal{R}(k) = \begin{bmatrix} b & A_eb & \ldots & (A_e)^{n-1}b \end{bmatrix}, \tag{3.10}$$

with $k = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$.
Except for closed-loop poles at the origin, arbitrary pole placement by means of periodic output feedback with period $n+1$ is possible if rank $\mathcal{R}(k_0) = n$ for $k_0 = \begin{bmatrix} p_{n-1}/q_{n-1} & p_{n-2}/q_{n-2} & \cdots & p_0/q_0 \end{bmatrix}$.

3.2 On generic pole placement by periodic output feedback

The ideas presented in the previous section can be readily extended to MIMO systems. Doing so we can formulate our problem statement in the next section.

3.2.1 Problem statement

Consider the discrete-time LTI system

$$x_{i+1} = Ax_i + Bu_i, \qquad y_i = Cx_i, \tag{3.11}$$

with $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m, y_i \in \mathbb{R}^p$ and $i \in \mathbb{Z}$, i.e. $\mathbb{Z}$ denotes the time-axis. Applying the periodic output feedback law with period T

$$u_{\ell T+i} = K_{T-i-1}y_{\ell T+i}, \tag{3.12}$$

where $i \in \{0, 1, \ldots, T-1\}$ and $\ell \in \mathbb{Z}$, leads to a periodic closed-loop system with period T. We assign to the periodic system a time-invariant system with time-interval T.

Defining $\zeta_\ell = x_{\ell T}$, the lifted closed-loop system is given by

$$\zeta_{\ell+1} = (A + BK_0C)(A + BK_1C) \cdot \ldots \cdot (A + BK_{T-1}C)\zeta_\ell := A_L\zeta_\ell. \tag{3.13}$$

The problem of pole assignment amounts to selecting T feedback gain matrices such that the characteristic polynomial of A_L equals some given monic polynomial of degree n.
Define the pole placement map φ as follows

$$\varphi : \mathbb{R}^{T(m+p)} \to \mathbb{R}^n, \tag{3.14}$$
$$\varphi(K_0, K_1, \ldots, K_{T-1}) = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}), \tag{3.15}$$

where $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ is the n -vector associated with the n^{th} order real monic polynomial

$$\alpha(\xi) = \alpha_0 + \alpha_1\xi + \ldots + \alpha_{n-1}\xi^{n-1} + \xi^n = det(\xi I - A_L), \tag{3.16}$$

with $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{R}$. If for every polynomial $\alpha(\xi)$ (3.16) there exist feedback gains $K_0, \ldots, K_{T-1}$ such that $\varphi(K_0, \ldots, K_{T-1}) = (\alpha_0, \ldots, \alpha_{n-1})$, then we say that system (3.11) can be arbitrarily pole assigned by T -periodic output feedback. By an argument on the dimensions, it is easily seen that $n \leq mpT$ is a necessary condition for arbitrary pole assignability. We know that this cannot be sufficient as well. Indeed, the case $T = 1$, corresponding to a time-invariant controller, would lead to the inequality $n \leq mp$, which by Theorem 2.2.2 is known not to be sufficient.

In the next section we consider this problem from a behavioral point of view. By extending the behavioral proof in section 2.3 to periodically time-varying controllers it is shown that for some choices for n, m, p, T generically $n < mpT$ is a sufficient condition for arbitrary pole placement.

3.2.2 Main results

Consider the discrete-time LTI system

$$R(\sigma)w = 0, \tag{3.17}$$

with $R(\xi) \in \mathbb{R}^{p\times(m+p)}[\xi]$. In (3.17), and from now on, σ stands for the shift operator. We take the behavior defined by (3.17) to be the set of all trajectories $w : \mathbb{Z} \to \mathbb{R}^{m+p}$, i.e. the time-axis is taken to be $\mathbb{Z}$ and the signal space is denoted by $\mathbb{R}^{m+p}$. Assume that R has McMillan degree n. Furthermore, assume (without loss of generality) that R is row reduced.
With system (3.17) we can associate a discrete-time LTI system of the same McMillan degree n, whose behavior is the set of trajectories $w^L : \mathbb{Z} \to \mathbb{R}^{T(m+p)}$. The association of (3.17) with such a system is captured by the notion of lifting. How lifting affects the representation (3.17) is presented in [12]. We recapitulate the procedure below. Write the matrix R as a sum of T matrices that contain powers of ξ that are taken modulo T. Thus, write

$$R(\xi) = R_0(\xi^T) + \xi^{-1}R_1(\xi^T) + \ldots + \xi^{-(T-1)}R_{T-1}(\xi^T). \tag{3.18}$$

Then the lifted system is described by a $Tp \times T(m+p)$ -system matrix

$$\begin{bmatrix} R_0(\sigma) & \sigma^{-1}R_{T-1}(\sigma) & \cdots & \sigma^{-1}R_1(\sigma) \\ R_1(\sigma) & R_0(\sigma) & \cdots & \sigma^{-1}R_2(\sigma) \\ \vdots & \vdots & \ddots & \vdots \\ R_{T-1}(\sigma) & R_{T-2}(\sigma) & \cdots & R_0(\sigma) \end{bmatrix} w^L = 0, \tag{3.19}$$

with

$$w_i^L = \begin{pmatrix} w_{Ti+1} \\ w_{Ti+2} \\ \vdots \\ w_{Ti+T} \end{pmatrix}. \tag{3.20}$$

The lifted variable w^L thus takes it values in the space $\mathbb{R}^{T(m+p)}$.
Each of the matrices R_i, $i = 0, \dots, T-1$, has the same dimensions as R has. Consider the periodic control law with period T

$$K_r(\sigma)w = 0, \tag{3.21}$$

with $r = 0, 1, \dots, T-1$. Interconnection of the lifted system and the T -periodic controller yields the controlled lifted system

$$\begin{bmatrix} R_0(\sigma) & \sigma^{-1}R_{T-1}(\sigma) & \cdots & \sigma^{-1}R_1(\sigma) \\ R_1(\sigma) & R_0(\sigma) & \cdots & \sigma^{-1}R_2(\sigma) \\ \vdots & \vdots & \ddots & \vdots \\ R_{T-1}(\sigma) & R_{T-2}(\sigma) & \cdots & R_0(\sigma) \\ K_0 & \bullet & \cdots & \bullet \\ \bullet & K_1 & \cdots & \bullet \\ \vdots & \ddots & \ddots & \vdots \\ \bullet & \cdots & \cdots & K_{T-1} \end{bmatrix} w^L = 0, \tag{3.22}$$

where the $\bullet$'s stand for $m \times (m+p)$ zero-matrices.
Let us denote the controlled lifted system-matrix in (3.19) by $R^L(\sigma)$, and denote the controller with T blocks on the diagonal by $\bar{K}$, i.e. $\bar{K} = diag(K_0, \dots, K_{T-1})$. Then the controlled lifted system is written in short hand notation

$$\begin{bmatrix} R^L(\sigma) \\ \bar{K} \end{bmatrix} w^L = 0. \tag{3.23}$$

Due to this technique of lifting, the controlled system is time-invariant. This enables us to define the characteristic polynomial of the controlled system. The characteristic polynomial of (3.23) is defined to be the monic polynomial having the same roots as

$$det \begin{bmatrix} R^L(\xi) \\ \bar{K} \end{bmatrix}. \tag{3.24}$$

Define

$$det \begin{bmatrix} R^L(\xi) \\ \bar{K} \end{bmatrix} = \alpha_0 + \alpha_1\xi + \dots \alpha_{n-1}\xi^{n-1} + \alpha_n\xi^n. \tag{3.25}$$

