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Equivalences between behavioral representations and controllability in terms of driving variable representations

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Abstract

In this paper, we deal with the relationship between a rational kernel representation and an output nulling representation that a behavior admits. Similarly we deal with the relationship between a rational image representation and a driving variable representation that a behavior admits. We will establish sufficient conditions on a realization of a proper rational matrix such that the behavior induced by the kernel of this matrix is equal to the behavior of the output nulling representation yielded by this realization. Also we will look under which assumptions these conditions are also necessary.

We will establish necessary and sufficient conditions on a realization of a proper rational matrix such that the behavior induced by the image of this matrix is equal to the behavior of the driving variable representation yielded by this realization. Finally necessary and sufficient conditions are found under which the (full/external) behavior induced by a driving variable represenation is controllable.

1 Introduction

In this paper we deal with representations of linear differential systems like rational kernel (image) representations, and state representations such as driving variable representations and output nulling representations. The state representations that we consider in this paper are more generic and a natural starting point as compared to the well-known input/state/output (I/S/O) representations. Though we deal with representations in this paper, emphasis is laid on the behavior of the system. A behavior admits various representations and the above mentioned representations are some of them. We do not impose a priori input-output partition on the system variables.

One goal of this paper is, given a behavior represented by a proper rational kernel representation, find necessary and sufficient conditions under which a realization of this matrix yields an output nulling representation of this behavior.

Also, given a controllable behavior represented by a proper rational image representation, we want to find necessary and sufficient conditions under which a realization of this matrix yields a driving variable representation of this behavior. This question also results in finding conditions for controllability on the constant real matrices of a driving variable representation of a given behavior as shown in Section 6.

The paper is organized as follows: we will start with Section 2 wherein we discuss about the notation used in this paper. Then in Section 3 some canonical decompositions of linear systems will be discussed and in Section 4 we introduce the basic concepts related to linear differential systems and further discusses necessary definitions, theorems and various representations of behaviors that will be used in obtaining the main results. In Section 5 we will focus on solving the first problem we mentioned above and in Section 6 we will focus on the second problem.

2 Notation

In this section we introduce the notation that will be used in this paper. We denote by $\mathbb{R}[\xi]^{m\times n}$ as the set of polynomial matrices with dimension $m \times n$ with argument ξ. Similary we denote by $\mathbb{R}(\xi)^{m\times n}$ as the set of rational matrices with dimension $m \times n$ with argument ξ. Often we write $\mathbb{R}[\xi]^{\bullet\times\bullet}$, $\mathbb{R}[\xi]^{m\times\bullet}$, $\mathbb{R}[\xi]^{\bullet\times n}$ if dimension is clear from the context. The notation $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ is used for the set of all infinitely differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. The polynomial matrix $P \in \mathbb{R}[\xi]^{m\times n}$ is said to be *left prime* over $\mathbb{R}[\xi]$ if $P(\lambda)$ has rank m for all $\lambda \in \mathbb{C}$. For a given matrix $A \in \mathbb{R}^{m\times n}$ we denote by $\sigma(A)$ as the spectrum A.

We denote by $\mathcal{R}$ the reachable subspace, $\mathcal{N}$ the unobservable subspace, $\mathscr{S}^*$ the strongly reachable subspace and $\mathscr{V}^*$ the weakly unobservable subspace of the state space of the dynamical system Σ (see also Definition 4.1). We will write $\oplus$ for the direct sum of linear subspaces.

$\mathbb{R}(\xi)_{\mathcal{P}}^{\bullet\times\bullet}$ denotes the set of all proper rational matrices. In this paper we will consider only left coprime and right coprime factorizations over the ring $\mathbb{R}[\xi]$. We call a factorization $G = P^{-1}Q$ with $P, Q \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ a *left coprime factorization* over the ring of polynomials if $\begin{bmatrix} P & Q \end{bmatrix}$ is left prime over $\mathbb{R}[\xi]$ and $\det(P(\xi)) \neq 0$. Similary we define a *right coprime factorization* analogously. Consider a rational matrix $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{\bullet\times\bullet}$ then we call the quadruple (A, B, C, D) of real constant matrices a *realization of* G if $G(\xi) = C(\xi I - A)^{-1}B + D$ [2]. Given the quadruple (A, B, C, D) we define the system matrix as

$$P_\Sigma(\xi) = \begin{bmatrix} \xi I - A & -B \\ -C & -D \end{bmatrix}.$$

3 Canonical decompositions for linear systems

In this section we will discuss some canonical decompositions for linear systems. These decompositions will be useful in proving the main results in this paper. First we start with the controllable form, then the finer Kalman decomposition and the nine-fold decomposition will be discussed.

