Charles Sturm, Joseph Liouville, and Their Theory:

Through the Prism of Liouville's 1837 Paper 'Solution Nouvelle d'un problème d'analyse...'

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Wiskunde

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May 1998

Preface

Writing may be a solitary pursuit, but producing this thesis required a great deal of assistance. I would like to express my gratitude to both of my advisors, Prof.dr.ir. A. Dijksma and Dr. J. A. van Maanen, for their time, guidance, and many suggestions.

A very appreciative thank you goes to my parents-in-law, Jan and Yvonne Greve, for the frequent use of their computer. The completion of this thesis must be especially gratifying for Jan who, with his own son having also completed his degree this year, finally gets his study back. Many thanks also to my fellow student Alja Vrieling for her help with, well, practically everything here at the university over the past three years.

Finally, there would be no thesis or graduation with out the love and support of my wife, Sylvia— thank you for making it possible for me to study.

Eric Burniston
Groningen, May 1998

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Chapter 1

Analysis in the Nineteenth Century

At the dawn of the nineteenth century, the relatively new field of analysis had created a great deal of interest and seen fantastic advances, yet these developments lacked the rigor necessary to fully establish these results. Perhaps it is predictable and sensible that the rapid development of a field with so many physical applications followed first these physical models before mathematicians turned their attentions towards more theoretical aspects. In the view of mathematical historian Morris Kline, the nineteenth century saw the reintroduction of the rigorous proof. He comments, "From about 200BC to about 1870 almost all of mathematics rested on an empirical and pragmatic basis. The concept of a deductive proof from explicit axioms had been lost sight of." For example, "Fourier's work makes a modern analysist's hair stand on end; and as far as Poisson was concerned, the derivative and integral were just shorthand for the difference quotient and the finite sum."[1] Given this, it was inevitable that the most basic problems concerning analytical development, centering around questions involving infinity and continuity, would have to be addressed in a more rigorous manner. The successful application of this rigor to analysis was likely the leading development of the nineteenth century. Still, developments in complex variables and set theory broadened the field, and as in the previous century, physical problems led to advances in areas such as differential equations, vector analysis, and probability. It was, however, the rigorous formulation of the tenants of analysis that solidified the advances of the previous century and provided the direction for the future.

One of the first to tackle these questions was Silvestre-Francois Lacroix. In the late eighteenth century and the early nineteenth century, this French mathematician published a three volume work, *Traité du calcul differentiel et du calcul integral*. His teaching work at the École Polytechnique, where he was in 1799 ap-

[1] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, New York: Oxford, 1972, p. 1024.

pointed to Joseph-Louis Lagrange's old position, later led him to condense this work into the single volume *Traité elementaire du calcul differentiel et du calcul integral*. In these works, Lacroix advocated using the limit as the basic concept underlying differential calculus. In particular, Lacroix wrote, "the differential calculus is the finding of the limit of the ratios of the simultaneous increments of a function and of the variables on which it depends."[2] Yet, Lacroix's commitment to this approach was not complete. He was convinced that all functions could be expressed as series, except at a finite number of points, and he used the Taylor series representation to develop differentiation formulas for various functions. In fact, Lacroix felt that his work was an important reconciliation of the limit method and that of Lagrange which relied on power series. Instead, perhaps the most important consequence of Lacroix's work was that it settled the controversy in England regarding limits. *Traite elementaire...* was translated into English in 1816, and through the publication of texts such as this, the Leibniz notation and the doctrine of limits replaced the method of fluxions and interpretations thus establishing a single notation and method throughout the mathematical community. Historian Carl Boyer comments, "The year 1816, in which Lacroix's shorter work was translated into English, marks an important period of transition, because it witnessed the triumph in England of the methods used on the Continent. This particular point in the history of mathematics marks a new epoch for a far more significant reason, for in the very next year the Czech priest Bernhard Bolzano published a short work with a long title- *Rein analytischer Beweis des Lehrsatzes dass zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reele Wurzel der Gleichung liege-* which indicated the rise of the period of mathematical rigor in all braches of the subject."[3]

Bolzano worked at the fringes of the established mathematical community, and consequently, his work did not receive the attention that it deserved. Despite this relative obscurity, Bolzano's ideas were advanced. In *Rein analytischer Beweis...*, he was motivated by a desire to rigorously prove the intermediate value theorem, "that between any two values of the unknown quantity which give results of opposite sign [when substituted in a continuous function $f(x)$] there must always lie at least one real root of the equation [$f(x) = 0$]."[4] To do so, Bolzano had to give a self-described "correct definition" of the type of function for which the theorem would hold. He wrote, "A function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits if [when] x is some such value, the difference $f(x+w) - f(x)$ can be made smaller than any given quantity provided w can be taken as small as we please."[5] This important definition predated Cauchy's similar formulation by

[2]Victor Katz, *A History of Mathematics: An Introduction*, New York: Harper-Collins , 1993, p. 637.

[3]Carl Boyer, *A History of the Calculus and Its Conceptual Development*, New York: Dover, 1959, p. 266.

[4]Steve B. Russ, 'A translation of Bolzano's paper on the intermediate value theorem', *Historia Mathematica*, 7, 1980, p. 159.

[5]ibid., p. 162.

four years, but Bolzano's relative isolation prevented wide-spread dissemination of his ideas. It was Cauchy, then, who was the leading figure in the movement towards increased analytical rigor.

Augustin-Louis Cauchy, the most prolific mathematician of the nineteenth century, authored some 800 books and articles on almost all branches of mathematics. Certainly among the most important of these are his three books introducing rigor into calculus. According to Boyer, through these books, *Cours d'analyse de l'École Polytechnique* (1821), *Resume des leçons sur le calcul infinitesimal* (1823), and *Leçons sur le calcul differentiel* (1829), "Cauchy did more than anyone else to impress upon the subject the character which it bears at the present time."[6] Cauchy's definition of a limit appeared in *Cours d'analyse.* Concerning limits, he wrote, "When the values successively attributed to the same variable approach indefinitely a fixed value, eventually differing from it by as little as one could wish, that fixed value is called the limit of all the others."[7] The superiority of Cauchy's definition is not immediately apparent. However, an analysis of his use of this definition shows that he "not only deals with both the dependent and independent variables, but also translates his statement arithmetically by use of the language of inequalities."[8] Thus, while the wording is not exactly the modern conception of a limit, its use is very close, and this, along with Cauchy's definition of continuity, form the basis on which he develops his calculus texts *Resume des lecons sur le calcul infinitesimal and Lecons sur le calcul differentiel.*

Like Bolzano, Cauchy defines continuity not at a point, but over an interval. He wrote, "Let $f(x)$ be a function of [the real] variable x and suppose that this function has a unique and finite value for each value of x in a given interval. If, to a value of x in this interval, one adds an infinitesimal increment h, the function itself increases by the difference $f(x+h) - f(x)$; this depends on both the new variable h and the value of x. Given this, the function $f(x)$ will be a continuous function of the variable x in the interval when, for each value of x in the interval, the magnitude of the difference $f(x+h)-f(x)$ decreases indefinitely with that of h."[9] As with limits, Cauchy's definition (and Bolzano's) represented a significant advance. The concept of continuity had now been given a precise mathematical meaning based on the concept of the limit. "Newton (implicitly) and Leibniz (explicitly) based the validity of the calculus on the assumption that, by a vague sense of continuity, limiting states would obey the same laws as the variables approaching them."[10] Given these powerful definitions, it is clear that Cauchy now had the tools to develop calculus more rigorously, and in fact, he did exactly this. His consideration of the derivative improved on earlier work by Euler and Lagrange by incorporating the new definition of limits. Of even greater consequence, though, was Cauchy's treatment of integration.

Resisting the convention of defining integration as simply the inverse of dif-

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[6] Boyer, p. 271.
[7] Garrett Birkhoff, *A Source Book in Classical Analysis*, Cambridge: Harvard, 1973, p. 2.
[8] Katz, p. 639.
[9] Birkhoff, p. 2.
[10] Boyer, p. 277.

Figure 1.1: A.L. Cauchy (1789-1857).

ferentiation, Cauchy defined the integral as the limit of a sum. Cauchy's reasons for doing so seem to be varied. He certainly realized that there were many situations where an anti-derivative could not be used at the endpoint of an interval, and moreover, an anti-derivative may not exist for every function. Further, his work in complex analysis reinforced his conceptions regarding defining integration in this way.[11] Having defined the integral in this way, Cauchy was able to prove a mean value theorem for integrals,

$$\int_a^x f(x)dx = (x-a)f[a+\theta(x-a)],$$

and then using this, he formulated and gave the first truly acceptable proof of the Fundamental Theorem of Calculus. In Cauchy's own words, "The definite integral of $f(x)dx$, taken between the limits a and b, is thus really the difference between the values which the function having the differential $f(x)dx$ takes at $x = a$ and $x = b$."[12] In his varied work towards unifying what had been a disparate field, Cauchy did make some errors, including failing to recognize the distinction between convergence and uniform convergence and continuity and uniform continuity. However, Cauchy's work helped to create a well-devised,

[11] Katz, p. 648.
[12] Birkhoff, pp. 9-11.

logical foundation for the field. Even today, the basic logical structure Cauchy erected provides the framework within which rigorous calculus is considered.[13]

Cauchy also made significant contributions to another very important area, that of complex analysis. The advances made in this field during the nineteenth century were substantial, leading historian Morris Kline to suggest that "from the standpoint of technical development, complex function theory was the most significant of the new creations."[14] The other leading mathematician of the nineteenth century, Carl Friedrich Gauss, also did much in this area. In particular, Gauss helped to popularize the use of the plane to geometrically represent complex numbers in no small part by producing four different proofs of the fundamental theorem of algebra during his career, each of which relied in some way on a geometric interpretation of complex numbers. A revealing and indicative look at the different ways in which Gauss and Cauchy worked can be had by studying the development of the theorem which states that if $f(z)$ (where z is complex) is never infinite within the enclosed regions of two different curves with the same end points a and b then the integral $\int_a^b f(z)dz$ has the same value over each curve. Gauss proved important result, calling it in his private journal a "very beautiful theorem." But being slow and often reluctant to publish his work, he never produced a published proof of this theorem. The prolific Cauchy did publish a proof of this in 1825, and today the theorem bears his name.[15] Another important mathematician working with complex functions was Bernhard Riemann. His work, and the earlier work of Cauchy, led to the Cauchy-Riemann equations,

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x}$$

and

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y},$$

which are the characteristic properties of complex functions. Riemann also contributed much in the area of integration of complex and real function; he led the way into the field of topology by creating Riemann surfaces as a way of representing two-variable complex functions; and he helped to extend the results of Joseph Fourier and P.G. Lejeune Dirichlet concerning the convergence of series. Moreover, this last development would pave the way for Karl Weierstrass to provide important results concerning uniform convergence and continuity, finally helping to 'correct' one of the errors made in Cauchy's work. Significantly, the work done in complex analysis reinforced the rigorous development of calculus by extending it to complex functions. This and the complete arithmetization of analysis achieved in the last half of the century finally moved analysis away from its geometric origins.

[13] John Fauvel and Jeremy Gray, *The History of Mathematics: A Reader*, London: MacMillan, 1990, p. 572.

[14] Kline, p. 1023.

[15] Katz, p. 667.

In retrospect, it seems odd that even by the middle of the century there was only an intuitive understanding of the indispensable real numbers. According to the historian Carl Boyer, analysis at mid-century "was still encumbered with geometric intuition."[16] Realizing this, Weierstrass, the leading mathematician of the second half of the century, attempted to restructure the calculus so that it was based completely on number. To overcome the problems inherent in a loosely defined number system, Weierstrass sought to define irrational numbers independently from a limit process. His success in this was typical of his work. He did not publish the result— Weierstrass published infrequently— instead, he presented the result in his lectures. A great teacher, Weierstrass' contributions to mathematics were considerable not only for the substantial body of original work (much of it presented years later by his former students), but also for his influence on the mathematicians of the following generation. Extending Weierstrass' work with the irrational numbers was Richard Dedekind. His research and that of his close friend Georg Cantor helped to solidify conceptions regarding the real numbers. Dedekind described his famous cut:

> If now any separation of the system R (a discontinuous domain of rational numbers) into two classes A_1, A_2 is given which possesses only this characteristic property that every number a_1 in A_1 is less than every number a_2 in A_2, then for brevity we shall call such a separation a cut and designate it by (A_1, A_2). We can then say that every rational number produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different; this cut possesses, besides, the property that either among the numbers of the first class there exist a greatest or among the numbers of the second class a least number. And conversely, if a cut possesses this property, then it is produced by this greatest or least rational number. but it is easy to show that there exist infinitely many cuts not produced by rational numbers.[17]

Dedekind called such cuts the irrational numbers, and the set of all cuts was the real numbers. Cantor's approach to the feal numbers differed from the elegant work presented by Dedekind. He introduced the idea of a fundamental sequence (now called a Cauchy sequence) and used it to establish a correspondence between every fundamental sequence of rational numbers and a real number. The work of Cantor led him to a consideration of set theory, in which he pioneered the study of cardinality of infinite sets. Two sets were of the same power if a one-to-one correspondence could be established between the members of the sets. This work led to the Cantor-Dedekind axiom, that the points on a line can be put into one-to-one correspondence with the real numbers. Together, Cantor and Dedekind helped to create a numerical basis for analysis by clearly devising a rigorous way of defining number sets. Katz notes, "It was this work, together with the work of Weierstrass and his school, which enabled calculus to

[16] Carl Boyer and Uta Merzbach, *A History of Mathematics*, New York: Wiley, 1989, p. 627.

[17] Fauvel and Gray, p. 576.

be placed on a firm foundation beginning with the basic notions of set theory. It also showed that calculus had an existence independent of the physical work of motion and curves , the world used by Newton to create the subject in the first place."[18]

Other work in analysis during the century certainly deserves mention, beginning with advances in vector analysis. The divergence theorem, relating an integral over a solid to an integral over a bounding surface, was worked on by Lagrange and Gauss before a proof of the general case was offered by the Russian mathematician Mikhail Ostrogradsky.[19] The theorem proven by George Green relating an integral over the boundary curve also represented an important development. In the area of ordinary differential equations, Charles Sturm and Joseph Liouville produced what Boyer and Merzbach called "perhaps the best-known French analytic work of mid-century..., dealing with the theory of second-order ordinary differential equations with boundary conditions."[20] The work of these two would help to lead to, half a century later, the birth of spectral theory. Sturm and Liouville's work owed much to the earlier work of Joseph Fourier. Fourier's investigations into the theory of heat led him to many profound discoveries involving functions and series representations of functions. The novelty of his discoveries inspired many others, including Sturm and Liouville, to study related topics. Still other work done in differential equations included that of a former student of Weierstrass, Lazarus Fuchs. He described and worked extensively on what became known as the Fuchsian theory of linear differential equations, a concentrated study of solutions in the neighborhood of singular points. Fuchs described the problem in a paper published in 1866, "In the present condition of science the problem of the theory of differential equations is not so much to reduce a given differential equation to quadratures, as to deduce from the equation itself the behavior of its integrals at all points of the plane, that is, for all values of the complex variable."[21] The numerous advances achieved during the nineteenth century were certainly broad in scope, but the most important development in analysis during this century was not the broadening but the crucial formulation of a rigorous basis that led to a deeper understanding of the basic concepts in analysis.

Writing in the first half of the twentieth century, historian Eric Bell divided the development of analysis into five periods, each dominated by a mathematician or two. The first two periods, with l'Hospital and then Euler leading the way, saw rapid development of the field. The third period, the last of the eighteenth century, was led by Lagrange and characterized by a recognition that the calculus was in an unstable state. The two periods belonging entirely to the nineteenth century were dominated first by Cauchy and Gauss, and then later by Weierstrass. The first period helped to develop the concept of rigor; Bell claims that Gauss was the modern originator of rigorous math, and Cauchy was the first modern rigorist to gain a following. Finally, Weierstrass (and his

[18] Katz, p. 665.
[19] Katz, pp. 676-677.
[20] Boyer and Merzbach, p. 637.
[21] Kline, 721.

period) represents the progress made in the field since the time of confusion at the start of the century. "The general trend from 1700 to 1900," Bell writes, "was toward a stricter arithmetization of the three basic concepts of calculus: number, function, limit."[22] The choices Bell makes for the leading mathematicians and the period trends are subject to debate, but it is harder to find an argument against his view of the general trend in mahtematics. The nineteenth century was dominated by the rigorization of calculus which freed analysis from its geometric underpinnings. Number systems were given a proper form by Weierstrass, Cantor, and Dedekind. Limits were well-defined by Cauchy and Bolzano, and the continuity of functions and convergence of series were tackled by Cauchy, Bolzano, and Weierstrass. These advances led, at last, to the field of analysis having as its basis a concrete development of its own central ideas.

[22] Eric Bell, *The Development of Mathematics*, New York, McGraw-Hill, 1940, p. 261-262.

Chapter 2

Differential Equations in the Nineteenth Century

The French mathematician Joseph Fourier wrote, "The profound study of nature is the most fertile source of mathematical discoveries."[1] Not surprisingly, Fourier devoted much of his career to a study of differential equations, in which he could fully explore connections between interesting physical problems and mathematics. Like analysis in general, the nineteenth century saw differential equations in the nineteenth century given a more rigorous foundation. However, perhaps due to its very practical nature and the attitudes of many mathematicians working in the area, rigor was a bit slower to come to this field. Given the rough and occasionally proximate nature of at least sections of much of the important work, many mathematicians working in the field may have nodded in agreement when hearing Josef Maria Hoene-Wronski characterize the Paris Academy of Sciences criticism of his lack of rigor as "pedantry which prefers means to the end."[2] Yet, the nineteenth century mathematical climate demanded increased rigor, and it was gradually achieved. Many of the century's greatest mathematicians– Fourier, Cauchy, Gauss, Riemann, and Weierstrass– studied differential equations and contributed significantly to the field. Undoubtedly, the allure of differential equations was powerful, witnessed not only by this list of great mathematicians making significant contributions, but also by cases like that of George Boole. In the opinion of his biographer, the superb algebraist Boole "placed a futile over-emphasis on differential equations and their solution and, by returning to this topic in the last few years of his life, he missed a golden opportunity of placing his greatest discoveries in their true mathematical context of abstract algebra."[3] Thus, there was clearly significant interest, and consequently, many spectacular advances were achieved during the

[1] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, New York: Oxford, 1972, p. 671.

[2] ibid., p. 619.

[3] Desmond MacHale, *George Boole: His Life and Work*, Dublin: Boole Publishing, 1985, p. 226.

Figure 2.1: Joseph Fourier (1768-1830).

century. These advances came in both partial and ordinary differential equations, and were most often inspired by what Fourier termed "the profound study of nature."

It was Fourier's fascination with heat diffusion that inspired his development of the series that bears his name. The problem involving heat flow was an important one for both industry and science, and as a result, many scientists were involved in its study. Fourier submitted his first work on the topic to the Paris Academy in 1807, but it was rejected. Reworking and rewriting his ideas, Fourier in 1822 published *Théorie analytique de la chaleur (Analytic Theory of Heat)*. This important book began by considering the temperature distribution v in a homogeneous and isotropic body as a function of x, y, z, and t. Using established physical principles, Fourier proved that v must satisfy the equation

$$k^2 \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

where k^2 is a constant depending on the material. Then, Fourier worked specific problems. He considered the special case of a rectangular lamina infinite in the positive x-direction, having width 2 in the y-direction, and the edge $x = 0$ having a constant temperature 1, while the edges $y = \pm 1$ have a constant temperature of 0. Under these assumptions, using the method of separation of variables ($v = \phi(x)\psi(y)$), Fourier differentiated this v with respect first to x and then to

y before substituting it into the equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

At an equilibrium point for the temperature, $\frac{\partial v}{\partial t} = 0$, Fourier thus obtained the equation:

$$\phi''(x)\psi(y) + \phi(x)\psi''(y) = 0$$

or

$$\frac{\phi(x)}{\phi''(x)} = -\frac{\psi(x)}{\psi''(x)} = A,$$

where A is a constant. Solutions to these equations are

$$\phi(x) = ae^{mx}, \psi(y) = b\cos ny,$$

with $m^2 = n^2 = \frac{1}{A}$. Then, he noticed that m must be negative or else the temperature would tend toward infinity as x assumed large positive values.

