UNIVERSITY OF GRONINGEN

Implementation of a quantum logic gate with one single laser-cooled trapped barium ion

by

J.J. van Dijk

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Supervisors: Prof. Dr. R.G.E. Timmermans, KVI
Dr. O.O. Versolato, KVI

Contents

1 Introduction

The successful implementation of a quantum computer would be a breakthrough in computer science. For example the factorization of a product of two large prime numbers is one though cookie for conventional computers, but using a special algorithm designed for a quantum computer, this would become a piece of cake. Currently modern technology is far from fully functioning quantum computers. The biggest factorial calculated so far is 15, which was correctly factorized in three and five. So where to start?

A quantum computer is a circuit of quantum operators, where specific combinations perform specific tasks. This thesis will focus on the controlled-NOT (CNOT) quantum logic gate, because it is essential to quantum computing. Any quantum circuit can be created by combining CNOT gates and single qubit-rotations [1]. A CNOT gate operation is similar to a classical XOR-gate. The exact workings will be discussed in section 2.

It is possible to create a CNOT logic gate, by trapping a single ion and creating an entanglement between the motional state of the ion in the trap and an internal-state of the ion. This is a two quantum bit (qubit) system, with the two motional states as one qubit and two internal electron states as the other qubit.

Monroe et al. succeeded in creating a CNOT gate with a trapped $^{9}Be^{+}$ ion, but the gate is only realized by applying three pulses of Raman beams to the ion. Here the electron is first excited to a third auxiliary state, before transitioning to the other qubit state [2]. Monroe et al. did show that it is theoretically also possible to create a CNOT gate without using an auxiliary state. This has four major advantages [3]:

1. The possibility to use other ions that do not have a third electronic ground state available to be used as an auxiliary level, such as $^{138}Ba^{+}$.

2. It prevents leaks to other levels from the auxiliary level, making the transition more stable. This may be important in quantum error-correction schemes.

3. It can greatly reduce the sensitivity of the gate to magnetic field fluctuations. For $^{138}Ba^{+}$ it should be possible to make the transition between the two levels insensitive to the magnetic-field in the first order.

4. Without the auxiliary state only one laser pulse is required, which makes the gate faster and is available at lower cost.

This reduced gate has been created between the motional and internal-state qubits of a trapped $^{9}Be^{+}$ ion [4]. But the advantage of $^{138}Ba^{+}$ over

$^{9}Be^{+}$ is the extremely stable $5D_{5/2}$ state, with a lifetime of over 30 seconds [5]. So this thesis will explore the possibly of creating a similar gate using the method described by Monroe et al. and the by them recommended $^{138}Ba^{+}$ ion.

2 Quantum Computing

2.1 Quantum Bit

Information in computers is stored into bits. A bit is a small piece of information with a discrete value, conventionally labeled 0 or 1. In electronics these levels are often decoded by voltages, a low voltage indicating 0 and a high voltage indicating 1.

This information can also be stored in a quantum mechanical system, where the bit is represented by two distinguishable quantum states of the system. An example of a quantum bit, or qubit, is the spin of an electron, which has two possible states, up and down. The advantage of labeling quantum mechanical states as bits lies in to possibility to create superpositions of these states and therefore of the qubits [1]. Something which is impossible with classical bits, but gives a major advantage with certain algorithms.

As mentioned in the introduction, Peter Shor gave an example of an algorithm that could take advantage of this new way of processing data. Resulting into Shor's algorithm to find the prime factors of an integer, working exponentially faster than most classical factoring algorithms. To successfully apply this algorithm would have major implications for cryptography, which relies on the vast amount of time it currently takes to factorize the product of two large (more then 300 digits) prime numbers.

2.2 Controlled-not

A controlled NOT or CNOT is a 2-qubit operation. This gate has two inputs and two outputs. The control-qubit remains unchanged and when the control-qubit is zero, the target qubit also remains unchanged. But when the control-qubit is one, the target qubit is inverted.

control-qubit	target-qubit	result
0	0	00
0	1	01
1	0	11
1	1	10

Table 1: Truth table of CNOT gate [1]

3 Trapping of an ion

3.1 Energy of a trapped ion

The Hamiltonian of the motion of a single trapped ion is very similar to the Hamiltonian of a static potential harmonic oscillator [6]:

$$\hat{H}^{(m)} = \frac{\hat{p}^2}{2m} + V(t), \tag{1}$$

where $\hat{p}$ and m are the impulse and mass of the ion respectively and

$$V(t) = \frac{m\omega_{\text{rf}}^2}{8}\left[a_x + 2q_x cos\left(\omega_{\text{rf}}t\right)\right]\hat{x}^2 = \frac{m}{2}W(t)\,\hat{x}^2, \tag{2}$$

where rf represents the radial frequency of the potential applied to the ring electrode, a_x and q_x are dimensionless trap parameters and $\hat{x}$ the Cartesian coordinates of the particle. The normalized wave function for the ground state of a single trapped ion is

$$\Psi(x,t) = \left(\frac{m\nu}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{u(t)}}\exp\left[\frac{im}{2\hbar}\frac{\dot{u}(t)}{u(t)}x^2\right], \tag{3}$$

where $u(t)$ has replaced the operator $\hat{x}$ and is approximated by

$$\beta_x \approx \sqrt{a_x + q_x^2/2}, \qquad \nu \approx \beta_x\omega_{\text{rf}}/2,$$

$$u(t) \approx \exp(i\nu t)\,\frac{1 + (q_x/2)\cos\left(\omega_{\text{rf}}t\right)}{1 + q_x/2}. \tag{4}$$

The complete derivation of the Hamiltonian and the wave function is shown by Leibfried et al. [6] and the proof of the normalization is found in appendix A.