3.1 Controllable form

From [4] we know that in case of output nulling and driving variable representations we can apply a transformation on the state variable x such that the quadruple (A, B, C, D) is in *controllable form:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D = D. \tag{1}$$

Here the pair (A_{11}, B_1) is controllable.

3.2 The Kalman decomposition

For I/S/O representations we know the following. Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^p$ is the output. Furthermore the constant matrices satisfy $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$.

We know that $X_1 := \mathcal{R} \cap \mathcal{N}$ is a subspace of $\mathbb{R}^n$. Let $\mathscr{B}_1$ be a basis for this subspace. Then it is possible to choose a basis $\mathscr{B}_2$ such that the subspace X_2 spanned by $\mathscr{B}_2$ satisfies $X_1 \oplus X_2 = \mathcal{R}$. Also it is possible to construct a basis $\mathscr{B}_3$ such that the subspace $X_3 = \text{span}(\mathscr{B}_3)$ satisfies $X_1 \oplus X_3 = \mathcal{N}$. Having this we finally construct the basis $\mathscr{B}_4$ such that the subspace $X_4 = \text{span}(\mathscr{B}_4)$ satisfies $X_1 \oplus X_2 \oplus X_3 \oplus X_4 = \mathbb{R}^n$. Then from [3] we know that there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ defined as

$$T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix},$$

where the columns of T_i are the elements of $\mathscr{B}_i$, $i = 1, 2, 3, 4$. The transformed system (with $\hat{x} = Tx$) is of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \\ y &= \hat{C}\hat{x} + \hat{D}u, \end{aligned} \tag{3}$$

where $\hat{A} = T^{-1}AT, \hat{B} = T^{-1}B, \hat{C} = CT$, and $\hat{D} = D$. Moreover $\hat{A}, \hat{B}, \hat{C}$ are of the form

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \hat{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \hat{C} = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix}. \tag{4}$$

We state the properties of $\hat{A}, \hat{B}, \hat{C}$:

1. The subsystem $(A_{22}, B_2, C_2, \hat{D})$ is reachable and observable. This means that the pair (A_{22}, B_2) is controllable and the pair (C_2, A_{22}) is observable.
2. The subsystem $\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} 0 & C_2 \end{bmatrix}, \hat{D} \right)$ is controllable.
3. The subsystem $\left(\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \begin{bmatrix} C_2 & C_4. \end{bmatrix}, \hat{D} \right)$ is observable.

For a more detailed explanation on the Kalman Decomposition we refer to [3].

3.3 Nine-fold decomposition

Combining the results from the Kalman decomposition and the so called Morse canonical decomposition H. Aling and J.M. Schumacher have come to the nine-fold decomposition [1]. Similary as in the Kalman decomposition we may choose bases $\mathscr{B}_i, i = 1, \ldots, 9$ such that the subspaces $Y_i := \text{span}(\mathscr{B}_i), i = 1, \ldots, 9$ satisfy

- $\mathcal{R} = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6$
- $\mathcal{N} = Y_1 \oplus Y_2 \oplus Y_7$
- $\mathscr{S}^* = Y_1 \oplus Y_5 \oplus Y_6$
- $\mathscr{V}^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_5 \oplus Y_7 \oplus Y_8$
- $\mathbb{R}^n = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$.

Then there exists transformations on $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ where $\hat{A} \in \mathbb{R}^{n \times n}, \hat{B} \in \mathbb{R}^{n \times m}, \hat{C} \in \mathbb{R}^{p \times n}, \hat{D} \in \mathbb{R}^{p \times m}$ such that transformed system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is of the form

