Generalized Fresnel Distributions

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Introduction

The wide context of this article is given by the research on the Feynmann integral. The mathematical definition of the Feynmann-integral has been a problem for over 50 years. Because of the analogy of the Feynmann integral with the Wiener measure we first look at that.
We will have a look at the relation between the heat diffusion equation and the Wiener measure. After that we will see the connection between the heat diffusion equation and the Schrödinger equation and discuss the Fresnel distributions.

The heat diffusion equation is:

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u \tag{0.1}
\]

with \( u(t, x) = u_t(x), x \in \mathbb{R}^n \). For simplicity we take \( n = 1 \). We define

\[
G_t(x) = \frac{Y(t)e^{-x^2/2t}}{\sqrt{2\pi t}}
\]

We use the physical notation \( \delta(x) \) in stead of \( \delta \) to avoid confusion with the \( \delta(t) \). Now \( G_t \) is a fundamental solution:

\[
\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) G_t = \delta(t)\delta(x)
\]

One can see this by recognizing that \( \int G_t(x)dx = 1 \) for all \( t > 0 \) and \( G_t(x) = \frac{1}{\sqrt{t}} G_1(\frac{x}{\sqrt{t}}) \).
Therefore \( G_t(x) \) is an approximation for \( \delta(x) \) for \( t \searrow 0 \). We define \( G_0(x) = \delta(x) \).
The solutions \( u_t \) of the Cauchy problem of the heat diffusion equation with initial value \( u_0 \) are given by

\[
u_t = G_t * u_0
\]

for \( t > 0 \). Then \( \lim_{t \searrow 0} u_t = G_0 * u_0 = u_0 \).

Now we can define the \( W_\sigma \). Let \( \sigma \) be a subdivision of \([0, T] \):

\( \sigma = 0 = t_1 < \ldots < t_n < T \). Then we can write \( \mathbb{R}^n = \mathbb{R}^\sigma \) and we define \( W_\sigma \) on \( \mathbb{R}^\sigma \) by

\[
W_\sigma = G_{t_n-t_{n-1}}(x_n - x_{n-1}) \ldots G_{t_2-t_1}(x_2 - x_1)G_t(x_1)dx_1 \ldots dx_n \tag{0.2}
\]

We abbreviate: \( W_\sigma = G_\sigma(x)dx_1 \ldots dx_n \). Then \( W_\sigma \) is a probability measure on \( \mathbb{R}^\sigma \), because \( \int_{\mathbb{R}} G_t(x)dx = 1 \) for all \( t > 0 \).
Let $\pi_\sigma : C([0, T]) \rightarrow \mathbb{R}^\sigma$ be the projection-map $\pi_\sigma(X) = (X(t_1), \ldots, X(t_n)) \in \mathbb{R}^\sigma$.

If $\sigma \leq \sigma'$, we define $\pi_{\sigma'} : \mathbb{R}^{\sigma'} \rightarrow \mathbb{R}^\sigma$ to be the projection. Then $W_\sigma = \pi_{\sigma'}(W_{\sigma'})$. This means that the set $\{\pi_\sigma\}$ is a projective system. Now we can ask the question: given the projective system of measures $W_\sigma$ on $\mathbb{R}^\sigma$, does there exist a measure $W$ on $C([0, T])$ such that $W_\sigma = \pi_\sigma(W)$, for all $\sigma$? If it does, $W$ is called the projective limit. The answer is given by the next theorem.

**Theorem [Wiener]** There exists a unique probability measure $W$ on $C([0, T])$ such that

$$\pi_\sigma W = W_\sigma, \ \forall \sigma.$$ 

We call $W$ the Wiener measure.

Now we switch to the Schrödinger equation:

$$\frac{1}{i} \frac{\partial \varphi}{\partial t} = \frac{1}{2} \Delta \varphi \quad (0.3)$$

The difference with the heat diffusion equation is the fraction $\frac{1}{i}$ in front of it. This corresponds with the substitution $t \rightarrow it$. The analogue of $W_\sigma$ is $F_\sigma$:

$$F_\sigma = G_{i(t_n-t_{n-1})}(x_n - x_{n-1}) \cdots G_{i(t_2-t_1)}(x_2 - x_1)G_{it_1}(x_1)dx_1 \ldots dx_n =$$

$$\prod_{k=1}^m \frac{1}{\sqrt{2\pi i(t_k - t_{k-1})}} e^{\frac{i}{2} \sum_{k=1}^n \frac{(x_k - x_{k-1})^2}{i(t_k - t_{k-1})}} dx = c_\sigma e^{i Q_\sigma(x)} dx,$$

with $Q_\sigma$ a quadratic form. If there is a potential in the Schrödinger equation, the $Q_\sigma$ is replaced by $Q_\sigma + V_\sigma$ with $V_\sigma$ a potential. In this article we restrict ourselves to polynomial potentials.

We will see in chapter 1 that the $F_\sigma$ are summable distributions. They are called Fresnel distributions. Moreover the set $\{F_\sigma\}$ is a projective system. Now the great question is: does there exist some kind of projective limit

$$F = \lim_{\sigma \to \infty} F_\sigma$$

such that $\pi_\sigma F = F_\sigma$, for all $\sigma$. If it does, $F$ is called the Feynmann integral. What is $F$ like? To answer this difficult question, one needs to know more about the $F_\sigma$.

In chapter 2 we will get more information about the sum order of the Fresnel distributions. In the case of potentials we get distributions $e^{i P}$ with $P$ more general polynomials, which we call the Generalized Fresnel Distributions. Therefore we try to find out for which polynomials $P$ the distribution $e^{i P}$ is summable. And if so, what is the sum order? This is the central question in this article.
Chapter 1

Summable Distributions and Gauss-Fresnel distributions on \( \mathbb{R}^n \)

In this chapter we will treat the class of summable distributions on \( \mathbb{R}^n \): \( \mathcal{D}'(\mathbb{R}^n) \). This space is in fact the case \( p = 1 \) of the spaces \( \mathcal{D}'_{L^p}(\mathbb{R}^n) \) defined by L. Schwartz for \( 1 \leq p \leq \infty \) in [TD]. Because we only consider this case, we abbreviate \( \mathcal{D}'_{L^1}(\mathbb{R}^n) = \mathcal{D}'(\mathbb{R}^n) \). One can find this theory in [TD, Ch.VI, §8] and more detailed in [TH1, Ch1].

Furthermore, we take a look at the Gauss-Fresnel distributions, which turn out to be summable distributions.

1.1 Summable Distributions

Definition We define the spaces \( \mathcal{B} \) and \( \mathcal{B}^\ast \) as follows:

\[
\mathcal{B} = \mathcal{B}(\mathbb{R}^n) = \{ \varphi \in \mathcal{E}(\mathbb{R}^n) : D^k\varphi \in L^\infty(\mathbb{R}^n) \forall k \in \mathbb{Z}_+ \},
\]

\[
\mathcal{B}^\ast = \mathcal{B}(\mathbb{R}^n) = \{ \varphi \in \mathcal{E}(\mathbb{R}^n) : D^k\varphi \in C_0(\mathbb{R}^n) \forall k \in \mathbb{Z}_+ \}.
\]

The topologies of these spaces are the natural topologies induced by the (semi-)norms:

\[
p_m(\varphi) = \sup_{|k| \leq m} \| D^k\varphi \|_\infty
\]

(1.1)

Proposition 1.1

1. The space \( \mathcal{B}(\mathbb{R}^n) \) is a Fréchet space (i.e. metrizable and complete).

2. \( \mathcal{B}(\mathbb{R}^n) \) is the closure of \( \mathcal{D}(\mathbb{R}^n) \) in \( \mathcal{B}(\mathbb{R}^n) \).

Proof The first statement follows from the completeness of \( \mathcal{E}(\mathbb{R}^n) \) and the fact that the semi-norms on \( \mathcal{B}(\mathbb{R}^n) \) are norms.

To prove 2. it is sufficient to verify that \( \mathcal{D} \) is dense in \( \mathcal{B} \). Let \( \alpha \in \mathcal{D} \) be a function between 0 and 1, equal to 1 on the unit ball and define \( \alpha_n(x) = \alpha(\frac{x}{n}) \). Then, if \( \varphi \in \mathcal{B} \), the functions \( \alpha_n\varphi \) converge to \( \varphi \) uniformly on compact sets. Namely, \( |\alpha_n(x)\varphi(x) - \varphi(x)| = 0 \) for \( |x| < n \).
But given $\epsilon > 0$, $\exists N$ such that $|\alpha_n(x)\varphi(x) - \varphi(x)| \leq 2|\varphi(x)| \leq \epsilon$ for $|x| > N$, because $\varphi \in C_0$. So $|\alpha_n(x)\varphi(x) - \varphi(x)| \leq \epsilon$, $\forall x \in \mathbb{R}^n$, for $n > N$, and the convergence is uniform on $\mathbb{R}^n$.

Similarly, for the uniform convergence of the derivatives of $\varphi$ one uses Leibniz' rule and the fact that $D^k(\alpha_n)$ goes to 0 uniformly for $n \to \infty$.

**Definition** The space of summable distributions, $\mathcal{D}'(\mathbb{R}^n)$, is defined to be the dual of $\mathcal{B}(\mathbb{R}^n)$, equipped with the strong dual topology.

Because $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n)$ densely, $\mathcal{D}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. It follows from this definition that a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is summable if and only if there exist $m \in \mathbb{N}$ and $M \geq 0$ such that:

$$|(T, \varphi)| \leq M p_m(\varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \quad (1.2)$$

Namely, for $\varphi \in \mathcal{B}$, there exists a sequence $\varphi_n$ in $\mathcal{D}$ such that $\varphi_n \to \varphi$ in $\mathcal{B}$. So (1.2) implies that $|(T, \varphi_n)| \leq M p_m(\varphi_n) \quad \forall n$, and passing to the limit this gives equation (1.2) for $\varphi \in \mathcal{B}$. So $T \in \mathcal{D}'$ is a continuous linear form on $\mathcal{B}$, i.e. $T \in \mathcal{D}'_L$.

**Proposition 1.2** If $T \in \mathcal{D}'_L$ then

1. $D^kT$, defined by

$$\langle D^kT, \varphi \rangle = (-1)^{|k|} \langle T, D^k\varphi \rangle \quad \text{for} \quad \varphi \in \mathcal{D} \quad (1.3)$$

is summable.

2. for $\psi \in \mathcal{B}$, the distribution $\psi T$ defined by

$$\langle \psi T, \varphi \rangle = \langle T, \psi \varphi \rangle \quad \text{for} \quad \varphi \in \mathcal{D} \quad (1.4)$$

is summable.

3. The maps $T \mapsto D^kT$ and $T \mapsto \psi T$ are continuous from $\mathcal{D}'_L(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.

4. The formulas in 1 and 2 are valid for all $\varphi \in \mathcal{B}$ and $\psi \in \mathcal{B}$.

**Proof** Take $m$ and $M$ such that $|(T, \varphi)| \leq M p_m(\varphi) \quad \forall \varphi \in \mathcal{D}$. Then for $\varphi \in \mathcal{D}$,

$$|(D^kT, \varphi)| = \langle T, D^k\varphi \rangle \leq M p_m(D^k\varphi) \leq M p_{m+|k|}(\varphi).$$

This implies 1.

Similar for $\psi \in \mathcal{B}, \varphi \in \mathcal{D}$,

$$|(\psi T, \varphi)| = \langle T, \psi \varphi \rangle \leq M p_m(\psi \varphi) \leq M p_m(\varphi),$$

where the last inequality follows from Leibniz' rule. This proves the second statement.

For the continuity of the two maps in 3. take a sequence $T_n$ in $\mathcal{D}'_L$, such that $T_n \to 0$ in $\mathcal{D}'_L$ for $n \to \infty$. Then for all $\varphi \in \mathcal{B}$, $\langle D^kT_n, \varphi \rangle = (-1)^{|k|} \langle T_n, D^k\varphi \rangle \to 0$ for $n \to \infty$. And for $\psi \in \mathcal{B}$, $\langle \psi T_n, \varphi \rangle = \langle T_n, \psi \varphi \rangle \to 0$, for $n \to \infty$. 

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To prove 4. take, for $\varphi \in \mathcal{B}$, a sequence $\varphi_n$ in $\mathcal{D}$ such that $\varphi_n \to \varphi$ in $\mathcal{B}$ for $n$ to $\infty$. Then $D^k\varphi_n \to D^k\varphi$ in $\mathcal{B}$. Now it follows from the continuity of $T$ on $\mathcal{B}$ that $\langle D^kT, \varphi_n \rangle = (-1)^{|k|}[T, D^k\varphi_n] \to (-1)^{|k|}[T, D^k\varphi]$ and on the other hand $\langle D^kT, \varphi_n \rangle \to \langle D^kT, \varphi \rangle$. So the formula in 1 holds for $\varphi \in \mathcal{B}$. One proves the validity of the formula in 2 for $\varphi \in \mathcal{B}$ similarly.

Example An important example of summable distributions is the space of distributions with compact support. Because $\mathcal{B} \subset \mathcal{E}$ with dense image, one has

\[ E' \subset \mathcal{D}'_L. \]

Definition The **summability order** of a summable distribution $T$ is the smallest number $m$ such that the inequality (1.2) holds. (Frequently, we will abbreviate this to sum order($T$) or even s.o.($T$).)

**Proposition 1.3** If $T$ is a summable distribution with sum order $m$, then $\text{s.o.}(\psi T) \leq m$ for all $\psi \in \mathcal{B}$.

**Proof** For $\varphi \in \mathcal{D}$ we have $|\langle \psi T, \varphi \rangle| = |\langle T, \psi \varphi \rangle| = M_{p_m}(\psi \varphi) \leq M_{p_m}(\varphi)$.

One can define spaces $B^{(m)}$ resp. $\mathcal{B}^{(m)}$ to be the spaces of functions whose derivatives up to order $m$ are in $L^\infty$ resp. $C_0$. These spaces are again Banach spaces when equipped with the topology induced by the norms (1.1). If we define similarly the space $\mathcal{D}^{(m)}$ of functions with compact support whose derivatives up to order $m$ are continuous, then $\mathcal{D}^{(m)}$ is dense in $\mathcal{B}^{(m)}$ (one can prove this similar to the proof of proposition 1.1). And because $\mathcal{D}^{(m)} \subset \mathcal{D}$ we also have $\mathcal{D}$ is dense in $\mathcal{B}^{(m)}$. When we now define $\mathcal{D}'^{(m)}_L$ as the dual of $\mathcal{B}^{(m)}$ and use $\mathcal{D}'^{(0)}_L = \mathcal{M}_b$ (the space of bounded measures) we have the continuous injections:

\[ \mathcal{M}_b(\mathbb{R}^n) \subset \mathcal{D}'^{(m)}_L(\mathbb{R}^n) \subset \mathcal{D}'_L(\mathbb{R}^n) \]

It also follows from (1.2) that $\mathcal{D}'_L$ is the union of the subspaces $\mathcal{D}'^{(m)}_L$, and sum order($T$) is the smallest number such that $T \in \mathcal{D}'^{(m)}_L$.

Remember that the order of a distribution in $\mathcal{D}'(\mathbb{R}^n)$ is the smallest number $N$ such that $\forall K \subset \mathbb{R}^n, K$ compact, $\exists M \geq 0$

\[ |\langle T, \varphi \rangle| \leq M_{p_N}(\varphi) \quad \forall \varphi \in \mathcal{D}_K(\mathbb{R}^n). \]

So a summable distribution has finite order and the following inequality holds:

\[ \text{order}(T) \leq \text{sum order}(T) \]

(1.5)

Note that for $T \in \mathcal{E}'$ the equality holds.

If $T$ belongs to $\mathcal{D}'^{(m)}_L(\mathbb{R}^n)$ then $D^kT \in \mathcal{D}'^{(m+|k|)}(\mathbb{R}^n)$. So if $T = \mu \in \mathcal{M}_b(\mathbb{R}^n) = \mathcal{D}'^{(0)}(\mathbb{R}^n)$ then $D^k\mu = D^k\mu \in \mathcal{D}'^{(m)}(\mathbb{R}^n)$ for $|k| \leq m$. Thus the derivatives of bounded measures are again summable distributions. The converse of this is stated in the next theorem.
Theorem 1.4 [TD, Ch.6, §8] Let \( T \in \mathcal{D}'(\mathbb{R}^n) \). Then the following conditions are equivalent:

1. \( T \) belongs to \( \mathcal{D}'_L(\mathbb{R}^n) \).
2. \( T \) is a finite sum of derivatives of bounded measures on \( \mathbb{R}^n \).
3. \( T \) is a finite sum of derivatives of \( L^1(\mathbb{R}^n) \)-functions.
4. For every \( \alpha \in \mathcal{D}(\mathbb{R}^n) \), \( \alpha \ast T \) belongs to \( M_b(\mathbb{R}^n) \).
5. For every \( \alpha \in \mathcal{D}(\mathbb{R}^n) \), \( \alpha \ast T \) belongs to \( L^1(\mathbb{R}^n) \).

Proof 3. \( \implies \) 2. because \( L^1 \subseteq M_b \).
2. \( \implies \) 1. because of the reasoning above.
1. \( \implies \) 4. Let \( B = \{ \varphi \in \mathcal{D} : \| \varphi \|_{\infty} \leq 1 \} \).

Lemma 1.5 \( B_0 \) is dense in the unit ball of \( C_0 \).

Proof of the lemma
Let \( f \in C_0 \), \( \| f \|_{\infty} \leq 1 \). Then if \( \beta \in \mathcal{D}, \beta \geq 0, \int \beta = 1, \beta \ast f \) belongs to the unit ball of \( C_0 \). If we take \( \beta_n \in \mathcal{D} \) a standard approximation of \( \delta \), then \( \beta_n \ast f \) belongs to the unit ball of \( C_0 \) and \( \beta_n \ast f \to f \). Namely,

\[
|\beta_n \ast f(x) - f(x)| = \left| \int \beta_n(t)f(x-t)dt - f(x) \right| = \left| \int \beta_n(t)[f(x-t) - f(x)]dt \right|
\]

\[
\leq \int_{|t| \leq \delta} \beta_n(t)|f(x-t) - f(x)|dt + \int_{|t| > \delta} \beta_n(t)|f(x-t) - f(x)|dt
\]

\[
\leq \varepsilon \int_{|t| \leq \delta} \beta_n(t)dt + \varepsilon \leq 2\varepsilon, \text{ by the uniform continuity of } f.
\]

Let \( \alpha \in \mathcal{D} \), \( 0 \leq \alpha \leq 1 \), and \( \alpha = 1 \) on the unit ball, and define \( \alpha_n(x) = \alpha(x/\delta) \). Then \( \alpha_n(\beta_n \ast f) \) belongs to \( B_0 \) and converges to \( f \) uniformly on compact sets. But being dominated by a \( C_0 \)-function, namely \( |f| \), the convergence is uniform. This proves the lemma.

Consider the functions \( \tilde{\alpha} \ast \varphi \), with \( \alpha \in \mathcal{D} \) fixed, and \( \varphi \in B_0 \). Because \( D^k(\tilde{\alpha} \ast \varphi) = D^k\tilde{\alpha} \ast \varphi \) and \( \| D^k(\tilde{\alpha} \ast \varphi) \|_{\infty} \leq \| D^k\tilde{\alpha} \|_1 \| \varphi \|_{\infty} \leq \| D^k\tilde{\alpha} \|_1 \) for \( \varphi \in B \), these functions form a bounded subset of \( B_0 \).

Therefore, by the summability of \( T \),

\[
(\alpha \ast T, \varphi) = (T, \tilde{\alpha} \ast \varphi)
\]

are bounded numbers for all \( \varphi \in B_0 \), so

\[
\sup_{\varphi \in B_0} |(\alpha \ast T, \varphi)| < +\infty
\]

Now with the use of the lemma, \( \alpha \ast T \) extends to a continuous linear form on the unit ball of \( C_0 \), and therefore to the whole space \( C_0 \). So \( \alpha \ast T \) belongs to the dual of \( C_0 \), that is \( M_b \).