The expected value for the energy is

$$\langle E\rangle = \int_{-\infty}^{\infty}\Psi^* H\Psi\,dx = \left[\left(\frac{\dot{u}(t)}{u(t)}\right)^2 + W(t)\right]\frac{u^*(t)\,u(t)\,\hbar}{4\nu} - \frac{ih}{2}\frac{\dot{u}(t)}{u(t)}, \tag{5}$$

and is time-dependent. In order to compare the energy to the energy of a harmonic oscillator we calculate it for $t = 0$. By definition

$$u(0) = u^*(0) = 1 \quad \text{and} \qquad \dot{u}(0) = i\nu, \tag{6}$$

the approximations (4) and

$$a_x \approx 0, \tag{7}$$

we get

$$\langle E_0 \rangle \approx \frac{\hbar\omega_{\text{rf}}}{2}\frac{1}{4\sqrt{2}}\left[4+q_x\right], \tag{8}$$

as shown in appendix C.

3.2 Interaction with electromagnetic fields

The internal structure of the ion can be represented by the energy levels $|g\rangle$ and $|e\rangle$ with energy difference $\hbar\omega = \hbar\left(\omega_e - \omega_g\right)$ [6]. The corresponding two-level Hamiltonian is

$$\begin{aligned}
\hat{H}^{(e)} &= \hbar\left(\omega_g \left|g\right\rangle \left\langle g\right| + \omega_e \left|e\right\rangle \left\langle e\right|\right) \\
&= \hbar\frac{2\omega_g + \omega_e - \omega_e}{2} \left|g\right\rangle \left\langle g\right| + \hbar\frac{2\omega_e + \omega_g - \omega_g}{2} \left|e\right\rangle \left\langle e\right| \\
&= \left(\hbar\frac{\omega_e + \omega_g}{2} - \hbar\frac{\omega_e - \omega_g}{2}\right) \left|g\right\rangle \left\langle g\right| \\
&\quad + \left(\hbar\frac{\omega_e + \omega_g}{2} + \hbar\frac{\omega_e - \omega_g}{2}\right) \left|e\right\rangle \left\langle e\right| \\
&= \hbar\frac{\omega_e + \omega_g}{2} \left|g\right\rangle \left\langle g\right| - \hbar\frac{\omega_e - \omega_g}{2} \left|g\right\rangle \left\langle g\right| \\
&\quad + \hbar\frac{\omega_e + \omega_g}{2} \left|e\right\rangle \left\langle e\right| + \hbar\frac{\omega_e - \omega_g}{2} \left|e\right\rangle \left\langle e\right| \\
&= \hbar\frac{\omega_e + \omega_g}{2} \left(\left|g\right\rangle \left\langle g\right| + \left|e\right\rangle \left\langle e\right|\right) + \hbar\frac{\omega}{2} \left(\left|e\right\rangle \left\langle e\right| - \left|g\right\rangle \left\langle g\right|\right).
\end{aligned} \tag{9}$$

The Hamiltonian $\hat{H}^{(e)}$ can be expressed using spin-1/2 algebra. Using the mapping

$$\begin{aligned}
\left|g\right\rangle \left\langle g\right| + \left|e\right\rangle \left\langle e\right| \mapsto \hat{I}, \qquad & \left|g\right\rangle \left\langle e\right| + \left|e\right\rangle \left\langle g\right| \mapsto \hat{\sigma}_x, \\
i\left(\left|g\right\rangle \left\langle e\right| - \left|e\right\rangle \left\langle g\right|\right) \mapsto \hat{\sigma}_y, \qquad & \left|e\right\rangle \left\langle e\right| - \left|g\right\rangle \left\langle g\right| \mapsto \hat{\sigma}_z.
\end{aligned} \tag{10}$$

With this mapping

$$\hat{H}^{(e)} = \hbar\frac{\omega_e + \omega_g}{2}\hat{I} + \hbar\frac{\omega}{2}\hat{\sigma}_z, \tag{11}$$

which reduces to

$$\hat{H}^{(e)} = \hbar\frac{\omega}{2}\hat{\sigma}_z, \tag{12}$$

by rescaling the energy to suppress the state-independent energy contribution.