$$\left[\begin{array}{ccccccccc|cc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} & A_{19} & B_{11} & B_{12} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} & A_{29} & 0 & B_{22} \\ 0 & 0 & A_{33} & A_{34} & A_{35} & A_{36} & 0 & A_{38} & A_{39} & 0 & B_{32} \\ 0 & 0 & A_{43} & A_{44} & A_{45} & A_{46} & 0 & A_{48} & A_{49} & 0 & B_{42} \\ 0 & 0 & A_{53} & A_{54} & A_{55} & A_{56} & 0 & A_{58} & A_{59} & B_{51} & B_{52} \\ 0 & 0 & A_{13} & A_{14} & A_{15} & A_{66} & 0 & A_{18} & A_{19} & 0 & B_{62} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{77} & A_{78} & A_{79} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{88} & A_{89} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{99} & 0 & 0 \\ \hline 0 & 0 & 0 & C_{14} & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 \\ 0 & 0 & C_{23} & C_{24} & C_{25} & C_{26} & 0 & C_{28} & C_{29} & 0 & D_{22} \end{array}\right]. \tag{5}$$

Where D_{22} is nonsingular. Observe that the quadruple (A, B, C, D) is also in Kalman form and that

- $X_1 = Y_1 \oplus Y_2$
- $X_2 = Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6$
- $X_3 = Y_7$
- $X_4 = Y_8 \oplus Y_9$.

4 Behaviors and their representations

In this section we introduce behaviors and their representations. Also some basic properties of behaviors are listed. We start with the definition of a dynamical system.

Definition 4.1. *A **dynamical system** Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ where $\mathbb{T} \subseteq \mathbb{R}$ is called the time axis, $\mathbb{W}$ is called the signal space and the behavior $\mathfrak{B}$ is a subset of $\mathbb{W}^{\mathbb{T}}$ which is the collection of all maps from $\mathbb{T}$ to $\mathbb{W}$.*

In this paper we deal with linear time-invariant differential (dynamical) systems (LTIDS). A LTIDS is defined as follows.

Definition 4.2. *A dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is called a LTIDS if there exists a polynomial matrix $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ such that*

$$\mathfrak{B} = \ker R(\frac{d}{dt}) := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\}. \tag{6}$$

We then call the set of differential equations $R(\frac{d}{dt})w = 0$ a **representation** of the behavior $\mathfrak{B}$. We can represent a behavior $\mathfrak{B}$ of a LTIDS in different ways than the one mentioned above.

One such representation is called an output nulling representation. In an **output nulling representation** we look at all infinitely often differentiable functions (w, x) that satisfy the equations

$$\begin{aligned} \dot{x} &= Ax + Bw, \\ 0 &= Cx + Dw, \end{aligned} \tag{7}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Here x is a state variable that satisfies the property of state and w is the manifest variable. We define the full behavior of an output nulling representation as

$$\mathfrak{B}_{ON}(A, B, C, D) := \{(w, x) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n) \mid (7) \text{ holds } \}. \tag{8}$$

If we project this behavior onto w we call this the external behavior of the output nulling representation. We define it as

$$\mathfrak{B}_{ON}(A, B, C, D)_{ext} := \{w \mid \exists x \text{ such that } (w, x) \in \mathfrak{B}_{ON}(A, B, C, D)\}. \tag{9}$$

Another such representation is a driving variable representation. In a **driving variable representation** we look at all (w, x, v) that satisfy equations (10).

$$\begin{aligned} \dot{x} &= Ax + Bv, \\ w &= Cx + Dv, \end{aligned} \tag{10}$$

where x satisfies the property of state, v is an auxillary variable called the driving variable, and w is the manifest variable. We define the full behavior as

$$\mathfrak{B}_{DV}(A, B, C, D) := \{(w, x, v) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m) \mid (10) \text{ holds } \}, \tag{11}$$

and the external behavior as

$$\mathfrak{B}_{DV}(A, B, C, D)_{ext} := \{w \mid \exists (x, v) \text{ such that } (w, x, v) \in \mathfrak{B}_{DV}(A, B, C, D)\}. \tag{12}$$

Instead of representing behaviors as the solutions of the differential equations using the quadruple (A, B, C, D) we can look at solutions of differential equations using one (polynomial) matrix instead. This is called a polynomial kernel representation.

