Matrices depending on parameter, with focus on some resonant Hamiltonian cases

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Abstract

This paper presents an algorithm for constructing a universal unfolding of matrices that respect to a certain structure, like given by a symplectic form or by a reversing involution. Our methods are inspired by Gantmacher and Arnold. Applications concern linear Hamiltonian or linear reversible systems in $1 : 1$ and $1 : -1$ resonance. The latter resonance is studied both in the semi-simple and generic case.

1 Introduction

1.1 Setting of the Problem

In an n–dimensional system of ordinary differential equations $\dot{x} = f(x)$, where $x \in \mathbf{R}^n$, we consider an equilibrium point $x = 0$. Here we can expand in (local) coordinates

$$\dot{x} = f(x) = Ax + O(|x|^2). \tag{1}$$

Transportation of the system $\dot{x} = f(x)$ along a (local) coordinate change $y = \Phi(x)$, with $\Phi(0) = 0$, then leads to a new expansion

$$\dot{y} = D\Phi(0)f(x) = D\Phi(0)f(\Phi^{-1}(y)) = By + O(|y^2|). \tag{2}$$

So again expanding

$$\Phi(x) = D\Phi(0)x + O(|x|^2), \tag{3}$$

we see from (1), (3) and (2) that the linear parts have the following relationship

$$B = SAS^{-1},$$

where $S = D\Phi(0)$. In other words, the linear parts are *similar* in the sense of linear maps. This idea has been exploited by Arnold [Ar1], when studying matrices depending on parameters.

The general question is what changes in the dynamics when perturbing or deforming the system (1). For this it is essential first to understand this problem at the level of linear systems. For future purposes we mention that this approach is also helpful when studying relative equilibria.

It is known that Arnold's theory [Ar2] also holds in many settings where the systems have to respect a certain structure. Examples are symplectic or volume preserving structures or symmetries. A key mathematical tool is the fact that the systems (1) have to belong to a certain Lie sub–algebra of vector fields. Then the matrices A belong to the corresponding Lie sub–algebra of matrices. In the symplectic case we would have $A \in sp(2n, R)$, the Lie algebra of infinitesimally symplectic matrices in n degrees of freedom. In such a setting the transformations Φ also will be required to preserve the structure, which in the symplectic example, leads to the fact that $S \in Sp(2n, R)$, the corresponding Lie group of symplectic transformations.

Our present interest concerns examples in this symplectic setting where the linear part contains strong resonances. The simplest cases of this occur for $n = 2$ and when the spectrum of A is purely imaginary, $\pm i\lambda_1$ and $\pm i\lambda_2$, where we take $\lambda_{1,2} > 0$. A resonance is a relation of the form

$$k_1\lambda_1 + k_2\lambda_2 = 0, \tag{4}$$

with $k_1, k_2 \in \mathbf{Z}$. The adjective 'strong' here refers to the fact that $k_{1,2}$ are 'small' integers. Our particular interest is where such a resonance (4) exists with $k_{1,2} = \pm 1$: the 'strongest' resonances possible. Important then is whether A is semi-simple or not, we aim to study all cases. For background see, e.g., Hoveijn [H], van der Meer [VdM] and de Jong [J].

1.2 Outline and Results

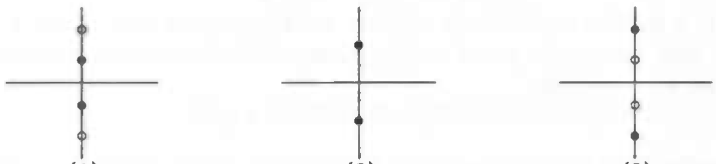
In section 2, several definitions, such as of a Lie group G acting on a manifold M and the G-orbit $O(x)$ of an element $x \in M$, are given. Also unfoldings of elements $x \in M$ are introduced, which can be seen as deformations or perturbations, modulo the action of G. These unfoldings depend on a finite number of deformation parameters, and therefore are sometimes called families. To capture a full neighbourhood of $x \in M$, the unfolding needs to be versal, and it turns out that the corresponding subset of M is transversal to the orbit $O(x)$. If the number of deformation parameters is minimal in this respect, we speak of mini-versal or universal unfoldings.

In the section 3 we focus on the space $gl(n, \mathbf{C})$ of matrices, under the general action of $GL(n, \mathbf{C})$. This means that matrices on the same orbit are similar. It turns out that the concept of centralizer is strongly connected to that of universal unfolding. This leads to a straightforward construction of universal unfoldings. Moreover it turns out that these ideas have simple generalizations to some interesting subspaces such as Lie algebra's of $gl(n, \mathbf{C})$, which makes it possible treat linear Hamiltonian systems or linear reversible systems by the same method.

In section 4, we shall introduce the symplectic vector space, on which linear Hamiltonian functions are defined. In section 4.4, the group $Sp(2n, \mathbf{R})$ of all symplectic matrices is introduced. Each symplectic matrix preserves the standard symplectic structure and is therefore volume-preserving. Since there is a close connection between the Lie algebra $sp(2n, \mathbf{R})$ of the symplectic group and linear Hamiltonian systems, we study the linear universal families (unfoldings) of elements from this algebra. A practical construction of a linear universal family of $A \in sp(2n, \mathbf{R})$ is demonstrated in section 4.4.

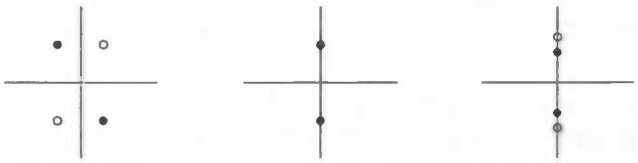
In the final section, our main result is presented. Here we study the eigenvalue distribution of a linear Hamiltonian in $1 : \pm 1$ resonance due to small deformation parameters. In all cases, we found three possible eigenvalue distributions (or types). In the case of the $1 : 1$ resonance the eigenvalues remain on the imaginary axis, and in the case of the $1 : -1$ resonance the eigenvalues escape from the imaginary axis. We showed the eigenvalues of a universal family of the $1 : \cdots : 1$ resonance also do not escape from the imaginary axis. The results are given by the following figures. Although in the latter case there is a distinction between the generic and semisimple case, they both have the same types.

In the appendix we also discuss the universal unfoldings of the linear reversible systems in $1 : \pm 1$ resonance. The eigenvalue distribution of such a unfolding is the same as universal unfoldings of a Hamiltonian $1 : -1$ resonance. By contrast to the Hamiltonian systems, there is no distinction between the $1 : 1$ and $1 : -1$ resonances.



(1) (2) (3)

Figure 1: Three possible types of the vector field of a Hamiltonian in $1 : 1$ resonance, depending on values of the parameters.



$\mu > 0$ $\mu = 0$ $\mu < 0$

Figure 2: Different types of a Hamiltonian vector field in $1 : -1$ according to the sign of the parameter μ.

2 Finite Dimensional Systems

2.1 The jet space

Let us consider the smooth mapping $f : \mathbf{R}^n \mapsto \mathbf{R}$ with $f(0) = 0$. By the *k-jet* $j^k(f)$ we mean the part of the Taylor series of f at $x = 0$, obtained by removing all terms of degree $> k$. For instance, take $f = \sin x$, then the 3-jet of f is: $j^3(f)(x) = x - \frac{1}{6}x^3$.

Let $f : \mathbf{R}^n \mapsto \mathbf{R}^p$ be a smooth mapping and $f(0) = 0$. By $C^\infty(n,p)$ we denote the set containing all such mappings. When $n = p$, we simply write $C^\infty(n)$. The k-jet of f is defined by

$$j^k(f) = (j^k(f_1), j^k(f_2), ..., j^k(f_p)), \tag{5}$$

where f_j are the components of f relative to the standard coordinate system of $\mathbf{R}^p$. We say that two mappings f and $g \in C^\infty(n,p)$ are *k-equivalent*, if both

k-jet are identical, i.e., $j^k(f) = j^k(g)$. By the jet space $J^k(n,p)$ we denote the corresponding set of equivalence classes. That is, $J^k(n,p)$ contains all smooth mapping $f \in C^\infty(n,p)$ each of whose components is a polynomial of degree $\leq k$. It is clear that $J^k(n,p)$ is a vector space. The elements of a jet space are called *k-jets*.

2.2 Equivalence classes

Recall that two germs $f, g\ : (\mathbf{R^n}, 0) \mapsto (\mathbf{R^p}, 0)$ are equivalent when there exits a pair (h,k) of invertible germs with $h : (\mathbf{R^n},0) \mapsto (\mathbf{R^n},0)$ and $k : (\mathbf{R^p},0) \mapsto (\mathbf{R^p},0)$, such that

$$f \circ h = k \circ g \tag{6}$$

There is a similar equivalence relation between two k-jets f and g in $J^k(n,p)$. We say that two s-jets f and g are *equivalent*, if there exists a pair of invertible germs (h,k), such that

$$j^s(f \circ h) = j^s(k \circ g). \tag{7}$$

Suppose that the components of two s-jets f and g are homogeneous polynomial of degree s and f,g are equivalent. It can be shown that

$$j^s(f \circ h) = f \circ j^1(h) \tag{8}$$

Similarly, we have:

$$j^s(k \circ f) = j^1(k) \circ g \tag{9}$$

It follows that

$$f \circ j^1(h) = j^1(k) \circ g. \tag{10}$$

Observe that $j^1(h), j^1(k)$ are invertible linear mappings and thus are elements of the general linear groups $GL(n, \mathbf{R})$) and $GL(p, \mathbf{R})$ respectively. It is also obvious that if we have (10), then f, g are certainly equivalent. Hence, two s-jets f and g, each of whose components is a homogeneous polynomial of degree s, are equivalent if and only if there exits a pair $(h,k) \in GL(n,\mathbf{R}) \times GL(p,\mathbf{R})$ such that,

$$f \circ h = k \circ g. \tag{11}$$

We denote the set of all $C^\infty(n,p)$-mappings, each of whose components is a homogeneous polynomials of degree s, by $H^s(n,p)$. Observe that $H^s(n,p)$ is a subspace of the jet space $J^s(n,p)$ and, moreover, that

$$J^s(n,p) = \oplus_{i=1}^s H^i(n,p).$$

2.3 Group Action

A group G is said to be a *Lie group* if G is a finite dimensional smooth manifold with a smooth group structure. The latter means that the group multiplication $(x,y) \mapsto x \cdot y$ and the group inversion $x \mapsto x^{-1}$ both are smooth.

Example 1 The general linear groups $GL(n,\mathbf{R}), GL(n,\mathbf{C})$ are Lie groups.

Let G be a Lie group and M a smooth manifold.

Definition 1 [Group action] *An action of G on M is a smooth mapping $\Psi : G \times M \mapsto M$ with the following properties:*

1. *If id is the identity of G, then* $\Psi(id, x) = x$, *for all* $x \in M$;

2. *If* $g, h \in G$, *then* $\Psi(g \cdot h, x) = \Psi(g, \Psi(h, x))$ *for all* $x \in M$.

By *the orbit through a point* $x \in M$ under the group action Ψ we mean the set:

$$O(x) = \{\Psi(g, x) : g \in G\}. \tag{12}$$

Remarks.

1. Take a fixed $x \in M$. The natural mapping $\Psi_x : G \mapsto O(x)$ is defined by

$$\Psi_x(g) = \Psi(g, x), g \in G. \tag{13}$$

   Obviously, we have that $O(x) = \Psi_x(G)$.

2. The action $\Psi : G \times M \mapsto M$ induces an equivalent relation

$$x \sim y \Leftrightarrow x \in O(y).$$

3. For our applications in this paper we will require that all orbits are smooth submanifolds. By Gibson [Gib] (p.222-225) all case studies of our interest in this paper meet this requirement.

Example 2 [Rotation in a plane] The action of the group of rotations $G = SO(2)$ on the plane $\mathbf{R}^2$. Define $\Psi : (G, \mathbf{R}^2) \mapsto \mathbf{R}^2$ by:

$$(\theta, x) \mapsto R_\theta x, \tag{14}$$

with R_θ the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It is easy to check that the mapping Ψ is group action. The orbit through $x \in \mathbf{R}^2$ under this action is given by the set: $O(x) = \{R_\theta x : R_\theta \in SO(2)\} = \{R_\theta x : 0 \leq \theta \leq 2\pi\}$. Clearly, except for the origin, this set is a circle. See Figure-3.

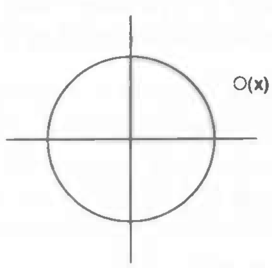


Figure 3: The orbit $O(x)$ in $\mathbf{R}^2$ under the rotation.

Example 3 [**The natural action**] Consider the direct product $G = GL(n, \mathbf{R}) \times GL(p, \mathbf{R})$, then G is a Lie group. Consider the mapping $\Psi : G \times H^d(n,p) \mapsto H^d(n,p)$ defined by

$$\Psi((h,k), f) = k \circ f \circ h^{-1}. \tag{15}$$

Then Ψ is a group action of G on $H^d(n,p)$ which is called *the natural action.* The orbits through $f \in H^d(n,p)$ under Ψ is given by the image $\Psi_x(G)$, i.e. by

$$\{k \circ f \circ h^{-1} : (h,k) \in GL(n, \mathbf{R}) \times GL(p, \mathbf{R})\}. \tag{16}$$

Example 4 [**The general action**] Consider the group $GL(n, \mathbf{F})$ of all invertible $n \times n$-matrices over the field $\mathbf{F}$, where $\mathbf{F} = \mathbf{R}, \mathbf{C}$. The mapping $\Psi : GL(n, \mathbf{F}) \mapsto gl(n, \mathbf{F})$ defined by

$$(Q, A) \mapsto QAQ^{-1}$$

is an action and is a special case of the natural action. Each orbit in the space $gl(n, \mathbf{F})$ is a similarity class.

2.4 Examples of computing orbits

Consider the vector space $H^k(n,p)$ of $C^\infty(n,p)$-mappings each of whose components is a homogeneous polynomial of degree k. In general it is no easy task to find all orbits in $H^k(n,p)$ under the natural action. Here we only give three relatively easy examples.

Remark. The zero form $f = 0$ always forms an orbit $O(0)$, the *zero-orbit.* Indeed, it is easy to see that

$$g \in O(0) \Leftrightarrow g = 0.$$

Example 5 [**The case $H^1(n,1)$**] Let $f \in H^1(n,1)$, i.e.,

$$f(x) = a \cdot x, \; a, x \in R^n.$$

Here the dot means the inner product. In this way, one identifies an element $f \in H^1(n,1)$ by the vector $a \in \mathbf{R}^n$. When $a = 0$, then $f = 0$ which is the zero-orbit. Suppose that $a \neq 0$, then $f \neq 0$. Every non-zero vector $b \in \mathbf{R}^n$ can be transformed to a by a (nonsingular) linear transformation, it follows that every non-zero form $g \in H^1(n,1)$ is equivalent with f. Summarizing we only have orbits, namely, the zero-orbit and the orbit of all non-zero forms.