With the polynomial $\alpha(\xi)$ we can associate the vector $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1}$. Then we can define the pole assignment map

$$\begin{aligned} &\varphi : \mathbb{R}^{m \times T(m+p)} \to \mathbb{R}^{n+1}, \\ &\varphi(K_0, \dots, K_{T-1}) = (\alpha_0, \alpha_1, \dots, \alpha_n). \end{aligned} \tag{3.26}$$

The pole placement map φ is a mapping to the real $n+1$ -dimensional vector space by letting the coefficients of (3.26) correspond with a vector in $\mathbb{R}^{n+1}$.

Definition 3.2.1 The system $R(\sigma)w = 0$ is said to be arbitrary pole assignable by T -periodic feedback if for any given monic polynomial $\alpha(\xi)$ of degree $\leq n$ there exists a T -periodic control law (3.21) such that the characteristic polynomial of the controlled system is equal to $\alpha(\xi)$.

We are now ready to formulate the main conjecture of this report.

Conjecture 3.2.2 The set of systems (3.17) with $n < mpT$ that can be pole assigned arbitrarily by periodically time-varying feedback with period T is a generic set.

We prove the conjecture for some specific choices for the parameters n, m, p and T. Since every state-space model can be written in input- output representation, Conjecture 3.2.2 can be formulated and proved for state-space models as well. As explained in the previous chapter, the state-space result can be regarded as a corollary of Conjecture 3.2.2. So, it is sufficient to focus on the behavioral part of the proof.

Proof The proof given in section 2.3 can be extended to periodically time-varying controllers. To prove Conjecture 3.2.2 we show an equivalent statement, i.e. we show that generically, the pole assignment map is surjective.

Consider the Taylor expansion of the pole placement map φ around a point $\bar{K}^\star \in \mathbb{R}^{m\times T(m+p)}$

$$\begin{aligned}\varphi(\bar{K}^\star + \Delta) &= det\begin{bmatrix} R^L(\xi) \\ \bar{K}^\star + \Delta \end{bmatrix} \\ &= det\begin{bmatrix} R^L(\xi) \\ \bar{K}^\star \end{bmatrix} + \sum_{i=0}^{T-1}\sum_{j=1}^{m}\sum_{k=1}^{m+p} \Delta^i_{jk}\bar{R}^L_{ijk}(\xi) + \text{ higher order terms in } \Delta.\end{aligned} \tag{3.27}$$

In (3.27) we used (2.16). The disturbance $\Delta = diag(\Delta^0, \Delta^1, \dots, \Delta^{T-1})$ is an element of $\mathbb{R}^{m\times T(m+p)}$, so $\Delta^i \in \mathbb{R}^{m\times(m+p)}, i = 0, \dots, T-1$. Δ^i_{jk} is the element in row j and column k of the matrix Δ^i. $\bar{R}^L_{ijk}$ is computed by multiplying the factor $(-1)^{Tp+i(2m+p)+j+k}$ by the minor of the $(Tp + im + j, k + i(m+p))^{th}$ element of the matrix $\begin{bmatrix} R^L(\xi) \\ \bar{K}^\star \end{bmatrix}$.

Let $\bar{K}^\star$ have the properties

a.1 $det\begin{bmatrix} R^L(\xi) \\ \bar{K}^\star \end{bmatrix} = 0,$

a.2 the set of polynomials $\bar{R^L}_{ijk}(\xi)$ for $i = 0, \dots, T-1$, $j = 1 \dots m$ and $k = 1 \dots m+p$ spans the space of real polynomials of polynomials of degree $\leq n$.

We can prove that if $\bar{K}^\star$ has the properties described above, the pole placement map is generically surjective. We successively follow the next three steps

1. By examining the Taylor expansion of φ, we show that if $\bar{K}^\star$ has the properties a.1 and a.2 then φ is surjective.
2. Then we construct a dependent controller having property a.1.
3. Finally, we establish that the set of systems for which the constructed controller has the properties a.1 and a.2 is a generic set.

1. By Theorem B.0.5 and Lemma B.0.6, the image of φ contains an open neighborhood of the origin. This implies that φ is surjective, since every polynomial of degree $\leq n$ can be reached by pre-multiplying $\bar{K}^\star$ by a suitable matrix.

2. Now we construct a (non-trivial) controller $\bar{K}^\star$ with period T, $\bar{K}^\star := diag(\bar{K}_0^\star, \ldots, \bar{K}_{T-1}^\star)$, such that $\varphi(\bar{K}^\star) = 0$. We use the first row of blocks of (3.19) to construct a dependent controller. The first row of blocks of R^L is given by

$$\begin{bmatrix} R_0 & \xi^{-1}R_{T-1} & \ldots & \xi^{-1}R_1 \end{bmatrix}. \tag{3.28}$$

From now on we assume that n is as large as possible, i.e. $n = mpT - 1$. The highest power of ξ found in the last row of R is $\mu_p < mT$. Then, because of (3.18), the highest power of ξ in the last row of R_0 is at most $m-1$, and the highest powers of ξ in the last row of $R_1, \ldots, R_{T-1}$ are at most m. Hence, the last row of (3.28) contains powers of ξ up to at most $m-1$. Let

$$r_i^0 + r_i^1\xi + \ldots + r_i^{m-1}\xi^{m-1} \tag{3.29}$$

denote the last row of $\xi^{-1}R_{T-i}$, for $i = 1, \ldots, T-1$ or $R_i, i = 0$.
Let us define

$$K_0^\star = \begin{bmatrix} r_0^0 \\ r_0^1 \\ \vdots \\ r_0^{m-1} \end{bmatrix}, K_1^\star = \begin{bmatrix} r_1^0 \\ r_1^1 \\ \vdots \\ r_1^{m-1} \end{bmatrix}, \ldots\ldots, K_{T-1}^\star = \begin{bmatrix} r_{T-1}^0 \\ r_{T-1}^1 \\ \vdots \\ r_{T-1}^{m-1} \end{bmatrix}. \tag{3.30}$$