The general solution of the original equation was then $v = ae^{-nx}\cos ny$, and by employing the original boundary conditions, Fourier demonstrated that n must be an odd multiple of $\frac{\pi}{2}$. Thus, the series solution is:

$$v = a_1 e^{-\pi x/2}\cos(\frac{\pi y}{2}) + a_2 e^{-3\pi x/2}\cos(\frac{3\pi y}{2}) + a_3 e^{-5\pi x/2}\cos(\frac{5\pi y}{2}) + ...$$

Working intuitively, Fourier was able to find the values for the coefficients under the additional constraint that $v = 1$ when $x = 0$. These values, $a_1 = \frac{4}{\pi}, a_2 = -\frac{4}{3\pi}, a_3 = \frac{4}{5\pi}, ...$, implied that

$$\cos u - \frac{1}{3}\cos 3u + \frac{1}{5}\cos 5u - \frac{1}{7}\cos 7u + ... = \frac{\pi}{4}$$

with $u = \frac{\pi y}{2}$ or $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$ since y is between -1 and 1. Fourier realized that this work was unique; he commented, "As these results appear to depart from the ordinary consequences of the calculus, it is necessary to examine them with care and to interpret them in their true sense."[4] Fourier believed that this meant considering the equation

$$y = \cos u - \frac{1}{3}\cos 3u + \frac{1}{5}\cos 5u - \frac{1}{7}\cos 7u + ...$$

as belonging "to a line which having u for the abscissa and y for the ordinate, is composed of separated straight lines, each of which is parallel to the axis and equal to [π]. These parallels are situated alternately above and below

[4]Victor Katz, *A History of Mathematics: An Introduction*, New York: Harper-Collins , 1993, p. 652.

the axis at the distance $[\frac{\pi}{4}]$."[5] Historian Victor Katz notes that Fourier was not truly concerned with whether this 'curve' (Fourier made it continuous by adding parallel line segments at each discontinuity.) represented a 'function' or a 'continuous function'. This was not relevant to Fourier because "he was interested in a physical problem and probably conceived of this solution in geometrical terms, where he could draw a 'continuous' curve without worrying about whether it represented a 'function'."[6]

From there, Fourier considered representing functions as a trigonometric series. Beginning with the equation $f(x) = \sum_{v=1}^{\infty} b_v \sin vx$, for $0 < x < \pi$, Fourier proceeded with steps the historian Kline characterized as "bold and ingenious, though again often questionable."[7] This led him to the formula for the coefficients,

$$b_v = \frac{2}{\pi}\int_0^{\pi} f(x)\sin vsds.$$

This equation was similar to what Alexis-Claude Clairut, Daniel Bernoulli, and Leonhard Euler had found in the previous century. At this point, though, Fourier made observations that allowed him to take his theory further. In particular, each b_v could be interpreted geometrically as the area under the curve $y = \frac{2}{\pi}f(x)\sin vx$ with x between 0 and π. This idea allowed Fourier to credibly claim that his series representation was valid for all types of functions, not only those infinitely differentiable. To demonstrate this, Fourier calculated the first few coefficients for many different types of functions, finding that his series assumed the same values as the function itself on the interval $(0, \pi)$. Finally, Fourier found that his earlier sine representation could lead to a cosine representation of $f(x) = \frac{a_0}{2} + \sum_{v=1}^{\infty} a_v \cos vx$, and elementary facts concerning even and odd functions led to a representation for any $f(x)$ between $-\pi$ and π of

$$f(x) = \frac{a_0}{2} + \sum_{v=1}^{\infty} a_v \cos vx + \sum_{v=1}^{\infty} b_v \sin vx,$$

with $a_v = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos vsds$ and $b_v = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin vsds$. Fourier did not give a proof that this series would represent every function; instead, he rested his claim on geometrical evidence and its success in solving physical problems. He wrote, "Nothing has appeared to us more suitable than geometrical constructions to demonstrate the truth of the new results and to render intelligible the forms which analysis employs for their expressions."[8] Fourier's ideas substantially advanced both partial differential equations and the debate concerning the conception of a function. Moreover, this ground breaking work served as inspiration and a starting point for work done by many other mathematicians throughout the rest of the century.

[5] Joseph Fourier, *The Analytic Theory of Heat*, translated by Alexander Freeman, Cambridge: Cambridge University Press, 1878.

[6] Katz, p. 653.

[7] Kline, p. 675.

[8] ibid., p. 677.

Gustav Peter Dirichlet, for instance, was strongly influenced by what Fourier had done. Working in the years immediately following the publication of *Théorie analytic de la chaleur*, Dirichlet recognized some of the holes in Fourier's work, including the absence of a proof that an 'arbitrary' function would converge to a Fourier series. While unable to supply this proof, he did, within a few years, find sufficient conditions on the function which would assure that it converged. Further, Dirichlet's continued interest in this area led him to a class of partial differential equation problems that now bear his name. These problems involve solving Laplace's equation in a region with given boundary values. Cauchy and Siméon-Denis Poisson also were quite intrigued with Fourier's results. In the years succeeding the presentation and publication of Fourier's ideas, both mathematicians (as well as Fourier himself) worked on 'Fourier integrals'. Fourier integrals arose from a desire to express solutions in closed form. Closed solutions are given in terms of elementary functions and integrals of such functions. Independently, Fourier, Poisson, and Cauchy worked on this problem with similar findings. In 1816, Cauchy was awarded the prize from the Paris Academy for his paper 'Théorie de la propogation des ondes'. This paper investigated waves on the surface of a fluid, and in so doing developed the equations

$$F(x) = \int_0^\infty \cos mx f(m)dm, \tag{2.1}$$

$$f(m) = \frac{2}{\pi}\int_0^\infty \cos mu F(u)du, \tag{2.2}$$

and

$$F(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty \cos mx \cos mu F(u)du dm. \tag{2.3}$$

Cauchy had thus derived the Fourier transform from $f(m)$ to $F(m)$ (2.1), the inverse transform (2.2), and the Fourier double integral representation of $F(x)$ (2.3). Later the same year, Poisson– who was a rather unfriendly rival of Fourier's– published results similar to Cauchy's.

Mention of the involvement of Cauchy, the leading rigorist of the nineteenth century, in the field of differential equations begs the question of when rigor was introduced into the theory. As we have seen, many of the advances were a direct consequence of work done to solve physical problems, and mathematician Garrett Birkhoff comments, "The theory of partial differential equations hardly existed before 1840." To be sure, there had been numerous advances in the century preceding that date, "but the first *general* existence theorem about partial differential equation was not proved until 1842."[9] In his paper supplying this proof, 'Mémoire sur l'integration...,' Cauchy wrote, "Can one generally integrate a partial differential equation of any order whatsoever, or even a system of such equations? As I remarked at the next to the last session [of the Paris Academy], this is a problem whose importance is incontestable, but which

[9] Birkhoff, p. 318.

is totally unresolved. Now, with the help of the fundamental theorem established [here] I not only solve the problem in question, but also bound the error committed on truncating certain power series representing the expansions of solutions after...a number of terms."[10] In this paper, Cauchy worked with the initial-value problem (now called the Cauchy problem) for systems of first-order partial differential equations. That is, he considered

$$\frac{\partial u_i}{\partial t} = F_i\left(t, x_1, x_2, ..., x_n; u_1, u_2, ..., u_m; \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_m}{\partial x_n}\right),$$

with the initial 'condition' that in the t-plane:

$$t = 0 : u_i(0, x_1, ..., x_n) = \omega_i(x_1, ..., x_n), \text{ for } i = 1, ..., m.$$

By assuming that F_i and ω_i were analytic, Cauchy was able to obtain a unique power series solution that was locally convergent. Relying on his recently created Method of Majorants, which he had developed and used to prove a similar existence theorem for initial value problems involving ordinary differential equations, Cauchy was able to complete this proof. In 1874, one of Weierstrass' students, Sonia Kowalewski, generalized Cauchy's proof to cover cases of any order. Consequently, the existence theorem bears both their names. Without question, much of the lack of rigor in the field can be attributed to the inclinations of the mathematicians occupied with solving specific physical problems; however, there is some evidence suggesting institutional bias may have played a role in slowing the rigorization of the theory of differential equations. Specifically, Cauchy's teaching at the École Polytechnique included his treatment of calculus in the first year and differential equations in the second year. While his rigorous texts, *Cours d'Analyse* and *Résumé...*, formed the basis of Cauchy's teaching in the first year, there is some indication that Cauchy was not free to teach the second year subject as he desired. Katz notes that "Cauchy was reproached by the directors of the school. He was told that, because the École Polytechnique was basically an engineering school, he should use class time to teach applications of differential equations rather than to deal with questions of rigor."[11] Cauchy, it is believed, never published his course notes for this subject because they were contrary to his own views on how the subject should be handled.

Another significant development during the century involved advances made in potential theory. In the late eighteenth century, Laplace showed that the potential V satisfied the equation $\Delta V = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$. However, Poisson in 1813 noticed that Laplace's equation "holds when the attracted point lies outside the solid under consideration, or even when, this body being hollow, the attracted point is situated in the interior cavity... It is nonetheless not superfluous to observe that it no longer holds if the attracted point is an interior

[10]Birkhoff, p.319.
[11]Katz, p. 650.

point of the solid."[12] Poisson then demonstrated that in regions occupied by matter, the proper equation is:

$$\Delta V = 4\pi\rho,$$

where ρ is the local density. The self-taught British mathematician George Green carefully studied the work of Poisson, and in 1828 he published the privately printed booklet *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* This work began by deriving from Laplace's equation an amazing theorem. Green found that if U and V are two continuous functions of x, y, and z with derivatives not infinite at any point of an arbitrary body, then

$$\iiint U\Delta V dv + \iint U\frac{\partial V}{\partial n}d\sigma = \iiint V\Delta U dv + \iint V\frac{\partial U}{\partial n}d\sigma, \tag{2.4}$$

where n is the surface normal (directed inwards) of the arbitrary body, and $d\sigma$ is a surface element. From there, Green demonstrated that rather than boundary conditions, he could stipulate only that V and its first derivatives be continuous inside the body. Then, Green represented V inside the body in terms of $\bar{V}$, the value V assumes on the boundary of the surface and another function U, where U satisfies Laplace's equation in the interior; U is 0 on the surface; and at a fixed arbitrary point P inside the body, U becomes infinite. Once U is found, then V can be determined using the equation

$$4\pi V = -\iint \bar{V}\frac{\partial U}{\partial n}d\sigma,$$

where the integral is over the surface of the body, and the partial derivative of U with respect to n is the derivative of U perpendicular to the surface, directed inwards. A little more than a decade after Green's publication, Gauss still further advanced potential theory by producing a rigorous proof of Poisson's equation in his 1840 paper 'Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs— und Abstossungs— kräfte' (General theorems on attractive and repulsive forces which act according to the inverse square of the distance').

In the eighteenth century, ordinary differential equations were the direct consequence of attempts to solve physical problems, and partial differential equations arose as a tool for handling especially complicated problems. "In the nineteenth century," Kline claims, "the roles of these two subjects were somewhat reversed. The efforts to solve partial differential equations by the method of separation of variables led to the problem of solving ordinary differential equations. Moreover, because the partial differential equations were expressed in various coordinate systems the ordinary differential equations that resulted were strange ones and not solvable in closed form."[13] One of the differential

[12] Birkhoff, p. 343.
[13] Kline, p. 709.

equations that resulted form the method of separation of variables is

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

for a parameter n. This equation is known as the Bessel equation after Friedrich Wilhelm Bessel. Bessel, who like Gauss worked as both a mathematician and an astronomer, considered this question as part of his study of the motions of the planets. He gave the first of two independent solutions of this equation for each n. [14] The solution Bessel obtained was

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(nu - x\sin u)du.$$

Subsequent to this finding, Bessel obtained a series solution for $J_n(x)$, and then later showed in 1818 that $J_0(x)$ had infinitely many real zeros. Six years later, he produced results that included a recursion formula for $J_n(x)$. In particular,

$$xJ_{n+1}(x) - 2nJ_n(x) + xJ_{n-1}(x) = 0.$$

Bessel, however, worked strictly with real variables; only later work would establish his results for complex variables. The questions left unanswered by Bessel (including the second solution to his second-order equation) spurred further work in this area. Moreover, the original question involving separation of variables that motivated Bessel inspired others like Gauss, Gabriel Lamé, and Joseph Liouville to consider this type of problem. This created branches of study which included the Bessel functions as well as spherical and ellipsoidal functions.

Liouville's interest in differential equations and these separation of variables problems led him to investigate the equations connected with various physical problems. In fact, together with the Swiss born mathematician Charles Sturm, he developed a class of problems known today as Sturm-Liouville problems. Almost all of their work was published in the first two editions of what was perhaps the century's most important journal. Edited by Liouville, the *Journal de mathématiques pures et appliquées* featured Sturm and Liouville's results in its 1836 and 1837 editions. Sturm and Liouville considered problems like the heat equation or the vibrating string for which there are certain necessary boundary conditions. In each case, the resulting partial differential equation is resolved through separation of variables into two or more ordinary differential equations. The boundary conditions on the original solution then become boundary conditions on an ordinary differential equation. In other words, the original partial equation leads to an equation of the form:

$$\frac{d}{dx}(k(x)\frac{dV}{dx}) + (g(x)r - l(x))V(x) = 0, x \in (a,b)$$

with the imposed boundary conditions

$$kV'(a) - hV(a) = 0,$$

[14] Carl Neumann and Hermann Hankel's work resulted in the second solution.

$$kV'(b) + HV(b) = 0.$$

In these equations k, g, and l are given functions, h and H are given positive constants, and r is a parameter. Solutions of the problem could be found, then, for certain values of this parameter. These values are now called the eigenvalues while the solution for a particular eigenvalue is termed the eigenfunction. Sturm and Liouville primarily considered three aspects of these types of problems. These areas included: properties of the eigenvalues, qualitative behavior of the eigenfunctions, and expansion of arbitrary functions into an infinite series of eigenfunctions. Of these areas Sturm worked primarily with the first two, while Liouville considered the third (and in so doing contributed to the first two).[15] Sturm and Liouville were able to demonstrate various fundamental properties for these problems. First, the problem will have non-zero solutions only when the parameter takes on one of an infinite sequence of positive numbers increasing towards infinity. Moreover, for each parameter in this sequence of possible non-zero parameter values r_n, the solutions are multiples of one function q_n which can be normalized by solving $\int_a^b \rho q_n^2 dx = 1$. These functions q_n were also shown to be orthogonal.

Further, every twice differentiable function $f(x)$ which satisfied the boundary conditions could be expanded as:

$$f(x) = \sum_{n=1}^{\infty} c_n q_n,$$

with $c_n = \int_a^b \rho f(x) q_n(x) dx$. However, Kline notes that the proof for this last assertion was inadequate. He writes, "One difficulty was the matter of the completeness of the set of eigenfunctions... Also, the question of whether the series converges to $f(x)$, whether uniform, or in some more general sense, was not covered though Liouville did give convergence proofs in some cases."[16] Despite these failings, the importance of the theory extended beyond solutions to the problems inspiring Sturm and Liouville. Indeed, Birkhoff comments, "The general principle that one can expand 'any' periodic function into Fourier series is a special case of a far more general principle; that of expansion into eigenfunctions of any linear boundary-value problem... The first major generalization of the theory of Fourier series (more accurately, of sine series) was provided by Sturm-Liouville theory."[17] In many ways, Sturm-Liouville theory was typical of nineteenth century differential equation. That is, Sturm and Liouville were motivated by a physical problem to produce exceptional results that were not completely rigorous.

One more important area of study involving nineteenth century differential equations was the consideration of regular singularities of linear ordinary differential equations in the complex plane. The initiator of this study was Lazarus

[15] Jesper Lützen, *Joseph Liouville, 1809-1882: Master of Pure and Applied Mathematics*, New York, Springer-Verlag, 1990, p. 422.

[16] Kline, p. 717.

[17] Birkhoff, p. 258.

Fuchs, who had in turn been motivated by the papers of Riemann and the lectures on Abelian functions given by Weierstrass.[18] In 'Zur Theorie der linearen Differentialgleiehungen', Fuchs began, "In the theory of differential equations, there is currently less concern about reducing a given differential equation to quadratures than about determining from the given differential equation itself the behavior of solutions in the [complex] plane, that is, for all values of the independent variable. Analysis shows how to determine a function when behavior is known in the neighborhood of all [singular] points where it is discontinuous or multiple-valued. Hence, the essential task in the integration of a given differential equation is to determine the location of these points and the behavior of solutions in their neighborhood.[19] Then, Fuchs pushed the study of differential equations in a new direction. In particular, he considered linear differential equations of the type

$$y^{(n)} + p_1(z)y^{(n-1)} + \ldots + p_n(z)y = 0,$$

where $p_i(z)$ are single-valued analytic functions of a complex variable z having at most a finite number of poles, none of order exceeding its index. Fuchs proved that there exists a basis of solutions to this equation of the form

$$y_j = (z-c)^j F_j(z),$$

with $F_j(z)$ analytic in most cases, although in certain cases it was shown to have logarithmic terms. Fuchs' work was further extended and refined by Georg Frobenius in the last part of the century. Almost without question, the most important aspect of the work of Fuchs and Frobenius) was in extending the theory of differential equations to complex function theory. In so doing, they succeeded in further broadening the scope of the study of differential equations.

The study of differential equations in the nineteenth century was, most often, motivated by physical problems. However, like analysis as a whole, rigor slowly came to the theory of differential equation. That this process was a bit slower is not surprising given these close ties to real world problems. Indeed, many mathematicians questioned why it was necessary to prove a solution exists to a mathematical problem based on a physical problem with an obvious solution. Further, many were concerned primarily with the result not the process, and consequently, it is not uncommon to see loose demonstrations and incomplete proofs in even the best papers presented during the century. Still, many of the advances made were astounding. Prophesizing near the close of the eighteenth century, Lagrange wrote, "It appears to me...that the mine [of mathematics] is already very deep and that unless one discovers new veins it will be necessary sooner of later to abandon it."[20] Given Lagrange's interest in differential equations, he certainly would have been amazed and gratified to see the varied developments of the next century. Through systematic study, the field of differential equations was substantially broadened. Physical problems were, as

[18]Carl Boyer and Uta Merzbach, *A Histoy of Mathematics*, New York: Wiley, 1989, p. 626.
[19]Birkhoff, p. 283.
[20]Kline, p. 623

always, at its heart, but the development of its theory had been responsible for breakthroughs in other areas including series representation, complex function theory, and a closer analysis of functions themselves. Certainly, these and other advances often owed their inspiration to physical problem, yet just as obviously, the significance of these solutions transcended the importance of the original problem.

Chapter 3

Sturm, Liouville, and Their Joint Work

Like Sturm-Liouville theory, the following all belong to nineteenth century analysis: The Cauchy-Riemann equations, the Bolzano-Weierstrass theorem, the Cauchy-Lipschitz theorem, the Hamilton-Jacobi Equation, the Riemann-Roch theorem, the Riemann-Stieltjes integral, the Cauchy-Schwarz inequality, the Cauchy-Goursat theorem, and the Schwarz-Christoffel transformation. Each of these mathematical ideas bears the name of two mathematicians. However, unlike their nineteenth century counterparts, Sturm-Liouville theory is anachronistic in a remarkable way. While each of the other named theorems, integrals, or equations bears the names of two men, none of these pairs worked together. Each of the ideas were developed simultaneously, but independently, or the concept was developed by one man and later improved by the other. For instance, proof of the Bolzano-Weierstrass theorem was first presented by Bolzano in 1817. Almost a half century later, Weierstrass' demonstration modified and improved Bolzano's proof by basing the reasoning on a rigorous system of real numbers. Atypically for their time, Sturm and Liouville worked cooperatively to develop their ideas. Sturm's interest in the subject was encouraged by Liouville. And, after Liouville learned of Sturm's work, he took up in earnest the study of this same subject. They authored one article on this subject together, and they commented on each other's work. Yet these two friends were not only well ahead of their time in terms of their collaborative, synergistical work style, the results produced by Sturm and Liouville were both advanced and complete enough to stand almost untouched for forty years. That Sturm-Liouville theory was 'rediscovered' then as an important early example motivating spectral theory only reinforces the importance of their work. The achievements of both Sturm and Liouville, however, go well beyond this theory. Their lives and careers, then, provide a revealing look at the interests and personalities that made the production of Sturm-Liouville theory possible.

3.1 Charles Sturm

Perhaps one of the greatest testaments to Sturm's abilities was his success, as an outsider, in rising through the highly politicized French mathematical system of the early nineteenth century. Born in Geneva on 29 September 1803, Charles-François was the son of the arithmetic teacher Jean-Henri Sturm and Jeanne-Louise-Henriette Gremay. His early educational interests were the classics. Alongside his developing abilities in both Latin and Greek, Sturm made it a practice to attend Lutheran sermons each week to perfect his German. At sixteen, however, he ended this study of classic literature, turning his attention to mathematics. Attending the Geneva Academy, Sturm heard lectures in mathematics by Simon l'Huillier and physics lectures given by Pierre Prevost and Marc-Auguste Pictet. L'Huillier was especially encouraging of the young Sturm, loaning books and offering advice.[1] At the Academy, Sturm met Daniel Colladon, an aspiring physicist who quickly became Sturm's closest friend. Sturm had completed his studies by 1823, and he moved just outside of Geneva to take a private tutoring position. The job had limited demands, allowing him to begin writing and publishing mathematical articles. The subject of this early work was geometry, and it appeared in the pages of the journal by J.D. Gergonne *Annales de mathématiques pures et appliquées.* His tutoring position had still other advantages. Through this job, Sturm met the wealthy and the well-connected, and when his employer, the de Staël family, traveled to Paris later this same year, Sturm was brought along. Moreover, one of the relatives of this family and now an acquaintance of Sturm's, Duke Victor de Broglie, arranged meetings for Sturm during this trip with some of the leading mathematicians in Paris.

The family and Sturm stayed for a half year in Paris, during which time Sturm began to make contacts in the Parisian mathematical community. Sturm met the mathematical physicist Francois Arago and was subsequently invited to his house. In a letter to Colladon from Paris, Sturm wrote, "I have two or three times been among the group of scientists he invites to his house every Thursday, and there I have seen the leading scientists, Laplace, Poisson, Fourier, Gay-Lussac, Ampère, etc. Mr. de Humboldt, to whom I was recommended by Mr. de Brogllie, has shown an interest in me; it is he who brought me to this group. I often attend the meetings of the Institiut that take place every Monday."[2]

Upon his return to Switzerland, Sturm and Colladon devoted themselves to scientific research. The compression of liquids was the topic set by the Paris Academy as its 1825 mathematics and physics grand prize, and this proved sufficient motivation for the two. They devised a plan to measure the speed of sound in water, using nearby Lake Geneva as a testing site. They would derive the coefficient of compressibility of water from physical tests, and compare this result with theoretical values obtained from work with Poisson's formula for the

[1] Pierre Speziali, 'Charles-François Sturm', in Charles Coulston Gillispie (ed.), *Dictionary of Scientific Biography*, **13**, New York: Scribners, 1975, p. 127.

[2] Pierre Speziali, *Charles-François Sturm (1803-1855) Documents Inédits.* Paris: Palais de la Découverte Paris, 1964, p. 15.