Example 6 [**The case $H^2(n,1)$**] We take $n = 2$ and let $f \in H^2(n,1)$. Then $f(x,y) = ax^2 + 2bxy + cy^2 =< X, A_f X >$, with $X = (x,y)^T$ and

$$A_f = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

In this way each element $f \in H^2(n,1)$ is naturally identified with a symmetric matrix A_f. In fact,$f \mapsto A_f$ an isomorphism between $H^2(n,1)$ and the vector space $sym(n, \mathbf{R})$ of all real symmetric matrices of order n. Let $g(x,y) = \langle X, A_g X \rangle$, where $A_g \in sym(2, \mathbf{R})$ and $X = (x,y)^T$. If f, g are equivalent then there is $h \in GL(2, \mathbf{R})$ and a real number $k \neq 0$ such that

$h^T A_f h = kA_g$, i.e., A_f is *congruent* to kA_g. By Sylvester's Law this is exactly the case when A_f, kA_g have the same rank and the same index. Note that $rank(kA_g) = rank(A_g)$, but $index(kA_g) = index(sign(k)A_g)$. Thus, we have the following statement

Proposition 2 *Two mappings $g, f \in H^2(2,1)$ are on the same orbit under the natural action if and only if $rank(A_f) = rank(A_g)$ and $index(A_f) = index(\pm A_g)$.*

To find the different types of orbits we only have to study the rank and the index of A_f. We introduce the diagonal matrices $J_{r,p}$, whose rank is r and whose index is p. By Proposition-2 such matrices can be chosen freely as long as the rank and index remain unchanged. Moreover, $J_{r,p}, J_{r,r-p}$ are on the same orbit, since $index(J_{r,p}) = index(-J_{r,r-p})$.

- **Rank(A_f)=2:** By Proposition-2 it follows that f lies on the orbit through $g_p = \langle X, J_{2,p}X\rangle$ where $p = 0, 1$ and 2. Since $index(J_{2,2}) = index(-J_{2,0})$, g_2 and g_0 lie on the same orbit. Hence, if A_f has rank 2, there are just two type of orbits through f: $O_{2,2} = O_{2,0} = O(g_2)$ and $O_{2,1} = O(g_1)$. We can take:
$$J_{2,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J_{2,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
In this way we find: $O_{2,2} = O(x^2 + y^2)$ and $O_{2,1} = O(x^2 - y^2)$.

- **Rank(A_f)=1:** In this case f is equivalent with $g = \langle X, J_{1,1}X\rangle$ or $g = \langle X, J_{1,0}X\rangle$. But they both lie on the same orbit. Hence, if rank(A_f)=1, then there is only one orbit: $O_{1,1} = O(g)$. Here we may take:
$$J_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
So we have: $O_{1,1} = O(x^2)$.

- **Rank(A_f)=0:** The zero-orbit $O_{0,0} = \{0\}$.

Hence, there are four orbits: $O_{2,2}, O_{2,1}, O_{1,1}, O_{0,0}$, in $H^2(2,1)$ under the natural action.

Now it is easy to obtain a relationship between the values of a, b, c and the orbits. We know that $H^2(2,1)$ can also be identified with $\mathbf{R^3}$ by

$$ax^2 + bxy + cy^2 \mapsto (a, b, c).$$

Suppose that $f \in O_{1,1}$, i.e. rank(A_f)=1 and index(A_f) = 1. Since rank(A_f) < 2, we have $det(A_f) = 0$. Hence, $ac - b^2 = 0$. It is clear that $(a, b, c) \neq 0$. Hence, the cone without vertex $\{(a, b, c) \in \mathbf{R}^3 : b^2 = ac\}$ in the *abc*-space corresponds to the orbit $O_{1,1}$. the vertex $(a, b, c) = 0$ corresponds to $O_{0,0}$ and so on. See Figure-4.

Since every quadratic form f can be written as $\langle X, A_f X\rangle$, with $X \in \mathbf{R^n}$, the whole above analysis can be easily generalized for $n \geq 2$.

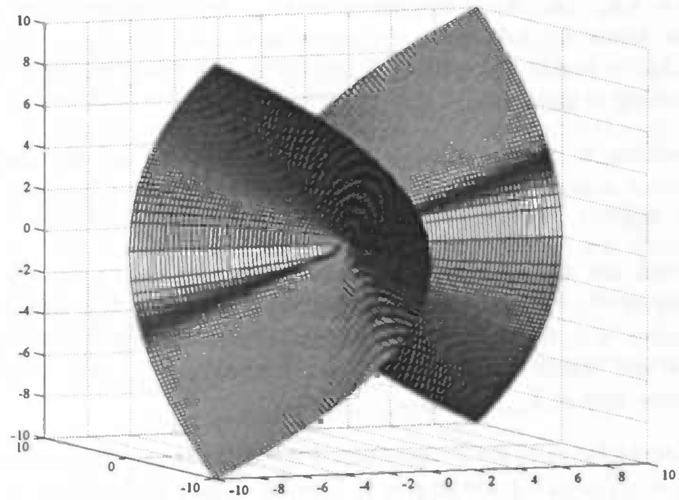


Figure 4: The surface of $b^2 = ac$ in the parameter space

2.5 The tangent space to an orbit

Let G be a Lie group, S be a smooth manifold and $x \in S$. In order to study a neighbourhood of x in S, it turns out useful to consider the tangent space of the orbit $O(x)$. Let us recall some notations, see Definition-1 and Remarks after it. If the action is denoted by $\Psi : G \times S \to S$, then the natural mapping Ψ_x is given by

$$\Psi_x : g \in G \mapsto \Psi(g, x) \in O(x).$$

We restrict to the cases when all orbits are smooth sub-manifolds of S (see Remark on p.5). Under this restriction one can show that the map Ψ_x is submersive (Gibson [Gib]), i.e. the rank of the differential of Ψ at each point $g \in G$ is equal to the dimension of the tangent space to the orbit $O(x)$ at $y = \Psi_x(g)$:

$$rank(D_g\Psi_x) = dim(T_yO(x)).$$

In particular if we take $g = id$ (the identity), we have

$$rank(D_{id}\Psi_x) = dim(T_xO(x)). \tag{17}$$

It follows that

$$T_xO(x) = D_{id}\Psi_x(T_{id}G). \tag{18}$$

In the following examples we use this principle to calculate the tangent space to an orbit.

Example 7 [The tangent spaces in $H^d(n,p)$] We return to the examples of Sec. 2.3. Consider the natural action Ψ of Example-3. Let $f \in H^d(n,p)$ and Ψ_f be the natural mapping of Ψ, i.e.

$$\Psi_f(h, k) = k \circ f \circ h^{-1}.$$

Since orbits under the natural action in $H^d(n,p)$ are all smooth sub-manifolds of $H^d(n,p)$ (Remark on p.5), by (18) one has,

$$T_f O(f) = D_{id}\Psi_f(T_{id}G), \tag{19}$$

where $id = (id_n, id_p)$. Observe that

$$T_{id}G = T_{id_n}GL(n,\mathbf{R}) \times T_{id_p}GL(p,\mathbf{R}) = gl(n,\mathbf{R}) \times gl(p,\mathbf{R}).$$

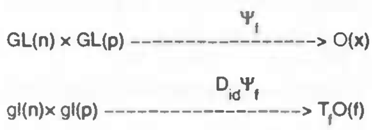


Figure 5: The differential of the natural mapping Ψ_f.

In order to really compute the image of $D_{id}\Psi_f$, consider a smooth curve $C : \mathbf{R} \mapsto G$ in the space G by

$$t \mapsto (id_n + tB_n, id_p + tB_p),$$

where $(B_n, B_p) \in gl(n,\mathbf{R}) \times gl(p,\mathbf{R})$. After a few steps of calculation one shows that

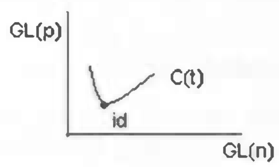


Figure 6: A curve $C(t)$ in the space G.

Proposition 3 [Tangent space of an orbit in $H^d(n,p)$] *The tangent space to the orbit $O(f)$ under the natural action in $H^d(n,p)$ at the point f is spanned by $\{x_j \frac{\partial f}{\partial x_i}\}_{1 \le i,j \le n}$ and $\{f_j e_i\}_{1 \le i,j \le p}$.*

In the special case when $p = 1$, by Euler $f = c\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$, where $c \in \mathbf{R}$. Hence, the tangent space to the orbit $O(f)$ is spanned by $\{x_j \frac{\partial f}{\partial x_i}\}$, when $p = 1$.

Example 8 [The space $H^2(2,1)$] Given the element $f(x,y) = x^2 + y^2$ of the orbit $O_{2,2} = O(x^2 + y^2)$ in $H^2(2,1)$. By Proposition-3 the tangent space to $O_{2,2}$ at $f(x,y)$ is given by $span(x^2, y^2, xy)$, which is just the space $H^2(2,1)$ self.

2.6 Stable Orbits

Returning to the general setting. Consider the action $\Psi : G \times M \mapsto M$ of the Lie group G on the smooth manifold M. Recall that $O(x)$ is the orbit through the point x in M and from now we assume that all orbits are smooth sub-manifolds of M.

Definition 4 [Stability of orbits] *The orbit $O(x)$ is stable if there is a neighbourhood $B(x)$ of $x \in O(x)$ in M such that $y \in O(x)$, for all $y \in B(x)$.*

A consequence of this definition is that if $z \in O(x)$ is stable, then all points of $O(x)$ are stable. Therefore we call $O(x)$ stable if x is.
The *codimension* of x is defined as the co-dimension of the orbits $O(x)$:

$$cod(x) = dim(M) - dim(T_x O(x)).$$

Suppose that x is stable, i.e., $O(x)$ is stable. It follows that $O(x)$ is open in M, is equivalent to the condition that $cod(x) = cod(O(x)) = 0$. Summarizing,

Theorem 5 [Stability versus codimension] *In the above context, the point $x \in M$ is stable if and only if $cod(x) = 0$, i.e., if and only if $O(x)$ is open in M.*

Example 9 Consider the orbit $O(x^2) \in H^2(2,1)$ of Example-6. It is easy to check that $cod(x^2) = 1$. By Theorem-5 the orbit $O(x^2)$ is not stable. Indeed, let $\tilde{f} = x^2 + \epsilon g$ with $g \in H^2(2,1)$ and ϵ small. By choosing a suitable g (for instance $g = y^2$) we get $det(A_{\tilde{f}}) \neq 0$. This means that $rank(A_{\tilde{f}}) = 2$. Hence the deformation $\tilde{f}$ of x^2 no longer is on the orbit $O(x^2)$. On the other hand the orbits $O(x^2 \pm y^2)$ are stable, since $cod(x^2 \pm y^2) = 0$.

2.7 Transversal Unfolding

In Example-9 we study the unstable element $f = x^2$, by considering deformations $\tilde{f} = x^2 + \epsilon y^2$. This deformation turns out to be stable for all $\epsilon \neq 0$. A natural question may be, whether this is the general situation, namely that an unstable element can be made stable by a suitable deformation. To study this question and related problems, now the general concept of *unfolding* is introduced.

Remark. Although for each element $f \in H^2(n,1)$ one can stabilize the orbit $O(f)$ by a small deformation, we observe that it is not always possible to to do this. A good example is the instability of orbits in $gl(n, \mathbf{C})$ under the general action. The reason is that the eigenvalues are invariants of the general action.

The above discussion leads to the following definition

Definition 6 [(Transversal) Unfolding] *Let the Lie group G acts smoothly on the manifold M and assume that all orbits are smooth sub-manifolds of M. A c-unfolding of x is a germ $U : (\mathbf{R}^c, 0) \mapsto (M, x)$. U is said to be transversal if*

$$T_0 U(R^c) + T_x O(x) = T_x M. \qquad (20)$$

Note that, in the case of transversality, $c \geq cod(x)$. A transversal c-unfolding U is said to be a *minimal transversal unfolding*, simply *mini-unfolding*, when $c = cod(x)$. We restrict to the case when M is a linear space. It is our aim to construct such a mini-unfolding in M. Take a linear c-unfolding at $x \in M$

$$U(\epsilon) = x + \sum_{i=1}^{c} \epsilon_i Y_i, \; Y_i \in M$$

A brief computation shows that

$$span\{Y_i\} + T_x O(x) = M. \tag{21}$$

Hence, if one finds a set $\{Y_i\}$ which forms a basis for a complement of $T_x O(x)$, then our goal has been reached. This leads to one of the following theorem.

Theorem 7 [Linear mini-unfolding, [Gib]] *Let M be a linear space with the action Ψ and assume that all orbits in M under Ψ are smooth sub-manifolds of M. Suppose that the set $\{Y_i\}$ is a basis of a complement of $T_x O(x)$. Then, the linear unfolding*

$$U(\epsilon) = x + \sum_{i=1}^{c} \epsilon_i Y_i \tag{22}$$

is a mini-unfolding and $c = cod(x)$.

Example 10 [Mini-unfolding in $H^3(2,1)$] Consider the vector space $H^3(2,1)$ with the natural action. Every element in this space has the form:

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

and therefore the set $\{x^3, x^2y, xy^2, y^3\}$ is a basis of $H^3(2,1)$.

First, let us look at the orbit $O(x^2y)$. The tangent space of $O(x^2y)$ at $f = x^2y$ is given by

$$span(x^3, x^2y, xy^2).$$

The element y^3 clearly is a complement of the tangent space $T_f(x^2y)$ in $H^3(2,1)$. Thus, the orbit $O(xy^2)$ is not stable. We take the linear 1-unfolding: $U(\epsilon) = x^2y + \epsilon y^3$ and by the above discussion this is a mini-unfolding. It is easy to show that if we take $\epsilon \neq 0$ then the orbit through $U(\epsilon)$ is stable. Also see the figure below.

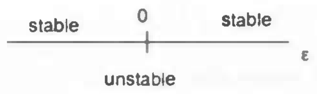


Figure 7: Regions of stability.

2.8 Universal unfolding

We return to the general setting of the smooth action $\Psi : G \times M \to M$. Let $x \in M$ and consider a transversal s-unfolding $U : (\mathbf{R}^s, 0) \to (M, x)$. By continuity, for $|\epsilon|$ sufficiently small, any point $x' = U(\epsilon)$, is in a given neighbourhood $B(x)$ of x. On the other hand, by transversality and application of the Inverse Function Theorem (cf. Rudin []), for sufficiently small $B(x)$, there exists a neighbourhood $B(0)$ of 0 in $\mathbf{R}^s$, a submanifold $F \subset G$ containing $id \in G$ and a submersion

$$F \times U(B(0)) \to B(x).$$

We call $U(B(0))$ a local *cross-section* of $O(x)$. If U is mini-unfolding and $s = cod(x)$, the submersion is a diffeomorphism and we call $S_m = U(B(0))$ a *mini-section*. In that case the image $U(B(0))$ and the orbit $O(x)$ define a product structure on a full neighbourhood of x. Also observe that the mapping $U : B(0) \mapsto S_m$ is invertible. See Figure-8.

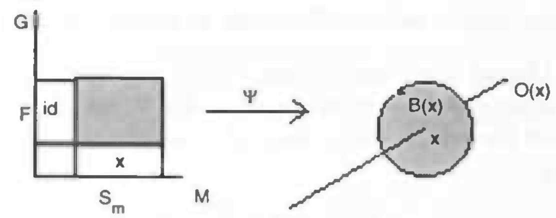


Figure 8: A product structure on a neighbourhood of x.

Transversal unfoldings apparently are related to cross-sections that are transversal to the orbit. In this section we address the problem how to relate different transversal unfoldings to each other by means of the group action. This leads to the concept of universality. To this end we need some notations concerning the relations between unfoldings.

Let U be a m-unfolding of $x \in M$ and $H : (\mathbf{C}^n, 0) \mapsto (\mathbf{C}^m, 0)$. The mapping $V = U \circ H$ then is a n-unfolding of x. We say that V is *induced* from U by H. Consider another m-unfolding S of x. The unfoldings U, S are *equivalent*, if there exists a m-unfolding $g : (\mathbf{C}^m, 0) \mapsto (G, id)$ of $id \in G$ such that

$$S(\epsilon) = \Psi(g(\epsilon), U(\epsilon)).$$

See Figure-9.

