rijksuniversiteit
groningen

Faculteit Wiskunde en
Natuurwetenschappen

Stability of reset controllers

Bachelor Thesis in Applied Mathematics

August 2010

Student: F. de Roo

Supervisor: Prof. dr. A. J. van der Schaft

Abstract

The stability of reset control is not always determined by the stability of the underlying LTI system without reset action. In this report it is first shown by an example how a reset action can destabilize a stable LTI feedback system. Then stability analysis of reset compensators in feedback interconnection are given using dissipativity and Lyapunov theory. The results are based on the fact that the reset compensator has the same $\mathcal{L}_2$-gain as the base compensator if the storage function satisfies a certain condition. The report ends with an application of reset control: cruise control with constant external disturbance.

Keywords: Reset control, Stability analysis, Dissipativity theory, $\mathcal{L}_2$-gain, Lyapunov theory.

Contents

Chapter 1

Introduction

One of the most applied ideas in the design of controllers is integral feedback. In this control scheme the error of some to-be-regulated variable is integrated over time and this integrated quantity is fed back for adjusting the dynamics. This results in many situations in a robust stabilization around the desired value. Nevertheless, the controlled system may have an undesirable overshoot behavior, which is due to the fact that even if the error is zero then still a control action is undertaken (since this action is based on the integral of the error signal). Thus a logical idea is to reset the value of the integrated error once the error becomes zero at some time instant. This is the concept of reset control.

In general, the reset controller is a linear time-invariant (LTI) system whose states, or subsets of states, reset to zero whenever its input and output satisfy certain conditions. One of the main disadvantages of reset controllers (also called reset compensators) is that the stability of the feedback system is not guaranteed by the stability of the underlying LTI system without reset action. It is in fact possible that the reset action destabilizes a stable LTI feedback system.

In this report we investigate the stability of reset controllers. In Chapter 2 'Example of a de-stabilizing reset action' we will show in an example how a reset action can destabilize an asymptotically stable LTI feedback system. This example is an extended version of the example in [4]. To say something more about the stability of reset controllers in general, reset control system theory from [1] and dissipativity theory from [2] are exposed in Chapter 3 'Preliminaries and problem formulation'. Futhermore, Chapter 3 recalls a few notions and results about Lyapunov stability theory. With the use of the theory obtained in the previous chapter, stability conditions for the feedback system are derived in Chapter 4 'Main results'. In Chapter 5 'Application: Cruise control' an application of reset control to cruise control is given. With the use of Matlab and Simulink the mathematical model of cruise control is simulated such that the stability of cruise control can be determined when reset control is applied. Chapter 6 'Conclusion' summarizes the most important results and conclusions from this report. In 'Bibliography' one can find the used references and A 'First Appendix' and B 'Second Appendix' contain respectively the simulation from Simulink and the results from the simulation.

Chapter 2

Example of a de-stabilizing reset action

One of the main disadvantages of reset compensators is that the stability of the feedback system for the compensator without reset action does not always guarantee the stability of the feedback system for the compensator with reset. In fact, the reset action can destabilize a stable LTI feedback system which is shown in the example 2.3 which is an extended version of the example in [4]. But first we need some theory about how a transfer function gives rise to a state-space representation with matrices A, B, C and D, which is done using the *controllable canonical form.*

2.1 Controllable canonical form

Lemma 2.1. *Consider a transfer function $G(s)$ and assume that it is a proper rational function. Then there exist an $n \times n$ matrix A, an $n \times 1$ matrix B, a $1 \times n$ matrix C and a 1×1 matrix D such that*

$$G(s) = C(sI - A)^{-1} + D$$

and the state-space representation of the corresponding single-input single-output system is given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Proof. For the proof of this lemma we refer to [3]. □

Construction 2.2. *Construction of matrices A, B, C and D when G(s) is given.*
Because $G(s)$ is a proper rational function, we can write $G(s) = \frac{\overline{q}(s)}{\overline{p}(s)}$ where $\overline{p}(s) = s^n + \overline{p}_{n-1}s^{n-1} + \ldots + \overline{p}_0$ and $\overline{q}(s) = \overline{q}_n s^n + \overline{q}_{n-1}s^{n-1} + \ldots + \overline{q}_0$ such that $\deg(\overline{q}(s)) \leq \deg(\overline{p}(s)) = n$.

1. If $\deg(\overline{q}(s)) < \deg(\overline{p}(s))$ (i.e., $\overline{q}_n = 0$), then take $D = 0$.
   Then, for $i = 0, 1, \ldots, n-1$, let

$$\begin{aligned} p(s) &= s^n + p_{n-1}s^{n-1} + \ldots + p_0, \qquad && p_i = \overline{p}_i \quad \text{and} \\ q(s) &= q_{n-1}s^{n-1} + \ldots + q_0, && q_i = \overline{q}_i. \end{aligned}$$

2. If $\deg(\overline{q}(s)) = \deg(\overline{p}(s))$ (i.e., $\overline{q}_n \neq 0$), then take $D = \overline{q}_n$.
   Now the transfer function is given by

$$G(s) = \overline{q}_n + \frac{(\overline{q}_{n-1} - \overline{q}_n\overline{p}_{n-1})s^{n-1} + \ldots + (\overline{q}_0 - \overline{q}_n\overline{p}_0)}{\overline{p}(s)} = \overline{q}_n + \frac{q(s)}{p(s)}$$

   such that, for $i = 0, 1, 2, \ldots, n-1$,

$$\begin{aligned} p(s) &= s^n + p_{n-1}s^{n-1} + \ldots + p_0, \qquad && p_i = \overline{p}_i \quad \text{and} \\ q(s) &= q_{n-1}s^{n-1} + \ldots + q_0, && q_i = (\overline{q}_i - \overline{q}_n\overline{p}_i). \end{aligned}$$

Then the matrices A, B and C are formed as follows

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} q_0 & q_1 & \cdots & q_{n-1} \end{pmatrix}.$$

(This formation of matrices A, B and C is not unique, i.e., other triples of matrices A, B and C are possible that correspond to the same transfer function.)

2.2 Example: De-stabilizing reset action

Example 2.3.
In this example, we consider a negative-feedback system consisting of a reset compensator and a single-input, single-output, linear time-invariant plant, with transfer function

$$G(s) = \frac{(3+\beta)s + 1}{s^2 + 3s - \beta}. \tag{2.1}$$

Here we have $\overline{q}(s) = (3+\beta)s + 1$ and $\overline{p}(s) = s^2 + 3s - \beta$. Because $\deg(\overline{q}(s)) < \deg(\overline{p}(s))$, we take D=0. With $q(s) := \overline{q}(s)$ and $p(s) := \overline{p}(s)$ given it follows immediately that $q_1 = 3 + \beta$, $q_0 = 1$, $p_1 = 3$ and $p_0 = -\beta$. Hence a state space realization is given as

$$A = \begin{pmatrix} 0 & 1 \\ \beta & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3+\beta \end{pmatrix}.$$

The state-space representation of the closed-loop system obtained by the feedback interconnection of the plant with the compensator corresponding to the integrator-action, then becomes:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ \beta & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_c(t) \\ y(t) &= \begin{pmatrix} 1 & 3+\beta \end{pmatrix} x(t) \\ \dot{x}_c(t) &= -y(t), \quad t_i < t < t_{i+1} \\ x_c(t_i) &= 0 \end{aligned}$$

where $x(t)$ is the plant state and $x_c(t)$ is the integrator's state. The set $\{t_i : t_i < t_{i+1}, i = 1, 2, ...\}$ is defined as the ordered set of all zero-crossing times τ for which $y(\tau) = 0$. The action of the integrator is simple: it integrates, except for the zero-crossing times where the integrator's state is set to zero.

Between two zero-crossing times, the full state $\overline{x} = (x, x_c)^T$ behaves as the LTI system

$$\dot{\overline{x}} = \overline{A}\overline{x}$$

with solution

$$\overline{x}(t) = e^{\overline{A}t}\overline{x}(t_i) = e^{\overline{A}t}\begin{pmatrix} x(t_i) \\ 0 \end{pmatrix}, \quad t \in [t_i, t_{i+1}].$$

Hence, the plant state behaves as the LTI system

$$x(t) = P(t - t_i)x(t_i), \quad t \in [t_i, t_{i+1}] \tag{2.2}$$

where

$$P(t) = \begin{pmatrix} I & 0 \end{pmatrix} e^{\overline{A}t}\begin{pmatrix} I \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{A} = \begin{pmatrix} A & b \\ -c & 0 \end{pmatrix}.$$

So, in this example

$$\overline{A} = \begin{pmatrix} 0 & 1 & 0 \\ \beta & -3 & 1 \\ -1 & -3-\beta & 0 \end{pmatrix},$$

which has eigenvalues $\lambda_{1,2,3} = -1$, eigenvector $v_1 = (1, -1, -2-\beta)^T$ and generalized eigenvectors $v_2 = (0, 1, 1)^T$ and $v_3 = (1, -1, -1-\beta)^T$. Because all the eigenvalues of $\overline{A}$ have negative real part for all β, the negative-feedback system given by

$$\dot{\overline{x}} = \overline{A}\overline{x} \tag{2.3}$$

$$y(t) = cx(t)$$

is asymptotically stable. So, the system without reset action is asymptotically stable for all β.