The total Hamiltonian $\hat{H}$ of the system can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}^{(i)} = \hat{H}^{(m)} + \hat{H}^{(e)} + \hat{H}^{(i)}, \tag{13}$$

where $\hat{H}_0$ is the free Hamiltonian and $\hat{H}^{(i)}$, the Hamiltonian of the interactions mediated by the applied light fields, is

$$\hat{H}^{(i)} = (\hbar/2)\,\Omega\left(|g\rangle\langle e| + |e\rangle\langle g|\right) \times \left[e^{i(k\hat{x}_s - \omega t + \phi)} + e^{-i(k\hat{x}_s - \omega t + \phi)}\right] [6]. \tag{14}$$

In the spin-1/2 algebra we can re express

$$|e\rangle\langle g| \mapsto \hat{\sigma}_+ = 1/2\left(\hat{\sigma}_x + i\hat{\sigma}_y\right), \tag{15}$$

$$|g\rangle\langle e| \mapsto \hat{\sigma}_- = 1/2\left(\hat{\sigma}_x - i\hat{\sigma}_y\right), \tag{16}$$

resulting into

$$\hat{H}^{(i)} = (\hbar/2)\,\Omega\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \times \left[e^{i(k\hat{x}_s - \omega t + \phi)} + e^{-i(k\hat{x}_s - \omega t + \phi)}\right]. \tag{17}$$

For simplicity the problem is restricted to one dimension. The transformation of $\hat{H}^{(i)}$ into the interaction picture with $\hat{U}_0 = \exp\left[-(i/\hbar)\,\hat{H}_0 t\right]$ is

$$\begin{aligned}
\hat{H}_{\text{int}} &= \hat{U}_0^\dagger \hat{H}^{(i)} \hat{U}_0 \\
&= \left(e^{-(i/\hbar)\hat{H}_0 t}\right)^\dagger (\hbar/2)\,\Omega\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \\
&\quad \times \left[e^{i(k\hat{x} - \omega t + \phi)} + e^{-i(k\hat{x} - \omega t + \phi)}\right] e^{-(i/\hbar)\hat{H}_0 t} \\
&= e^{(i/\hbar)\hat{H}^{(m)} t} e^{(i/\hbar)\hat{H}^{(e)} t} (\hbar/2)\,\Omega\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \\
&\quad \times \left[e^{i(k\hat{x} - \omega t + \phi)} + e^{-i(k\hat{x} - \omega t + \phi)}\right] e^{-(i/\hbar)\hat{H}^{(m)} t} e^{-(i/\hbar)\hat{H}^{(e)} t},
\end{aligned} \tag{18}$$

which can be simplified to

$$\begin{aligned}
\hat{H}_{\text{int}} &= (\hbar/2)\,\Omega\left(\hat{\sigma}_+ e^{i\omega_0 t} + \hat{\sigma}_- e^{-i\omega_0 t}\right) e^{(i/\hbar)\hat{H}^{(m)} t} \\
&\quad \left[e^{i(k\hat{x} - \omega t + \phi)} + e^{-i(k\hat{x} - \omega t + \phi)}\right] e^{-(i/\hbar)\hat{H}^{(m)} t}.
\end{aligned} \tag{19}$$

See equation 64 of Leifried et al. for more details on this derivation [6].

4 Implementation of ^{138}Ba$^+$

4.1 Energy levels of ^{138}Ba$^+$

The considered energy levels are shown in figure 1 on page 9. Table 2 shows the incredibly long lifetime of the D states of ^{138}Ba$^+$ making them very suitable for our CNOT gate. And figure 1 shows that the D level can be attained by pumping the electron into the P shell that decays very quickly. The transition we will further look into is between $5D_{5/2}$ and $6S_{1/2}$.

4.2 A controlled-not two-qubit gate

A trapped ^{138}Ba$^+$ ion is cooled to its motional $|n = 0\rangle$ ground state in the ion trap. The other motional state that will be considered is $|n = 1\rangle$. The internal states that will be used are $6S_{1/2}$ and $5D_{5/2}$, referred to as $|\downarrow\rangle$ and $|\uparrow\rangle$ respectively. The energy of $|0\rangle$ is $\langle E_0\rangle \approx 0.785\frac{\hbar\omega}{2}$ and the energy of $|1\rangle$ is $\langle E_0\rangle \approx 0.785\frac{3\hbar\omega}{2}$, with $\omega \approx 8 \times 10^5$ according to the trap specifications. The energy difference between $|\downarrow\rangle$ and $|\uparrow\rangle$ is $\hbar\omega_0$ with $\omega_0 = 1.701 \times 10^{14} s^{-1}$.

The ion will be exposed to a traveling-wave electric field configured in such a way that, when the ion is in the motional state $|1\rangle$ a transition will a occur from $|1\rangle|\downarrow\rangle$ to $|1\rangle|\uparrow\rangle$ and from $|1\rangle|\uparrow\rangle$ to $|1\rangle|\downarrow\rangle$. But when the ion is in the motional ground state $|0\rangle$ the transitions will not occur, thus creating a CNOT gate.