Consider an arbitrary unfolding $K : (\mathbf{C}^s, 0) \to (M, x)$. It is our aim to relate K to some mini-unfolding of x, using the product structure. For $|\epsilon|$ sufficiently small we have $K(\epsilon) \in \Psi(F \times S_m)$. Let us then write

$$\Psi^{-1}(K(\epsilon)) = (g, x') \in F \times S_m.$$

We now construct local maps $K_F : (\mathbf{C}^s, 0) \to (F, id)$ and $K_S : (\mathbf{C}^s, 0) \to (S_m, x)$ requiring that

$$g = K_F(\epsilon) \text{ and } x' = K_S(\epsilon).$$

From Figure-10, the existence of K_s, K_F is evident. Observe that, since $K(\epsilon) = \Psi(K_F(\epsilon), K_S(\epsilon))$, K and K_S are equivalent unfoldings. Moreover, K_S can be viewed as induced from a mini-unfolding

$$U : (\mathbf{C}^c, 0) \to (S_m, x)$$

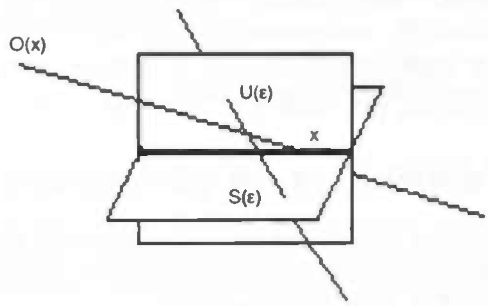


Figure 9: Two equivalence unfoldings.

by a local map $H : (\mathbf{C}^s, 0) \to (\mathbf{C}^c, 0)$, so where $K_S = U \circ H$. Indeed, let U be a mini-unfolding giving the mini-section S_m and choose $H = U^{-1} \circ K_S$. Summarizing we state,

Proposition 8 *For any s-unfolding K, there exist a mini-unfolding U, a parameter transformation H and a s-unfolding K_F of $id \in G$ such that*

$$K(\epsilon) = \Psi(K_F(\epsilon), U \circ H(\epsilon)). \tag{23}$$

By mean of (23), we say that the mini-unfolding U is *morphic* to K by the morphism (K_F, H).

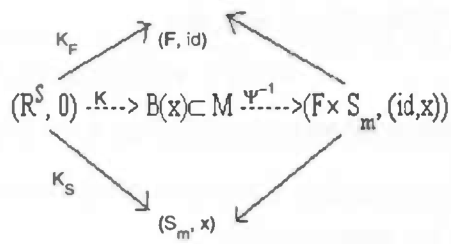


Figure 10: Construction of the s-unfolding K_S which is equivalent to any s-unfolding.

Definition 9 [Versal Unfolding] *A r-unfolding U is a* versal *unfolding of x if for any n-unfolding U' of x there exists a morphism (g, H), for which*

$$U'(\epsilon) = \Psi(g(\epsilon), U \circ H(\epsilon)).$$

where $g : (\mathbf{C}^n, 0) \mapsto (G, id)$ and $H : (\mathbf{C}^n, 0) \mapsto (\mathbf{C}^r, 0)$.

A versal unfolding is called *universal* if r is minimal with respect to the property of versality. By Proposition-8, mini-unfoldings are versal, even universal. It can also be shown that the converse is true. In general we have the following

Theorem 10 [Versality versus transversality, [Gib]] *Consider the Lie group G acting on the manifold M, and assume that all orbits in M are smooth submanifolds of M. For unfoldings U of $x \in M$ transversality is equivalent to versality and mini-transversality to universality.*

3 Matrices depending on parameters

In section 2 *transversal* and *(uni)versal unfolding* have been introduced. In this section we apply these concepts to the matrix space $gl(n, \mathbf{C}) \cong \mathbf{C}^{n \times n}$ under the general action of $GL(n, \mathbf{C})$. Also see Example-4.
Our main aim is to develop an algorithm for constructing a linear universal unfolding of a matrix, using the centralizer, compare to [Arl], section 30. In order to preserve certain structures we adapt these ideas to appropriate subspaces of $gl(n, \mathbf{C})$. Examples are several Lie subalgebras such the special algebra $sl(n, \mathbf{R})$ or the symplectic algebra $sp(2n, \mathbf{R})$, but also the reversible case $gl_{-R}(n, \mathbf{R})$ (with R is a linear involution). Note that the latter example is slightly out of the Lie algebra setting.

3.1 Preliminaries

A family of matrices is the image of a smooth mapping

$$A : (\mathbf{C}^r, 0) \mapsto (gl(n, \mathbf{C}), A(0)) \text{ defined by } \lambda \mapsto A(\lambda).$$

Clearly, $A(\lambda)$ is an r-unfolding of the matrix $A(0)$.

Consider two r-unfoldings $A_1, A_2 \in gl(n, \mathbf{C})$ of $A(0)$. Suppose that $A_1(\lambda)$ and $A_2(\lambda)$ are equivalent. By definition it follows that there exists an r-unfolding $Q : (\mathbf{C}^r, 0) \mapsto (GL(n, \mathbf{C}), id)$ such that

$$A_1(\lambda) = \Psi(Q(\lambda), A_2(\lambda)).$$

where Ψ is the general action. Recall that $\Psi : GL(n, \mathbf{C}) \times gl(n, \mathbf{C}) \mapsto gl(n, \mathbf{C})$ is defined by

$$(Q, A) \mapsto QAQ^{-1}.$$

Hence, A_1, A_2 are *equivalent* when

$$A_1(\lambda) = Q(\lambda) A_2(\lambda) Q^{-1}(\lambda).$$

All conclusions of Section 2 apply to this setting. In view of Theorem-10 our search will be for mini-unfoldings in $gl(n, \mathbf{C})$.

3.2 The tangent space to $O(A)$ in $gl(n, \mathbf{C})$

Given $A \in gl(n, \mathbf{C})$, let $\Psi_A : GL(n, \mathbf{C}) \mapsto gl(n, \mathbf{C})$ be the natural mapping of the general action Ψ. In order to determine the codimension of $O(A)$ we need to compute its tangent space. We already mentioned that $O(A)$ is a smooth sub-manifold of $gl(n, \mathbf{C})$. Hence, we compute $D_{id}\Psi_A(gl(n, \mathbf{C})) = T_A O(A)$. To this end take a smooth curve $C : \mathbf{R} \mapsto gl(n, \mathbf{C})$ given by $t \mapsto id + tB$, where $B \in gl(n, \mathbf{C})$ and $|t|$ is small. Then

$$D_{id}\Psi_A(B) = \frac{d}{dt} C(t) A C(t)^{-1}|_{t=0} = BA - AB = [B, A]. \tag{24}$$

It follows that $D_{id}\Psi_A(gl(n,\mathbf{C})) = \text{span}([e_{i,j}, A] : i,j = 1,\ldots,n\})$, where $\{e_{i,j}\}$ is the standard basis of $gl(n,\mathbf{C})$.

Theorem 11 [The tangent space to an orbit] *Let $A \in gl(n,\mathbf{C})$. The tangent space to the orbit $O(A)$ at A under the general action Ψ is given by*

$$T_AO(A) = \{[B,A] : B \in gl(n,\mathbf{C})\}. \tag{25}$$

Recall that it is our aim to find a mini-unfolding of A, and this can be done by finding a complementary to $T_AO(A)$. Theorem-11 provides a possibility to solve this problem in a systematic way. To this end consider a scalar product $\langle , \rangle : gl(n,\mathbf{R}) \times gl(n,\mathbf{R}) \to \mathbf{C}$ defined by

$$\langle X,Y\rangle = tr(XY^*).$$

Let $T_AO(A)^\perp$ be the orthogonal complement of $T_AO(A)$ with respect to $\langle , \rangle$. Suppose that $X \in T_AO(A)^\perp$, then we have:

$$< P,X >= tr(PX^*) = 0 \tag{26}$$

for any $P \in T_AO(A)$. From Theorem-11 we know that there exists a $B \in gl(n,\mathbf{C})$ for which $P = [B,A]$. So we have

$$tr([B,A]X^*) = tr(BAX^* - ABX^*) = tr(AX^*B - X^*AB) = tr([A,X^*]B).$$

This implies that $[X,A^*] = [A,X^*] = 0$. For any $A \in gl(n,\mathbf{C})$ we denote

$$Z(A) = \{X \in gl(n,\mathbf{C}) : [X,A] = 0\},$$

which is called the *centralizer of* A in $gl(n,\mathbf{C})$. It is easy to see $Z(A)$ is a linear subspace of $gl(n,\mathbf{C})$, and

$$Z(A^*) = Z(A)^* = \{X \in gl(n,\mathbf{C}) : [X^*,A] = 0\}.$$

Proposition 12 *In the above context. Given $A \in gl(n,\mathbf{C})$, $T_AO(A)^\perp$ is defined as above. Then,*

1. $T_AO(A)^\perp = Z(A^*)$;
2. $cod(A) = dim(Z(A))$.

Proof. By the above discussion,

$$X \in T_AO(A)^\perp \Leftrightarrow X \in Z(A^*).$$

The second part follows from the fact that $dim(Z(A)) = dim(Z(A^*))$. □

3.3 Construction of the Centralizer

A way to find a mini-unfolding of an element $A \in gl(n,\mathbf{C})$ is to find a complement of the tangent space $T_AO(A) \subset gl(n,\mathbf{C})$. Proposition-12 from Section 3.2 shows that it is worthwhile to study the centralizer. For more background see Gantmacher [Gan].

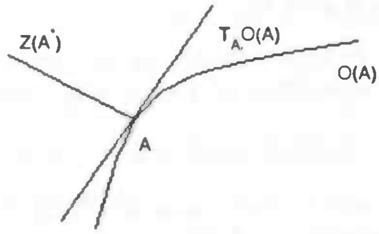


Figure 11: Tangent space of the orbit $O(A)$ at A and the centralizer $Z(A^*)$.

Let us take a matrix in (complex) Jordan normal form:

$$J = diag\{J_1(\lambda_1), J_2(\lambda_2), ..., J_s(\lambda_s)\}$$

where $J_i(\lambda_i)$ is the Jordan block corresponding to the eigenvalue λ_i. Let n_i be the order of $J_i(\lambda_i)$. We may assume that $n_1 \geq n_2 ... \geq n_s$. From [Gan] it follows that a matrix Y commutes with J if and only if Y has the following properties:

1. The matrix Y is in the following form

$$Y = \begin{pmatrix} Y_{1,1} & ... & Y_{1,s} \\ ... & ... & ... \\ Y_{s,1} & .... & Y_{s,s} \end{pmatrix}, \tag{27}$$

where $Y_{i,j} \in \mathbf{C}^{n_i \times n_j}$ are matrix blocks.

2. Each blocks $Y_{i,j}$ is one of the four types presented below:

I. $Y_{i,j} = 0$, if $\lambda_i \neq \lambda_j$;

II. In the case when $\lambda_i = \lambda_j$:

(1). if $n_i = n_j$ then
$$Y_{i,j} = T_{n_i};$$

(2). if $n_i > n_j$ then
$$Y_{i,j} = \begin{bmatrix} T_{n_j} \\ 0 \end{bmatrix};$$

(3). if $n_i < n_j$ then
$$Y_{i,j} = [0, T_{n_i}].$$

Where T_n is an upper triangular square matrix defined by

$$T_n = T_n(\mu) = \begin{pmatrix} \mu_1 & \mu_2 & ... & \mu_n \\ 0 & \mu_1 & \mu_2 & ... \\ ... & ... & ... & ... \\ 0 & ... & ... & \mu_1 \end{pmatrix}, \tag{28}$$

where $\mu_i \in \mathbf{C}$ are independent parameters.

Hence, all elements of $Z(J)$ is in the form of (27).

As one sees, each block $Y_{i,j}$ is a family of matrices. By the *dimension of* $Y_{i,j}$ we mean the dimension of the parameter space, i.e. the number of independent parameters. Since $Y = \{Y_{i,j}\}$, it follows that

$$dim(Y) = \sum_{i,j} dim(Y_{i,j}) - p,$$

where p denotes the number of dependent parameters from different blocks.

Remarks.

1. In the space $gl(n, \mathbf{C})$, all parameters from different blocks are independent, hence in this case one has $p = 0$. Later on when keeping preservation of structures into account, the number p generally is *not* equal to zero.

2. As said before that under the general action in the space $gl(n, \mathbf{C})$ all orbits are instable. We give another proof of this using the above construction: Indeed, let J be a Jordan normal form of $A \in \mathbf{C}$ and $Y \in Z(J)$. By property (2.II) of Y it is clear that $dim(Y_{i,i}) > 0$ for all $1 \leq i \leq s$. It directly follows:
$$cod(A) = dim(Y) > 0.$$

It is easy to see that $dim(Y) = Z(J)$. Hence, in order to compute $dim(Z(J))$ one has to know all $dim(Y_{i,j})$.

First, let us take a look at a simpler case where all eigenvalues of J are equal. By construction $Y_{i,j} \neq 0$, $1 \leq i, j \leq s$, see property (2) above. Now by property (2b) one sees that
$$dim(Y_{i,j}) = min(n_i, n_j).$$

Recall that $n_i \geq n_j$ if $i \leq j$. Hence,

$$\sum dim(Y_{i,j}) = (\sum_{i<j} + \sum_{i=j} + \sum_{i>j}) dim(Y_{i,j}).$$

It follows that in $gl(n, \mathbf{C})$:

$$dim(Z(J)) = \sum_{i,j} dim(Y_{i,j}) = \sum_{k=1}^{s} (2k-1) n_k. \tag{29}$$

This can be easily seen by the table below, where the dimension of each block $Y_{i,j}$ is listed at the corresponding position (i, j). The sum over all numbers in the table then gives the result.

n_1	n_2	n_3	...	n_s
n_2	n_2	n_3	...	n_s
n_3	n_3	n_3	...	n_s
...	...	...	...	n_s
n_s	...	...	...	n_s

Remark. It is known that each $A \in gl(n, \mathbf{C})$ can be reduced to a Jordan normal form J_A by a change the of basis, say, $A = QJ_AQ^{-1}$ ($Q \in GL(n, \mathbf{C})$). It is easy to see that

$$Z(A) = \{QYQ^{-1} : Y \in Z(J_A)\} = QZ(J_A)Q^{-1}, \tag{30}$$

which tells us that $dim(Z(A)) = dim(Z(J))$, and which makes it possible to consider only Jordan normal forms.

Now we are ready to compute $dim(Z(J))$, where $J \in gl(n, \mathbf{C})$ is a Jordan normal form having distinct eigenvalues of various multiplicity. In this case, one can split J into different blocks according to the values of distinct eigenvalues. More precisely, consider the Jordan normal form

$$J = diag\{J_1(\lambda_1), \ldots, J_s(\lambda_s)\},$$

where $J_i(\lambda_i)$ are Jordan blocks. We rearrange the blocks such that

$$J = diag\{B_1(\lambda_1), \ldots, B_k(\lambda_k)\},\ k \leq s$$

where B_i are matrix blocks of the distinct eigenvalues, and $\lambda_i \neq \lambda_j$ if $i \neq j$. Since the eigenvalues of a block $B_i(\lambda_i)$ all coincide with λ_i, we treat the blocks separately, using (29). The sum over all blocks gives the final result.

