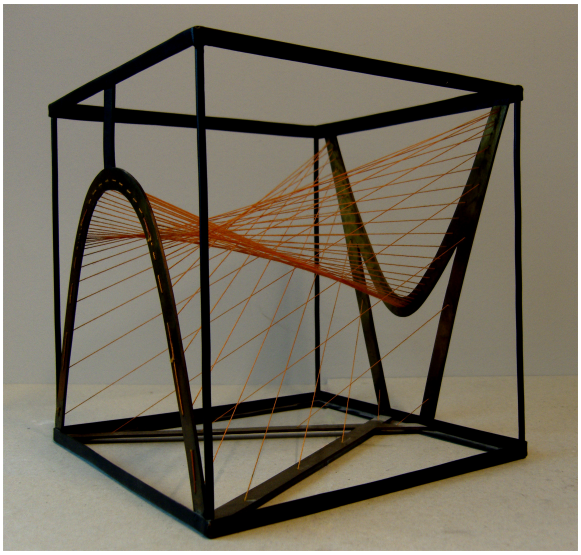
university of groningen

faculty of mathematics and natural sciences

# Tangent planes to discriminant surfaces

## Erik Weitenberg



Bachelor Thesis in Mathematics

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# Tangent planes to discriminant surfaces

**Summary**

The Rijksuniversiteit Groningen has a number of string models of discriminant surfaces. In this thesis we study these discriminant surfaces and identify the models. We begin by defining the discriminant for arbitrary polynomials and we recall some of its properties.

Next, we study the family of tangent planes to such discriminant surfaces. A result is proved that expresses them in terms of the polynomial that defines the discriminant surface itself. This results in a ruling for discriminant surfaces corresponding to a certain family of polynomials. For the surfaces depicted by the string models, it turns out that the ruling is the set of tangent lines to one singular curve in the surface.

Finally, we identify the rulings and each of these singular curves in each of the string models.

Bachelor Thesis in Mathematics
Author: Erik Weitenberg
Supervisor: prof. dr. Jaap Top
Date: January 2010

Institute of Mathematics and Computing Science
P.O. Box 407
9700 AK Groningen
The Netherlands

The photographs on the cover page and in chapter 3 are by courtesy of prof. dr. Marius van der Put.

# Contents

# Chapter 1

# Introduction

In the new RUG building, the Bernoulliborg, one can find on the third floor a display case containing iron frames in which thin thread has been strung. The thread forms surfaces in the frame. The frames are quite old, some of them have even have broken strings and aren't recognizable at all anymore. Most are in good condition, though, due to professor M. van der put, who restored many of the string models. Thanks to the work of Irene Polo-Blanco [9], we know a lot about them. However, for three of the models the work in [9] does not show conclusively what surface they represent (compare [9, p. 105]).

Many are also mentioned in old catalogs, such as the set published by one Martin Schilling and the one published by Walther Dyck [3]. These books also contain descriptions of what the models represent. These are somewhat terse, but they contain some interesting statements about the models, or rather, about the surfaces the models represent.

Our goal is to understand and identify the remaining three models in the RUG collection. We will first elaborate a bit on their history, and brush up on the concept of a discriminant. Then, we will investigate a claim in one of Schilling's descriptions of a model. Lastly, we will try to match each of the models with the surface it represents.

## 1.1 Background

On May 27, 1893, during a meeting of the Dutch Royal Academy of Sciences (Koninklijke Akademie van Wetenschappen, KNAW), the Groningen geometry professor P. H. Schoute presented three string models of surfaces. The minutes (in Dutch) of this meeting also contain a description of their properties: the models represent the discriminant of third, fourth or sixth

degree polynomials, and divide space in two, three and four connected parts, respectively. These parts correspond to the number of real zeroes of the corresponding polynomials [8]. Schoute also wrote a quite clear description in German of the models for Walther Dyck's catalog [3, Nachtrag, Abt.I, pp. 25-28].[1]

A second catalog, this one by Martin Schilling,[2]also describes string models representing the discriminant of a polynomial [12, 11]. One of them shows a fifth-degree polynomial's discriminant—the degree omitted by Schoute—and was made by Mary Emily Sinclair as part of her Master's thesis [14, 13]. She was directed to made it for the Mathematical Seminar of the University of Chicago [11] by Oscar Bolza, and would later become the first woman to receive a Ph.D in mathematics from the University of Chicago.

Schilling's catalog also contained two models derived from fourth-degree polynomials. They were made by Roderich Hartenstein, who designed the models in Göttingen under the direction of Felix Klein [4, 7]. Hartenstein has also written a text for Schilling describing the models and their properties in greater detail [5].

Strangely, Schillings catalog makes no mention of Schoute's models, even though Klein and Hartenstein were most likely well aware of their existence. This may be because Schoute left out a part of the discriminant's zero set in his model (as we will see in section 3.2), or because some of his models seem to be slightly skewed (as is visible in figure 3.3).

## 1.2 The discriminant

**Definition.** The *discriminant* of a polynomial is an expression in its coefficients which gives information about the zeroes of the polynomial. More specifically, it vanishes if the polynomial has a multiple zero. For a monic polynomial $f(x)$ with zeroes $r_i$ counted with multiplicity (so that $f(x) =$

[1]The dates of publication for these catalogues are somewhat confusing. For example, the text about Sinclair's model in Schilling's catalog [11] mentions it was published in 1908, while the catalog itself was published in 1904, according to the front cover. The same goes for the text on Hartenstein's models, which was published in 1909 in the same catalog.

Dyck's catalog [3] has the same peculiarity: while the catalog is dated 1892, it contains a text about a fifth-degree polynomial's discriminant by G. Kerschensteiner, who mentions that Schoute has presented his models in the 1893 KNAW meeting.

[2]The version of Schilling's two works in the Groningen library is actually one book at the time of writing: the second part [11] is simply appended to the first part [12], and not separately referenced in the library system.

$\prod(x - r_i)$ holds), we can define the discriminant as

$$\Delta_f = \prod_{i<j}(r_i - r_j)^2.$$

In this chapter, some useful properties of the discriminant will be discussed, as well as how to compute it.

### 1.2.1 Properties

When considering a polynomial $f_{(a_i)_i}(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, the set of points $(a_0, \ldots, a_{n-1})$ for which $\Delta_f = 0$ forms a hypersurface in $n$-space.

It is important to remember that the zero set of $\Delta_f$ can also be described as the image of $\{(x, a_0, \ldots, a_n) : f = \frac{df}{dx} = 0\}$ under the projection $(x, a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_n)$.

If $f_{(a_i)_i}$ has a zero of higher order than two, or multiple double zeroes, then $(a_0, \ldots, a_{n-1})$ is a singular point of $\{\Delta_f = 0\}$, and in fact, these are the only singular points [1]. Studying these singularities can tell us a lot about the polynomial used to generate it, and most models of surfaces depict the singularities. We will further study this matter in chapter 3.

### 1.2.2 Calculation

Since the discriminant is often used to get information about the zeroes of a polynomial, it makes sense to try and find a way to compute it without using the actual zeroes. To accomplish this, the concept of a resultant will be useful.

The resultant of a pair of polynomials is an expression much like the discriminant. It vanishes precisely when the two polynomials used to compute it share a zero. In fact, up to a constant factor it is just the product of squares of differences of zeroes. We can use this notion, since if a certain polynomial has a multiple zero, its derivative will have this zero as well.

The benefit of this resultant is that it can also be much more pleasantly computed as the determinant of the Sylvester matrix—on which we will elaborate in a moment—of the two polynomials. If $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$, the Sylvester matrix $S(f, g)$ has size $(m+n) \times (m+n)$,

and

$$S(f,g) := \begin{bmatrix} a_n & a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & a_n & \cdots & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & b_m & \cdots & b_1 & b_0 \end{bmatrix},$$

$$R(f,g) := \det S(f,g).$$

The discriminant of a monic polynomal $f$ of degree $n$ satisfies [2, p. 118]

$$\Delta_f = (-1)^{n(n-1)/2} R(f, f').$$

A corollary of this is that $\Delta_f$ is a polynomial in the coefficients of $f$. Additionally, we have obtained a definition for the discriminant of a non-monic polynomial.