Polynomial/Rational kernel representation. From Definition 4.2 it is known that for a LTIDS $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ there exists a polynomial matrix $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ such that

$$\mathfrak{B} = \ker R(\frac{d}{dt}) := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\}. \tag{13}$$

Further, from [8], it is known that for any rational matrix $G \in \mathbb{R}(\xi)^{\bullet \times \bullet}$ there exists a left coprime factorization over $\mathbb{R}[\xi]$. Recall that a factorization $G = P^{-1}Q$ with $P, Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ is a left coprime factorization over $\mathbb{R}[\xi]$ if $\begin{bmatrix} P & Q \end{bmatrix}$ is left prime over $\mathbb{R}[\xi]$ and $\det(P(\xi)) \neq 0$. Now we define

$$\mathfrak{B} = \ker G(\frac{d}{dt}) := \ker Q(\frac{d}{dt}). \tag{14}$$

We denote by $\mathfrak{L}^{\mathtt{w}}$ the set of all linear differential systems with $\mathtt{w}$ variables. Controllability is a very important property of behaviors. It is defined in Definition 4.3.

Definition 4.3. *A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be **controllable** if for any two trajectories $w_1, w_2 \in \mathfrak{B}$ there exists $t_1 \geq 0$ and a trajectory $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t \leq 0$ and $w(t) = w_2(t)$ for $t \geq t_1$.*

Apart from admitting kernel representations, controllable behaviors also admit **rational and polynomial image representations** as follows. Let $R \in \mathbb{R}[\xi]^{\mathtt{w} \times \mathtt{l}}$ be a polynomial matrix, we define

$$\mathfrak{B} = \operatorname{im} R(\frac{d}{dt}) := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists l \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \text{ s.t. } w = R(\frac{d}{dt})l\}. \tag{15}$$

Further it is also shown in [8] that a controllable behavior admits a **rational image representation**. Consider a rational matrix $G \in \mathbb{R}(\xi)^{\mathtt{w} \times \mathtt{l}}$ with a left coprime factorization $G = P^{-1}Q$, then we define

$$\mathfrak{B} = \operatorname{im} G(\frac{d}{dt}) := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists l \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \text{ s.t. } P(\frac{d}{dt})w = Q(\frac{d}{dt})l\}. \tag{16}$$

Observability in the behavioral framework is defined as follows [5]:

Definition 4.4. *Let $(\mathbb{R}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ be a LTIDS. Trajectories in $\mathfrak{B}$ are partitioned as (w_1, w_2) with $w_i : \mathbb{R} \to \mathbb{W}_i, i = 1, 2$. We say that w_2 is **observable** from w_1 if $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$ implies $w_2 = w_2'$.*

In case we consider the quadruple (A, B, C, D) we define (strongly) controllability and (strongly) observability as follows.

Definition 4.5. *Consider the quadruple (A, B, C, D) with $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}$*

- *The pair (A, B) is called **controllable** if $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$.*
- *The pair (C, A) is called **observable** if $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.*
- *The quadruple (A, B, C, D) is called **strongly observable** if the pair $(C + DF, A + BF)$ is observable for all $F \in \mathbb{R}^{m \times n}$.*
- *The quadruple (A, B, C, D) is called **strongly controllable** if the pair $(A + GC, B + GD)$ is observable for all $G \in \mathbb{R}^{n \times p}$.*

If we consider a behavior $\mathfrak{B}$ induced by a kernel representation we define minimality in the following way [5].

Definition 4.6. *Let* $\mathfrak{B} = \ker R(\frac{d}{dt})$ *where* $R \in \mathbb{R}[\xi]^{m\times n}$. *Then the representation* $R(\frac{d}{dt})w = 0$ *of* $\mathfrak{B}$ *is called* ***minimal*** *if every other representation has at least* m *rows, that is, if* $\ker R(\frac{d}{dt}) = \ker R'(\frac{d}{dt})$ *for some* $R' \in \mathbb{R}[\xi]^{m'\times n}$ *then* $m' \geq m$.

Using Theorem 3.6.4 of [5] we can check if a given representations of a LTIDS is minimal.

Theorem 4.7. *Let* $\mathfrak{B} = \ker R(\frac{d}{dt})$, *where* $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$. *Then* R *has full row rank if and only if* $R(\frac{d}{dt})w = 0$ *is a minimal representation of* $\mathfrak{B}$.