Theorem 13 [Centralizer of matrices in Jordan normal form] *Let $J = diag\{J_1(\lambda_1), \ldots, J_s(\lambda_s)\}$, where $J_i(\lambda_i) \in gl(n_i, \mathbf{C})$ are Jordan blocks, and assume that $n_1 \geq \ldots \geq n_s$. The centralizer $Z(J)$ is given by the family (27). Moreover,*

$$dim(Z(J)) = \sum_{i=1}^{k} dim(Z(B_i)) = \sum_{i=1}^{k} \sum (n_1 + 3n_2 + 5n_3 + \cdots).$$

Generalization of the definition

In Section 3.2 we only introduced the centralizer of a matrix A in the space $gl(n, \mathbf{C})$. For later purposes it is convenient to generalize this definition for any linear subspace $V(n, \mathbf{C})$ as follows

$$Z_V(A) = \{X \in V : [A, X] = 0\}$$

the centralizer of A in the space V. An direct but useful observation is that

$$Z_V(A) = Z(A) \cap V,$$

which is a linear subspace of V. When $V = gl(n, \mathbf{F})$, ($\mathbf{F} = \mathbf{C}, \mathbf{R}$), we simply write $Z_V(A) = Z(A)$.

Gantmacher's construction

Let V be a linear subspace of $gl(n, \mathbf{C})$ and $A \in V$. Summarizing the above discussion we have the following general algorithm for constructing the centralizer $Z_V(A)$ of A in V:

1. Determine the Jordan normal J_A of A by Jordan decomposition $A = QJ_AQ^{-1}$;
2. Determine the centralizer $Z(J_A)$ in $gl(n, \mathbf{C})$;
3. Compute $Z(A)$, using $Z(A) = QZ(J_A)Q^{-1}$;
4. Compute $Z_V(A)$, using $Z_V(A) = Z(A) \cap V$.

Remark. We call the above algorithm *Gantmacher's construction*, because it is based on Gantmacher [Gan].

Example 11 [Distinct, simple eigenvalues] Let $A \in gl(n, \mathbf{C})$ be a diagonal matrix and

$$A = diag\{\lambda_1, ..., \lambda_n\},\ \lambda_i \in \mathbf{C},$$

Hence, A has n Jordan blocks $J_i(\lambda_i)$. Let $Y \in Z(A)$. Then, $Y = \{Y_{i,j}\}$, where $Y_{i,j}$ are matrix blocks and $1 \leq i, j \leq n$. Suppose that $\lambda_i \neq \lambda_j$ if $i \neq j$, by construction we then have $Y_{i,j} = 0$ if $i \neq j$ and $Y_{i,i} = \epsilon_i$, where $\epsilon_i \in \mathbf{C}$. Hence, $Y \in Z(A)$ must be a diagonal matrix given by

$$Y = \begin{pmatrix} \epsilon_1 & 0 & ... & 0 \\ 0 & \epsilon_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & \epsilon_n \end{pmatrix}.$$

By Theorem-13 it follows that the centralizer is given by

$$Z(A) = \{U \in \mathbf{C}^{\mathbf{n} \times \mathbf{n}} : U = diag\{\epsilon_1, ..., \epsilon_n\}\},$$

where $\epsilon_i \in \mathbf{C}$ are parameters, and $dim(Z(A)) = n$.

Example 12 [Real matrices with complex eigenvalues] Let $A \in gl(n, \mathbf{R})$ be given by

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $b \neq 0$. A brief calculation shows that $A = QJ_AQ^{-1}$, where

$$Q = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \quad \text{and} \quad J_A = \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix}.$$

Suppose that $X \in Z(A)$. By Example-11, it follows that

$$X = Q \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} Q^{-1} = \frac{1}{2} \begin{pmatrix} \epsilon_1 + \epsilon_2 & i(\epsilon_1 - \epsilon_2) \\ -i(\epsilon_1 - \epsilon_2) & \epsilon_1 + \epsilon_2 \end{pmatrix}.$$

If we restrict to $gl(n, \mathbf{R})$, then X is a real matrix, and $\epsilon_1 = \bar{\epsilon_2}$. Now let $\mu_1 = Re(\epsilon_1)$ and $\mu_2 = Im(\epsilon_1)$, then $\mu_i \in \mathbf{R}$. It follows,

$$Z(A) = \{\begin{pmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix} : \mu_i \in \mathbf{R}\}. \tag{31}$$

Example 13 [Semi-simple matrices] Consider a diagonal Jordan normal form J

$$J = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where $\alpha \in \mathbf{C}$. As we see J has two Jordan blocks $J_1(\alpha) = J_2(\alpha) = \{\alpha\}$. Again by Theorem-13, it follows that the centralizer of J is given by

$$Z(J) = \{U \in gl(2, \mathbf{C}) : U = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix}, \epsilon_i \in \mathbf{C}\}. \tag{32}$$

This example can easily be generalized to the case where

$$J = diag\{\alpha, ..., \alpha\} \in gl(n, \mathbf{C}),$$

in which case

$$Z(J) = \{U \in gl(n, \mathbf{C}) : U = \{\epsilon_{ij}\}, \epsilon_{ij} \in \mathbf{C}\}, \tag{33}$$

implies that $cod(J) = n^2$.

Example 14 [The complex structure J] Consider the matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

with Jordan normal form

$$D = \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix}$$

Similar to Example-12 we here have $J = QDQ^{-1}$ with

$$Q = \begin{pmatrix} I_n & I_n \\ -iI_n & iI_n \end{pmatrix}.$$

In order to compute the centralizer $Z(D)$, one splits D in two blocks: $J_1(i) = iI_n$ and $J_2(-i) = -iI_n$. By Example-13 and Theorem-13 it directly follows that

$$Z(D) = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in gl(n, \mathbf{C})\}.$$

Similar to the analysis of Example-12, one shows that the centralizer of J in the space $gl(n, \mathbf{R})$ is given by

$$Z(J) = \{\begin{pmatrix} U_1 & -U_2 \\ U_2 & U_1 \end{pmatrix} : U_i \in gl(n, \mathbf{R})\}, \tag{34}$$

which implies that $cod(J) = dim(Z(J)) = 2n^2$ in $gl(2n, \mathbf{R})$.

3.4 Construction of linear universal unfoldings

3.4.1 Linear universal unfolding in $gl(n, \mathbf{C})$

A construction of a linear mini-unfolding of each $A \in gl(n, \mathbf{C})$ is provided by Theorem-7. By Theorem-10 and Proposition-12, it follows

Proposition 14 [Linear universal unfolding in $gl(n, \mathbf{C})$] *The family*

$$U(\epsilon) = A + \sum_{i}^{r} \epsilon_i Y_i \tag{35}$$

is a universal unfolding of $A \in gl(n, \mathbf{C})$, where $r = cod(A)$ and $\{Y_i\}$ is a basis of $Z(A^*)$.

Example 15 [Distinct eigenvalues] Consider the diagonal matrices A, U from Example-11. It is obvious that $A + U$ is a universal unfolding of A.

Example 16 [Non-simple eigenvalues] Let A be in Jordan normal form:

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \qquad Z(A^*) = \{\begin{pmatrix} \epsilon_1 & 0 \\ \epsilon_2 & \epsilon_1 \end{pmatrix} : \epsilon_i \in \mathbf{C}\}.$$

By the discussion above the following family of matrices is a universal unfolding:

$$U(\epsilon) = \begin{pmatrix} \alpha + \epsilon_1 & 1 \\ \epsilon_2 & \alpha + \epsilon_1 \end{pmatrix}.$$

Example 17 [Semi-simple matrices] Consider the semi-simple matrix J of Example-13. By this example one sees that the family

$$U(\epsilon) = \begin{pmatrix} \alpha + \epsilon_1 & \epsilon_2 \\ \epsilon_2 & \alpha + \epsilon_3 \end{pmatrix} \tag{36}$$

is a universal unfolding of J.

3.4.2 Linear universal unfoldings in $V \subset gl(n, \mathbf{C})$

We now return to setting where a certain structure has to be respected, for motivation see section 1.1 and the beginning of section 3. We consider a certain $\mathbf{R}$-linear subspace $V \in gl(n, \mathbf{C})$, that contains the matrix A preserving the structure at hand. We like to extend the construction of section 3.4.1, where neither the deformations nor the allowable transformations take us out of V. To this end we also introduce a Lie subgroup $G \subset GL(n, \mathbf{C})$, considering the restriction of the general action

$$\Psi|_{G \times V} : G \times V \to V, \text{ given by } (Q, A) \mapsto QAQ^{-1}.$$

Of course, the restriction $\Psi|_{G \times V}$ must be well-defined, which implies that some restrictions must be made for G and V.

Theorem 15 *In the above situation assume that*

P1. V is invariant under the action, i.e. $\Psi(G, V) \subset V$;

P2. for all $X, Y \in V$, $[X,Y] \subset T_{id}G$

P3. if $X \in V$ *then* $X^* \in V$.

Then, the linear family

$$U(\epsilon) = A + \sum_i^c \epsilon_i Y_i,$$

where $\{Y_i\}$ *is a basis of* $Z_V(A^*)$*, is a universal unfolding of* A *in* V.

Remarks.

1. The first condition ensures that the action is well-defined, which implies that all tangent spaces of orbits lie inside V.
2. Note that the the first two conditions of Theorem-15 are satisfied, when V is the Lie algebra of G. Hence, the theorem holds, whenever the Lie algebra of G in invariant under $*$, i.e., $G^* \subset G$. For instance, the classical Lie algebras $gl(n, \mathbf{C}), gl(n, \mathbf{R}), sl(2n, \mathbf{R})$, $so(2n, \mathbf{R})$ and $sp(2n, \mathbf{R})$.
3. In fact, the three conditions of Theorem-15 are necessary for constructing universal unfoldings using Gantmacher's construction. The condition (P1) is obviously required. We show that if one of the last two conditions is not satisfied, then the theorem fails.
   Consider the Lie group $G = \{id\}$ and $V = gl(n, \mathbf{R})$. It is not hard to see that G, V do not satisfy (P2), but it do satisfy (P1) and (P3). Since $O(A) = \{A\}$ and $T_A O(A) = \{0\}$ for all $A \in gl(n, \mathbf{R})$, it follows that V is the complement of $T_A O(A)$, which in general is not equal to $Z_V(A^*)$. Accordingly, Theorem-15 is not true.
   As a second counter-example, we consider the Lie group G and its Lie algebra V

   $$G = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R}\}, \quad V = \{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in \mathbf{R}\}.$$

   Obviously, in this case only the condition (P3) is not satisfied. By direct computation, it is not hard to verify that the theorem fails too in this case.

Example 18 **[Reversible case]** Let $R \in GL(n, \mathbf{R})$ be an involution, i.e., $R^2 = id$. Assume that R is diagonal. We consider the subspace

$$V = gl_{-R}(n, \mathbf{R}) = \{X \in gl(n, \mathbf{R}) : XR = -RX\},$$

and the Lie group

$$G = GL_R(n, \mathbf{R}) = \{X \in gl(n, \mathbf{R}) : XR = RX\}.$$

We show that for this choice of V and G the conditions of Theorem-15 hold.

P1. Let $Q \in G$ and $A \in V$. It is easy to show that $QAQ^{-1} \in V$, since $RQAQ^{-1} = -QARQ^{-1} = -QAQ^{-1}R$.

P2. Let $X, Y \in V$,then

$$R[X,Y] = RXY - RYX = XYR - YXR = [X,Y]R.$$

P3. Let $X \in V$, then $X^* \in V$.

We prove Theorem-15 by two lemma's, keeping the above setting.

Lemma 16 *Given $A \in V$, then the tangent space $T_AO(A)$ of the orbit $O(A)$ at A is given by*

$$T_AO(A) = \{[B,A] : B \in T_{id}G\}.$$

Proof. Define a curve $C(t) = e^{tB}$, $B \in T_{id}G$. When $|t|$ is sufficient small, $C(t) \in G$ and $C(t) = id + tB + o(t^2)$. Thus,

$$D_{id}\Psi_A(B) = [B,A].$$

It follows that $T_AO(A) = D_{id}\Psi_A(T_{id}G) = \{[B,A] : B \in T_{id}G\}$. □

Again, introduce the scalar product $\langle , \rangle : V \times V \to C$ on V, and

$$\langle X, Y \rangle = tr(XY^*),\ X, Y \in V.$$

Lemma 17 *Let $A \in V$ and $T_AO(A)^\perp$ be the orthogonal complement of $T_AO(A)$ with respect to the inner product $\langle , \rangle$. Then*

$$T_AO(A)^\perp = Z_V(A^*).$$

Proof. In the case when $A = 0$, $Z_V(A^*) = V$ and $T_AO(A) = \{0\}$. So in this trivial case the claim is true.
Now suppose that $A \neq 0$. It is easy to show that $Z_V(A^*) \subset T_AO(A)^\perp$.
We show the converse $T_AO(A)^\perp \subset Z_V(A^*)$. Let $X \in T_AO(A)^\perp$ and $[B,A] \in T_AO(A), B \in T_{id}G$. Then we have

$$\langle [B,A], X \rangle = tr([B,A]X^*) = tr([A,X^*]B) = 0,\ \forall B \in T_{id}G. \quad (37)$$

By the property (P3) and (P4) of V, one can choose $B = [A, X^*]$, and it follows that $[X, A^*] = [A, X^*] = 0$, i.e., $X \in Z_V(A^*)$.

□

Lemma-16 and Lemma-17 directly imply Theorem-15. We give a few examples of application of Theorem-15.

Example 19 [Simple Eigenvalues] Let $A \in gl(2n + m, \mathbf{R})$ $(n, m \geq 0)$ be a matrix with simple eigenvalues. In order to compute a universal unfolding of the matrix A, it is enough to look at the (real) Jordan normal form J_A of A. Since A is simple, J_A is given by a diagonal matrix of blocks: $J_A = \{T_1, ..., T_n, D_1, ..., D_m\}$, where $T_i \in gl(2, \mathbf{R})$ and $D_i \in \mathbf{R}$, and they respectively are given by:

$$T_i = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix},\ D_i = \{c_i\}, \qquad a_i, b_i, c_i \in \mathbf{R}.$$

Since the eigenvalues of A are simple, there is no need to consider the possibility that $T_i = T_j$ or $D_i = D_j$, when $i \neq j$. Note also that $b_i \neq 0$. Now by Example-11 and Example-12 one can easily see that a universal unfolding is given by:

$$U = J_A + diag\{T'_1, ..., T'_n, D'_1, ..., D'_m\}, \tag{38}$$

where

$$T'_i = \begin{pmatrix} \epsilon_i & -\mu_i \\ \mu_i & \epsilon_i \end{pmatrix}, \; D'_i = \alpha_i; \; \epsilon_i, \mu_i, \alpha_i \in \mathbf{R}.$$

Example 20 **[The Lie algebra $sl(n, \mathbf{R})$]** The Lie algebra $sl(n, \mathbf{R})$ is given by

$$sl(n, \mathbf{R}) = \{X \in gl(n, \mathbf{R}) : tr(X) = 0\}.$$

Let $A \in sl(2, \mathbf{R})$, and is given by

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \; \lambda \in \mathbf{R}.$$

In the case where $\lambda \neq 0$, then

$$U = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}, \epsilon \in \mathbf{R}.$$

is a universal family of A, and $cod(A) = 1$.
Now suppose $\lambda = 0$. In this case it turns out that $cod(A) = 3$. Indeed, a universal unfolding is given by

$$U = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & -\epsilon_1 \end{pmatrix}, \epsilon_i \in \mathbf{R}.$$

4 Symplectic Structure and Hamiltonian Systems

4.1 The Lie algebras induced by a Lie group

Recall that a Lie group is a smooth manifold with a smooth group structure. For instance, the general linear group $GL(n, \mathbf{R})$ is a Lie group. We give a formal definition of a Lie algebra.

Definition 18 **[Lie algebra]** *Let V be a vector space. A Lie algebra is a pair (V, L), where L is a bilinear and skew-symmetric operator which maps $V \times V$ into V and furthermore,*

$$L(x, L(y, z)) + L(z, L(x, y)) + L(y, (z, x)) = 0.$$ [1]

for all $x, y, z \in V$.