This is also one of the methods used by computer algebra systems to calculate the discriminant.

*Example.* The polynomial $f(x) = x^2 + bx + c$ has derivative $2x + b$, so

$$R(f, f') = \det \begin{bmatrix} 1 & b & c \\ 2 & b & 0 \\ 0 & 2 & b \end{bmatrix} = 1 \cdot b^2 - 2 \cdot (b^2 - 2c) = 4c - b^2.$$

From this we see that $\Delta_f = b^2 - 4c$.

Alternatively, write $x^2 + bx + c = (x - r_1)(x - r_2)$. Then $r_1 + r_2 = -b$ and $r_1 r_2 = c$, so $(r_1 - r_2)^2 = (r_1 + r_2)^2 - 4r_1 r_2 = b^2 - 4c$.

## 1.3 On the Sylvester matrix

To illustrate the way the Sylvester matrix is a suitable tool when trying to calculate a discriminant, we will elaborate on it briefly.

Suppose $f$ and $g$ are polynomials over a field $K$, with

$$f = a_n x^n + \cdots + a_0,$$
$$g = b_m x^m + \cdots + b_0,$$

$a_n \neq 0 \neq b_m$, and let

$$P_{<m} \oplus P_{<n} = \{(a,b) : a, b \in K[x], \deg(a) < m, \deg(b) < n\},$$
$$P_{<n+m} = \{a \in K[x] : \deg(a) < n + m\}$$

be two vector spaces, each of dimension $n+m$.

Now, consider the map $\phi : P_{<m} \oplus P_{<n} \to P_{<n+m}$, with $\phi(a,b) = af + bg$. This map is linear, and therefore we can express it as a matrix with respect to the bases

$$\begin{array}{ll} \{(x^{m-1},0),\dots,(1,0),(0,x^{n-1}),\dots,(0,1)\} & \text{for } P_{<m} \oplus P_{<n}, \text{ and} \\ \{x^{n+m-1},\dots,1\} & \text{for } P_{<n+m}: \end{array}$$

$$[\phi] = \begin{bmatrix} a_n & \ddots & 0 & b_m & \ddots & 0 \\ a_{n-1} & \ddots & \vdots & b_{m-1} & \ddots & \vdots \\ \vdots & \ddots & 0 & \vdots & \ddots & 0 \\ a_0 & \ddots & a_n & b_0 & \ddots & b_m \\ 0 & \ddots & a_{n-1} & 0 & \ddots & b_{m-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ddots & a_0 & 0 & \ddots & b_0 \end{bmatrix} = S(f,g)^T.$$

This means that $\det[\phi] = \det S(f,g)$.

Now, to find out when $\phi$ is invertible (i.e. when $\det[\phi] \neq 0$), we consider its kernel. Suppose $(a,b) \in \ker\phi$, i.e. $af + bg = 0$. If $\operatorname{ggd}(f,g) = 1$, then $g|a$ and $f|b$, and $a = b = 0$ because otherwise, their degrees would be too high. If $\operatorname{ggd}(f,g) = d$ and $\deg(d) > 0$, then

$$\frac{g}{d}f - \frac{f}{d}g = 0, \quad \text{so} \quad \left(\frac{g}{d}, -\frac{f}{d}\right) \in \ker\phi.$$

In other words, $\det[\phi] = 0$ if and only if $\operatorname{ggd}(f,g) \neq 1$, which happens precisely when $f$ and $g$ share a zero.

While this does not show that the discriminant and the determinant of $S(f,f')$ are the same, it does show that they both vanish if $f$ has a double zero. As we will see in the next chapter, this is the property we are most interested in.

# Chapter 2

# Tangent hyperplanes to $(\Delta = 0)$

In the text in the *Catalog of mathematical models* [12], string models displaying the discriminant surface of several polynomials are described:

> **Models nr. 2 and 3.** The general equation of fourth degree allows itself by a simple transformation to be put in the form:
>
> $$f(t) = t^4 + 6a_2t^2 + 4a_3t + a_4 = 0.$$
>
> If we interpret $a_2, a_3, a_4$ as Cartesian space coordinates $x, y, z$, this equation represents a family of planes with parameter $t$. The envelope of this family of planes is a developable surface of degree five, the 'discriminant surface of the equation'.
>
> *Translation by me,* [11, section XXXIII], *see figure 2.1*

The same section also contains a description of two other models: one of the same surface as the first, but displaying two elements of the aforementioned family of planes as well; and one of a discriminant surface corresponding to a polynomial of degree five instead of four.

This suggests that for polynomials $f(t) = t^n + x_{n-1}t^{n-1} + \cdots + x_0$ of degree $n \geq 2$, the (hyper)planes formed by fixing $t$ to fixed values and considering the equation $f = 0$ of degree one in the variables $x_{n-1}, \ldots, x_0$, are tangent to the discriminant (hyper)surface $\Delta_f = 0$.

Before we dive into this, we should try to understand what this statement means. To accomplish that, we will first verify it for a simpler case.

Serie XXXIII. 95

**Modelle Nr. 2 u. 3.** Die allgemeine Gleichung vierten Grades lässt sich durch eine einfache Transformation in die Form überführen:

$$f(t) = t^4 + 6a_2\ t^2 + 4a_3\ t + a_4 = 0.$$

Deutet man $a_2$, $a_3$, $a_4$ als rechtwinkliche Raumkoordinaten $x$, $y$, $z$, so stellt diese Gleichung eine Schar von Ebenen mit dem Parameter $t$ dar. Die Enveloppe dieser Ebenenschar ist eine abwickelbare Fläche fünfter Ordnung, die „Discriminantenfläche der Gleichung". Die Fläche hat in ihrer Symmetrieebene als Doppelcurve eine Parabel und ihre Rückkehrkante, deren Punkte als Schnitt je dreier unendlich benachbarten Ebenen bestimmt sind, wird durch die Gleichungen gegeben:

$$x = -t^2,\ y = 2t^3,\ z = -3t^4.$$

Die Fläche zerlegt den ganzen Raum in drei Gebiete, von deren Punkten aus vier, zwei oder keine Schmiegungsebene an die Rückkehrcurve gelegt werden können, entsprechend den Zahlen reeller Wurzeln, die bei einer Gleichung vierten Grades auftreten können. Die allgemeinen Punkte der Discriminantenfläche entsprechen Gleichungen mit einer reellen Doppelwurzel, und die Punkte der Rückkehrcurve Gleichungen mit einer dreifachen Wurzel, die Spitze der Rückkehrcurve endlich der Gleichung $t^4 = 0$ mit der vierfachen Wurzel null.

Diese Verhältnisse werden durch das erste Modell veranschaulicht.

Das zweite Modell enthält ausser der Discriminantenfläche noch zwei ihrer Schmiegungsebenen, die den Werten $\pm t_0$ entsprechen. Hierdurch wird der ganze Raum in neun wesentlich verschiedene Gebiete geteilt, die einen Überblick über die Gleichungen vierten Grades im Hinblick auf die Anzahl der reellen Wurzeln zwischen $\pm t_0$ gestatten.

Eine ausführliche Abhandlung wird beigefügt.

Veröffentlicht 1908 u. 1909.

Figure 2.1: Text from [11, section XXXIII]

## 2.1 An introductory example

Our example concerns $f = t^2 + xt + y \in \mathbb{Z}[x, y, t]$ (in which we think of $x$, $y$ and $t$ as real variables). Regarded as a polynomial in $t$, its discriminant, as we know by now, is $x^2 - 4y$. The discriminant 'hypersurface' is the set of points where the discriminant is zero, which is just the parabola $y = \frac{1}{4}x^2$. The family of surfaces we would like, if you will, to be tangent to this parabola is $\{V_t\} = \{\{(x, y) : t^2 + xt + y = 0\}\}$, parametrized by $t$.