Figure 3.1: Charles Sturm (1803-1855). Drawing based on an 1822 line drawing by his friend Daniel Colladon.

speed of sound. The work proved less successful than either man had hoped, and during the testing process, Colladon had the misfortune of badly injuring his hand. Undeterred, the two traveled to Paris in December 1825. There, they hoped to attend physics courses, complete the work from their experiment, and submit their paper to the Paris Academy. During this time, Sturm and Colladon were able to hear lectures at the Sorbonne and the Collège de France from Ampère and Gao-Lussac in physics and Lacroix and Cauchy in mathematics.[3] Moreover, Ampère offered the two the use of his physics laboratory. Their paper was submitted, but it did not receive the prize.

Luckily for Sturm and Colladon, the Academy judged no paper to be worthy of the award, and the topic was now the subject of the 1827 prize. It was also at this time that the two friends visited Fourier, who, impressed by their abilities, suggested Colladon measure the thermal conductivity of different materials while Sturm could make a detailed study of harmonic analysis.[4] In 1826 Sturm and Colladon received appointments as assistants to Ampère, who proposed the two men collaborate on a major project of theoretical and experimental physics. However, this was never realized. Sturm's interests were principally mathematical, and consequently, after 1827, his collaborative work with Colladon ended. Before this, though, Colladon returned to Geneva in November 1826 for further

[3]Speziali, 'Charles-François Sturm', p. 127.
[4]Speziali, *Charles-François Sturm (1803-1855) Documents Inédits*, p. 18.

testing on Lake Geneva. When he returned to Paris the following year, he and Sturm rewrote their paper and submitted it to the prize committee. This paper, 'Mémoire sur la compression des liquides et la vitesse du son dans l'eau,' won Sturm and Colladon the 3000 franc grand prize, a sum that allowed each of them to remain in Paris.

From this point, Sturm's mathematical career moved forward quickly. He briefly resumed his investigations into geometry. Opinions regarding his output in this field are mixed. Mathematician Maxime Bôcher dismisses them as a collection of "minor papers,"[5] while Sturm's biographer Pierre Speziali claims that "he made a valuable, original contribution. The essential features of his work in this area were incorporated in later works on geometry, often without mention of their origin."[6] Sturm's admiration of Fourier led him towards research topics also favored by Fourier. Thus Sturm began to study both the theory of heat and solutions of numerical equations. By 1829, Ampère had helped him to become the head editor of mathematics of *Bulletin des sciences et de l'industrie.* More significantly, in May of that year he submitted to the Paris Academy his first truly important mathematical paper. This paper, a working draft of his 1835 publication 'Mémoire sur la résolution des équations numériques', presents his theorem for determining the number of real roots of a polynomial on an interval. In the theorem, Sturm lets $V(x) = 0$ be an equation of arbitrary degree with distinct roots. Further, let V_1 be the derivative of V. Then, one continues as if finding the greatest common divisor of V and V_1, except the sign of the remainders must be changed when they are used as divisors. Further, call $Q_1, ..., Q_{r-1}$ the quotients and $V_2, ..., V_{r-1}$ the remainders with V_r a constant. One now has the following equations:

$$V = V_1Q_1 - V_2$$

$$V_1 = V_2Q_2 - V_3$$

$$.....$$

$$V_{r-2} = V_{r-1}Q_{r-1} - V_r.$$

Sturm's theorem then states: For arbitrary a and b, $a < b$, let M be the number of changes in the signs of the sequence of functions $V, V_1, V_2, ..., V_{r-1}, V_r$ for $x = b$ and let N be the total sign changes for this same sequence when $x = a$. The difference $M - N$ will be equal to the number of real roots of the equation $V = 0$ between a and b.

This theorem was based in part on similar research done by Fourier, but according to Sturm it was not developed as part of Sturm's studies in algebra. Instead, Sturm wrote that it was a fortuitous by-product of his own researches into second-order linear differential equations.[7] That this theorem was credited

[5] Maxime Bôcher, 'The Published and Unpublished Work of Charles Sturm on Algebraic and Differential Equations', *Bulletin of the American Mathematical Society*, **18**,1911-1912, p. 1.

[6] Speziali, 'Charles-François Sturm', p. 128-129.

[7] Charles Sturm, 'Mémoire sur les équations différentielles linéaires du second ordre', *Journal de mathématiques pures et appliquées*,**1**, 1836, p. 186.

to Sturm created some controversy. Cauchy had also developed a method for determining the number of real solutions, and it had appeared several years before Sturm's work. Cauchy reminded others of his priority, writing in 1837 that "M. Poisson gave a report that verified, as to an equation of arbitrary degree, I was the first to have developed methods by which it is possible to find rational functions with coefficients whose signs show the number of real roots between given limits."[8] However, Cauchy's method was quite different from that developed by Sturm, and it involved significantly longer calculations. Summarizing this dispute years later mathematician Charles Hermite (1822-1901) wrote, "Sturm's theorem had the good fortune of immediately becoming classic and of finding a place in teaching that it will hold forever. His demonstration, which utilizes only the most elementary considerations, is a rare example of simplicity and elegance."[9]

Following this, in 1833 Cauchy demonstrated a method for finding the number of imaginary roots of an equation. Interestingly, Sturm also studied this problem. He was joined in these considerations by Liouville, with whom he was at the time working on second-order differential equations. In 1836 and 1837, the two men co-authored a pair of articles, again finding a method that was shorter and more basic than that presented a few years earlier by Cauchy.[10] By this time, despite his status as an outsider, Sturm had established a good reputation within the Parisian mathematics community. However, his career advancements were retarded by factors outside of his control. One of the leading historians of French mathematics during this period, Ivor Grattan-Guinness, writes that the system in early nineteenth century France "was very competitive, and indeed pursuit by many men of the same post was [common]." This phenomenon was the result of the general practice of a single man holding many posts. Grattan-Guinness comments, "The French evolved a system which foreigners had the good sense to avoid: *cumul*, as it was called, the accumulation of several appointments simultaneously."[11] Moreover, foreigners were banned from permanent positions at certain state schools.

Luck played an important part in Sturm's rise. The revolution of July 1830 removed the Catholic monarchy from power, and foreigners and Protestants such as Sturm were now eligible to be appointed to all positions. Despite this ultra-competitive atmosphere, Sturm quickly received an appointment as professor of *mathématiques spéciales* at the Collège Rollin. Then in March 1833, Sturm was granted French citizenship. Soon thereafter, he was offered positions at the Geneva Academy and the University of Ghent, but he turned both down, preferring to remain in France. When Ampère died, a seat fell vacant on the prestigious Paris Académie des Sciences, and Lacroix nominated Sturm for the

[8] Bruno Belhoste, *Augustin-Louis Cauchy*, New York: Springer-Verlag, 1991, p. 329

[9] Speziali, 'Charles-François Sturm', p. 129.

[10] Joseph Liouville and Charles Sturm, 'Démonstration d'un théorème de M. Cauchy relatif aux racines imaginairds des équations', *Journal de mathématiques pures et appliquées*, **1**, 1836, pp. 278-289 and 'Note sur un héorème de M. Cauchy relatif aux racines des équations simultanées', *Comptes Rendus de l'Académie des Sciences*, 4, pp. 720-739.

[11] Ivor Grattan-Guinness, 'Some Puzzled Remarks on Higher Education in Mathematics in France, 1795-1840', *History of Universities*, 7, 1988, pp. 211-212.

seat. In the ensuing election, Sturm won easily, helped by fellow candidates Liouville and Jean-Marie-Constant Duhamel who withdrew shortly before the balloting, thereby giving Sturm the victory.

Now an insider, Sturm began to accumulate his collection of appointments. He became in 1838 a *répétiteur* of analysis for courses given by Liouville at the École Polytechnique. Two years later, he replaced Duhamel as the second professor in analysis and mechanics at this school. Also in 1840, Sturm was appointed to succeed Poisson as the chair in mechanics at the Faculté des Sciences. These various positions meant that Sturm had to spend increasing amounts of time preparing courses. At the École, for instance, he was responsible for lectures covering differential and integral calculus and rational mechanics. Regarded as a fine lecturer, Sturm won praise for his personality as well as his knowledge.[12] Evidence of his devotion to his teaching comes in the fine design of his course notes. Published posthumously as *Cours d'Analyse de l'École Polytechnique* and *Cours d'Mécanique de l'École Polytechnique*, Sturm's books set a standard. Used in courses for many years, the books were reprinted many times; *Cours d'Analyse* had fourteen editions with the last reprint in 1910. Yet, despite an increasingly heavy work load, Sturm continued important research.

The most significant of this research involved questions inspired by investigations the theory of heat. This research led Sturm, fortuitously, to his theorem concerning the number of real roots of an algebraic equation, and direct consideration of this same problem resulted in his contribution to what is now known as Sturm-Liouville theory. In 1829, Sturm published his first works on this subject. These papers are now lost, but preserved short summaries indicate that Sturm had already begun to develop many of the ideas of the theory.[13] In his first major memoir, published in the first volume of the new journal edited by Liouville, *Journal de mathématiques pures et appliquées*, Sturm wrote:

> The theory explained in this memoir on linear differential equations of the form
>
> $$L\frac{d^2V}{dx^2} + M\frac{dV}{dx} + NV = 0$$
>
> corresponds to a completely analogous theory which I have previously made concerning linear second-order equations of finite differences of the form
>
> $$LU_{i+1} + MU_i + NU_{i-1} = 0$$
>
> where i is a variable index replacing the continuous variable x; L, M, N are functions of this index i and of another undetermined quantity m which is subject of certain conditions. It is while studying the properties of a sequence of functions $U_0, U_1, U_2, \ldots$, related by a similar system of equations as that above, that I found my theorem on the determination of the number of real roots of a numeric

[12] Speziali, 'Charles-François Sturm', p. 128.

[13] See the bibliography for the titles of these papers.

equation between two arbitrary limits, which is contained as a particular case in the theory which I just indicated here. It turns into the one which is the subject of this memoir by passing from finite differences to the infinitely small differences.[14]

Basing his considerations on the title of one of these papers from 1829, 'Sur la distribution de la chaleur dans un assemblage de vases,' Bôcher convincingly reconstructs how Sturm could have been led to the above difference equation and thus his breakthroughs in two separate fields from his consideration of the heat equation.[15] It was from this early work, then, that Sturm was able, in 1833, to submit to the Paris Academy a paper outlining all of his major results.

This paper formed only an outline, however, including results, but not proofs. Full publication of these results was delayed for several years. This was at least in part because Sturm's friend Liouville had asked to print these results in the inaugural volume of his journal. Thus it was 1836 before Sturm made a detailed presentation of his results. This delay in publication may have proved somewhat regrettable. Sturm's two memoirs were overly long (81 and 72 pages respectively), the papers were not concise (often including several proofs of a single point), and the major results lacked the proper emphasis. This was perhaps a consequence of the delay, although Sturm's own excitement involving the novelty of his results may have influenced his style. He proudly proclaimed at the start of the first paper, "The principle on which the theorems which I develop rests has never been used in analysis as far as I know."[16] Indeed, Sturm's results were ground breaking. He studied the properties of solutions to differential equations of the form

$$\frac{d}{dx}(k(x)\frac{dV}{dx}) + (g(x)r - l(x))V(x) = 0, \; x \in (a,b) \tag{3.1}$$

with the imposed boundary conditions

$$kV'(a) - hV(a) = 0 \tag{3.2}$$

$$kV'(b) + HV(b) = 0. \tag{3.3}$$

Here k, g, and l are given positive functions, h and H are positive constants, and r is a parameter. With his papers completed before Liouville had even begun his, Sturm, it should be noted, was the driving force behind the creation of this theory. Of the three major areas involved in this research: properties of the eigenvalues, qualitative behavior of the eigenfunctions, and the expansion of functions into series of eigenfunctions, Sturm's researches were confined to the first two. A general sketch of Sturm-Liouville theory was given in the preceding chapter; however, it is worthwhile to emphasize some of Sturm's most important results.

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[14]Charles Sturm, 'Mémoire sur les équations différentielles linéaires du second ordre', p. 186.

[15]Bôcher, 'The published and unpublished work of Charles Sturm...', pp. 8-13.

[16]Sturm, 'Mémoire sur les équations différentielles linéaires du second ordre', p. 107.

Included among these were comparison theorems. These literally compared the number of zero points on an interval (a, b) for solutions $u_i(x)$, $i = 1, 2$ of equations $\frac{d}{dx}\left(K_i\frac{dy}{dx}\right) + G_i y = 0$ with boundary conditions $K_i u_i'(a) = h_i u_i(a)$. Now, if on the interval $G_2 \geq G_1$, $K_2 \leq K_1$, and $h_2 < h_1$, then u_2 vanishes and changes sign at least as many times as u_1; and if one lists the roots of u_1 and u_2 in increasing order from a, the roots of u_1 are greater than the roots of u_2 of the same order. Sturm also proved oscillation theorems. He showed that on the same interval (a, b), if r_1 and r_2 are two consecutive values of r corresponding, respectively, to the characteristic functions u_1 and u_2 that satisfy $Ku_i'(x) + hu_i(x) = 0$ for $x = b$, then u_2 has one more root on the interval than u_1.[17] Finally, Sturm proved some remarkable results involving (in modern terminology) the eigenvalues and eigenvectors of a Sturm-Liouville problem (a problem of the form (3.1) with the boundary conditions (3.2) and (3.3)). In particular, he showed that the eigenvalues are all real and positive, and that they are infinite in number without multiple roots. Further, the eigenfunctions corresponding to these eigenvalues were demonstrated to be orthogonal to one another.

Sturm never again developed anything approaching these results. The research production in the later part of his career slowed, perhaps as a consequence of his greater academic responsibilities. It is possible that this gradual strain of a difficult, accumulated workload weighed too heavily on Sturm, the conscientious teacher and dedicated researcher. Around 1851, his health began to deteriorate, and he was forced to arrange for substitute lecturers at the Sorbonne and at the École Polytechnique. Sturm suffered a nervous breakdown, became obese, and lost all interest in intellectual pursuits. Ordered by his doctors to move to the country and rest, Sturm did so and recovered briefly. He resumed some teaching duties in 1853, but his health deteriorated again soon after, this time with additional complications. His illness progressively worsened, and on 18 December 1855, Sturm died in Paris. Two days later, Sturm was buried in the cemetery at Montparnasse. Among those eulogizing Sturm was his friend, Joseph Liouville. Liouville said that Sturm was "a second Ampère: candid like him, indifferent to wealth and the vanities of the world." He movingly concluded, "Adieu, Sturm, adieu."[18]

3.2 Joseph Liouville

Borrowing characterizations from considerations of Russian literature, it is common to characterize a writer with a broad ranging style as a fox (for instance, Pushkin and Chekov are foxes) while an author who has a more focused view is called a hedgehog (Dostoevsky is one). Almost certainly, one could make a strong case for Sturm as a hedgehog. Like Fourier, he centered his career on

[17] Sturm, 'Mémoire sur les équations différentielles linéaires du second ordre', p. 139-143.

[18] Eric Prouhet, 'Notice sur la vie et les travaux de Ch. Sturm,' in Charles Sturm, *Cours d'Analyse de l'École Polytechnique*, Paris: Gauther-Villars Imprimeur-Libraire, 1880, **1**, pp. XX,XXII.

work involving the theory of heat and algebraic equations. Liouville, however, would certainly be categorized as a fox. The breadth and scope of his studies are astounding, especially since alongside his teaching and research work, Liouville founded and then edited (for almost forty years) the most important French mathematical journal of the century. At the moment that Liouville stood over the grave of his friend Sturm, emotionally bidding him 'adieu', he was still only 46. Despite this relatively young age, he had by the time of Sturm's death accomplished a great deal in a wide array of mathematical fields. Moreover, his career would continue, albeit with less substantial results in the later years, for another quarter century.

The second child of a middle class family, Joseph Liouville was born on 24 March 1809 in Saint-Omer, Pas de Calais. He developed an early interest in mathematics, studying even in his adolescence the leading French research journal of the time, *Annales de Mathématiques Pures et Appliquées*. At the age of sixteen, Liouville was accepted into the École Polytechnique, which despite its status as an engineering school featured what was probably the best mathematics training in France. At the École, Liouville further cultivated his interest in mathematics, both through the program there and by way of independent reading. The professors in analysis and mechanics at the time were Cauchy and Ampère, with the two men teaching the classes in alternating years. Arriving in an odd year, 1825, Liouville's class had Ampère as their professor. While there may have been some disappointment in missing Cauchy's lectures, it was perhaps a blessing in disguise. Ampère presentation of the subjects did not differ much from Cauchy's and Ampère became a big supporter of Liouville's.[19] Also, by some reports, Cauchy was a poor teacher.[20]

Liouville completed his studies at the École Polytechnique in 1827, and was then faced with the decision of where to complete his engineering studies. After some deliberation, he chose the École des Ponts et Chaussées. The program there included both theoretical and practical training. While Liouville had little complaint with the former, he described the mandatory trainee voyages as a "waste [of] your time."[21] Indeed, Liouville bristled against the practical demands of the engineering profession, far preferring studies involving pure mathematics. It was this feeling that led Liouville to leave the École des Ponts et Chaussées before his training was complete. Having just married, Liouville in 1830 left for Paris with his new wife to pursue a scientific career.

Liouville arrived there well prepared. He had during his time in the Ponts et Chaussées already begun some research work; he submitted seven papers to the Paris Academy between the years 1828 and 1830. Two of these papers dealt with mathematical analysis, three with the theory of heat, and the other two treated the theory of electricity. These papers were solid but not especially noteworthy, and the reviewers, Fourier and Poisson, elected not to write an official report on them. Persevering, Liouville submitted some of these papers to various

[19]Lützen, *Joseph Liouville*, p. 8

[20]Grattan-Guinness, 'Grandes Écoles, Petite Université: Some puzzled remarks...', pp. 214-215.

[21]Lützen,*Joseph Liouville*, p. 10.

Figure 3.2: Joseph Liouville (1809-1882) in 1859.

scientific journals, and several were in fact published. With this publication, Liouville began to establish a reputation in the Paris scientific community, and the following year, he was offered a position as *répétiteur* for Claude-Louis Mathieu's course in analysis and mechanics at the École Polytechnique. From there, Liouville's academic career progressed relatively quickly. He had as career goals two principal aspirations: attaining a professor position at one of the prestigious schools and election to the Paris Academy.[22] Both of these would eventually be achieved, but not without some drama and effort. Liouville was an unsuccessful candidate for election to the Paris Academy in both 1833 and 1836. In 1833, he was simply defeated, losing to the more experienced Count Libri-Carucci. However, in the second election, Liouville made a glowing speech on the Academy floor praising the work of his friend and fellow candidate Sturm. Then, on the day of the election, Liouville and another candidate, Duhamel, pulled their names from consideration, thus handing the election to Sturm. His third candidacy, in 1839, was finally successful.

Liouville also was denied in his initial attempt to become a professor of analysis and mechanics at the École Polytechnique. The death of Navier in 1836 left the position open, and Liouville, Duhamel, and Auguste Comte all applied. Though the selection committee ranked Liouville highest of the three applicants, it was Duhamel who would receive the post. To increase his chances

[22]Lützen, *Joseph Liouville*, p. 41

at other appointments, Liouville padded his academic credentials by earning his doctorate in 1836. His thesis was an unremarkable collection of two previously published papers.[23] The failed first attempt to become a professor at the École Polytechnique was overcome two years later when the other professorship for analysis and mechanics fell vacant and Liouville was appointed. Between the first *répétiteur* position and his appointment as professor in 1838, Liouville accumulated various posts at other institutions. In 1833, Liouville took a position at École Centrale des Arts et Manufactures. His initial concerns that the program at this school would be too technical proved correct. After years of battling both students and the school's administration, Liouville left in 1838. He was able to leave because he had in March 1837 obtained another position–teacher at the Collège de France. He held this post until 1843, when he resigned in protest over the appointment of Count Libri-Carucci, who had become his bitter rival. He returned in 1851 and remained until retiring in 1879. Then, after his appointment as professor at the École Polytechnique and his election to the Paris Academy, Liouville was appointed to the Bureau des Longitudes in 1840. A seat there was almost equal in prestige to Academy membership. Additionally, this achievement had the added advantage of being a paid position, offering compensation roughly equal to the salary earned as a professor. Finally, years later, Liouville accumulated his final academic post, succeeding the late Sturm as the chair of rational mechanics at the Facultè des Sciences. Liouville held this professorship from 1857 until 1874.

Alongside all of these posts, Liouville began the *Journal de mathématiques pures et appliquées*, which became known simply as Liouville's journal. From its start in 1836, the journal featured top mathematical work. This first volume, for example, included articles written by Ampère, Jacobi, Lamé, Coriiolis, Sturm, and Liouville himself. Liouville thus threw himself into the time-consuming task of editing the journal; however, he also continued with his research work and the teaching and various other responsibilities associated with his academic positions. The inspiration for creation of the journal lay in the demise of France's two mathematics journals in the early 1830's.[24] Consequently, it became inordinately difficult to have results published. One remaining outlet of publication, the Academy, soon found itself with such a backlog that one of Liouville's papers submitted in winter 1832-33 was not published until 1838.[25] Liouville recognized this and hoped to fill the void. The establishment and subsequent success of this enterprise firmly established Liouville's reputation throughout the international mathematics community. Certainly, this new prestige helped him in his career, but this was the result, not the aim. Indeed, it is a bit surprising that Liouville, at the time a 26 year-old *répétiteur*, could succeed in such a lofty undertaking. It is a tribute to his mathematical skill, his self-confidence, and his editorial ability that he would be so overwhelmingly successful.

[23]ibid., p. 43.