The mapping $L : V \times V \mapsto V$ is called a *Lie operator* of V. A well-known example of a Lie algebra is the matrix space $gl(n, \mathbf{R})$ with as Lie operator the commutator:

$$L(A, B) = AB - BA = [A, B].$$

[1] This is called the Jacobi identity.

Let $G \in GL(n, \mathbf{C})$ be a Lie group. We show that $T_{id}G$ is a Lie algebra, which is called the Lie algebra of G. We denote by Ψ_x the natural mapping of the action Ψ (see Sec. 2.5). Note that the tangent space $T_{id}G$ to G at the identity is a vector space. We take $V = T_{id}G$. It is easy to see that the differential $D_{id}\Psi_x$ of the natural mapping Ψ_x at the identity maps V into itself. Define $L : V \times V \to V$ by

$$(x, y) \mapsto D_{id}\Psi_x(y) = [y, x], \ \forall x, y \in V.$$

It follows that V equipped with L is a Lie algebra.

A linear subspace $V \subset gl(n, \mathbf{F})$ ($\mathbf{F} = \mathbf{R}, \mathbf{C}$) is said to be *Lie sub-algebra* of $gl(n, \mathbf{F})$ if V is a Lie algebra, i.e. if V is closed under the commutator $[,]$. The tangent space $(T_{id}G, [,])$ is therefore a Lie sub-algebra of $gl(n, \mathbf{F})$, if $G \subset GL(n, \mathbf{F})$ is a Lie group.

Example 21 [The general linear group $GL(n, \mathbf{R})$] Since $GL(n, \mathbf{R})$ is a Lie group and $T_{id}G = gl(n, \mathbf{R})$, the matrix space $gl(n, \mathbf{R})$ of all real $(n \times n)$ matrices with the commutator bracket is the Lie algebra of $GL(n, \mathbf{R})$.

Example 22 [The special linear group $SL(n, \mathbf{R})$] The Lie group $SL(n, \mathbf{R})$ is given by:

$$SL(n, \mathbf{R}) = \{A \in gl(n, \mathbf{R}) : \det(A) = 1\}.$$

Let $sl(n, \mathbf{R})$ be the Lie algebra of G, i.e. $sl(n, \mathbf{R}) = T_{id}G$. Consider a curve $C : \mathbf{R} \mapsto G$ by $t \mapsto e^{tX}$ in the space G. Clearly, this curve goes through the identity and furthermore

$$e^{tTr(X)} = \det(e^{tX}) = 1.$$

It follows that $Tr(X) = 0$. The converse can be easily shown. Hence,

$$sl(n, \mathbf{R}) = \{X \in gl(n, \mathbf{R}) : \mathrm{Tr}(X) = 0\}.$$

Example 23 [The special orthogonal group $SO(n, \mathbf{R})$] The group $SO(n, \mathbf{R})$ is the set of all real orthogonal matrices with the determinant is 1. Observe that $SO(n, \mathbf{R})$ is a smooth manifold with smooth group structure, and hence a Lie group. So the tangent space $T_{id}SO(n, \mathbf{R})$ is the natural Lie algebra in which the Lie operator is defined by the bracket $[,]$.
Similar to the previous example, one can show that the Lie algebra $so(n, R)$ of $SO(n, \mathbf{R})$ is given by:

$$so(n, R) = \{X \in gl(n, \mathbf{R}) : X = -X^T\}. \tag{39}$$

4.2 Symplectic structure on a manifold

4.2.1 Symplectic vector space

Consider a vector space V of dimension $2n$, and a 2-form ω on V.

Definition 19 *The pair (V, ω) is said to be a symplectic vector space if the 2-form ω is closed and non-degenerate i.e. if $d\omega = 0$ and*

$$\omega(x, y) = 0, \ \forall y \in V \Leftrightarrow x = 0.$$

This 2-form is called a *symplectic structure on* V.

Remark. The dimension of V is set to be even due to the fact that the symplectic structure ω is skew symmetric.

Recall that the dual space V^* of V is the vector space of all linear mappings from V into $\mathbf{R}$. For each element $x \in V$ one can define the mapping

$$\omega_x : y \in V \mapsto \omega(x, y) \in \mathrm{R} \tag{40}$$

Clearly ω_x is an element of V^*. Define the isomorphism $\omega^* : V \mapsto V^*$ by

$$\omega^* : x \mapsto \omega_x \tag{41}$$

Since ω is non-degenerate, it easily follows that $ker(\omega^*) = \{0\}$, therefore ω^* is an isomorphism.

Now let the matrix $(A_{i,j}) = (\omega(p_j, p_i))$ be the matrix representation of ω in the ordered basis $\mathbf{p} = (p_1, ..., p_{2n})$. A minor exercise shows that ω is non-degenerate if and only if A is non-singular. Furthermore, we have $rank(A) = rank(\omega^*)$.

Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be coordinates of the points x, y with respect to the local coordinate system $\mathbf{p}$. Then the symplectic structure is given by:

$$\omega(x, y) = \sum_{i,j=1}^{2n} -A_{ij} x_i y_j,$$

where A is the matrix representation of ω in the system $\mathbf{p}$. By elementary linear algebra there exists a coordinate system relative to which the nonsingular and skew-symmetric matrix A can be reduced to the form:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{42}$$

with I_n the identity matrix of order n. The symplectic structure ω on V corresponding to J is given by

$$\omega(x, y) = \sum_{i=1}^{2n} x_i y_{n+i} - x_{n+i} y_i = \langle Jx, y \rangle, \forall x, y \in V, \tag{43}$$

i.e, $\omega = \sum_{i=1}^{2n} dp_i \wedge dp_{n+i}$. We call this ω *the standard symplectic structure* on V in the coordinate system $\mathbf{p}$. The system $\mathbf{p}$ is a *canonical system* on V.

The standard symplectic form on the space $\mathbf{R^{2n}}$ is one of our main objects in this section and therefore we study it more extensively. Let ω be the standard symplectic structure on $\mathbf{R^{2n}}$ and define

$$\omega^n = \omega \wedge \omega ... \wedge \omega \ (n \text{ times }).$$

It is easy to see that ω^n is a $2n$-form on $\mathbf{R^{2n}}$. Thus ω^n must be given by

$$\omega^n = k dp_1 \wedge dp_2 \wedge ... \wedge dp_{2n} \tag{44}$$

where k is a constant. Since the exterior product $dp_1 \wedge dp_2 \wedge ... \wedge dp_{2n}$ defines a volume element on $\mathbf{R^{2n}}$, so does the $2n$-form ω^n.

Remark. The constant k in (44) can be explicitely determined. See also Abraham and Marsden [Abr]. However, we don't need it in this paper.

Definition 20 [Linear symplectic transformation] *Consider the symplectic vector spaces $(V_i, \omega_i), (i = 1, 2)$. A linear transformation $f : V_1 \mapsto V_2$ is said to be symplectic if*

$$\omega_2(f(x_1), f(x_2)) = \omega_1(x_1, x_2), \tag{45}$$

for all $x_1, x_2 \in V_1$.

Remark. Recall that the *pull-back* f^* of $f : V_1 \mapsto V_2$ which is a mapping that takes a k-form λ on V_2 to a k-form $f^*\lambda$ on V_1 by the following relationship:

$$f^*\lambda(x_1, ..., x_k) = \lambda(f(x_1), ..., f(x_k)), x_i \in V_1.$$

Let λ be a p-form and γ be a q-form on V_2. Then, it is not hard to show that

$$f^*(\lambda \wedge \gamma) = (f^*\lambda) \wedge (f^*\gamma). \tag{46}$$

Using this identity, one easily shows that a symplectic transformation is volume-preserving.

We return to the standard symplectic structure ω on $\mathbf{R^{2n}}$. Let $f : \mathbf{R^{2n}} \mapsto \mathbf{R^{2n}}$ be a linear symplectic mapping, i.e. $f^*\omega = \omega$. By 46 one directly has $f^*\omega^n = \omega^n$, i.e. f is volume-preserving.

Theorem 21 [Volume-preserving on R^{2n}] *Each symplectic transformation $f : \mathbf{R^{2n}} \mapsto \mathbf{R^{2n}}$ is volume-preserving.*

Remark. A more general version of this claim is the following:

Theorem 22 *Consider the symplectic vector spaces $(V_i, \omega_i), (i = 1, 2)$ and a linear symplectic transformation $f : V_1 \mapsto V_2$. Then f is volume-preserving.*

4.3 The Lie algebra of the symplectic group

As mentioned before we restrict to the case of the symplectic vector space $M = \mathbf{R^{2n}}$ with the standard symplectic structure $\omega(x, y) = \langle Jx, y\rangle$ on it. By the definition of symplectic transformation and Theorem-21 we have:

Corollary 23 [Symplectic matrix] *A matrix $A \in gl(2n, \mathbf{R})$ is symplectic if and only if*

$$A^t JA = J,$$

where J is given by (42). Moreover, $det(A) = 1$.

We denote the set of all symplectic matrices of order n by $Sp(2n, \mathbf{R})$. Clearly, $Sp(2n, \mathbf{R}) \subset GL(2n, \mathbf{R})$. An easy calculation shows that if $A, B \in Sp(2n, \mathbf{R})$, then the inverse A^{-1} and the product AB are symplectic. It follows that $Sp(2n, \mathbf{R})$ is a group with a smooth group structure. Furthermore, the space $Sp(2n, \mathbf{R})$ is a sub-manifold of $GL(2n, \mathbf{R})$. Hence, $Sp(2n, \mathbf{R})$ is Lie group.

Let $g = sp(2n, \mathbf{R})$ be the Lie algebra of $G = Sp(2n, \mathbf{R})$. By introducing the curve $C(t) = e^{tX}$, $X \in g$ in G, one easily shows that

$$X^T J = -JX. \tag{47}$$

Since $J^T = -J$, one has $X^T J = -(JX)^T$. Thus, by (47) the matrix JX is symmetric. It is also not hard to show that the converse is true. Summarizing, we state (cf. [J]):

Theorem 24 [The infinitesimally symplectic Lie algebra] *The Lie algebra of the Lie group $Sp(2n, \mathbf{R})$ is given by:*

$$sp(2n, \mathbf{R}) = \{X \in gl(2n, \mathbf{R}) : JX = (JX)^T\} \tag{48}$$

and the matrix X can be explicitly given by the block matrix:

$$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \tag{49}$$

where $A, B, C \in gl(n, \mathbf{R})$ and B, C are symmetric matrices.

Proof. Let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A, B, C, D \in gl(n, \mathbf{R})$. A direct computation of JX and $(JX)^T$ implies (49). □

Consider the system

$$\dot{x} = Ax, \; x \in \mathbf{R}^{2n}, \tag{50}$$

where $A \in gl(2n, \mathbf{R})$. Let (50) be Hamiltonian, i.e. there is a Hamiltonian function H such that $A = -J\nabla H$. Recall that ∇H, the Hessian of H, is a symmetric matrix. It follows that $JA = \nabla H$ is symmetric. By Theorem-24, $A \in sp(2n, \mathbf{R})$. Conversely, suppose that $A \in sp(2n, \mathbf{R})$. It is not hard to show that in this case (50) is a linear Hamiltonian system. See also [CLW].

Theorem 25 [Linear Hamiltonian systems versus $sp(2n, \mathbf{R})$] *The linear system $\dot{x} = Ax$ with $x \in \mathbf{R}^{2n}$ and $A \in gl(2n, \mathbf{R})$ is Hamiltonian if and only if $A \in sp(2n, \mathbf{R})$.*

4.4 Universal unfoldings of elements of $sp(2n, \mathbf{R})$

We try to construct a universal unfolding of an arbitrary element of the Lie algebra $sp(2n, \mathbf{R})$. Suppose that $S \in sp(2n, \mathbf{R})$, i.e. by Theorem-24,

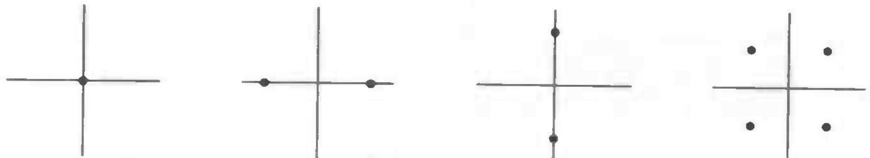
$$S^T = JSJ.$$

Let $\lambda \in \mathbf{C}$ a eigenvalue of S. Note that $det(J) = 1$, hence one has

$$\det(\lambda - S) = \det(J(\lambda - S)J) = \det(-\lambda - S^T) = \det(-\lambda - S).$$

Consequently, $-\lambda$ is a eigenvalue of S too. Note that the characteristic polynomial of S is real, so also $\bar{\lambda}$ and $-\bar{\lambda}$ belong to the spectrum of S.

Theorem 26 [Distribution of eigenvalues] *If $\lambda \in C$ an eigenvalue of the matrix $S \in sp(2n, R)$, then $-\lambda, \bar{\lambda}$ and $-\bar{\lambda}$ are also eigenvalues (See Figure-12).*



Type T^0 Type T^1 Type T^2 Type T^3

Figure 12: Eigenvalue distributions of the four different blocks.

Because of Theorem-26, a real normal form of $S \in sp(2n, \mathbf{R})$ can be given by:

$$J_S = diag\{T_1^1, ..., T_s^1, T_1^2, ..., T_t^2, T_1^3, ..., T_p^3\}.^2 \tag{51}$$

where the different types of blocks are given by:

$$T^1 = \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix}; \quad T^2 = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}; \quad T^3 = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & -a & -b \\ 0 & 0 & b & -a \end{pmatrix},$$

where a, b, c, d are non-zero real numbers. The zero-type T^0 is the (2×2) zero matrix. We are already familiar with types T^0, T^1, T^2. Indeed, by Example-15 and Example-19 we already know how to construct a universal unfolding of those types. The type T^3 can be reduced to a diagonal matrix in $gl(4, \mathbf{C})$ with four distinct eigenvalues. So Example-15 is again applicable in this case.

Let us take a close look at the case when $n = 2$. Since the eigenvalues of $A \in sp(2n, \mathbf{R})$ come in 4-tuples. we then have only seven *types* of real Jordan normal forms, namely

$$T^0T^0, T^0T^1, T^0T^2, T^1T^1, T^1T^2, T^2T^2 \text{ and } T^3.$$

By *type* T^iT^j we mean the distribution of eigenvalues in the complex plane of the matrix $J_S = \text{diag}\{T^i, T^j\}$.

Next we examine universal families (unfoldings) of some types for $n = 2$ and obtain insight in the question how (deformation) parameters affect types. In the next examples we denote the centralizer of A in $sp(2n, \mathbf{R})$ by $Z_{sp}(A)$. Furthermore, by $cod(A)$ in the present context we only mean the codimension of the orbit $O(A)$ in $sp(2n, \mathbf{R})$, except when explicitly stated otherwise.

Before proceeding, we make two observations, namely

Remarks.

1. Recall that the centralizer of $A \in sp(2n, \mathbf{R})$ can be obtained by the relation:

$$Z_{sp}(A) = Z(A) \cap sp(2n, \mathbf{R}). \tag{52}$$

[2]Different superscript means different type of block.

2. Consider $A \in sp(2n, \mathbf{R})$, and let the diagonal matrix J given by

$$J = diag\{\lambda_1, -\lambda_1, \ldots, \lambda_n, -\lambda_n\}$$

be a normal form of A^T, where $\lambda_i \neq 0$. We know that a diagonal matrix $D = D(\mu)$ depending on parameters μ is a universal unfolding of J in $gl(2n, \mathbf{C})$. Thus, the family

$$U = A + QDQ^{-1}$$

is a universal unfolding of A in the same space. The eigenvalues of U will be close to the eigenvalues of J, when the values of parameters are sufficient small. Hence, the type of A will remain the same after adding a small deformation. From this point of view, universal unfoldings of the types T^3 and T^1T^2 remain of the same type, when the parameters are small.