Let us start by describing a tangent line to the parabola in a point $(x_0, y_0)$:

$$\begin{aligned} y &= \frac{dy}{dx}(x_0) \cdot (x - x_0) + y_0 \\ &= \frac{1}{2}x_0 x - \frac{1}{4}x_0^2. \end{aligned}$$

This doesn't yet look like much until we rewrite the equation for $V_t$:

$$y = -xt - t^2$$

If we replace $x_0$ by $-2t$ in the equation for the tangent lines, we indeed obtain the equation for $V_t$. In other words, for any $t$, $V_t$ is a tangent line to the discriminant curve at the point $(-2t, t^2)$.

Proving a more general result will be harder, since calculating the discriminant explicitly for polynomials of arbitrary degree is quite unpleasant. Fortunately, there is a way around that.

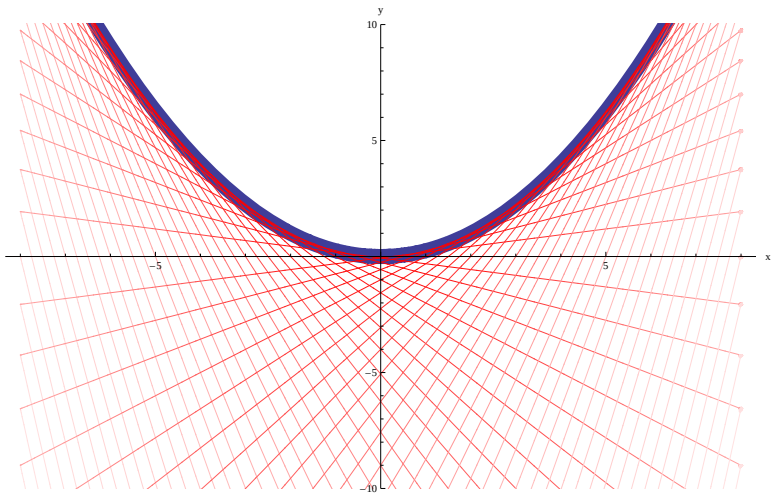



Figure 2.2: For every $x_0$, a $t$ exists such that $t^2 + xt + y$ describes a line tangent to the discriminant curve in $x_0$.

## 2.2 A family of tangent hyperplanes

**Theorem 1.** *Suppose $n \geq 2$ and let*

$$f(x_0, \ldots, x_{n-1}, t) = t^n + x_{n-1}t^{n-1} + \cdots + x_0 \in \mathbb{C}[x_0, \ldots, x_{n-1}, t],$$

*and let $\Delta(x_0, \ldots, x_{n-1})$ be the discriminant of $f$ considered as a (monic) polynomial in $t$. For every $t \in \mathbb{C}$, the hyperplane defined by*

$$V_t = \{\mathbf{p} \in \mathbb{C}^n : f(\mathbf{p}, t) = 0\}$$

*is either a tangent hyperplane to the hypersurface*

$$W = \{\mathbf{p} \in \mathbb{C}^n : \Delta(\mathbf{p}) = 0\}$$

*or is contained in such a tangent plane.*

Since calculating the discriminant directly for polynomials of any order greater than three can become quite time-consuming, we will use another way to identify the values of $x_i$ that lead to multiple zeroes.

**Lemma 2** (Projection from $\mathbb{C}^{n+1}$)**.** *The zero set $W$ of the discriminant can also be described in terms of*

$$W^* = \{(\mathbf{p}, t_0) \in \mathbb{C}^{n+1} : f(\mathbf{p}, t_0) = f'(\mathbf{p}, t_0) = 0\},$$

*where $f'$ denotes the derivative of $f$ with respect to $t$. More precisely, $W = \{\mathbf{p} \in \mathbb{C}^n : \exists t_0 : f(\mathbf{p}, t_0) = f'(\mathbf{p}, t_0) = 0\}$ is the image of $W^*$ under the projection $(\mathbf{p}, t) \mapsto \mathbf{p}$.*

*Proof.* The discriminant of $f$ is zero if and only if $f$ has a zero with multiplicity larger than 1. For any zero $t_0$ with multiplicity $k$, write $f(t) = (t - t_0)^k g(t)$ for a polynomial $g$ such that $g(t_0) \neq 0$. Then $f'(t) = k(t - t_0)^{k-1}g(t) + (t - t_0)^k g'(t) = k(t - t_0)^{k-1}g(t)$, and $f'(t_0) = 0 \Leftrightarrow k > 1$. □

This projection allows us to re-interpret the tangent plane to the discriminant surface as well.

**Definitions.** The *variety* corresponding to an ideal $\mathfrak{a}$ in $\mathbb{C}[x_0, \ldots, x_{n-1}]$ is the intersection of the zero sets of all polynomials in $\mathfrak{a}$, and is denoted $V(\mathfrak{a})$. It is a subset of $\mathbb{C}^n$.

The *ideal $I(V)$ of a variety $V$* is the ideal formed by all polynomials that vanish on $V$.

The *radical* of an ideal $\mathfrak{a}$ in a commutative ring $R$ is the ideal

$$\sqrt{\mathfrak{a}} = \{x \in R : x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

In the case of $R = \mathbb{C}[x_0, \ldots, x_n]$, $\sqrt{\mathfrak{a}} = I(V(\mathfrak{a}))$ [10, Nullstellensatz].

*Example.* To better understand these notions, consider a polynomial $u(x, y) = x^2$. It generates an ideal $(x^2) \in \mathbb{C}[x, y]$, which in turn leads to a variety $V = \{(x, y) : x^2 = 0\}$. This variety may also be expressed as $\{(x, y) : x = 0\}$, which is a more helpful description since precisely all multiples of the polynomial $v(x, y) = x$ vanish on $V$. Therefore, $I(V) = \sqrt{(x^2)} = (x)$.

**Lemma 3** (Notation for tangent hyperplanes)**.** *The space tangent to $W^*$ at a point $\mathbf{q} = (\xi_0, \ldots, \xi_{n-1}, \tau)$ can be expressed as*

$$T_{\mathbf{q}}W^* = \{\mathbf{q} + \mathbf{r} : f(\mathbf{q} + \lambda\mathbf{r}), f'(\mathbf{q} + \lambda\mathbf{r}) \in \lambda^2\mathbb{C}[\lambda]\}.$$

*Proof.* Let $\mathfrak{f} = (f, f')$, the ideal in $\mathbb{C}[x_0, \ldots, t]$ generated by $f$ and $f'$. By definition,

$$T_{\mathbf{q}}W^* = \{\mathbf{q} + \mathbf{r} : \forall g \in I(W^*),\ g(\mathbf{q} + \lambda\mathbf{r}) \in \lambda^2\mathbb{C}[\lambda]\}.$$

By the Nullstellensatz (explained nicely in [10, §5.6]), we know that $I(W^*) = \sqrt{\mathfrak{f}}$. We will now show that $\sqrt{\mathfrak{f}} = \mathfrak{f}$, for if that is the case, the lemma follows.

Every point in $V(\mathfrak{f})$ may be written as $(x_0, x_1, x_2, \ldots, t)$, with uniquely determined $x_0$ and $x_1$ for all choices of $x_2, \ldots, t$. Therefore, every polynomial $g \in \sqrt{\mathfrak{f}}$ has the property that for any $x_2, \ldots, t$ and $x_0$ and $x_1$ chosen such that $f = f' = 0$, $g(x_0, \ldots, t) = 0$.