From the similarity transformation $T^{-1}\overline{A}T = J$, where J is the Jordan form of $\overline{A}$ and $T = (v_1, v_2, v_3)$, we can write

$$\overline{A} = TJT^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ -2-\beta & 1 & -1-\beta \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -\beta & 1 & -1 \\ 1 & 1 & 0 \\ 1+\beta & -1 & 1 \end{pmatrix}.$$

To calculate $P(t)$ is easy now, because the exponential of $\overline{A}$ is the exponential of the Jordan form premultiplied by T and postmultiplied by T^{-1}:

$$\begin{aligned}
P(t) &= \begin{pmatrix} I & 0 \end{pmatrix} e^{\overline{A}t}\begin{pmatrix} I \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \end{pmatrix} Te^{Jt}T^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ -2-\beta & 1 & -1-\beta \end{pmatrix} e^{-t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\beta & 1 & -1 \\ 1 & 1 & 0 \\ 1+\beta & -1 & 1 \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} (\beta+1)t^2/2 + t + 1 & -t^2/2 + t \\ -(\beta+1)t^2/2 + \beta t & t^2/2 - 2t + 1 \end{pmatrix}.
\end{aligned}$$

We defined the zero-crossing times t_i as the instances when $y(t_i) = 0$, so from the second equality from the state space description it immediatly follows that $y(t_i) = cx(t_i) = 0$ for all t_i zero-crossing times. So, using $cx(t_i) = 0$ in combination with (2.2) it follows that the zero-crossing times t_i must satisfy:

$$\begin{aligned}
cx(t_1) &= 0 \\
cx(t_2) &= cP(t_2 - t_1)x(t_1) = cP(\tau_1)x(t_1) = 0 \\
cx(t_3) &= cP(t_3 - t_2)x(t_2) = cP(t_3 - t_2)P(t_2 - t_1)x(t_1) = cP(\tau_2)P(\tau_1)x(t_1) = 0 \\
&\vdots
\end{aligned}$$

where $\tau_i = t_{i+1} - t_i$ (for $i = 1, 2, ...$). If $\tau_1 = \tau_2 = ...$, then we say that the zero-crossing times t_i are periodically-spaced.

To determine the stability of the closed-loop system we need to characterize the zero-crossing times. It turns out that there are two cases to be distinguished: there is at most one zero-crossing time or the zero-crossing times are infinite in number and periodically spaced.

1. Suppose that for all $\tau > 0$, c is not an pre-eigenvector of $P(\tau)$. In this case there exists at most one zero-crossing time. Indeed, suppose that there are two zero-crossing times t_1 and t_2, i.e.,

   $$\begin{aligned}
   cx(t_1) &= 0 \\
   cx(t_2) &= cP(t_2 - t_1)x(t_1) = cP(\tau_1)x(t_1) = 0 \quad (t_2 > t_1, i.e., \tau_1 > 0).
   \end{aligned}$$

   From the first equality $cx(t_1) = 0$ it follows that $x(t_1) \in \ker(c)$. Hence, $\ker(c) = \text{span}\{x(t_1)\}$ (because c is a two-dimensional row vector and $x(t_1)$ is a two-dimensional column vector). Then, c^T and $x(t_1)$ are orthogonal which together with $\ker(c) = \text{span}\{x(t_1)\}$ implies that the only vectors which are orthogonal to $x(t_1)$ are multiples of c^T.

   From the second equality $cP(\tau_1)x(t_1) = 0$ it follows that $(cP(\tau_1))^T$ is orthogonal to $x(t_1)$, so $(cP(\tau_1))^T$ must be a multiple of c^T:

   $$(cP(\tau_1))^T = \mu c^T \quad \text{for some} \quad \mu \in \mathbb{R} \quad \text{which implies} \quad cP(\tau_1) = \mu c.$$

   Now, c is a pre-eigenvector of $P(\tau_1)$ associated to μ, for $\tau_1 = t_2 - t_1 > 0$, which is contradicting with the assumption that c is not a pre-eigenvector of $P(\tau)$ for all $\tau > 0$. Hence, $x(t_1) = 0$ which together with $x_c(t_1) = 0$ (by definition) implies $x(t) = 0$ and $x_c(t) = 0$ for all $t > t_1$ contradicting the existence of zero-crossing time t_2. Hence, there exists at most one zero-crossing time t_1, for which $x(t_1) = 0$.

2. Now suppose τ is the smallest number for which c is an pre-eigenvector of $P(\tau)$, such that

   $$cP(\tau) = \lambda c$$

   where λ is the eigenvalue associated to c. Furthermore, assume that there exists at least one zero-crossing time t_1, i.e., $c(x(t_1)) = 0$.

   - Let $cP(\tau_1)x(t_1) = 0$, where τ_1 is the smallest number for which this holds.

(a) For $\tau_1 = \tau$ this is satisfied:
$cP(\tau_1)x(t_1) = cP(\tau)x(t_1) = \lambda cx(t_1) = \lambda{\cdot}0 = 0$.

(b) Does there exist a $\tau_1 < \tau$ such that $cP(\tau_1)x(t_1) = 0$ is satisfied?
We know $cx(t_1) = 0$, so $x(t_1) \in \ker(c)$. Hence, $\ker(c) = \mathrm{span}\{x(t_1)\}$. Then, as stated above, c^T and $x(t_1)$ are orthogonal which together with $\ker(c)$ =span$\{x(t_1)\}$ implies that the only vectors which are orthogonal to $x(t_1)$ are multiples of c^T.
From $cP(\tau_1)x(t_1) = 0$ it follows that $(cP(\tau_1))^T$ is orthogonal to $x(t_1)$, so $(cP(\tau_1))^T$ must be a multiple of c^T:

$$(cP(\tau_1))^T = \mu c^T \quad \text{for some} \quad \mu \in \mathbb{R} \quad \text{which implies} \quad cP(\tau_1) = \mu c.$$

In addition, τ is the smallest number such that $cP(\tau) = \lambda c$ for some λ, so $\tau_1 = \tau$ (and $\mu = \lambda$).
Hence, τ is the smallest τ_1 for which $cP(\tau_1)x(t_1) = 0$.

- Let $cP(\tau_2)P(\tau)x(t_1) = 0$, where τ_2 is the smallest number for which this holds.

(a) For $\tau_2 = \tau$ this is satisfied:
$cP(\tau_2)P(\tau)x(t_1) = cP(\tau)P(\tau)x(t_1) = \lambda cP(\tau)x(t_1) = \lambda\lambda cx(t_1) = \lambda^2{\cdot}0 = 0$.

(b) Does there exist a $\tau_2 < \tau$ such that $cP(\tau_2)P(\tau)x(t_1) = 0$ is satisfied?
We know $cP(\tau)x(t_1) = 0$, so $P(\tau)x(t_1) \in \ker(c) = \mathrm{span}\{x(t_1)\}$ which means that $P(\tau)x(t_1)$ is a linear combination of $x(t_1)$, i.e.,

$$P(\tau)x(t_1) = \alpha x(t_1) \quad \text{for some} \quad \alpha \in \mathbb{R}. \tag{2.4}$$

Then, c^T and $P(\tau)x(t_1)$ are orthogonal which together with $\ker(c) = \mathrm{span}\{x(t_1)\}$ implies that the only vectors which are orthogonal to $x(t_1)$ (or a multiple of it) are multiples of c^T. From $cP(\tau_2)P(\tau)x(t_1) = 0$ it follows that $(cP(\tau_2))^T$ is orthogonal to $P(\tau)x(t_1) = \alpha x(t_1)$, so $(cP(\tau_2))^T$ must be a multiple of c^T:

$$(cP(\tau_2))^T = \nu c^T \quad \text{for some} \quad \nu \in \mathbb{R} \quad \text{which implies} \quad cP(\tau_2) = \nu c.$$

In addition, τ is the smallest number such that $cP(\tau) = \lambda c$ for some λ, so $\tau_2 = \tau$ (and $\nu = \lambda$).
Hence, τ is the smallest τ_2 for which $cP(\tau_2)P(\tau)x(t_1) = 0$.

Continuing the argument that τ is the smallest τ_i such that $cP(\tau_i)P(\tau)^{i-1}x(t_1) = 0$ for $i = 3, 4, \ldots$, shows $\tau = \tau_1 = \tau_2 = \ldots$. Hence, we conclude that the zero-crossing times t_i are periodically spaced with period τ. Finally, since $x(t_1) \neq 0$ and $P(\tau)$ is nonsingular, then

$$x(t_i) = P(\tau)x(t_{i-1}) \neq 0 \quad \text{for} \quad i = 2, 3, \ldots \quad \text{(from (2.2))}$$

Concluding, the zero-crossing times t_i are infinite in number.

The previous discussion can be summarised as follows (cf. [Lemma 1, 4]):

1. If c is not a pre-eigenvector of $P(\tau)$ for all $\tau > 0$, then there exists at most one zero-crossing time t_1, for which $x(t_1) = 0$.

2. If c is a pre-eigenvector of $P(\tau)$ for some $\tau > 0$, then the zero-crossing times are infinite in number and periodically spaced. Moreover, the smallest such τ is the spacing of the zero-crossing times.

