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Index Theory

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Abstract

Is it possible to comb the hairs of a billiard ball? In other words, can a singularity-free vector field exist on the 2-sphere? This is a question that can be answered using index theory. First, the definition of an index is discussed, after which we will prove that a vector field on the sphere must have singularities and that the sum of indices is independent of the vector field on the sphere. Finally, we will consider other manifolds, both homeomorphic and not homeomorphic to the sphere.

Contents

1 Index

In this section, we will cover the definition of winding number and index, with their properties.

1.1 Winding Number

For a proper definition of the index, we need the notion of winding number (Fulton [2], page 20). Take a closed curve gamma $\gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}$, then the winding number is defined as the "net" number of times γ goes (winds) around the origin, with the counterclockwise direction as positive. To be precise, the winding number of γ reads as:

$$W(\gamma, 0) = \frac{1}{2\pi} \int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{1}$$

According to the Stokes formula, the winding number is well-defined for a smooth γ (Broer, [1], page 126), so $W(\gamma, 0)$ is independent of parametrization.

For example, let $\gamma_k(t) \in \mathbb{C} \cong \mathbb{R}^2$ and let

$$\gamma_k(t) = e^{2\pi ikt}, t \in [0,1], k \in \mathbb{Z},$$

then $W(\gamma_k, 0) = k$.

1.2 The index of a singularity

Given a vector field V on a surface U and assume that V has isolated singularities that form a set Z. Then V maps as follows: $V : U \backslash Z \to \mathbb{R}^2 \backslash \{0\}$. Now we can define the index of V, at point $p \in Z$. Take a small circle C around p and restrict V to this circle (denoted by V_C), such that $V_C \subseteq U \setminus Z$ and C only contains p. We then have a mapping from C to $\mathbb{R}^2$, with a winding number. We define the index to be (Fulton [2], page 97):

$$\mathrm{Index}_p V = W(V|_C, 0) \tag{2}$$

Since the winding number is independent of parametrization, so is the index. However, the index is only independent of the radius of C, as long as the number of singularities inside V_C remains the same inside V_C. If p is no singular point, then $\mathrm{Index}_p V = 0$.

We now have a vector field $V(a,b) = (a(x,y), b(x,y))$. The index is the winding number over a circle restricted to the vector field, which will give us:

$$\text{Index}_p V = \frac{1}{2\pi} \int_C \frac{adb - bda}{a^2 + b^2} \tag{3}$$

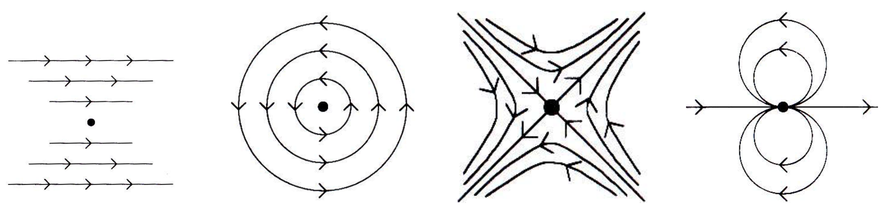
For computing the index, we will first substitute x and y in a and b, after that, we will substitute $\cos t$ for x and $\sin t$ for y. The latter will often be the same: $x = \cos t$, $dx = -\sin t\, dt$ $y = \sin t$, $dy = \cos t\, dt$.

Let D be a closed disk with boundary circle C and let V be a vector field on D without singularities at the boundary, then (Fulton [2], page 100):

$$W(V|_C) = \sum_{p \in D} \text{Index}_p V \tag{4}$$

A qualitative approach: Instead of using the integral formula, let an arrow follow C once, tangent to the vectors of V_C as it goes. The index is the number of times the arrow revolves.

Examples with various indices. Of the following examples, all singularities are found at the origin. The integral formula is used to calculate the index and the qualitative approach is left to the reader, with the images found below. (Fulton [2], page 98).



(a) Index 0 (b) Index 1 (c) Index -1 (d) Index 2

Examples of indices

(a) $V(x,y) = (x^2 + y^2, 0)$. This field has a singularity with an index equal to zero, which can be seen by the parametrization ($a = x^2 + y^2$, $b = 0$, so also $db = 0$).

(b) $V(x,y) = (-y, x)$. This vector field has a singularity with an index equal to one. The integral formula becomes (after parametrization):

$$\frac{1}{2\pi}\int_0^{2\pi} dt = 1$$

(c) $V(x,y) = (y,x)$. Saddle points have an index equal to minus one. From the integral formula, we get:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{-\sin^2 t - \cos^2 t}{\cos^2 t + \sin^2 t} dt = -1$$

(d) $V(x,y) = (x^2 - y^2, 2xy)$ (let $z = x + iy$, then $V = (x,y) = (Re(z^2), Im(z^2))$. This vector field has a singularity with an index equal to two. The resulting parametrization will give:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{2(\cos^2 t - \sin^2 t)^2 + 8\cos^2 t \sin^2 t}{(\cos^2 t - \sin^2 t)^2 + 4\cos^2 t \sin^2 t} dt = 2$$

2 Indices on the 2-sphere

In this section, we will look at the existence of singularities of a vector field on the 2-sphere.