Because of the diagonal structure of the controller,

$$det \begin{bmatrix} R^L \\ \bar{K}^\star \end{bmatrix} = 0 \tag{3.31}$$

holds. This is easily seen as follows. The p^{th} row of R^L can be made equal to the zero row, by successively subtracting $(r_0^0 + r_1^0 + \ldots + r_{m-1}^0), \xi(r_0^1 + r_1^1 + \ldots + r_{m-1}^1), \ldots, \xi^{m-1}(r_0^{T-1} + r_1^{T-1} + \ldots + r_{m-1}^{T-1})$ from the p^{th} row of R^L. □

Remark 3.2.3 Because of the assumption that $n = mpT - 1$, we have exactly as many row vectors r_i (3.29) as rows in the controller $\bar{K}^\star = diag(\bar{K}_0^\star, \ldots, \bar{K}_{T-1}^\star)$. If n is strictly smaller than $mpT - 1$, we need to define extra rows for the construction of a dependent controller. In [19], these extra vectors were taken to be unit vectors (see (2.21)). At this stage, it is not clear which choice for vectors would be suitable for the periodic controller. Intuitively, we could expect that if generic systems with $n = mpT - 1$ can be pole assigned arbitrarily, then we would also have arbitrary pole assignability for generic systems with a less number of states, i.e. $n < mpT - 1$. □

3. Finally, we show that the set of systems for which the constructed controller (3.30) has properties a.1 and a.2 is a generic set. We only consider a few specific cases, i.e. we make choices for n, m, p, T. For each case we prove that the set

$$\bar{S} := \{R \mid \text{ (a.2) holds for } \bar{K}^\star \text{ defined in(3.30)} \subset \mathbb{R}^{p(m+p)} \tag{3.32}$$

is a generic set. First, observe that the complement $\bar{S}^c$ is an algebraic variety. Indeed, $\bar{K}^\star$ is a linear expression in R^L, and hence it is linear in R. If the polynomials $\bar{R}_{ijk}(\xi)$ do not span the space of all polynomials of degree $\leq n$, then some polynomial in the coefficients of R must be zero. Then there exists a polynomial in $p(m+p)$ variables, such that its set of zeros is equal to $\bar{S}^c$. Remains to prove that $\bar{S}^c$ is a proper algebraic variety. So, we have to specify an element of $\bar{S}$, with the restriction that the spanning minors $\bar{R}_{ijk}(\xi)$ appear in the diagonal blocks corresponding to the diagonal blocks of the controller. Because of the complexity (first R is lifted, then interconnected with a controller having a diagonal structure), specifying an element is not at all trivial. Assuming $n = mpT - 1$ and making choices for the parameters n, m, p and T, we provide polynomial matrices which are elements of $\bar{S}$. So, in those cases, the specification of an element of $\bar{S}$ completes the proof that for those cases, arbitrary pole assignment by periodic output feedback holds. Expressions for the relevant minors are easily obtained by using Lemma 2.3.2.

Case 1: m=1, p=1 The equality $n = mpT - 1$ reduces to $n = T - 1$, which is consistent with [3], where it is stated that at least a period of $n + 1$ is needed for arbitrary pole assignment of an n^{th} order system. We provide a matrix R for which $\bar{K}^*$ has the property a.2.

$$R(\xi) = \begin{bmatrix} 1 + \xi^{T-1} & \xi + \ldots + \xi^{T-1} \end{bmatrix}. \tag{3.33}$$

Then the T matrices $R_0, \ldots, R_{T-1}$ (3.18) are

$$\begin{aligned} R_0 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ R_1 &= \begin{bmatrix} \xi & \xi \end{bmatrix}, \\ R_2 &= \begin{bmatrix} 0 & \xi \end{bmatrix}, \\ &\vdots \\ R_{T-1} &= \begin{bmatrix} 0 & \xi \end{bmatrix}. \end{aligned} \tag{3.34}$$

Consequently, the lifted system matrix R^L is given by

$$R^L(\xi) = \begin{bmatrix} 1 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1 \\ \xi & \xi & 1 & 0 & & 0 & 1 & 0 & 1 \\ \vdots & \vdots & & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \xi & 0 & \xi & \ldots & 1 & 0 & 0 & 1 \\ 0 & \xi & 0 & \xi & \ldots & \xi & \xi & 1 & 0 \end{bmatrix}. \tag{3.35}$$

By (3.30) the T controllers are

$$K_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, K_1 = \ldots = K_{T-2} = \begin{bmatrix} 0 & 1 \end{bmatrix}, K_{T-1} = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{3.36}$$

Computing all the minors, we get the following polynomials on the diagonal blocks (up to minus signs):

$$\begin{array}{lcc} \text{Block 1} & 0 & \xi^{T-2} - \xi^{T-3} + \xi^{T-4} - \ldots + \xi^2 - \xi - 1 \\ \text{Block 2} & 2\xi & 0 \\ \text{Block 3} & -2\xi^2 + 2\xi & 0 \\ \text{Block 4} & 2\xi^3 - 2\xi^2 + 2\xi & 0 \\ \vdots & \vdots & \vdots \\ \text{Block T-1} & -2\xi^{T-2} + \ldots - 2\xi^2 + 2\xi & 0 \\ \text{Block T} & \xi^{T-1} - \xi^{T-2} + \ldots - \xi^2 + \xi & -\xi^{T-1} + \xi^{T-2} - \ldots + \xi^2 - \xi \end{array} \tag{3.37}$$

It is easily seen that these polynomials in fact span the space of all polynomials of degree $\leq T - 1 = n$.
□

Case 2: p=1, T=2 In this case the McMillan degree equals $n = 2m - 1$. The matrix

$$R(\xi) = \begin{bmatrix} 1 & \xi + \xi^2 & \cdots & \xi^{2m-3} + \xi^{2m-2} & \xi^{2m-1} \end{bmatrix} \tag{3.38}$$

leads to the lifted system matrix

$$R^L(\xi) = \begin{bmatrix} 1 & \xi & \cdots & \xi^{m-1} & 0 & 0 & 1 & \cdots & \xi^{m-2} & \xi^{m-1} \\ 0 & \xi & \cdots & \xi^{m-1} & \xi^m & 1 & \xi & \cdots & \xi^{m-1} & 0 \end{bmatrix}, \tag{3.39}$$

which would, by (3.30), lead to the controlled lifted system matrix

$$\begin{bmatrix} 1 & \xi & \cdots & \xi^{m-1} & 0 & 0 & 1 & \cdots & \xi^{m-2} & \xi^{m-1} \\ 0 & \xi & \cdots & \xi^{m-1} & \xi^m & 1 & \xi & \cdots & \xi^{m-1} & 0 \\ 1 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots & \vdots & & & & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & & \vdots & \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3.40}$$