Divide $g$ by $f$ with respect to $x_0$ to obtain $q_1$ and $r_1$ such that $g = q_1 f + r_1$ and $\deg_{x_0}(r_1) = 0$. Divide $r_1$ by $f'$ with respect to $x_1$ to get $g = q_1 f + q_2 f' + r_2$, where $\deg_{x_1}(r_2) = 0$, i.e. $r_2 \in \mathbb{C}[x_2, \ldots, t]$. Because $g = f = f' = 0$ for any choice of $x_2, \ldots, t$ and corresponding $x_1$ and $x_0$, $r_2$ must also be zero for all these choices, therefore it is the zero polynomial. This means that $g \in \mathfrak{f}$, so $\sqrt{\mathfrak{f}} \subseteq \mathfrak{f}$, which completes the proof since $\sqrt{\mathfrak{f}} \supseteq \mathfrak{f}$ by definition. □

*Proof of theorem 1.* We claim that, given a fixed $t$ and a hyperplane $V_t \in \mathbb{C}^n$, $V_t \subseteq T_{\mathbf{p}}W$ for any $\mathbf{p} \in \mathbb{C}^n$ for which $t$ is a double zero of $f(\mathbf{p})$.

If $\mathbf{p}$ is a singular point of $W$, $T_{\mathbf{p}}W = \mathbb{C}^n$ and the claim is trivial. If not, our claim is reduced to $V_t = T_{\mathbf{p}}W$, because both are linear and have the same dimension. In particular, all points $\mathbf{p} \in W$ corresponding to the same double zero $t$ have the same $V_t$ as, or in, their tangent hyperspace.

To prove this claim, we will consider $T_{\mathbf{q}}W^*$ and project it back to $W$ as $T_{\mathbf{p}}W$, and show that each $V_t$ is exactly equal to a $T_{\mathbf{p}}W$.

For any $\mathbf{q} = (\xi_0, \ldots, \xi_{n-1}, \tau) \in W^*$, we know from lemma 3 that points $\mathbf{q} + \mathbf{r} \in T_{\mathbf{q}}W^*$ are described by the two conditions $f(\mathbf{q} + \lambda\mathbf{r}), f'(\mathbf{q} + \lambda\mathbf{r}) \in \lambda^2\mathbb{C}[\lambda]$. Expanding a Taylor series for both of these functions gives us

$$\begin{aligned} f(\mathbf{q} + \lambda\mathbf{r}) &\equiv f(\mathbf{q}) + \lambda \operatorname{grad}(f)(\mathbf{q}) \cdot \mathbf{r} \mod \lambda^2\mathbb{C}[\lambda], \\ f'(\mathbf{q} + \lambda\mathbf{r}) &\equiv f'(\mathbf{q}) + \lambda \operatorname{grad}(f')(\mathbf{q}) \cdot \mathbf{r} \mod \lambda^2\mathbb{C}[\lambda]. \end{aligned}$$

Since $f(\mathbf{q}) = f'(\mathbf{q}) = 0$ and using $\mathbf{r} = (r_0, \ldots, r_{n-1}, r_n)$, our constraint reduces to

$$0 = \sum_{i=0}^{n-1} \tau^i r_i,$$
$$r_n = \frac{1}{f''(\mathbf{q})} \sum_{i=0}^{n-2} (i+1)\tau^i r_{i+1}.$$

Here we assume $\mathbf{q}$ is a nonsingular point on $W^*$, which implies $f''(\mathbf{q}) \neq 0$ (as we remarked in §1.2.1).

We can now consider the projection of $T_{\mathbf{q}}W^*$ to $C^n$, $T_{\mathbf{p}}W$, which we obtain by leaving out the last coordinate. This is still a tangent space to $W$, and it is given by

$$T_{\mathbf{p}}W = \left\{ \mathbf{p} + \mathbf{r} \in \mathbb{C}^n : \mathbf{p} \in W \wedge \exists\, \tau \text{ double zero of } f(\mathbf{p}),\ \sum_{i=0}^{n-1} \tau^i r_i = 0 \right\}$$

Consider $V_t$ for a fixed $t = \tau$:

$$V_\tau = \{\boldsymbol{\pi} \in \mathbb{C}^n : f(\boldsymbol{\pi}, \tau) = 0\}$$

Let $\mathbf{p}$ be any point such that $f(\mathbf{p})$ has a double zero in $\tau$. Express any point $\boldsymbol{\pi}$ in $\mathbb{C}^n$ relative to $\mathbf{p}$ by setting $\mathbf{r} = \boldsymbol{\pi} - \mathbf{p}$ such that $\boldsymbol{\pi} = \mathbf{p} + \mathbf{r}$. For fixed $\tau$, $\mathbf{p}$ is fixed as well.

$$V_\tau = \{\mathbf{p} + \mathbf{r} : f(\mathbf{p} + \mathbf{r}, \tau) = 0\} = \{\mathbf{p} + \mathbf{r} : f(\mathbf{p}, \tau) + \sum_{i=0}^{n-1} \tau^i r_i = 0\}$$

Since $f(\mathbf{p}) = 0$, this is exactly the definition of $T_{\mathbf{p}}W$. □

## 2.3 Ruled discriminant surfaces

If the polynomial $f$ is of the form $t^n + x_k t^k + x_1 t + x_0$ with $n > k > 1$, the possible choices for $\mathbf{p}$ (the coefficient vector that results in a double zero at a certain fixed $t$) are limited. Because $f(\mathbf{p}, t) = f'(\mathbf{p}, t) = 0$,

$$\begin{aligned} x_1 &= -(nt^{n-1} + x_k k t^{k-1}), \text{ and} \\ x_0 &= -(t^n + x_k t^k + x_1 t) = (n-1)t^n + x_k(k-1)t^k. \end{aligned} \tag{2.1}$$

This describes a family of lines $l_t$ on $W$, given by

$$l_t = \left\{ \begin{pmatrix} 0 \\ -nt^{n-1} \\ (n-1)t^n \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -kt^{k-1} \\ (k-1)t^k \end{pmatrix} ; \lambda \in \mathbb{C} \right\}$$

and for a given $t$, $V_t$ is tangent to $W$ in each point on $l_t$. Also, since $W$ is nonsingular almost everywhere (this is a general property of algebraic varieties [6, chapter 1, theorem 5.3]), we can find for every point on it an $l_t \subset W$ that passes through this point. Therefore, $W$ is a ruled surface in this case.

In fact, this can be extended to any case where $x_2, \ldots, x_{n-1}$ are all linearly dependent on some new variable. If $f$ has more independent coefficients but does contain $x_1 t + x_0$, we obtain a 'ruling' with (hyper)planes instead of lines.

This property is quite useful when one wants to model the surface using strings in a frame. Conversely, the configuration of the lines in a string model may tell us a lot about the surface it represents.

For certain choices of $f$, the surface contains a curve of points corresponding to a polynomial with a triple zero. The tangent lines to smooth points on this curve are also precisely the lines in $W$ corresponding to a constant double zero, which are again the rules of the surface. We will discuss this in depth for the surfaces modeled by Schoute in the next chapter.