4. \( \iff \) 5. The inclusion \( L^1 \subseteq M_b \) gives us the \( \iff \) direction. For the opposite direction, note that \( \alpha \ast T \) is \( C^\infty \) and therefore belongs to \( L^1 \).
4. \implies 3. If $\alpha * T \in M_b$, the numbers:

$$
(\phi * T, \alpha) = (\alpha * T, \varphi)
$$

are, for fixed $\alpha \in D$, bounded for $\varphi \in B_0 \subset (\text{unit ball of } C_0)$. This implies that the distributions $\phi * T$ are bounded in $D'$ for $\varphi \in B_0$.

Let $A \subset D'$ be a bounded set, i.e. $\sup_{\alpha \in B} |(T, \alpha)| \leq M_B$, $\forall T \in A$, $\forall B \subset D$ bounded (that is $B \subset D_K$ for some compact $K$ and $B$ bounded in $D_K$). Then, because $D_K$ is a Fréchet space, we can use the Uniform Boundedness Principle for Fréchet spaces:

$$
\sup_{T \in A} |(T, \alpha)| < +\infty, \forall \alpha \in D_K.
$$

So the linear forms $T : D_K \rightarrow \mathbb{C}$, $T \in A$, with $K$ the unit ball say, are equicontinuous: $\exists m \in \mathbb{N}$, $\forall M > 0$, such that

$$
|T, \alpha) | \leq M p_m(\alpha), \forall \alpha \in D_K, \forall T \in A.
$$

Therefore $T$ extends to a linear form on $D_K^{(m)}$ for all $T \in A$ and the extension still satisfies the estimate (1.9).

In the present case $A = \{ \phi * T, \varphi \in B_0 \}$. So the distributions $\phi * T$ extend to $D_K^{(m)}$ and still satisfy the inequality (1.7). This means that the numbers in (1.6) make sense and are bounded for $\varphi \in B_0$ and $\alpha \in D_K^{(m)}$. But this means that $\alpha * T$ is a bounded linear form on $B_0$ and therefore on $C_0$, so it belongs to $M_b$, even for $\alpha \in D_K^{(m)}$. And because $\alpha * T$ is $C^{(m)}$ for $\alpha \in D_K^{(m)}$, $\alpha * T \in L^1$.

For $l \in \mathbb{N}$ sufficiently large ($l > m + \frac{3}{2}$), $\Delta^l$ has a fundamental solution $E$ in $C^{(m)}$. Because the Laplace operator is elliptic, so also hypo-elliptic, this solution $E$ is $C^{\infty}$ on the complement of $\{0\}$. If $\gamma \in D_K$, is 1 on a neighborhood of the origin, the function $\alpha = \gamma E$ belongs to $D_K^{(m)}$ and has the property

$$
\Delta^l \alpha = \delta + \zeta
$$

where $\zeta \in D$ and has support in the complement of a neighborhood of the origin. Therefore,

$$
T = \delta * T = \Delta^l \alpha * T - \zeta * T.
$$

And since $\alpha * T$ and $\zeta * T$ belong to $L^1$ (this is statement 5.) $T$ is a sum of derivatives of $L^1$-functions. This finishes the proof of the theorem.

**Remark** Using the Hahn-Banach theorem one gets information about the order of the derivatives in the representation as sums of derivatives of measures of a general summable distribution $T$. In the one dimensional case, let $T \in D_L^{(m)}(\mathbb{R})$, i.e. $|T, \varphi) | \leq M_{L_m}(\varphi) = M \sup_{k \leq m} \| \varphi^{(k)} \|_{\infty}$, $\forall \varphi \in D(\mathbb{R})$. Using the map $\varphi \mapsto (\varphi, \varphi', \ldots, \varphi^{(m)})$ on $D(\mathbb{R}) \rightarrow (C_0(\mathbb{R}))^{m+1}$, we define the bounded map $l$ on the subspace $A = \{(\varphi, \varphi', \ldots, \varphi^{(m)}) \mid \varphi \in D\}$ of $C_0^{m+1}$ by $l(\varphi, \varphi', \ldots, \varphi^{(m)}) = (T, \varphi)$. By the Hahn-Banach theorem, this map $l$ can be extended to a map $L$ defined on $C_0^{m+1}$, such that $L|_A = l$. Now we use Riesz' theorem to conclude that $L \sim (\tilde{L}_{\mu_0}, \tilde{L}_{\mu_1}, \ldots, \tilde{L}_{\mu_m})$ with $\tilde{\mu}_k \in M_b(\mathbb{R})$ i.e. for $\varphi \in D$,

$$
(T, \varphi) = L(\varphi, \varphi', \ldots, \varphi^{(m)}) = \sum_{k=0}^{m-1} \tilde{\mu}_k (\varphi^{(k)}), \varphi = \sum_{k=0}^{m} (-1)^k \tilde{\mu}_k^{(k)} \varphi^{(k)}.
$$
Theorem 1.6 The topology of $D'_L(\mathbb{R}^n)$ is the weakest topology for which the maps $T \mapsto \alpha \ast T \in L^1$ are continuous: $T_i \to T$ in $D'_L(\mathbb{R}^n)$ if and only if $\alpha \ast T_i \to \alpha \ast T$ in $L^1(\mathbb{R}^n)$ for all $\alpha \in D(\mathbb{R}^n)$. The proof of this theorem can be found in [TH1, § 1.1].

1.2 The duality with $B$
Consider a bounded subset of $B(\mathbb{R}^n)$, i.e. a set $B \subset B(\mathbb{R}^n)$ such that

$$\sup_{\varphi \in B} p_m(\varphi) < +\infty \ \forall m \in \mathbb{N}$$

Then by Ascoli's theorem $B$ is relatively compact in the space $E(\mathbb{R}^n)$. The compact closure of $B$ in $E(\mathbb{R}^n)$ is contained in $B(\mathbb{R}^n)$. Let $K$ be the set of bounded subsets of $B(\mathbb{R}^n)$ which are closed in $E(\mathbb{R}^n)$. Then the sets $K \in K$ are compact in $E(\mathbb{R}^n)$ and every bounded set $B$ is $B(\mathbb{R}^n)$ is contained in a set $K \in K$.

Theorem 1.7 Let $f : K \to Y$ be a continuous map, where $K$ is a compact space and $Y$ a Hausdorff space. Assume that $f$ is bijective. Then $f$ is a homeomorphism, i.e. $f^{-1} : Y \to K$ is continuous too.

Proof Let $g = f^{-1} : Y \to K$, $g^{-1}(F) = f(F)$ for $F \subset K$. Take $F$ a closed set in $K$, then $F$ is compact. So $f(F)$ is compact because $f$ is continuous. Therefore $g^{-1}(F)$ is closed and it follows that $g$ is continuous, i.e. $f$ is a homeomorphism.

Corollary 1.8 Let $K$ be a compact space and $X = K$ as linear space with a weaker Hausdorff topology. Then $X = K$ also as topology.

Proof The identity map $Id : K \to X$ is continuous and bijective on $K$. So by the theorem the inverse identity map $Id^{-1} : X \to K$ is continuous too. This means that if $O \subset K$ is open, then $Id(O) = O \subset X$ is open. So the topology of $K$ is not stronger than that of $X$.

It follows from this corollary that on $K$, and therefore on $B$, the topology induced by $E$ coincides with any weaker Hausdorff topology: for instance with the topology of uniform convergence on compact sets or even the topology of pointwise convergence.

Consider a sequence $\varphi_i$ in $B(\mathbb{R}^n)$ and let $\varphi \in B(\mathbb{R}^n)$. 

Definition We define \( \varphi_i \to \varphi \) pseudo-topologically in \( B(\mathbb{R}^n) \) if

a. there exists \( B \subset B(\mathbb{R}^n) \) bounded such that \( \varphi_i \in B \) for all \( i \),

b. \( \varphi_i \to \varphi \) in \( E(\mathbb{R}^n) \).

As remarked, the condition b. may for instance be replaced by:

b'. \( \varphi_i \to \varphi \) uniformly on compact sets.

Lemma 1.9 \( \mathcal{D} \) is dense in \( B \) with respect to the pseudo-topology. Given \( \varphi \in B \), there exists a sequence \( \varphi_n \) in \( \mathcal{D} \) such that \( \varphi_n \to \varphi \) pseudo-topologically for \( n \to \infty \), i.e. \( \{\varphi_n\} \) is bounded in \( B \) and \( \varphi_n \to \varphi \) in \( E \) for \( n \to \infty \).

Proof Let \( \alpha \in \mathcal{D} \) with \( \alpha(x) = 1 \) on the unit ball, and \( 0 \leq \alpha \leq 1 \). Let \( \alpha_n(x) = \alpha(\frac{x}{n}) \) and define \( \varphi_n = \alpha_n \varphi \). Then \( D^k \alpha_n(x) = \frac{1}{n^k} (D^k \alpha)(\frac{x}{n}) \), therefore \( \|D^k \alpha_n\|_\infty \leq \frac{1}{n^k} \|D^k \alpha\|_\infty \leq \|D^k \alpha\|_\infty \) for all \( n \).

Using this together with Leibniz’ rule one gets that the \( \varphi_n \) are bounded in \( B \). But \( \varphi_n(x) = \varphi(x) \), for all \( x \) with \( |x| \leq n \) and so \( \varphi_n \to \varphi \) in \( E \). This implies that \( \varphi_n \to \varphi \) in \( B \) pseudo-topologically.

If \( T \in \mathcal{D}'(\mathbb{R}^n) \), and \( T = \sum D^k \mu_k \) is a representation of \( T \) as sum of derivatives of bounded measures, we have, if \( \varphi \in B(\mathbb{R}^n) \)

\[
\langle T, \varphi \rangle = \sum \langle D^k \mu_k, \varphi \rangle = \sum (-1)^{|k|} \langle \mu_k, D^k \varphi \rangle. \tag{1.13}
\]

From this it follows that \( T \) is continuous on \( B \) equipped with the pseudo-topology. Namely, if \( \varphi_n \to \varphi \) pseudo-topologically

\[
\langle \mu_k, D^k \varphi_n \rangle \to \langle \mu_k, D^k \varphi \rangle
\]

by the Dominated Convergence Theorem of Lebesgue.

Definition \( T \in \mathcal{D}'(\mathbb{R}^n) \) has the bounded convergence property if \( T \) is continuous on \( B(\mathbb{R}^n) \) equipped with the pseudo-topology, i.e. the restriction of \( T \) to bounded subsets of \( B(\mathbb{R}^n) \) is continuous for the topology induced by \( E(\mathbb{R}^n) \).

Theorem 1.10 If \( T \) is a summable distribution, then \( T \) has a unique linear extension to \( B(\mathbb{R}^n) \) having the bounded convergence property.

Proof For \( \varphi \in B \), let \( (\varphi_n)_{n \in \mathbb{N}} \) be a sequence in \( B \), such that \( \varphi_n \to \varphi \) in the pseudo-topology. We then define

\[
\langle T, \varphi \rangle = \lim_{n \to \infty} \langle T, \varphi_n \rangle
\]

where we define \( T \) by (1.13).

Then \( T \) has the bounded convergence property again by the Dominated Convergence Theorem. Because \( T \) is uniquely defined on \( B \) the right-hand side does not depend on the representation (1.13). Therefore \( T \) is uniquely defined. \( \blacksquare \)
In particular, one can take \( \varphi = 1 \), and define the total mass of a summable distribution \( T \):

\[ \langle T, 1 \rangle. \]

**Proposition 1.11** The extension of \( T \in \mathcal{D}'_L(\mathbb{R}^n) \) to \( B(\mathbb{R}^n) \) is compatible with multiplication by functions in \( B(\mathbb{R}^n) \) and with differentiation:

\[
\langle \psi T, \varphi \rangle = \langle T, \psi \varphi \rangle \quad \forall \varphi, \psi \in B(\mathbb{R}^n) \tag{1.14}
\]

\[
\langle D^k T, \varphi \rangle = (-1)^{|k|} \langle T, D^k \varphi \rangle \quad \forall \varphi \in B(\mathbb{R}^n) \tag{1.15}
\]

**Proof** If \( \varphi \in \hat{B} \) these equalities are correct because \( \mathcal{D} \) is dense in \( \hat{B} \) and \( T \) is continuous.

If \( \varphi_n \to \varphi \in B \) pseudo-topologically, i.e. remaining bounded in \( \hat{B} \) and converging in \( \mathcal{E} \), we also have in the pseudo-topology \( \psi \varphi_n \to \psi \varphi \) and \( D^k \varphi_n \to D^k \varphi \). This yields

\[
\langle \psi T, \varphi \rangle = \lim_{n \to \infty} \langle \psi T, \varphi_n \rangle = \lim_{n \to \infty} \langle T, \psi \varphi_n \rangle = \langle T, \psi \varphi \rangle.
\]

Passing to the limit is justified because \( \psi T \) belongs to \( \mathcal{D}'_L \). Similar for the derivative:

\[
\langle D^k T, \varphi \rangle = \lim_{n \to \infty} \langle D^k T, \varphi_n \rangle = \lim_{n \to \infty} (-1)^{-|k|} \langle T, D^k \varphi_n \rangle = (-1)^{-|k|} \langle T, D^k \varphi \rangle
\]

because \( D^k T \in \mathcal{D}'_L \).

**Theorem 1.12** Let \( L : B \to C \) be a linear form which has the bounded convergence property. Then there exists a unique \( T \in \mathcal{D}'_L(\mathbb{R}^n) \) such that \( L(\varphi) = \langle T, \varphi \rangle \) for all \( \varphi \in B \).

**Proof** First note that \( L \) maps bounded subsets of \( B \) to bounded sets in \( C \). If this were not so there would exist a balanced bounded set \( B \subset B \) and a sequence \( (\varphi_n) \) in \( B \) such that \( |L(\varphi_n)| \geq n \). But then \( \frac{1}{n} \varphi_n \) belongs to \( B \) and goes to zero in the pseudo-topology. On the other hand \( L(\frac{1}{n} \varphi_n) \) does not go to 0. This contradicts the linearity of \( L \).

Let \( T \) be the restriction of \( L \) to \( \hat{B} \). Then if \( \varphi = \lim_{n \to \infty} \varphi_n \) pseudo-topologically, we have \( L(\varphi) = \lim_{n \to \infty} L(\varphi_n) = \lim_{n \to \infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle \), for all \( \varphi \in B \).

**Theorem 1.13**

1. If \( L \) is a continuous linear form on \( \mathcal{D}'_L(\mathbb{R}^n) \) there exists a unique function \( \varphi \in B \), such that \( L(T) = \langle T, \varphi \rangle \).
   
   Briefly: the dual of \( \mathcal{D}'_L(\mathbb{R}^n) \) is \( B(\mathbb{R}^n) \), the bidual of \( B(\mathbb{R}^n) \) is \( B(\mathbb{R}^n) \).

2. The given topology of \( B(\mathbb{R}^n) \) equals the topology of uniform convergence on bounded subsets of \( \mathcal{D}'_L(\mathbb{R}^n) \) and the strong dual topology of \( \mathcal{D}'_L(\mathbb{R}^n) \) equals the topology of uniform convergence on bounded subsets of \( B(\mathbb{R}^n) \).
The proof of this theorem can be found in [TH1, §1.2, §1.3].

We have seen now that one can alternatively define the space $D'_L(\mathbb{R}^n)$ as the space of linear forms on $B(\mathbb{R}^n)$ having the bounded convergence property. This is the second definition of summable distributions.

We can conclude that the linear topological vector space $D'_L(\mathbb{R}^n)$ can be completely characterized in terms of the space $B(\mathbb{R}^n)$. Moreover the operations of differentiation and multiplication on $D'_L(\mathbb{R}^n)$ can be defined by (1.14) and (1.15).

1.3 Operations on summable distributions

In this section we will define image distributions, Fourier transforms, direct products and convolution products of summable distributions.

1.3.1 Image distributions

For linear maps $u : \mathbb{R}^n \to \mathbb{R}^k$, we define the image of a summable distribution $T \in D'_L(\mathbb{R}^n)$ under $u$ by the formula:

$$ (u(T), \varphi) = (T, \varphi \circ u) $$

(1.16)

Let $\varphi \in B(\mathbb{R}^k)$. Then $\psi = \varphi \circ u$ is bounded, and by the chain rule, has bounded derivatives i.e. $\psi$ belongs to $B(\mathbb{R}^n)$. That means that $(T, \varphi \circ u)$ makes sense. Moreover, if $B \subset B(\mathbb{R}^k)$ is a bounded subset, the set of composites $B \circ u$ is bounded in $B(\mathbb{R}^n)$. Thus if $\varphi_i$ tends to 0 in $B$ (say pointwise) $\varphi_i \circ u$ tends to zero in $B \circ u$ and $(T, \varphi_i \circ u)$ tends to 0, because $T$ has the bounded convergence property on $B(\mathbb{R}^n)$. So $u(T)$ has the bounded convergence property on $B(\mathbb{R}^n)$. Therefore $u(T) \in D'_L(\mathbb{R}^k)$ by the second definition.

Note that if $\varphi \in \hat{B}(\mathbb{R}^k)$, then $\varphi \circ u$ does not in general belong to $\hat{B}(\mathbb{R}^n)$ so one can not define the image directly by the first definition and transposition. For example $u : \mathbb{R}^n \to \mathbb{R} : u(x) = 0$, then $(\varphi \circ u)(x) = \varphi(0)$, $\forall x \in \mathbb{R}^n$, and $\varphi \circ u \notin B(\mathbb{R}^n)$ if $\varphi(0) \neq 0$.

Since the derivatives of order $\leq m$ of $\varphi \circ u$ only involve derivatives of order $\leq m$ of $\varphi$ and $u$, we have:

$$ \text{sum order}(u(T)) \leq \text{sum order}(T) $$

(1.17)

1.3.2 Fourier transformations

For $\xi \in \mathbb{R}^n$ let $e_\xi(x) = e^{-ix\xi}$ where $x\xi = \sum_{j=1}^n x_j \xi_j$. Then

$$ D^k e_\xi = (-i\xi)^k e_\xi $$

where as usual $\xi^k = \xi_1^k, \ldots, \xi_n^k$.

It follows that $e_\xi$ belongs to $B(\mathbb{R}^n)$. Moreover if $\xi$ remains on a bounded subset of $\mathbb{R}^n$, $e_\xi$ describes a bounded subset of $B(\mathbb{R}^n)$. It follows that we can define the Fourier-transformation.
\( F(T) \) of a summable distribution by:

\[
F(T)(\xi) = \langle T, e_\xi \rangle. \tag{1.18}
\]

If \( \xi_n \) tends to \( \xi \), \( e_{\xi_n} \) remains bounded and converges in the space \( \mathcal{E}(\mathbb{R}^n) \) to \( e_\xi \). Thus \( F(T) \) is a continuous function.

Clearly we have \( \mathcal{S}(\mathbb{R}^n) \subset \mathcal{L}(\mathbb{R}^n) \), \( S \) being dense in \( \mathcal{L} \) because \( \mathcal{D} \) is dense. By transposition we get the continuous inclusion:

\[
\mathcal{D}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \tag{1.19}
\]

i.e. summable distributions are temperate. An application of Fubini’s theorem shows that \( F(T) \) defined above is also the Fourier-transform in the sense of temperate distributions, i.e.

\[
\langle F(T), \varphi \rangle = \langle T, F(\varphi) \rangle \quad \forall \varphi \in \mathcal{S}. \tag{1.20}
\]

In particular \( T \) is uniquely determined by its Fourier transform, or characteristic function.

From the representation \( T = \sum \mathbb{D}^k \mu_k \), with \( \mu_k \in \mathcal{M}_b \), it follows that

\[
F(T)(\xi) = \sum (-i\xi)^k F(\mu_k) \tag{1.21}
\]

which shows that \( F(T) \) is a continuous function having at most polynomial growth.

More precisely this shows:

\[
T \in \mathcal{D}'_{(m)}(\mathbb{R}^n) \implies |F(T)(\xi)| \leq M(1 + |\xi|)^m \quad \forall \xi \in \mathbb{R}^n. \tag{1.22}
\]

In the next section we will see that the converse is not true: not every distribution whose Fourier transform is continuous with polynomial growth is summable.