Theorem 1 (hairy ball). *On the 2-sphere, the sum of indices is equal to 2, regardless of the chosen vector field.*

For the proof, we will first flatten out the sphere to the plane by stereographic projection (Fulton [2], page 103):

$$\begin{aligned} \Phi : \mathbb{S}^2 \backslash \{NP\} &\to \mathbb{R}^2 \\ V \qquad & \quad \Phi_*(V), \end{aligned} \tag{5}$$

where $NP = (0,0,1)$. This map is continuous, so $\Phi_*(V)$ on $\mathbb{R}^2$ is continuous if V on $\mathbb{S}^2$ is continous.

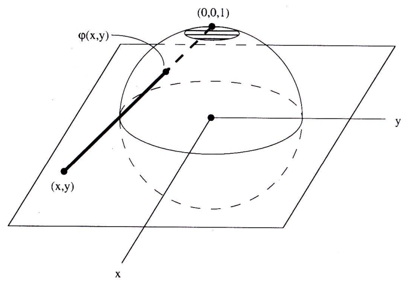


figure 2.1: stereographic projection

Proof. Assume, by contradiction, that V has no singularities on $\mathbb{S}^2$, thus $\Phi_*(V)$ has no singularities on $\mathbb{R}^2$. First, consider a small disc D at NP, with boundary C, then $W(V|_C) = 0$. If the disk is small enough, we can consider the vectors to be almost parallel. Using the stereographic projection, $C' = \Phi(C)$ is a very large circle in $\mathbb{R}^2$, with $W(\Phi_*(V)) = 0$.

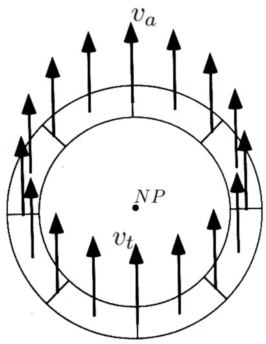


figure 2.2: V near NP

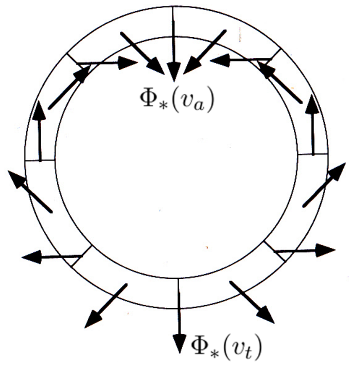


figure 2.3: $\Phi_*(V)$

However, let v_t be the vector pointing at C towards NP and v_a be the vector pointing away from NP, then $\Phi_*(v_t)$ will point towards infinity and $\Phi_*(v_a)$ will point towards the origin. Since $\Phi_*(V)$ is continuous, it will look like the one in figure 2.3. Here we see that $W(\Phi * (V)|_{C'}) = 2 \neq 0$, concluding the contradiction. Furthermore, let D' be the region inside C' (e.g. $C' = \partial D'$), then:

$$\sum_{q \in D'} \mathrm{Ind}_p \Phi_*(V) = 2,$$

so

$$\sum_{p \in \mathbb{S}^2 \setminus D} \mathrm{Ind}_p V = 2.$$

Since $\sum_{p \in D} \mathrm{Ind}_p V = 0$, we now know that $\sum_{p \in \mathbb{S}^2} \mathrm{Ind}_p V = 2$.

Now that we have proven that $\sum_{p \in \mathbb{S}^2} \mathrm{Ind}_p V$, we only need to prove that this holds for any vector field. Let W be an arbitrary vector field on $\mathbb{S}^2$ with a finite number of isolated singularities and let $p \in \mathbb{S}^2$ be a point where W is not zero. Let D be a small disk at p, with boundary C. If the disk is small enough, the vectors are almost parallel (figure 2.2). Now the rest of the proof is the same. $\square$

This theorem is known as the hairy ball theorem: consider a ball covered in small hairs, then it is impossible to comb the hairs without creating a cowlick. The theorem was proven by L.E.J. Brouwer (Broer [1], page 106) for any n-dimensional sphere, when n is even. If n is odd, the sum of indices is zero, i.e. a singularity-free vector field can exist.

3 Other manifolds

3.1 Triangulation of $\mathbb{S}^2$

Let M be a manifold with v vertices, e edges and f faces, then the Euler characteristic $\chi(M)$ is defined as (Fulton [2], page 113):

$$\chi(M) = v - e + f \tag{6}$$

Theorem 2. *For any manifold M, with v vertices, e edges and f faces, homeomorphic to $\mathbb{S}^2$, the following holds:*

$$\chi(M) = v - e + f = 2$$

To prove the theorem, we will first triangulate the sphere: decompose $\mathbb{S}^2$ into a small surfaces homeomorphic to triangles, that fit together along the edges. Other polygonals can be used too, as long as they fit.