Example 24 [Types T^0T^0, T^0T^1 and T^1T^1] Consider a matrix A given by

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}, \quad a, b \in \mathbf{R}. \tag{53}$$

Note that each of the types T^0T^0, T^0T^1 and T^1T^1 can be obtained from A by choosing suitable values for a and b.

- **(Type T^0T^0** : $a = b = 0$) In this case $A = \{0\}$. Since every $B \in gl(4, \mathbf{R})$ commutes with the zero matrix it follows that $Z_{gl}(O) = gl(4, \mathbf{R})$ (cf Example-13). Hence, by (52) one has

$$Z_{sp}(O) = sp(4, \mathbf{R}).$$

Therefore $cod(O) = dim(sp(4, \mathbf{R})) = 10$ and each unfolding

$$U_{00} : (\mathbf{R}^{10}, 0) \mapsto (sp(4, \mathbf{R}), 0)$$

with maximal rank is a universal unfolding of the zero matrix. In this case, all seven types can occur by changing parameters , since $U(\epsilon)$ can belong to each type.

- **(Type T^0T^1** : $a = 0$ and $b \neq 0$) The matrix A has the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}, \quad b \neq 0. \tag{54}$$

A universal unfolding is given by

$$U_{01} = A + \begin{pmatrix} \mu_1 & 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 & 0 \\ \mu_4 & 0 & -\mu_1 & 0 \\ 0 & 0 & 0 & -\mu_3 \end{pmatrix}, \quad \mu_i \in \mathbf{R}, \tag{55}$$

which implies that $cod(A) = 4$, and the eigenvalues are

$$\pm(\mu_3 + b), \pm\sqrt{\mu_1^2 + \mu_4\mu_2}.$$

The possible types of U_{01} are sketched in Figure-15.

$y > 0,\ T^1T^1$ $\qquad$ $y = 0,\ T^0T^1$ $\qquad$ $y < 0,\ T^1T^2$

Figure 13: Possible types of U_{01} according to $y = \sqrt{\mu_1^2 + \mu_4\mu_2}$.

- **(Type T^1T^1 : $a \neq 0$ and $b \neq 0$)** First consider the case where $a \neq b$, and A then has four distinct non-zero eigenvalues. By Remark above, this is not a interesting case.
  We study the case where $a = b$, A is in the form:

$$A = diag\{a, a, -a, -a\},$$

where $a \neq 0$. The centralizer of $A^T = A$ in $gl(2n, \mathbf{R})$ is given by (cf. Example-13):

$$Z(A) = \{\begin{pmatrix} \mu_1 & \mu_2 & 0 & 0 \\ \mu_3 & \mu_4 & 0 & 0 \\ 0 & 0 & \mu_5 & \mu_6 \\ 0 & 0 & \mu_7 & \mu_8 \end{pmatrix} : \mu_i \in \mathbf{R}\}. \tag{56}$$

Together with (52) it follows that,

$$Z_{sp}(A) = \begin{pmatrix} \mu_1 & \mu_2 & 0 & 0 \\ \mu_3 & \mu_4 & 0 & 0 \\ 0 & 0 & -\mu_1 & -\mu_3 \\ 0 & 0 & -\mu_2 & -\mu_4 \end{pmatrix}. \tag{57}$$

It follows directly that $cod(A) = 4$ and the family

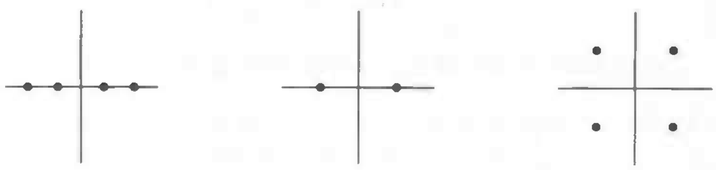
$$U_{11} = \begin{pmatrix} a + \mu_1 & \mu_2 & 0 & 0 \\ \mu_3 & a + \mu_4 & 0 & 0 \\ 0 & 0 & -a - \mu_1 & -\mu_3 \\ 0 & 0 & -\mu_2 & -a - \mu_4 \end{pmatrix}. \tag{58}$$

is a universal unfolding of A.

We assume that the parameters μ_i are small compared to the constant a. The four eigenvalues of U_{11} are given by

$$\pm x \pm \frac{1}{2}\sqrt{y},$$

with $x = a + \frac{1}{2}(\mu_1 + \mu_4)$ and $y = (\mu_1 - \mu_4)^2 + 4\mu_2\mu_3$. There are three possible situations sketched in the picture below.



$y > 0,\ T^1T^1$ $\quad$ $y = 0,\ T^1T^1$ $\quad$ $y < 0,\ T^3$

Figure 14: Possible types of the universal unfolding U_{11}.

Example 25 **[Type T^0T^2]** Let us consider the matrix A of the form

$$A = \begin{pmatrix} 0 & a & a & 0 \\ -a & 0 & 0 & -a \\ a & 0 & 0 & a \\ 0 & a & -a & 0 \end{pmatrix}, \ a \in \mathbf{R}.$$

Obviously, one can take $a = 1$. The (complex) Jordan form J of A then is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \ a, b \in \mathbf{R}.$$

This case is very similar to the type T^0T^1. Compare Figure-15.

$y > 0,\ T^1T^2$ $\quad$ $y = 0,\ T^0T^2$ $\quad$ $y < 0,\ T^2T^2$

Figure 15: Possible types of U_{02} according to $y = \sqrt{\mu_1^2 + \mu_4\mu_2}$.

Example 26 **[Type T^2T^2]** Here we deal with a real matrix given by

$$A = \begin{pmatrix} 0 & -b & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{pmatrix}, (b \neq 0) \in \mathbf{R}. \tag{59}$$

With a little help of the computer program Maple, it can be checked that $A^T = QDQ^{-1}$ with

$$Q = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ i/2 & -i/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ i/2 & -i/2 & i/2 & -i/2 \end{pmatrix}, \text{ and } D = diag\{ia, -ia, ia, -ia\}.$$

Now the centralizer $Z(D)$ can be determined and is given by

$$Z(D) = \{\begin{pmatrix} s_1 & 0 & s_2 & 0 \\ 0 & s_3 & 0 & s_4 \\ s_5 & 0 & s_6 & 0 \\ 0 & s_7 & 0 & s_8 \end{pmatrix}, s_i \in \mathbf{C}\}$$

By Theorem-?? one quickly finds $Z(A^T)$. Finally, using (52) and doing some calculations by computer one finds that

$$Z_{sp}(A^T) = \{\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & 0 \\ -\mu_2 & \mu_1 & 0 & \mu_3 \\ \mu_4 & 0 & -\mu_1 & \mu_2 \\ 0 & \mu_4 & -\mu_2 & -\mu_1 \end{pmatrix}, \mu_i \in \mathbf{R}\}. \tag{60}$$

Therefore, $cod(A) = 4$ and the family

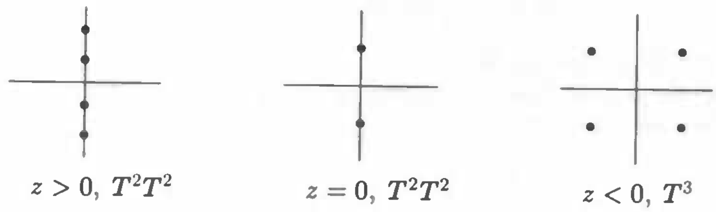
$$U_{22} = A + Y, \; Y \in Z_{sp}(A^T)$$

is a linear universal unfolding.

Define $z = \mu_1^2 + \mu_3\mu_4$ and $x = z - (\mu_2 - b)^2$. Then, the eigenvalues of U_{22} have the form:

$$\pm\sqrt{x \pm 2\sqrt{-(\mu_2 - b)^2 z}}$$

If the parameters μ_i are small, then $(\mu_2 - b)^2 > 0$ and x is real negative. It is not hard to see that the sign of z determines the distribution of the eigenvalues. See Figure-16. We will come back to such a case again, when studying the linear Hamiltonian in the resonance $1 : -1$. In that case one will find a similar diagram to Figure-16.



$z > 0, \; T^2T^2$ $\qquad$ $z = 0, \; T^2T^2$ $\qquad$ $z < 0, \; T^3$

Figure 16: Types of U_{22}, according to the sign of z.

Example 27 [Universal unfolding of J^T] Consider the matrix J defined by (42). It is easy to see that $J^T = -J$, and $J^T \in sp(2n, \mathbf{R})$. For future purposes we are interested in constructing $Z_{sp}(J)$. Recall that the centralizer of $Z(J)$ in $gl(n, \mathbf{R})$ is known already by (34) in Example-14. By (52) one has

$$Z_{sp}(J) = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A^T = -A, B^T = B\}, \tag{61}$$

where $A, B \in gl(n, \mathbf{R})$. The dimension of $Z_{sp}(J)$ can be easily determined, since

$$dim(Z_{sp}(J)) = dim(A) + dim(B) = (\frac{n(n+1)}{2} - n) + \frac{n(n+1)}{2} = n^2.$$

Recall that $dim(A)$ is the number of independent parameters in A, similarly for $dim(B)$. It follows directly that $cod(J) = dim(Z_{sp}(J)) = n^2$.

Theorem 27 [Universal Unfolding of the matrix J^T] *A universal unfolding of the matrix J in the Lie algebra $sp(2n, \mathbf{R})$ is given by:*

$$U_J = \begin{pmatrix} A & I_n + B \\ -I_n - B & A \end{pmatrix}, \tag{62}$$

where $A, B \in gl(n, \mathbf{R})$ are the parameters matrices and $A = -A^T, B = B^T$. Moreover, in the space $sp(2n, \mathbf{R})$ one has $cod(J^T) = n^2$.

Remark. In this example, one can obtain the centralizer of J in the space $sp(2n, \mathbf{R})$ by a direct computation (see [J]), since if $X \in Z_{sp}(J)$ then $XJ = JX$. This equation can easily be solved.

5 Application to linear Hamiltonian in strong resonance

We continue with linear Hamiltonian systems defined on the symplectic vector space $\mathbf{R}^{2n}$ equipped with the standard symplectic structure

$$\omega = \sum_{i=1}^{2n} dp_i \wedge dp_{n+i}.$$

In a coordinate system (p, q) in $\mathbf{R}^{2n}$ a quadratic Hamiltonian function is given by

$$H(x) = \frac{1}{2}\langle x, A_H x\rangle, x = (p, q)^T$$

where $A_H = \nabla H \in sym(2n, \mathbf{R})$.
Let X_H be the vector field corresponding to H and then $X_H = -J\nabla H = -JA_H$. The canonical system defined by H then is given by

$$\dot{x} = X_H(x) = (-JA_H)x. \tag{63}$$

Let us take

$$A_H = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix},$$

where $\Lambda = diag\{\lambda_1, ..., \lambda_n\}, \lambda_i \in \mathbf{R}$. It follows that

$$H(p, q) = \frac{1}{2}\sum_{i=1}^{n} \lambda_i (p_i^2 + q_i^2),\ \lambda_i \in \mathbf{R} \tag{64}$$

and the linear vector field

$$X_H = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}. \tag{65}$$

Obviously, the eigenvalues of X_H are given by $\{\pm i\lambda_1, \ldots, \pm i\lambda_n\}$.
In the above setting, the system (63) is given by

$$\begin{cases} \dot{p}_i = \lambda_i q_i \\ \dot{q}_i = -\lambda_i p_i \end{cases}. \tag{66}$$

This is a system of n un-coupled linear oscillations. The numbers λ_i at the diagonal of Λ are called the *frequencies* of the system. The set of frequencies coincides with the spectrum of the vector field X_H. The system is in resonance whenever there is a relation of the form

$$\sum_{i=1}^{n} k_i \lambda_i = 0, \tag{67}$$

where $k_i \in \mathbf{Z}$ and $(k_1, ..., k_n) \neq 0$. For instance, in the case when $n = 2$, the resonance relation is given by $k_1\lambda_1 + k_2\lambda_2 = 0$, and we say that a Hamiltonian system (66) is in $k_1 : -k_2$ resonance. A system is said to be in strong resonance, when k_1, k_2 are small integers. The strongest resonances occur for $k_{1,2} = \pm 1$.

In the following subsections we handle Hamiltonian systems in the strongest resonances and universal families (unfoldings) of such systems. One main concern is to find the regions in the parameter space in which the obtained universal families are in the strongest resonance, i.e., where $k_{1,2} = \pm 1$.

5.1 The $1:1$ resonance

The Hamiltonian function H is said to be in *the* $1 : \cdots : 1$ *resonance*, if the λ_i's in (64) are all equal. For simplicity, we set all $\lambda_i = 1$, i.e., $\lambda = I_n$. Such a Hamiltonian has the form

$$H(p,q) = \tfrac{1}{2} < \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} > . \tag{68}$$

By (65) the linear vector field X_H equals $-J$, where J is given by (42). In order to construct a universal unfolding of X_H, we compute the centralizer $Z_{sp}(X_H^*) = Z_{sp}(X_H^T)$ of the vector field X_H. But this we have already done, since $Z_{sp}(X_H^*) = Z_{sp}(J)$ (see Theorem-27). Hence, the family U_J of (62) is a linear universal unfolding of the field $X_H = -J = J^T$ and the corresponding linear Hamiltonian H_1 to U_J is given by

$$H_1 = \frac{1}{2} < \begin{pmatrix} p \\ q \end{pmatrix}, JU_J \begin{pmatrix} p \\ q \end{pmatrix} > .$$

By Theorem-27, $cod(X_H) = n^2$. So, we have proved the following theorem (cf. [J]).

Theorem 28 [Universal unfolding of the $1 : \cdots : 1$ resonance] *Consider a Hamiltonian function H in $1 : \cdots : 1$ resonance, and let $X_H = -J$ be the vector field corresponding to H. Then the family (62) is a universal unfolding of X_H. Therefore the corresponding quadratic Hamiltonian H_1 has the form:*

$$H_1 = \tfrac{1}{2} < \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} I_n + B & -A \\ A & I_n + B \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} >, \tag{69}$$

where $A, B \in gl(n, \mathbf{R})$ and $A = -A^T$, $B = B^T$. Furthermore, $cod(X_H) = n^2$.

Let us consider the case when $n = 2$. In this case one may take

$$A = \begin{pmatrix} 0 & \mu_1 \\ -\mu_1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} -1 + \epsilon + \mu_2 & \mu_3 \\ \mu_3 & -1 + \epsilon - \mu_2 \end{pmatrix}, \tag{70}$$

where μ_i is a small real number close to 0, and ϵ is close to 1, since the sum $-1+\epsilon+\mu_2$ is close to 0. The vector field X_H is then given by

$$X_H = \begin{pmatrix} 0 & \mu_1 & \epsilon+\mu_2 & \mu_3 \\ -\mu_1 & 0 & \mu_3 & \epsilon-\mu_2 \\ -\epsilon-\mu_2 & -\mu_3 & 0 & \mu_1 \\ -\mu_3 & -\epsilon+\mu_2 & -\mu_1 & 0 \end{pmatrix}. \tag{71}$$

With a computer program (for instance Maple) one can find the eigenvalues of X_H to be

$$\pm i(\epsilon \pm r),$$

where $r^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$.

Observe that r is a small positive number since μ_i are small nonzero numbers and ϵ is positive, since it is close to 1. It turns out that we have only one possibility concerning the distribution of the eigenvalues in the complex plane (see Figure-19):

- ($r \neq 0$) the eigenvalues of X_H are $\pm i(\epsilon + r)$, $\pm i(\epsilon - r)$;
- ($r = 0$) X_H has two pairs of double eigenvalues, namely $\pm i\epsilon$.