For \( u : \mathbb{R}^n \to \mathbb{R}^k \) a linear map, and \( 'u \) its transpose, we have by this definition:

\[
F(u(T))(\xi) = \langle u(T), e_\xi \rangle = \langle T, e_\xi \circ u \rangle = \langle T, e_\eta(\xi) \rangle = F(T)('u(\xi)) \quad \forall \xi \in \mathbb{R}^k. \tag{1.24}
\]

1.3.3 Direct products

Let \( X = \mathbb{R}^p, Y = \mathbb{R}^q \). Let \( T \in \mathcal{D}'_L(X) \) and \( S \in \mathcal{D}'_L(Y) \). Then the direct product \( T \otimes S \) is summable and one has:

\[
\langle T \otimes S, \varphi \otimes \psi \rangle = \langle T, \varphi \rangle \langle S, \psi \rangle \quad \forall \varphi \in \mathcal{B}(X) \quad \forall \psi \in \mathcal{B}(Y). \tag{1.23}
\]

For \( \Phi \in \mathcal{B}(X \times Y) \) we have:

\[
\langle T \otimes S, \Phi \rangle = \langle T, \theta \rangle, \tag{1.24}
\]

with \( \theta(x) = \langle S, \Phi_x \rangle \) where \( \Phi_x(y) = \Phi(x, y) \).

For the proof of the summability of \( T \otimes S \) note that the map \( x \mapsto \Phi_x \) is \( C^\infty \) from \( X \) to \( \mathcal{E}(Y) \). Since \( \Phi \in \mathcal{B}(X \times Y) \), \( \Phi_x \) belongs to \( \mathcal{B}(Y) \) for all \( x \in X \). Moreover the functions \( \Phi_x \) form a bounded subset of \( \mathcal{B}(Y) \) and therefore the function \( \theta = \langle S, \Phi_x \rangle \) is a well defined function belonging to \( \mathcal{B}(X) \). So the formula (1.24) makes sense.
If \( \Phi \) remains on a bounded subset of \( B(X \times Y) \), the corresponding functions \( \Theta \) remain bounded in \( B(X) \). So if \( \Phi_i \to \Phi \) pseudo-topologically the corresponding \( \Theta_i \to \Theta \) pseudo-topologically. This means that \( T \otimes S \) has the bounded convergence property, i.e. \( T \otimes S \) is summable, satisfying (1.23). Moreover, the restriction to \( D(X \times Y) \) is the tensor product in the usual sense.

If \( \Phi \in B^{(n+m)}(X \times Y) \), it follows that \( x \mapsto \Phi_x \) belongs to \( B^{(n)}(X) \). So, if \( T \in D_L^{(n)}(X) \) and \( S \in D_L^{(m)}(Y) \), the function \( \Theta = \langle S, \Phi_x \rangle \in B^{(n)}(X) \) and \( \langle T, \Theta \rangle \) make sense. It follows that

\[
\text{sum order}(T \otimes S) \leq \text{sum order}(T) + \text{sum order}(S). \quad (1.25)
\]

The condition that \( x \mapsto \Phi_x \) belongs to \( B^{(n)}(X) \) does not involve all derivatives of order \( n + m \) of \( \Phi \), and so the above inequality may be in some cases a strict inequality. We will see examples where this is the case in the next chapter.

### 1.3.4 Convolution products

For \( T, S \in D_L'(\mathbb{R}^n) \) we define the convolution product \( T \ast S \) as the image of the direct product \( T \otimes S \) under the linear map \( (x, y) \mapsto x + y \). Thus, for \( \Phi \in B(\mathbb{R}^n) \)

\[
\langle T \ast S, \Phi \rangle = \langle T \otimes S, \Psi \rangle \quad (1.26)
\]

where \( \Psi(x, y) = \Phi(x + y) \). It follows that \( T \ast S \) is summable. Note that there is no condition on the supports of \( T \) and \( S \).

Since \( e_\xi(x + y) = e_\xi(x)e_\xi(y) \) we have by (1.18) \( \mathcal{F}(T \ast S)(\xi) = \langle T \ast S, e_\xi \rangle = \langle T \otimes S, e_\xi \rangle = \langle T, e_\xi \rangle \langle S, e_\xi \rangle = \mathcal{F}(T)(\xi) \mathcal{F}(S)(\xi) \). This yields

\[
\mathcal{F}(T \ast S) = \mathcal{F}(T) \mathcal{F}(S). \quad (1.27)
\]

### 1.4 The class \( \mathcal{F}(O_M) \)

**Definition** We define the class \( O_M(\mathbb{R}^n) \) to be the space of functions \( f \in \mathcal{E}(\mathbb{R}^n) \) such that \( f \) and all its derivatives have at most polynomial growth.

**Theorem 1.14** Every \( T \in \mathcal{F}(O_M) \) is summable. More precisely we have the continuous inclusion:

\[
\mathcal{F}(O_M)(\mathbb{R}^n) \subset D_L'(\mathbb{R}^n). \quad (1.28)
\]

Moreover, if \( P \) is a polynomial and \( T \) belongs to \( \mathcal{F}(O_M)(\mathbb{R}^n) \), we have \( PT \in D_L'(\mathbb{R}^n) \). Conversely, if \( PT \) belongs to \( D_L'(\mathbb{R}^n) \) for all polynomials \( P \), we have \( T \in \mathcal{F}(O_M) \).
Proof It is clear that \( T \in \mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n) \iff \mathcal{F}(T) \in \mathcal{O}_M \iff \mathcal{F}(T) = \hat{T} \in \mathcal{O}_M \). Let \( \alpha \in \mathcal{D}(\mathbb{R}^n) \), and assume \( \hat{T} \in \mathcal{O}_M \). Then \( \mathcal{F}(\alpha * T) = \hat{\alpha} \hat{T} \in \mathcal{S}(\mathbb{R}^n) \). It follows that \( \alpha * T \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n) \), which implies that \( T \in \mathcal{D}'(\mathbb{R}^n) \) by theorem 1.4.

If \( T_i \) goes to 0 in \( \mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n) \), i.e. \( \hat{T}_i \) goes to 0 in \( \mathcal{O}_M \), \( \alpha \hat{T}_i \) goes to 0 for all \( \alpha \in \mathcal{D} \), and so \( \alpha * T_i \) goes to 0 in \( \mathcal{S} \) and in \( L^1 \). Therefore by theorem 1.6 \( T_i \) goes to 0 in \( \mathcal{D}'(\mathbb{R}^n) \).

This proves the continuous inclusion (1.28).

Let \( P \) be a polynomial and \( T \in \mathcal{F}(\mathcal{O}_M) \), then \( \mathcal{F}(PT) = D \hat{T} \) where \( D \) is a differential operator with constant coefficients. If follows that \( \mathcal{F}(PT) \) is a derivative of a distribution in \( \mathcal{O}_M \), so it belongs to \( \mathcal{O}_M \). Therefore \( PT \in \mathcal{F}(\mathcal{O}_M) \subset \mathcal{D}'(\mathbb{R}^n) \).

Conversely, if \( PT \) belongs to \( \mathcal{D}'(\mathbb{R}^n) \) for all polynomials \( P \), it follows that \( D \hat{T} \) is continuous with polynomial growth for all differential operators \( D \). This implies that \( T \) belongs to \( \mathcal{O}_M \). So we have \( T \in \mathcal{F}(\mathcal{O}_M) \).

Remark The Fourier transform of a summable distribution is continuous, but not in general \( C^\infty \). Thus \( \mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n) \) is a proper subset of \( \mathcal{D}'(\mathbb{R}^n) \).

1.5 Gauss-Fresnel distributions

Let \( G_\lambda = \frac{1}{(2\pi \lambda)^{n/2}} e^{-\frac{|\xi|^2}{2\lambda}} \) for \( \Re(\lambda) \geq 0 \) and \( \lambda \neq 0 \), the root being determined so as to coincide with the positive root if \( \lambda > 0 \). Let \( G_0 = \delta \). We know that if \( \lambda > 0 \)

\[
\mathcal{F}(\frac{1}{(2\pi \lambda)^{n/2}} e^{-\frac{|\xi|^2}{2\lambda}}) = e^{-\lambda|\xi|^2/2}. \tag{1.29}
\]

Now for all \( \lambda \) with \( \Re(\lambda) \geq 0 \), \( e^{-\lambda|\xi|^2/2} \) belongs to \( \mathcal{O}_M \).

Theorem 1.15 \( G_\lambda \) belongs to \( \mathcal{D}'(\mathbb{R}^n) \) for all \( \lambda \) with \( \Re(\lambda) \geq 0 \). Moreover the map \( \lambda \mapsto G_\lambda \in \mathcal{D}'(\mathbb{R}^n) \) is continuous on the closed half plane, and holomorphic in the interior of \( \mathbb{C}_+ \).

Proof \( \mathcal{F}(G_\lambda) = e^{-\lambda|\xi|^2/2} \in \mathcal{O}_M \) for all \( \lambda \) with \( \Re(\lambda) \geq 0 \). Using theorem 1.14 this implies that \( G_\lambda = \mathcal{F}(\mathcal{F}(G_\lambda)) \in \mathcal{F}\mathcal{O}_M \subset \mathcal{D}'(\mathbb{R}^n) \).

Moreover it is clear that the map \( \lambda \mapsto e^{-\lambda|\xi|^2/2} = \mathcal{F}(G_\lambda) \) is continuous and holomorphic on \( \mathbb{C}_+ \). Therefore the composition with the linear map \( \mathcal{F} \), i.e. \( \lambda \mapsto G_\lambda \) is holomorphic too.

Corollary 1.16 If \( P \) is any polynomial on \( \mathbb{R}^n \), then \( PG_\lambda \) belongs to \( \mathcal{D}'(\mathbb{R}^n) \) and the map \( \lambda \mapsto PG_\lambda \) is holomorphic for \( \Re(\lambda) > 0 \) and continuous for \( \Re(\lambda) \geq 0 \).

Proof We have \( \mathcal{F}(PG_\lambda) = De^{-\lambda|\xi|^2/2} \in \mathcal{O}_M \), where \( D \) is a differential operator with constant coefficients, which maps \( \mathcal{O}_M \) continuously into itself.

We will calculate the sum order of several Fresnel distributions in Chapter 2. In the third chapter we try to find out which polynomials \( P \) lead to summable distributions \( e^{ip} \).
Chapter 2

Fresnel distributions

In this chapter we will compute the sum order of several Fresnel distributions, as preparation to compute the summability order of the distributions $e^{iP}$ (with $P$ a general polynomial) which maybe infinite. Some theorems and special cases of this chapter are already written down in [TH1, Ch. 2].

2.1 The one dimensional case

In this section we exclusively deal with $\mathbb{R}$, so instead of writing $D(\mathbb{R})$, we just write $D$, etc.

Proposition 2.1 [TH1] The sum order of $e^{ix^2}$ is exactly 2.

Proof $\int e^{ix^2} \varphi(x) dx = (\int_{-\infty}^{-1} + \int_{1}^{+\infty} + \int_{1}^{+\infty})(e^{ix^2} \varphi(x) dx)$. The middle integral can be estimated by $2\|\varphi\|_{\infty}$. The first integral is equivalent to the last one, by reflection $x \mapsto -x$. So it is sufficient to consider only $\int_{1}^{+\infty} e^{ix^2} \varphi(x) dx$.

\[
\begin{align*}
\int_{1}^{+\infty} e^{ix^2} \varphi(x) dx & = \int_{1}^{+\infty} x e^{ix^2} \varphi(x) x dx = c^{st} \int_{1}^{+\infty} d \frac{e^{ix^2} \varphi(x)}{x} dx = \\
c^{st} \int_{1}^{+\infty} e^{ix^2} \frac{d}{dx} \left( \frac{\varphi(x)}{x} \right) dx + c^{st} \varphi(1) & = c^{st} \int_{1}^{+\infty} x e^{ix^2} \frac{1}{x} d \frac{\varphi(x)}{x} dx + c^{st} \varphi(1) + c^{st} \varphi'(1) = \\
c^{st} \int_{1}^{+\infty} e^{ix^2} \frac{d}{dx} \left[ \frac{1}{x} \left( \frac{\varphi'(x)}{x} - \frac{\varphi(x)}{x^2} \right) \right] dx + c^{st} \varphi(1) + c^{st} \varphi'(1) = \\
c^{st} \int_{1}^{+\infty} e^{ix^2} \left( \frac{\varphi''(x)}{x^2} - 2 \frac{\varphi'(x)}{x^3} + \frac{\varphi(x)}{x^4} \right) dx + c^{st} \varphi(1) + c^{st} \varphi'(1) = \\
c^{st} \int_{1}^{+\infty} e^{ix^2} \left( \frac{\varphi''(x)}{x^2} - 3 \frac{\varphi'(x)}{x^3} + \frac{3 \varphi(x)}{x^4} \right) dx + c^{st} \varphi(1) + c^{st} \varphi'(1)
\end{align*}
\]
Because \( \frac{1}{x^4} \) is integrable over \([1, +\infty)\) for \( k > 1 \),

\[
|\int_1^{+\infty} e^{ix^2} \varphi(x) dx| \leq M_0 (\|\varphi\|_\infty + \|\varphi'\|_\infty + \|\varphi''\|_\infty) = M_{p_2}(\varphi).
\]

And

\[
|\int e^{ix^2} \varphi(x) dx| \leq M_{p_2}(\varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}). \tag{2.1}
\]

So \( s.o.(e^{ix^2}) \leq 2 \).

For the converse estimate, we must contradict the inequality:

\[
|\int e^{ix^2} \varphi(x) dx| \leq M_{p_1}(\varphi) \tag{2.2}
\]

for some \( \varphi \in \mathcal{D} \).

Let \( \alpha \in \mathcal{D} \) be such that \( 0 \leq \alpha(x) \leq 1, \alpha(x) = 1 \) for \( |x| \leq 1 \) and \( \alpha(x) = 0 \) for \( |x| \geq 2 \). Let \( \beta \in \mathcal{B} \) with support in \( [\frac{1}{2}, +\infty) \), \( 0 \leq \beta(x) \leq 1 \), \( \beta(x) = 1 \) on \([1, +\infty)\).

Let \( \alpha_n(x) = \alpha(\frac{x}{n}) \beta(x), \) then \( \alpha_n \in \mathcal{D} \). Now define \( \varphi(x) = \frac{1}{x} e^{-ix^2} \) and \( \varphi_n = \alpha_n \varphi \), then \( \varphi_n \in \mathcal{D} \) and \( \varphi_n(x) = \varphi(x) \) on \([1, n]\).

Then \( \|\varphi_n\|_\infty \leq M_1 < +\infty \). Furthermore, because \( \varphi'_n(x) = \alpha'_n(x) \varphi(x) + \alpha_n(x) \varphi'(x) = \alpha'_n(x) \frac{1}{2} e^{-ix^2} + \alpha_n(x) (\frac{1}{2} e^{-ix^2} - 2ie^{-ix^2}) \), it also follows that \( \|\varphi'_n\|_\infty \leq M_2 < +\infty \). So \( p_1(\varphi_n) = \max(\|\varphi_n\|_\infty, \|\varphi'_n\|_\infty) \leq M < +\infty \), i.e. the \( p_1(\varphi_n) \) are uniformly bounded in \( n \).

On the other hand, the integral

\[
\int e^{ix^2} \varphi_n(x) dx = \int_{\frac{1}{2}}^{+\infty} e^{ix^2} \varphi_n(x) = \int_{\frac{1}{2}}^{+\infty} e^{ix^2} \alpha_n(x) \frac{1}{x} e^{-ix^2} dx =
\]

\[
\int_{\frac{1}{2}}^{+\infty} \alpha_n(x) \frac{1}{x} dx \geq \int_{1}^{n} \frac{1}{x} dx \geq \log(n) + e^{st}
\]

diverges to \( +\infty \), for \( n \to \infty \). So the left-hand side in (2.2) with \( \varphi_n \) in stead of \( \varphi \) diverges for \( n \to \infty \), while the right-hand side is uniformly bounded in \( n \). This means that the inequality (2.2) does not hold for all \( \varphi \in \mathcal{D} \), i.e. \( s.o.(e^{ix^2}) > 1 \).

This finishes the proof of the proposition: \( s.o.(e^{ix^2}) = 2 \).

**Proposition 2.2** For \( E_k(x) = x^k e^{ix^2} \), the following holds for all \( \varphi \in \mathcal{D} \),

1. \( |\int_1^{+\infty} E_k(x) \varphi(x) dx| \leq M_{p_0}(\varphi) \) for \( k \leq -2 \).

2. \( |\int_1^{+\infty} E_{-1}(x) \varphi(x) dx| \leq M_{p_1}(\varphi) \).

3. \([TH1]\) \( s.o.(E_k) = k + 2 \), for \( k \geq 0 \).
Proof The first statement follows directly from the fact that \( x^k \in L^1([1, +\infty)) \), for \( k < -1 \).

For \( \varphi \in \mathcal{D} \), then

\[
\int_{1}^{\infty} x^{-1} e^{ix^2} \varphi(x) \, dx = \int_{1}^{\infty} xe^{ix^2} \frac{\varphi(x)}{x^2} \, dx = c^{st} \int_{1}^{\infty} \frac{d}{dx} \left[ e^{ix^2} \varphi(x) \right] \, dx =
\]

\[
c^{st} \int_{1}^{\infty} e^{ix^2} \frac{d}{dx} \left[ \frac{\varphi(x)}{x^2} \right] \, dx + c^{st} \varphi(1) = c^{st} \int_{1}^{\infty} e^{ix^2} \left( \frac{\varphi'(x)}{x^2} - 2 \frac{\varphi(x)}{x^3} \right) \, dx + c^{st} \varphi(1)
\]

So

\[
\left| \int_{1}^{\infty} x^{-1} e^{ix^2} \varphi(x) \, dx \right| \leq M_{p1}(\varphi).
\]

For the proof of 3. we again write \( \int E_k(x) \varphi(x) \, dx = (\int_{-\infty}^{1} + \int_{1}^{\infty}) (E_k(x) \varphi(x) \, dx) \), and note that the second integral can be estimated by \( M\|\varphi\|_{\infty} \) and the first and last are equivalent.

Because 3. holds for \( k = 0 \) it is sufficient to prove by induction that

\[
\left| \int_{1}^{\infty} x^k e^{ix^2} \varphi(x) \, dx \right| \leq M_{p_k+2}(\varphi), \quad \forall \varphi \in \mathcal{D} \quad \text{(2.3)}
\]

and that

\[
\text{s.o.}(E_k) > k + 1 \text{ for } k \geq 0. \quad \text{(2.4)}
\]

Let's assume (2.3) for \( k \), then for \( k + 1 \):

\[
\int_{1}^{\infty} x^{k+1} e^{ix^2} \varphi(x) \, dx = \int_{1}^{\infty} xe^{ix^2} x^k \varphi(x) \, dx =
\]

\[
c^{st} \int_{1}^{\infty} \frac{d}{dx} \left[ e^{ix^2} x^k \varphi(x) \right] \, dx = c^{st} \int_{1}^{\infty} e^{ix^2} \frac{d}{dx} \left[ x^k \varphi(x) \right] \, dx + c^{st} \varphi(1) =
\]

\[
c^{st} \int_{1}^{\infty} e^{ix^2} (kx^{k-1} \varphi(x) + x^k \varphi'(x)) \, dx + c^{st} \varphi(1) =
\]

\[
c^{st} \int_{1}^{\infty} x^{k-1} e^{ix^2} \varphi(x) \, dx + c^{st} \int_{1}^{\infty} x^k e^{ix^2} \varphi'(x) \, dx + c^{st} \varphi(1)
\]

Now, the induction hypothesis yields:

\[
\left| c^{st} \int_{1}^{\infty} x^{k-1} e^{ix^2} \varphi(x) \, dx \right| \leq M_{1p_{k-1}+2}(\varphi)
\]

and

\[
\left| c^{st} \int_{1}^{\infty} x^k e^{ix^2} \varphi'(x) \, dx \right| \leq M_{2p_{k+2}}(\varphi') \leq M_{2p_{k+2}+1}(\varphi)
\]

So, we have

\[
\left| \int_{1}^{\infty} x^{k+1} e^{ix^2} \varphi(x) \, dx \right| \leq M_{p_{k+1}+2}(\varphi)
\]
which finishes the proof of the inequality (2.3) for all $k \geq 0$.