With Lemma 2.3.2 it is easily seen that most of the minors are zero. The only polynomials $\bar{R}_{ijk}(\xi)$ unequal to zero are the polynomials $\bar{R}_{0jk}(\xi), j = 1, \ldots, m,\ k = m + 1$ and the polynomials $\bar{R}_{1jk}(\xi), j = 1, \ldots, m,\ k = m + 2$. These are the polynomials in the last column of the first block and those in the first column of the second block. The first block contains the polynomials $1, \xi, \ldots, \xi^{m-1}$, while the second block contains $\xi^m, \xi^{m+1}, \ldots, \xi^{2m-1}$. These obviously span the space of all real polynomials of degree $\leq n$. □

Case 3: p=1, T=3 The equality $n = mpT - 1$ reduces to $n = 3m - 1$ and the matrix

$$R(\xi) = \begin{bmatrix} 2 + \xi^{3m-1} & \xi + \xi^2 + \xi^3 & \cdots & \xi^{3m-5} + \xi^{3m-4} + \xi^{3m-3} & \xi^{3m-2} + \xi^{3m-1} \end{bmatrix} \tag{3.41}$$

is an element of $\bar{S}$. The controlled lifted system matrix is given by

$$\begin{bmatrix} & R^L(\xi) & \\ K_0^* & & \\ & K_1^* & \\ & & K_2^* \end{bmatrix} = \tag{3.42}$$

$$\begin{bmatrix}
2 & \xi & \cdots & \xi^{m-1} & 0 & 0 & 1 & \cdots & \xi^{m-2} & \xi^{m-1} & \xi^{m-1} & 1 & \cdots & \xi^{m-2} & \xi^{m-1} \\
\xi^m & \xi & \cdots & \xi^{m-1} & \xi^m & 2 & \xi & \cdots & \xi^{m-1} & 0 & 0 & 1 & \cdots & \xi^{m-2} & \xi^{m-1} \\
0 & \xi & \cdots & \xi^{m-1} & \xi^m & \xi^m & \xi & \cdots & \xi^{m-1} & \xi^m & 2 & \xi & \cdots & \xi^{m-1} & 0 \\
2 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & & & \vdots & \vdots & & & & \vdots & \vdots & & & & \vdots \\
\vdots & & \ddots & & \vdots & \vdots & & & & \vdots & \vdots & & & & \vdots \\
0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & \vdots & \vdots & & \ddots & & \vdots & \vdots & & & & \vdots \\
\vdots & & & & \vdots & \vdots & & & \ddots & \vdots & \vdots & & & & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots & \vdots & & \ddots & & \vdots \\
\vdots & & & & \vdots & \vdots & & & & \vdots & \vdots & & & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 1
\end{bmatrix}$$

Most of the minors $\bar{R}_{ijk}$ are equal to zero. The first $(m \times (m+1))$ block of minors corresponding to the elements in that block is given by:

$$\begin{matrix}
0 & \cdots & 0 & -2\xi^{2m-1} - 8 \\
0 & \cdots & 0 & \xi(-2\xi^{2m-1} - 8) \\
\vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \xi^{m-1}(-2\xi^{2m-1} - 8)
\end{matrix} \tag{3.43}$$

So the first m columns contain the zero polynomials and only the last column contains non-zero polynomials. The second $(m \times (m+1))$ block of minors also contains m^2 zero polynomials, and one column with non-zero polynomials:

$$\begin{matrix}
2\xi^{2m-1} + 4\xi^m & 0 & \cdots & 0 \\
\xi(2\xi^{2m-1} + 4\xi^m) & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
\xi^{m-1}(2\xi^{2m-1} + 4\xi^m) & 0 & \cdots & 0
\end{matrix} \tag{3.44}$$

and the last block equals:

$$\begin{matrix}
-2\xi^{2m} + 4\xi^m & 0 & \cdots & 0 & 2\xi^{2m} - 4\xi^m \\
\xi(-2\xi^{2m} + 4\xi^m) & 0 & \cdots & 0 & \xi(2\xi^{2m} - 4\xi^m) \\
\vdots & \vdots & & \vdots & \vdots \\
\xi^{m-1}(-2\xi^{2m} + 4\xi^m) & 0 & \cdots & 0 & \xi^{m-1}(2\xi^{2m} - 4\xi^m)
\end{matrix} \tag{3.45}$$

Case 4: p=1, T=4 In this case the McMillan degree equals $n = 4m - 1$. An element of $\bar{S}$ (3.32) is the matrix

$$R(\xi) = [\, 2+\xi^{4m-1} \quad \xi+\xi^2+\xi^3+\xi^4 \quad \cdots\cdots \quad \xi^{4m-7}+\xi^{4m-6}+\xi^{4m-5}+\xi^{4m-4} \quad \xi^{4m-3}+\xi^{4m-2}+\xi^{4m-1} \,]. \tag{3.46}$$

Writing $R(\xi)$ as a sum of matrices (3.18), then

$$\begin{aligned} R_0 &= \begin{bmatrix} 2 & \xi & \cdots\cdots & \xi^{m-1} & 0 \end{bmatrix}, \\ R_1 &= \begin{bmatrix} \xi^{m-1} & \xi & \cdots\cdots & \xi^{m-1} & \xi^m \end{bmatrix}, \\ R_2 &= \begin{bmatrix} 0 & \xi & \cdots\cdots & \xi^{m-1} & \xi^m \end{bmatrix}, \\ R_3 &= \begin{bmatrix} 0 & \xi & \cdots\cdots & \xi^{m-1} & \xi^m \end{bmatrix}. \end{aligned} \tag{3.47}$$

Though in this case the controlled lifted system matrix is larger than the one in the previous case, (3.41), they are similar.
The result of the calculation of all the minors corresponding to the elements in the 4 diagonal blocks is found below (all polynomials should be multiplied by -1 if m is odd).

$$\begin{array}{ccccc} & & \text{Block 1} & \text{Block 2} & \\ 0 & \cdots & 0 \quad -2\xi^{3m-1}-4\xi^{2m-1}-16 & 4\xi^{2m-1}+8\xi^m & 0 \ \cdots \ 0 \\ 0 & \cdots & 0 \quad \xi(-2\xi^{3m-1}-4\xi^{2m-1}-16) & \xi(4\xi^{2m-1}+8\xi^m) & 0 \ \cdots \ 0 \\ \vdots & & \vdots \qquad \vdots & \vdots & \vdots \qquad \vdots \\ 0 & \cdots & 0 \quad \xi^{m-1}(-2\xi^{3m-1}-4\xi^{2m-1}-16) & \xi^{m-1}(4\xi^{2m-1}+8\xi^m) & 0 \ \cdots \ 0 \end{array} \tag{3.48}$$