Similar to nonsingulairty of real constant matrices, polynomial matrices admit a property called unimodularity and unimodular matrices are defined as follows:

Definition 4.8. *A matrix* $U \in \mathbb{R}[\xi]^{n\times n}$ *is called* ***unimodular*** *if there exists a matrix* $P \in \mathbb{R}[\xi]^{n\times n}$ *such that* $UP = I$, *equivalently if* $det(U(\xi))$ *is a nonzero constant.*

The following theorem from [5] is well known in case of kernel representations which relates two minimal kernel represtations of a given behavior.

Theorem 4.9. *Let* $R \in \mathbb{R}[\xi]^{m\times n}$ *and* $U \in \mathbb{R}[\xi]^{m\times m}$, *define* $R' = UR \in \mathbb{R}[\xi]^{m\times n}$. *Then*

- $\ker(R(\frac{d}{dt})) \subset \ker(R'(\frac{d}{dt}))$
- *If* U *is unimodular, then* $\ker(R(\frac{d}{dt})) = \ker(R'(\frac{d}{dt}))$.

The following theorem gives a condition under which a behavior induced by a polynomial kernel representation is controllable.

Theorem 4.10. *A behavior defined by* $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ *with* $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ *is controllable if and only if* $R(\lambda)$ *has the same rank for all* $\lambda \in \mathbb{C}$. *Equivalently* R *is left prime over* $\mathbb{R}[\xi]$.

For image representations it is shown in [8] that $\mathfrak{B}$ admits a rational image representation if and only if it is controllable.

We will now review some results from [4]. In the next two theorems the system that is considered is assumed to be in controllable form (1). Using Theorem 4.11 we can always express the external behavior of an output nulling representation as a kernel representation.

Theorem 4.11. *Let* $\mathfrak{B}_{ON}(A, B, C, D)$ *be the full behavior induced by the output nulling representation* (7). *Assume that* A, B, C *are as in* (1). *Let* $L_2^{-1}L_1 = C_1(\xi I - A_{11})^{-1}$ *and* $K_2^{-1}K_1 = (L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ *be left coprime factorizations over* $\mathbb{R}[\xi]$. *Then*

$$\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker K_2(L_1B_1 + L_2D)(\frac{d}{dt}).$$

The following theorem gives a polynomial kernel representation admitted by the external behavior of a driving variable representation, but first we need the following definition.

Definition 4.12. *Consider* $R \in \mathbb{R}[\xi]^{n\times\bullet}$. *Then the polynomial matrix* $Q \in \mathbb{R}[\xi]^{\bullet\times n}$ *is called a minimal left annihilator (MLA) of* R *if*

1. Q *is left annihilator of* R, *i.e.* $QR = 0$.
2. *any left annihilator of* R *is a multiple of* Q, *i.e.* $Q_1R = 0$ *implies* $Q_1 = XQ$ *for some polynomial matrix* X.

Theorem 4.13. *Let* $\mathfrak{B}_{ON}(A, B, C, D)$ *be full behavior induced by the driving variable respresentation* (10). *Assume* (A, B, C) *are in controllable form* (1) *and let* $G(\xi) = C(\xi I - A)^{-1}B + D$ *be the tranfer function of* (10). $L_2^{-1}L_1 = C_1(\xi I - A_{11})^{-1}$ *and* $K_2^{-1}K_1 = L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ *are left coprime factorizations over* $\mathbb{R}[\xi]$. *Then*

$$\mathfrak{B}_{DV}(A, B, C, D)_{ext} = \ker (QK_2L_2)(\frac{d}{dt}),$$

where Q *is any MLA of* $K_2(L_1B_1 + L_2D)$.

Using Theorem 4.11 and 4.13, in [4] the following theorems were proven.

Theorem 4.14. *Let $G \in \mathbb{R}[\xi]_{\mathcal{P}}^{p\times m}$ have full row rank. Let $G(\xi) = C(\xi I - A)^{-1}B + D$ be a realization of G where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. If (A, B) is a controllable pair then $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$.*

Theorem 4.15. *Let $G \in \mathbb{R}[\xi]_{\mathcal{P}}^{p\times m}$ have full column rank. Let $G(\xi) = C(\xi I - A)^{-1}B + D$ be a realization of G where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. If (A, B) is a controllable pair then $\mathfrak{B}_{DV}(A, B, C, D)_{ext} = \text{im } G(\frac{d}{dt})$.*

Observe that Theorem 4.14 and Theorem 4.15 give already sufficient conditions on (A, B, C, D) on the problems we want to solve.