For the proof of (2.4), define $\varphi(x) = \frac{1}{x^{k+1}} e^{-ix^2}$ and $\alpha_n$ as in the proof of proposition 2.1. Then, if we define $\varphi_n(x) = \alpha_n(x) \varphi(x)$, $\varphi_n \in \mathcal{D}$.

By Leibniz' rule
\[
\left(\frac{d}{dx}\right)^m \varphi_n = \sum_{i=0}^m \binom{m}{i} \left(\frac{d}{dx}\right)^i \alpha_n \left(\frac{d}{dx}\right)^{m-i} \varphi.
\]

And
\[
\left(\frac{d}{dx}\right)^i \alpha_n(x) = \frac{1}{n!} \left(\frac{d}{dx}\right)^i \alpha \left(\frac{x}{n}\right)
\]
implies that
\[
\|\left(\frac{d}{dx}\right)^i \alpha_n\|_\infty \leq \|\left(\frac{d}{dx}\right)^i \alpha\|_\infty
\]
thus the derivatives of $\alpha_n$ are uniformly bounded in $n$. It follows that
\[
\|\left(\frac{d}{dx}\right)^m \varphi_n\|_\infty \leq M_0 \sup_{|k| \leq m} \sup_{x \in [\frac{1}{2}, \infty)} |\varphi^{(k)}(x)|.
\]

Now
\[
\left(\frac{d}{dx}\right)^i \varphi = \left(\frac{d}{dx}\right)^i \left[\frac{1}{x^{k+1}} e^{-ix^2}\right] = \sum_{j=0}^i \binom{i}{j} \left(\frac{d}{dx}\right)^j \left[\frac{1}{x^{k+1}}\right] \left(\frac{d}{dx}\right)^{i-j} [e^{-ix^2}].
\]
This is a finite linear combination of terms
\[
\frac{1}{x^r} e^{-ix^2}
\]
where $k + 1 - l \leq r \leq k + 1 + l$.

If $0 \leq l \leq k + 1$, then $0 \leq r \leq 2k + 2$ and $\frac{1}{x^r} e^{-ix^2}$ remains bounded on $[\frac{1}{2}, \infty)$. From this it follows:
\[
\sup_{|l| \leq k+1} \sup_{x \in [\frac{1}{2}, \infty)} |\varphi^{(l)}(x)| < M_1 < +\infty
\]
and therefore
\[
p_{k+1}(\varphi_n) = \sup_{i \leq k+1} \|\left(\frac{d}{dx}\right)^i \varphi_n\|_\infty < M_2 < +\infty
\]
uniformly in $n$.

If $s.o.(E_k) \leq k + 1$ then
\[
|\int_1^{\infty} x^k e^{ix^2} \varphi(x) dx| \leq M_{p_{k+1}}(\varphi), \forall \varphi \in \mathcal{D}
\]
in particular for the $\varphi_n$. The right-hand side of this inequality is, for $\varphi_n$, uniformly bounded in $n$, while the left-hand side,
\[
\int_1^{\infty} x^k e^{ix^2} \varphi_n(x) dx = \int_1^{\infty} x^k e^{ix^2} \alpha_n(x) \frac{1}{x^k + 1} e^{-ix^2} dx = \\
\int_1^{\infty} \alpha_n(x) \frac{1}{x} dx \geq \int_1^{n} \frac{1}{x} dx = \mathcal{O} \log(n)
\]
diverges. So (2.4) has been proved now. This finishes the proof of 3.
2.2 Two dimensions

In this section we wish to calculate the sum order of \( e^{i(x^2-y^2)} \) and \( P(x, y)e^{i(x^2-y^2)} \) with \( P \) a polynomial.

**Definition** We define on \( \mathbb{R}^2 \) the differential operator \( D_{a,b} = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \).

**Proposition 2.3** For \( f, g \) general functions on \( \mathbb{R}^2 \) and \( \varphi \in \mathcal{D}(\mathbb{R}^2) \) the following holds:
1. \( D_{a,b}(fg) = (D_{a,b}f)g + f(D_{a,b}g) \) (product rule)
2. \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_{a,b} \varphi(x, y)dy = -a \int_{-\infty}^{\infty} \varphi(\alpha, y)dy - b \int_{-\infty}^{\infty} \varphi(x, \beta)dx \) (integration rule)

The simple proof is left to the reader.

**Proposition 2.4** The sum order of the distribution \( T \) defined by \( (T, \varphi) = \iint e^{i(x^2-y^2)} \varphi(x, y)dxdy \) is 3.

**Proof** The proof is in two steps. We first prove that

\[
\text{s.o.}(e^{i(x^2-y^2)}) \leq 3. \tag{2.5}
\]

And after that we will prove that

\[
|\iint e^{i(x^2-y^2)} \varphi(x, y)dxdy| \leq Mp_2(\varphi) \tag{2.6}
\]

does not hold for all \( \varphi \in \mathcal{D} \). Then it follows that the summability order of \( T \) is exactly 3.

Let's abbreviate \( E(x, y) = e^{i(x^2-y^2)} \). Then \( D_{1,-1}E(x, y) = 2i(x+y)E(x, y) \). We first consider the quadrant \([1, \infty) \times [0, \infty)\), where \( x + y \) is not zero. So we can write:

\[
\int_{0}^{\infty} dy \int_{1}^{\infty} E(x, y)\varphi(x, y)dx = \int_{0}^{\infty} dy \int_{1}^{\infty} (x+y)E(x, y)\frac{\varphi(x, y)}{x+y}dx =
\]

\[
c^st \int_{0}^{\infty} dy \int_{1}^{\infty} D_{1,-1}E(x, y)\frac{\varphi(x, y)}{x+y}dx =
\]

\[
c^st \int_{0}^{\infty} dy \int_{1}^{\infty} D_{1,-1}[E(x, y)\frac{\varphi(x, y)}{x+y}]dx -
\]

\[
c^st \int_{0}^{\infty} dy \int_{1}^{\infty} E(x, y)D_{1,-1}[\frac{\varphi(x, y)}{x+y}]dx =
\]

(we abbreviate: \( B_1 \) is the first integral)

\[
B_1 + c^st \int_{0}^{\infty} dy \int_{1}^{\infty} E(x, y)D_{1,-1}[\frac{\varphi(x, y)}{x+y}]dx =
\]

\[
B_1 + c^st \int_{0}^{\infty} dy \int_{1}^{\infty} E(x, y)\left(\frac{D_{1,-1}\varphi(x, y)}{x+y} - \frac{\varphi(x, y)D_{1,-1}(x+y)}{(x+y)^2}\right)dx =
\]

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\begin{align*}
B_1 + c^t \int_0^\infty dy \int_1^\infty E(x,y) \frac{D_{1,-1} \varphi(x,y)}{x+y} dx = \\
B_1 + c^t \int_0^\infty dy \int_1^\infty (x+y) E(x,y) \frac{D_{1,-1} \varphi(x,y)}{(x+y)^2} dx = \\
B_1 + c^t \int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)] \frac{D_{1,-1} \varphi(x,y)}{(x+y)^2} dx = \\
B_1 + c^t \int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)] \frac{D_{1,-1} \varphi(x,y)}{(x+y)^2} dx - \\
c^t \int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1} \frac{D_{1,-1} \varphi(x,y)}{(x+y)^2} dx = \\
&\text{(again, } B_2 \text{ is the first integral)} \\
B_1 + B_2 + c^t \int_0^\infty dy \int_1^\infty \frac{D_{1,-1} \varphi(x,y)}{(x+y)^2} dx = \\
B_1 + B_2 + c^t \int_0^\infty dy \int_1^\infty E(x,y) \left( \frac{(D_{1,-1})^2 \varphi(x,y)}{(x+y)^2} - \frac{2D_{1,-1} \varphi(x,y)D_{1,-1}(x+y)}{(x+y)^3} \right) dx = \\
B_1 + B_2 + c^t \int_0^\infty dy \int_1^\infty (x+y) E(x,y) \frac{(D_{1,-1})^2 \varphi(x,y)}{(x+y)^3} dx = \\
B_1 + B_2 + c^t \int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)] \frac{(D_{1,-1})^2 \varphi(x,y)}{(x+y)^3} dx = \\
B_1 + B_2 + c^t \int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)] \frac{(D_{1,-1})^2 \varphi(x,y)}{(x+y)^3} dx - \\
c^t \int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1} \frac{(D_{1,-1})^2 \varphi(x,y)}{(x+y)^3} dx = \\
&\text{(} B_3 \text{ is an abbreviation for the first integral)} \\
B_1 + B_2 + B_3 + c^t \int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1} \left( \frac{(D_{1,-1})^2 \varphi(x,y)}{(x+y)^3} \right) dx = \\
B_1 + B_2 + B_3 + c^t \int_0^\infty dy \int_1^\infty E(x,y) \left( \frac{(D_{1,-1})^3 \varphi(x,y)}{(x+y)^3} - \frac{3(D_{1,-1})^2 \varphi(x,y)D_{1,-1}(x+y)}{(x+y)^4} \right) dx = \\
B_1 + B_2 + B_3 + c^t \int_0^\infty dy \int_1^\infty E(x,y) \frac{(D_{1,-1})^3 \varphi(x,y)}{(x+y)^3} dx.
\end{align*}
Now, because \( x + y > \sqrt{x^2 + y^2} = r \) on this quadrant, \( \frac{1}{(x+y)^2} \leq \frac{1}{r^2} \in L^1([1, \infty) \times [0, \infty)) \).
This implies that we can estimate:

\[
| \int_0^\infty dy \int_1^\infty E(x, y) \varphi(x, y) dx | \leq |B_1| + |B_2| + |B_3| + c^st \|((D_{1,-1})^3 \varphi)\|_{\infty} \int_0^\infty dy \int_1^\infty \frac{1}{(x+y)^3} dx \leq |B_1| + |B_2| + |B_3| + M_0p_3(\varphi)
\]

For the calculation of the \( B_k \)'s, we use the integration rule from the proposition above.

\[
B_1 = \int_0^\infty \int_1^\infty D_{1,-1}[E(x, y)] \varphi(x, y) dx =
\]

\[
c^st \int_0^\infty E(1, y) \varphi(1, y) dy + c^st \int_1^\infty E(x, 0) \varphi(x, 0) dx =
\]

\[
c^st \int_0^\infty e^{i(1+y^2)} \varphi(1, y) dy + c^st \int_1^\infty e^{ix^2} \varphi(x, 0) dx =
\]

\[
c^st \int_0^\infty \frac{1}{1+y} e^{iy^2} \varphi(1, y) dy + c^st \int_1^\infty \frac{1}{x} e^{ix^2} \varphi(x, 0) dx.
\]

From proposition 2.2 we know that s.o.(\( \frac{1}{1+y^2} \)) = s.o.(\( \frac{1}{2} e^{ix^2} \)) = 1. This implies that

\[
|B_1| \leq M_1p_1(\varphi).
\]

For \( B_2 \) we have

\[
B_2 = \int_0^\infty \int_1^\infty \frac{D_{1,-1}[E(x, y)]}{(x+y)^2} dx dy =
\]

\[
c^st \int_0^\infty E(1, y) \frac{D_{1,-1}\varphi(1, y)}{(1+y)^2} dy + c^st \int_1^\infty E(x, 0) \frac{D_{1,-1}\varphi(x, 0)}{x^2} dx =
\]

\[
c^st \int_0^\infty e^{i(1+y^2)} \frac{D_{1,-1}\varphi(1, y)}{(1+y)^2} dy + c^st \int_1^\infty e^{ix^2} \frac{D_{1,-1}\varphi(x, 0)}{x^2} dx =
\]

\[
c^st \int_0^\infty e^{iy^2} \frac{D_{1,-1}\varphi(1, y)}{(1+y)^2} dy + c^st \int_1^\infty e^{ix^2} \frac{D_{1,-1}\varphi(x, 0)}{x^2} dx.
\]

So

\[
|B_2| \leq M_1p_0(\varphi).
\]

And the last one:

\[
B_3 = \int_0^\infty \int_1^\infty \frac{(D_{1,-1})^2\varphi(x, y)}{(x+y)^3} dx dy =
\]

\[
c^st \int_0^\infty E(1, y) \frac{(D_{1,-1})^2\varphi(1, y)}{(1+y)^3} dy + c^st \int_1^\infty E(x, 0) \frac{(D_{1,-1})^2\varphi(x, 0)}{x^3} dx =
\]

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\[ c^* \int_0^\infty e^{i(1+y^2)} \frac{(D_{1,-1})^2 \varphi(x,y)}{(1+y)^3} \, dy + c^* \int_1^\infty e^{ix^2} \frac{(D_{1,-1})^2 \varphi(x,0)}{x^3} \, dx = \]

\[ c^* \int_0^\infty e^{iy^2} \frac{(D_{1,-1})^2 \varphi(1,y)}{(1+y)^3} \, dy + c^* \int_1^\infty e^{ix^2} \frac{(D_{1,-1})^2 \varphi(x,0)}{x^3} \, dx. \]

It follows that

\[ |B_3| \leq M_3 p_2(\varphi). \]

This yields

\[ |\int_0^\infty \int_1^\infty E(x,y) \varphi(x,y) \, dx \, dy| \leq M p_3(\varphi). \]

For the integral over \([0, \infty) \times [1, \infty)\) we can use the reflection \(x \mapsto y, y \mapsto x\) and write:

\[ \int_1^\infty dy \int_0^\infty e^{i(x^2-y^2)} \varphi(x,y) \, dx = \int_0^\infty dy \int_1^\infty e^{-i(x^2-y^2)} \varphi(y,x) \, dx. \]

Similar to the calculations above, the last integral can be estimated by \(M p_3(\varphi)\). Namely, the minus sign in the exponent only causes minus signs in the constants in front of the integrals, so the absolute values remain the same.

We have now proved that

\[ |\int \int (0,\infty)^2 \setminus ((0,1)^2 \int e^{i(x^2-y^2)} \varphi(x,y) \, dx \, dy| \leq M p_3(\varphi). \]

For the other three quadrants we use reflections \(x \mapsto -x\) and \(y \mapsto -y\). This then results in:

\[ |\int \int \mathbb{R}^2 \setminus([-1,1]^2 \int e^{i(x^2-y^2)} \varphi(x,y) \, dx \, dy| \leq M p_3(\varphi). \]

Because \([-1,1]^2\) is compact, the integral over this square can be estimated by \(M p_0(\varphi)\), so we have now proved:

\[ |\langle T, \varphi \rangle| \leq M p_3(\varphi) \quad \text{(2.7)} \]

i.e. \(\text{s.o.}(T) \leq 3\) and this is (2.5).

To prove that (2.6) does not hold for all \(\varphi \in D\), we need a sequence \(\varphi_n \in D\) such that

\[ p_2(\varphi_n) < M, \text{ uniformly in } n \quad \text{(2.8)} \]

and

\[ \int \int e^{i(x^2-y^2)} \varphi_n(x) \, dx \, dy \rightarrow +\infty \text{ for } n \rightarrow \infty. \quad \text{(2.9)} \]

When we have found such a sequence, it is clear that (2.6) does not hold for the \(\varphi_n\), so \(\text{s.o.}(T) = 3\).

Let \(\alpha \in D(\mathbb{R}^2)\) be such that \(0 \leq \alpha \leq 1\), \(\alpha(x,y) = 1\) for \((x,y) \in B(1) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}\) and \(\alpha(x,y) = 0\) for \((x,y) \in (B(2))^c\). Take \(\beta \in B(\mathbb{R}^2)\), with \(\text{supp}(\beta) \subset Q = \ldots\)
Define \( \alpha_n(x, y) = \alpha(\frac{x}{n}, \frac{y}{n}) \beta(x, y) \), then \( \alpha_n \in \mathcal{D} \).

When we now define
\[
\varphi(x, y) = \frac{e^{i(y^2 - x^2)}}{(x + y)^2},
\]
the sequence
\[
\varphi_n(x, y) = \alpha_n(x, y) \varphi(x, y)
\]
satisfies (2.8) and (2.9).

To see (2.8) note that \( \alpha \in \mathcal{D} \), so the derivatives \( D^k(\alpha_n(x, y)) = D^k(\alpha(\frac{x}{n}, \frac{y}{n})) = \frac{1}{n^{k+1}} D^k \alpha(x, y) \)
are uniformly bounded in \( n \).

From this it follows that
\[
p_2(\varphi_n) \leq M \sup_{|k| \leq 2} \sup_{(x, y) \in Q} |D^k \varphi(x, y)|
\]
by Leibniz' rule. To majorize this we need to consider the partial derivatives \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2} \) and \( \frac{\partial^2}{\partial y \partial x} \) of \( \varphi \).

\[
\frac{\partial}{\partial x} \varphi(x, y) = \frac{-2ix e^{i(y^2 - x^2)}}{(x + y)^2} - \frac{2e^{i(y^2 - x^2)}}{(x + y)^3}
\]
So \( \sup_{(x, y) \in Q} \left| \frac{\partial}{\partial x} \varphi(x, y) \right| < +\infty \). The \( \frac{\partial}{\partial y} \)-derivative is equivalent to this one, and therefore also bounded on \( Q \).

\[
\frac{\partial^2}{\partial x^2} \varphi(x, y) = c_1 \frac{x^2 e^{i(y^2 - x^2)}}{(x + y)^2} + c_2 \frac{e^{i(y^2 - x^2)}}{(x + y)^2} + c_3 \frac{xe^{i(y^2 - x^2)}}{(x + y)^3} + c_4 \frac{e^{i(y^2 - x^2)}}{(x + y)^4}
\]
And because all these functions are bounded on \( Q \), \( \sup_{(x, y) \in Q} \left| \frac{\partial^2}{\partial x^2} \varphi(x, y) \right| < +\infty \). And similar for the \( \frac{\partial^2}{\partial y^2} \)-derivative.

\[
\frac{\partial^2}{\partial y \partial x} \varphi(x, y) = c_1 \frac{xy e^{i(y^2 - x^2)}}{(x + y)^2} + c_2 \frac{xe^{i(y^2 - x^2)}}{(x + y)^3} + c_3 \frac{ye^{i(y^2 - x^2)}}{(x + y)^3} + c_4 \frac{e^{i(y^2 - x^2)}}{(x + y)^4}
\]
Because \( \frac{xy}{(x + y)^2} \leq \frac{1}{2} \) on \( Q \), all these functions are bounded on \( Q \).

So (2.8) holds.

The integral sequence is
\[
\int_{\frac{1}{2}}^{\infty} \int_{\frac{1}{2}}^{\infty} e^{i(x^2 - y^2)} \alpha_n(x, y) e^{i(y^2 - x^2)} \frac{1}{(x + y)^2} \, dx \, dy =
\int_{\frac{1}{2}}^{\infty} \int_{\frac{1}{2}}^{\infty} \alpha_n(x, y) \frac{1}{(x + y)^2} \, dx \, dy \geq \int_{\frac{1}{2}}^{\frac{3}{2}} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{(x + y)^2} \, dx \, dy \geq \log(n) + c^*.
\]
So (2.9) holds.

This finishes the proof of the proposition.
Theorem 2.5 For \( p \in \mathbb{Z}_+ \), \( q \in \mathbb{Z}_+ \), the summability order of the distribution \( T_{p,q} \), defined by
\[
(T_{p,q}, \varphi) = \iint x^p y^q e^{i(x^2-y^2)} \varphi(x,y) dxdy
\]
is \( p + q + 3 \).

Proof To prove that s.o.(\( T_{p,q} \)) \( \leq p + q + 3 \), we use induction on \( p + q \).

If \( p + q = 0 \), this is exactly the proposition above.