[24]These two journals were Joseph Gergonne's *Annales de mathématiques pures et appliquées* and *Bulletin des sciences mathématiques, astonomiques, physiques et chimiques* edited by Férussac.

[25]Lützen, *Joseph Liouville*, p. 36.

Liouville had a broad range of mathematical interests which doubtless proved invaluable in his capacity as an editor. However, he was not merely conversant in these varied fields; he made significant contributions to each of them. His work in algebra provides a particularly illustrative example of the working style of Liouville, especially as editor and teacher. His best known contribution in this area, indeed one of Liouville's most celebrated results, is the proof he supplied in 1844 of the existence of transcendental numbers. This demonstration followed his earlier (1840) unsuccessful studies attempting to show that e is transcendental. Liouville proved that all algebraic numbers have certain characteristics when they are expanded as continued fractions. In finding a class of numbers lacking this characteristic, Liouville uncovered the first transcendental numbers. Thus, the specific example of this type of number given in his work, now called Liouville's number, $\sum 10^{-n!}$, could not be a root of an equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 = 0.$$

However, Liouville viewed this as only a partial success; he continued his investigations of e and also tried to show that π was a transcendental number.[26] That it was one of Liouville's former students, Hermite, who eventually showed that e was transcendental reflects Liouville's own interest in this field and his ability as a teacher. Moreover, Hermite's investigations led to Lindemann's proof shortly before Liouville's death that e^x is irrational when x is a non-zero real or complex algebraic number. Thus, $e^{i\pi} = -1$ implies π is transcendental. Liouville, then, had provided this important first proof as well as the teaching and motivation that made further breakthroughs possible. This was also the pattern for his association with Galois theory. About a decade after Evariste Galois'(1811-1832) tragic early death, some of Galois' friends asked Liouville to review his work. Recognizing the importance of this, Liouville sought to edit and then publish it in his journal. From 1843 until 1846, Liouville studied Galois theory in detail, even giving private lectures on the subject. Then, the contents of Galois' manuscripts were published in the October and November 1846 issues of his *Journal de mathématiques*. In a study of Liouville's notebooks, Lützen has found that Liouville was able to fill many of the gaps in Galois' group theory.[27] That Liouville had developed such a comprehensive understanding of Galois theory certainly contributed to the clarity of the private lectures. In this way, by publishing this masterpiece and conducting the first seminars in this theory, Liouville did much to promote the development of group theory and modern algebra.

Liouville's contributions to other areas were more direct. In particular, in the field of analysis he published over one hundred papers during his long career. According to biographer René Taton, "It was in mathematical analysis that Liouville published the greatest number and the most varied of his works."[28]

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[26]ibid., p. 526.

[27]Lützen, *Joseph Liouville*, p. 131.

[28]René Taton, 'Joseph Liouville', in Charles Coulston Gillispie (ed.), *Dictionary of Scientific Biography*, 8, New York: Scribners, 1973, p. 383.

Early in his career, he considered questions involving differentiation of arbitrary order. That is, assuming a function $f(x)$ could be represented by a series of exponentials

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x},$$

then he could define its derivative of order s as

$$D_s f(x) = \sum_{n=0}^{\infty} c_n a_n^s e^{a_n x}.$$

By making such a definition, Liouville could extend the ordinary differential calculus of integral indexes, but it lacked generality as not every function admits such a series expansion.

In the 1840's, Liouville studied functions having two periods. These considerations led him to Liouville's theorem: A bounded, analytic function must be a constant. In actuality, it was Cauchy who in 1843 first published Liouville's theorem; however, Lützen argues that "Liouville justly deserves the honor of having his name attached to the theorem [because he] was the first to discover the fundamental importance of the theorem, and Liouville probably had arrived at the general form of Liouville's theorem before Cauchy.[29] Other noteworthy results in the field of complex analysis included the proofs written together with Sturm involving the improvement of Cauchy's method for finding the number of complex roots of a given equation.

Of greater significance still was Liouville's contribution to Sturm-Liouville theory. The bulk of this contribution centered on the expansion of arbitrary functions into series of eigenfunctions; however, Liouville also frequently commented on the work Sturm had done involving the properties of the eigenvalues and the behavior of the eigenfunctions. Liouville's interest in the subject dates back to the papers submitted to the Paris Academy between 1828 and 1830. Three of these dealt with the theory of heat, and an examination of a surviving paper reveals partial development of some of the elements of Sturm-Liouville theory.[30] His discovery of Sturm's work, then, prompted Liouville to once more consider these questions. Indeed, Liouville built on Sturm's work, with the various discoveries of Sturm lending rigor to Liouville's results. These results are principally contained in the three-part series entitled, 'Mémoire sur le dévelopment des fonctions ou parties de fonctions en séries, dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable.'

In these papers, published in 1836 and 1837, Liouville hoped to rigorously demonstrate that an arbitrary function could be represented as a series of the eigenfunctions of a Sturm-Liouville system. After the first paper, in which he laid out the series itself, Liouville devoted most of the second and third

[29]Lützen, *Joseph Liouville*, p. 543.

[30]Lützen, 'The emergence of Sturm-Liouville theory,' *Archive for History of Exact Sciences*, 1984, pp. 337-338

papers to the question of convergence of the arbitrary function to this series. In particular, Liouville was able to demonstrate that for eigenfunction solutions V_n to a Sturm-Liouville problem (3.1)-(3.3), the series

$$F(x) = \sum_n \frac{\int_a^b g(x)V_n(x)f(x)\mathrm{d}x}{\int_a^b g(x)V_n^2(x)dx} V_n(x) \tag{3.4}$$

converges for a large class of functions $f(x)$. Sturm had cautioned that he did not believe his results could be generalized to differential equations of order higher than two,[31] but Liouville was unconvinced of this. Considerations of $\frac{du}{dt} = \frac{d^3u}{dx^3}$ led Liouville to a separated third order equation. Liouville worked diligently on this problem in 1837 and 1838 without considerable success; however, these failures demonstrated to him the importance of the criterion of self-adjointness possessed by the system (3.1)-(3.3). While he could show that the eigenvalues and eigenfunctions V_n of the equation

$$\frac{d^3V}{dx^3} + rV(x) = 0, \; x \in (0,1)$$

with boundary conditions

$$V(0) = V'(0) = 0,$$
$$V(1) = 0$$

satisfied most of the conditions Sturm had proved for the case of a second order equation, the question of orthogonality and the development of a convergent series remained open. Liouville worked around this problem by developing adjoint boundary conditions. Then, he could show that

$$\int_0^1 V_n(x)U_m(x)dx = 0, \text{ for } m \neq n$$

where $U_m(x)$ is an eigenfunction of the equation

$$\frac{d^3U}{dx^3} - rU(x) = 0, \; x \in (0,1)$$

with boundary conditions

$$U(0) = 0,$$
$$U(1) = U'(1) = 0.^{32}$$

Liouville continued his investigations into this theory, but he did not publish many of these results after his 1838 paper 'Premier mémoire sur la théorie des équations differentielles linéaires, et sur le développement des fonctions en

[31] Sturm, 'Mémoire sur les équations différentielles du second ordre', p. 107

[32] Joseph Liouville, 'Sur l'inegration de l'equation $\frac{du}{dt} = |fracd^3udx^3$', *Journal de l'École Polytechnique*, 15, 1837, pp. 104-106.

séries'. French mathematician Jean Dieudonné claims that Sturm and Liouville's "remarkable results were to form the pattern of spectral theory."[33] Lützen goes further still. He believes that Liouville's attempted generalization to higher order equations and a few minor papers he published on integral equations during the 1840's demonstrated "that he was an even greater pioneer in this field than...is usually acknowledged."[34]

Over Liouville's later career, less needs to be said. After the revolution of 1848, Liouville had a brief foray into politics. A candidate for the Constituting Assembly in the election of 23 April 1848, Liouville was elected as a moderate republican. He stood again for this seat in the National Assembly the following year, but his party had fallen out of favor and he lost. This defeat affected Liouville deeply. He began to frequently interrupt his mathematical work in his notebooks with quotations from philosophers or poets, and his personal relations, even with old friends, suffered.[35] The deaths of two of his closest friends, Sturm and Peter Gustav Lejeune Dirichlet (1859), contributed further to these feelings of isolation. Still, he continued his work in the various academic posts and edited his journal until 1874. However, Liouville's research certainly suffered. After the year 1857, he focused his work exclusively on specific and often not altogether interesting problems in number theory. Gradually this production slowed and by 1880, Liouville had given up the last of his teaching positions. He died in Paris on 8 September 1882. Despite this abandonment of fields other than number theory in the twilight of his career, Liouville's contributions to mathematics are far ranging. In Lützen's biography, for instance, there are chapters detailing his work in differentiation of arbitrary order, integration in finite terms, fluid mechanics, transcendental numbers, doubly periodic functions, Galois theory, potential theory, mechanics, geometry, and Sturm-Liouville theory. Yet alongside his substantial mathematical achievements, Liouville had a very long, successful tenure as editor of the most important journal in France and a distinguished academic career. One immediately sees the contrasts with the professional and intellectual focus of Sturm. Yet, despite their differences, Sturm and Liouville, the hedgehog and the fox, were able to maintain a close friendship and a working relationship atypical of the cut-throat scientific community in which they lived.

3.3 The Friendship and Cooperation Between Sturm and Liouville

Liouville first met Sturm perhaps as early as the late 1820's, but most likely, their friendship really began to develop after Liouville came to Paris in 1830. As young scientists, the two men found themselves in the same informal academic discussion groups, and in this way, Liouville struck up a friendship with both

[33] Jean Dieudonné, *History of Functional Analysis*, North-Holland: Amsterdam, 1981, p. 21

[34] Lützen, 'The birth of spectral theory...', p. 1661

[35] Lützen, *Joseph Liouville*, p. 148.

Sturm and Sturm's best fried, Daniel Colladon. In the winter of 1832-33, Colladon was forced to miss several of his lectures at the École Centrale des Artes et Manufactures and asked Liouville to substitute for him. Liouville did so, and not long afterwards was recommended by Colladon for the chair of rational mechanics at the school, the job Colladon was leaving to return to Geneva. In his September 1833 letter thanking Colladon for his efforts in securing this job for him, Liouville inquired solicitously about Sturm. He wrote:

> Also, send me some news about Sturm. This year he has once more recoiled from the agrégation examination.[36] I fear that he will one day be taken ill because he has lost his time in this way, leaving aside his memoirs because he says he wants to be aggregated and neglecting the agrégation under the pretext of working on his memoirs. Before leaving Paris, you must scold him rather severely. No one has such a great influence on his state of mind as you have, and if you do not succeed in curing him nobody will.[37]

It was also 1833 when both Liouville and Sturm were for the first time candidates for the Paris Academy. However, both lost with Sturm failing even to receive a vote.

Their second attempt at Academy election provided a clear indication of their friendship. This came in 1837, and in a rather unexpected speech on the academy floor shortly before the election, Liouville praised Sturm. He announced, "Sturm's research [on Sturm-Liouville problems] has appeared in my journal. I am proud that I have been the first to do justice to these two beautiful memoirs, which the impartial future will rank on the same level as Lagrange's most beautiful memoirs."[38] Then, even more shockingly, Liouville withdrew his candidacy on the day of the election, thereby handing the election to Sturm. Clearly, Liouville had from the beginning of his candidacy planned not to win himself, but to help Sturm garner election. Without this help, it seems highly unlikely that Sturm could have won the seat, especially given his showing three years earlier. In this same time period, Liouville also supported Sturm against Cauchy's charges of priority in the case of Sturm's theorem. He did so in the two articles written with Sturm in 1836 and 1837 concerning the number of complex solutions to an algebraic equation.

The two men also co-authored another article in 1837 on Sturm-Liouville theory. The article gives a different proof of the convergence of a function $f(x)$ to the series (3.4) than Liouville had given in his work. This relatively minor piece, however, was not their only cooperative effort concerning the development of their theory. Remarks in their work hint at the collaborative nature of this undertaking, and in one case at least, Sturm expressly glossed over an error made by Liouville. In Liouville's first memoir of 1836, he incorrectly formulated a convergence criterion in a theorem. Without comment, Sturm corrected the

[36] The French state exam for a teaching position.

[37] Lützen, *Joseph Liouville*, p. 29.

[38] ibid., p. 46.

error when he restated this theorem in his own work later the same year. Moreover, they frequently praised one another's accomplishments. A glance through the 1836 and 1837 volumes of *Journal de mathématiques pures et appliquées* finds, for instance, Liouville discussing the "beautiful theorem of Mr. Sturm" or the "great care" that Sturm had taken "developing the properties...in his two memoirs." Thus, Sturm and Liouville maintained a close, supportive working relationship. Moreover, it may well have been the case that the contrasting styles of the two men helped to facilitate the theory's development. Sturm handled the initial breakthroughs, establishing properties for eigenvalues and eigenfunctions, while Liouville expanded on Sturm's results to generalize the theory.

After their successful collaboration, Sturm and Liouville remained close friends while pursuing their own, different research interests. As Liouville had done for him, Sturm actively supported Liouville's candidacy for the Paris Academy in 1839. This proved especially important as Liouville at the time was engaged in a dispute with another Academy member, Count Libri-Carucci. In the weeks before the election, Libri made some negative remarks concerning Liouville's character and ability; hearing of this, Sturm responded on 20 May on the floor at the Academy.

> In the memoir which he read during the last meeting of the Academy [Mr. Libri]...attacked some works by [Mr. Liouville]. Mr. Libri has the advantage over Mr. Liouville of being a member of the Academy, and he has chosen to accuse Mr. Liouville of making errors at a moment where the latter has presented himself as a candidate for the Astronomy section. It would be unfortunate if the silence of the Commission, who had been asked to decide the controversy, was interpreted in a way unfavorable to Mr. Liouville.

When Libri began to respond to these comments, Sturm became enraged and walked out of the meeting.[39] It was Sturm's impassioned support, then, that secured Liouville's election to the Academy. Soon afterwards, Sturm was appointed as one of the two professors of analysis and mechanics at the École Polytechnique. The other professor in this field was Liouville, and so for the next decade the two were colleagues at this prestigious institution. This pairing ended in 1851 when Sturm's failing health forced him out of teaching. Four years later, Sturm died, and Liouville, so moved by his close friend's passing, decided to honor him by delivering a stirring, personal eulogy. During a trip to Paris in 1826, the Norwegian mathematician Niels Abel had observed, "Everybody works for himself without concern for others. All want to instruct, and nobody wants to learn. The most absolute egotism reigns everywhere."[40] Thus the cooperative work of Sturm and Liouville at this time was certainly remarkable. Such close partnerships were clearly quite uncommon in the early

[39] Lützen, *Joseph Liouville*, pp. 56-57.

[40] Oystein Ore, *Niels Henrik Abel: Mathematician Extraordinary*, London: Chelsea, 1957, p. 147.

nineteenth century French system, and their theory forms an enduring tribute to their friendship and their respective mathematical abilities. To get a still better sense of this theory, we now turn our attention toward a very representative 1837 article by Liouville considering a particular problem from the theory of heat.

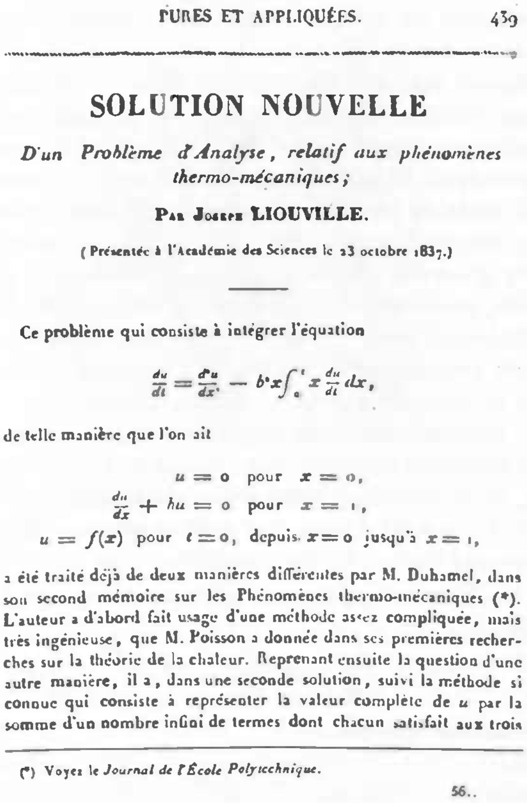
Chapter 4

The Contents and History of Liouville's 1837 'Solution Nouvelle...'

On 23 October 1837, Joseph Liouville presented to the Paris Academy of Sciences a paper detailing how he solved a particular partial differential equation. This paper subsequently appeared in the second volume of the journal he had begun the previous year. This article, entitled 'Solution nouvelle d'un problème d'analyse, relatif aux phènomènes thermo-mécaniques', illustrated how results published by Liouville and fellow mathematician Charles Sturm could be applied to a specific problem. Sturm and Liouville's general results, of which this article is just an example, appeared almost entirely in the pages of Liouville's *Journal de mathématiques pures et appliquées* during the years 1836 and 1837. Ultimately, their work developed an entire theory for solving a particular class of problems. These problems were partial differential equations (often motivated by physical problems such as the vibrating string or as is the case in this article, heat transfer) with certain boundary conditions that, when solved using separation of variables, become systems of ordinary differential equations. The properties and nature of the solutions to these systems were studied by Sturm and Liouville, and their results were both thorough and advanced. Consequently, it was more than forty years before mathematicians again turned their attention to this theory. One can get an idea of the elegance and sophistication of this work by looking at this example of a solution presented by Liouville.

Liouville begins the paper by stating the problem for which he has discovered this method of solution, and he briefly discusses earlier work done in this area. He writes:

> This problem consists of the integral equation
>
> $$\frac{du}{dt} = \frac{d^2u}{dx^2} - b^2x\int_0^1 x\frac{du}{dt}dx, \tag{4.1}$$



PURES ET APPLIQUÉES. 439

SOLUTION NOUVELLE

D'un Problème d'Analyse, relatif aux phénomènes thermo-mécaniques;

PAR JOSEPH LIOUVILLE.

(Présentée à l'Académie des Sciences le 23 octobre 1837.)

Ce problème qui consiste à intégrer l'équation

$$\frac{du}{dt} = \frac{d^2u}{dx^2} - b^2x\int_0^1 x\frac{du}{dt}\,dx,$$

de telle manière que l'on ait

$$u = 0 \text{ pour } x = 0,$$
$$\frac{du}{dx} + hu = 0 \text{ pour } x = 1,$$
$$u = f(x) \text{ pour } t = 0, \text{ depuis } x = 0 \text{ jusqu'à } x = 1,$$

a été traité déjà de deux manières différentes par M. Duhamel, dans son second mémoire sur les Phénomènes thermo-mécaniques (*). L'auteur a d'abord fait usage d'une méthode assez compliquée, mais très ingénieuse, que M. Poisson a donnée dans ses premières recherches sur la théorie de la chaleur. Reprenant ensuite la question d'une autre manière, il a, dans une seconde solution, suivi la méthode si connue qui consiste à représenter la valeur complète de u par la somme d'un nombre infini de termes dont chacun satisfait aux trois

(*) Voyez le *Journal de l'École Polytechnique.*

56..

Figure 4.1: The first page of Joseph Liouville's article 'Solution Nouvelle d'un problème d'analyse, relatif aux phènomènes thermo-mécaniques'.

with the stated conditions

$$u = 0 \text{ for } x = 0, \tag{4.2}$$

$$\frac{du}{dx} + hu = 0 \text{ for } x = 1, \tag{4.3}$$

$$u = f(x), \text{ for } t = 0, \text{ between } x = 0 \text{ and } x = 1 \tag{4.4}$$

This problem had been treated in two different ways by M. Duhamel this past summer in his second memoir concerning thermo-mechanics.[1] This author's approach uses a rather complicated yet most ingenious method that has as

[1] Jean-Marie-Constante Duhamel, 'Second mémoire sur les phénomènes thermo-méchaniques', *Journal de l'École Polytechnique*, 15, 1837, pp. 1-57.

its basis the earlier work in the theory of heat done by M. Poisson.[2] Here, Liouville refers to the work of Jean-Marie-Constant Duhamel (1797-1872), who is most famous for the principle in partial differential equations bearing his name. Duhamel derived this principle while researching the theory of heat, a subject upon which he published a number of papers in the early 1830's. One of the most influential appeared in 1833, and its results can be generalized to the Duhamel principle. In this paper, he considers the temperature distribution in a solid with variable boundary temperature. Duhamel studied the cases where the surface temperature could be reduced to a constant. In such a situation, he could, by generalizing a solution of Fourier's and applying the principle of superposition, substitute for the original temperature function. In place of this function, Duhamel used the sum of a constant and an integral of the rate of change of the temperature function. By so doing, Duhamel was able to reduce the original problem to a second, simpler problem.[3]

In 1837, Duhamel authored his second memoir concerning a new method of solution for problems involving heat transfer. In this paper, Duhamel examines the cooling of an arbitrary solid with consideration of the heat gained due to contraction of the solid during this cooling process. He then applies these ideas to the sphere and derives for the particular "case where the surface of the sphere is exposed to the action of its environment which has a constant temperature" the equation

$$\frac{du}{dt} = A^2 \frac{d^2u}{d\rho^2} - B^2 \rho \int_0^R \rho \frac{du}{dt} d\rho,$$

with the boundary conditions

$$u = 0 \text{ for } x = 0,$$

and

$$\frac{du}{d\rho} + hu = 0 \text{ for } \rho = R.$$ [4]

Obviously, these are the equations used by Liouville in (4.1)-(4.3). Duhamel then, as Liouville notes in this paper, produced solutions that have the same basic form as those found in Liouville's work with Sturm. That is, the solutions are of the form:

$$f(x) = H_1 V_1 + H_2 V_2 + \ldots + H_n V_n + \ldots$$

with the H_i's constants and the V_i's functions of x.