4.3 Configuration of the traveling-wave

The energy of the transition between the motional states is $\hbar\omega$ treating it as harmonic oscillator, neglecting earlier calculations for simplicity. For the transition between the internal states the energy is $\hbar\omega_0$. Exposing the ion to a traveling-wave electric field $\mathbf{E}(\mathbf{z}) = \mathbf{E}_0 cos\left(\mathbf{k} \cdot \mathbf{z} - \omega_L t + \phi\right)$, a coupling will be created between $|0\rangle|\downarrow\rangle$ and $|1\rangle|\uparrow\rangle$ by setting $\omega_L = \omega_0 + \omega$. The coupling between the internal and motional states is controlled by the Lamb-Dicke parameter, $\eta = (\mathbf{k} \cdot \hat{\mathbf{z}})\, z_0$. The CNOT gate can be realized with a single pulse tuned to $\omega_L = \omega_0$ that couples the states $|n\rangle|\downarrow\rangle$ and $|n\rangle|\uparrow\rangle$ [3].

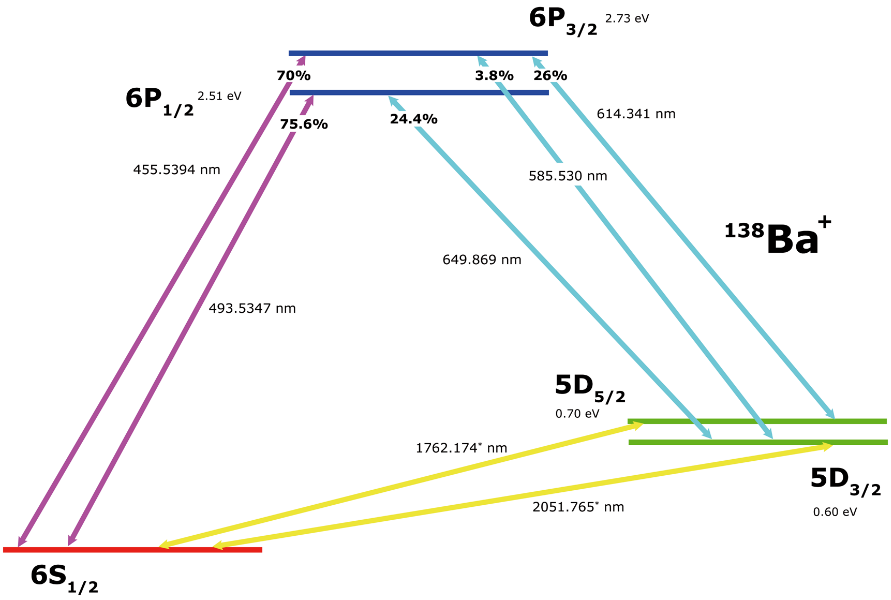

Figure 1: Based on appendix D the percentage show the branching ratios of the decay of $^{138}Ba^{+}$. The numbers next to the transitions are the wavelengths associated with them. The energy of every level with respect to $6S_{1/2}$ is calculated from the wavelengths and shown next to the level in eV.

Energy level	Lifetime	
	Theory	Experiment
$6P_{3/2}$	6.27 ns	6.32 (10) ns
$6P_{1/2}$	7.83 ns	7.90 (10) ns
$5D_{5/2}$	30.3(4) s	32.0 (4.6) s
$5D_{3/2}$	81.5(1.2) s	79.8 (4.6) s

Table 2: The theoretical and experimental values of the lifetime of the P and D states of $^{138}Ba^{+}$ [5]. The table clearly shows that the D level of $^{138}Ba^{+}$ is very stable.

4.4 Transition mechanism

How is it possible for a photon with energy $\hbar\omega$ to cause a transition between $|\downarrow\rangle$ and $|\uparrow\rangle$ only when the ion is in motional state $|1\rangle$? Because of the coupling between the internal and motional state the transition will take place in state $|0\rangle$ for all photons with energy $\hbar\omega_0 \pm \delta_0$ and in state $|1\rangle$ for all photons with energy $\hbar\omega_0 \pm \delta_1$. Where $\delta_{0,1}$ is the change in energy due to the motion of the ion in the trap. When $\delta_0 < \delta_1$ a photon of energy satisfying $\hbar\omega_0 - \delta_1 \leq \hbar\omega < \hbar\omega_0 - \delta_0$ or $\hbar\omega_0 + \delta_0 > \hbar\omega \geq \hbar\omega_0 + \delta_1$ will make the ion act as a controlled-NOT.

4.5 Using ^{138}Ba$^+$

Our goal is to create a ^{138}Ba$^+$ ion controlled-NOT, CNOT, gate. Cirac en Zoller created a CNOT gate using two internal and two external energy levels of ^{9}Be$^+$. For this configuration a third, auxiliary, internal energy level was required to make the transition work. This transition required three laser pulses [3].

According to Monroe et al. this CNOT gate can be simplified to a configuration requiring only one laser pulse, without the need for an auxiliary energy level. This creates the possibility to use ^{138}Ba$^+$ as a CNOT gate, because ^{138}Ba$^+$ does not have a third ground state that can serve as an auxiliary energy level. The use of only one pulse will give a much faster transition. [3].

When implementing the CNOT gate, the target qubit $|S\rangle$ is spanned by the $6S_{1/2}$ and $5D_{5/2}$ states of a single ^{138}Ba$^+$ ion. These will be labeled as $|\downarrow\rangle$ and $|\uparrow\rangle$ respectively. The difference in frequency equals $\omega_0 \approx 1.701 \times 10^{14} s^{-1}$. The control qubit $|n\rangle$ is spanned by the first two motional states ($|0\rangle$ en $|1\rangle$) of the trapped ion. The energy difference between the states is equivalent to the $\omega \approx 0.785\omega_{rf} = 6.28 \times 10^5 s^{-1}$.