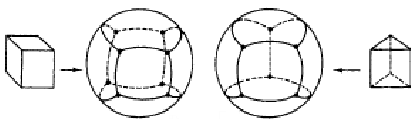


figure 3.1: decompositions of the sphere for the cube and prism ([3], page 370)

Having made a triangulation of the sphere, we will construct a vector field V such that we find a singularity with index 1 at each vertex, one with index -1 on each edge and one with index 1 in each face. Then:

$$2 = \sum_{p \in M} \mathrm{Ind}_p V = 1 \cdot v + (-1) \cdot e + 1 \cdot f = \chi(M)$$

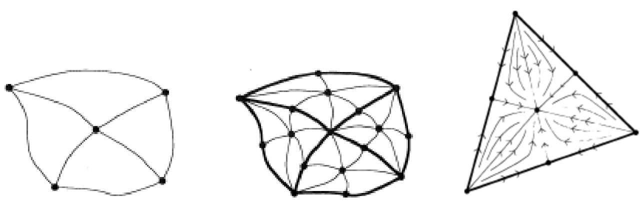


figure 3.2: triangulation

3.2 Singularities on the torus

While a singularity-free vector field cannot exist on $\mathbb{S}^2$, it can on $\mathbb{T}^2$ (hairs on a doughnut can be combed without creating a cowlick).

Theorem 3 (Fulton [2], page 107)**.** *for any continuous vector field with singularities V on $\mathbb{T}^2$,*

$$\sum_{p\in\mathbb{T}^2} Ind_p V = 0 \tag{7}$$

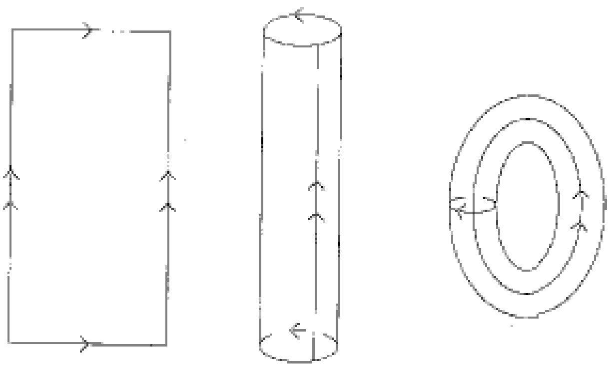


figure 3.3: construction of a torus

Proof. The torus can be constructed by taking a square or rectangle and identifying the opposite edges. Conversely, a rectangle R can be constructed by cutting the torus open twice. Let V be a continuous vector field on $\mathbb{T}^2$, with isolated singularities. Then, cut $\mathbb{T}^2$ open along a meridian and circle of longitude that have no singularities. Then we have a vector field V' that is continuous on R, including its edges. Since the vector field is exactly the same on the opposite edges,

$$0 = W(V'|_{\partial R}) = \sum_{q\in R} \mathrm{Ind}_q V' = \sum_{p\in\mathbb{T}^2} \mathrm{Ind}_p V.$$

□

Using triangulation, we now know that for any manifold M homeomorfic to $\mathbb{T}^2, \chi(M) = 0$.

3.3 Manifolds with more holes

Now that we have proven that $\sum_{p\in\mathbb{S}^2} \mathrm{Ind}_p V = 2$ and that $\sum_{p\in\mathbb{T}^2} \mathrm{Ind}_p V = 0$ for a vector field V, we wish to find a generalization.

Theorem 4 (Poincaré-Hopf). *Let X be a sphere with g handles, or doughnut with g holes. For any vector field V with singularities on X,*

$$\sum_{p\in X} Ind_p V = 2 \tag{8}$$

The proof for this theorem (Fulton [2], page 108) shall be omitted. The theorem was proven by Henri Poincaré for two dimensional manifolds and generalized by Heinz Hopf.
Like the torus and the sphere, a manifold M with g holes can be triangulated, so $\chi(M) = 2 - 2g$ (Fulton, [2], page 114).

4 conclusion

In this thesis, we have described the definition of an index and used index theory to show that a singularity-free vector field cannot exist on the 2-sphere. Furthermore, sum of indices of a vector field V, with a finite number of singularities, on the 2-sphere always equals 2. If we triangulate the 2-sphere, we have shown that any manifold M (with v vertices, e edges and f faces) homeomorphic with the 2-sphere has an Euler characteristic of 2.
Whereas a singularity-free vector field cannot exist on the 2-sphere, it can on the torus (compare combing the hairs of a ball and combing the hairs of a doughnut), so the sum of indices of a vector field on a torus always equals zero. Using triangulation, it becomes clear that a manifold homeomorphic with the torus has an Euler characteristic of 0.
For further reading, the reader is referred to the Poincaré-Hopf theorem (Fulton, [2], page 108), which describes the relation between the number of holes in a manifold and the sum of indices of a vector field on that manifold.

References

[1] H. W. Broer. *Meetkunde en Fysica.* Epsilon Uitgraven, Utrecht, 1999.

[2] Willian Fulton. *Algebraic Topology.* Springer-Verlag, 1995.

[3] Robert O'Neill. *Elementary Differential Geometry.* Academic Press, 2006.