While Liouville credits Duhamel for attempting to find a solution to this problem, he also cautions, "But he tried neither to demonstrate the convergence of the series he developed nor to establish in a conclusive manner that the proposed series converges to the sum $f(x)$ at least between the limit points

[2] Joseph Liouville, 'Solution nouvelle d'un problème d'analyse, relatif aux phènomènes thermo-mécaniques', *Journal de mathématiques pures et appliquées*, 2, 1837, p. 439.

[3] Sigalia Dostrovsky, 'Jean-Marie-Constant Duhamel', in Charles Coulston Gillispie (ed.), *Dictionary of Scientific Biography*, 4, New York: Scribners, 1973, p. 555.

[4] Duhamel, p. 49.

Figure 4.2: J.M.C. Duhamel (1797-1872).

$x = 0$ and $x = 1$."[5] Liouville writes that he therefore considers "the problem in its entirety in order to base a solution on the previously developed rigor. This solution rests on the principles developed in previous memoirs published with M. Sturm." The method of solution presented by Liouville, then, is given to "make known the superiority of our methods."[6] This, though, is likely not the only reason that Liouville has chosen to work with this problem. During the 1830's, Duhamel and Liouville were freqent, though amicable, competitors both for a seat on the Paris Academy and for a professorship at the École Polytechnique, where they both worked as *répétiteurs*. Duhamel's publication in the École's journal certainly could not have escaped Liouville's attention and interest, and his decision to apply his "superior" theory may well have been at least partially motivated by a desire to advance his postition against Duhamel in this fiercely competitive French system.

Liouville begins his work by again stating the basic problem of equation (4.1) with boundary conditions (4.2)-(4.4). If

$$u = Ve^{-rt},$$

then these equations can be replaced by the ordinary differential equation:

$$\frac{d^2V}{dx^2} + r\left(V + b^2x\int_0^1 xV\,dx\right) = 0, \tag{4.5}$$

with the conditions

$$V = 0 \text{ for } x = 0 \tag{4.6}$$

[5] Liouville, p. 440.
[6] ibid.

and

$$\frac{dV}{dx} + hV = 0 \text{ for } x = 1 \tag{4.7}$$

Moreover, if the integral in (4.5) is treated as a constant C, the equation becomes:

$$\frac{d^2V}{dx^2} + r\left(V + b^2Cx\right) = 0,$$

This equation yields the solution

$$V = A\sin(x\sqrt{r}) + B\cos(x\sqrt{r}) - b^2Cx, \tag{4.8}$$

with A and B arbitrary constants and $C = \int_0^1 xV\,dx$. Solving for C, one then has

$$C = \frac{A\alpha}{b^2}$$

with

$$\alpha = \left(\frac{3b^2}{b^2+3}\right)\left(\frac{\sin\sqrt{r} - \sqrt{r}\cos\sqrt{r}}{r}\right).^{7}$$

Liouville then comments, "Since the A is arbitrary, we can take it to be equal to $\frac{1}{\sqrt{r}}$. Therefore $B = 0$, $C = \frac{\alpha}{b^2\sqrt{r}}$, [and] we conclude

$$V = \frac{\sin x\sqrt{r} - \alpha}{\sqrt{r}}.\text{"}^{8} \tag{4.9}$$

Differentiation of V with respect to x yields:

$$\frac{dV}{dx} = \cos(x\sqrt{r}) - \frac{\alpha}{\sqrt{r}} \tag{4.10}$$

and the substitution of the expressions (4.9) and (4.10) into (4.7) gives for $x = 1$:

$$\cos\sqrt{r} + \frac{h\sin\sqrt{r}}{\sqrt{r}} - \frac{\alpha(h+1)}{\sqrt{r}} = 0. \tag{4.11}$$

This equation, claims Liouville, "possesses infinitely many zero points (eigenvalues), all real, positive, and not equal to one another."[9] He then proceeds with a proof of the fact that these eigenvalues are positive. Now, if one considers the Taylor series expansions of cosine and sine, then equation (4.11) can be transformed. Through manipulation of the resulting series expression, Liouville shows that if r is a negative, real eigenvalue, then the series obtained for

$$\cos\sqrt{r} + \frac{h\sin\sqrt{r}}{\sqrt{r}} \text{ and } \alpha(h+1)\sqrt{r}$$

---

[7] Boundary condition (4.6) implies $B = 0$. Then, $C = \int_0^1 x(A\sin(x\sqrt{r}) - b^2Cx)dx$. This means $C + \frac{1}{3}b^2C = A[\frac{1}{r}\sin(\sqrt{r}) - \frac{1}{\sqrt{r}}\cos(\sqrt{r})]$. Thus, we have $C = \frac{3A}{b^2+3}(\frac{\sin(\sqrt{r}) - \sqrt{r}\cos(\sqrt{r})}{r})$.

[8] Liouville, p. 442.

[9] Liouville, p. 443.

Figure 4.3: Siméon-Denis Poisson (1781-1840).

"should be equal to one another, but it cannot since the term-by-term comparison reveals that the terms of the second series are slightly larger than the corresponding terms of the first." Thus the eigenvalues must be positive "at least while the coefficient h is (as is the supposition) between -1 and $+\infty$."[10]

Interestingly, this is one of the only ideas in Sturm-Liouville theory not developed by one of the principals. Instead, as early as 1823, Siméon-Denis Poisson had proven that the eigenvalues must be exclusively real.[11] Moreover, later (1826 and 1835) Poisson supplied proof of orthogonality relationships between eigenfunctions. Poisson's contributions were duly noted by Sturm when he supplied his own proofs of these theorems in his work.[12] This work of Poisson was also important in that it provided (as Liouville claims in this paper) a basis for the work done by mathematicians like Duhamel, and also to some extent, the research done by Sturm and Liouville. However, Liouville's biographer, Jesper Lützen, maintains, "Poisson's researches are of a limited scope compared with the gigantic advances in the field made by Sturm and Liouville within two years of the publication of Poisson's last results."[13]

While Poisson's research somewhat influenced the later work done by Sturm and Liouville, it was Joseph Fourier who was the most influential. In his *Théorie*

[10] ibid., p. 444.

[11] Siméon-Denis Poisson, 'Mémoire sur la distribution de la chaleur dans les corps solides' and 'Second mémoire sur la distribution de la chaleur dans les corps solides', both in *Journal l'École Polytechnique*, **12**, pp. 1-144 and pp. 249-402.

[12] Charles Sturm, 'Mémoire sur les équations différentielles du second ordre', *Journal de mathématiques pures et appliquées*, **1**, 1836, pp. 106-186.

[13] Jesper Lützen, *Joseph Liouville, 1809-1882: Master of Pure and Applied Mathematics*, New York, Springer-Verlag, 1990, p. 434.

analytique de la chaleur (1822), Fourier was the first to provide a complete solution of partial differential equations using, by and large, the method of separation of variables.[14] Further, he began or significantly furthered research in areas such as the determination of eigenvalues and eigenfunctions, the determination of trigonometric series expansion of functions, and orthogonality relationships. All of this is , of course, at the heart of Sturm and Liouville's work. Beyond this, though, there is the matter of Fourier's personal involvement in Sturm's development. It was at Fourier's suggestion upon Sturm's arrival in Paris in 1826 that young Sturm study questions related to his own investigations of the theory of heat.[15] This directly shaped the course of Sturm's research, and since Liouville's involvement in the subject came primarily through Sturm, it can be claimed that Fourier was the main source of inspiration for Sturm-Liouville theory, both in terms of ideas and encouragement.

Returning to the solution, Liouville has at this point already shown that if the eigenvalues are real, they must also be positive. In the ensuing section, he demonstrates that the eigenfunctions must be real valued and each eigenvalue has only one corresponding eigenfunction.[16] He does so by first considering two solutions V and V' for equations (4.1) and (4.6). By combining equations of V and V', he eventually arrives at the equation

$$\int_0^1 VV'dx + b^2 \int_0^1 xVdx \int_0^1 xV'dx = 0 \tag{4.12}$$

which results in a contradiction. Specifically, if there is a complex eigenvalue, then it must be the case that its conjugate is also a zero point. Then, if V is the eigenfunction corresponding to the complex eigenvalue, "V takes the form $M + Ni$ where M and N are not identically zero,...[and] it is easy to see that $V' = M - Ni$; consequently, the formula (4.12) gives us the absurd result

$$\int_0^1 \left(M^2 + N^2\right) dx + b^2 \left(\int_0^1 xMdx\right)^2 + b^2 \left(\int_0^1 xNdx\right)^2 = 0.\text{"}^{17}$$

Liouville also demonstrates that the eigenvalues are simple by pointing out that if there were multiple values of a zero point r, then another equation would be similarly contradicted. He concludes, "The zero points of the equation (4.11) are therefore real, positive, and not equal to each other."[18]

Having proven these basic properties of the problem's eigenvalues, Liouville now attempts to determine the values of these points. For convenience, he sets

[14]Jesper Lützen, 'The solution of partial differential equations by separation of variables: a historical survey', in E. Phillips (ed.), *Studies in the History of Mathematics*, 26, Mathematical Association of America, 1987, p. 242.

[15]Pierre Speziali, 'Charles-François Sturm,' in Charles Coulston Gillispie (ed.), *Dictionary of Scientific Biography*, 13, New York: Scribners, 1975, p. 127.

[16]Liouville uses the expression *"inégales entre elles"* (not equal to one another) to describe eigenvalues now called simple.

[17]Liouville, p. 446.

[18]ibid.

$\rho = r^2$. This transforms equation (4.11) into:

$$\cos\rho + \frac{h\sin\rho}{\rho} - \alpha(h+1)\rho = 0, \tag{4.13}$$

and here, it suffices to consider only positive values for ρ since the eigenvalues must be real. Liouville first shows that for each n ther must be one eigenvalue ρ between each of

$$(2n+1)\frac{\pi}{2} - \frac{\pi}{4} \text{ and } (2n+1)\frac{\pi}{2} + \frac{\pi}{4}, (2n+3)\frac{\pi}{2} - \frac{\pi}{4} \text{ and } (2n+3)\frac{\pi}{2} + \frac{\pi}{4}, \ldots$$

"But there do not exist any between

$$(2n+1)\frac{\pi}{2}+\frac{\pi}{4} \text{ and } (2n+3)\frac{\pi}{2}-\frac{\pi}{4}, \text{ between } (2n+3)\frac{\pi}{2}+\frac{\pi}{4} \text{ and } (2n+5)\frac{\pi}{2}-\frac{\pi}{4}, \text{ etc.}$$"[19]

Then, he proves that if one has $\rho = (2n+1)\frac{\pi}{2} + \frac{B_n}{n}$, then the values of B_n form a sequence which is bounded, and consequently, there are infinitely many eigenvalues.[20]

At this point, Liouville returns to the question of solutions to the original problem. He notes that "Each one of the particular integrals

$$V_1 e^{-r_1 t}, V_2 e^{-r_2 t}, \ldots, V_n e^{-r_n t}, \ldots.$$

...satisfies simultaneously the equations (4.1), (4.2), and (4.3)."[21] Further, for constant H's, the equation

$$u = H_1 V_1 e^{-r_1 t} + H_2 V_2 e^{-r_2 t} + \ldots + H_n V_n e^{-r_n t} + \ldots.$$

also satisfies the three equations. However, if boundary condition (4.4) (that is, $u = f(x)$ for $t = 0$ between $x = 0$ and $x = 1$) is to hold, then the values of the H_j's must be determined. To do so, first recall that since $t = 0$,

$$f(x) = H_1 V_1 + H_2 V_2 + \ldots + H_n V_n + \ldots.$$

By reexpressing the equation, $\int_0^1 VV'dx + b^2 \int_0^1 xVdx \int_0^1 xV'dx = 0$ as

$$\int_0^1 V\left(V' + b^2 x \int_0^1 xV'dx\right)dx = 0,$$

"we have

$$V' + b^2 x \int_0^1 xV'dx = -\frac{d^2V'}{dx^2} = \sqrt{r'}\sin(x\sqrt{r'})$$

this formula is therefore

$$\int_0^1 V\sin(x\sqrt{r})dx = 0.$$"[22]

[19] ibid., pp. 447-448.
[20] ibid., pp. 448-449.
[21] ibid., p. 449.
[22] Liouville, p. 450

It is easy to determine the H's from this last equation, Liouville writes, because "by virtue of this formula, all of the coefficients H_j vanish except those with index n."[23] This allows one to write

$$\int_0^1 f(x)\sin(x\sqrt{r_n})dx = H_n \int_0^1 V_n \sin(x\sqrt{r_n})dx. \tag{4.14}$$

Clearly, this last result is a consequence of what is now called orthogonality. This idea is central to the development of Sturm-Liouville theory, but as mentioned earlier, it is one of the only main ideas used in the creation of this theory not originally proven by Sturm or Liouville. Since Poisson's first proof involving this idea had appeared only a decade before, it is not surprising both that Liouville had no name for this process (the term orthogonal had not yet come into usage) and that he did not regard any of this as a trivial consequence.

From (4.14), one sees immediately that

$$H_n = \frac{\int_0^1 f(x)\sin(x\sqrt{r_n})\mathrm{d}x}{\int_0^1 V_n \sin(x\sqrt{r_n})dx}.$$

This, of course, means that

$$f(x) = \sum_n \left\{ \frac{V_n \int_0^1 f(x)\sin(x\sqrt{r_n})dx}{\int_0^1 V_n \sin(x\sqrt{r_n})dx} \right\} \tag{4.15}$$

and consequently,

$$u = \sum_n \left\{ \frac{V_n e^{-r_n t} \int_0^1 f(x)\sin(x\sqrt{r_n})dx}{\int_0^1 V_n \sin(x\sqrt{r_n})dx} \right\}. \tag{4.16}$$

The last section of Liouville's paper is devoted to studying the convergence of the series (4.15) and (4.16). On the preceding pages of this same volume of the *Journal de mathématiques pures et appliquées*, Liouville had provided two proofs of convergence. Applying the same basic method used in these proofs to this problem, Liouville attempts to show that the series on the right-hand side of (4.15)(he calls this $F(x)$) is actually equal to the function $f(x)$. It is worth noting that Liouville, in contrast to modern style, does not merely direct the reader to the theorem proved earlier in the volume; instead, he demonstrates it once again for this problem. This, perhaps, is a direct consequence of both the different style of the early nineteenth century and the fact that his paper was intended for presentation to the Paris Academy. That this problem was being independently submitted no doubt gave Liouville extra incentive to author a piece in which the reasoning was self-contained. Now, returning to the question of convergence, Liouville writes, "By virtue of the equation

$$\int_0^1 V \sin(x\sqrt{r'})dx = 0$$

[23] ibid.

this integration makes all of the terms except one disappear, and gives

$$\int_0^1 F(x)\sin(x\sqrt{r_n})dx = \int_0^1 f(x)\sin(x\sqrt{r_n})dx$$

or

$$\int_0^1 [F(x) - f(x)]\sin(x\sqrt{r_n})dx = 0."^{24}$$

Making use of an independent variable z, he considers the fraction

$$\frac{\sin(x\sqrt{z})}{\sqrt{z}\omega(z)},$$

where $\omega(z) = \frac{dV}{dx} + hV$. Liouville claims, "Through known methods, this fraction can be decomposed into infinitely many partial fractions, and the resulting series expression is

$$\frac{\sin(x\sqrt{z})}{\sqrt{z}\omega(z)} = \sum\left\{\frac{\sin(x\sqrt{r_n})}{(z - r_n)\sqrt{r_n}\omega'(r_n)}\right\}.^{25}$$

Hence,

$$\sin(x\sqrt{z}) = \sqrt{z}\omega(z)\sum\left\{\frac{\sin(x\sqrt{r_n})}{(z - r_n)\sqrt{r_n}\omega'(r_n)}\right\},$$

and therefore

$$\int_0^1 [F(x) - f(x)]\sin(x\sqrt{z})dx = \sqrt{z}\omega(z)\sum\left\{\frac{\int_0^1 [F(x) - f(x)]\sin(x\sqrt{r_n})dx}{(z - r_n)\sqrt{r_n}\omega'(r_n)}\right\}.$$

Each of the terms of the series on the right-hand side is zero, so the general equation becomes

$$\int_0^1 [F(x) - f(x)]\sin(x\sqrt{z})dx = 0.$$

Then, Liouville writes, "Since z was indeterminate...we have

$$[F(x) - f(x)]\sin(x\sqrt{z}) = 0 \text{ for } x \in (0, 1],$$

and that gives, in general, $F(x) = f(x)$ for $x \in (0, 1]$." However, since $\sin(x\sqrt{z})$ is zero for $x = 0$, it need not be the case that $F(0) = f(0)$. "This last equation, in which the first part is always zero could not be true when $f(0)$ is equal to zero, which we effectively allow."[26] Thus, the interval converges on the interval with only this minor possible exception. Finally, in the last section of his paper, Liouville is able to show that the series in (4.16) converges to u. He completes this proof by considering a series of terms each one greater than the

[24] Liouville, p. 453.

[25] See Appendix A for Liouville's possible development of this series.

[26] Liouville, p. 454.

corresponding term of the function u's series representation. By demonstrating that this series converges, Liouville writes that the series equal to u "is also convergent."[27] Despite these affirmations, Liouville's proofs of convergence are flawed. At the basis of Liouville's claim that these series converge is his claim that a function $\psi(x) = F(x) - f(x)$ is identically zero on an interval $[a, b]$ if it has infinitely many zero points there. Even for continuous functions this is not true. For example,

$$\psi(x) = x \sin \frac{1}{x}$$

has infinitely many zeros on $(0, 1]$, but it is clearly not identically zero there. It would hold for an analytic function, yet it is difficult to show that $F(x)$ is analytic even if $f(x)$ is. According to Lützen, who has made a thorough study of Liouville's notebooks, Liouville later realized this problem, discussing it in his notes in 1838 and 1840.[28] However, he was unable to resolve this difficulty.

This paper was originally presented by Liouville to the Paris Academy in 1837 to "make known the superiority" of the methods of Sturm and Liouville. As such, Liouville certainly wanted to highlight all of the areas of research the two men explored. According to the biographer Lützen, these areas included: the properties of eigenvalues, the qualitative behavior of the eigenfunctions, and the expansion of arbitrary functions in an infinite series of eigenfunctions.[29] Even more than Sturm, it was Liouville who was best qualified to give this overview of their work. As editor of *Journal de mathématiques pures et appliquées*, Liouville originally encouraged Sturm to collect and publish his research. Moreover, while Sturm's work was concentrated in the first two areas, Liouville offered commentary on the research done by Sturm throughout his investigations of the expansion of functions into eigenfunction series. As expected, all of the research topics Lützen cites are presented in 'Solution nouvelle...' giving members of the Academy of Sciences in 1837, or even the modern reader, a clear idea of the advances contained in their work.

[27]Liouville, p. 456

[28]Lützen, 'Sturm and Liouville's work on ordinary linear differential equations: The emergence of Sturm-Liouville theory', *Archive for the History of Exact Sciences*, 29, 1983, p. 348.

[29]Lützen, p. 423.

Chapter 5

Subsequent Development of Sturm-Liouville Theory

Liouville's paper discussed in the previous chapter highlights the advances Sturm and Liouville had made. Yet, it was also the last of the articles published by the two men during the very productive years of 1836 and 1837. Hereafter, save Liouville's continued investigations of applications to higher order problems and his consideration of difficulties with the convergence of his eigenfunction series, the both Sturm and Liouville stopped their research in this area. In fact, this work was not again seriously considered for forty years. Certainly this is a tribute to the completeness of the results achieved by Sturm and Liouville, but it is also a commentary on the mathematics of the early and mid-nineteenth century. In short, the tools needed to take their theory further had simply not yet been devised. However, when this theory was 'rediscovered', it played an important role in the development of spectral theory, integral equations, and operator theory.

By 1880, consideration of problems involving vibrating rods and plates led Lord Rayleigh and G. Kirchhoff to develop theorems similar to those Sturm had proven for higher-order equations. Meanwhile, mathematicians worked to supply proofs for continuity, differentiability, and uniformity that Sturm and Liouville had not provided. For instance, Maxime Bôcher did much during the 1890s along these lines for Sturm's theorems. Also during this period, Ulisse Dini and Heinrich Heine tried to use residue theory to supply a correct proof for Liouville's theorem involving the convergence of the eigenfunction series. In general, then, the most concentrated area of research emerging from this period involved finding generalizations of Sturm-Liouville theory to functions of several variables. Nicholas Bourbaki writes, "In the second half of the nineteenth century, the main effort of the analysists is towards the extension of the Sturm-Liouville theory to functions of several variables,to which led notably the study of partial differential equations of elliptic type from mathematical physics, and the boundary problems that are naturally associated with them."[1]

Inspired by the vibrating membranes equation

$$Lu = \Delta u + \lambda u = 0, \tag{5.1}$$

this theory was slowly advanced during the period 1880-1910. In chapter 2, Green's formula was introduced. In his famous 1828 paper, George Green was able to hypothesize, but not prove, that there should be what Riemann later called a Green's function. By substituting for u in (2.4) a function $G(M,P)$ with certain properties,[2] then the resulting equation is

$$V(M) = \frac{1}{4\pi}\iint V(P)\frac{\partial G}{\partial n}(M,P)d\sigma. \tag{5.2}$$

Moreover, Green could show that for $M \neq P$, $G(M,P) = G(P,M)$. It was more than forty years, however, before the proof of the existence of Green's function was given by Herman Armandus Schwarz. Using this result, Henri Poincaré in the mid-1890's was able to prove a general theorem on expansion in eigenfunctions for the Laplace operator. Further, Poincaré derived from $\Delta u + \lambda u = f(x,y)$ the equation

$$u(x) + \lambda \int K(x,y)u(y)dy = f(x).$$

This work, along with that of Erik Ivar Fredholm and Vito Volterra, helped to begin investigations towards developing a theory for these integral equations.