In the first case $y(t)$ has at most one zero-crossing time. After this zero-crossing time, the closed-loop system behaves as the system in (2.3) which is asymptotically stable if and only if $\overline{A}$ has eigenvalues in the left half plane. In the second case, where the zero-crossing times are infinite in number and periodically spaced, the plant state satisfies the state equation

$$x(t_{i+1}) = P(\tau)x(t_i) \quad i = 1, 2, \ldots \qquad \text{(from (2.2))}$$

which is asymptotically stable if the eigenvalues of $P(\tau)$ are contained in the open unit disk.

The stability conditions are now characterized by the number of zero-crossing times that occur, which is dependent of c being a pre-eigenvalue of $P(\tau)$ (for some $\tau > 0$) or not (cf. [Theorem 1, 4]), i.e.,

1. When c is not a pre-eigenvalue of $P(\tau)$ for all $\tau > 0$, then the closed-loop system is asymptotically stable if and only if $\overline{A}$ has eigenvalues in the left half plane.

2. The closed-loop system is asymptotically stable if and only if all the eigenvalues of $P(\tau)$ are contained in the open unit disk, where $\tau > 0$ is the smallest number for which c is a pre-eigenvector of $P(\tau)$.

For arbitrary t, $P(t)$ has eigenvalues and corresponding pre-eigenvectors

$$\lambda_{+-} = \tfrac{1}{4}(4 - 2t + 2t^2 + \beta t^2 \pm t\sqrt{36 + 16\beta - 8t - 4\beta t + 4t^2 + 4\beta t^2 + \beta^2 t^2})$$

$$v_{+-} = ((-6 + \beta t \pm \sqrt{36 + 16\beta - 8t - 4\beta t + 4t^2 + 4\beta t^2 + \beta^2 t^2})/2(-2 + t), \quad 1).$$

For $\tau = 2$, we have

$$P(2) = \begin{pmatrix} (2\beta + 5)e^{-2} & 0 \\ -2e^{-2} & -e^{-2} \end{pmatrix}$$

which has eigenvalues $\lambda_1 = (2\beta + 5)e^{-2}$ with corresponding pre-eigenvector $v_1 = (1 \quad 0)$ and $\lambda_2 = -e^{-2}$ with corresponding pre-eigenvector $v_2 = (1 \quad 3 + \beta)$. In this example, $\tau = 2$ is the smallest positive number for which c is a pre-eigenvector of $P(\tau)$ for all β. Thus, the integrator resets every 2 seconds, independent of β.

In this example we wanted to show that a reset action can destabilize a stable LTI feedback system. Following the theorem given above, the closed-loop system is unstable if and only if at least one of the eigenvalues of $P(\tau)$ are on- or outside the unit disk for $\tau = 2$ (in this example). The largest eigenvalue of $P(2)$ is $\lambda_1 = (2\beta + 5)e^{-2}$, so if β is chosen such that

$$\mid (2\beta + 5)e^{-2} \mid \geq 1,$$

then the system is unstable.

Thus, the negative-feedback system without reset action is asymptotically stable for all β, while the system with reset action is unstable for any $\beta \notin (-6.1945, 1.1945)$. Hence, the reset action destabilizes the asymptotically stable LTI feedback system if β is chosen such that it is outside the interval $(-6.1945, 1.1945)$.

Chapter 3

Preliminaries and problem formulation

3.1 Description of reset control systems

Consider the state space systems Ω:

$$\begin{aligned} \dot{x} &= f(x,u) \qquad u \in \mathbb{R} \\ y &= h(x) \qquad y \in \mathbb{R} \end{aligned} \tag{3.1}$$

where $x \subset \mathcal{X}$ is the n-dimensional state.

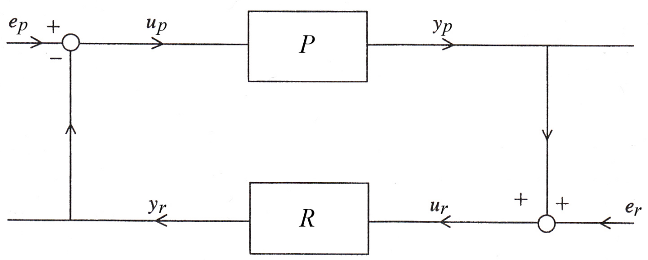

Figure 3.1: Standard feedback configuration

Consider a negative-feedback system Σ with plant P and reset compensator R as given in Figure 3.1. The closed loop system Σ is described by the equations

$$\begin{aligned} u_p &= e_p - y_r, \quad y_p = P(u_p) \\ u_r &= e_r + y_p, \quad y_r = R(u_r) \end{aligned}$$

where $u_i \in U_i$ and $y_i \in Y_i$ for $i = p, r$ and $U_p = Y_r$, $U_r = Y_p$.
The plant P is of the form of Ω as given above, i.e.,

$$P: \quad \begin{aligned} \dot{x}_p &= f(x_p, u_p) \\ y_p &= h(x_p) \end{aligned} \tag{3.2}$$

where $\mathcal{X}$ is n_p-dimensional. Futhermore, the reset compensator R, consisting of an LTI compensator (the so-called base linear compensator) together with a reset action, is given by the following differential equation plus a reset law:

$$R: \quad \begin{array}{rcl} \dot{x}_r &=& A_r x_r + B_r u_r, \\ x_r^+ &=& A_\rho x_r, \\ y_r &=& C_r x_r \end{array} \qquad \begin{array}{rcl} u_r &\neq& 0 \\ u_r &=& 0 \\ && \end{array} \tag{3.3}$$

where n_r is the dimension of the state x_r. Here x_r^+ denotes the state of the reset compensator immediately after the reset-action, i.e., x_r^+ is the notation for the value $x_r(t+\tau)$ with $\tau \to 0^+$. The diagonal matrix A_ρ in (3.3) has diagonal elements equal to zero for the state components to be reset and equal to one for the rest of the compensator states. Now n_ρ is the dimension of the reset subspace and $n_{\overline{\rho}}$ is defined as the dimension of the non-reset subspace, such that $n_r = n_\rho + n_{\overline{\rho}}$. R will be referred to as a *full reset compensator* if the diagonal matrix $A_\rho = 0$ (or $n_\rho = n_r$, $n_{\overline{\rho}} = 0$) and as a *partial reset compensator* if A_ρ has a non-zero diagonal element. The complete feedback system Σ, consisting of plant P and reset compensator R, evolves in a continous fashion in time intervals in which the reset law is not applied, while the system undergoes a jump when the resetting law is applied.

To avoid problems, it is assumed that the solutions of (3.3) are time regularized which means that the reset law is switched off for a time interval of length $\Delta_m > 0$ after each reset time. Thus, time regularization has as a consequence that for any input u_r only a finite number of reset times on any finite time interval will exist. Now the reset compensator R is described as follow:

$$\begin{array}{rclrcl} \dot{\Delta} &=& 1 & \dot{x}_r &=& A_r x_r + B_r u_r, \\ \Delta^+ &=& 0 & x_r^+ &=& A_\rho x_r, \\ &&& y_r &=& C_r x_r \end{array} \qquad \begin{array}{rclcrcl} u_2 &\neq& 0 & \text{or} & \Delta &<& \Delta_m \\ u_r &=& 0 & \text{and} & \Delta &\geq& \Delta_m \\ &&&&&& \end{array} \tag{3.4}$$

with zero initial conditions: $\Delta(0) = 0$, $x_r(0) = 0$.

3.2 Main results of dissipative system theory

Define the supply rate as a function

$s : U \times Y \to \mathbb{R}$

Definition 3.1. A state space system Ω as in (3.1) is said to be dissipative with respect to the supply rate s if there exists a function $S : \mathcal{X} \to \mathbb{R}^+$, called the storage function, such that for all $x_0 \in \mathcal{X}$, all $t_1 \geq t_0$, and all input functions u

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt \tag{3.5}$$

where $x(t_0) = x_0$, and $x(t_1)$ is the state of Ω at time t_1 resulting from initial condition x_0 and input function $u(\cdot)$.

The supply rate

$s(u, y) = \frac{1}{2}(\gamma^2 \parallel u \parallel^2 - \parallel y \parallel^2), \quad \gamma \geq 0$

is an important choice for a supply rate. According to def 3.1, Ω is dissipative with respect to this supply rate if and only if there exist $S \geq 0$ such that for all $T \geq 0$, $x(0)$ and $u(\cdot)$:

$$\frac{1}{2}\int_0^T (\gamma^2 \parallel u(t) \parallel^2 - \parallel y(t) \parallel^2)dt \geq S(x(T)) - S(x(0)) \geq -S(x(0))$$

and thus

$$\int_0^T \parallel y(t) \parallel^2 dt \leq \gamma^2 \int_0^T \parallel u(t) \parallel^2 dt + 2S(x(0)).$$

Definition 3.2. A state space system Ω with $U = \mathbb{R}$, $Y = \mathbb{R}$, has $\mathcal{L}_2$-gain $\leq \gamma$ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}(\gamma^2 \parallel u \parallel^2 - \parallel y \parallel^2)$.
The $\mathcal{L}_2$-gain is defined as $\gamma(\Omega) = \inf\{\gamma \mid \Omega \text{ has } \mathcal{L}_2\text{−gain} \leq \gamma\}$.

To say something about the stability of the feedback system Σ later on, we will concentrate on stability of the equilibria of Σ. We will do this using Lyapunov stability theory since there is a direct link between dissipativity and Lyapunov stability. So, first we need a few notions and results from Lyapunov stabiliy theory.