To create the CNOT gate the laser has to be tuned to $\omega_L = \omega_0$, coupling states $|n\rangle|\downarrow\rangle$ and $|n\rangle|\uparrow\rangle$ with their Rabi frequency

$$\Omega_{n,n} = g_j e^{-\eta^2/2} L_n\left(\eta^2\right). \tag{20}$$

For the states $|n\rangle = |0\rangle$ en $|n\rangle = |1\rangle$ this is

$$\Omega_{0,0} = g_j e^{-\eta^2/2} = \frac{\vec{\mu}_j \cdot \vec{E}_0}{2\hbar} e^{-\eta^2/2} \tag{21}$$

$$\Omega_{1,1} = g_j e^{-\eta^2/2}\left(1-\eta^2\right) = \frac{\vec{\mu}_j \cdot \vec{E}_0}{2\hbar} e^{-\eta^2/2}\left(1-\eta^2\right). \tag{22}$$

To realize a CNOT gate, the condition $\Omega_{1,1}/\Omega_{0,0} = (2k+1)/2m$ has to be met with $m > k \geq 0$ [3]. Therefore η must be set to

$$\eta = \sqrt{1 - \frac{2k+1}{2m}} \tag{23}$$

4.6 Problems with ^{138}Ba$^+$

A possible transition of ^{138}Ba$^+$ is shown schematically in fig. 2 on page 12. The barium ions are trapped by a combination of oscillating and non-oscillating electric fields. The changing electric field caused a magnetic field, so we have to consider the Zeeman effect. The energy of states with $M_F \neq 0$ will depend linearly on a weak magnetic field because of this. Because ^{138}Ba$^+$ has a nuclear spin angular momentum of $I = 0$ $6S_{1/2}$ and $5D_{5/2}$ will depend on the magnetic field in the first order. As shown in table 3, as a consequence of $I = 0$, both levels don't have a state with an integer F, resulting in no available hyperfine states with $M_F = 0$.

This is why we consider two alternatives. The first alternative, shown in fig. 2, is to disregard the higher metastable $5D_{5/2}$ state. The internal part of the CNOT gate could use the two Zeeman levels of the $6S_{1/2}$ state, using a Raman transition. The second alternative, shown in fig. 2, is to switch from ^{138}Ba$^+$ to ^{137}Ba$^+$, because ^{138}Ba$^+$ to ^{137}Ba$^+$ has $I = 3/2$. Which results into levels $6S_{1/2}$ and $5D_{5/2}$ having states that *are* first order independent to the magnetic field strength. This is the alternative we will explore on further.

J \ I	1/2	5/2
0	$F = 1/2$	$F = 5/2$
3/2	$F = \{1, 2\}$	$F = \{1, 2, 3, 4\}$

Table 3: Possible total angular momenta (F) with certain nuclear spin (I) and electronic angular momentum (J)