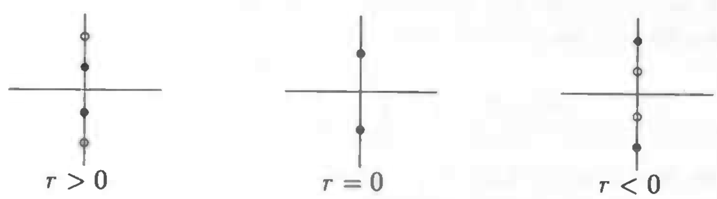


Figure 17: The eigenvalue distribution of the vector field of a Hamiltonian in 1 : 1 resonance.

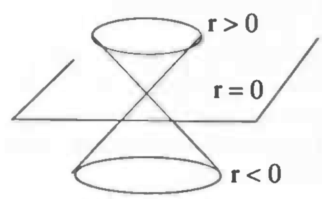


Figure 18: The graph $r^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$ in the 4-dimensional parameter spaces (μ_1, μ_2, μ_3, r).

Observe that the eigenvalues of the universal unfolding X_H remain on the imaginary axis. In fact, we even have

Theorem 29 *Consider a Hamiltonian vector field X in the form:*

$$X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{72}$$

where $A, B \in gl(n, \mathbf{R})$ and $A = -A^T, B = B^T$. Then X has only purely imaginary or zero eigenvalues.

Proof. Note that JX is symmetric. It follows that JX has only real eigenvalues. Let $\lambda \in \mathbf{R}$ be such an eigenvalue and v be the eigenvector of λ, i.e. $JXv = \lambda v$. From this it follows that

$$X^2 v = -\lambda JXJv = -\lambda JXv = -\lambda^2 v.$$

□

Since the universal unfolding X_H has the form of (72), Theorem-29 explains the purely imaginary eigenvalue distribution. Note that the corresponding Hamiltonian function H always has a minimum at $(p,q) = 0$. Also note that application of Theorem-29 yields the eigenvalue distribution of (62) without explicit computation.
As an example of the latter, consider the case $n = 3$, where the universal unfolding U has three pairs of eigenvalues, say, $\pm\lambda_1, \pm\lambda_2$ and $\pm\lambda_3$. According to Theorem-29, these eigenvalues are all on the imaginary axis, which leaves the following possibilities:

(1). All three eigenvalues are distinct, i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3$;

(2). Only two of these eigenvalues are equal;

(3). The three pairs coincide.

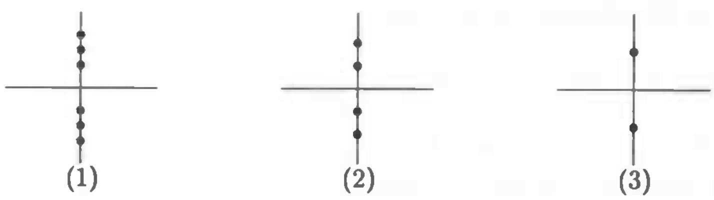


Figure 19: The eigenvalue distribution for $n = 3$.

5.2 The resonance $1 : -1$, semi-simple case

Consider a Hamiltonian function

$$H_{1:-1}(p,q) = \alpha(p_1^2 + q_1^2) - \alpha(p_2^2 + q_2^2), \ (\alpha \neq 0) \in \mathbf{R}$$

corresponding to the linear vector field

$$X_{1:-1} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}.$$

Then, by definition $H_{1:-1}$ is a linear Hamiltonian in $1 : -1$ resonance. Without loss of generality we may take $\alpha = 1$. It is easy to see that the (complex) Jordan normal form of $X_{1:-1}$ is

$$J_X := diag\{i, -i, i, -i\}.$$

Hence, $X_{1:-1}$ belongs to type T^2T^2 (see Example-26). However, here the centralizer $Z_{sp}(X_{1:-1})$ is a bit different from the family (60), since the transformation matrix in this case is different. After some calculations it follows that the centralizer given by

$$Z_{sp}(X^T_{1:-1}) = \{\begin{pmatrix} 0 & x_1 & x_3 & x_2 \\ x_1 & 0 & x_2 & x_4 \\ -x_3 & x_2 & 0 & -x_1 \\ x_2 & -x_4 & -x_1 & 0 \end{pmatrix}, x_i \in \mathbf{R}\}. \tag{73}$$

So $cod(X_{1:-1}) = 4$ and a universal unfolding of $X_{1:-1}$ is given by

$$U_X = X_{1:-1} + B,\ B \in Z_{sp}(X^T_{1:-1}).$$

Theorem 30 [Universal unfolding of $X_{1:-1}$] *Let $X_{1:-1}$ be the linear vector field defined above and $H_{1:-1}$ the linear Hamiltonian function corresponding to $X_{1:-1}$. Then the family*

$$U = \begin{pmatrix} 0 & \mu_1 & \mu_3+1 & \mu_2 \\ \mu_1 & 0 & \mu_2 & \mu_4-1 \\ -\mu_3-1 & \mu_2 & 0 & -\mu_1 \\ \mu_2 & -\mu_4+1 & -\mu_1 & 0 \end{pmatrix}, \mu_i \in \mathbf{R} \tag{74}$$

is a universal unfolding of $X_{1:-1}$ and hence $cod(X_{1:-1}) = 4$. The corresponding family of Hamiltonians is given by:

$$H_U(p,q) = \frac{\mu_3'}{2}(p_1^2+q_1^2) + \frac{\mu_4'}{2}(p_2^2+q_2^2) + \mu_1(p_1q_2+p_2q_1) + \mu_2(q_1q_2 - p_1p_2),$$

where $\mu_3' = \mu_3 + 1, \mu_4' = \mu_4 - 1$.

The eigenvalues of the universal family U are given by

$$\pm\sqrt{\alpha+\sqrt{\beta}},\ \pm\sqrt{\alpha-\sqrt{\beta}},$$

where $\alpha = (\mu_2^2+\mu_1^2) - \frac{1}{2}(\mu_3'^2+\mu_4'^2)$ and $\beta = (\mu_3'-\mu_4')^2(\frac{1}{4}(\mu_3'+\mu_4')^2 - (\mu_1^2+\mu_2^2))$. Let us analyze the eigenvalues more closely. Notice that the parameters μ_i ($i = 1,..,4$) all are small real values. This means that α is real negative and close to -1 and the number $(\mu_3'-\mu_4')^2$ is always real positive, since μ_3', μ_4' are close to the numbers $1, -1$ respectively. Another observation is that β is close to zero. As we shall see, the sign of β will play a crucial role in this case. Since $(\mu_3'-\mu_4')^2$ is positive, the sign of β is entirely determined by the sign of

$$z = (\mu_3+\mu_4)^2 - 4(\mu_1^2+\mu_2^2). \tag{75}$$

There are there possible cases concerning the eigenvalues of the family U:

- ($z > 0$) $x = \alpha \pm \sqrt{\beta}$ is real negative and hence $\sqrt{x}$ is purely imaginary. It follows that U has four distinct eigenvalues on the imaginary axis;
- ($z = 0$) here U has two distinct eigenvalues $\pm i\sqrt{|\alpha|}$;
- ($z < 0$) in this case $x = \alpha \pm i\sqrt{|\beta|}$ is complex with non-zero real part and therefore $\sqrt{x}$ is complex with (non-zero) real part.

From the figure below one sees that the deformed Hamiltonian system is in $1:-1$ resonance whenever $z = 0$.

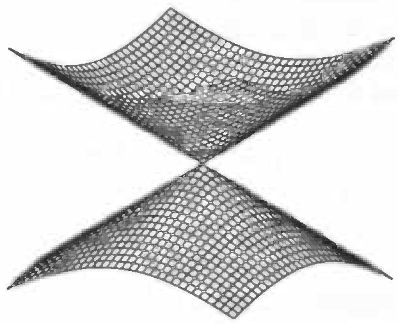


Figure 20: The graph $z = 0$ in the parameter space $(\mu_1, \mu_2, \mu_3 + \mu_4)$.

Figure 21: Distribution of eigenvalues according to the sign of z.

5.3 The 1 : −1 resonance, generic case

In this section we study the linear vector field

$$X^*_{1:-1} = \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ \epsilon & 0 & 0 & -a \\ 0 & \epsilon & a & 0 \end{pmatrix}, \epsilon, a \in \mathbf{R} \setminus \{0\}. \tag{76}$$

Let H be the quadratic Hamiltonian function corresponding to (76), then

$$H(p,q) = a(p_1 q_2 - p_2 q_1) - \frac{\epsilon}{2}(p_1^2 + p_2^2) \tag{77}$$

The block ϵI_2 in (76) is the nilpotent part of $X^*_{1:-1}$. It is easy to see that the eigenvalues of $X^*_{1:-1}$ are $\pm ia$, $\pm ia$. Therefore, the Hamiltonian H is in $1 : -1$ resonance.

In the case $\epsilon = 0$, where $X^*_{1:-1}$ has no nilpotent part, $X^*_{1:-1}$ is semi-simple. If $\epsilon \neq 0$, then $X^*_{1:-1}$ is called *generic*. In this case we are presently interested.

Remark. The essential difference of generic and the semi-simple case lies in the structure of the (complex) Jordan normal form of $X^*_{1:-1}$.

$$J_{\epsilon=0} = diag\{ia, ia, -ia, -ia\}$$

$$J_{\epsilon\neq 0} = diag\{\begin{pmatrix} ia & 1 \\ 0 & ia \end{pmatrix}, \begin{pmatrix} -ia & 1 \\ 0 & -ia \end{pmatrix}\}.$$

By section 3.3, the centralizer of a matrix is determined by the structure of the (complex) Jordan normal form. Hence, the centralizers $Z_{sp}(X^{\bullet}_{1:-1})$ in the two cases will generally be different.
By the Jordan normal form $J_{\epsilon\neq 0}$ and Theorem-13, it follows that,

$$Z((X^{\bullet}_{1:-1})^T) = \{ \begin{pmatrix} x_1 & -x_2 & \epsilon x_3 & -\epsilon x_4 \\ x_2 & x_1 & \epsilon x_4 & \epsilon x_3 \\ 0 & 0 & x_1 & -x_2 \\ 0 & 0 & x_2 & x_1 \end{pmatrix}, x_i \in \mathbf{R}\}. \tag{78}$$

Applying (52), it immediately follows that

$$Z_{sp}((X^{\bullet}_{1:-1})^T) = \{ \begin{pmatrix} 0 & x_2 & \epsilon x_3 & 0 \\ -x_2 & 0 & 0 & \epsilon x_3 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & -x_2 & 0 \end{pmatrix}, x_i \in \mathbf{R}\}, \tag{79}$$

and therefore $cod(X^{\bullet}_{1:-1}) = 2$.

Theorem 31 [Universal Unfolding of the generic matrix $X^{\bullet}_{1:-1}$] *Consider the vector field $X^{\bullet}_{1:-1}$ given by (76) and its Hamiltonian function H given by (77). A universal unfolding of $X^{\bullet}_{1:-1}$ is given by*

$$U^{\bullet} = \begin{pmatrix} 0 & -a+\mu_1 & \epsilon\mu_2 & 0 \\ a-\mu_1 & 0 & 0 & \epsilon\mu_2 \\ \epsilon & 0 & 0 & -a+\mu_1 \\ 0 & \epsilon & a-\mu_1 & 0 \end{pmatrix}, \ \mu_i \in \mathbf{R} \tag{80}$$

and hence $cod(X^{\bullet}_{1:-1}) = 2$. The family of quadratic Hamiltonians corresponding to $U^{\bullet}$ is given by:

$$H^{\bullet}(p,q) = (a-\mu_1)(p_1q_2 - p_2q_1) - \frac{\epsilon}{2}(p_1^2 + p_2^2) + \frac{\epsilon\mu_2}{2}(q_1^2 + q_2^2).$$

It turns out that the distribution of the eigenvalues of the universal family $U^{\bullet}$ entirely determined by the sign of μ_2. Let us take a look at these eigenvalues:

$$\pm(i\alpha \pm \sqrt{\beta}) \tag{81}$$

where $\alpha = (a - \mu_1)$ and $\beta = \epsilon^2\mu_2$.

Keeping in mind that the all parameters μ_i are small real numbers and the sign of β is the same as that of μ_2. There are three possible configurations (also Figure-2):

- ($\mu_2 > 0$) i.e. $\beta > 0$. It follows that $U^{\bullet}$ has four complex eigenvalues with nonzero real part;
- ($\mu_2 = 0$) $U^{\bullet}$ has only two distinct eigenvalues, namely $\pm i\alpha$;
- ($\mu_2 < 0$) Let $\sqrt{\beta} = i\gamma$, $\gamma \in \mathbf{R}$. Then $U^{\bullet}$ has two pairs of purely imaginary eigenvalues, namely $\pm i(\alpha \pm \gamma)$.

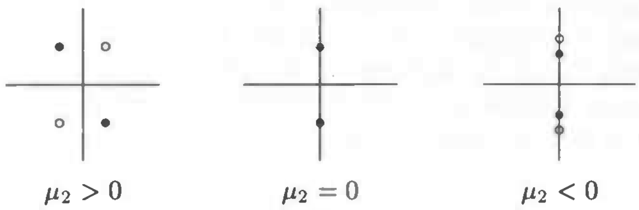


Figure 22: Different distributions according to the sign of μ_2.

We see that the deformed Hamiltonian is in the 'strongest' resonance if $\mu_2 = 0$.

Remark. In [VdM] similar diagrams to Figure-2 are found (Figure-1.2, p.18). There the distribution depends on the sign of $\epsilon\mu_2$. However, the two diagrams are consistent after a re-parameterization.

A Reversible systems in 1 : 1 resonance

In this appendix we apply the above techniques to construct universal families of a linear reversible vector field. For more background about reversible systems see [BH], [BCV] and [Sev].

A.1 Some definitions

Consider the (smooth) vector field X on a smooth manifold M. A dynamical system is given by

$$\dot{x} = X(x), x \in M \tag{82}$$

This system is said to be *reversible*, if there exists a involution $R : M \mapsto M$ (i.e., with $R^2 = id$) such that

$$R_*(X) = -X. \tag{83}$$

This means that the involution R taking (82) into a system with reversed time. Such a vector field X is called *reversible with respect to R*.
Since we are interested in linear systems, by now on we assume that $X \in gl(n, \mathbf{R})$ and $M = \mathbf{R^n}$. Suppose that X is reversible under a linear involution R. By (83) and by the fact that

$$R_*(X(x))(y) = (D_x R)(X(x)), y = R(x),$$

it follows that $-X = RXR^{-1}$. Therefore, in accordance to the above definitions, we have

Definition 32 [Reversible matrices] The matrix $A \in gl(n, \mathbf{R})$ is (infinitesimally) reversible with respect to the linear involution $R \in Gl(n, \mathbf{R})$, if

$$-A = RAR^{-1}. \tag{84}$$

We denote the set of all real reversible matrices with respect to a involution R by $gl_{-R}(n, \mathbf{R})$. For a fixed $R \in GL(n, \mathbf{R})$, the set $gl_{-R}(n, \mathbf{R})$ is a linear subspace of $gl(n, \mathbf{R})$, however it is not a Lie sub-algebra, since $gl_{-R}(n, \mathbf{R})$ is not closed under the commutator $[,]$. Consequently, there exists no natural Lie group corresponding to $gl_{-R}(n, \mathbf{R})$. In order to well-define orbits (i.e the equivalent classes by similarity) in the space $gl_{-R}(n, \mathbf{R})$ one needs a Lie group S equipped with the general action Ψ under which $gl_{-R}(n, \mathbf{R})$ is invariant.