Assume that s.o.(\( T_{p,q} \)) \( \leq p + q + 3 \) for \( 0 \leq p + q \leq m \). Then if \( p + q = m + 1 \), (we may assume that \( p \neq 0 \)), it follows
\[
(T_{p,q}, \varphi) = \iint x^p y^q e^{i(x^2-y^2)} \varphi(x,y) dxdy = \iint x e^{i(x^2-y^2)} x^{p-1} y^q \varphi(x,y) dxdy =
\]
\[
\begin{align*}
& c \iint \frac{\partial}{\partial x} e^{i(x^2-y^2)} x^{p-1} y^q \varphi(x,y) dxdy = \\
& c \iint e^{i(x^2-y^2)} x^{p-2} y^q \frac{\partial}{\partial x} \varphi(x,y) dxdy = \\
& c \iint e^{i(x^2-y^2)} x^{p-2} y^q \varphi(x,y) dxdy + c \iint e^{i(x^2-y^2)} x^{p-1} y^q \frac{\partial}{\partial x} \varphi(x,y) dxdy
\end{align*}
\]
(2.11) for \( p \neq 1 \). For \( p = 1 \) (2.11) reduces to the second integral, which is dominating in sum order. By the induction hypothesis
\[
|\langle T_{p,q}, \varphi \rangle| \leq M_1 p_{p-2+q+3} + M_2 p_{p-1+q+3} \left( \frac{\partial \varphi}{\partial x} \right) \leq M_{p+q+3}(\varphi)
\]
So s.o.(\( T_{p,q} \)) \( \leq p + q + 3 \).

To prove that the sum order is exactly 3, we consider the sequence \( \varphi_n \) in \( D \), defined by \( \varphi_n(x,y) = \alpha_n(x,y) \varphi(x,y) \). With \( \alpha_n \) the same as in the proof of the proposition above and
\[
\varphi(x,y) = \frac{e^{i(y^2-x^2)}}{(x+y)^{p+q+2}}.
\]
Now we have to prove (similar to (2.8) and (2.9)) that
\[
p_{p+q+2}(\varphi_n) < M, \text{ uniformly in } n
\]
(2.12)
and
\[
\int \int x^p y^q e^{i(x^2-y^2)} \varphi_n(x) dxdy \to +\infty \text{ for } n \to \infty.
\]
(2.13)
By a similar argument the analogue of (2.10) holds:
\[
p_{p+q+2}(\varphi_n) \leq M \sup_{|k| \leq p+q+2} \sup_{(x,y) \in Q} |D^k \varphi(x,y)|
\]
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Using Leibniz' rule again we find
\[
D^k \varphi(x, y) = D^k \left( \frac{1}{(x+y)^{p+q+2}} e^{i(y^2 - x^2)} \right) = \\
\sum_{0 \leq l \leq k} \binom{k}{l} D^l \frac{1}{(x+y)^{p+q+2}} D^{k-l} e^{i(y^2 - x^2)} = 
\]
And the terms
\[
D^l \frac{1}{(x+y)^{p+q+2}} D^{k-l} e^{i(y^2 - x^2)}
\]
are of the form
\[
\sum_{t+s \leq k-l} \frac{c_{t,s}}{(x+y)^{p+q+2+|l|}} x^t y^s e^{i(y^2 - x^2)}
\]
Now because
\[
\frac{x^t y^s}{(x+y)^u} \leq \frac{r^r s^s}{r^n} = r^{r-s-u},
\]
where \( r = \sqrt{x^2 + y^2} \), the terms in the last sum are uniformly bounded if \( t + s - (p + q + 2 + |l|) \leq 0 \). We have
\[
t + s - (p + q + 2 + |l|) \leq |k - l| - (p + q + 2 + |l|) \leq |k| - (p + q + 2) \leq 0
\]
if \( |k| \leq p + q + 2 \). So \( D^k \varphi \) is uniformly bounded for \( |k| \leq p + q + 2 \), i.e.
\[
\rho_{p+q+2}(\varphi_n) < M.
\]
This proves (2.12). The integral sequence in (2.13) is divergent, namely
\[
\int_1^{\infty} \int_1^{\infty} x^p y^q e^{i(x^2 - y^2)} \varphi_n(x) dx dy = \\
\int_{1/2}^{\infty} \int_{1/2}^{\infty} x^p y^q e^{i(x^2 - y^2)} \alpha_n(x, y) \left( \frac{1}{x+y} \right)^{p+q+2} e^{i(y^2 - x^2)} dx dy = \\
\int_1^{\infty} \int_1^{\infty} x^p y^q \alpha_n(x, y) \left( \frac{1}{x+y} \right)^{p+q+2} dx dy \geq \\
\int_1^{1/2} \int_{1/2}^{1/2} x^p y^q \left( \frac{1}{x+y} \right)^{p+q+2} dx dy
\]
(we substitute \( x = yt \), and \( dx = y dt \))
\[
\int_{1/2}^{1/2} dy \int_{1/2}^{1/2} (yt)^p y^q \left( \frac{1}{yt+y} \right)^{p+q+2} y dt = \int_{1/2}^{1/2} dy \int_{1/2}^{1/2} \left( \frac{t^p}{(t+1)^{p+q+2}} \right)^{p+q+2} dt = \\
\int_{1/2}^{1/2} \frac{1}{y} dy \int_{1/2}^{1/2} \left( \frac{t^p}{(t+1)^{p+q+2}} \right) dt \rightarrow +\infty
\]
for \( n \to \infty \). This ends the proof of the theorem. ■

Corollary 2.6 The summability order of \( P(x, y) e^{i(x^2 - y^2)} \) is \( d^0(P) + 3 \).

In the next section we use these results together with the desintegration formula to calculate the sum order of \( e^{iP} \) with \( P \) a non-degenerate quadratic form on \( \mathbb{R}^n \).
2.3 Non-degenerate quadratic forms on \( \mathbb{R}^m \)

We will determine the sum order of \( e^{iP} \) with \( P \) a non-degenerate quadratic form on \( \mathbb{R}^m \), \( P(x) = (Ax, x) \), \( \det(A) \neq 0 \). First we reduce to the case that \( A \) is positive definite, later on we take \( A \) general. For the integral calculations we will need the desintegration formula. This is stated in the first proposition of this section.

**Proposition 2.7** (Desintegration Formula) Let \( f \) be a radial function on \( \mathbb{R}^m \). Then

\[
\int_{\mathbb{R}^m} f(|x|) dx = \int_0^\infty dr \int_{|x|=r} f(|x|) ds_r(x) = \omega_m \int_0^\infty r^{m-1} f(r) dr
\]

where

\[
\omega_m = 2 \frac{\pi^{m/2}}{\Gamma(m/2)}.
\]

One can prove this by using polar coordinates.

**Proposition 2.8** Let \( L : \mathbb{R}^m \to \mathbb{R}^n \) be an invertible and linear map and \( T \) a summable distribution on \( \mathbb{R}^n \) with sum order \( m_0 \). Then the distribution \( T \circ L \) is summable and has sum order \( m_0 \).

**Proof** Let \( \phi \in D(\mathbb{R}^n) \). Then \( (T \circ L, \phi) = J^{-1}(T, \phi \circ L^{-1}) \), where \( J \) is the determinant of the Jacobi matrix of \( L \). So \( |(T \circ L, \phi)| \leq M_0 p_{m_0}(\phi \circ L^{-1}) \). Now for \( D \) of first order \( D(\phi \circ L^{-1}) = (D\phi \circ L^{-1})L^{-1} \), and by induction, using the productrule, it follows that \( D^k(\phi \circ L^{-1}) = (D^k\phi \circ L^{-1})(L^{-1})^k \).

This implies that \( \|D^k(\phi \circ L^{-1})\|_\infty \leq c \|D^k\phi \circ L^{-1}\|_\infty = c \|D^k\phi\|_\infty \).

So \( p_{m_0}(\phi \circ L^{-1}) \leq M p_{m_0}(\phi) \) and

\[
|(T \circ L, \phi)| \leq M_1 p_{m_0}(\phi) \ \forall \phi \in D,
\]

i.e. \( T \circ L \) is summable and has sum order smaller or equal to \( m_0 \), the sum order of \( T \). Because this holds for all linear and invertible maps \( L \) and all summable distributions \( T \), the converse also holds. Namely write \( S = T \circ L \), then \( S \) is summable and \( \text{s.o.}(T) \leq \text{s.o.}(S \circ L^{-1}) \leq \text{s.o.}(S) = \text{s.o.}(T \circ L) \).

**Theorem 2.9** [TH1] Let \( P \) be a positive definite quadratic form on \( \mathbb{R}^m \). Then the summability order of \( T : \varphi \mapsto \int e^{iP(x)}\varphi(x) dx \) is \( m + 1 \).

**Proof** There exists a positive definite matrix \( A \) such that \( P(x) = (Ax, x) \). Thus there exists an orthogonal matrix \( U \), such that \( U^tAU = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \) with \( \lambda_k > 0 \), for \( k = 1, \ldots, m \). If \( W = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_m}) \), then \( W^tDW = I \). So \( W^tU^tAUW = I \). Define \( L = UW \), \( J = \det(L) \) then

\[
\langle T, \varphi \rangle = \int_{\mathbb{R}^m} e^{i(Ax, x)} \varphi(x) dx = J \int_{\mathbb{R}^m} e^{i(Ly, y)} \varphi(Ly) dy =
\]

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\[ J \int_{\mathbb{R}^m} e^{i(L^*A_y,v)} \varphi(Ly) \, dy = J \int_{\mathbb{R}^m} e^{i(f_0,v)} \varphi(Ly) \, dy = \]

\[ J \int_{\mathbb{R}^m} e^{|y|^2} \varphi(Ly) \, dy = J \langle T_0, \varphi \circ L \rangle \]

where \( T_0 \) is the distribution \( \varphi \mapsto \int e^{i|x|^2} \varphi(x) \, dx \). Because \( L \) is an invertible linear transformation, the previous proposition applies: \( \text{s.o.}(T) = \text{s.o.}(T_0) \).

Let \( B \) be the unit ball in \( \mathbb{R}^m \).

\[ \langle T_0, \varphi \rangle = \left( \int_B + \int_{B^c} \right) e^{|x|^2} \varphi(x) \, dx. \]

Because \( B \) is compact the first integral can be estimated by \( M_{p_0}(\varphi) \). The second integral is equal to

\[ \int_{B^c} e^{|x|^2} \varphi(x) \, dx = \omega_m \int_1^\infty r^{m-1} \Phi(r) \, dr \]

where \( \Phi(r) = \int_{|x|=r} \varphi(x) \, ds_r(x) = \int_{|y|=1} \varphi(ry) \, ds_1(y) \). Moreover

\[ \Phi^{(n)}(r) = \int_{|y|=1} \left( \sum_{k=1}^m \frac{\partial}{\partial x_k} \right)^n \varphi(ry) \, ds_1(y) \]

implies that \( \Phi \) belongs to \( \mathcal{D}(\mathbb{R}) \) and \( p_n(\Phi) \leq e^{\epsilon_t} p_n(\varphi) \). By proposition 2.2

\[ | \int_{B^c} e^{|x|^2} \varphi(x) \, dx | \leq M_{p_{m+1}}(\Phi) = M_{p_{m+1}}(\varphi) \forall \varphi \in \mathcal{D}(\mathbb{R}^m). \]

Therefore the sum order of \( T_0 \) is at most \( m+1 \). To see that it is not smaller than \( m+1 \), consider a sequence of radial functions \( \varphi_n \) with support in \((B(\frac{1}{2}))^c \). Then \( \varphi_n(x) = \Phi_n(r) \) if \( |x| = r \), \( \text{supp}(\Phi_n) \subset \left[ \frac{1}{2}, \infty \right) \). Let \( \alpha_n \in \mathcal{D}(\mathbb{R}) \) be as in the proof of proposition 2.1: \( 0 \leq \alpha_n \leq 1 \) and \( \alpha_n(x) = 1 \) for \( 1 \leq x \leq n \). Define

\[ \Phi_n(r) = \alpha_n(r) e^{-ir^2} r^{-m}. \]

Then the derivatives \( D^k \varphi_n \) are linear combinations of derivatives of \( \Phi_n \) up to order \( |k| \) with bounded coefficients. For example

\[ \frac{\partial \varphi_n}{\partial x_i}(x) = \Phi'_n(r) \frac{\partial r}{\partial x_i} = \Phi'_n(r) \frac{x_i}{r} = O(\Phi'_n(r)). \]

The derivatives up to order \( m \) of \( \Phi_n \) are uniformly bounded in \( n \), so \( p_m(\varphi_n) \) is uniformly bounded in \( n \). On the other hand, the integral sequence diverges:

\[ \int_{\mathbb{R}^m} e^{|x|^2} \varphi_n(x) \, dx \geq \int_{1 \leq |x| \leq n} e^{|x|^2} e^{-i|x|^2} \frac{|x|^m}{|x|^m} \, dx = \]

\[ \omega_m \int_1^n r^{m-1} e^{ir^2} e^{-ir^2} \, dr = \omega_m \int_1^n \frac{1}{r} \, dr \to +\infty \]

for \( n \to \infty \). This implies that the summability order of \( T_0 \) is exactly \( m+1 \).
Corollary 2.10 Let $P$ be a negative definite quadratic form on $\mathbb{R}^m$. Then the summability order of $T : \varphi \mapsto \int e^{iP(x)}\varphi(x)dx$ is $m + 1$.

Remark Because $e^{i|x|^2} = e^{ix_1^2} \ldots e^{ix_m^2} = e^{ix_1^2} \otimes \ldots \otimes e^{ix_m^2}$, we know from the previous chapter that $\text{s.o.}(e^{i|x|^2}) \leq \text{s.o.}(e^{ix_1^2}) + \ldots + \text{s.o.}(e^{ix_m^2}) = m \text{s.o.}(e^{ix_1^2}) = 2m$. But we have proved in the theorem that $\text{s.o.}(e^{i|x|^2}) = m + 1 < 2m$ for $m > 1$. So we see that we have found examples of the strict inequality in (1.25).

Theorem 2.11 The summability order of the distribution $T : \varphi \mapsto \int e^{iP(x)}\varphi(x)dx$ with $P$ a non-degenerate quadratic form on $\mathbb{R}^m$ is $m + 1$.

Proof For $P$ positive or negative definite this is the previous theorem. Assume $P$ not to be definite. Then by a similar argument as in the previous theorem, $\text{s.o.}(T) = \text{s.o.}(T_0)$, where $T_0 : \varphi \mapsto \int e^{i(Dx,x)}\varphi(x)dx$, with $D = \text{diag}(1,\ldots,1,-1,\ldots,-1)$ ($k$ times 1 and $l$ times -1).

Denote $\eta = (x_1,\ldots,x_k)$, $\xi = (x_{k+1},\ldots,x_m)$ and $x = (\eta,\xi)$. Furthermore, $r^2 = ||\eta||^2 = \sum_{i=1}^k |x_i|^2$ and $r^2 = ||\xi||^2 = \sum_{i=k+1}^m |x_i|^2$. We again use the desintegration formula, but now successively on two variables.

\[ \langle T_0, \varphi \rangle = \int_{\mathbb{R}^m} e^{i(Dx,x)}\varphi(x)dx = \int_{\mathbb{R}^l} \int_{\mathbb{R}^k} e^{i(||\eta||^2-||\xi||^2)}\varphi(\eta,\xi)d\eta = \int_{\mathbb{R}^l} e^{-i||\xi||^2} \int_0^\infty r^{k-1} e^{ir^2}\Phi_{\xi}(r)dr, \]

where

\[ \Phi_{\xi}(r) = \int_{||\xi||=r} \varphi(t,\xi)ds_1(t) = \int_{||\xi||=1} \varphi(rt,\xi)ds_1(t). \]

Continuing,

\[ \langle T_0, \varphi \rangle = \int_0^\infty \int_0^\infty r^{k-1} e^{ir^2} e^{-i^2} \Phi(r,\rho)drd\rho, \]

with

\[ \Phi(r,\rho) = \int_{||u||=\rho} \Phi_u(r)ds_1(u) = \int_{||u||=1} \Phi_{uu}(r)ds_1(u) = \int_{||u||=1} \int_{||t||=1} \varphi(rt,pu)ds_1(t)ds_1(u). \]

Now we use the results from the previous section, and know that $r^{k-1} e^{i(r^2-\rho^2)}$ has sum order $k - 1 + l - 1 + 3 = k + l + 1 = m + 1$ on $\mathbb{R}^2$, which yields an estimate:

\[ |\langle T_0, \varphi \rangle| \leq M_{m+1}(\Phi) \tag{2.14} \]

Now we need a relation between $p_{m+1}(\Phi)$ and $p_{m+1}(\varphi)$.

\[ \frac{\partial}{\partial r}\Phi(r,\rho) = \int_{||u||=1} \int_{||t||=1} \sum_{i=1}^k t_i \frac{\partial}{\partial \eta_i} \varphi(rt,pu)ds_1(t)ds_1(u), \]

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\[
\frac{\partial^n}{\partial r^n} \Phi(r, \rho) = \int_{\|u\|=1} \int_{\|t\|=1} \left( \sum_{i=1}^k t_i \frac{\partial}{\partial \eta_i} \right)^n \varphi(rt, \rho u) ds_1(t) ds_1(u),
\]

And this gives us
\[
\left( \frac{\partial}{\partial r} \right)^n_1 \left( \frac{\partial}{\partial \rho} \right)^n_2 \Phi(r, \rho) = \int_{\|u\|=1} \int_{\|t\|=1} \left( \sum_{i=1}^k t_i \frac{\partial}{\partial \eta_i} \right)^n \left( \sum_{j=1}^l u_j \frac{\partial}{\partial \xi_j} \right)^n \varphi(rt, \rho u) ds_1(t) ds_1(u).
\]

From this the relation follows:
\[
p_{m+1}(\Phi) \leq M_{p_{m+1}}(\varphi).
\]

So (2.14) can be replaced by
\[
|\langle T_0, \varphi \rangle | \leq M_{p_{m+1}}(\varphi) \forall \varphi \in D(\mathbb{R}^m),
\]
i.e. s.o.(T) \leq m + 1.

To see that the sum order of T_0 is not smaller than \( m + 1 \), consider the sequence of 'biradial' functions \( \varphi_n = \varphi_n(\eta, \xi) \) in \( D(\mathbb{R}^m) \) defined by \( \varphi_n(\eta, \xi) = \Phi_n(\|\eta\|, \|\xi\|) \), where \( \Phi_n \) is similar to the function \( \varphi_n \) used in the proof of theorem 2.5:
\[
\Phi(r, \rho) = \frac{1}{(r + \rho)^m} e^{i(r^2 - \rho^2)}, \quad \Phi_n(r, \rho) = \alpha_n(r, \rho) \Phi(r, \rho),
\]
where \( \alpha_n \) is the same function as before. We also use \( \varphi(\eta, \xi) = \Phi(\|\eta\|, \|\xi\|) \). We know from that proof that \( p_m(\Phi_n) \) is uniformly bounded in \( n \) and that the integrals diverge:
\[
\langle T_0, \varphi_n \rangle = \int \int r^{k-1} \rho^{l-1} e^{i(r^2 - \rho^2)} \Phi_n(r, \rho) dr d\rho = \int \int r^{k-1} \rho^{l-1} e^{i(r^2 - \rho^2)} \frac{\alpha_n(r, \rho)}{(r + \rho)^m} e^{i(r^2 - \rho^2)} dr d\rho \nrightarrow \infty,
\]
for \( n \rightarrow \infty \). So it remains to prove that
\[
p_m(\Phi_n) < M \Rightarrow p_m(\varphi_n) < \bar{M}.
\]

Define again \( Q = (1, 2)^2 \), then
\[
p_m(\varphi_n) = \sup_{|k| \leq m} \sup_{\|\eta\|, \|\xi\| \in Q} \left| D^k \varphi_n(\eta, \xi) \right| \leq M \sup_{|k| \leq m} \sup_{\|\eta\|, \|\xi\| \in Q} \left| D^k \varphi(\eta, \xi) \right| \leq M \sup_{\|\eta\|, \|\xi\| \in Q} |D^k \Phi(\|\eta\|, \|\xi\|)|.
\]

The derivatives \( D^k \Phi(\|\eta\|, \|\xi\|) \) with \( |k| \leq m \) are linear combinations of
\[
\frac{\partial^k}{\partial r^k} \varphi(\|\eta\|, \|\xi\|) D^a \|\eta\| \quad D^b \|\xi\|.
\]
with \( t + s \leq m \) and \( 0 < |\alpha| \leq m, 0 < |\beta| \leq m \). Because \( p_m(\Phi) \) is bounded on \( Q \), the first term can be estimated by a constant. The norm derivatives are treated in the following lemma.