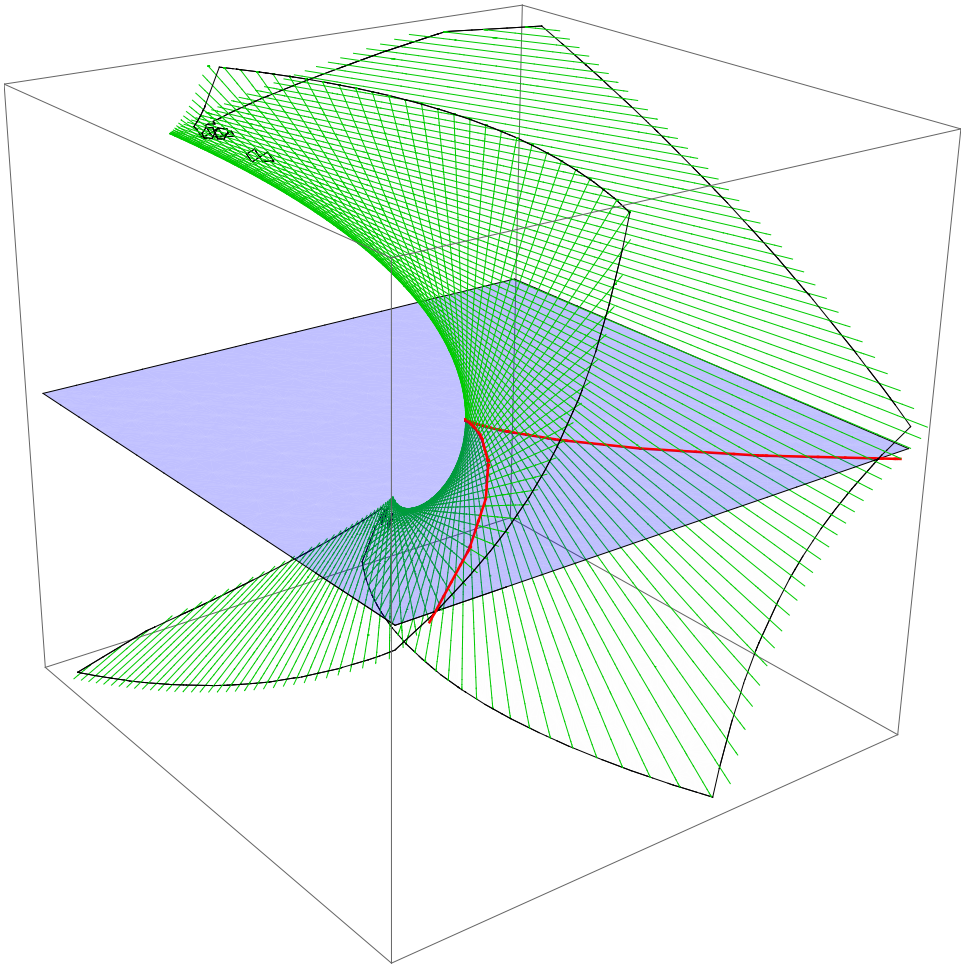


Figure 2.3: A ruling of part of the discriminant's zero set for $f = t^3 + x_2t^2 + x_1t + x_0$. The blue plane $x_2 = 0$ displays for every $t$ the 'start' of the line parametrized by (2.1).

# Chapter 3

# Identifying string models

With the material from the previous chapter, we can identify and describe existing models of discriminant surfaces from the collection of the RUG.

Schoute describes three string models of discriminant surfaces [3, 8]. All three of them match a string model in the RUG collection. From this we conclude that the Groningen collection contains the original models presented by Schoute at the KNAW-meeting on May 27, 1893. We will discuss them individually.

While we do not have the model Sinclair describes in her Master's thesis [14], her discription gives us a fairly good idea of what it must have looked like, so we will discuss it as well.

## 3.1 Schoute's first model

According to Schoute's description, this model depicts the discriminant surface belonging to $f(t) = t^3 + 3xt^2 + 3yt + z$ and divides the space in two parts. The strings in the model are rules of the surface. As explained in section 2.3, the rule belonging to the real number $t$ is parametrized by

$$x(s) = s, \qquad y(s) = -t^2 - 2st, \qquad z(s) = 2t^3 + 3st^2, \qquad s \in \mathbb{R}. \tag{3.1}$$

At every point that corresponds to a triple zero of $f$, the surface is singular. This can be seen as a ridge in the surface. The model depicts this area quite well, as can be seen in figure 3.1.

It is also interesting to consider the singularity corresponding to a triple zero. It can be parametrized by solving $f(t) = (t - s)^3$ for $x$, $y$ and $z$:

$$x(s) = -s, \qquad y(s) = s^2, \qquad z(s) = -s^3, \qquad s \in \mathbb{R}. \tag{3.2}$$

Interestingly, this curve also provides a ruling for the surface: the rules are the curve's tangent lines almost everywhere. This property also holds for Schoute's other two models, but we will prove the result here to benefit from the relatively simple parametrizations. The proof for the other models is analogous.

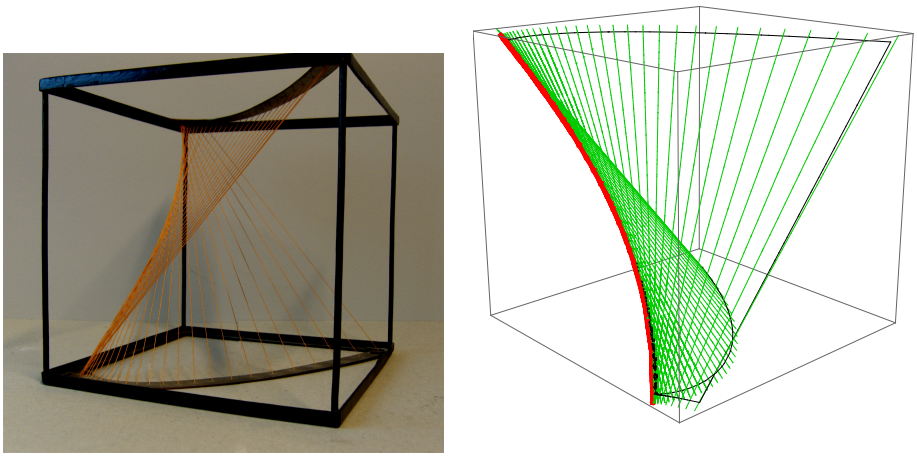
*Proof.* We already have the ruling in (3.1). For a fixed $t$, we may reparametrize the line belonging to $t$ by substituting $-s-t$ for $s$:

$$x(s) = -t - s, \qquad y(s) = t^2 + 2st, \qquad z(s) = -t^3 - 3st^2, \qquad s \in \mathbb{R},$$

which is the tangent line to the curve in (3.2) at the point $(-t, t^2, -t^3)$. $\square$

A remark about this property is in order. If $f$ is not of the form $\cdots + x_1 t + x_0$, the rules may not all be tangent to the curve. For example, consider $f(t) = t^4 + xt^3 + yt^2 + z$. Here, $z = 0$ does belong to the discriminant surface, but its rules are not tangent lines to the curve of points for which $f$ has a triple zero. Another example is $f(t) = t^5 + xt^3 + yt^2 + z$. Here, a curve of points with a triple zero exists, but the line $(x, 0, 0)$ also corresponds to a triple zero.

Schoute remarks that this model is the easiest to depict when the planes $x = \pm 10$ are chosen as boundaries. However, because the singular curve (3.2) in the surface is perfectly vertical at the origin while the ridge in the model is not, I suspect that this model displays the surface as rotated by an angle of about $\frac{\pi}{8}$ radians.



(a) The real model (b) Plot of the ruled surface

Figure 3.1: The first string model mentioned by Schoute. The singularity corresponding to a triple zero is marked in red.

## 3.2 Schoute's second model

The second model is much like the first in terms of the formulae involved, but divides the space in three partitions instead of two, as was Schoute's intent.