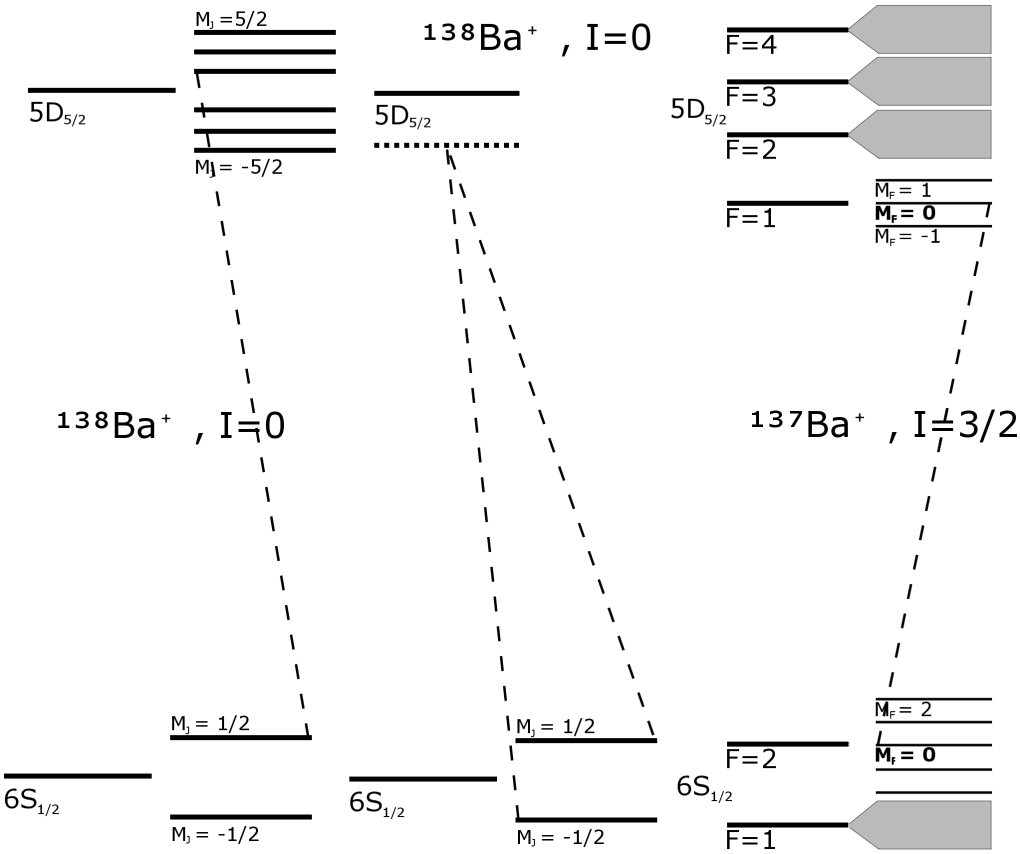

Figure 2: The left and middle shows possible energy transition of ^{138}Ba^{+}. All S and D levels depend on the magnetic field (B), because there is no $M_F = 0$ level. The diagram in the middle shows the possibility of a Raman transition. The diagram on the right shows the possibility of using ^{137}Ba^{+}, which (as marked in bold) has energy levels that *are* independent of weak magnetic fields.

4.7 $^{137}Ba^+$ as an alternative

Just as with $^{138}Ba^+$, the reduced controlled-NOT gate can be achieved in a single pulse by setting η such that $\Omega_{1,1}/\Omega_{0,0} = (2k+1)/2m$ when $\omega_L = \omega_0$. In this section we will discuss the effect of detuning on this ratio. Is this system very sensitive, rendering it unpractical, or is this something that can be dealt with in an experimental setting?

The generalized Rabi frequency is given by eq. 24, with the Rabi frequency $\Omega_{n,n}$ at resonance, the detuning Δ and the decay rate Γ [6].

$$\chi_{n,n} = \sqrt{\Omega_{n,n}^2 + \Delta^2 + \Gamma^2} \tag{24}$$

Typical values for Ω, η and Γ are

$$\Omega_{0,0} = 2\pi \times 92\text{kHz}[4]$$

$$\eta = \sqrt{1 - \frac{2\times 0+1}{2\times 1}} = \sqrt{1/2}[3]$$

$$\Gamma = 1/32\text{s}.$$

The derivative of $\Omega_{1,1}/\Omega_{0,0}$ (eq. 26) shows that the ratio is first order independent of Δ.

$$\frac{\partial}{\partial\Delta}\left(\frac{\chi_{1,1}}{\chi_{0,0}}\right) = \frac{\partial}{\partial\Delta}\sqrt{\frac{\Omega_{1,1}^2+\Delta^2+\Gamma^2}{\Omega_{0,0}^2+\Delta^2+\Gamma^2}} \tag{25}$$

$$= \frac{\left(\Omega_{0,0}^2 - \Omega_{1,1}^2\right)\Delta}{\sqrt{\frac{\Omega_{1,1}^2+\Gamma^2+\Delta^2}{\Omega_{0,0}^2+\Gamma^2+\Delta^2}}\left(\Omega_{0,0}^2+\Gamma^2+\Delta^2\right)^2} \tag{26}$$

To gain a better understanding we will expand the ratio in $\frac{\Delta}{\Gamma}$. This results in the following third order Taylor approximation.

$$\sqrt{\frac{\Omega_{1,1}^2+\Gamma^2}{\Omega_{0,0}^2+\Gamma^2}} + \frac{\left(\Omega_{0,0}^2+\Gamma^2\right)\sqrt{\frac{\Omega_{1,1}^2+\Gamma^2}{\Omega_{0,0}^2+\Gamma^2}}\left(\frac{\Gamma^2}{\Omega_{0,0}^2+\Gamma^2} - \frac{\Gamma^2\left(\Omega_{1,1}^2+\Gamma^2\right)}{\left(\Omega_{0,0}^2+\Gamma^2\right)^2}\right)\frac{\Delta}{\Gamma}^2}{2\left(\Omega_{1,1}^2+\Gamma^2\right)} \tag{27}$$

To prevent uncertainty in the frequency the ion is pumped with a lot of power and the decay rate of the $5D_{5/2}$ level is very slow, or put differently: $\Omega_{0,0} >> \Gamma$. This allows for the approximation $\Omega_{0,0}^2 + \Gamma^2 \approx \Omega_{0,0}^2$. With this approximation and $\Omega_{1,1} = \frac{1-\eta^2}{2}\Omega_{0,0}$ eq. 27 can be simplified to

$$1 - \eta^2 + \frac{1}{2}\left(\frac{1}{1-\eta^2} - 1 + \eta^2\right)\frac{\Delta}{\Omega_{0,0}}^{2} \tag{28}$$

This approximation yield that the ratio $\Omega_{1,1}/\Omega_{0,0}$ still equals $1 - \eta^2$ when there is no detuning ($\Delta = 0$). And the effect of detuning is indeed not linear, but quadratic. Filling in a sample values for η and a detuning of 1% ($\Delta/\Omega_{0,0} = 0.01$), eq. 29 shows that the ratio only shifts a little. This should not be a big problem when trying to create a CNOT gate with $^{138}Ba^{+}$.