Finally, Poincaré used complex analysis to help determine eigenvalues, a result that was improved on by W. Stekloff in 1898. Moreover, Stekloff gave the first rigorous proof that a twice differentiable function satisfying the separated, self-adjoint boundary conditions used by Sturm and Liouville could be expanded in a Fourier series. Then, in 1904, using this theory as well as the model of one of Liouville's attempted proofs of convergence, Adolf Kneser demonstrated the expansion into eigenfunctions given by Liouville for any piecewise continuous fucntion of bounded variation.[3]

There was even greater interest, however, in the integral equations Poincaré had helped to popularize. Extending the work of Poincaré, Volterra, and Fredholm was David Hilbert. Of this work done by Hilbert, mathematician Hans Freudenthal writes that "his most important contribution to analysis is integral equations, dealt with in a series of papers from 1904 to 1910."[4] The leading mathematician of this era, Hilbert had broad interests in many fields. In his famous speech of 1900 given to the International Congress of Mathematicians on

[1] Nicholas Bourbaki, *History of Mathematics*, Berlin: Springer-Verlag, 1989, p. 209.

[2] The properties include: $G = 0$ on the surface; at a fixed point in the interior, $G(M,P) - \frac{1}{MP}$ remains bounded when P tends to M; and in the interior $\Delta G = 0$.

[3] Jesper Lützen, *Joseph Liouville, 1809-1882: Master of Pure and Applied Mathematics*, New York: Springer-Verlag, 1990, p. 473.

[4] Hans Freudenthal, 'David Hilbert', in *Dictionary of Scientific Biography*, Charles Coulston Gillispie, ed., vol. 7, New York: Scribners, 1973, p. 349.

Figure 5.1: David Hilbert (1862-1943).

mathematical problems, he issued a challenge to find solutions to twenty-three open questions. He said, "The conviction of the solvability of any mathematical problem is a strong incentive in our work; it beckons us: this is the problem, find its solutions." One of the areas he cited in his speech was "general boundary value problems," and not long after this challenge, Hilbert began his investigations in this area.[5] According to Kline, "One of the most noteworthy achievements in Hilbert's work, which appears in the 1904 and 1905 papers, is the formulation of Sturm-Liouville boundary-value problems of differential equations as integral equations."[6] The style preferred by Hilbert was to demonstrate an idea for a relatively simple example before generalizing. Thus, in Hilbert's work here, he first considers a specific Sturm-Liouville problem. That is, he examines

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u + \lambda u = 0 \tag{5.3}$$

on the interval $[a, b]$ with the boundary conditions $u(a) = u(b) = 0$. For this problem, Hilbert could show that the eigenvalues and eigenfunctions of the Sturm-Liouville problem corresponded to the eigenvalues and eigenfunctions of the integral equation

$$\varphi(x) - \lambda \int_a^b G(x, \xi)\varphi(\xi)d\xi = 0, \tag{5.4}$$

[5] ibid., p. 351-352.

[6] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, New York: Oxford, 1972, p. 1069.

where $G(x,\xi)$ is the Green's function for the differential equation (5.3) of the Sturm-Liouville problem. Moreover, it was Hilbert who invented and popularized the term "spectrum" as he established a general spectral theory for symmetric kernels k. In 1906, he proved a version of what later came to be known as the Hilbert-Schmidt theorem. Erhard Schmidt's contribution involved simplified and generalized these results the following year. Hilbert and Schmidt demonstrated that any continuous function of the form

$$f(x) = \int_a^b G(x,\xi)\varphi(\xi)d\xi$$

can be expanded in a Fourier series of eigenfunctions of (5.4). Lützen comments, "When applied to the Sturm-Liouville problem, this theorem almost immediately gives the expansion theorem for twice differentiable functions, which satisfy the boundary conditions."[7]

To aid in these investigations in spectral theory, Hilbert turned to sequences for which the sum of its squared terms is finite. By regarding functions as given by their Fourier coefficients (c_p),[8] Hilbert noted the equivalence (and thus the importance) of the condition $\sum_p |c_p|^2 < \infty$ and the convergence of the eigenfunction series $\sum_p c_p\varphi_p$. Freudenthal notes, "With a fresh start Hilbert then coordinatized functions and entered the space of number sequence with convergent square sums."[9] Yet despite considerable geometric intuition, it was not Hilbert who first established that this formed a space wherein functions could be seen as points. Instead, this step was taken by Schmidt and Maurice Fréchet.[10] Crucial to the development of these spaces as a mathematical tool was the theorem of Friedrich Riesz and Ernst Fischer. In 1807, the two men, working independently, established the one-to-one correspondence between the functions in L^2 and the square summable sequences of their Fourier coefficients. Since functions could now be viewed as points in a Hilbert space, the integral $\int_a^b k(t,s)f(s)ds$ could be seen as an operator sending the function f to some other function or to itself. In other words, a more abstract approach, operator theory, could be used.

After 1910, an increasing interest in L^2 convergence made the Hilbert space invaluable. Most significantly, by using operator theory Riesz and then later John von Neumann and Hermann Weyl were able to better the approach while also expanding and generalizing the results achieved by Hilbert and others in the first decade of the century. Questions of convergence were reexplored, the idea that the eigenfunctions could form a basis for the space L^2 was considered, and the Hilbert space was fully axiomized. Sturm-Liouville theory remained, and indeed still remains, a potent motivator of this research. For instance, Weyl considered more general Sturm-Liouville problems (5.3) that were subject only

[7] Lützen, p. 474.

[8] $c_p = \int_a^b \varphi_p(s)f(s)ds$ is the Fourier coefficient of f with respect to the system of eigenfunctions $(\varphi_p(s))$.

[9] Freudenthal, p. 350.

[10] Kline, p. 1082.

to the restriction that $p(x) > 0$ for $x \in [a, b]$.[11] Much of the work discussed here is applied in the following chapter as the problem Liouville considered in 1837 is tackled with these more modern techniques.

[11] Jean Dieudonné, *History of Functional Analysis*, Amsterdam: North-Holland Publishing Company, 1981, p. 164-166.

Chapter 6

A Modern Approach to Liouville's 1837 'Solution Nouvelle...'

In the preceding chapters, we first saw how Liouville in his 1837 article 'Solution nouvelle d'un problème d'analyse, relatif aux phénomènes thermo-mécaniques', approached the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - b^2 x \int_0^1 x \frac{\partial u}{\partial t} dx, \tag{6.1}$$

with the stated boundary conditions

$$u = 0 \text{ for } x = 0, \tag{6.2}$$

$$\frac{\partial u}{\partial x} + hu = 0 \text{ for } x = 1, \tag{6.3}$$

and

$$u = f(x) \text{ for } t = 0, \text{ between } x = 0 \text{ and } x = 1. \tag{6.4}$$

Then, in the previous chapter, various developments of Sturm-Liouville theory were surveyed.

We now consider this problem from a modern standpoint, using the methods and theorems discussed in chapter 5. Beginning in the same way as Liouville, if we take

$$u = ve^{-rt},$$

with v a function of x, then equation (6.1) and its boundary conditions (6.2) and (6.3) can be reexpressed as this integro-differential boundary problem:

$$\frac{d^2 v}{dx^2} + r\left(v + b^2 x \int_0^1 xv dx\right) = 0, \tag{6.5}$$

with

$$v(0) = 0 \tag{6.6}$$

and

$$v'(1) + hv(1) = 0. \tag{6.7}$$

Furthermore, we assume, as Liouville does, that $h > -1$.

6.1 The Operator H

Consider the operator $H : M \subset L^2[0,1] \to L^2[0,1], Hy = -y''$, where functions $u \in M$ have the following properties:

1. $u \in L^2[0,1] \bigcap C^1[0,1]$
2. $u' \in AC[0,1]$
3. u satisfies boundary conditions (6.6) and (6.7).

This is an unbounded, self-adjoint linear operator that is injective and positive. H is clearly linear, and that H is self-adjoint follows from the fact that for a Sturm-Liouville problem $Ly = 0$ over the interval $[a,b]$, the separated, self-adjoint boundary conditions are

$$y'(a) - my(a) = 0$$

and

$$y'(b) + My(b) = 0,$$

for positive constants m and M.[1]

Moreover, it can be shown that H is both injective and positive.

H is injective.

Proof: Suppose $Hy = 0$. Then $-y'' = 0 \Rightarrow y = ax + b$, for some $a, b \in \mathbf{C}$. Now, $y(0) = 0$ implies $b = 0$, and the other boundary condition $y'(1) + hy(1) = 0$ gives $a + ah = a(h+1) = 0 \Rightarrow a = 0$. Thus, $y = 0$ and H is injective.

H is a positive operator.

Proof: $\langle , \rangle$ represents the inner product on $L^2[0,1]$ defined as $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$. Then, for $y \in M$, $\langle Hy, y\rangle = \langle -y'', y\rangle = \int_0^1 -y''\overline{y}dx = -y'\overline{y}|_0^1 + \int_0^1 |y'|^2 dx = h|y(1)|^2 + \int_0^1 |y'|^2 dx$. Now, notice that $\langle 1, y'\rangle = \int_0^1 y'dx = y(1) \Rightarrow |y(1)|^2 \leq \left(\int_0^1 |y'|dx\right)^2 \leq \int_0^1 |y'|^2 dx$. Then for $h > -1, \langle Hy, y\rangle = h|y(1)|^2 + \int_0^1 |y'|^2 dx \geq 0$, and by definition, H is a positive operator.

From these last two proofs, one sees the importance of defining $h > -1$. The operator H is injective when $h \neq -1$, and H is positive follows from the fact that $h > -1$.

[1] Edward L. Ince, *Ordinary Differential Equations*, New York: Dover, 1956, pp. 215-218.

6.2 The Operators C, $C^{1/2}$, and $H - rC$

A second operator $C : L^2[0,1] \to L^2[0,1], Cf = f + b^2 x\langle f, x\rangle$ is defined so that the equation

$$(H - rC)f = 0 \tag{6.8}$$

is equivalent with equation (4.5) given by Liouville. This linear operator C is invertible, bounded, positive, and self-adjoint. To show that C is bounded, notice:

$$\begin{aligned}
||Cf|| &= ||f + b^2 x\langle f, x\rangle|| \\
&\leq ||f|| + b^2||x||^2||f|| \\
&= ||f|| + \frac{b^2}{3}||f|| \\
&= \frac{3+b^2}{3}||f||
\end{aligned}$$

Thus, C is bounded. Further, C is positive.

Proof:

$$\begin{aligned}
\langle Cf, f\rangle &= \langle f + b^2 x\langle f, x\rangle, f\rangle \\
&= \langle f, f\rangle + b^2\langle x\langle f, x\rangle, f\rangle \\
&= ||f||^2 + b^2\,|\langle f, x\rangle|^2
\end{aligned}$$

Hence, C is positive, and we also have the stronger conclusion, $\langle Cf, f\rangle > 0$ for all $f \neq 0$.

Finally, C is a self-adjoint operator.

Proof: For arbitrary $f, g \in L^2[0,1]$,

$$\begin{aligned}
\langle Cf, g\rangle &= \langle f + b^2 x\langle f, x\rangle, g\rangle \\
&= \langle f, g\rangle + \langle b^2 x\langle f, x\rangle, g\rangle \\
&= \langle f, g\rangle + b^2\langle x\langle f, x\rangle, g\rangle \\
&= \langle f, g\rangle + b^2\langle f, x\rangle\langle x, g\rangle \\
&= \langle f, g\rangle + \langle f, b^2 x\langle g, x\rangle\rangle \\
&= \langle f, g + \gamma x\langle g, x\rangle\rangle \\
&= \langle f, Cg\rangle.
\end{aligned}$$

Therefore, C is a self-adjoint operator.

Moreover, for the decomposition $L^2[0,1] = \text{span}\{x\} \oplus \text{span}\{x\}^\perp$, the matrix representation of C can be computed. Notice,

$$Cx = x + b^2 x\langle x, x\rangle = x + b^2 x\int_0^1 |x|^2 dx = x + \frac{1}{3}b^2 x = \frac{b^2+3}{3}x$$

and

$$Cx^{\perp} = x^{\perp} + b^2 x\langle x^{\perp}, x\rangle = x^{\perp}.$$

Thus,

$$\text{mat } C = \begin{pmatrix} \frac{b^2+3}{3} & 0 \\ 0 & \text{I} \end{pmatrix}.$$

Given this matrix representation of C, it is obvious that C^{-1} exists, and it is easily determined that:

$$\text{mat } C^{-1} = \begin{pmatrix} \frac{3}{b^2+3} & 0 \\ 0 & \text{I} \end{pmatrix}.$$

Morover $C > 0$ for non-zero f, which implies that $C^{1/2}$ and $C^{-1/2}$ exist. So,

$$\text{mat } C^{1/2} = \begin{pmatrix} \sqrt{\frac{b^2+3}{3}} & 0 \\ 0 & \text{I} \end{pmatrix}.$$

Then, from this matrix representation, it can quickly be computed that the matrix representation of $C^{-1/2}$ must be

$$\text{mat } C^{-1/2} = \begin{pmatrix} \sqrt{\frac{3}{b^2+3}} & 0 \\ 0 & \text{I} \end{pmatrix}.$$

The given decomposition of $L^2[0,1]$ means that for every $f \in L^2[0,1]$, $f = \alpha x + g$, where $\alpha = 3\langle f, x\rangle$ and $g \perp x$. In other words, $g = f - 3x\langle f, x\rangle$. Then,

$$f = \begin{pmatrix} 3x\langle f, x\rangle \\ f - 3x\langle f, x\rangle \end{pmatrix}$$

and

$$C^{-1} f = \text{mat } C^{-1} \begin{pmatrix} 3x\langle f, x\rangle \\ f - 3x\langle f, x\rangle \end{pmatrix} = \begin{pmatrix} \frac{9}{b^2+3} x\langle f, x\rangle \\ f - 3x\langle f, x\rangle \end{pmatrix}.$$

So, the inverse of C is C^{-1}, and $C^{-1} f = f - \frac{3b^2}{b^2+3} x\langle f, x\rangle$. Also,

$$C^{1/2} f = \text{mat } C^{1/2} \begin{pmatrix} 3x\langle f, x\rangle \\ f - 3x\langle f, x\rangle \end{pmatrix} = \begin{pmatrix} 3x\sqrt{\frac{b^2+3}{3}}\langle f, x\rangle \\ f - 3x\langle f, x\rangle \end{pmatrix}.$$

Therefore,

$$C^{1/2} f = f + \left(3\sqrt{\frac{b^2+3}{3}} - 3\right) x\langle f, x\rangle. \tag{6.9}$$

Further, notice that the inverse of this last operator is

$$C^{-1/2} f = f + \left(3\sqrt{\frac{3}{b^2+3}} - 3\right) x\langle f, x\rangle,$$

because obviously for all $f \in L^2[0,1], C^{1/2}C^{-1/2}f = C^{-1/2}C^{1/2}f = f$. Then, by definition, the inverse of $C^{1/2}$ is $C^{-1/2}$.

The significance of these last two operators becomes clearer with the observation that they play an important role in transforming (6.8), the equation equivalent with Liouville's. In particular, if $g = C^{1/2}f$, then $(H - rC)f = 0$ becomes

$$\left(HC^{-1/2} - rC^{1/2}\right) g = 0$$

or

$$\left(C^{-1/2}HC^{-1/2} - r\right) g = 0.$$

And, for $r = 1/\lambda$,

$$C^{1/2}H^{-1}C^{1/2}g = \lambda g.$$

6.3 Determination of H^{-1}

After defining H and showing that it is invective, and thus has an inverse, it is natural to try and determine this inverse H^{-1}. Let y_1 be a function which is a solution to the first boundary condition and y_2 satisfy the second boundary condition. Further, let $c = (pW)^{-1}$, where W is the Wronskian of (y_1, y_2), and p is the coefficient function $p(x)$ in the Sturm-Liouville operator $Ly = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y$.

Now, according to theorem IV-6-1 in the text by Gohberg and Goldberg, the integral operator G with the Green's function

$$g(x,s) = \begin{cases} cy_2(s)y_1(x), & 0 \le x \le s \\ cy_1(s)y_2(x), & s \le x \le 1 \end{cases}$$

is compact and self-adjoint, and it satisfies: $HGf = f$ on $L^2[0,1]$ and $GHy = y$, $y \in M$.[2] Here, the y_1 and y_2 in the Green's function refer to functions which are solutions of the equation $Hy = 0$, with y_1 also satisfying the first boundary condition (6.6) and y_2 satisfying the second boundary condition (6.7). Specifically, one therefore must find y_1 such that $Hy_1 = 0$ and $y_1(0) = 0$ and y_2 such that $Hy_2 = 0$ and $y_2'(1) + hy_2(1) = 0$.
For $i = 1,2 : Hy_i = -y_i'' = 0 \Rightarrow y_i = a_i x + b_i$. Therefore,

$$y_1(0) = 0 \Rightarrow b_1 = 0 \Rightarrow y_1 = a_1 x$$

and

$$y_2'(1) + hy_2(1) = 0 \Rightarrow a_2 + ha_2 + hb_2 = 0 \Rightarrow -\frac{h+1}{h}a_2 = b_2 \Rightarrow y_2 = a_2 x - \frac{h+1}{h}a_2.$$

[2] Gohberg and Goldberg, p. 145-146.

So,

$$\begin{aligned} y_1 &= x \\ y_2 &= h(x-1)-1 \end{aligned}$$

may be chosen as solutions. Moreover, in this case, $p(x) = -1$ and the Wronskian is

$$W = \det \begin{pmatrix} x & h(x-1)-1 \\ 1 & h \end{pmatrix} = h+1.$$

Hence, $c = \frac{-1}{h+1}$. Thus, H^{-1} is the integral operator with the kernel

$$h(t,s) = \begin{cases} \frac{-1}{h+1}(h(t-1)-1)s, & 0 \le t \le s \\ \frac{-1}{h+1}(h(s-1)-1)t, & s \le t \le 1 \end{cases},$$

and

$$H^{-1}f = \int_0^1 h(t,s)f(s)ds. \tag{6.10}$$

6.4 The Kernel Function of the Operator $\tilde{H} = C^{1/2}H^{-1}C^{1/2}$

From this last equation (6.10) and (6.9),

$$H^{-1}C^{1/2}f = \int_0^1 h(t,s)\left[f(s) + \gamma s\langle f,x\rangle\right]ds,$$

where $\gamma = \left(3\sqrt{\frac{b^2+3}{3}} - 3\right)$. Then,

$$\begin{aligned} H^{-1}C^{1/2}f &= \int_0^1 h(t,s)f(s)ds + \int_0^1 h(t,s)\gamma s\langle f,x\rangle ds \\ &= \int_0^1 h(t,s)f(s)ds + \gamma\langle f,x\rangle \int_0^1 h(t,s)sds. \end{aligned}$$

Again using equation (6.9), $C^{1/2}H^{-1}C^{1/2}f$ can now be computed. That is, $C^{1/2}H^{-1}C^{1/2}f = C^{1/2}\left(H^{-1}C^{1/2}f\right)$. This means

$$\begin{aligned} C^{1/2}H^{-1}C^{1/2}f &= C^{1/2}\left(\int_0^1 h(t,s)f(s)ds + \gamma\langle f,x\rangle \int_0^1 h(t,s)sds\right) \\ &= \int_0^1 h(t,s)f(s)ds + \gamma\langle f,x\rangle \int_0^1 h(t,s)sds + \\ &\quad \gamma t\left[\int_0^1\int_0^1 h(t,s)f(s)tdsdt + \gamma\langle f,x\rangle \int_0^1\int_0^1 h(t,s)stdsdt\right] \end{aligned}$$

$$= \int_0^1 \Big[h(t,s) + \gamma s \int_0^1 u h(t,u)du + \gamma t \int_0^1 v h(v,s)dv + \gamma^2 st \int_0^1\int_0^1 h(v,u)uvdudv\Big] f(s)ds.$$

Since $h(t,s)$ is the kernel of the self-adjoint integral operator H^{-1}, it follows that $h(t,s) = \overline{h(s,t)}$. Moreover, if $k(t,s) = \gamma^2 st \int_0^1\int_0^1 h(v,u)uvdudv$, then

$$\begin{aligned}
\overline{k(s,t)} &= \overline{\gamma^2 st \int_0^1\int_0^1 h(v,u)uvdudv} = \gamma^2 st \int_0^1\int_0^1 \overline{h(v,u)}uvdudv \\
&= \gamma^2 st \int_0^1\int_0^1 \overline{h(u,v)}vudvdu = \gamma^2 st \int_0^1\int_0^1 h(v,u)uvdudv = k(t,s).
\end{aligned}$$

Finally, let $g(t,s) = \gamma s \int_0^1 u h(t,u)du$. Then,

$$\begin{aligned}
\overline{g(s,t)} &= \overline{\gamma t \int_0^1 u h(s,u)du} = \gamma t \int_0^1 u\overline{h(s,u)}du \\
&= \gamma t \int_0^1 u h(u,s)du = \gamma t \int_0^1 v h(v,s)dv \\
&= g(s,t).
\end{aligned}$$

Thus, from

$$\begin{aligned}
\int_0^1 \tilde{h}(t,s)f(s)ds &= \int_0^1 \Big[h(t,s) + \gamma s \int_0^1 u h(t,u)du + \\
&\quad \gamma t \int_0^1 v h(v,s)dv + \gamma^2 st \int_0^1\int_0^1 h(t,s)stdsdt\Big] f(s)ds \qquad (6.11) \\
&= \int_0^1 [h(t,s) + g(t,s) + g(s,t) + k(t,s)]\, f(s)ds, \qquad (6.12)
\end{aligned}$$

it follows that

$$\begin{aligned}
\overline{\tilde{h}(t,s)} &= \overline{h(t,s) + g(t,s) + g(s,t) + k(t,s)} \\
&= \overline{h(t,s)} + \overline{g(t,s)} + \overline{g(s,t)} + \overline{k(t,s)} \\
&= h(s,t) + g(t,s) + g(s,t) + k(s,t).
\end{aligned}$$

Hence, it can be concluded that $\tilde{h}(t,s) = \overline{\tilde{h}(s,t)}$. This means that $\tilde{H} = C^{1/2}H^{-1}C^{1/2}$ is a self-adjoint integral operator with the continuous kernel function $\tilde{h}(t,s)$. Further, $\tilde{H}$ is a compact operator. Compactness follows from the fact that H^{-1} is compact and $C^{1/2}$ is a bounded operator on $L^2[0,1]$. Twice applying the following theorem from Gohberg and Goldberg's text yields the desired result.