Consider a continuously differentiable storage function S associated to a system of the form of Ω. Since S is continuously differentiable, (3.5) is equivalent to the *differential dissipation inequality*:

$$S_x(x)f(x, u) \leq s(u, h(x)), \quad \text{for all } x,u \tag{3.6}$$

where

$$S_x(x) = (\frac{\partial S}{\partial x_1}(x), \ldots, \frac{\partial S}{\partial x_n}(x)).$$

The differential dissipation inequality is obtained by dividing (3.5) by $t_1 - t_0$ and let $t_1 \to t_0$.

Theorem 3.3. *Direct method of Lyapunov*
Let x^* be an equilibrium of $\dot{x} = F(x)$, that is $F(x^*) = 0$ and thus $x(t; x^*) = x^*$. Let $V : \mathcal{X} \to \mathbb{R}^+$ be a continuously differentiable function with

$$V(x^*) = 0, \quad V(x) > 0, \quad x \neq x^* \tag{3.7}$$

such that

$$\dot{V}(x) := V_x(x)F(x) \leq 0, \quad \forall x \in \mathcal{X} \tag{3.8}$$

then x^* is a stable equilibrium. If moreover

$$\dot{V}(x) < 0, \quad \forall x \in \mathcal{X}, x \neq x^*,$$

then x^* is an asymptotically stable equilibrium, which is globally asymptotically stable if V is proper (that is, the sets $\{x \in \mathcal{X} \mid 0 \leq V(x) \leq c\}$ are compact for every $c \in \mathbb{R}^+$).

The function V in the theorem is called a *Lyapunov function* if (3.7) and (3.8) are satisfied. Note that the theorem can also be applied to any neighborhood $\hat{\mathcal{X}}$ of x^*.

The direct method of Lyapunov is applicable to every system which is of the form of Ω. If we want to apply this theorem to a system of the form as Σ with plant P and reset compensator R, then the following adapted version of the direct method of Lyapunov must be applied.

Theorem 3.4. *Adapted version of the direct method of Lyapunov*
Let x^* be an equilibrium of $\dot{x} = F(x)$, that is $F(x^*) = 0$ and thus $x(t; x^*) = x^*$. Furthermore, let the system undergo the reset action $x^+ = Kx$. Let $V : \mathcal{X} \to \mathbb{R}^+$ be a continuously differentiable function with

$$V(x^*) = 0, \quad V(x) > 0, \quad x \neq x^* \tag{3.9}$$

such that

$$\dot{V}(x) := V_x(x)F(x) \leq 0, \quad \forall x \in \mathcal{X} \tag{3.10}$$

as well as

$$V(Kx) \leq V(x), \quad \forall x \in \mathcal{X} \tag{3.11}$$

then x^* is a stable equilibrium. If moreover

$$\dot{V}(x) < 0, \quad \forall x \in \mathcal{X}, x \neq x^*,$$

then x^* is an asymptotically stable equilibrium, which is globally asymptotically stable if V is proper (that is, the sets $\{x \in \mathcal{X} \mid 0 \leq V(x) \leq c\}$ are compact for every $c \in \mathbb{R}^+$).

Chapter 4

Main results

In proposition 1, 2, 3 and 4 of [1] it has been shown what the necessary and sufficient conditions are for a feedback system with reset action to be stable using passivity theories. In this chapter we are going to do something similiar to find sufficient conditions for stability of the closed loop system Σ, using the theory given in the previous chapter. First we need to show that the reset compensator R has $\mathcal{L}_2$-gain $\leq \gamma$ underlying if the base compensator has $\mathcal{L}_2$-gain $\leq \gamma$, whereupon we will use the small gain theorem to determine the storage function of the closed loop system Σ. At the end of this chapter the stability conditions are determined.

4.1 $\mathcal{L}_2$-gain of the reset compensator

Theorem 4.1. *A reset compensator R given by (3.4), has $\mathcal{L}_2$-gain $\leq \gamma$ for some $\gamma \geq 0$, if its base compensator has $\mathcal{L}_2$-gain $\leq \gamma$ with storage function $S(x)$ satisfying*

$$S(A_\rho x) \leq S(x) \qquad \forall x \in \mathbb{R}^{n_r}.$$

Proof. Assume that the base compensator has $\mathcal{L}_2$-gain $\leq \gamma$. Then, according to definition 3.2, the base compensator must be dissipative with respect toe the supply rate $s(u,y) = \frac{1}{2}(\gamma^2 \parallel u \parallel^2 - \parallel y \parallel^2)$ for the storage function S, such that the following equation is satisfied

$$S(x(T)) \leq S(x(0)) + \int_0^T s(u(t), y(t))dt \qquad \forall u, \forall x(0) \in \mathbb{R}^{n_r}, \forall T \geq 0. \tag{4.1}$$

Furthermore, the storage function $S(x)$ is required to satisfy

$$S(A_\rho x) \leq S(x) \qquad \forall x \in \mathbb{R}^{n_r}$$

from which, at the reset time t_i, it follows that

$$S(x(t_i^+)) = S(A_\rho x(t_i)) \leq S(x(t_i)). \tag{4.2}$$

For a given input $u(t)$, by time regularization there is a finite number of reset times t_i, $i = 1, 2, ..., k$, in the interval $[0, T]$ where $T \in (t_k, t_{k+1}]$. In the interval $[x_k^+, T]$ the reset compensator equals its base compensator with initial condition $x(t_k^+)$, so equation (4.1) is still satisfied in this case:

$$S(x(T)) \leq S(x(t_k^+)) + \int_{t_k}^T s(u(t), y(t))dt. \tag{4.3}$$

Actually, the previous argument is true for every interval without reset action $[t_{i-1}^+, t_i]$ for $i = 1, 2, ...k$, i.e.,

$$S(x(t_i)) \leq S(x(t_{i-1}^+)) + \int_{t_{i-1}}^{t_i} s(u(t), y(t))dt \tag{4.4}$$

and

$$S(x(t_1)) \leq S(x(0)) + \int_0^{t_1} s(u(t), y(t))dt. \tag{4.5}$$

If we now combine conditions (4.2), (4.3), (4.4) and (4.5) then it follows that for the reset compensator

$$\begin{aligned}
S(x(T)) &\leq S(x(t_k^+)) + \int_{t_k}^{T} s(u(t), y(t))dt \qquad \text{(condition (4.3))} \\
&\leq S(x(t_k)) + \int_{t_k}^{T} s(u(t), y(t))dt \qquad \text{(condition (4.2) for } t_i = t_k) \\
&\leq S(x(t_{k-1}^+)) + \int_{t_{k-1}}^{t_k} s(u(t), y(t))dt + \int_{t_k}^{T} s(u(t), y(t))dt \quad \text{(condition (4.4))} \\
&= S(x(t_{k-1}^+)) + \int_{t_{k-1}}^{T} s(u(t), y(t))dt \\
&\vdots \qquad\qquad\qquad\qquad \text{(repeating conditions (4.2) and (4.4))} \\
&\leq S(x(t_1^+)) + \int_{t_1}^{T} s(u(t), y(t))dt \\
&\leq S(x(t_1)) + \int_{t_1}^{T} s(u(t), y(t))dt \quad \text{(condition (4.2) for } t_i = t_1) \\
&\leq S(x(0)) + \int_{0}^{t_1} s(u(t), y(t))dt + \int_{t_1}^{T} s(u(t), y(t))dt \quad \text{(condition (4.5))} \\
&= S(x(0)) + \int_{0}^{T} s(u(t), y(t))dt.
\end{aligned}$$

So, according to what is stated above, the reset compensator satisfies:

$$0 \leq S(x(T)) \leq S(x(0)) + \int_0^T s(u(t), y(t))dt. \tag{4.6}$$

In particular, (4.6) also holds for the supply rate $s(u, y) = \frac{1}{2}(\gamma^2 \parallel u \parallel^2 - \parallel y \parallel^2)$. So,

$$\begin{aligned}
0 &\leq S(x(0)) + \int_0^T (\frac{1}{2}\gamma^2 \parallel u \parallel^2 - \frac{1}{2} \parallel y \parallel^2)dt \\
\Rightarrow \int_0^T \parallel y \parallel^2 dt &\leq \gamma^2 \int_0^T \parallel u \parallel^2 dt + 2S(x(0)).
\end{aligned}$$

So the reset compensator is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}(\gamma^2 \parallel u \parallel^2 - \parallel y \parallel^2)$, from which it can be concluded that the reset compensator has $\mathcal{L}_2$-gain $\leq \gamma$. □

The condition for the storage function

$$S(A_\rho x) \leq S(x) \qquad \forall x \in \mathbb{R}^{n_r}$$

can be satisfied in several ways. For instance, if R is a full reset compensator (i.e., A_ρ is a zero matrix) and the storage function satisfies $S(0) = 0$, then

$$S(A_\rho x) = S(0) \leq S(x)$$

because $S(x) \geq 0$ by definition. For a partial reset compensator, the condition is satisfied if A_ρ is given as the projection on the first vector component and $S(x)$ is quadratic of the following form

$$S(x) = x^T \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} x$$

where Q_{11}, Q_{22} are positive definite matrices with respective dimensions $\mathbb{R}^{n_{\bar{\rho}} \times n_{\bar{\rho}}}$, $\mathbb{R}^{n_\rho \times n_\rho}$. In this case

$$S(A_\rho x) - S(x) = -x^T \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix} x \leq 0, \quad \forall x \in \mathbb{R}^{n_r}.$$

Note that there are other possibilities than these for storage functions to satisfy the condition $S(A_\rho x) \leq S(x)$; these two given above are just examples of such storage functions.