$$\Omega_{1,1}/\Omega_{0,0} = \frac{1}{2} + \frac{3}{4}0.01^2 = 0.500075 \tag{29}$$

5 Conclusion

Monroe et al. created a CNOT gate with a $^{9}Be^{+}$ ion in an Paul trap [2]. This gate operated with three laser pulses and needed besides the two energy levels for the qubit a third auxiliary level to make the transition. But two years later Monroe et al. showed that this gate could also be created using only one laser pulse and without the use of an auxiliary level [3]. This opened the door for other ions that don't have an auxiliary level available. An excellent candidate would be, $^{138}Ba^{+}$, because the qubit could consist of two states, making the transition first order insensitive to changes in the magnetic field of the trap.

In section 3 we analyzed the energy levels of an ion in a Paul trap. Specifically those of $^{138}Ba^{+}$, which showed to be an outstanding candidate because of the extremely long lifetime of the D shell energy levels. In section 4 we further looked into the transition mechanism of the CNOT gate and tried to select suitable states to use for the transition. On closer inspection $^{138}Ba^{+}$ with $I = 0$ had no hyperfine states with $M_F = 0$, making all states dependent on weak magnetic field, rendering $^{138}Ba^{+}$ unsuitable for use as a CNOT gate contrary to the recommendation of Monroe et al. The changing magnetic field in the trap would make it very hard to set the laser to the right frequency.

Looking for alternatives we found a different barium isotope. $^{137}Ba^{+}$ has a nuclear spin of $I = 3/2$, and therefore both the S and D shell have several integer F states with $M_F = 0$, which *are* independent to weak magnetic fields. Changes in the magnetic field only had a second order effect on the detuning. With a detuning of 1% resulting in a change in the ratio of Rabi frequency's of only 0.015%. This is a stability that can be reached in an experimental setting, making $^{137}Ba^{+}$ a viable candidate for a CNOT gate in a Paul trap. And this would certainly be recommended further research.

A Normalization

Proof that equation (3) is normalized.

$$\begin{aligned} \Psi(x,t) &= \left(\frac{m\nu}{\pi\hbar}\right)^{1/4} \frac{1}{\{u(t)\}^{1/2}} \exp\left[\frac{im}{2\hbar}\frac{\dot{u}(t)}{u(t)}x^2\right], & (30) \\ \Psi^*\Psi &= \left(\frac{m\nu}{\pi\hbar}\right)^{1/2} \frac{1}{\{u^*u\}^{1/2}} \exp\left[\frac{im}{2\hbar}x^2\left(\frac{\dot{u}}{u} - \frac{\dot{u}^*}{u^*}\right)\right], & (31) \end{aligned}$$

$$u^*\dot{u} - u\dot{u}^* = 2i\nu[6] \Leftrightarrow \frac{\dot{u}}{u} - \frac{\dot{u}^*}{u^*} = \frac{2i\nu}{u^*u}, \tag{32}$$

$$\begin{aligned} \Rightarrow \Psi^*\Psi &= \left(\frac{m\nu}{\pi\hbar}\right)^{1/2} \frac{1}{\{u^*u\}^{1/2}} \exp\left[-\frac{m\nu}{\hbar}\frac{1}{u^*u}x^2\right], & (33) \\ \int_{-\infty}^{\infty} \Psi^*\Psi dx &= \left(\frac{m\nu}{\pi\hbar}\right)^{1/2} \frac{1}{\{u^*u\}^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{m\nu}{\hbar}\frac{1}{u^*u}x^2\right] dx \\ &= \left(\frac{m\nu}{\pi\hbar}\right)^{1/2} \frac{1}{\{u^*u\}^{1/2}} \frac{\pi^{1/2}}{\left\{\frac{m\nu}{\hbar}\frac{1}{u^*u}\right\}^{1/2}} \\ &= \sqrt{\frac{m\nu\pi\hbar u^*u}{\pi\hbar u^*um\nu}} = 1. & (34) \end{aligned}$$

So

$$\int_{-\infty}^{\infty} \Psi^*\Psi dx = 1, \tag{35}$$

therefore (3) is normalized.