> *Suppose K and L are compact operators in $L(H_1, H_2)$. Then, if $A \in L(H_3, H_1)$ and $B \in L(H_2, H_3)$, then KA and BK are compact.*[3]

[3] Gohberg and Goldberg, p. 84.

Then, $H^{-1}C^{1/2}$ is a compact operator, and thus $\tilde{H} = C^{1/2}H^{-1}C^{1/2}$ must also be compact.

Finally, a connection can be established between the kernel functions of $\tilde{H}$ and H^{-1}. For arbitrary $f, g \in L^2[0,1]$, $\langle \tilde{H}f, g\rangle = \langle C^{1/2}H^{-1}C^{1/2}f, g\rangle = \langle C^{1/2}$ is self-adjoint.) $\langle H^{-1}C^{1/2}f, C^{1/2}g\rangle = \int_0^1\int_0^1 h(t,s)C^{1/2}f(s)ds\overline{C^{1/2}g(t)}dt =$ $\int_0^1\int_0^1 h(t,s)C_s^{1/2}f(s)dsC_t^{1/2}\overline{g(t)}dt$. This implies that

$$\tilde{h}(t,s) = C_s^{1/2}C_t^{1/2}h(t,s). \tag{6.13}$$

6.5 Eigenfunctions and Eigenvalues

6.5.1 The Existence of the Eigenfunctions and Eigenvalues

The Spectral Theorem states:

> *Suppose A is a compact self-adjoint operator on a Hilbert space H. There exist an orthonormal system $\varphi_1, \varphi_2, \ldots$ of eigenvectors of A and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$ such that for all $x \in H$,*
>
> $$Ax = \sum_k \lambda_k \langle x, \varphi_k\rangle \varphi_k.$$
>
> *If $\{\lambda_k\}$ is an infinite sequence, then it converges to zero.*[4]

Since it has already been shown that $\tilde{H} = C^{1/2}H^{-1}C^{1/2}$ is a compact, self-adjoint operator, there must then exist a system of eigenfunctions $\varphi_1, \varphi_2, \ldots$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$ for $\tilde{H}$ as described in the Spectral Theorem. The first assertion of the Spectral Theorem is a consequence of results proved by Sturm (and Poisson) in the early nineteenth century. These two men demonstrated the orthogonality of such a system of eigenfunctions. Further, Sturm also showed that the eigenvalues of a self-adjoint operator are real. So, the λ_n's are all real and $\langle \varphi_n, \varphi_m\rangle = 0$, for all $n \neq m$.[5]

6.5.2 Connections between the Basic Systems for H and $\tilde{H}$

Consider now the connection between this basic system for $\tilde{H}$ and the solutions to the original problem $(H - rC)f = 0$. In particular, the functions

$$\psi_n = C^{-1/2}\varphi_n$$

are solutions to the original problem, and they therefore correspond with the solutions V_n in Liouville's paper. One sees this by studying the basic system

[4]Gohberg and Goldberg, p. 113.

[5]See Gohberg and Goldberg, p.108, or Charles Sturm, 'Mémoire sur les équations différentielles linéaires du second ordre,' *Journal de mathematiques pures et appliques*, 1, 1836, pp. 106-186.

(φ_n, λ_n) for $\tilde{H}$. As a basic system, it holds that for all $n \in \mathrm{N}$,

$$\tilde{H}\varphi_n = \lambda_n \varphi_n,$$

or $C^{1/2}H^{-1}C^{1/2}\varphi_n = \lambda_n\varphi_n$. This means $C^{1/2}H^{-1}C^{1/2}C^{1/2}\psi_n = \lambda_n C^{1/2}\psi_n$. Or, $H^{-1}C\psi_n = \lambda_n\psi_n \Leftrightarrow r_n C\psi_n = H\psi_n$, where $r_n = 1/\lambda_n$.

Thus, $(H - r_n C)\,\psi_n = 0$, and as conjectured $\psi_1, \psi_2, \ldots$ forms a system of solutions to the original problem corresponding to $r_1, r_2, \ldots$, the parameter values. These ψ_n's also satisfy the boundary conditions (6.6) and (6.7). One sees this by noticing that for all n, φ_n satisfies these conditions, and since $C^{-1/2}$ is a linear operator, $\psi_n = C^{-1/2}\varphi_n$ must do so as well. In particular, $\varphi_n(0) = 0 \Rightarrow C^{-1/2}\varphi_n(0) = C^{-1/2}(0) = 0$ and $\varphi_n'(1) + h\varphi_n(1) = 0 \Rightarrow C^{-1/2}\varphi_n'(1) + hC^{-1/2}\varphi_n(1) = C^{-1/2}\left(\varphi_n'(1) + h\varphi_n(1)\right) = C^{-1/2}(0) = 0$.

6.6 Series Representations of the Kernel Functions of $\tilde{H}$ and H^{-1}

It has been established that there are basic systems (φ_n, λ_n) for H and (ψ_n, λ_n) for $\tilde{H}$. For these two integral operators, it can be shown that there exist series of eigenvalues and eigenfunctions which converge absolutely and uniformly to their respective kernel functions. To do so, a theorem originally proved in 1909 by the British mathematician James Mercer is applied.[6] Mercer's theorem states:

Let k be continuous on $[a, b] \times [a, b]$. Suppose that for all $f \in L^2[a, b]$,

$$\int_a^b \int_a^b k(t, s) f(s) \overline{f(t)} ds dt \geq 0.$$

If (φ_n, λ_n) is a basic system of eigenvectors and eigenvalues of the integral operator with kernel function k, then for all t and s in $[a, b]$,

$$k(t, s) = \sum_j \lambda_j \varphi_j(t) \overline{\varphi_j(s)}.$$

This series converges absolutely uniformly on $[a, b] \times [a, b]$.[7]

The kernel function for the operator $\tilde{H}$ given in (6.11) is clearly continuous on $[0, 1] \times [0, 1]$, thus before Mercer's theorem can be applied, it remains to be shown that $\tilde{H}$ is a positive operator.

Proof: It has already been proven that H is positive. Then, the eigenvalues for H are all positive as well. This means that the eigenvalues for H^{-1} must also be positive, and H^{-1} itself is a positive

[6] James Mercer, 'Functions of positive and negative type and their connection with the theory of integral equations, *Trans. London Phil. Soc.*, **209**, 1909, pp. 415-446.

[7] This version of Mercer's theorem is taken from the text by Gohberg and Goldberg, p. 136.

operator. By definition, $\langle H^{-1}f, f\rangle \geq 0$, for all $f \in L^2[0,1]$. Hence, $\langle \tilde{H}f, f\rangle = \langle C^{1/2}H^{-1}C^{1/2}f, f\rangle =$ ($C^{1/2}$ is self-adjoint.) $\langle H^{-1}C^{1/2}f, C^{1/2}f\rangle = \langle H^{-1}\left(C^{1/2}f\right), C^{1/2}f\rangle \geq 0$ because $C^{1/2}f \in L^2[0,1]$.

Applying Mercer's theorem for the operator $\tilde{H}$, one sees that for all $t, s \in L^2[0,1]$,

$$\tilde{h}(t,s) = \sum_j \lambda_j \varphi_j(t)\overline{\varphi_j(s)} = \sum_j \lambda_j C_t^{1/2}\psi_j(t) C_s^{1/2}\overline{\psi_j(s)} \tag{6.14}$$

is absolutely uniformly convergent on the square $[0,1] \times [0,1]$. Moreover, given the connection established in (6.13), $\tilde{h}(t,s) = C_s^{1/2}C_t^{1/2}h(t,s)$, it is clear that:

$$h(t,s) = \sum_j \lambda_j \psi_j(t)\overline{\psi_j(s)} \tag{6.15}$$

However, it remains to be demonstrated that this series (6.15) converges absolutely uniformly as well.

To demonstrate that the series in (6.15) converges absolutely uniformly, begin by noting the following inequality.

$$\begin{aligned} |\psi_j(t)| &= \left|C_t^{-1/2}C_t^{1/2}\psi_j(t)\right| \\ &= \left|C_t^{1/2}\psi_j(t) + \tilde{\gamma}x\langle C_t^{1/2}\psi_j(t), t\rangle\right| \\ &\leq \left|C_t^{1/2}\psi_j(t)\right| + |\tilde{\gamma}|\left|\langle C_t^{1/2}\psi_j(t), t\rangle\right|. \end{aligned} \tag{6.16}$$

Further, since the series in (6.14) is absolutely uniformly convergent, the equivalent Cauchy convergence criterion is satisfied. This means that for all $\epsilon > 0$, there exists a N dependent only on the ϵ such that for all $n, m > N$ the following holds:

$$\sum_{j=n}^{m} \lambda_j |C_t^{1/2}\psi_j(t)\overline{C_s^{1/2}\psi_j(s)}| \leq \epsilon,$$

for all $t, s \in [0,1]$. Consider now the series $\sum_{j=n}^{m} \lambda_j|\psi_j(t)\overline{\psi_j(s)}|$. With the help of (6.16), the following inequality is obtained:

$$\begin{aligned} \sum_{j=n}^{m} \lambda_j|\psi_j(t)\overline{\psi_j(s)}| &\leq \sum_{j=n}^{m} \lambda_j \left[|C_t^{1/2}\psi_j(t)| + |\tilde{\gamma}||\langle C_t^{1/2}\psi_j(t), t\rangle|\right] \\ &\quad \left[|C_s^{1/2}\psi_j(s)| + |\tilde{\gamma}||\langle C_s^{1/2}\psi_j(s), t\rangle|\right] \\ &\leq \sum_{j=n}^{m} \lambda_j \left[|C_t^{1/2}\psi_j(t)C_s^{1/2}\psi_j(s)| + \right. \\ &\quad |\tilde{\gamma}| \int_0^1 |C_s^{1/2}\psi_j(s)C_t^{1/2}\psi_j(t)|t\,dt + \\ &\quad |\tilde{\gamma}| \int_0^1 |C_t^{1/2}\psi_j(t)C_s^{1/2}\psi_j(s)|s\,ds + \end{aligned}$$

$$|\tilde{\gamma}|^2 \int_0^1 \int_0^1 |C_s^{1/2}\psi_j(s)C_t^{1/2}\psi_j(t)|stdtds\Big]$$

$$\begin{aligned}
&\leq \epsilon + |\tilde{\gamma}| \int_0^1 \epsilon tdt + \\
&\quad |\tilde{\gamma}| \int_0^1 \epsilon sds + |\tilde{\gamma}|^2 \int_0^1 \int_0^1 \epsilon tsdsdt \\
&= \epsilon + |\tilde{\gamma}|\epsilon + \frac{1}{4}|\tilde{\gamma}|^2\epsilon \\
&= k\epsilon, \text{ where } k \text{ is a constant.}
\end{aligned}$$

Thus, for any $\epsilon_0 (= k\epsilon) > 0$ there exists a $N(\epsilon_0)$ such that for all $n, m > N$,

$$\sum_{j=n}^{m} |\lambda_j \psi_j(t)\overline{\psi_j(s)}| < \epsilon_0.$$

This, of course, means that the sum in (6.15) is absolutely uniformly convergent on the interval $[0, 1]$ to the kernel function $h(t, s)$.

6.7 Convergence of an Eigenfunction Series to an Arbitrary Function

It would be helpful to establish some criterion involving the eigenfunctions and eigenvalues of H for an arbitrary function f to be in M (the domain of H). Beginning indirectly, we consider, for the moment, the compact, self-adjoint operator $\tilde{H}$.

It has already been proven that $\tilde{H}$ is an injective operator, and hence, $\ker \tilde{H} = 0$. Now, a solution $y = \tilde{H}x$ exists if and only if conditions (i) and (ii) in the following theorem are satisfied.

> *Let $K \in L(H)$ be a compact, self-adjoint operator with a basic system of eigenvectors and eigenvalues (φ_n, λ_n). Given $y \in H$, the equation $Kx = y$ has a solution if and only if*
>
> *(i)* $y \perp \ker K$
>
> *and*
>
> $$\text{(ii)} \sum_n \frac{1}{\lambda_n^2} |\langle y, \varphi_n \rangle|^2 < \infty.$$
>
> *Every solution is of the form*
>
> $$x = u + \sum_n \frac{1}{\lambda_n} \langle y, \varphi_n \rangle \varphi_n, u \in \ker K.^8$$

[8] Gohberg and Goldberg, p. 120.

However, in this case ($K = \tilde{H}$) we have $y \perp 0 = \ker \tilde{H}$, and therefore a solution $y = \tilde{H}x, y \in L^2[0,1]$ exists if and only if

$$\sum_n \frac{1}{\lambda_n^2} |\langle y, \varphi_n \rangle|^2 < \infty.$$

This is equivalent with

$$\sum_n r_n^2 |\langle y, C^{1/2}\psi_n \rangle|^2 < \infty.$$

And, since $C^{-1/2}$ is self-adjoint, the following condition is also equivalent:

$$\sum_n r_n^2 |\langle C^{-1/2}y, C\psi_n \rangle|^2 < \infty. \tag{6.17}$$

Now, $y \in$ range $\tilde{H}$ or $y = C^{1/2}H^{-1}C^{1/2}x$ if and only if (6.17) holds. Then, $C^{-1/2}y = H^{-1}C^{1/2}x$, and for $f = C^{-1/2}y$, one has

$$\sum_n r_n^2 |\langle f, C\psi_n \rangle|^2 < \infty,$$

which is satisfied if and only if $f = H^{-1}C^{1/2}x$, or more simply, $f \in$ domain H. In other words,

$$f \in M \Leftrightarrow \sum_n r_n^2 |\langle f, C\psi_n \rangle|^2 < \infty. \tag{6.18}$$

Also given in the theorem quoted above is that every solution of the equation $y = \tilde{H}x$ is of the form

$$x = u + \sum_n \frac{1}{\lambda_n} \langle y, \varphi_n \rangle \varphi_n.$$

Expressed in terms of r_n, f, and ψ_n, this is

$$x = \sum_n r_n \langle f, C\psi_n \rangle C^{1/2}\psi_n,$$

since in this case $u = 0$. Now, $y = \tilde{H}x = C^{1/2}H^{-1}C^{1/2}x \Rightarrow f = C^{-1/2}y = H^{-1}C^{1/2}x$, and

$$C^{1/2}x = \sum_n r_n \langle f, C\psi_n \rangle C\psi_n.$$

Finally, since $f = H^{-1}\left(C^{1/2}x\right)$, it must be the case that

$$Hf = \sum_n r_n \langle f, C\psi_n \rangle C\psi_n. \tag{6.19}$$

Thus, we have (6.18), the desired criterion for $f \in M$, and the expansion of Hf in eigenfunctions ψ_n.

6.7.1 Convergence to the Function f

Since $f \in M$, one can consider a function $g = Hf$. Obviously, from this equation $f = H^{-1}g$ or $f(t) = \int_0^1 h(t,s)g(s)ds$. Moreover, there is an absolutely uniformly convergent series representation for $h(t,s)$. So,

$$
\begin{aligned}
f(t) &= \int_0^1 \sum_j \lambda_j \psi_j(t)\overline{\psi_j(s)}g(s)ds \\
&= \sum_j \lambda_j \psi_j(t) \int_0^1 \overline{\psi_j(s)}g(s)ds \\
&= \sum_j \lambda_j \psi_j(t)\langle g, \psi_j\rangle \\
&= \sum_j \lambda_j \psi_j(t)\langle Hf, \psi_j\rangle \\
(H \text{ is self-adjoint.}) \quad &= \sum_j \lambda_j \psi_j(t)\langle f, H\psi_j\rangle \\
(H\psi_n - r_n C\psi_n = 0.) \quad &= \sum_j \lambda_j \psi_j(t)\langle f, r_j C\psi_j\rangle \\
(r_n \in \mathbf{R}) \quad &= \sum_j \psi_j(t)\langle f, C\psi_j\rangle \\
&= \sum_j \psi_j(t) \int_0^1 f(s)\overline{C_s\psi_j(s)}ds
\end{aligned} \tag{6.20}
$$

Therefore, this series in (6.20) converges absolutely uniformly and it converges pointwise to $f(t)$.

6.7.2 Bases for $L^2[0,1]$

Consider the basic system (φ_n, λ_n) of the operator $\tilde{H}$. The Spectral Theorem, presented earlier, states that for every $x \in L^2[0,1]$,

$$\tilde{H}x = \sum_k \lambda_k \langle x, \varphi_k\rangle \varphi_k. \tag{6.21}$$

Further, it has also already been shown that this operator is compact, self-adjoint, and injective.

Since $\tilde{H}\varphi_n = \lambda_n \varphi_n$, obviously

$$\varphi_n = \frac{1}{\lambda}\tilde{H}\varphi_n.$$

Now, clearly φ_n is in the range of $\tilde{H}$, and since for all $x \in L^2[0,1]$ the equation (6.21) from the Spectral Theorem holds, this implies that span φ_n is dense in range $\tilde{H}$. From theorem II-12-1 in Gohberg and Goldberg,[9] it follows that

$$\overline{\mathrm{span}}\,\varphi_n = \overline{\mathrm{range}\,\tilde{H}} = \ker \tilde{H}^{\perp}.$$

Since $\tilde{H}$ is injective, $\ker \tilde{H} = 0$ and thus $\overline{\mathrm{span}}\,\varphi_n = \ker \tilde{H}^{\perp} = L^2[0,1]$. Then, (φ_n) forms an orthonormal basis for $L^2[0,1]$.

A Riesz Basis is defined to be an orthonormal basis transformed by a bounded invertible operator. It is called a basis equivalent to an orthonormal basis.[10] $C^{\pm 1/2}$ are both bounded invertible operators, so from the orthonormal basis (φ_n), one obtains the Riesz bases $(\psi_n) = (C^{-1/2}\varphi_n)$ and $(\chi_n) = (C^{1/2}\varphi_n)$. The vectors in these two Riesz bases have a special biorthogonal property because $C^{1/2}$ is self-adjoint. Since $\delta_{nm} = \langle \varphi_n, \varphi_m \rangle$, the following must hold:

$$\delta_{nm} = \langle \varphi_n, C^{1/2}C^{-1/2}\varphi_m \rangle = \langle C^{1/2}\varphi_n, C^{-1/2}\varphi_m \rangle = \langle \psi_n, \chi_n \rangle.$$

Finally, it can be demonstrated that from a Riesz basis (ψ_n), another equivalent basis $\left(\frac{\psi_n}{\|\psi_n\|}\right)$ of unit vectors can be formed. This follows from

$$1 = \|\varphi_n\| = \|C^{-1/2}\psi_n\| \le \|C^{-1/2}\|\|\psi_n\|$$

and

$$\|\psi_n\| = \|C^{1/2}\varphi_n\| \le \|C^{1/2}\|\|\varphi_n\| = \|C^{1/2}\|,$$

which together imply

$$\sup_n |\psi_n| \le \|C^{-1/2}\| \text{ and } \inf_n |\psi_n| \ge \|C^{1/2}\|.$$

Thus, both (ψ_n) and (χ_n) can be normalized and these normalized sets of vectors also form bases of $L^2[0,1]$.

6.7.3 Comparison with Liouville's Series

Actually, (6.20) is the same as the series (4.15) developed by Liouville; the difference is in the calculation of the eigenvectors. Specifically, each of Liouville's V_n's differ from the corresponding ψ_n used in this paper by a factor of a real constant. Hence, if it is assumed that $\psi_n = k_n V_n, k_n \in \mathbf{R}$, then it can be shown that the two series are identical.

To see this, recall that Liouville's series is

$$f(x) = \sum_n \left\{ \frac{V_n \int_0^1 f(x)\sin(x\sqrt{r_n})dx}{\int_0^1 V_n \sin(x\sqrt{r_n})dx} \right\},$$

[9]Gohberg and Goldberg, p. 81: *If $A \in L(H)$ is self-adjoint, then* $\ker A^{\perp} = \overline{\mathit{range}\ A}$.

[10]Gohberg and M.G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Providence, RI: American Mathematical Society, 1969, p. 309-311.

where the V_n's are Liouville's eigenfunctions. In particular,

$$V_n = \frac{\sin(x\sqrt{r_n}) - \alpha(r_n)x}{\sqrt{r_n}},$$

and obviously, these V_n's also satisfy the system (6.5) — (6.7). This means in modern terminology, $HV_n - r_nCV_n = 0$. Therefore,

$$CV_n = \frac{1}{r_n}HV_n = \frac{1}{r_n}\frac{r_n}{\sqrt{r_n}}\sin(x\sqrt{r_n}) = \frac{\sin(x\sqrt{r_n})}{\sqrt{r_n}}.$$

Now, suppose that $\psi_n = k_nV_n, k_n \in \mathbf{R}$. Since the functions φ_n are orthonormal, the operator $C^{1/2}$ is self-adjoint, and $\varphi_n = C^{1/2}\psi$, several earlier results are restated to make clear that:

$$\delta_{nm} = \langle\varphi_n, \varphi_m\rangle = \langle C\psi_n, \psi_m\rangle = \langle k_nCV_n, k_mV_m\rangle.$$

Then, $\langle k_nCV_n, k_mV_m\rangle = 0$ if $n \neq m$. And,

$$1 = \langle k_nCV_n, k_nV_n\rangle = k_n^2\int_0^1 \frac{\sin(x\sqrt{r_n})}{\sqrt{r_n}}\overline{V_n(x)}dx.$$

The result is, of course, $k_n^2 = \left(\int_0^1 \frac{\sin(x\sqrt{r_n})}{\sqrt{r_n}}\overline{V_n(x)}dx\right)^{-1}$.