4.2 Stability of the closed loop system Σ

Now consider the closed loop system Σ of Figure 1.1 and suppose that plant P has $\mathcal{L}_2$-gain $\leq \gamma_p$ and that the base compensator of R has $\mathcal{L}_2$-gain $\leq \gamma_r$ such that the full reset compensator R has $\mathcal{L}_2$-gain $\leq \gamma_r$. Denote the storage functions of P, R by S_p, S_r resulting in the dissipation inequalities

$$\begin{aligned} S_p(x_p(t_1)) - S_p(x_p(t_0)) &\leq \frac{1}{2}\int_{t_0}^{t_1} (\gamma_p^2 \parallel u_p(t) \parallel^2 - \parallel y_p(t) \parallel^2) dt, \\ S_r(x_r(t_1)) - S_r(x_r(t_0)) &\leq \frac{1}{2}\int_{t_0}^{t_1} (\gamma_r^2 \parallel u_r(t) \parallel^2 - \parallel y_r(t) \parallel^2) dt. \end{aligned}$$

Consider now the feedback interconnection of Σ (with $e_p = e_r = 0$)

$$u_p = -y_r, \quad u_r = y_p,$$

and assume $\gamma_p \cdot \gamma_r < 1$ (*small gain condition*). Then we can find an α such that

$$\gamma_p < \alpha < \frac{1}{\gamma_r}.$$

It follows immediately that if the small gain condition is satisfied that for $S(x) := S_p(x_p) + \alpha^2 S_r(x_r)$:

$$\begin{aligned} S(x(t_1)) - S(x(t_0)) &\leq \frac{1}{2}\int_{t_0}^{t_1} (\gamma_p^2 \parallel u_p \parallel^2 - \parallel y_p \parallel^2 + \alpha^2\gamma_r^2 \parallel u_r \parallel^2 - \alpha^2 \parallel y_r \parallel^2)dt \\ &= \frac{1}{2}\int_{t_0}^{t_1} (\gamma_p^2 \parallel -y_r \parallel^2 - \parallel y_p \parallel^2 + \alpha^2\gamma_r^2 \parallel y_p \parallel^2 - \alpha^2 \parallel y_r \parallel^2)dt \\ &= \frac{1}{2}\int_{t_0}^{t_1} [(\alpha^2\gamma_r^2 - 1) \parallel y_p \parallel^2 + (\gamma_p^2 - \alpha^2) \parallel y_r \parallel^2]dt \\ &= -\frac{1}{2}\int_{t_0}^{t_1} (\epsilon_1 \parallel y_p \parallel^2 + \epsilon_2 \parallel y_r \parallel^2)dt \end{aligned}$$

where $\epsilon_1 = 1 - \alpha^2\gamma_r^2 > 0$ and $\epsilon_2 = \alpha^2 - \gamma_p^2 > 0$. This is known as the *small gain theorem.*

The state of the closed-loop system of Σ is given by the state of the plant P and the state of the reset compensator R, i.e., $x = (x_p, x_r)^T$. The differential equation $\dot{x} = F(x)$ of the closed-loop of Σ is now given by (where $u_p = -y_r$, $u_r = y_p$)

$$\frac{d}{dt}\begin{pmatrix} x_p \\ x_r \end{pmatrix} = \begin{pmatrix} f(x_p, -C_r x_r) \\ A_r x_r + B_r h(x_p) \end{pmatrix} \tag{4.7}$$

implying that $F : \mathbb{R}^{n_p} \times \mathbb{R}^{n_r} \rightarrow \mathbb{R}^{n_p} \times \mathbb{R}^{n_r}$.

We assume that the storage function S_p and S_r are continuously differentiable functions implying that the storage function S is continuously differentiable (S is a linear combination of S_p and S_r). Then, it follows that for the storage function $S(x)$ of the closed loop system Σ, the differential dissipation inequality (3.6) is satisfied:

$$S_x(x)F(x) \leq -\frac{1}{2}(\epsilon_1 \parallel y_p \parallel^2 + \epsilon_2 \parallel y_r \parallel^2) \leq 0 \tag{4.8}$$

because ϵ_1, $\epsilon_2 > 0$ by the small gain theorem.

Now we can determine the stability conditions for the closed-loop system of Σ by determine the stability of the equilibruim $x^* = 0$, using theorem 3.4.

Theorem 4.2. *Let plant P have $\mathcal{L}_2$-gain $\leq \gamma_p$ and let the base compensator of R have $\mathcal{L}_2$-gain $\leq \gamma_r$ with $\gamma_p \cdot \gamma_r < 1$, and denote the storage functions of P, R by S_p, S_r. Assume that $S_r(A_\rho x_r) \leq S_r(x_r)$, $\forall x_r \in \mathbb{R}^{n_r}$. Furthermore, assume $S_p(0) = 0$, $S_p(x) > 0$ if $x_p \neq 0$ and $S_r(0) = 0$, $S_r(x) > 0$ if $x_r \neq 0$. Then the closed-loop system is stable with Lyapunov function*

$$V(x) := S(x) = S_p(x_p) + \alpha^2 S_r(x_r). \tag{4.9}$$

If moreover

$$\dot{V}(x) < 0, \quad \forall x \in \mathcal{X}, x \neq 0,$$

then the closed-loop system is asymptotically stable and globally asymptotically stable if V is proper.

Proof. We can show that this theorem is valid, by showing it is actually a special case of theorem 3.4. The first part includes the conditions of theorem 4.1 and the small gain theorem such that they are satisfied here. By assuming that the storage functions of P and R have both their minimum in the origin, the storage function $S(x)$ (which is a linear combination of S_p an S_r) has its minimum in the origin. Thereby, $x^* = 0$ is now a equilibrium of $\dot{x} = F(x)$ of the closed-loop system Σ. So if we now choose the Lyapunov function $V(x) = S(x)$, then

- $V(x^*) = S(x^*) = S(0) = 0, \quad V(x) = S(x) > 0, x \neq x^*, \quad$ ((3.9) is satisfied)
- $\dot{V}(x) = \dot{S}(x) = S_x(x)F(x) \leq 0, \quad \forall x \in \mathcal{X}, \quad$ ((3.10) is satisfied)
- $S(x^+) = S(A_\rho x) = S_p(x_p) + \alpha^2 S_r(A_\rho x_r) \leq S_p(x_p) + \alpha^2 S_r(x_r) = S(x)$,
  so $V(x^+) = S(x^+) \leq S(x) = V(x), \quad$ ((3.11) is satisfied)

from which it follows that the conditions of theorem 3.4 for $x^* = 0$ being a stable equilibrium are satisfied, concluding that the closed-loop system is stable. The conditions for the closed-loop system to be asymptotically stable and globally asymptotically stable follow immediately from the conditions for the equilibrium $x^* = 0$ to be respectively asymptotically stable and globally asymptotically stable, according to theorem 3.4 □

Chapter 5

Application: Cruise control

Reset control can be applied to cruise control, which is a system that automatically controls the speed of a motor vehicle. Before we apply the reset control to the problem, we first take a look at the mathematical modelling of cruise control.

5.1 Mathematical model of cruise control

A simple model of a motor vehicle is given by

$$m\dot{v} = -R(v) + bu$$

where m and v are respectively the mass and the speed of the vehicle, $R(v)$ is the friction depending on v, b is a constant and u is the input variable. Suppose that we want to regulate the speed v of the vehicle to a desired value v^* by using the input variable $u \in [0, 1]$.
First try a *proportional error regulation*

$$bu(t) = -k_p e(t)$$

where $e = v - v^*$ is the error, which results in the differential equation

$$m\dot{v} = -R(v) - k_p e.$$

If we take for simplicity the linear friction $R(v) = cv$, the equilibrium solution becomes:

$$0 = -cv - k_p e = -cv - k_p(v - v^*)$$

which implies

$$v = \frac{k_p}{c + k_p} v^*.$$

In general $\frac{k_p}{c+k_p} \neq 1$, so $v \neq v^*$, which means that the cruise control does not work properly. Only if $k_p \gg c$, then $v \approx v^*$.
Because a proportional error regulation does not work properly, we try to solve the problem using *integral feedback*, i.e.,

$$bu(t) = -k_p e(t) - k_i \int_0^t e(s)ds$$

which leads to the system of differential equations

$$\begin{aligned} m\dot{v} &= -R(v) - k_i r - k_p e \\ \dot{r} &= e \end{aligned}$$

where r is the position of the motor vehicle. If we again take $R(v) = cv$, then the equilibrium solution is given by:

$$\begin{aligned} 0 &= -cv - k_i r - k_p e \\ 0 &= e = v - v^*. \end{aligned}$$

From the second equality it follows that $v = v^*$, meaning the cruise control works. We can write the system of differential equations with respect to the state $e = v - v^*$, i.e.,

$$\begin{aligned} m\dot{e} &= -c(e + v^*) - k_i r - k_p e \\ \dot{r} &= e. \end{aligned}$$

If we define $r^* = -\frac{c}{k_i}v^*$, $e^* = 0$ and $\xi = r - r^*$, then we arrive at the standard form of the system of differential equations:

$$\begin{aligned} m\dot{e} &= -ce - k_i\xi - k_p e \\ \dot{\xi} &= e \end{aligned}$$

which has as equilibrium $(e, \xi) = (0, 0)$. In matrix notation, the standard form is given by

$$\frac{d}{dt}\begin{pmatrix} e \\ \xi \end{pmatrix} = \begin{pmatrix} -\frac{c+k_p}{m} & -\frac{k_i}{m} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e \\ \xi \end{pmatrix}. \tag{5.1}$$

Now we can choose k_i and k_p such that the equilibruim is asymptotically stable, that is, such that the eigenvalues of the 2×2 matrix are in the open left half-plane. The advantage of asymptotically stable systems, i.e., for which the equilibrium is asymptotically stable, is that the system is robust against internal disturbances. Indeed, after the occurrence of any internal disturbance leading to a new value of the state vector, this effect will converge to zero.