Schoute normalizes his polynomial as $f(t) = t^4 + 6xt^2 + 4yt + z$. This leads to the following ruling:

$$x(s) = s, \qquad y(s) = -t^3 - 3st, \qquad z(s) = 3t^4 + 6st^2, \qquad s \in \mathbb{R}.$$

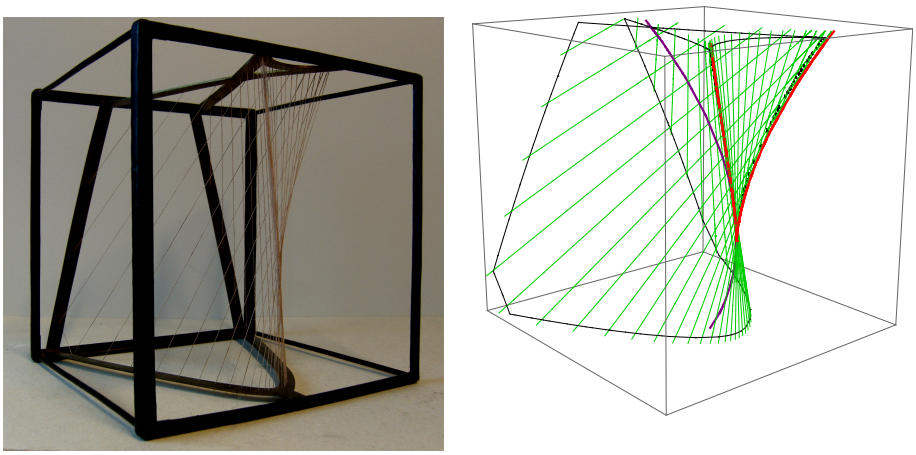
At every point that corresponds to a triple zero of $f$, as well as every point that corresponds to two double zeroes, the surface is singular. The former case can be identified as a ridge along a cuspidal curve in the surface. The latter corresponds to all points in the surface in which two rules intersect; this set of points forms one half of a parabola. The model depicts these areas quite well, as can be seen in figure 3.2.

We can parametrize the two singularities as

$$x(s) = -s^2, \qquad y(s) = 2s^3, \qquad z(s) = -3s^4, \qquad s \in \mathbb{R}$$

for the ridge corresponding to a triple zero, and

$$x(s) = -\frac{1}{3}s^2, \qquad y(s) = 0, \qquad z(s) = s^4, \qquad s \in \mathbb{R}$$



(a) The real model (b) Plot of the ruled surface

Figure 3.2: Schoute's second string model. Singularities are marked by a red line for the points corresponding a triple zero, and a purple line for points corresponding to two double zeroes.

for the half of a parabola corresponding to two double zeroes. These parametrizations can be obtained by solving $f(t) = (t-s)^3(t-a)$ resp. $f(t) = (t-s)^2(t-a)^2$ for $x$, $y$ and $z$; $a$ is eliminated by the constraint on the coefficient of $t^3$.

The other half of this parabola, obtained for $s \in i\mathbb{R}$, also results in real values for $x$, $y$ and $z$ and is this part of the set of points where $\Delta = 0$. Schoute has not made this visible in the model; he does discuss it, but since it corresponds to imaginary zeroes, he does not consider it part of the discriminant surface [3, Nachtrag, Abt. I, p. 26]. Klein [7] and Hartenstein [5] do mention this curve as part of the discriminant surface.

The first singularity (corresponding to a triple zero) acts like the one in model 1, in that the rules are tangent to it, and thus, that the surface is equal to the set of tangent lines to this one curve. The proof is analogous, but this time there is one exception: the tangent line to the singular curve at the origin does not exist, and the line $(x, y, z) = (s, 0, 0)$ is not strictly tangent to the curve. This is easily solved by taking the closure of the surface to get the missing rule, and by taking the limit from either end to get the missing tangent line.

## 3.3 Schoute's third model

According to Schoute, the last model shows a sixth-degree discriminant surface. It corresponds to the polynomial $f(t) = t^6 - 15t^4 + 15xt^2 + 6yt + z$. As explained in section 2.3, the rules now satisfy

$$x(s) = s, \quad y(s) = -t^5 + 10t^3 - 5st, \quad z(s) = 5t^6 - 45t^4 + 15st^2, \quad s \in \mathbb{R}. \tag{3.3}$$

Singular curves occur in those coordinates corresponding to polynomials with triple zeroes or two pairs of double zeroes. The triple zeroes are again on a set of ridges, indicated in red in figure 3.3(b). One may obtain a parametrization in much the same way as with the second model:

$$x(s) = -s^2(s^2 - 6), \quad y(s) = 4s^3(s^2 - 5), \quad z(s) = 45s^4 - 10s^6, \quad s \in \mathbb{R}.$$

The pairs of zeroes lie on this curve, indicated in purple,

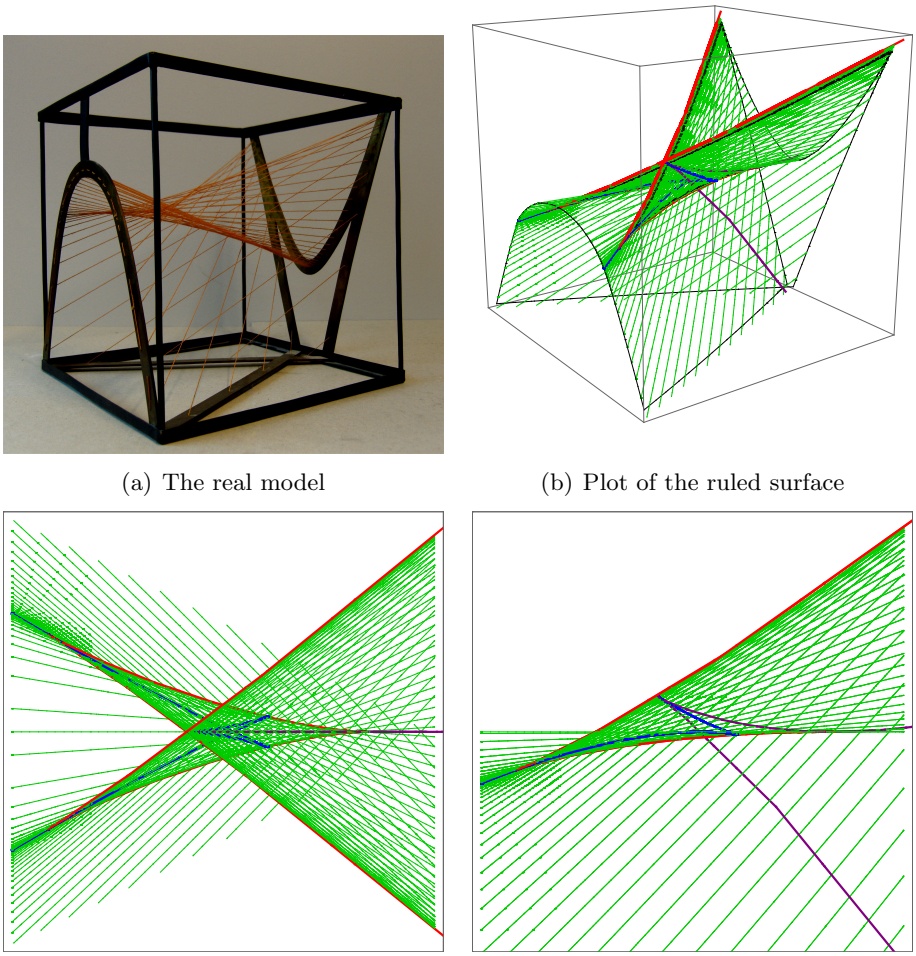
$$x(s) = -1/5s^2(-10 + s^2), \quad y(s) = 0, \quad z(s) = s^4(-15 + 2s^2), \quad s \in \mathbb{R} \cup i\mathbb{R} \tag{3.4}$$

(of which only the part for $s \in \mathbb{R}$ lies inside the surface), as well as on the

curves that satisfy the following system, indicated in blue:

$$\begin{aligned} x &= \frac{1}{5}(u^4+v^4)+\frac{4}{5}uv(u^2+v^2-5)-u^2-v^2+u^2v^2 \\ y &= -uv(u+v)(u^2+v^2+uv-5), \\ z &= u^2v^2(u^2+3uv+v^2), \\ 15 &= 2u^2+2v^2+uv \quad \text{and} \quad x,y,z\in\mathbb{R}. \end{aligned}$$

This curve also has parts outside of the surface. These are, as with the second model, points on rules for complex zeroes of $f$ that happen to be real themselves. A parametrization for these may be obtained by allowing $t$



(a) The real model (b) Plot of the ruled surface

(c) Close-up from above (d) Close-up from side

Figure 3.3: Schoute's third string model. Singularities are marked by red curves for triple zeroes and blue and purple curves for two pairs of double zeroes.