**Lemma 2.12** For \(|k| > 0\),

\[
D^k \|\eta\| = \sum c_i \eta_{i}^l / \|\eta\|^{n_i},
\]

where the sum is finite and the multi-index \( l_i \geq 0, n_i \geq 0, |l_i| \leq n_i, \) for all \( i \).

**Proof of the lemma**

We prove the lemma with the use of induction. For \(|k| = 1\), \( D^k = \frac{\partial}{\partial \eta_j} \) and

\[
\frac{\partial}{\partial \eta_j} \|\eta\| = \frac{\eta_j}{\|\eta\|}.
\]

Assume that for \(|k| \leq k_0\) the lemma holds, then for \(|k| = k_0 + 1\), \( D^k = \frac{\partial}{\partial \eta_j} D^{k-e_j} \) for some \( j, 1 \leq j \leq m \). We apply the induction hypothesis to \( D^{k-e_j} \):

\[
D^k \|\eta\| = \frac{\partial}{\partial \eta_j} D^{k-e_j} \|\eta\| = \frac{\partial}{\partial \eta_j} \sum c_i \|\eta\|^{n_i} =
\]

\[
(\|l_i\| \leq n_i \text{ for all } i)
\]

\[
\sum c_i \frac{\partial}{\partial \eta_j} \|\eta\|^{n_i} = \sum c_i \|\eta\|^{n_i - e_j} + \tilde{c}_i \|\eta\|^{n_i + e_j}
\]

where \( \tilde{c}_i = 0 \) if \( (l_i)_j = 0 \). In this sum we compare: \( |l_i - e_j| = |l_i - 1| \leq n_i - 1 < n_i \) and \( |l_i + e_j| = |l_i| + 1 \leq n_i + 1 < n_i + 2 \). This proves the lemma.

It follows from the lemma that \( D^\alpha \|\eta\| \) and \( D^\beta \|\xi\| \) are bounded on \( Q \) for \(|\alpha| > 0\) and \(|\beta| > 0\).

We have now proved that the derivatives \( D^k \Phi(\|\eta\|, \|\xi\|) \) are bounded on \( Q \) for \(|k| \leq m \) and therefore \( p_m(\varphi_n) \) is uniformly bounded in \( n \).

This proves that the summability order of \( T_0 \) is exactly \( m + 1 \). So the sum order of \( T \) is \( m + 1 \).
Chapter 3

Generalized Fresnel Distributions

In this chapter we try to find out which polynomials $P$ lead to summable distributions $e^{iP}$ on $\mathbb{R}^n$. We first consider the one dimensional case.

3.1 One dimension

For general polynomials $P$ in one dimension we have the following result:

**Proposition 3.1** [E.G.F. Thomas] Let $P : \mathbb{R} \to \mathbb{R}$ be a real polynomial, then

1. $e^{iP}$ is summable $\iff d^0(P) \geq 2$.

2. if $d^0(P) = 2$, s.o.($e^{iP}$) = 2.

3. if $d^0(P) \geq 3$, s.o.($e^{iP}$) = 1.

**Proof** If $d^0(P) = 0$, then $e^{iP} = c^t$, so $\mathcal{F}(e^{iP}) = c\delta$ is not continuous, therefore $e^{iP}$ is not summable. Similarly, if $d^0(P) = 1$, $\mathcal{F}(e^{iP}) = \mathcal{F}(e^{i(a+bx)}) = c\mathcal{F}(e^{ibx}) = c\delta_{-b}$ is also not continuous, and therefore not summable. For polynomials $P$ of degree 2 $e^{iP}$ can be written as $ce^{i(a(x+b))^2}$ by completing the square. By proposition 2.1, using a translation over $-b$, for these polynomials $e^{iP}$ has sum order 2.

For $P$ with $d^0(P) \geq 3$, take $a$ large enough, such that $|P'(x)| > 0$, for $|x| > a$. Because $[-a,a]$ is compact the integral of $e^{iP}\varphi$ over this interval can again be estimated by $M_{p0}(\varphi)$, and we need only consider the integral over $[a,\infty)$. For $\varphi \in D$, we have

$$\int_a^\infty e^{iP(x)}\varphi(x)dx = \int_a^\infty P'(x)e^{iP(x)}\frac{\varphi(x)}{P'(x)}dx =$$

$$c^st\int_a^\infty \frac{d}{dz}[e^{iP(x)}] \frac{\varphi(x)}{P'(x)}dx = c^st\int_a^\infty e^{iP(x)} \frac{d}{dz}\frac{\varphi(x)}{P'(x)}dx + c^st\varphi(a) =$$

$$c^st\int_a^\infty e^{iP(x)} \left( \frac{\varphi'(x)}{P'(x)} - \frac{\varphi(x)P''(x)}{(P'(x))^2} \right)dx + c^st\varphi(a).$$
If $d^\circ(P) = n \geq 3$, $d^\circ(P_0) = 1 - n \leq -2$, and $d^\circ(P_0^{-1}(P_0)) = n - 2 + 2(1 - n) = -n \leq -3$. So $P_0$ and $P_0^{-1}(P_0)$ are integrable on $[a, \infty)$, therefore

$$\left| \int_a^\infty e^{iP(x)} \varphi(x) dx \right| \leq M_{P_0}(\varphi)$$

and $s.o.(e^{iP(x)}) = 1$, because it cannot be zero (that would imply that $e^{iP(x)}$ is integrable in the usual sense, taking $\varphi_n(x) = \alpha(x^2)$ with $\alpha$ from the proof of proposition 2.1). This proves the proposition.

Definition We define the round up map $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ by $\lceil a \rceil = n$ with $n - 1 < a \leq n$.

**Theorem 3.2** Let $E_{k,p}(x) = x^ke^{iP(x)}$ and $n = d^\circ(P) \geq 2$. Then

1. $\left| \int_1^\infty E_{k,p}(x) \varphi(x) dx \right| \leq M_{P_0}(\varphi)$ for $k \leq -2$.
2. $\left| \int_1^\infty E_{-1,p}(x) \varphi(x) dx \right| \leq M_{P_1}(\varphi)$.
3. $s.o.(E_{k,p}) = \lceil \frac{k + 2}{n - 1} \rceil$ for $k \geq 0$.

**Proof** The first statement is clear: $E_{k,p}$ is integrable over $[1, \infty)$ in that case. For 2. take $\varphi \in \mathcal{D}$ and $a$ large enough such that $|P'(x)| > 0$ for $|x| > a$. Then

$$\int_a^\infty E_{-1,p}(x) \varphi(x) dx = \int_a^\infty \frac{1}{x} e^{iP(x)} \varphi(x) dx = \int_a^\infty P'(x) e^{iP(x)} \frac{\varphi(x)}{xP'(x)} dx =$$

$$c^s \int_a^\infty \frac{d}{dx}(e^{iP(x)}) \frac{\varphi(x)}{xP'(x)} dx = c^s \int_a^\infty e^{iP(x)} \frac{d}{dx} \frac{\varphi(x)}{xP'(x)} dx + c^s \varphi(a) =$$

$$c^s \int_a^\infty e^{iP(x)} \left( \frac{\varphi(x)}{xP'(x)} - \frac{\varphi(x)P''(x)}{x^2P'(x)} \right) dx + c^s \varphi(a).$$

Now considering the degrees: $d^\circ(P_0^{-1}(P_0)) = -n \leq -2$, $d^\circ(P_0^{-1}(P_0)) = -n - 1 \leq -3$ and $d^\circ(P_0^{-1}(P_0)) = -n - 2 - 2(n - 1) = -n - 1 \leq -3$, we see that all these rational functions are integrable on $[a, \infty)$. Because $[1, a]$ is compact the integral over this interval can be estimated by $M_{P_0}(\varphi)$. This results in:

$$\left| \int_1^\infty E_{-1,p}(x) \varphi(x) dx \right| \leq M_{P_1}(\varphi).$$

This proves 2.

We prove the third statement using induction on $k$. For $k = 0$, we know that $s.o.(e^{iP}) = 2$ if $n = 2$ and $s.o.(e^{iP}) = 1$ if $n \geq 3$ from proposition 3.1. So we have to check whether $\lceil \frac{2}{n - 1} \rceil = 2$ for $n = 2$ and $\lceil \frac{2}{n - 1} \rceil = 1$ for $n \geq 3$. The first equality is obvious. For the second note that $0 < \frac{2}{n - 1} \leq \frac{1}{2} = 1$ if $n \geq 3$. So then $\lceil \frac{2}{n - 1} \rceil = 1$. Now, assume that $s.o.(E_{l,p}) = \lceil \frac{2}{n - 1} \rceil$ for $0 \leq l < k - 1$. Then we also have that $s.o.(Q(x)e^{iP(x)}) = \lceil \frac{2}{n - 1} \rceil$
if \( d^0(Q) \leq k - 1 \). Again, we only consider the interval \([a, \infty)\) which is sufficient by compactness of \([-a, a]\) and equivalence to \((-\infty, -a]\). We now have for \( \varphi \in \mathcal{D} \)

\[
\int_a^\infty E_k,p \varphi(x)dx = \int_a^\infty e^{iP(x)} x^k \varphi(x)dx = \int_a^\infty P'(x) e^{iP(x)} \frac{x^k \varphi(x)}{P'(x)} dx = c^{st} \int_a^\infty \frac{d}{dx}(e^{iP(x)} \frac{x^k \varphi(x)}{P'(x)}) dx = \int_a^\infty e^{iP(x)} \frac{d}{dx}(x^k \varphi(x)) dx + c^{st} \varphi(a) =
\]

\[
e^{iP(x)} \frac{x^k \varphi(x)}{P'(x)} + \frac{x^k \varphi(x) P''(x)}{(P'(x))^2} dx + c^{st} \varphi(a).
\]

The degrees of the rational functions are \( d^0(\frac{x^{k-1}}{P'(x)}) = k - 1 - (n - 1) = k - n < k, \)
\( d^0(\frac{x^k}{P'(x)}) = k - (n - 1) = k - n + 1 < k \) and \( d^0(\frac{x^k P''(x)}{(P'(x))^2}) = k + n - 2 - 2(n - 1) = k - n < k. \)
So we can apply the induction hypothesis to all three:

\[
s.o.(e^{iP(x)} \frac{x^{k-1}}{P'(x)}) = \left\lfloor \frac{k - n + 2}{n - 1} \right\rfloor
\]

\[
s.o.(e^{iP(x)} \frac{x^k}{P'(x)}) = \left\lfloor \frac{k - n + 1 + 2}{n - 1} \right\rfloor
\]

\[
s.o.(e^{iP(x)} \frac{x^k P''(x)}{(P'(x))^2}) = \left\lfloor \frac{k - n + 2}{n - 1} \right\rfloor.
\]

Therefore \( s.o.(E_k,p) = \max(\left\lfloor \frac{k - n + 2}{n - 1} \right\rfloor, \left\lfloor \frac{k - n + 1 + 2}{n - 1} \right\rfloor, \left\lfloor \frac{k - n + 2}{n - 1} \right\rfloor) = \left\lfloor \frac{k + 2 - (n - 1)}{n - 1} \right\rfloor + 1 = \left\lfloor \frac{k + 2}{n - 1} \right\rfloor + 1 = \left\lfloor \frac{k + 2}{n - 1} \right\rfloor. \) So the statement is true for \( k \). This proves 3.

### 3.2 Symmetric polynomials in the two dimensional case

**Proposition 3.3** For \( E_a(x, y) = e^{i(x^2 + y^2 + axy)} \) the following holds:

1. \( s.o.(E_a) = 3, |a| \neq 2. \)

2. \( E_{-2} \) and \( E_2 \) are not summable.

**Proof** Consider the matrix \( A = \begin{pmatrix} 1 & \frac{a}{2} \\ \frac{a}{2} & 1 \end{pmatrix} \)

Then

\[
E_a(x, y) = e^{i(A(\xi), \xi)}.
\]

We can apply theorem 2.11 if \( \det(A) \neq 0 \), that is if \( |a| \neq 2. \) So 1. follows directly from theorem 2.11.

For the proof of 2. we introduce new coordinates: \( u = x + y, v = x - y. \) If \( a = 2, \) then
\[ E_a(x, y) = e^{i(x+y)^2} = e^{iu^2}. \] If \( a = -2 \), then \( E_a(x, y) = e^{i(x-y)^2} = e^{iv^2} \). So (in the first case for example)

\[
\iint E_a(x, y) \varphi(x, y) dx dy = \iint e^{i(x+y)^2} \varphi(x, y) dx dy = 2 \iint e^{iu^2} \varphi\left(\frac{1}{2}(u + v), \frac{1}{2}(u - v)\right) du dv
\]

We see that by a linear transformation of coordinates the distribution only depends on one variable. So it cannot be summable.

**Theorem 3.4** Let \( k \in \mathbb{R} \) and let \( T_k \) be the distribution \( \varphi \mapsto \int_{\mathbb{R}^2} e^{i(x^2+y^2+kx^2y^2)} \varphi(x, y) dx dy \) for \( \varphi \in \mathcal{D}(\mathbb{R}^2) \). Then

1. \( s.o.(T_k) = 1 \) for \( k > 0 \).
2. \( s.o.(T_0) = 3 \).
3. \( s.o.(T_k) = 1 \) for \( k < 0 \).

**Proof** It is sufficient to consider \([1, \infty) \times [0, \infty)\) because the polynomial is symmetric. Namely, using the reflections \( x \mapsto -x, \ y \mapsto -y \) and \( x \mapsto y \) and \( y \mapsto x \) one gets the same result for the other quadrants in \( \mathbb{R}^2 \setminus [-1,1]^2 \). We abbreviate \( E_k(x, y) = e^{i(x^2+y^2+kx^2y^2)} \) and \( B = (x + y)(1 + xy) \). Then \( D_{1,1}E_1 = c^{st}BE_1 \).

To prove the first statement, consider the case \( k = 1 \). Let \( \varphi \in \mathcal{D}(\mathbb{R}^2) \).

\[
\int_0^\infty dy \int_1^\infty E_1(x, y) \varphi(x, y) dx = \int_0^\infty dy \int_1^\infty BE_1(x, y) \frac{\varphi(x, y)}{B} dx = \\
c^{st} \int_0^\infty dy \int_1^\infty D_{1,1}(E_1(x, y)) \frac{\varphi(x, y)}{B} dx = \\
c^{st} \int_0^\infty dy \int_1^\infty D_{1,1}(E_1(x, y)) \frac{\varphi(x, y)}{B} dx + c^{st} \int_0^\infty dy \int_1^\infty E_1(x, y)D_{1,1} \left( \frac{\varphi(x, y)}{B}\right) dx = \\
c^{st} \int_0^\infty dy \int_1^\infty E_1(1, y) \frac{\varphi(1, y)}{B(1, y)} dy + c^{st} \int_1^\infty dy \int_0^\infty E_1(x, 0) \frac{\varphi(x, 0)}{B(x, 0)} dx + \\
c^{st} \int_0^\infty dy \int_1^\infty E_1(x, y)D_{1,1} \left( \frac{\varphi(x, y)}{B}\right) dx = \\
c^{st} \int_0^\infty dy \int_1^\infty E_1(x, y)D_{1,1} \left( \frac{\varphi(x, y)}{B}\right) dx + c^{st} \int_1^\infty dy \int_0^\infty E_1(x, y)D_{1,1} \left( \frac{\varphi(x, y)}{B}\right) dx.
\]

We call these integrals respectively \( I, II \) and \( III \).

\[
|I| \leq c^{st} \int_0^\infty \frac{\varphi(1, y)}{(1+y)^2} dy \leq Mp_0(\varphi)
\]

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and

$$|II| \leq MP_1(\varphi)$$

by proposition 2.2. To estimate III we first have to calculate $D_{1,1,1}$. 

$$D_{1,1,1}(x+y)(1+xy) = \frac{\varphi}{(x+y)(1+xy)} - \frac{\varphi D_{1,1,1}((x+y)(1+xy))}{(x+y)^2(1+xy)^2} =$$

$$\frac{D_{1,1,1} \varphi}{(x+y)(1+xy)} - \frac{2\varphi (1+xy)}{(x+y)^2(1+xy)^2} - \frac{\varphi (x+y)^2}{(x+y)^2(1+xy)^2} =$$

$$\frac{D_{1,1,1} \varphi}{(x+y)(1+xy)} - \frac{2\varphi}{(x+y)^2(1+xy)^2} - \frac{\varphi}{(1+xy)^2}.$$

This yields

$$III = c^{st} \int_0^\infty \int_1^0 E_1(x,y) D_{1,1,1} \left( \frac{\varphi(x,y)}{B} \right) dx =$$

$$c^{st} \int_0^\infty \int_1^0 E_1(x,y) D_{1,1,1} \varphi(x,y) \frac{dx}{(x+y)(1+xy)} +$$

$$c^{st} \int_0^\infty \int_1^0 E_1(x,y) \frac{\varphi(x,y)}{(x+y)^2(1+xy)^2} dx +$$

$$c^{st} \int_0^\infty \int_1^0 E_1(x,y) \frac{\varphi(x,y)}{(1+xy)^2} dx.$$

We will call these integrals respectively $III_1$, $III_2$ and $III_3$. 