If we apply a constant external disturbance d to the system, we obtain the following differential equation

$$m\dot{e} = -ce - k_i\xi - k_p e + d \tag{5.2}$$

where the desired controller action for $e = 0$ is $k_i\xi = d$. If now the reset-action $\xi^+ = 0$ is applied, then at switching times

$$\begin{aligned} m\dot{e} &= d, \quad e^+ = 0 \\ \dot{\xi} &= e, \quad \xi^+ = 0 \end{aligned}$$

where the switching times are defined as follows: If the error of the problem becomes zero, then we put the feedback controller ξ also to zero; the times t at which this happens are called switching times. Note that the reset-action $\xi^+ = 0$ might not be the best reset-value because we have a constant disturbance d. Because of this constant disturbance d, the equilibrium becomes $(e, \xi) = (0, \frac{d}{k_i})$. So, probably the reset-action $\xi = \frac{d}{k_i}$ where d is known, gives a more stable solution.

Before we apply the reset action, we would like to know if the reset action does not destabilize an asymptotically stable feedback system. In theorem 4.2 of Chapter 4 'Main results' we found stability conditions for the feedback system with reset action, which we are going to check here. Note that the system without reset action being asymptotically stable gives restrictions on the variables c, k_p, k_i and m. For our convenience we take c and m equal to one such that only k_p and k_i are variable and determine the stability of the system. According to theorem 4.2 the plant P and the reset compensator R have to satisfy the small gain theorem.

The plant P is given by

$$m\dot{v} = -cv + u$$

with corresponding transfer function

$$G(s) = \frac{1}{ms + c}.$$

The $\mathcal{L}_2$-gain for the plant P is defined as follows

$$\gamma_p = \max\{| G(i\omega) | : \omega \in \mathbb{R}\}.$$

We can determine the $\mathcal{L}_2$-gain of the plant by taking a look at the bode diagram in Figure 5.1. We see that the magnitude of the plant has maximum 0 dB, which is equal to one. We could also determine this analytically: the transfer function $G(i\omega)$ attains its maximum for $\omega = 0$ (the smaller the denominator , the larger the complete quotient), such that the maximum becomes $\frac{1}{c}$ which is one for $c = 1$. So, we found $\gamma_p = \frac{1}{c}$.

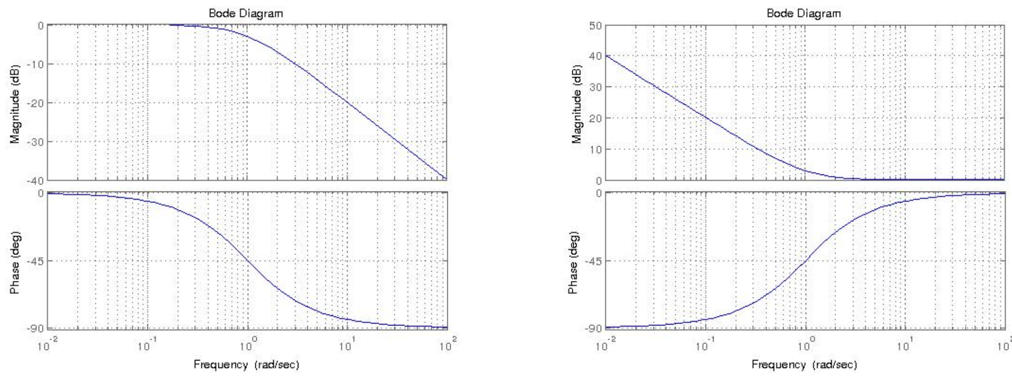

Figure 5.1: Bode diagrams of the plant P (left) and the reset compensator R (right).

The reset compensator R is given by

$$u(t) = k_p e(t) + k_i \int_0^t e(s)ds$$

(which becomes negative for negative integral feedback) which is equivalent to

$$\dot{u}(t) = k_p \dot{e}(t) + k_i e(t).$$

The correspoding transfer function (from $e(t)$ to $u(t)$) is then given by

$$H(s) = \frac{k_p s + k_i}{s},$$

so the $\mathcal{L}_2$-gain for the reset compensator is defined as

$$\gamma_r = \max\{\mid H(i\omega) \mid : \omega \in \mathbb{R}\}.$$

We can determine the $\mathcal{L}_2$-gain of the reset compensator R by again taking a look at the bode diagram in Figure 5.1. We see that the magnitude of the reset compensator has an infinitely large maximum. This follows immediately from the transfer function $H(i\omega)$, since for large ω the transfer function converges to k_p and for small ω it converges to $\frac{k_i}{i\omega}$ which becomes infinitely large. So $\gamma_r \to \infty$ for $\omega \to 0$.

It follows now immediately that theorem 4.2 can not be applied, since the small gain condition is not satisfied, i.e.,

$$\gamma_p \cdot \gamma_r \not< 1 \quad \text{since} \quad \gamma_r = \infty.$$

Although the theorem obtained in the previous section cannot be applied, the closed-loop system *does* remain stable under the reset action since both the motor vehicle model and the base compensator can be shown to be *passive*, and thus the main theorem derived in [1] can be applied.

To find out what happens, we simulate the cruise control problem with Matlab and Simulink.

5.2 Simulation results

For all simulations, the mass m and the constant c have been chosen equal to one (i.e., $m \equiv 1$, $c \equiv 1$). Furthermore, the initial value for the error e and the controller state ξ have been chosen respectively equal to 10 and -2, during all simulations. The time-interval over which will be simulated is $[0, 10]$, for all simulations. The values of k_p, k_i and d have been varied to see what the influence of these values are on the problem, especially when the reset-action is applied. We already know that the values of k_p and k_i influence the stability of the system, i.e., the system is asymptotically stable if all the eigenvalues of the 2×2 matrix in (5.1) are in the open left half plane, stable if all the eigenvalues are in the closed left half plane and unstable if it is not stable. In general, the system is required to be asymptotically stable. The following values of k_p and k_i have been used:

$$\text{1)} \quad k_p = 1, k_i = 1 \quad \text{such that} \quad \lambda_{1,2} = -1 \tag{5.3}$$

$$\text{2)} \quad k_p = 2, k_i = 10 \quad \text{such that} \quad \lambda_{1,2} = -1.5 \pm 2.78i \tag{5.4}$$

$$\text{3)} \quad k_p = 2, k_i = 2 \quad \text{such that} \quad \lambda_1 = -2, \lambda_2 = -1. \tag{5.5}$$

Note: with these values of k_p and k_i, the system is asymptotically stable.
We choose these three cases to represent the three possible asymptotically stable systems with respectively two the same eigenvalues, two imaginary eigenvalues (where the eigenvalues are each others complex conjugate) and two disjoint eigenvalues.

In Figure A.1 the Simulink implementations of the standard form and the system with external disturbance d of the cruise-control problem have been represented. These implementations have been used as a basis for the Simulink implementation in Figure A.2, in which three systems with respectively no reset, reset-action $\xi^+ = 0$ and reset-action $\xi^+ = d/k_i$, are compared. Note that to all the systems in Figure A.2 the external disturbance d is still applied. Furthermore, we speak of overshoot when the transitory values of the error exceed zero. If the transitory values of the error are lower than zero, we speak of undershoot.

In Figure B.1, B.2 and B.3 we see respectively the cases (5.3), (5.4) and (5.5) where constant disturbance $d = 1$ is applied.

- The red line in Figure B.1 represents the case with no reset and values k_p and k_i as in (5.3). The error has some undershoot, but it converges nicely to zero so this system is asymptotically stable.

  But if we now apply the reset action $\xi^+ = 0$, which is represented by the blue line, we see that as soon as the error e becomes zero, the controller state ξ is also put to zero. If we compare the system with no reset and the system with reset action $\xi^+ = 0$, we see that for the system with no reset the error has some undershoot, while for the the system with reset action $\xi^+ = 0$ the error has some overshoot. Furthermore, we see that the system with no reset converges as fast as the system with reset. Only the magnitude of the undershoot of the system with no reset is larger than the magnitude of the overshoot of the system with reset. Here, the system with reset action $\xi^+ = 0$ does not work better than the system without reset.

  If the reset-action $\xi^+ = d/k_i$ is applied, which is represented by the green line, we see that as soon as the error becomes zero, it stays zero. This is a direct consequence of the choice of reset-action, since the reset-action $\xi^+ = d/k_i$ represents the equilibrium solution. So, what actually happens is that as soon as the error becomes zero, the error and controller reached their equilibrium and consequently stayed there. For a cruise controller, the latter is exactly what we want: as soon as we reach the desired speed, we stay at that speed without first any under- or overshoot as what happens if respectively no reset or the reset-action $\xi^+ = 0$ is applied.