Block 3

$$\begin{array}{ccc} -2\xi^{3m-1}-4\xi^{2m}+4\xi^{2m-1}+8\xi^m & 0 \ \cdots \ 0 \\ \xi(-2\xi^{3m-1}-4\xi^{2m}+4\xi^{2m-1}+8\xi^m) & 0 \ \cdots \ 0 \\ \vdots & \vdots \qquad \vdots \\ \xi^{m-1}(-2\xi^{3m-1}-4\xi^{2m}+4\xi^{2m-1}+8\xi^m) & 0 \ \cdots \ 0 \end{array} \tag{3.49}$$

Block 4

$$\begin{array}{ccc} 2\xi^{3m}-4\xi^{2m}+8\xi^m & 0 \ \cdots \ 0 & -2\xi^{3m}+4\xi^{2m}-8\xi^m \\ \xi(2\xi^{3m}-4\xi^{2m}+8\xi^m) & 0 \ \cdots \ 0 & \xi(-2\xi^{3m}+4\xi^{2m}-8\xi^m) \\ \vdots & \vdots \qquad \vdots & \vdots \\ \xi^{m-1}(2\xi^{3m}-4\xi^{2m}+8\xi^m) & 0 \ \cdots \ 0 & \xi^{m-1}(-2\xi^{3m}+4\xi^{2m}-8\xi^m) \end{array} \tag{3.50}$$

Like the previous case, the last block contains two columns with non-zero polynomials. One column is a multiple of the other one. So we have $4m$ linearly independent polynomials and they span the space of all real polynomials of degree $\leq 4m - 1 = n$. This result can be easily extended for cases with $p = 1$ and larger T.

Case 5: p=1, T=5 In this case the McMillan degree equals $5m - 1$. We argue that the following $R(\xi)$ is an element of $\bar{S}$.

$$R(\xi) = [\, 2+\xi^{5m-1} \quad \xi+\xi^2+\xi^3+\xi^4+\xi^5 \quad \cdots\cdots \quad \xi^{5m-9}+\xi^{5m-8}+\xi^{5m-7}+\xi^{5m-6}+\xi^{5m-5} \quad \xi^{5m-4}+\xi^{5m-3}+\xi^{5m-2}+\xi^{5m-1} \,] \tag{3.51}$$

Then the matrices R_0, R_1, R_2, R_3 and R_4 are given by

$$\begin{aligned} R_0 &= [2 \quad \xi \quad \cdots\cdots \quad \xi^{m-1} \quad 0] \\ R_1 &= [\xi^{m-1} \quad \xi \quad \cdots\cdots \quad \xi^{m-1} \quad \xi^m] \\ R_2 &= [0 \quad \xi \quad \cdots\cdots \quad \xi^{m-1} \quad \xi^m] \\ R_3 &= [0 \quad \xi \quad \cdots\cdots \quad \xi^{m-1} \quad \xi^m] \\ R_4 &= [0 \quad \xi \quad \cdots\cdots \quad \xi^{m-1} \quad \xi^m] \end{aligned} \tag{3.52}$$

The controlled lifted system matrix is very similar to the one in Case 4. The blocks of minors also have a similar structure in the sense that the non-zero polynomials are found in similar places: Non-zero polynomials of the first block appear in the last column. In the next 3 blocks, the first column consists of non-zero elements, while the last block has two non-zero columns, viz. the first and the last. Again, the polynomials in the last block are equal up to a factor -1. So we have $5m$ linearly independent polynomials, just enough to span the space of all polynomials of degree $\leq 5m - 1 = n$. □

Examining the cases 3, 4 and 5, we see a similarity in the matrices $R(\xi)$ that are elements of $\bar{S}$:

- All the powers of ξ occur only once, except for ξ^{Tm-1}, which is found in both the first and the last element of $R(\xi)$
- The first element is the sum of 2 and ξ to the power $n = Tm - 1$,
- The last element is a sum of $T - 1$ terms, where the powers of ξ are increasing up to the power $n = Tm - 1$,
- The other elements are sums of T terms, with increasing powers of ξ.

These observations lead to a conjecture for the single output case, where $n = Tm - 1$.

Conjecture If $p = 1$ and $n = Tm - 1$ is the McMillan degree, then (3.53) is an element of $\bar{S}$.

$$\begin{aligned} R = [2 + \xi^{Tm-1} \quad \xi + \ldots + \xi^T \quad \xi^{T+1} + \ldots + \xi^{2T} \quad \cdots\cdots \\ \cdots \quad \xi^{T(m-1)-(T-1)} + \ldots + \xi^{T(m-1)} \quad \xi^{Tm-(T-1)} + \ldots + \xi^{Tm-1}] \end{aligned} \tag{3.53}$$

Remark 3.2.4 The case $m = 1$ (Case 1) is also captured by the conjecture. Though the matrix R is not exactly equal to the one presented in (3.33) (the first element is slightly different), the minors do span the space of all polynomials of degree $\leq n$. Case 2 is also concluded in the conjecture. Again, only the first element differs, and the polynomials span the space of all polynomials of degree $\leq n$. □

Case 6: p=2, T=2 In this case the equality $n = mpT - 1$ is reduced to $4m - 1$. The plant given by (3.54) is a matrix for which the dependent controller (3.30) has the properties a.1 and a.2.

$$R(\xi) = \begin{bmatrix} 1+\xi & \xi^2+\xi^3 & \cdots & \xi^{2m-4}+\xi^{2m-3} & \xi^{2m-2} & \xi^{2m-1} & \xi^{2m} \\ 1 & \xi+\xi^2 & \cdots & \xi^{2m-5}+\xi^{2m-4} & \xi^{2m-3}+\xi^{2m-2} & \xi^{2m-1} & 1 \end{bmatrix} \tag{3.54}$$

The matrices R_0 and R_1 are given by

$$\begin{aligned} R_0(\xi) &= \begin{bmatrix} 1 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & 0 & \xi^m \\ 1 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & 0 & 1 \end{bmatrix} \\ R_1(\xi) &= \begin{bmatrix} \xi & \xi^2 & \cdots & \xi^{m-1} & 0 & \xi^m & 0 \\ 0 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & \xi^m & 0 \end{bmatrix} \end{aligned} \tag{3.55}$$

The controlled lifted system would be

$$
\begin{bmatrix}
1 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & 0 & \xi^m & 1 & \xi & \cdots & \xi^{m-2} & 0 & \xi^{m-1} & 0 \\
1 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & 0 & 1 & 0 & 1 & \cdots & \xi^{m-3} & \xi^{m-2} & \xi^{m-1} & 0 \\
\xi & \xi^2 & \cdots & \xi^{m-1} & 0 & \xi^m & 0 & 1 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & 0 & \xi^m \\
0 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & \xi^m & 0 & 1 & \xi & \cdots & \xi^{m-2} & \xi^{m-1} & 0 & 1 \\
1 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & & & & & \vdots & \vdots & & & & & & \vdots \\
\vdots & & \ddots & & & & \vdots & \vdots & & & & & & \vdots \\
\vdots & & & \ddots & & & \vdots & \vdots & & & & & & \vdots \\
0 & \cdots & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & & & \vdots & \vdots & & \ddots & & & & \vdots \\
\vdots & & & & & & \vdots & \vdots & & & \ddots & & & \vdots \\
\vdots & & & & & & \vdots & \vdots & & & & \ddots & & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 & 0
\end{bmatrix}
\tag{3.56}
$$