Now, consider anew the series representation given in (6.20).

$$\begin{aligned}
f(x) &= \sum_n \langle f(x), C\psi_n\rangle\psi_n \\
&= \sum_n k_n^2\langle f(x), CV_n\rangle V_n \\
&= \sum_n \left\{\frac{V_n\int_0^1 f(x)\frac{\sin(x\sqrt{r_n})}{\sqrt{r_n}}dx}{\int_0^1 V_n\frac{\sin(x\sqrt{r_n})}{\sqrt{r_n}}dx}\right\} \\
&= \sum_n \left\{\frac{V_n\int_0^1 f(x)\sin(x\sqrt{r_n})dx}{\int_0^1 V_n\sin(x\sqrt{r_n})dx}\right\}
\end{aligned}$$

Thus, for $\psi_n = k_nV_n$, the series representation given in (6.20) is the same as that presented by Liouville (4.15).

6.8 The Uniqueness of the Solution $u(x,t)$

Assume that $u(x,t)$ is a solution of the original equation. Then,

$$\frac{\partial u}{dt} = \frac{\partial^2 u}{\partial x^2} - b^2x\int_0^1 x\frac{\partial u}{\partial t}dx,$$

with the boundary conditions

$$u(0,t) = 0,$$

$$\frac{\partial u}{\partial x}(1,t) + hu(1,t) = 0,$$

and

$$u(x,0) = f(x).$$

Now, for any function $u(x,t)$, the following equality holds:

$$2u\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) = \left(\frac{\partial u}{\partial t}\right)^2 - 2\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + 2\left(\frac{\partial u}{\partial x}\right)^2. \tag{6.22}$$

After integrating each side of this equation over $x \in [0,1]$ and $t \in [0,T]$, with $T \geq 0$, the left-hand side of (6.22) becomes:

$$\int_0^1 \int_0^T 2u\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) dxdt. \tag{6.23}$$

Moreover, since $u(x,t)$ is an arbitrary solution to the original problem, the previous expression (6.23) becomes:

$$\int_0^1 \int_0^T \left(2u(x,t)\left(-b^2 x \int_0^1 s\frac{\partial u}{\partial t}(s,t)ds\right)\right) dxdt. \tag{6.24}$$

Rearranging (6.24), one obtains

$$-2b^2 \int_0^T \int_0^1 xu(x,t)dx \int_0^1 s\frac{\partial u}{\partial t}(s,t)dsdt = -2b^2 \int_0^T q(t)q'(t)dt,$$

with $q(t) = \int_0^1 xu(x,t)dx$. So,

$$-2b^2 \int_0^T q(t)q'(t)dt = -b^2 q(t)^2 \big|_0^T = -b^2 \left[\left(\int_0^1 xu(x,T)dx\right)^2 - \left(\int_0^1 xu(x,0)dx\right)^2\right]$$

$$= b^2 \left[\left(\int_0^1 xf(x)dx\right)^2 - \left(\int_0^1 xu(x,T)dx\right)^2\right] \tag{6.25}$$

Meanwhile, the integral over $x \in [0,1]$ and $t \in [0,T]$ of the right-hand side of (6.22) is

$$\int_0^1 \int_0^T \left(\frac{\partial u}{\partial t}\right)^2 - 2\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + 2\left(\frac{\partial u}{\partial x}\right)^2 dxdt. \tag{6.26}$$

Then,

$$(6.26) = \int_0^1 u(x,t)^2\Big|_0^T dx - 2\int_0^T u\frac{\partial u}{\partial x}\Big|_0^1 dt + 2\int_0^1\int_0^T \left(\frac{\partial u}{\partial x}\right)^2 dxdt =$$

$$\int_0^1 \left[u(x,T)^2 - u(x,0)^2\right] dx - 2\int_0^T \left[u(1,t)\frac{\partial u}{\partial x}(1,t) + \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\right] dt.$$

From the boundary conditions (6.2) and (6.3), this double integral of the right-hand side is thus

$$\int_0^1 \left[u(x,T)^2 - f(x)^2\right] dx + 2\int_0^T \left[u(1,t)^2 + \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\right] dt. \qquad (6.27)$$

The two sides are once more equated; (6.25) = (6.27). And, from this, one quickly sees that

$$b^2\left(\int_0^1 xf(x)dx\right)^2 + \int_0^1 f(x)^2 dx =$$

$$\int_0^1 u(x,T)^2 dx + b^2\left(\int_0^1 xu(x,T)dx\right)^2 + 2\int_0^T \left[u(1,t)^2 + \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\right] dt. \qquad (6.28)$$

For $h \geq -1$ the integral $2\int_0^T \left[u(1,t)^2 + \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\right] dt$ is greater than zero. To prove this, notice that the demonstration is analogous to that presented in showing H to be a positive operator. Specifically, $u(1,t)^2 = \left(\int_0^1 \frac{\partial u}{\partial x} dx\right)^2 = \langle \frac{\partial u}{\partial x}, 1\rangle$ (Cauchy-Schwarz) $\leq \|\frac{\partial u}{\partial x}\|^2\|1\|^2 = \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx$.

Then, for $h \geq -1$, $2\int_0^T \left[u(1,t)^2 + \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\right] dt \geq 0$. From equation (6.28), for an arbitrary solution to the original problem, the following inequality is obtained:

$$b^2\left(\int_0^1 xf(x)dx\right)^2 + \int_0^1 f(x)^2 dx \geq \int_0^1 u(x,T)^2 dx + b^2\left(\int_0^1 xu(x,T)dx\right)^2. \qquad (6.29)$$

Given (6.29), it is easy to demonstrate that a solution to the original problem is unique. Suppose $u_1(x,t)$ and $u_2(x,t)$ are solutions to the system (6.1)-(6.4). Then, $\tilde{u}(x,t) = u_2(x,t) - u_1(x,t)$ is also a solution, but it has a special boundary condition, namely $\tilde{u}(x,0) = f(x) - f(x) = 0$. Now, $\tilde{u}(x,t)$ must satisfy (6.29), but one of the boundary conditions for $\tilde{u}$ is $f(x) = 0$. Thus, (6.29) becomes

$$\int_0^1 \tilde{u}(x,T)^2 dx + b^2\left(\int_0^1 x\tilde{u}(x,T)dx\right)^2 \leq 0.$$

This implies that for all $T \geq 0, \tilde{u}(x,T) = 0$ which means that $u_1 = u_2$, and therefore, the solution $u(x,t)$ must be unique.

Appendix A

Liouville's Determination of a Partial Fraction Representation with "Known Theory"

In attempting to prove the convergence of an arbitrary function $f(x)$ and a solution $u(x,t)$ to, respectively, his eigenfunction series (4.15) and (4.16), Liouville makes use of the equality

$$\frac{\sin(x\sqrt{z})}{\sqrt{z}\omega(z)} = \sum_{n=1}^{\infty} \left\{ \frac{\sin(x\sqrt{r_n})}{(z-r_n)\sqrt{r_n}\omega'(r_n)} \right\}, \tag{A.1}$$

with $\omega(z) = \cos\sqrt{z} + \frac{h\sin\sqrt{z}}{\sqrt{z}} - (h+1)\left(\frac{3b^2}{b^2+3}\right)\frac{\sin\sqrt{z}-\sqrt{z}\cos\sqrt{z}}{z\sqrt{z}}$. Of this expression he writes, "Through known methods, $[\frac{\sin(x\sqrt{z})}{\sqrt{z}\omega(z)}]$ can be decomposed into infinitely many partial fractions, and the resulting series expression is [(A.1)]."[1] Yet, what for Liouville was seemingly apparent looks to the modern reader less obvious. A fairly clear idea of how Liouville came to the formula (A.1) can be had by examining the theory developed to this point in the field of complex analysis and applying it to this problem. Thus, upon examination, the methods to which Liouville refers were relatively new ideas developed by Cauchy.

Today's version of Cauchy's residue theorem reads:

> *Let $f(z)$ be analytic in a simply connected domain D except for finitely many points α_j at which f may have isolated singularities. Let C be a simple, closed contour that lies in D and does not pass*

[1] Joseph Liouville, 'Solution nouvelle d'un probléme d'analyse, relatif aux phènoménes thermo-méchaniques', *Journal de mathématiques pures et appliquées*, 2, 1837, p. 453.

through any point α_j. Then

$$\int_C f(z)dz = 2\pi i \sum \text{ (residue of } f \text{ at } \alpha_j)$$

where the sum is extended over the points α_j that are inside C. [2]

While it was 1846 before Augustin-Louis Cauchy proved a general residue theorem approaching this one,[3] he had begun discussing and considering this topic in the early years of the century. As early as 1814, Cauchy had briefly mentioned residues, without naming them as such, in one of his first papers discussing complex functions. Then, in his 1825 paper 'Mémoire sur les intégrales définies prises entre des limites imaginaires', Cauchy further developed the concept of the integral residue. In this article, a work the historian Morris Kline calls "one of the most beautiful in the history of science," Cauchy investigated what happens when a function is discontinuous inside or on the boundary of a rectangle. By considering the integral over different paths on the rectangle, he obtained different values of the integral. From these different values, Cauchy obtained residues. Further, when a function had several poles, the sum of the residues must be taken to get the value of the integral over the rectangle. Then, by allowing the sides of the rectangle to become infinite in length, Cauchy could evaluate all of the residues. Despite the obvious applications to this problem, Liouville was most likely unaware of this paper because it was not published until 1874.[4] Instead, this concept, as yet unnamed, became widely known to the mathematical community in the four volume set *Exercises de mathématique*, published in the late 1820's. In this collection, Cauchy used the term *résidu intégral* for the first time, and he expanded still more on the ideas presented in the 1825 paper.[5]

With use of Cauchy's residue theorem, we can develop the same series representation used by Liouville. Consider first $\omega(z)$. If we define $c = -(h+1)\left(\frac{3b^2}{b^2+3}\right)$, then

$$\omega(z) = \cos\sqrt{z} + \frac{h\sin\sqrt{z}}{\sqrt{z}} + c\frac{\sin\sqrt{z} - \sqrt{z}\cos\sqrt{z}}{z\sqrt{z}}.$$

As in Liouville's paper, the zeros of ω are denoted by r_n. Then, for all $r_n > 0$, $\omega(r_n) = 0$. Consider now the equation $q(z) = \omega(z^2)$. The zeros of

$$q(z) = \cos z + \frac{h\sin z}{z} + c\frac{\sin z - z\cos z}{z^3} \tag{A.2}$$

[2] Norman Levinson and Raymond M. Redheffer, *Complex Variables*, San Francisco: Holden-Day Inc., 1970, p. 190.

[3] In his article 'Sur les intégrales dans lesquelles la fonction sous le signe $\int$ change brusquenment de valeur' Cauchy demonstrated that the formula holds when C is an arbitrary simple, closed contour; however, he still workedonly with poles.

[4] Morris Kline, *Mathematical Thought From Ancient to Modern Times*, New York: Oxford University Press, 1972, p. 637.

[5] Augustin-Louis Cauchy, *Oeuvres*, 2, Paris: Gauthier-Villars, 1888, 6, pp. 23-37.

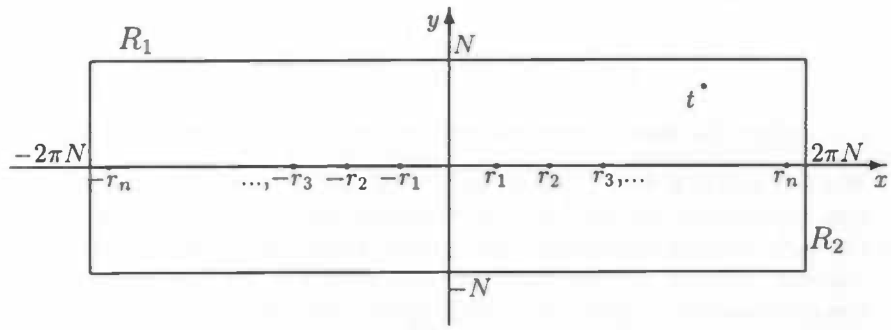


Figure A.1: The rectangle C_N.

are members of the set $\{\pm\sqrt{r_n}\,|\,r_n$ is a zero of $\omega(z)\}$. Moreover, $q'(z) = 2z\omega'(z^2)$. Suppose that

$$f(z) = \frac{\sin(xz)}{zq(z)}, \tag{A.3}$$

and the rectangle C_N pictured in figure A is such that $\pm\sqrt{r_j} \in C_N$ for $j \leq n$, but $\sqrt{r_{n+1}}$ and $-\sqrt{r_{n+1}}$ are outside the rectangle. Further, t is an arbitrary point inside C_N, not equal to any of the $\pm\sqrt{r_n}$. Now, we would like to evaluate the integral

$$\int_{C_N} \frac{f(z)}{z-t}.$$

By Cauchy's residue theorem,

$$\frac{1}{2\pi i}\int_{C_N} \frac{f(z)}{z-t} = (\text{residue at } t) + \sum_n (\text{residue at } \sqrt{r_n}) + \sum_n (\text{residue at } -\sqrt{r_n}). \tag{A.4}$$

Both of these residues can be readily computed. For t, the residue is simply $f(t)$, while the general term in both series of residues is $\frac{\sin(x\sqrt{r})}{\sqrt{r}(\sqrt{r}-t)q'(\sqrt{r})}$. Thus (A.4) becomes

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_N} \frac{f(z)}{z-t} &= f(t) + \sum_k \frac{\sin\left(x(\sqrt{r_k})\right)}{\sqrt{r_k}(\sqrt{r_k}-t)q'(\sqrt{r_k})} + \\ &\quad \sum_k \frac{\sin\left(x(-\sqrt{r_k})\right)}{-\sqrt{r_k}(-\sqrt{r_k}-t)q'(-\sqrt{r_k})}.\end{aligned} \tag{A.5}$$

So, we have

$$\int_{C_N} \frac{f(z)}{z-t}dz = f(t) + \sum_{k=1}^{n} \frac{\sin\left(x(\sqrt{r_k})\right)}{\sqrt{r_k}q'(\sqrt{r_k})(\sqrt{r_k}-t)} +$$

$$\sum_{k=1}^{n} \frac{\sin\left(x(-\sqrt{r_k})\right)}{-\sqrt{r_k}q'(-\sqrt{r_k})(-\sqrt{r_k}-t)}$$

$$= f(t) + \sum_{k=1}^{n} \frac{\sin\left(x(\sqrt{r_k})\right)}{\sqrt{r_k}(\sqrt{r_k}-t)2(\sqrt{r_k})\omega'(\sqrt{r_k})} +$$

$$\sum_{k=1}^{n} \frac{\sin\left(x(-\sqrt{r_k})\right)}{-\sqrt{r_k}(-\sqrt{r_k}-t)2(-\sqrt{r_k})\omega'(\sqrt{r_k})}$$

$$= f(t) + \sum_{k=1}^{n} \frac{\sin(x\sqrt{r_k})}{2r_k\omega'(\sqrt{r_k})(\pm\sqrt{r_k}-t)}\left(\frac{1}{\sqrt{r_k}-t} - \frac{1}{-\sqrt{r_k}-t}\right) +$$

$$\sum_{k=1}^{n} \frac{\sin(x\sqrt{r_k})}{2r_k\omega'(\sqrt{r_k})(\pm\sqrt{r_k}-t)}\left(\frac{1}{\sqrt{r_k}-t} - \frac{1}{-\sqrt{r_k}-t}\right)$$

$$= f(t) + \sum_{k=1}^{n} \frac{\sin(x\sqrt{r_k})2\sqrt{r_k}}{2r_k\omega'(\sqrt{r_k})(\sqrt{r_k}-t)(r_k-t^2)} +$$

$$\sum_{k=1}^{n} \frac{\sin(x\sqrt{r_k})2\sqrt{r_k}}{2r_k\omega'(\sqrt{r_k})(-\sqrt{r_k}-t)(r_k-t^2)}.$$

Then, by simplifying this last equation, (A.5) can now be written as

$$\frac{1}{2\pi i}\int_{C_N} \frac{f(z)}{z-t}dz = f(t) + \sum_{k=1}^{n} \frac{\sin(x\sqrt{r_k})}{\sqrt{r_k}\omega'(\sqrt{r_k})(r_k-t^2)}.$$

Looking at the integral on the left-hand side of (A.5), we consider what happens as the rectangle C_N becomes infinitely large. The integrand is

$$\frac{f(z)}{z-t} = \frac{\sin xz}{z(z-t)\left[\cos z + \frac{h\sin z}{z} + c\frac{\sin z - z\cos z}{z^3}\right]}, \tag{A.6}$$

or

$$\frac{f(z)}{z-t} = \frac{e^{ixz} - e^{-ixz}}{iz(z-t)\left[e^{iz} + e^{-iz} + \frac{ih(e^{-iz}-e^{iz})}{z} + c\frac{i(e^{-iz}-e^{iz})-z(e^{iz}+e^{-iz})}{z^3}\right]}.$$

A.0.1 The Integrand over R_1

Therefore, on R_1, $z = u + iv = u + iN$,

$$\frac{f(z)}{z-t} = \left[\frac{e^{ixu}e^{-xN} - e^{-ixu}e^{xN}}{i(u+iN)(u+iN-t)}\right]$$

$$\left[\frac{1}{e^{iu}e^{-N} + e^{-iu}e^{N} + \frac{ih(e^{-iu}e^{N}-e^{iu}e^{-N})}{u+iN} + c\frac{i(e^{-iu}e^{N}-e^{iu}e^{-N})-(u+iN)(e^{iu}e^{-N}+e^{-iu}e^{N})}{(u+iN)^3}}\right],$$

Combining (A.3) and (A.11),

$$\frac{\sin(x\sqrt{z})}{\sqrt{z}\omega z} = \sum_{k=1}^{\infty} \frac{\sin(x\sqrt{r_k})}{\sqrt{r_k}\omega'(\sqrt{r_k})\,(z - r_k)}.$$

This last equation is obviously (A.1), the one used by Liouville in 'Solution nouvelle...'

By factoring out $e^v + e^{-v}$ in the denominator of (A.8), and defining $q(v) = \frac{e^v - e^{-v}}{e^v + e^{-v}}$, we have

$$\left|\frac{f(z)}{z-t}\right| \leq \left|\frac{e^{xv} + e^{-xv}}{(2\pi N + iv)(2\pi N + iv - t)\,(e^v + e^{-v})\left(1 + \frac{ihq(v)}{2\pi N + iv} + c\frac{iq(v) + (2\pi N + iv)}{(2\pi N + iv)^3}\right)}\right|.$$

Since for $x \in [0,1]$, $\left|\frac{e^{xv}+e^{-xv}}{e^v+e^{-v}}\right| \leq 1$,

$$\left|\frac{f(z)}{z-t}\right| \leq \left|\frac{1}{(2\pi N + iv)(2\pi N + iv - t)\left(1 + \frac{ihq(v)}{2\pi N + iv} + c\frac{iq(v) + (2\pi N + iv)}{(2\pi N + iv)^3}\right)}\right|. \quad \text{(A.9)}$$

Further $|q(v)| = \left|\frac{e^v - e^{-v}}{e^v + e^{-v}}\right| \leq 1$, and therefore $1 + \frac{ihq(v)}{2\pi N + iv} + c\frac{iq(v) - (2\pi N + iv)}{(2\pi N + iv)^3} = 1 + 0\left(\frac{1}{N}\right)$ and from this and (A.9), one obtains:

$$\left|\frac{f(z)}{z-t}\right| = 0\left(\frac{1}{N^2},\right)$$

uniform in v. Hence,

$$\left|\int_{2\pi N}^{-2\pi N} \left.\frac{f(z)}{z-t}\right|_{z=2\pi N+iv} dv\right| = O\left(\frac{1}{N}\right).$$

Thus,

$$\lim_{n\to\infty}\left|\int_{2\pi N}^{-2\pi N} \left.\frac{f(z)}{z-t}\right|_{z=2\pi N+iv} dv\right| = 0. \quad \text{(A.10)}$$

From (A.7) and (A.10), it follows immediately that

$$\frac{1}{2\pi i}\int_{C_N} \frac{f(z)}{z-t} = 0$$

and therefore (A.5) is now

$$0 = f(t) + \sum_{k=1}^{\infty} \frac{\sin(x\sqrt{r_k})}{\sqrt{r_k}\omega'(\sqrt{r_k})\,(r_k - t^2)}.$$

Clearly this means

$$f(t) = -\sum_{k=1}^{\infty} \frac{\sin(x\sqrt{r_k})}{\sqrt{r_k}\omega'(\sqrt{r_k})\,(r_k - t^2)} = \sum_{k=1}^{\infty} \frac{\sin(x\sqrt{r_k})}{\sqrt{r_k}\omega'(\sqrt{r_k})\,(t^2 - r_k)}.$$

If one now takes $z = \sqrt{t}$, then this last equation becomes

$$f(\sqrt{z}) = \sum_{k=1}^{\infty} \frac{\sin(x\sqrt{r_k})}{\sqrt{r_k}\omega'(\sqrt{r_k})\,(z - r_k)}. \quad \text{(A.11)}$$

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