in (3.3) to be complex, while restricting $x$, $y$ and $z$ to $\mathbb{R}$. This results in the following values for $t$:

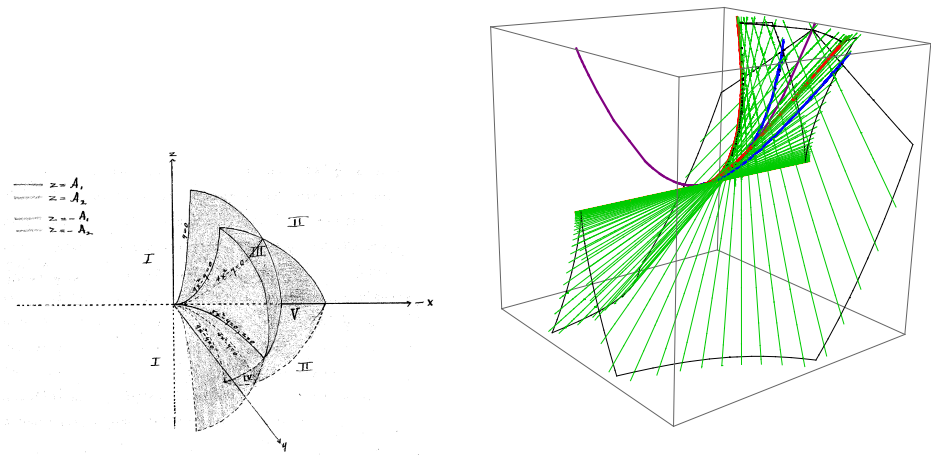
$$t = \pm\sqrt{\frac{3}{8}}\sqrt{3+\sqrt{-11+4s}} + i\sqrt{\frac{1}{8}}\sqrt{-25+5\sqrt{-11+4s}}, \qquad s \in \mathbb{R}.$$

The string model by Schoute shows neither these curves nor the continuation of (3.4) for $s \in i\mathbb{R}$. The choice of edges of the model makes it difficult to see that the curves are not entirely part of the surface, suggesting again that Schoute was not aware of their existence or disregarded them entirely.

Once again, it is the singularity corresponding to a triple zero that all rules are tangent to, and it is again possible to define the surface as the union of the tangent lines to this curve. This time, there are three singular points to contend with.

## 3.4 Sinclair's model

After Schoute had constructed three models for polynomials of degrees 3, 4 and 6, it seems natural to also build a model depicting the discriminant surface of a fifth-degree polynomial. This is exactly what professor O. Bolza asked Mary Sinclair to do for her Master's thesis [14].



(a) Sinclair's sketch of the surface, taken from her thesis [14]

(b) Plot of the ruled surface, boundaries as detailed in [14]

Figure 3.4: Sinclair's string model. Singularities are marked by red and orange lines for the points corresponding a triple zero, and purple and blue lines for points corresponding to two double zeroes.

Sinclair used $f(t) = t^5 + 10xt^3 + 5yt + z$, which also results in a ruled surface. The rules are given by

$$x(s) = s, \qquad y(s) = -t^4 - 6st^2, \qquad z(s) = 4t^5 + 20st^3, \qquad s \in \mathbb{R}.$$

Also, points corresponding to the imaginary $t = i\sqrt{5s}$ result in a polynomial with real coefficients and a double zero and form one half of a parabola, as we will see in a moment.

Triple zeroes are found on the entire line $(x, 0, 0)$ and on the curve

$$x(s) = -\frac{1}{3}s^2, \qquad y(s) = s^4, \qquad z(s) = -\frac{8}{3}s^5, \qquad s \in \mathbb{R}.$$

Two pairs of double zeroes are found on three curves. The first, corresponding to the zeroes $s$ (double), $-s$ (double) and 0, is the parabola

$$x(s) = -\frac{1}{5}s^2, \qquad y(s) = \frac{1}{5}s^4, \qquad z(s) = 0, \qquad s \in \mathbb{R} \cup i\mathbb{R},$$

where imaginary values of $s$ also yield polynomials with real coefficients; however, the rules of the surface only intersect the half of the parabola for which $s \in \mathbb{R}$. The second and third curves look a lot like each other, and correspond to the zeroes $s$ (double), $\frac{1}{2}s(\pm\sqrt{5} - 3)$ (double) and $s(1 \mp \sqrt{5})$:

$$x(s) = \frac{1}{4}(-3 + \sqrt{5})s^2, \quad y(s) = -\frac{1}{2}(-7 + 3\sqrt{5})s^4, \quad z(s) = (-11 + 5\sqrt{5})s^5,$$
$$x(s) = -\frac{1}{4}(3 + \sqrt{5})s^2, \quad y(s) = \frac{1}{2}(7 + 3\sqrt{5})s^4, \qquad z(s) = -(11 + 5\sqrt{5})s^5.$$

For both curves, the domain of $s$ is $\mathbb{R}$.

# Chapter 4

# Discussion

We have seen in chapter 2 that if one considers the set of points where the discriminant of a polynomial $f$ vanishes as a surface, the tangent planes to this surface are the planes described by $f(t) = 0$ for fixed $t$. For certain choices of $f$, we have concluded that for any given tangent plane, a line of points in the surface exists in all of which the tangent plane is tangent to the surface. This means that in these situations, the surface is a ruled one.

We have studied three models in current possession of the RUG, which are almost certainly the original ones documented by Schoute in the minutes of the 1893 KNAW meeting and in Walther Dyck's catalog of mathematical of physical models. The models were intended to use the discriminantal surface of polynomials of third, fourth and sixth degree to divide space into two, three or four connected parts. Sinclair has later constructed a model of the 'missing' discriminantal surface belonging to a fifth-degree polynomial.

While Klein and Hartenstein most likely knew about the models constructed by Schoute, Schilling used new models constructed and documented by Hartenstein. Perhaps they were unsatisfied with the fact that two of Schoute's models did not contain the singular curves that were not otherwise part of the ruled surface, or unhappy with the builder's workmanship (the sixth-degree model, for example, seems to be slightly skewed).

Each of the models contained a singular curve to which all rules were tangent, and we have seen that these discriminant surfaces can be defined in terms of tangent lines to these singular curves. It is probable that this property extends beyond the models we discussed, and that for any choice or for certain choices of $f$, the entire discriminant (hyper)surface is composed of (hyper)planes tangent to a certain singular subset of the hypersurface. Because of the way the proofs work, I have a feeling that it works for a broader spectrum of polynomials than we have considered thus far, but more abstraction or better notation is required to prove a more general result than we already have.

# Appendix A

# Mathematica-code for plots

To generate the plots used in the various figures in this paper, one might use the following *Mathematica*-code.

The code is certainly not the most graceful *Mathematica* ever written, but it can be used as a starting point or as a way to understand how the language can be used to generate images like those used in this paper. The code is written for version 6 of the program.