- The red line in Figure B.2 represents the case with no reset and values k_p and k_i as in (5.4). The error oscillates around zero and converges slowly. For a cruise control, this is not exactly what we want since it takes the system too long to converge to the desired speed.

  If now the reset action $\xi^+ = 0$ is applied, which is represented by the blue line, we see that the reset is applied periodical. This is an effect of the oscillating error, since every time the error becomes zero, the controller state ξ is also put to zero. For a cruise control this would mean that if the desired speed is reached, the controller destabilizes the system. So, this is not what we want.

  If the reset action $\xi^+ = d/k_i$ is applied, which is represented by the green line, we see that as soon as the error becomes zero, the error and controller reached their equilibrium and consequently stayed there. So this is again exactly what we want for cruise control.

- Figure B.3 represents the case with values k_p and k_i as in (5.5). We see that in this case exactly the same happens as in the case where the values k_p and k_i are as in (5.3). The same conclusion yield here too.

In Figure B.4, B.5 and B.6 we see respectively the cases (5.3), (5.4) and (5.5) where constant disturbance $d = 10$ is applied. So, the only difference to the latter is that the disturbance d is much bigger now.

- In Figure B.4 we only see one line which represents the case with no reset where the values k_p and k_i are as in (5.3). The error converges nicely to zero, without any oscillation. The reset-actions $\xi^+ = 0$ and $\xi^+ = d/k_i$ are never applied, since the error never crosses zero which is due to the fact that the external constant disturbance d is now much bigger than before. We can conclude that this system works properly for cruise-control.

- The lines in Figure B.5 represent the case with values k_p and k_i as in (5.4) with respectively no reset, reset action $\xi^+ = 0$ and reset action $\xi^+ = d/k_i$. Actually the same happens as in Figure B.2 but now the magnitude of the overshoot caused by the reset action $\xi^+ = 0$ is much bigger which is caused by the bigger external disturbance d. We can again conclude that for cruise control the system with no reset works properly, the system with reset action $\xi^+ = 0$ destabilizes the system and the system with reset action $\xi^+ = d/k_i$ is exactly what we want.

- Figure B.6 represents the case with values k_p and k_i as in (5.5) and is again the exactly the same as the case where the values k_p and k_i are as in (5.3), with large disturbance $d = 10$. So, we conclude that this system works properly for cruise control.

Conclusion: To make the cruise-control work best, we require the system to be asymptotically stable. We saw that for both disturbances $d = 1$ as for $d = 10$ that the error of system with no reset always converges to zero while the system with reset action $\xi^+ = 0$ can destabilize the asymptotically stable system. Furthermore, if the reset action $\xi^+ = d/k_i$ is applied, the system does exactly what we want for cruise control, i.e., as soon as the error becomes zero, it stays zero. So we can conclude that the system with no reset works properly while the system with reset action $\xi^+ = 0$ does not work at all in some cases and the system with reset action $\xi^+ = d/k_i$ works perfectly. So, the reset action $\xi^+ = d/k_i$ applied to an asymptotically stable system, is the best system for a cruise control.

Chapter 6

Conclusion

In this report, stability of reset controllers has been determined in several ways. In the example 'De-stabilizing reset action' we saw that a negative feedback system without reset action, which was asymptotically stable for all β, became unstable if the reset-action was applied for some β outside a specific interval. With the result that the reset compensator has the same $\mathcal{L}_2$-gain as the base compensator if the storage function satisfies a certain condition in combination with the small gain theorem, stability of the closed loop system has been shown as main result. The closed-loop system, consisting of plant P and reset compensator R, is stable with Lyapunov function equal to the storage function of the system if plant P and reset compensator R satisfy certain conditions. Furthermore, reset control can be applied to cruise control which is simulated with the use of Simulink and Matlab. The system for the cruise control is required to be asymptotically stable and works best if reset-action equal to the equilibrium solution of the controller is applied. As soon as the error of the cruise control becomes zero, the error and controller reach their equilibrium and consequently stay there.

A possible topic for further research is to consider what happens if higher-order differential equations occur in the state-space equations in the example 'De-stabilizing reset action', since the theory used can only be applied to second-order plants. Another topic for further research concerns the application cruise control: what happens if the external disturbance d is not a constant but variable and what if in addition, the disturbance is unknown? This topic should be concerned before the cruise control can be applied to the car, because disturbances are in general non-constant and unknown.

Bibliography

[1] J. Carrasco, A. Baños, A.J. van der Schaft, *A passivity-based approach to reset control systems stability*, Elsevier, Systems & control Letters 59 (2010); 18-24.

[2] A. van der Schaft, *$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinear Control*, Springer, second edition (2000); 31-52.

[3] G.J. Olsder, J.W. van der Woude, *Mathematical Systems Theory*, VSSD, third edition (2005), The Netherlands; 121-123.

[4] H. Hu, Y. Zheng, Y. Chait, C.V. Hollot, *On the zero-input stability of control systems with Clegg integrators*, Proceedings of the American Control Conference, Albuquerque, NM, 1997; 408-410.

Appendix A

First Appendix

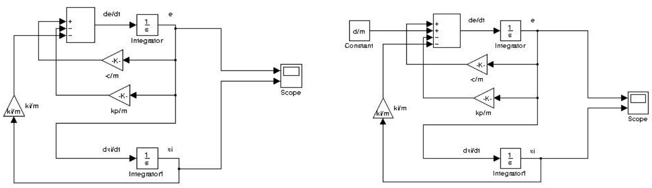

Figure A.1: Standard form (left) and system with external disturbance d (right).

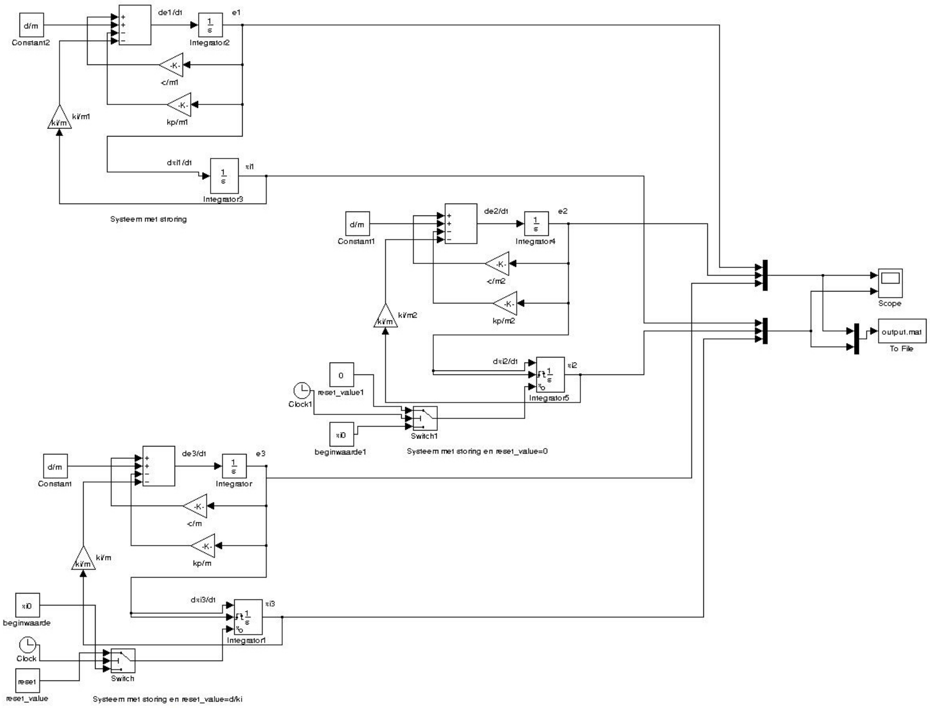

Figure A.2: Three systems with respectively no reset, with reset-action $\xi^+ = 0$ and with reset-action $\xi^+ = d/k_i$.

Appendix B

Second Appendix

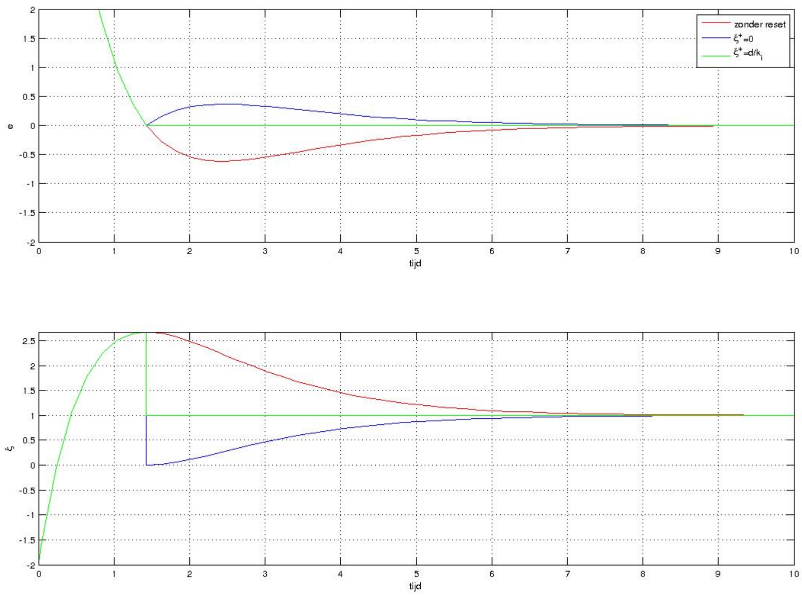

Figure B.1: System with values as in case 1 ($k_p = 1$, $k_i = 1$) with respectively no reset, with reset-action $\xi^+ = 0$ and with reset-action $\xi^+ = d/k_i$. Constant disturbance: $d = 1$.