Recall that the orbit through an element $A \in gl_{-R}(n, \mathbf{R})$ is given by

$$O(A) = \{QAQ^{-1} : Q \in S\}.$$

A necessary condition now is that $QAQ^{-1} \in gl_{-R}(n, \mathbf{R})$, i.e.

$$RQA(RQ)^{-1} = -QAQ^{-1}. \tag{85}$$

It is easy to check that if one chooses $RQ = QR$, condition (85) holds. Surprisingly, the set of all such Q's precisely is the centralizer $Z_{GL}(R)$ of R in $GL(n, \mathbf{R})$. It is not hard to show that $Z_{GL}(R)$ is a Lie group. Therefore we can take $S = Z_{GL}(G)$.

Remarks.

1. A brief calculation shows that the Lie algebra of $Z_{GL}(R)$ precisely is the centralizer $Z_{gl}(R)$ of R in the vector space $gl(n, \mathbf{R})$. Analogy to Theorem-11 the tangent space of the orbit $O(A)$ through $A \in gl_{-R}(n, \mathbf{R})$ is given by
$$T_A O(A) = \{[P, A] : P \in Z_{gl}(R)\}.$$
2. In [BCV] and [BH], $Z_{GL}(R)$ is given by the notation $GL_{+R}(n, \mathbf{R})$.

A.2 Basic facts about reversible matrices

In this section we will derive some basic properties of a reversible matrix. Let $A \in gl_{-R}(n, \mathbf{R})$, where $R \in Gl(n, \mathbf{R})$ is an involution. Since

$$-A = RAR^{-1},$$

it follows that $(-1)^n \det(A) = \det(A)$, i.e. n is even if $\det(A) \neq 0$. Put in other way: if all eigenvalues of A are non-zero then the dimension n of the space $gl_{-R}(n, \mathbf{R})$ of must be even.

Theorem 33 [Distribution of eigenvalues of a reversible matrix] If $\lambda \in \mathbf{C}$ belongs to the spectrum of $A \in gl_{-R}(n, \mathbf{R})$, so does $-\lambda, \bar{\lambda}$ and $-\bar{\lambda}$.

Proof. Suppose that λ is an eigenvalue of $A \in gl_{-R}(n, \mathbf{R})$. By the fact that $-A = RAR^{-1}$ it follows

$$\det(\lambda - A) = \det(R(\lambda + A)R^{-1}) = det(\lambda + A).$$

Hence $-\lambda$ is an eigenvalue of A. Since A is a real matrix, $\bar{\lambda}$ and $-\bar{\lambda}$ also are eigenvalues of A. □

This distribution property of a reversible matrix is similar to that of an infinitesimally symplectic matrix (cf. Theorem-26). Just like the latter case, here one has four different types of distribution of eigenvalues in the complex plane (see section 4.4, Figure-12), namely T^0, T^1, T^2 and T^3 (see sec.4.5).

By elementary linear algebra, by an appropriate choice of basis, we first simplify the form of the involution R and second – respecting to R – we can normalize the elements of $gl_{-R}(n,\mathbf{R})$. For a proof see [BCV]. We concentrate on the case when $n = 2k$ and where the involutions are given by

$$R_1 = diag\{1,-1,1,-1,...,1,-1\}$$
or
$$R_2 = diag\{1,-1,-1,1,...,(-1)^{k-1},(-1)^k\}.$$

Theorem 34 **[The semi-simple case]** *Let $X \in gl_{-R_1}(n,\mathbf{R})$, be semi-simple and have two distinct eigenvalues $i,-i$. Then X can be written as*

$$X = diag\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},...,\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}. \tag{86}$$

Theorem 35 **[The generic case]** *Let $X \in gl_{-R_2}(n,\mathbf{R})$ be non-simple (i.e., it has a non-zero nilpotent part) and have two distinct eigenvalues $i,-i$. Then X can be written as*

$$X = diag\{\begin{pmatrix} J_2 & I_2 & ... & I_2 \\ 0 & J_2 & ... & I_2 \\ 0 & ... & ... & ... \\ 0 & ... & ... & J_2 \end{pmatrix} \tag{87}$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and I_2 is the (2×2) identity matrix.

Remark. A consequence of Theorem-34 and Theorem-35 is that the representation of the vector field X can be chosen the same in the both resonances $1 : \pm 1$. In this sense there is no need to distinguish between the $1 : 1$ and $1 : -1$ resonances.

A.3 Universal unfoldings of element of $gl_{-R}(4,\mathbf{R})$

We study universal families of linear reversible vector fields X in $1 : 1$ resonance on $\mathbf{R}^4$. Consider an element $X \in gl_{-R}(4,\mathbf{R})$ with eigenvalues $\pm i$. By Theorem-34 and -35 we know that R can be chosen as follows:

$$\begin{aligned} R_1 &= diag\{1,-1,1,-1,...,1,-1\}, \\ R_2 &= diag\{1,-1,-1,1,...,(-1)^{n/2-1},(-1)^{n/2}\} \end{aligned} \tag{88}$$

and X can be given by (86) or (87) depending on whether X is semi-simple or not. Form now we restrict the involution R to be of R_1 or R_2. We call the former *the standard involution of type 1* and the latter *the standard involution of type 2.*

For later use we would like to know when a matrix $A \in gl(4,\mathbf{R})$ belongs to the vector space $gl_{-R}(4,\mathbf{R})$, here G is the standard involution of type 1 and 2. Direct computations lead to the following results:

Lemma 36 *A matrix $A \in gl(4,\mathbf{R})$ lies in the space $gl_{-R}(4,\mathbf{R})$ if and only if A is given by the following forms:*

- *(Type 1: $R = R_1$)*

$$A = \begin{pmatrix} 0 & x_1 & 0 & x_2 \\ x_3 & 0 & x_4 & 0 \\ 0 & x_5 & 0 & x_6 \\ x_7 & 0 & x_8 & 0 \end{pmatrix} \tag{89}$$

- *(Type 2: $R = R_2$)*

$$A = \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ x_3 & 0 & 0 & x_4 \\ x_5 & 0 & 0 & x_6 \\ 0 & x_6 & x_8 & 0 \end{pmatrix}, x_i \in \mathbf{R}. \tag{90}$$

A.3.1 Semi-simple case

In view of Theorem-34 we may take in this case

$$X = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}, \text{ with } J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is the same type of Example-26 from Section 4.4. There we already find the centralizer $Z_{gl}(X^T)$ of the transposed X^T in $gl(4, \mathbf{R})$. It is an easy observation that

$$Z_G(X^T) = Z_{gl}(X^T) \cap gl_{-R}(4, \mathbf{R}).$$

So, one obtains a linear universal family U of the vector field X.

Theorem 37 [Universal unfolding in semi-simple case] Given $X \in gl_{-R}(n, \mathbf{R})$ in $1:1$ resonance, where R is the standard involution of type 1. The following family U then is a linear universal unfolding of X

$$U = \begin{pmatrix} (1+\mu_1)J_2 & \mu_2 J_2 \\ \mu_3 J_2 & (1+\mu_4)J_2 \end{pmatrix}, \mu_i \in \mathbf{R}.$$

We rewrite the family U in a more convenient way, as

$$U = \begin{pmatrix} (1+\frac{1}{2}(\epsilon_1+\epsilon_2))J_2 & \epsilon_3 J_2 \\ \epsilon_4 J_2 & (1+\frac{1}{2}(\epsilon_2-\epsilon_1))J_2 \end{pmatrix}, \epsilon_i \in \mathbf{R},$$

the spectrum of which is give by

$$\left\{ \pm\frac{1}{2}\sqrt{x \pm 2(\epsilon_2+2)\sqrt{z}} \right\},$$

with $x = -\epsilon_1^2 - (\epsilon_2+2)^2 - 4\epsilon_3\epsilon_4$ and $z = \epsilon_1^2 + 4\epsilon_3\epsilon_4$. It follows that the distribution of U now depends on the sign of z. Note that the graph $z = 0$ in the space of parameters $(\epsilon_1, \epsilon_3, \epsilon_4)$ is a cone (see also Figure-4).

There are three types of distribution of eigenvalues of the family U according to different regions in the space of the parameters

- (Outside the cone: $z > 0$) Here $\sqrt{z}$ is real and small. Note that x is near to -2. Therefore, the spectrum of U consists four distinct purely imaginary eigenvalues;

- (On the cone: $z = 0$) Now U has two double but a conjugate pair; purely imaginary eigenvalues;
- (Insider the cone: $z < 0$) Here we have two conjugate pairs.

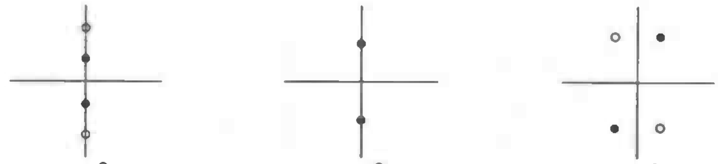


Figure 23: Distribution of eigenvalues according to regions in the parameter space.

Remark. The figure above is very similar to Figure-21.

A.3.2 Generic case

Here we deal with a vector field $X \in gl_{-R}(4, \mathbf{R})$ given by

$$X = \begin{pmatrix} J_2 & I_2 \\ 0 & J_2 \end{pmatrix}. \tag{91}$$

where R now is the standard involution of type 2. The case is very similar to the generic case of the $1:1$ resonance from Section 5.3 (see (76)). Notice that the linear vector field X here is the transposed of (76), if one sets $\epsilon = 1, a = 1$ in the latter. Hence, using (78), one easily obtains the centralizer

$$Z_{gl}(X^T) = \{ \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ -x_2 & x_1 & 0 & 0 \\ x_3 & x_4 & x_1 & x_2 \\ -x_4 & x_3 & -x_2 & x_1 \end{pmatrix}, x_i \in \mathbf{R}\}.$$

Thus, the centralizer of X^T in the vector space $gl_{-R}(4, \mathbf{R})$ is given by

$$\{ \begin{pmatrix} \mu_1 J_2 & 0 \\ \mu_2 I_2 & \mu_1 J_2 \end{pmatrix}, \mu_i \in \mathbf{R}\}.$$

Theorem 38 [Generic case] *Let the linear vector field X is given by (91). The family*

$$U = \begin{pmatrix} (1+\mu_1)J_2 & I_2 \\ \mu_2 I_2 & (1+\mu_1)J_2 \end{pmatrix}, \mu_i \in \mathbf{R}$$

is a linear universal unfolding of X and therefore $cod(X) = 2$.

The eigenvalues of the family U are given in the form:

$$\pm(\sqrt{\mu_2} \pm i(1+\mu_1))$$

Now it is not difficult to conclude that:

- in the case $\mu_2 > 0$, U has 4 distinct complex eigenvalues;
- if $\mu_2 = 0$, then U double conjugate pair $\pm i(1 + \mu_1)$;
- if $\mu_2 < 0$, U has two conjugate pairs.

$\mu_2 > 0$ $\mu_2 = 0$ $\mu_2 < 0$

Figure 24: Distribution of eigenvalues according to μ_2.

B Discussion

In this paper a systematical method, which is inspired by Gantmacher [Gan], is provided for constructing a universal unfoldings of a matrix. The technique is based on Theorem-7 [Gib] and the fact that a mini-unfolding also is a universal unfolding. This idea is also used by Arnold [Ar2] (Section 30), when studying universal unfoldings of elements of $gl(n, \mathbf{C})$. However, thanks to Theorem-15 the technique is also applicable to many subspaces V of $gl(n, \mathbf{C})$, which are perfect under the action group G.

There are also other techniques such as *SN-decomposition* (Vanderbauwhede [Van], Hoveijn [H]) introduced for construction of a universal unfoldings. The SN-decomposition is based on *Jordan-Chevally decomposition* of an endomorphism on a finite dimensional vector space V:

Theorem 39 [**Jordan-Chevally**] *Let $L(V)$ be the set of all linear mappings map V onto itself. Let $A \in L(V)$. Then there exist unique S, N in $L(V)$ such that $A = S + N$ and $SN = NS$ with S is semi-simple and N is nilpotent. Moreover, for any $B \in L(V)$, $B \in Z_V(A)$ implies that $B \in Z_V(S) \cap Z_V(N)$, where $B \in L(V)$.*

For more background we refer to [Hum]. Consider a Lie sub-algebra V of $gl(n, \mathbf{C})$ and $A \in V$. By Theorem-39 one has three conditions to determine an element $B \in Z_V(A)$, namely, $B \in V$, $B \in Z(S)$ and $B \in Z(N)$. Sometimes it is easier to compute the centralizers of S, N then performing a direct computation of $Z_V(A)$. However, if B is semi-simple then $N = 0$, i.e. the condition $B \in Z(N)$ will be lost. Hence, the SN-decomposition is more suitable for situations when B is not semi-simple.

Remark. Although the calculation of $Z(S), Z(N)$ might be easier than that of $Z(A)$, one still has to construct them. In this sense, the Gantmacher's construction provides a "more systematic" algorithm for constructions of the centralizers.

We shall give some applications of the SN-decomposition by considering a few examples.

Example 28 [Jordan block] Consider a Jordan block

$$A = \begin{pmatrix} \alpha & 1 & \cdots & 0 \\ 0 & \alpha & \cdots & 0 \\ 0 & \cdots & \alpha & 1 \end{pmatrix},$$

of the order n. The SN-decomposition of A then is

$$A = S + N = \alpha I_n + \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $X \in Z(A)$, i.e. $X \in Z(I_n)$ and $X \in Z(N)$. Note that $Z(I_n) = gl(n, \mathbf{C})$. So we conclude that $Z(A) = Z(N)$. It is not hard to show that

$$X_{i,j-1} - X_{i+1,j} = (XN - NX)_{i,j} = 0.$$

where $1 \geq i, j \geq n$ and $X_{i,j} = 0$ whenever it is not well-defined. It follows that

$$X = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ 0 & u_1 & u_2 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & u_1 \end{pmatrix}, u_i \in \mathbf{C},$$

which agrees with the Gantmacher's construction.

Example 29 [The 1 : −1 resonance (generic)] We consider the matrix

$$A = \begin{pmatrix} 0 & -1 & \epsilon & 0 \\ 1 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{92}$$

The SN-decomposition of A is obvious:

$$S = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & \epsilon I_2 \\ 0 & 0 \end{pmatrix}.$$

Here J_2 is the standard symplectic matrix and I_2 is the identity matrix.

We want to determine the centralizer $Z_{sp}(A)$ of A in the lie algebra $sp(2n, \mathbf{R})$. Let $Y \in Z_{sp}(A)$, i.e. $Y \in sp(2n, \mathbf{R}) \cap Z(S) \cap Z(N)$. Let us first study the generic case, i.e. $\epsilon \neq 0$.

By Theorem-24 the matrix Y has the form:

$$Y = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \tag{93}$$

where $A, B, C \in gl(n, \mathbf{R})$ and B, C are symmetric matrices. From the condition that $Y \in Z(N)$, it follows that

$$Y = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}, A = -A^T, B = B^T.$$

Since we also have that $Y \in Z(S)$, it turns out that $AJ = JA$ and $BJ = JB$, i.e.

$$A = \mu_1 J_2 \text{ and } B = \mu_2 I_2.$$

Thus, each element Y of $Z_{sp}(A)$ is given by:

$$Y = \begin{pmatrix} \mu_1 J_2 & \mu_2 I_2 \\ 0 & \mu_1 J_2 \end{pmatrix}.$$

Compare to the result (79).

In the generic case the SN-decomposition works very well and provides a fairly systematic way of construction of $Z_{sp}(A)$. However, in the semi-simple case, i.e. $\epsilon = 0$, the SN-decomposition has less to offer. Because in that case $N = 0$ and the semi-simple part $S = A$. So the decomposition provides no extra condition which makes the construction easier. Nevertheless, in this case the matrix A is fairly simple, so the construction of $Z_{sp}(A)$ can be done by hand.

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