The minors corresponding to the first diagonal block are

$$
\begin{matrix}
\xi^{2m} - \xi^m & 0 & \cdots & 0 & -\xi^{2m} + 2\xi^m + \xi - 1 & \xi^{2m} - \xi^m \\
\xi(\xi^{2m} - \xi^m) & 0 & \cdots & 0 & \xi(-\xi^{2m} + 2\xi^m + \xi - 1) & \xi(\xi^{2m} - \xi^m) \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
\xi^{m-1}(\xi^{2m} - \xi^m) & 0 & \cdots & 0 & \xi^{m-1}(-\xi^{2m} + 2\xi^m + \xi - 1) & \xi^{m-1}(\xi^{2m} - \xi^m)
\end{matrix}
\tag{3.57}
$$

and the minors corresponding to the second block are

$$
\begin{matrix}
\xi^{3m} - 2\xi^{2m} + \xi^m & 0 & \cdots & 0 & -\xi^{m+1} \\
\xi(\xi^{3m} - 2\xi^{2m} + \xi^m) & 0 & \cdots & 0 & \xi(-\xi^{m+1}) \\
\vdots & \vdots & & \vdots & \vdots \\
\xi^{m-1}(\xi^{3m} - 2\xi^{2m} + \xi^m) & 0 & \cdots & 0 & \xi^{m-1}(-\xi^{m+1})
\end{matrix}
\tag{3.58}
$$

If m is odd, then the polynomials should be multiplied by -1. These blocks contain exactly $4m$ linearly independent polynomials spanning the space of all polynomials of degree $\leq 4m - 1 = n$. □

Chapter 4

Conclusions

In this report we have taken the first steps towards proving that generically, $n < mpT$ is a sufficient condition for arbitrary pole assignment of LTI systems by periodically time-varying output feedback. The main conjecture of the present report states that the set of systems, with $n < mpT$, for which arbitrary pole assignability by T -periodic output feedback holds, is a generic set.

By adopting the behavioral approach and extending the behavioral proof by Willems [19], we proved that generically, the closed-loop poles of a n^{th} order SISO system can be arbitrarily assigned by $n+1$ -periodic output feedback. This result is in harmony with an earlier paper by Aeyels and Willems [3] which states that, under mild conditions, (almost) arbitrary pole assignability of n^{th} order SISO systems by $n+1$ periodic output feedback holds.
Moreover, we have extended our result to MIMO systems. With the use of the computer program Maple, we have demonstrated that for specific choices for the parameters n, m, p and T, $n < mpT$ implies generic arbitrary pole assignment by periodically time-varying output feedback. Based on the cases we have examined, we proposed the following conjecture for the single-output case: if $n < mT$, then the pole placement map is generically onto, i.e. generic arbitrary pole assignability by T -periodic output feedback holds.
Our results are not limited to single-output systems. We demonstrated that systems with 2 outputs are generically arbitrary pole assignable by 2 -periodic output feedback.

Since we have not found any counter example for our main conjecture, we expect that it is possible to complete our proof for the remaining cases. By doing so, we may establish that indeed $n < mpT$ is a sufficient condition for generic arbitrary pole placement of MIMO systems by periodically time-varying output feedback.

Appendix A

From state-space representation to behavioral equations

Consider the LTI system

$$\sigma x = Ax + Bu, \qquad y = Cx, \tag{A.1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$. Denote the smallest integer that is greater than or equal to $\frac{n}{p}$ by μ. Furthermore, let $p_2 := \mu p - n \geq 0$ and $p_1 := p - p_2$. If μ is equal to $\frac{n}{p}$ then $p_2 = 0$ and hence $p_1 = p$. Make a partition of the matrix C and the output vector y

$$y = \begin{bmatrix} y_1 \\ \hline y_2 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 \\ \hline C_2 \end{bmatrix}, \tag{A.2}$$

with $y_1 \in \mathbb{R}^{p_1}, y_2 \in \mathbb{R}^{p_2}, C_1 \in \mathbb{R}^{p_1 \times n}$ and $C_2 \in \mathbb{R}^{p_2 \times n}$. Observe that if $p_2 = 0$ then $y = y_1$ and $C = C_1$. Consider the matrix

$$S(A,B,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\mu-2} \\ C_1 A^{\mu-1} \end{bmatrix}. \tag{A.3}$$

If we know the matrices A, B, C, then we write S rather than $S(A,B,C)$. S is the matrix from which we derive the observability indices. It is a square matrix of size n which is invertible for a generic set of matrices A, B, C. Denote

$$\mathcal{M} := \{(A,B,C) \subset \mathbb{R}^{n^2+nm+pn} \mid S(A,B,C) \text{ is invertible}\}. \tag{A.4}$$

Notice that $\mathcal{M}$ is a generic set.

Denote the observability indices of system (A.1) by $\mu_1, \ldots, \mu_p$. For $(A,B,C) \in \mathcal{M}$, those indices are all equal *or* their value differs by one:

$$\begin{aligned} &\text{if } n \text{ divides } p \text{ then } \mu_1 = \ldots = \mu_p = \mu, \\ &\text{else } \mu_1 = \ldots = \mu_{p_1} = \mu \text{ and } \mu_{p_1+1} = \ldots = \mu_p = \mu - 1. \end{aligned} \tag{A.5}$$

The input-output behavior of system (A.1) can be described by a set of differential (continuous-time) or difference (discrete-time) equations. Let the manifest behavior $\mathcal{B}_m$ of system (A.1) be given by

$$\mathcal{B}_m := \{(y,u) \in \mathcal{F}^{\infty}(\mathbb{T},\mathbb{F}^{(m+p)} \mid \exists x \in \mathcal{F}^{\infty}(\mathbb{T},\mathbb{F}^{n}), \text{ and } \sigma x = Ax + bu,\ y = Cx \text{ holds}\}. \tag{A.6}$$

If σ stands for the differentiation operator, then the space $\mathcal{F}^{\infty}(\mathbb{T},\mathbb{F}^{n})$ is the space of all continuous functions f, infinitely often differentiable, $f : \mathbb{R} \to \mathbb{R}^n$. If σ stands for the shift operator, then the space $\mathcal{F}^{\infty}(\mathbb{T},\mathbb{F}^{n})$ is the space of all discrete-time functions g, $g : \mathbb{Z} \to \mathbb{Z}^n$ and all the functions $\sigma g, \sigma^2 g, \ldots$ are properly defined. Then there exists a polynomial matrix $R \in \mathbb{R}^{p\times(m+p)}[\xi]$ such that the behavior $\mathcal{B}_m$ equals the $\mathcal{F}^{\infty}$-solution of

$$R(\sigma)\begin{bmatrix} y \\ u \end{bmatrix} = 0. \tag{A.7}$$

(A.7) is the kernel representation of the manifest behavior.