## For figure 2.3

```
Show[
  {
    ContourPlot3D[Discriminant[t^3 + x t^2 + y t + z, t] == 0,
        {x, -5, 5}, {y, -5, 5}, {z, -5, 5}, Mesh -> None, Axes
        -> None, ContourStyle -> {Opacity[0.1], Green}],
    ContourPlot3D[x == 0, {x, -5, 5}, {y, -5, 5}, {z, -5, 5},
        ContourStyle -> {Blue, Opacity[0.5], BoundaryStyle ->
        None}, Mesh -> None],
    ParametricPlot3D[{0, -3 s^2, 2 s^3}, {s, -5, 5}, PlotStyle
        -> {Thick, Red}]
  }~Join~(
    ParametricPlot3D[{s, -3 #^2 - 2 s #, 2 #^3 + s #^2}, {s,
        -5, 5}, PlotStyle -> RGBColor[0, 0.8, 0]]& /@ Range[-5,
        5, 0.04]
  )
]
```

## For figure 3.1(b)

```
R = RotationMatrix[-Pi/8, {0, 1, 0}];
K = R.{{u}, {v}, {w}};
Show[
```

```
  {
    ContourPlot3D[0, {u, -1, 1}, {v, -1.4, 1.4}, {w, -1/2,
        1/2}, Axes -> None],
    ContourPlot3D[(Discriminant[t^3 + x t^2 + y t + z, t]
        /.Thread[Rule[{x, y, z}, Flatten[K]]] /. v -> (v - 1))
        == 0, {u, -1, 1}, {v, -1.4, 1.4}, {w, -1/2, 1/2}, Mesh
        -> None, ContourStyle -> {None}, MaxRecursion -> 5]
  }
  ~Join~
  (ParametricPlot3D[Inverse[R].{s, -3 #^2 - 2 s #, 2 #^3 + s
      #^2} + {0, 1, 0}, {s, -2, 2}, PlotStyle -> RGBColor[0, .8,
      0]] & /@ Range[-2, 2, .03])
  ~Join~
  {ParametricPlot3D[Inverse[R].{-3 s, 3 s^2, -s^3} + {0, 1, 0},
      {s, -2, 2}, PlotStyle -> {Thickness[0.01], Red}]}
]
```

## For figure 3.2(b)

```
g = 50;
R = RotationMatrix[Pi, {0, 1, 0}];
K = R.{{u}, {v}, {w}};
Show[
  {
    ContourPlot3D[0, {u, -.5, 1}, {v, -4, 4}, {w, -15, 5}, Mesh
        -> None, Axes -> None, ContourStyle -> {Opacity[0.1],
        Green}],
    ContourPlot3D[(Discriminant[t^4 + 6 x t^2 + 4 y t + z, t]
        /. Thread[Rule[{x, y, z}, Flatten[K]]]) == 0, {u, -.5,
        1}, {v, -4, 4}, {w, -15, 5}, Mesh -> None, ContourStyle
        -> {None}, MaxRecursion -> 5],
    ParametricPlot3D[Inverse[R].{-1/12 (u), 0, 1/16 (u)^2}, {u,
        -g, g}, PlotStyle -> {Thickness[0.005], Purple}],
    ParametricPlot3D[Inverse[R].{-3/48 (u)^2, 1/32 (u)^3,
        -3/256 (u)^4}, {u, -g, g}, PlotStyle ->
        {Thickness[0.005], Red}]
    }
    ~Join~
    (ParametricPlot3D[Inverse[R].{(s), -#^3 - 3 s #, (3 #^4 + 6
        s #^2)}, {s, -g, g}, PlotStyle -> RGBColor[0, 0.8, 0]] &
        /@ Range[-5, 5, .1])
]
```

## For figure 3.3(b)

```
Module[{v, K = {u, v, w}, b, t},
  range = ((5 Sign[#] Sqrt[Abs[#]]) & /@ Range[-3.5, 3.5,
      .01])~Join~{-.2, .2};
  Show[
    {
      ContourPlot3D[0, {x, -10, 2}, {y, -90, 90}, {z, -750,
          750}, Mesh -> None, Axes -> None],
```

```
        ParametricPlot3D[-{-s^2 (s^2 - 6), 4 s^3 (s^2 - 5), 45
            s^4 - 10 s^6}, {s, -10, 10}, PlotStyle -> {Red,
            Thickness[0.005]}, PlotPoints -> 100],
        ParametricPlot3D[-{-1/5 s (-10 + s), 0, s^2 (-15 + 2 s)},
            {s, -100, 100}, PlotStyle -> {Purple,
            Thickness[0.005]}, PlotPoints -> 100],
        ParametricPlot3D[v = .25 (-u - Sqrt[15 (8 - u^2)]);
            -{-u^2 + .2 u^4 - 4 u v + .8 u^3 v - v^2 + u^2 v^2 +
            .8 u v^3 + .2 v^4, 5 u^2 v - u^4 v + 5 u v^2 - 2 u^3
            v^2 - 2 u^2 v^3 -  u v^4, -15 u^2 v^2 + 3 u^4 v^2 + 4
            u^3 v^3 + 3 u^2 v^4}, {u, -20, 20}, PlotStyle ->
            {Blue, Thickness[0.005]}],
        ParametricPlot3D[v = .25 (-u + Sqrt[15 (8 - u^2)]);
            -{-u^2 + .2 u^4 - 4 u v + .8 u^3 v - v^2 + u^2 v^2 +
            .8 u v^3 + .2 v^4, 5 u^2 v - u^4 v + 5 u v^2 - 2 u^3
            v^2 - 2 u^2 v^3 - u v^4, -15 u^2 v^2 + 3 u^4 v^2 + 4
            u^3 v^3 + 3 u^2 v^4}, {u, -20, 20}, PlotStyle ->
            {Blue, Thickness[0.005]}],
        ParametricPlot3D[b := Sqrt[-25/8 + 5/8 Sqrt[4 s - 11]]; t
            := Sqrt[3 + 3 b^2/5] + b I; Chop[-{s, -t^5 + 10 t^3 -
            5 s t, 5 t^6 - 45 t^4 + 15 s t^2}], {s, 9, 10},
            PlotStyle -> {Blue, Thickness[0.005]}],
        ParametricPlot3D[b := Sqrt[-25/8 + 5/8 Sqrt[4 s - 11]]; t
            := - Sqrt[3 + 3 b^2/5] + b I; Chop[-{s, -t^5 + 10 t^3
            - 5 s t, 5 t^6 - 45 t^4 + 15 s t^2}], {s, 9, 10},
            PlotStyle -> {Blue, Thickness[0.005]}]
      }
      ~Join~
      (ParametricPlot3D[-{s, 10 #^3 - #^5 - 5 # s, 5 (-9 #^4 +
          #^6 + 3 #^2 s)}, {s, -2, 10}, PlotStyle -> RGBColor[0,
          .8, 0]] & /@ range)
    ]
  ]
```

## For figure 3.4(b)

```
Module[{range},
  range = ((#^3/25) & /@ Range[-5, 5, .1])~Join~{-.2, .2};
  Show[
    {
      ContourPlot3D[0, {x, -15, 15}, {y, -125, 125}, {z,
          -156.25, 156.25}, Mesh -> None, Axes -> None],
      ContourPlot3D[(Discriminant[t^5 + 10 u t^3 + 5 v t + w,
          t]) == 0, {u, -15, 15}, {v, -125, 125}, {w, -160,
          160}, Mesh -> None, ContourStyle -> {None},
          MaxRecursion -> 6],
      ParametricPlot3D[{-1/3 s^2, s^4, -8/3 s^5}, {s, -25, 25},
          PlotStyle -> {Red, Thickness[0.005]}],
      ParametricPlot3D[{s, 0, 0}, {s, -125, 125}, PlotStyle ->
          {Orange, Thickness[0.005]}],
      ParametricPlot3D[{-1/5 s, 1/5 s^2, 0}, {s, -625, 625},
          PlotStyle -> {Purple, Thickness[0.005]}],
```

```
      ParametricPlot3D[{1/4 (-3 + Sqrt[5]) s^2, -1/2 (-7 + 3
          Sqrt[5]) s^4, (-11 + 5 Sqrt[5]) s^5}, {s, -5, 5},
          PlotStyle -> {Blue, Thickness[0.005]}],
      ParametricPlot3D[{-1/4 (3 + Sqrt[5]) s^2, 1/2 (7 + 3
          Sqrt[5]) s^4, -(11 + 5 Sqrt[5]) s^5}, {s, -5, 5},
          PlotStyle -> {Blue, Thickness[0.005]}]
    }
    ~Join~
    (ParametricPlot3D[{s, -#^4 - 6 s #^2, 4 #^5 + 20 s #^3},
        {s, -125, 125}, PlotStyle -> RGBColor[0, .8, 0]] & /@
        range)
  ]
]
```

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