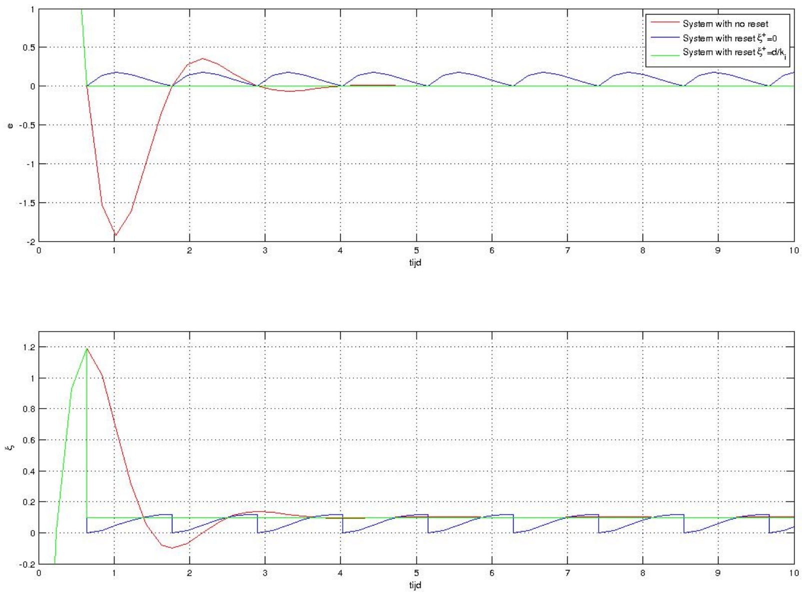

Figure B.2: System with variables as in case 2 ($k_p = 2$, $k_i = 10$) with respectively no reset, with reset-action $\xi^+ = 0$ and with reset-action $\xi^+ = d/k_i$. Constant disturbance: $d = 1$.

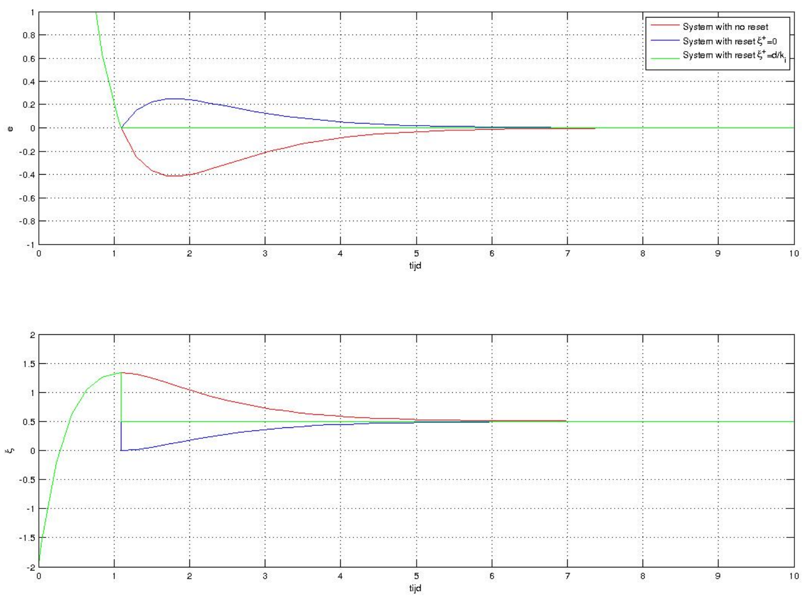

Figure B.3: System with values as in case 3 ($k_p = 2$, $k_i = 2$) with respectively no reset, with reset-action $\xi^+ = 0$ and with reset-action $\xi^+ = d/k_i$. Constant disturbance: $d = 1$.

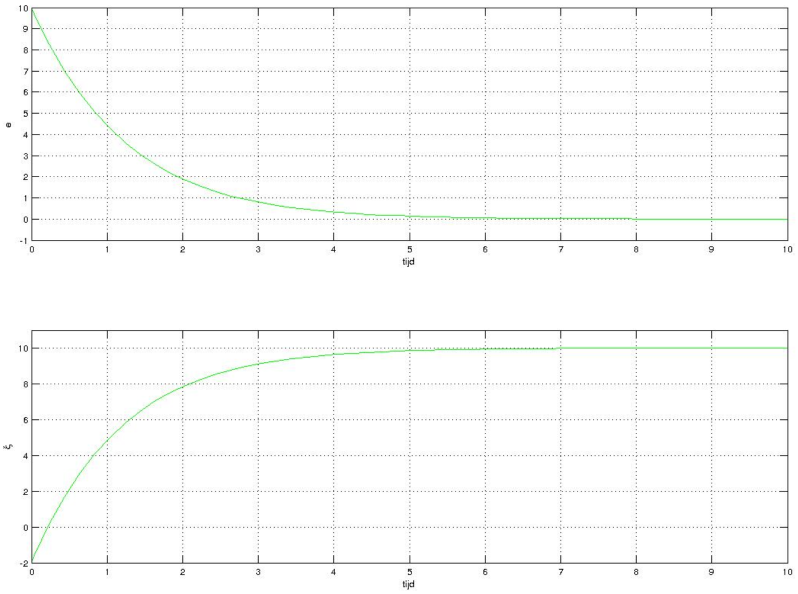

Figure B.4: System with values as in case 1 ($k_p = 1$, $k_i = 1$). Constant disturbance: $d = 10$.

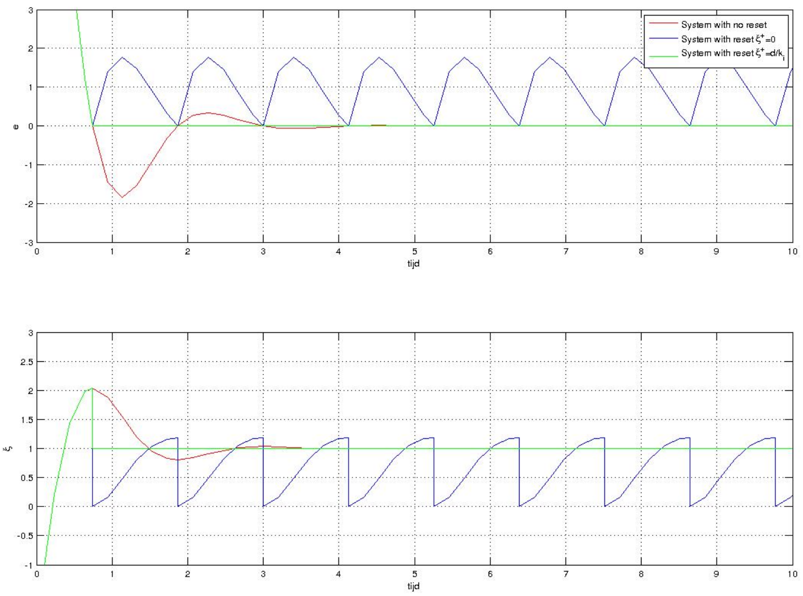

Figure B.5: System with values as in case 2 ($k_p = 2$, $k_i = 10$) with respectively no reset, with reset-action $\xi^+ = 0$ and with reset-action $\xi^+ = d/k_i$. Constant disturbance: $d = 10$.

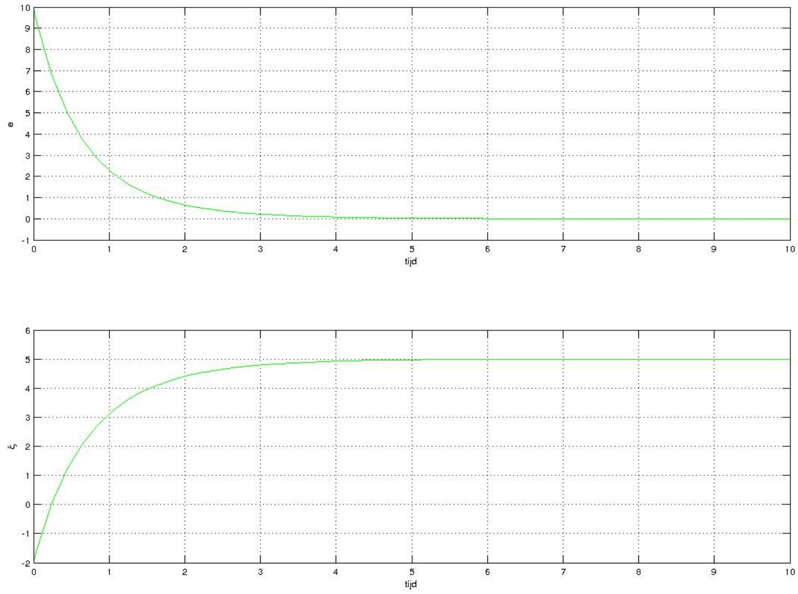

Figure B.6: System with values as in case 3 ($k_p = 2$, $k_i = 2$). Constant disturbance: $d = 10$.