For $III_1$ we calculate the integral:

$$\int_0^\infty \int_0^\infty \frac{1}{(x+y)(1+xy)} dx = \int_0^1 \int_1^0 \frac{1}{dx} \frac{1}{x} dx =$$

(Using the substitution $y = \frac{t}{x}$)

$$\int_0^\infty \int_0^\infty \frac{1}{1 + t \frac{x}{t} + \frac{1}{t}} dx = \int_0^\infty \int_1^0 \frac{1}{t(1+t) \frac{x}{t} + 1} \frac{1}{x} dx =$$

( Substitute $u = \frac{x}{\sqrt{t}}$)

$$\int_0^\infty dt \int_0^\infty \frac{1}{t(1+t) u^2 + 1} \sqrt{t} du \leq \int_0^\infty \frac{1}{\sqrt{t}(1+t)} dt \int_0^\infty \frac{1}{u^2 + 1} du < \infty.$$ 

This gives the estimate

$$|III_1| \leq MP_1(\varphi).$$

Now because

$$\left| \frac{1}{(x+y)^2(1+xy)} \right| \leq \left| \frac{1}{(x+y)(1+xy)} \right|$$

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on \([1, \infty) \times [0, \infty)\) we have the estimate

\[
|III_2| \leq M_{p_0}(\varphi).
\]

For \(III_3\) we need an extra integration by parts, because \(\frac{1}{(1+xy)^2}\) is not integrable on \([1, \infty) \times [0, \infty)\).

\[
III_3 = c^t \int_0^\infty dy \int_1^\infty E_1(x, y) \frac{\varphi(x, y)}{(1+xy)^2} dx =
\]

\[
c^t \int_0^\infty dy \int_1^\infty (2x + 2xy^2)e^{(x^2+y^2+z^2)} \frac{\varphi(x, y)}{2x(1+xy)^2(1+y^2)} dx =
\]

\[
c^t \int_0^\infty dy \int_1^\infty D_{1,0}E_1 \frac{\varphi(x, y)}{x(1+xy)^2(1+y^2)} dx =
\]

\[
c^t \int_0^\infty dy \int_1^\infty D_{1,0}(E_1(\varphi(x, y)) \frac{\varphi(x, y)}{x(1+xy)^2(1+y^2)} dx +
\]

\[
c^t \int_0^\infty dy \int_1^\infty E_1D_{1,0}(\varphi(x, y)) \frac{\varphi(x, y)}{x(1+xy)^2(1+y^2)} dx.
\]

Now we first calculate \(D_{1,0}\).

\[
\frac{1}{x(1+xy)^2(1+y^2)} = \frac{(1+xy)^2(1+y^2) + 2xy(1+xy)(1+y^2)}{x^2(1+xy)^2(1+y^2)^2} =
\]

\[
\frac{(1+xy)^2(1+y^2)}{x^2(1+xy)^2(1+y^2)^2} + \frac{2xy(1+xy)(1+y^2)}{x^2(1+xy)^2(1+y^2)^2} = \frac{1}{x^2(1+xy)^2(1+y^2)} + \frac{2y}{x(1+xy)^3(1+y^2)}.
\]

This yields

\[
III_3 = c^t \int_0^\infty dy \int_1^\infty D_{1,0}(E_1(\varphi(x, y)) \frac{\varphi(x, y)}{x(1+xy)^2(1+y^2)} dx +
\]

\[
c^t \int_0^\infty dy \int_1^\infty E_1(x, y) \frac{D_{1,0}\varphi(x, y)}{x(1+xy)^2(1+y^2)} dx +
\]

\[
c^t \int_0^\infty dy \int_1^\infty E_1(\varphi(x, y)) \frac{\varphi(x, y)}{x^2(1+xy)^2(1+y^2)} dx +
\]

\[
c^t \int_0^\infty dy \int_1^\infty E_1 \frac{y\varphi(x, y)}{x(1+xy)^3(1+y^2)} dx.
\]

Again we call these integrals \(III_{31}, III_{32}, III_{33}\) and \(III_{34}\) respectively.

\[
III_{31} = c^t \int_0^\infty e^{(1+2y^2)} \frac{\varphi(1,y)}{(1+y)^2(1+y^2)} dy.
\]
Because $\frac{1}{(1+y^2)}$ is integrable on $[0, \infty)$ this part can be estimated by

$$|III_{31}| \leq M_{p_0}(\varphi).$$

For $III_{32}$ we use the substitution $y = \frac{t}{x}$:

$$\int_0^\infty dy \int_1^\infty \frac{1}{x(1+xy)^2(1+y^2)} dx = \int_0^\infty dt \int_1^\infty \frac{1}{x(1+t)^2(1+\frac{t^2}{x^2})} \frac{1}{x} dx =$$

$$\int_0^\infty dt \int_1^\infty \frac{1}{(1+t)^2(x^2 + t^2)} dx \leq \int_0^\infty dt \int_1^\infty \frac{1}{(1+t)^2} \frac{1}{x^2} dx < \infty.$$ Therefore

$$|III_{32}| \leq M_{p_1}(\varphi).$$

Furthermore

$$\frac{1}{x^2(1+xy)^2(1+y^2)} \leq \frac{1}{x(1+xy)^2(1+y^2)}$$
on $[1, \infty) \times [0, \infty)$. Also

$$\frac{y}{x(1+xy)^3(1+y^2)} \leq \frac{1}{x(1+xy)^2(1+y^2)}$$
on $[1, \infty) \times [0, \infty)$ because $\frac{y}{1+xy} \leq 1$ on this area. These two inequalities then yield

$$|III_{33}| \leq M_{p_0}(\varphi), \quad |III_{34}| \leq M_{p_0}(\varphi).$$

We have now proved that

$$\left| \int_0^\infty dy \int_1^\infty e^{i(x^2+y^2+x^2y^2)} \varphi(x,y) dx \right| \leq M_{p_1}(\varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^2). \quad (3.1)$$

And therefore, as remarked in the beginning of the proof:

$$s.o.(e^{i(x^2+y^2+x^2y^2)}) = s.o.(T_1) = 1.$$ Let $k > 0$, then we use the linear transformation $(x,y) \mapsto (u,v) = (\sqrt{k}x, \sqrt{k}y)$. Now we can write $x^2 + y^2 + kx^2y^2 = \frac{u^2}{k} + \frac{v^2}{k} + \frac{u^2v^2}{k} = \frac{1}{k} (u^2 + v^2 + u^2v^2)$. Therefore

$$\int_0^\infty dy \int_1^\infty e^{i(x^2+y^2+kx^2y^2)} \varphi(x,y) dx = \frac{1}{\sqrt{k}} \int_0^\infty du \int_1^\infty \frac{1}{\sqrt{k}} e^{\frac{i}{k}(u^2+v^2+u^2v^2)} \varphi(u,v) du$$

by substituting $u$ and $v$.

The proof of an estimate similar to (3.1) is the same as for $k = 1$. Namely, the factor $\frac{1}{k}$ in the exponent causes some factor in $\mathcal{B}$ and can be put in the constant in front of the integral. This completes the proof of $I$. 

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The statement 2. is a special case of theorem 2.11.

To prove 3. it suffices to prove the case \( k = -1 \) by the reasoning above. This proof is in two steps. We first consider the quadrants and after that we will have a look at the neighborhoods of the axes.

We first consider the area \([2, \infty) \times [2, \infty)\). For this area we abbreviate \( B = (x + y)(xy - 1) \), so \( D_{1,1}E_{-1} = c^x B E_{-1} \).

\[
\int_2^\infty \int_2^\infty E_{-1}(x,y)\varphi(x,y)\,dxdy = \int_2^\infty \int_2^\infty B E_{-1}(x,y)\frac{\varphi(x,y)}{B}\,dxdy =
\]

\[
c^x \int_2^\infty \int_2^\infty D_{1,1}(E_{-1}(x,y))\frac{\varphi(x,y)}{B}\,dxdy =
\]

\[
c^x \int_2^\infty \int_2^\infty D_{1,1}(E_{-1}(x,y))\frac{\varphi(x,y)}{B}\,dxdy +
\]

\[
c^x \int_2^\infty \int_2^\infty E_{-1}(x,y)D_{1,1}\left(\frac{\varphi(x,y)}{B}\right)\,dxdy.
\]

We call these integrals respectively I and II. For I we use the integration rule for \( D_{1,1} \): \( I = c^x \int_2^\infty \int_2^\infty D_{1,1}(E_{-1}(x,y))\frac{\varphi(x,y)}{B}\,dxdy =
\]

\[
c^x \int_2^\infty \int_2^\infty (E_{-1}\frac{\varphi}{B})(x,2)\,dxdy + c^x \int_2^\infty (E_{-1}\frac{\varphi}{B})(2,y)\,dy =
\]

\[
c^x \int_2^\infty e^{i(4-3x^2)}\frac{\varphi(x,2)}{(x+2)(2x-1)}\,dx + c^x \int_2^\infty e^{i(4-3y^2)}\frac{\varphi(2,y)}{(2+y)(2y-1)}\,dy.
\]

Because the rational functions in these integrals are integrable on \([2, \infty)\), we have the estimate: \( |I| \leq Mp_0(\varphi) \).

We can write the integral II as a sum of integrals

\[
II = c^x \int_2^\infty \int_2^\infty E_{-1}(x,y)D_{1,1}\left(\frac{\varphi(x,y)}{B}\right)\,dxdy =
\]

\[
c^x \int_2^\infty \int_2^\infty E_{-1}(x,y)\frac{D_{1,1}(\varphi(x,y))}{B}\,dxdy + c^x \int_2^\infty \int_2^\infty E_{-1}(x,y)\frac{\varphi(x,y)D_{1,1}(B)}{B^2}\,dxdy.
\]

Let \( II_1 \) be the first integral and \( II_2 \) the second. Then for \( II_1 \) we estimate the following integral:

\[
\int_2^\infty \int_2^\infty \frac{1}{(x+y)(xy-1)}\,dxdy \leq \int_2^\infty \int_2^\infty \frac{4}{3} \frac{1}{(x+y)(xy)}\,dxdy
\]

\[
< c^x \int_2^\infty \int_2^\infty \frac{1}{x^\frac{3}{2}y^\frac{3}{2}}\,dxdy < \infty
\]
where we have used the inequalities \( xy - 1 \geq \frac{3}{4} xy \) and \( x + y \geq \sqrt{xy} \). This yields

\[ |II_1| \leq \mathcal{M}p_1(\varphi). \]

In \( II_2 \) we need the derivative \( D_{1,1}B \):

\[ D_{1,1}B = D_{1,1}((x + y)(xy - 1)) = 2(xy - 1) + (x + y)^2. \]

Therefore

\[ II_2 = c^* \int_{2}^{\infty} \int_{2}^{\infty} E_{-1}(x, y) \frac{\varphi(x, y)D_{1,1}B}{B^2} dy dx = \]

\[ c^* \int_{2}^{\infty} \int_{2}^{\infty} E_{-1}(x, y) \frac{\varphi(x, y)(xy - 1)}{(x + y)^2(xy - 1)^2} dy dx + \]

\[ c^* \int_{2}^{\infty} \int_{2}^{\infty} E_{-1}(x, y) \frac{\varphi(x, y)(x + y)^2}{(x + y)^2(xy - 1)^2} dy dx = \]

\[ c^* \int_{2}^{\infty} \int_{2}^{\infty} E_{-1}(x, y) \frac{\varphi(x, y)}{(xy - 1)^2} dy dx. \]

We call these integrals respectively \( II_{21} \) and \( II_{22} \). For \( II_{21} \) we mention that \( \frac{1}{(x + y)(xy - 1)} < \) on this area and the last function is integrable on \([2, \infty)^2\) as we have seen in \( II_1 \). This gives

\[ |II_{21}| \leq \mathcal{M}p_0(\varphi). \]

The function in \( II_{22} \) is also integrable:

\[ \int_{2}^{\infty} \int_{2}^{\infty} \frac{1}{(xy - 1)^2} dy dx = \int_{2}^{\infty} \frac{dy}{y} \int_{2}^{\infty} \frac{y(yx - 1)^{-2} dx}{y} = -\int_{2}^{\infty} (xy - 1)^{-1} \left[ \frac{dy}{y} \right] = \int_{2}^{\infty} \frac{1}{y(2y - 1)} dy < \infty. \]

From this it follows that

\[ |II_{22}| \leq \mathcal{M}p_0(\varphi). \]

Now we have proved that

\[ |\int_{2}^{\infty} \int_{2}^{\infty} E_{-1}(x, y)\varphi(x, y) dy dx| \leq \mathcal{M}p_1(\varphi). \]

By the reflections \( x \mapsto -x, y \mapsto -y, x \mapsto y \) and \( y \mapsto x \) we have

\[ |\int\int_{R} E_{-1}(x, y)\varphi(x, y) dy dx| \leq \mathcal{M}p_1(\varphi) \quad \forall \varphi \in D(\mathbb{R}^2) \quad (3.2) \]

where \( R = [\pm 2, \infty) \times [\pm 2, \infty) \). This finishes the first step.
For the neighborhoods of the axes we consider \([0, 2] \times [3, \infty)\). We now use the differential operator \(D_{1,-1}\) and \(B = (y-x)(xy+1)\). Then we have \(D_{1,-1}E_{-1} = c^t B E_{-1}\).

\[
\int_3^\infty dy \int_0^2 E_{-1}(x,y) \varphi(x,y) \, dx = \int_3^\infty dy \int_0^2 B E_{-1}(x,y) \frac{\varphi(x,y)}{B} \, dx =
\]

\[
c^t \int_3^\infty dy \int_0^2 D_{1,-1}(E_{-1}(x,y)) \frac{\varphi(x,y)}{B} \, dx =
\]

\[
c^t \int_3^\infty dy \int_0^2 D_{1,-1}(E_{-1}(x,y)) \frac{\varphi(x,y)}{B} \, dx +
\]

\[
c^t \int_3^\infty dy \int_0^2 E_{-1}(x,y) D_{1,-1} \frac{\varphi(x,y)}{B} \, dx.
\]

We call the first integral \(I\) and the second \(II\). By the integration rule we have

\[
I = c^t \int_3^\infty e^{i y^2 \frac{\varphi(0,y)}{y}} \, dy + c^t \int_3^\infty e^{i(4-3y^2)} \frac{\varphi(2,y)}{(y-2)(2y+1)} \, dy +
\]

\[
c^t \int_0^2 e^{i(3-8x^2)} \frac{\varphi(x,3)}{(3-x)(3x+1)} \, dx.
\]

By proposition 2.2 we can estimate these integrals:

\[
|I| \leq M p_1(\varphi).
\]

The integral \(II\) is equal to

\[
II = c^t \int_3^\infty dy \int_0^2 E_{-1}(x,y) D_{1,-1} \frac{\varphi(x,y)}{B} \, dx =
\]

\[
c^t \int_3^\infty dy \int_0^2 E_{-1}(x,y) \frac{D_{1,-1} \varphi(x,y)}{B} \, dx +
\]

\[
c^t \int_3^\infty dy \int_0^2 E_{-1}(x,y) \frac{\varphi(x,y)}{B} \frac{D_{1,-1} B}{B^2} \, dx.
\]

We need to calculate this derivative of \(B\).

\[
\frac{D_{1,-1} B}{B^2} = \frac{D_{1,-1} (y-x)(xy+1)}{(y-x)^2(xy+1)^2} = \frac{-(xy+1)-(xy+1)+(y-x)(y-x)}{(y-x)^2(xy+1)^2} =
\]

\[
\frac{-2(xy+1)+(y-x)^2}{(y-x)^2(xy+1)^2} = \frac{-2}{(xy+1)(y-x)^2} + \frac{1}{(xy+1)^2}.
\]

This yields

\[
II = c^t \int_3^\infty dy \int_0^2 E_{-1}(x,y) \frac{D_{1,-1} \varphi(x,y)}{B} \, dx +
\]

\[
c^t \int_3^\infty dy \int_0^2 E_{-1}(x,y) \frac{\varphi(x,y)}{(xy+1)^2} \, dx +
\]

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\[ c^t \int_0^\infty dy \int_0^2 E_{-1}(x, y) \frac{\varphi(x, y)}{xy + 1} dx. \]

We will abbreviate: \( I_1 \) is the first integral, \( I_2 \) the second and \( I_3 \) the last one. Then for \( I_1 \) we have a look at the following integral.

\[
\int_0^\infty dy \int_0^2 \frac{1}{B} dx = c^t \int_0^\infty dy \int_0^2 \frac{1}{(y - x)(xy + 1)} dx \leq \\
\int_0^\infty dy \int_0^2 \frac{1}{y(y - 2)(xy + 1)} dx = \int_0^\infty \frac{dy}{y(y - 2)} \int_0^2 \frac{ydx}{xy + 1} = \\
\int_0^\infty \frac{1}{y(y - 2)} \log(xy + 1)\big|_{x=0}^\infty dy = \\
\int_0^\infty \frac{1}{y(y - 2)} (\log(2y + 1) - \log 1)dy = \\
\int_0^\infty \frac{\log(2y + 1)}{y(y - 2)} dy \leq \int_0^\infty \frac{\sqrt{2y + 1}}{y(y - 2)} dy \leq \\
c^t \int_0^\infty \frac{\sqrt{y}}{y(y - 2)} dy = c^t \int_0^\infty \frac{1}{\sqrt{y(y - 2)}} dy < \infty.
\]

This gives

\[ |I_1| \leq M_{p_1}(\varphi). \]

By the inequality \( \frac{1}{(y-x)^2(xy+1)} \leq \frac{1}{(y-x)(xy+1)} \) on \([0, 2] \times [3, \infty)\) we also have:

\[ |I_3| \leq M_{p_0}(\varphi). \]

For \( I_2 \) we need to split up the integration area in \([0, \frac{1}{2}] \times [3, \infty)\) and \([\frac{1}{2}, 2] \times [3, \infty)\).

\[ I_2 = I_{21} + I_{22} = \\
c^t \int_0^\infty dy \int_0^{t/2} E_{-1}(x, y) \frac{\varphi(x, y)}{xy + 1} dx + c^t \int_0^\infty dy \int_0^{t/2} E_{-1}(x, y) \frac{\varphi(x, y)}{xy + 1} dx.
\]

On the first area we have an extra integration by parts. For this we use \( D_{0,1}E_{-1} = c^t y(1 - x^2)E_{-1} \).

\[
I_{21} = c^t \int_0^\infty dy \int_0^{t/2} y(1 - x^2)E_{-1}(x, y) \frac{\varphi(x, y)}{y(1 - x^2)(xy + 1)^2} dx = \\
c^t \int_0^\infty dy \int_0^{t/2} D_{0,1}(E_{-1}(x, y)) \frac{\varphi(x, y)}{y(1 - x^2)(xy + 1)^2} dx = \\
c^t \int_0^\infty dy \int_0^{t/2} D_{0,1}(E_{-1}(x, y)) \frac{\varphi(x, y)}{y(1 - x^2)(xy + 1)^2} dx + \\
\]

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\[
\int_3^\infty dy \int_0^{\frac{1}{2}} E_{-1}(x,y) D_{0,1}(\frac{\varphi(x,y)}{y(1-x^2)(xy+1)^2}) dx.
\]

We call the first integral \(I_{211}\) and the second \(I_{212}\). Then by the integration rule we have
\[
|I_{211}| = |c^st \int_0^{\frac{1}{2}} E_{-1}(x,3) \frac{\varphi(x,3)}{3(1-x^2)(3x+1)^2} dx| = \\
|c^st \int_0^{\frac{1}{2}} e^{i(\theta-8z^2)} \frac{\varphi(x,3)}{(1-x^2)(3x+1)^2} dx| \leq M \rho_0(\varphi).
\]

For the integral \(I_{212}\) we first calculate the derivative:
\[
D_{0,1}(\frac{1}{y(1-x^2)(xy+1)^2}) = \frac{(1-x^2)(xy+1)^2 + 2xy(1-x^2)(xy+1)}{y^2(1-x^2)(xy+1)^4} = \\
\frac{1}{y^2(1-x^2)(xy+1)^2} + \frac{2x}{y(1-x^2)(xy+1)^3}.
\]

This yields
\[
I_{212} = c^st \int_3^\infty dy \int_0^{\frac{1}{2}} E_{-1}(x,y) D_{0,1}(\frac{\varphi(x,y)}{y(1-x^2)(xy+1)^2}) dx = \\
c^st \int_3^\infty dy \int_0^{\frac{1}{2}} E_{-1}(x,y) \frac{D_{0,1} \varphi(x,y)}{y(1-x^2)(xy+1)^2} dx + \\
c^st \int_3^\infty dy \int_0^{\frac{1}{2}} E_{-1}(x,y) \frac{\varphi(x,y)}{y^2(1-x^2)(xy+1)^2} dx + \\
c^st \int_3^\infty dy \int_0^{\frac{1}{2}} E_{-1}(x,y) \frac{\varphi(x,y)}{y(1-x^2)(xy+1)^3} dx.
\]

We abbreviate: \(I_{2121}\) is the first one, \(I_{2122}\) the second and \(I_{2123}\) the last integral. For \(I_{2121}\) we calculate the following integral:
\[
c^st \int_3^\infty dy \int_0^{\frac{1}{2}} \frac{1}{y(1-x^2)(xy+1)^2} dx = c^st \int_3^\infty dy \int_0^{\frac{1}{2}} \frac{y dx}{y^2} \leq \\
c^st \int_3^\infty dy \int_0^{\frac{1}{2}} \frac{y dx}{\frac{3}{4}(xy+1)^2} = c^st \int_3^\infty dy \int_0^{\frac{1}{2}} \frac{y dx}{(xy+1)^2} = \\
c^st \int_3^\infty \frac{1}{y^2} \int_0^{\frac{1}{2}} \frac{y dx}{xy+1} = c^st \int_3^\infty \frac{1}{y^2} \left( \frac{1}{2y+1} - 1 \right) dy = \\
c^st \int_3^\infty \frac{1}{2y^3 + y^2} dy + c^st \int_3^\infty \frac{1}{y^2} dy.
\]

Because these integrals are finite we have the following estimate
\[
|I_{2121}| \leq M \rho_1(\varphi).
\]
For $II_{2122}$ we use the inequality
\[ \frac{1}{y^2(1-x^2)(xy+1)^2} \leq \frac{1}{y(1-x^2)(xy+1)^2} \text{ on } [0, \frac{1}{2}] \times [3, \infty). \]

Now the calculation of $II_{2121}$ gives
\[ |II_{2122}| \leq M_{p_0}(\varphi). \]

Similarly for $II_{2123}$ we use
\[ \frac{1}{y(1-x^2)(xy+1)^2} \leq \frac{1}{(1-x^2)(xy+1)^2} \]
because $\frac{x}{1+xy} \leq 1$ on this area. This yields
\[ |II_{2123}| \leq M_{p_0}(\varphi). \]

In all we now have
\[ |II_{21}| \leq M_{p_1}(\varphi). \]

For $II_{22}$ we look at
\[ \int_{\frac{1}{2}}^{\infty} \int_{\frac{1}{2}}^{2} \frac{1}{(xy+1)^2} \, dy \, dx = \int_{\frac{1}{2}}^{\infty} \int_{\frac{1}{2}}^{2} \frac{y}{(xy+1)^2} \, \frac{y}{dx} = \int_{\frac{1}{2}}^{\infty} \frac{1}{y(2y+1)} \left( \frac{1}{xy+1} \right) \, dy = \int_{\frac{1}{2}}^{\infty} \frac{1}{y(2y+1)} \, dy + c^* \int_{\frac{1}{2}}^{\infty} \frac{1}{y(\frac{1}{2}y+1)} \, dy. \]

These integrals are finite so
\[ |II_{22}| \leq M_{p_0}(\varphi). \]

We have now proved that
\[ |\int_{3}^{\infty} dy \int_{0}^{2} E_{-1}(x,y) \varphi(x,y) \, dx| \leq M_{p_1}(\varphi). \]

By the reflections this yields
\[ |\int \int_{R} E_{-1}(x,y) \varphi(x,y) \, dx| \leq M_{p_1}(\varphi) \tag{3.3} \]

where $R$ is one of the areas $[0,2] \times [\pm3,\infty), [-2,0] \times [\pm3,\infty), [\pm3,\infty) \times [0,2]$ and $[\pm3,\infty) \times [-2,0]$.