We use the matrix S to compute a kernel representation. Define

$$\bar{x} := Sx. \tag{A.8}$$

Then

$$\bar{x} = Sx = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\mu-2} \\ C_1A^{\mu-1} \end{bmatrix} x = \begin{bmatrix} Cx \\ C(Ax+Bu) - CBu \\ \vdots \\ CA^{\mu-3}(Ax+Bu) - CA^{\mu-3}Bu \\ C_1A^{\mu-2}(Ax+Bu) - CA^{\mu-2}Bu \end{bmatrix} = \cdots = \tag{A.9}$$

$$= \begin{bmatrix} y \\ \sigma y - CBu \\ \vdots \\ \sigma^{\mu-2}y - CB\sigma^{\mu-3}u - \ldots - CA^{\mu-3}Bu \\ \sigma^{\mu-1}y_1 - C_1B\sigma^{\mu-2}u - \ldots - C_1A^{\mu-2}Bu \end{bmatrix}. \tag{A.10}$$

Because S is invertible for a generic set of (A,B,C)'s, we can write $y = Cx = CS^{-1}\bar{x}$. Now we have a set of equations

$$y = CS^{-1}\bar{x}, \tag{A.11}$$
$$\sigma\bar{x} = SAS^{-1}\bar{x} + SBu. \tag{A.12}$$

The new system described by (A.11) and (A.12) contains $n+p$ equations. Only the last p equations are non-trivial and define the relations between y and u and the derivatives of y and u. The highest order of the derivatives is μ. If we define $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, we can compute $\mu+1$ matrices $R_0, R_1, \ldots, R_{\mu-1}, R_\mu \in \mathbb{R}^{p\times(m+p)}$ satisfying

$$R_0 w + R_1\frac{dw}{dt} + \ldots + R_{\mu-1}\frac{d^{\mu-1}w}{dt^{\mu-1}} + R_\mu\frac{d^\mu w}{dt^\mu} = 0. \tag{A.13}$$

This is just another way to write down the last p equations of (A.12). By observing this process of transforming a state-space representation to an input-output representation we can see that the matrices $R_{\mu-1}$ and R_μ have a structure:

$$R_{\mu-1} = \begin{bmatrix} M_{p_1\times p_1} & O_{p_1\times p_2} & M_{p_1\times m} \\ M_{p_2\times p_1} & I_{p_2\times p_2} & O_{p_2\times m} \end{bmatrix}, \qquad R_\mu = \begin{bmatrix} I_{p_1\times p_1} & O_{p_1\times p_2} & O_{p_1\times m} \\ O_{p_2\times p_1} & O_{p_2\times p_2} & O_{p_2\times m} \end{bmatrix}. \tag{A.14}$$

The matrices M have no specific structure. Because we have this knowledge of the matrices $R_{\mu-1}$ and R_μ, the number of unknowns in (A.13) is less than $(\mu+1)p(m+p)$, the total amount of parameters. Each of the matrices $R_0, R_1 \dots, R_{\mu-2}$ contains $p(m+p)$ unknowns, while $R_{\mu-1}$ contains $p_1(m+p)$ unknowns and R_μ is totally known. So the total number of unknowns in (A.13) is equal to $(\mu-1)p(m+p)+p_1(m+p) = n(m+p)$. We could say that $(R_0, R_1, \dots, R_{\mu-1}, R_\mu) \in \mathbb{R}^{n(m+p)}$. This implies that transforming system (A.1) into (A.13) can be associated with a map Ψ which maps a set (A, B, C) into a set $(R_0, R_1, \dots, R_{\mu-1}, R_\mu)$. Define this map:

$$\Psi : \mathcal{M} \to \mathbb{R}^{n(m+p)}. \tag{A.15}$$

Recall that $\mathcal{M}$ is a subset of $\mathbb{R}^{n^2+nm+pn}$.
For $(A, B, C) \in \mathcal{M}$, the matrix $S(A, B, C)$ has a non-zero determinant and then Ψ is a rational map. Notice that Ψ is surjective, since for every system in input-output representation for which (A.13)), (A.14) hold and $w = \begin{bmatrix} y \\ u \end{bmatrix}$, there exists (A, B, C) with (A.1), such that the behavior of the input-output representation is the external behavior of (A.1).

The latter implies that the corresponding transfer matrices are equal and hence both are strictly proper. Indeed, the LTI system (A.1) can also be written as

$$(\sigma I - A)x = Bu, \qquad y = Cx. \tag{A.16}$$

The transfer function equals $C(\xi I - A)^{-1}B$ and is always strictly proper. The kernel representation of the stat-space representation is also given by

$$\begin{bmatrix} R_1(\sigma) & R_2(\sigma) \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = 0, \tag{A.17}$$

and involves the transfer matrix $-R_1^{-1}(\xi)R_2(\xi)$. Since $C(\xi I - A)^{-1}B$ and $-R_1^{-1}(\xi)R_2(\xi)$ must be equal, $R_1^{-1}(\xi)R_2(\xi)$ must be strictly proper.

Appendix B

Image of φ contains open neighborhood of origin

In this appendix, we establish that the image of the pole placement map φ as defined in section 2.3.1 contains an open neighborhood of the origin. This is achieved by using a modified version of a corollary, stated here as Theorem B.0.5, which is an important application of the Implicit Function Theorem and can be found in [7].

Theorem B.0.5 Let E and F be two Banach spaces and let $x_0 \in E$ and V be a neighborhood of x_0. Furthermore, let $f : V \to F$ be a continuously differentiable mapping with $f(x_0) = y_0$. Assume that $f'(x_0)$ is a linear homeomorphism of E onto F. Then there exists an $U \subset V$, U open neighborhood of x_0, such that $f_{|U}$ is a homeomorphism of U onto an open neighborhood of $f(x_0) = y_0$.

We cannot use this corollary, because the derivative of $\varphi(0)$ is not necessarily a linear homeomorphism. That is why we need a modified version of Theorem B.0.5.

Lemma B.0.6 Let E and F be two Banach spaces and let $x_0 \in E$ and V be a neighborhood of x_0. Furthermore, let $f : V \to F$ be a continuously differentiable mapping with $f(x_0) = y_0$. Assume that $f'(x_0)$ is a surjective mapping of E onto F. Then there exists a subset $U \subset V$ such that $x_0 \in U$ and $f(U)$ is an open neighborhood of $f(x_0) = y_0$.