B Expected Energy

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \Psi^* H \Psi \, dx \\ &= \int_{-\infty}^{\infty} \Psi^* \left(\frac{m}{2}\left[\left(\frac{\dot{u}(t)}{u(t)}\right)^2 + W(t)\right]x^2 - \frac{ih}{2}\frac{\dot{u}(t)}{u(t)}\right) \Psi \, dx \\ &= \int_{-\infty}^{\infty} \Psi^* \frac{m}{2}\left[\left(\frac{\dot{u}(t)}{u(t)}\right)^2 + W(t)\right] x^2 \Psi \, dx - \int_{-\infty}^{\infty} \Psi^* \frac{i\hbar}{2}\frac{\dot{u}(t)}{u(t)} \Psi \, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{m}{2}\left[\left(\frac{\dot{u}(t)}{u(t)}\right)^2 + W(t)\right]\int_{-\infty}^{\infty}\Psi^* x^2 \Psi\, dx - \frac{i\hbar}{2}\frac{\dot{u}(t)}{u(t)}\int_{-\infty}^{\infty}\Psi^*\Psi\, dx \\
&= \frac{m}{2}\left[\left(\frac{\dot{u}(t)}{u(t)}\right)^2 + W(t)\right]\int_{-\infty}^{\infty}\Psi^* x^2 \Psi\, dx - \frac{i\hbar}{2}\frac{\dot{u}(t)}{u(t)} \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty}\Psi^* x^2 \Psi\, dx &= \left(\frac{m\nu}{\pi\hbar}\right)^{1/2}\frac{1}{\{u^*u\}^{1/2}}\int_{-\infty}^{\infty}\exp\left[-\frac{m\nu}{\hbar}\frac{1}{u^*u}x^2\right]x^2 dx \\
&= \left(\frac{m\nu}{\pi\hbar}\right)^{1/2}\frac{1}{\{u^*u\}^{1/2}}\frac{1}{2}\frac{\pi^{1/2}}{\left\{\frac{m\nu}{\hbar}\frac{1}{u^*u}\right\}^{3/2}} \\
&= \frac{1}{2\frac{m\nu}{\hbar}\frac{1}{u^*u}} \\
&= \frac{u^*u\hbar}{2m\nu} \qquad (37)
\end{aligned}
$$

$$
E = \left[\left(\frac{\dot{u}(t)}{u(t)}\right)^2 + W(t)\right]\frac{u^*(t)\,u(t)\,\hbar}{4\nu} - \frac{i\hbar}{2}\frac{\dot{u}(t)}{u(t)} \qquad (38)
$$

C Energy at $t = 0$

$$
\langle E_0\rangle = \langle E(t=0)\rangle = \left[\left(\frac{\dot{u}(0)}{u(0)}\right)^2 + W(0)\right]\frac{u^*(0)\,u(0)\,\hbar}{4\nu} - \frac{i\hbar}{2}\frac{\dot{u}(0)}{u(0)} \qquad (39)
$$

By definition (6) this reduces to

$$
\begin{aligned}
\langle E_0\rangle &= \left[(i\nu)^2 + W(0)\right]\frac{\hbar}{4\nu} - \frac{i\hbar}{2}i\nu \\
&= \left[-\nu^2 + W(0)\right]\frac{\hbar}{4\nu} + \frac{\hbar}{2}\nu \\
&= \frac{\hbar\nu}{2} - \frac{\hbar\nu}{4} + \frac{\hbar}{4\nu}W(0) \\
&= \frac{\hbar\nu}{4} + \frac{\hbar}{4\nu}W(0) \qquad (40)
\end{aligned}
$$

Using approximations (4),(7) and equation (2) to put

$$
\beta_x \approx \sqrt{a_x + q_x^2/2} \approx q_x/\sqrt{2}, \quad \nu \approx \frac{\beta_x\omega_{\mathrm{rf}}}{2} \approx \frac{q_x\omega_{\mathrm{rf}}}{2\sqrt{2}},
$$

$$W(0) = \frac{\omega_{\mathrm{rf}}^2}{4}\left[a_x + 2q_x\right] \approx \frac{q_x \omega_{\mathrm{rf}}^2}{2}, \tag{41}$$

into equation (40), giving

$$\begin{aligned}
\langle E_0 \rangle &\approx \frac{\hbar q_x \omega_{\mathrm{rf}}}{8\sqrt{2}} + \frac{2\sqrt{2}\hbar}{4 q_x \omega_{\mathrm{rf}}} \frac{q_x \omega_{\mathrm{rf}}^2}{2} \\
&= \frac{\hbar q_x \omega_{\mathrm{rf}}}{8\sqrt{2}} + \frac{\sqrt{2}\hbar \omega_{\mathrm{rf}}}{4} \\
&= \left[\frac{q_x}{4\sqrt{2}} + \frac{\sqrt{2}}{2}\right] \frac{\hbar \omega_{\mathrm{rf}}}{2} \\
&= \frac{\hbar \omega_{\mathrm{rf}}}{2} \frac{1}{4\sqrt{2}} \left[4 + q_x\right] \qquad (42)
\end{aligned}$$

D Energy levels of barium

Transition	Energy (eV)	Wavelength (nm) [7]		Branching Ratio
		Ritz	Observed	
$6P_{3/2} - 6S_{1/2}$	2.73 – 0	455.5394	455.5310	0.756 (46) [8]
$6P_{3/2} - 5D_{3/2}$	2.73 – 0.60	585.530	585.5298	0.0290 (15) [8]
$6P_{3/2} - 5D_{5/2}$	2.73 – 0.70	614.341	614.3413	0.2150 (64) [8]
$6P_{1/2} - 6S_{1/2}$	2.51 – 0	493.5347	493.5454	0.756 (12) [9]
$6P_{1/2} - 5D_{3/2}$	2.51 – 0.60	649.869	649.8693	0.244 (12) [9]
$5D_{5/2} - 6S_{1/2}$	0.70 – 0	1762.174 [1]		
$5D_{3/2} - 6S_{1/2}$	0.60 – 0	2051.765 [1]		

[1] Calculated from corresponding energy levels

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