Now the inequalities (3.2) and (3.3) and the compactness of the cross we have avoided finish the proof of 3.
3.3 Symmetric polynomials as exponents on $\mathbb{R}^m$

In this section we consider the generalized Fresnel distributions $e^{i(P+V)}$ with $P$ a polynomial of degree at most 2 (e.g. a quadratic form) and $V$ a special kind of polynomial potential.

**Theorem 3.5** The summability order of the distribution $T$ on $\mathbb{R}^m$ defined by

$$(T, \varphi) = \int_{\mathbb{R}^m} e^{i(P(x)+\sum_{k=1}^m \alpha_k x_k^2)} \varphi(x) dx$$

where $P$ is a polynomial of degree at most 2 and $\alpha_k \neq 0 \forall k$, is smaller or equal to $[\frac{m+1}{2p}]$ for $l \in \mathbb{N}$ and $l \geq 2$.

**Lemma 3.6** Denote on $\mathbb{R}^m$: $r = \|x\|$ and $A = \mathbb{R}^m \setminus B(1)$. Then the function $\frac{1}{r}$ belongs to $B(A)$.

**Proof of the lemma**

To prove this lemma we will prove the following statement:

$$D^k \frac{1}{r} = \sum c_i \frac{x_i^{l_i}}{r^{n_i}}$$

where the sum is finite and the $l_i$ are multi-indices with $l_i \geq 0$ and $n_i > 0$ and $|l_i| < n_i$ for all $i$.

We prove this statement by induction on $|k|$. For $k = 0$ we have $D^k \frac{1}{r} = \frac{1}{|x|}$. Therefore $l_1 = 0$ and $n_1 = 1$ and clearly we have $|l_1| < n_1$.

Now assume the statement is true for integers strictly smaller than $k$. Then for $k$ we have:

$$D^k \frac{1}{r} = D^k \frac{1}{r}$$

It now follows by the induction hypothesis on $D^k-e_j$ that

$$D^k \frac{1}{r} = \sum c_i \frac{x_i^{l_i-e_j}}{r^{n_i}} + c_i \frac{x_i^{l_i+e_j}}{r^{n_i+1}}$$

where $c_i = 0$ if $(l_i)_j = 0$. In this sum we again check the inequality: $|l_i - e_j| = |l_i| - 1 < n_i - 1 < n_i$ and $|l_i + e_j| = |l_i| + 1 < n_i + 1$. This proves the statement for $k$.

Now the lemma follows from the statement:

$$|\frac{x_i^{l_i}}{r^{n_i}}| \leq 1 \quad \forall x \in A.$$

Therefore $\frac{1}{r} \in B(A)$.
Proof of the theorem  First we abbreviate \( E_m = e^{i(P(x)+\sum_{k=1}^{m} \alpha_k x_k^2)} \), \( D_1 = \sum_{k=1}^{m} \frac{\partial}{\partial x_k} \) and \( l_0 = 2l - 1 \). The differential operator \( D_1 \) has similar properties to those of \( D_{a,b} \) in proposition 2.3. Then \( D_1 E_m = c^{st}(c_0 + \sum_{k=1}^{m} c_k x_k + 2l_0 \alpha_k x_k^0) \), where the \( c_k \in \mathbb{R} \), for \( k = 0, 1, \ldots, m \). We write \( B_m = c_0 + \sum_{k=1}^{m} c_k x_k + 2l_0 \alpha_k x_k^0 \).

Now we need to construct an integration area \( Q_m \). Define \( I_k = [0, \infty) \) for \( k \) with \( \alpha_k > 0 \) and \( I_k = (-\infty, 0] \) for \( k \) with \( \alpha_k < 0 \), \( k = 1, \ldots, m \). Similarly, we define \( J_k = [0, R] \) for \( k \) with \( \alpha_k > 0 \) and \( J_k = [-R, 0] \) for \( k \) with \( \alpha_k < 0 \), \( k = 1, \ldots, m \).

Now we take \( Q_m = (\prod_{k=1}^{m} I_k) \setminus (\prod_{k=1}^{m} J_k) \). To determine \( R \) we look at the conditions: \( B_m(x) \geq 1 \) on \( Q_m \) and \( R \geq 1 \). Given \( c_0, \ldots, c_m \) and \( \alpha_1, \ldots, \alpha_m \), there exists an \( R \) which satisfies these conditions.

Proposition 3.7 \( s.o. Q_m \left( \frac{E_m}{B_m} \right) \leq \max([\frac{m+1}{l_0}] - k, 0) \), for \( k \in \mathbb{Z}_+ \).

Proof of the proposition
We use induction on \( m \) to prove this proposition. For \( m = 1 \),

\[
\frac{E_1}{B_1} = \frac{e^{i(P(x)+\alpha_1 x^2)}}{(c_0 + c_1 x + 2l_0 \alpha_1 x^0)^k}.
\]

If \( k = 0 \), we have by theorem 3.2 \( s.o. (E_1) = 1 = \max([\frac{2}{l_0}], 0) \). If \( k > 0 \), \( s.o. Q_1 \left( \frac{E_1}{B_1} \right) = s.o. Q_1 \left( \frac{E_1}{(c_0 + c_1 x + 2l_0 \alpha_1 x^0)^k} \right) = 0 = \max([\frac{2}{l_0}] - k, 0) \).

From this the proposition follows for the case \( m = 1 \).

Now we assume that the proposition holds for all dimensions strictly smaller than \( m \).

Lemma 3.8 \( s.o. Q_m \left( E_m \right) \leq \max([\frac{m+1}{l_0}] - 1), s.o. Q_m \left( \frac{E_m}{B_m} \right) + 1 \).

Proof of the lemma
For \( \varphi \in D(\mathbb{R}^m) \) we have

\[
\int_{Q_m} E_m \varphi(x) dx = \int_{Q_m} B_m E_m \frac{\varphi(x)}{B_m} dx = c^{st} \int_{Q_m} D_1 (E_m) \frac{\varphi(x)}{B_m} dx =
\]

\[
c^{st} \int_{Q_m} D_1 (E_m) \frac{\varphi(x)}{B_m} dx + c^{st} \int_{Q_m} E_m D_1 (\frac{\varphi(x)}{B_m}) dx.
\]

We call the first integral \( I \) and the second \( II \). For \( I \) we first look at the last term in the sum: \( I^{(m)} = \int_{Q_m} \frac{\partial}{\partial x_m} (E_m \frac{\varphi(x)}{B_m}) dx \). We can write

\[
Q_m = \prod_{k=1}^{m-1} J_k \times [R, \infty) \cup (\prod_{k=1}^{m-1} I_k) \setminus (\prod_{k=1}^{m-1} J_k) \times [0, \infty) =
\]

\[
\prod_{k=1}^{m-1} J_k \times [R, \infty) \cup Q_{m-1} \times [0, \infty)
\]

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where the union is disjunct.

Writing \( \tilde{x} = (x_1, \ldots, x_{m-1}) \), this implies by Fubini’s theorem

\[
I^{(m)} = \int_{Q_m} \frac{\partial}{\partial x_m} (E_m \frac{\varphi(x)}{B_m}) dx = 
\]

\[
\int_{\prod_{i=1}^{m-1} J_k} d\tilde{x} \int_{R} \frac{\partial}{\partial x_m} (E_m \frac{\varphi(x)}{B_m}) dx_m + \int_{Q_{m-1}} d\tilde{x} \int_{0}^{\infty} \frac{\partial}{\partial x_m} (E_m \frac{\varphi(x)}{B_m}) dx_m = 
\]

\[
c^{st} \int_{\prod_{i=1}^{m-1} J_k} (E_m \frac{\varphi(x)}{B_m})(\tilde{x}, R)d\tilde{x} + c^{st} \int_{Q_{m-1}} (E_m \frac{\varphi(x)}{B_m})(\tilde{x}, 0)d\tilde{x} = 
\]

\[
c^{st} \int_{\prod_{i=1}^{m-1} J_k} (E_m \frac{\varphi(x)}{B_m})(\tilde{x}, R)d\tilde{x} + c^{st} \int_{Q_{m-1}} (E_{m-1} \frac{\varphi(x)}{B_{m-1}})(\tilde{x})d\tilde{x}.
\]

We abbreviate \( I_1^{(m)} \) is the first integral and \( I_2^{(m)} \) the second. We can estimate now:

\[
|I_1^{(m)}| \leq M p_0(\varphi)
\]

because the integration area here is compact and \( \frac{|E_m|}{B_m} \leq 1 \) on this area. In \( I_2^{(m)} \) we apply the induction hypothesis of the proposition. This yields: the sum order of \( I_2^{(m)} \) is smaller or equal to \( s.o.Q_{m-1}(\frac{|E_{m-1}|}{B_{m-1}}) \leq \max(\lfloor \frac{m}{2} \rfloor - 1, 0) \). In all we have that the summability order of \( I^{(m)} \) is smaller or equal to \( \max(\lfloor \frac{m}{2} \rfloor - 1, 0) \).

For the other \( m - 1 \) terms of \( I \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{m-1}} \right) \) we can do the same. In all we have that the sum order of \( I \) is smaller or equal to \( \max(\lfloor \frac{m}{2} \rfloor - 1, 0) \).

The integral \( II \) can be written

\[
II = c^{st} \int_{Q_m} E_m D_1(\frac{\varphi(x)}{B_m}) dx = 
\]

\[
c^{st} \int_{Q_m} E_m \frac{D_1 \varphi(x)}{B_m} dx + c^{st} \int_{Q_m} E_m \frac{\varphi(x)D_1 B_m}{B_m} dx.
\]

The first integral has sum order smaller or equal to \( s.o.Q_m(\frac{|E_m|}{B_m}) + 1 \). We write the second integral in a slightly different way:

\[
c^{st} \int_{Q_m} E_m \frac{\varphi(x)D_1 B_m}{B_m} dx.
\]

Now we consider the last fraction:

\[
\frac{D_1 B_m}{B_m} = \sum_{k=1}^{m} c_k + 2tt_0 x_k^{i_k} - 1 \sim \frac{1}{\| x \|} \in B(Q_m)
\]

by lemma 3.6. Now it follows by proposition 1.3 that the sum order of the second integral of \( II \) is smaller than the sum order of the first one.

We can conclude \( s.o.Q_m(\frac{|E_m|}{B_m}) \leq \max(\lfloor \frac{m}{2} \rfloor - 1, s.o.Q_m(\frac{|E_m|}{B_m}) + 1, 0) = \max(\lfloor \frac{m}{2} \rfloor - 1, s.o.Q_m(\frac{|E_m|}{B_m}) + 1) \).

This finishes the proof of the lemma.

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Lemma 3.9 For $k \in \mathbb{Z}_+$ such that $s.o.\left(\frac{E_m}{B_m}\right) > 0$, we have $s.o.Q_m\left(\frac{E_m}{B_m}\right) \leq \max(\lceil \frac{m}{l_0} \rceil - (k + 1), s.o.Q_m\left(\frac{E_{m+1}}{B_{m+1}}\right) + 1)$.

Proof of the lemma

For $\varphi \in D(\mathbb{R}^m)$ we have

$$
\int_{Q_m} E_m \frac{\varphi(x)}{B_m^k} dx = \int_{Q_m} B_m E_m \frac{\varphi(x)}{B_m^{k+1}} dx = c^t \int_{Q_m} D_1(E_m) \frac{\varphi(x)}{B_m^{k+1}} dx =
$$

$$
= c^t \int_{Q_m} D_1(E_m) \varphi(x) dx + c^t \int_{Q_m} E_m D_1\left(\frac{\varphi(x)}{B_m^{k+1}}\right) dx.
$$

We abbreviate: $I$ is the first integral and $II$ the second. Then as in the proof of the previous lemma the last term of $I$ (the $\frac{\varphi}{\partial_{x_m}}$-term) is

$$
I^{(m)} = \int_{Q_m} B_m \frac{\varphi(x)}{B_m^{k+1}}(\bar{x}, R) d\bar{x} + \int_{Q_m} E_{m-1}(\bar{x}) \frac{\varphi(\bar{x}, 0)}{B_m^{k+1}} d\bar{x}.
$$

We call the first integral $I_1^{(m)}$ and the second $I_2^{(m)}$. By arguments similar to those in the previous lemma we can estimate $|I_1^{(m)}| \leq M P_0(\varphi)$. Moreover we again apply the induction hypothesis of the proposition and get that the sum order of $I_2^{(m)}$ is smaller or equal to $\max(\lceil \frac{m}{l_0} \rceil - (k + 1), 0)$. Together with the other $m - 1$ terms we get that the sum order of $I$ is smaller or equal to $\max(\lceil \frac{m}{l_0} \rceil - (k + 1), 0)$.

The integral $II$ can be written

$$
II = c^t \int_{Q_m} E_m \frac{D_1 \varphi(x)}{B_m^{k+1}} dx + c^t \int_{Q_m} E_m \varphi(x) D_1 B_m d\bar{x}.
$$

The first integral has sum order smaller or equal to $s.o.Q_m\left(\frac{E_m}{B_m^{k+1}}\right) + 1$ and the summability order of the second integral is smaller.

Now we have proved $s.o.Q_m\left(\frac{E_m}{B_m}\right) \leq \max(\lceil \frac{m}{l_0} \rceil - (k + 1), s.o.Q_m\left(\frac{E_{m+1}}{B_{m+1}}\right) + 1, 0) = \max(\lceil \frac{m}{l_0} \rceil - (k + 1), s.o.Q_m\left(\frac{E_{m+1}}{B_{m+1}}\right) + 1)$.

This finishes the proof of the lemma.

We can now prove the proposition for general dimension $m$.

We have $B_m = (c_0 + \sum_{k=1}^{m} c_k x_k + 2l_{k0} x_k^{l_0})^k \sim ||x||_0^k$. Therefore, if $k l_0 \geq m + 1$ then $\frac{1}{B_m} \in L^1(Q_m)$. Define $k_0 = \lceil \frac{m+1}{l_0} \rceil$. Then $k_0 l_0 \geq m + 1$ and $k_0$ is the smallest integer $k$ for which $\frac{1}{B_m} \in L^1(Q_m)$. Therefore $s.o.Q_m\left(\frac{E_m}{B_m}\right) = 0$ for $k \geq k_0$.

To prove the proposition for dimension $m$, we first write it in a different form:

$$
s.o.Q_m\left(\frac{E_m}{B_m^{k_0-k}}\right) \leq \begin{cases} 0 & k < 0 \\ k & 0 \leq k \leq k_0. \end{cases}
$$

(3.4)
We prove (3.4) by induction on $k$. For $k = 0$ we have already proved this above. Assume that for integers strictly smaller than $k$ with $0 < k \leq k_0$ (3.4) holds. Then we have for $k$, using the second lemma,

\[
\text{s.o.}_{Q_m}(\frac{E_m}{B_m^{k_0-k}}) \leq \max(\frac{m}{l_0} - (k_0 - k + 1), \text{s.o.}_{Q_m}(\frac{E_m}{B_m^{k_0-k+1}}) + 1) \\
\leq \max(\frac{m}{l_0} - (k_0 - k + 1), k - 1 + 1) \\
= \max(\frac{m}{l_0} - (\lceil \frac{m+1}{l_0} \rceil - k + 1), k) \\
= \max(\frac{m}{l_0} - \lceil \frac{m+1}{l_0} \rceil + k - 1, k) \\
= k.
\]

For $k < 0$ we have that $|\frac{1}{B_m^{k_0-k}}| \leq |\frac{1}{B_m^{k_0}}| \in L^1(Q_m)$, so s.o.\(Q_m(\frac{E_m}{B_m^{k_0-k}}) = 0\). This proves (3.4) for all $k$ and therefore the proposition.

Taking $k = 0$ in the proposition yields s.o.\(E_m \leq \lceil \frac{m+1}{l_0} \rceil \) on $Q_m$. For the other quadrants we use the reflections $x_k \mapsto -x_k$. These reflections cause minus signs in the polynomial $P$, but because the proposition holds for any polynomial $P$ of degree at most 2, the sum order remains the same. This yields the theorem:

\[
\text{s.o.}(E_m) \leq \lceil \frac{m+1}{l_0} \rceil \tag{3.5}
\]

by compactness of $[-R, R]^m$.

Remark 1 The result of this theorem is not a precise equality yet. To prove the equality in (3.5) one needs to construct a sequence of test functions as we did before. By comparing this result with the other results in this article one would expect the equality in (3.5).

Remark 2 The case of Fresnel distributions with a potential of this form is a special case of this theorem. We see that the presence of the quadratic form makes no difference in the sum order. The part of the exponent with the highest order is dominating. This means that also degenerate quadratic forms with such a potential have this sum order.
Conclusions

In section 3.1 we have stated the most general result for the one dimensional case in theorem 3.2. We know for which polynomials $P$ on $\mathbb{R}$ the distribution $e^{iP}$ is summable and if so, we know the sum order. Moreover we know the sum order of $x^ke^{iP}$. This means that the one dimensional case is finished now.

For higher dimensions we now know the sum order of the Fresnel distributions. The sum order of $e^{iQ}$ on $\mathbb{R}^m$ is exactly $m + 1$ for $Q$ a non-degenerate quadratic form on $\mathbb{R}^m$. For more general polynomials we have considered $e^{i(P + V)}$ with $P$ a polynomial of degree at most 2 (for example a quadratic form) and $V$ a potential of the form $\sum \alpha_k x_k^2$. These distributions are summable. Moreover, we have seen: the higher the degree of $V$ the lower the sum order. This means that adding such a potential reduces the sum order. Furthermore, after addition the sum order does not depend on the quadratic form anymore. The potential is dominating.

For the case of two dimensions we have considered $P = x^2 + y^2 + kx^2y^2$ as a special case. We have seen that the sum order is equal to 1 for $k \neq 0$. For $k = 0$ the polynomial is a non-degenerate quadratic form and has therefore sum order 3. So, also in this special case we see: a polynomial of higher degree causes a lower sum order.

The sum order of Fresnel distributions on $\mathbb{R}^m$ diverges for $m \to \infty$. Also when we add a potential of the given form the sum order goes to $\infty$. A precise conclusion from this about the Feynman-integral can be found in [TH2, Thm 3.1].

It is clear that we have not given a full answer to the question: for which polynomials $P$ is $e^{iP}$ summable? We know that for a 'degenerate' polynomial $P$ (i.e. $P$ can be written as polynomial in less coordinates by a linear transformation), $e^{iP}$ is not summable, because its Fourier-transform is not continuous in that case. Conversely, we can ask: if $P$ is at least quadratic in all coordinates and $P$ is not degenerate, is $e^{iP}$ automatically summable then? A precise answer to this question has not been given yet.

Finally, to get more information about the Feynman integral one should try an other kind of potentials. For example $V = \sum e^{i|\tau|}$. Maybe one can get the sum order independent of the dimension by using other potentials. Then the sum order remains bounded for $m \to \infty$, which is important for taking the limit.
Bibliography