Proof Assume that all the assumptions in Lemma B.0.6 are met. The mapping $f' : V \to F$ is not necessarily injective, however, by making the domain smaller we can find a domain such that the restriction of f' to this domain is injective. Thus there exists a $W \subset V$ such that $f'(x_0)_{|W}$ is a homeomorphism. According to Theorem B.0.5 there exists a $U \subset W$, with U open in W and $x_0 \in U$, such that $f_{|U}$ is a homeomorphism of U onto an open neighborhood of $y_0 = f(x_0)$. □

It follows from Lemma B.0.6 that the image of φ contains an open neighborhood of the origin. By assumption that all the functions under consideration are as often differentiable as we need, $\varphi : \mathbb{R}^{m\times(m+p)} \to \mathbb{R}^{n+1}$ is continuously differentiable, and hence $\varphi' : \mathbb{R}^{m\times(m+p)} \to \mathbb{R}^{n+1}$ is a continuous mapping. Furthermore, we have that $\varphi(K^*) = 0$ by assumption P.1 and

$$\varphi'(K^*) = \left[\bar{R}^0_{11}\ \bar{R}^0_{12} \cdots \bar{R}^0_{m,m+p}\right] \in \mathbb{R}^{(n+1)\times m(m+p)}, \tag{B.1}$$

where $\bar{R}^0_{ij}, i = 1\ldots m, j = 1\ldots m+p$ written as $n+1$ -vector, i.e. we associate the coefficients of each polynomial $\bar{R}^0_{ij}$ with a vector in $\mathbb{R}^{n+1}$. By assumption (P.2), $\varphi(K^*)$ is a surjective mapping.

Let V be a neighborhood of K^*. According to Lemma B.0.6 there exists a $U \subset V$ such that $K^* \in U$ and $\varphi(V)$ is an open neighborhood of $\varphi(K^*) = 0$. So we have that the image of φ contains an open neighborhood of the origin. □

Appendix C

Maple programs

In this section we present the programs, written in Maple, which were written to compute lifted system-matrices, closed- loop systems and the relevant minors.

```
Lift:=proc(R,T)
   # Input: a p x q polynomial matrix R which represents a LTI system, and
   #        an integer T, which denotes the period of the periodic
   #        output feedback that is going to be applied.
   # LIFT computes the pT x qT which represents the lifted system.
  local i,j,k,l,p,q,L,RL,divide,curBlock,fact;
  p:=Matlab[dimensions](R)[1];
  q:=Matlab[dimensions](R)[2];
  RL:=matrix(p*T,q*T);
  L:=Lift2(R,T,p,q);
  for k from 1 to T do
     divide:=true;
     for l from 1 to T do
        curBlock:=irem(T+l-k,T)+1;
        divide:=(divide and curBlock>1);
        if divide
          then fact:=xi^(-1);
          else fact:=1;
        fi;
        for i from 1 to p do
           for j from 1 to q do
              RL[(l-1)*p+i,(k-1)*q+j]:=expand(fact*L[curBlock,i,j]);
           od;
        od;
     od;
  od;
  RL;
end;
```

```
Lift2:=proc(R,T,p,q)
    # Input: a p x q polynomial matrix R which represents a LTI system, and an
    #        integer T which denotes the period of the periodic output
    #        feedback that is going to be applied.
    # LIFT2 computes T matrices L[0],...,L[T-1] that contain powers of xi
    # that are taken modulo T. The sum of these matrices equals R.
  local i,j,k,L,deg,index;
    # L[h,i,j] is the (i,j)th element of L[h],h=0,...,T-1.
  L:=array(sparse,1..T,1..p,1..q);
  for i from 1 to p do
     for j from 1 to q do
        for k from 0 to degree(R[i,j],xi) do
           index:=irem(T-irem(k,T),T)+1;
           deg:=(index-1+k)/T;
           L[index,i,j]:=L[index,i,j]+coeff(R[i,j],xi,k)*xi^deg;
        od;
     od;
  od;
  L;
end;

Connect:=proc(L,T)
    # Input: a pT x qT lifted matrix L
    # CONNECT computes the controlled system, where the first pT rows
    # consists of L, and the last (q-p)T rows describe the
    # T- periodic controller. The controller is constructed in such
    # a way that it is a dependent controller. Hence, the computed
    # matrix C (the controlled system) has determinant zero.
  local i,j,m,p,q,v,w,K,C;
  p:=Matlab[dimensions](L)[1]/T;
  q:=Matlab[dimensions](L)[2]/T;
  m:=q-p;
    # initialization of K
  K:=matrix(m*T,q*T);
  for i from 1 to m*T do
     for j from 1 to q*T do
        K[i,j]:=0;
     od;
  od;
  v:=0;w:=0;
  while v<m*T do
    for i from 1 to m do
      for j from 1+w to q+w do
         K[i+v,j]:=coeff(L[p,j],xi,i-1);
      od;
    od;
    v:=v+m;w:=w+q;
  od;
  C:=stackmatrix(L,K);
end;
```

```
Pol:=proc(C,T,p)
    # Input: a qT x qT matrix C, which represents the controlled system
    #        of a lifted matrix RL and a T-periodic controller K. So C=[RL]
    #        The controller K has dimensions mT x qT.                [K ]
    # POL computes all the minors (C,i,j) where i=pT+1...pT+mT, and j=1...qT
    #                                         _
    # These minors are the polynomials R as defined in the Taylor expansion
    # of the pole placement map.
  local i,j,k,m,q,pol,mT,pT,qT;
  q:=Matlab[dimensions](C)[1]/T;
  m:=q-p;mT:=m*T;pT:=p*T;qT:=q*T;
  pol:=matrix(mT,qT);
  for i from 1 to mT do
     k:=pT+i;
     for j from 1 to qT do
        pol[i,j]:=sort(collect((-1)^(j+k)*det(minor(C,k,j)),xi));
     od;
  od;
  pol;
end;


Rang:=proc(M,T)
    # Input: a mT x qT matrix M, with q=m+p,                         _
    #        with on the diagonal blocks the polynomials R.
    # Associate these polynomials (mqT) with vectors. S is the resulting
    # matrix consisting of all those vectors.
    # RANG computes the rank of S.
  local i,j,m,p,q,v,w,deg,row,S;
  m:=Matlab[dimensions](M)[1]/T;
  q:=Matlab[dimensions](M)[2]/T;
  p:=q-m;mpT:=m*p*T;mqT:=m*q*T;
  v:=0;w:=0;row:=1;
    # S should consist of mqT rows, since we have mqT vectors. These vectors
    # should be elements of the mpT+1 dimensional vector space.
    # So S should be a mqT x mpT+1 matrix.
  S:=matrix(mqT,mpT+1);
  while v<m*T do
       for i from 1 to m do
          for j from 1+w to q+w do
             for deg from 0 to mpT do
                S[row,deg+1]:=coeff(M[i+v,j],xi,deg);
             od;
             row:=row+1;
          od;
       od;
       v:=v+m;w:=w+q;
  od;
  rank(S);
end;
```

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