5 Output nulling and rational kernel representations

In this section we address the questions that we posed in the introduction of this paper. Given a behavior and its kernel representation of it involving a proper and real rational matrix, the first problem involves finding necessary and sufficient conditions under which a realization of the above matrix yields an output nulling representations of the behavior. Throughout this section we assume that a realization of a real rational matrix G is given by the quadruple (A, B, C, D). In Section 2 we have already seen a sufficient condition such that $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$. Further in this section we search for a weaker sufficient condition than the one stated in Theorem 4.14.

We will start with a lemma that states a property of a left coprime factorization of rational matrix that has full row rank. More precisely:

Lemma 5.1. *Let $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ have full row rank. Let $G = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then Q has full row rank.*

Proof. Assume Q does not have full row rank then there exists a polynomial row vector η such that $\eta PG = \eta Q = 0$. Then there exists a rational row vector $\eta' = \eta P \neq 0$ such that $\eta' G = 0$. This contradicts with the fact that G has full row rank. Thus Q has full row rank. □

This lemma will be useful in proving the next lemma which states a property of the left coprime factorizations also used in Theorem 4.14.

Lemma 5.2. *Let $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{\bullet\times\bullet}$ have full row rank. Let (A, B, C, D) be a realization of G that satisfies (1). Let $L_2^{-1}L_1 = C_1(\xi I - A_{11})^{-1}$ and $K_2^{-1}K_1 = (L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ be left coprime factorizations over $\mathbb{R}[\xi]$. Then the following statements are equivalent:*

1. $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$.
2. $(L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ *is a polynomial matrix.*

Proof. $(1 \Rightarrow 2)$ Assume $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$. We know

$$\ker G(\frac{d}{dt}) = \ker (C_1(\xi I - A_{11}))B_1 + D)(\frac{d}{dt}) = \ker L_2^{-1}(L_1B_1 + L_2D)(\frac{d}{dt}).$$

It can be verified that $L_2^{-1}(L_1B_1 + L_2D)$ is also a left coprime factorization over $\mathbb{R}[\xi]$. From (14) it follows that

$$\ker G(\frac{d}{dt}) = \ker L_2^{-1}(L_1B_1 + L_2D)(\frac{d}{dt}) = \ker (L_1B_1 + L_2D)(\frac{d}{dt}), \tag{17}$$

and from Theorem 4.11 we have

$$\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker K_2(L_1B_1 + L_2D)(\frac{d}{dt}). \tag{18}$$

Since $G = L_2^{-1}(L_1B_1 + L_2D)$ is a left coprime factorization and G has full row rank it results from Lemma 5.1 that $L_1B_1 + L_2D$ has full row rank. Since $\ker K_2(L_1B_1 + L_2D)(\frac{d}{dt}) = \ker (L_1B_1 + L_2D)(\frac{d}{dt})$ we know from Theorem 3.6.2 from [5] that

$$L_1B_1 + L_2D = UK_2(L_1B_1 + L_2D) \tag{19}$$

for some unimodular matrix U. We prove now that K_2 is unimodular. Rewriting equation (19) we have

$$(I - UK_2)(L_1B_1 + L_2D) = 0. \tag{20}$$

Clearly $I - UK_2$ is a left annihilator (LA) of $L_1B_1 + L_2D$ but since this matrix has full row rank any LA must be equal to zero, thus $UK_2 = I$. By Definition 4.8 K_2 is unimodular, therefore $K_2^{-1}K_1 = (L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ is a polynomial matrix.

$(2 \Rightarrow 1)$ Assume $(L_1A_{12}+L_2C_2)(\xi I-A_{22})^{-1}$ is a polynomial matrix. Let $K_2 = I$ and $K_1 = (L_1A_{12}+L_2C_2)(\xi I-A_{22})^{-1}$. Then $K_2^{-1}K_1 = (L_1A_{12}+L_2C_2)(\xi I-A_{22})^{-1}$ is a left coprime factorization over $\mathbb{R}[\xi]$. Then the result follows from equations (17) and (18). □

From Theorem 4.14 we know that if the pair (A, B) is controllable we have $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$. Controllabiliy is however not necessary for $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$. The next lemma states a weaker condition on (A, B, C, D) such that $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$.

Lemma 5.3. *Let $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{p\times m}$ and let the quadruple (A, B, C, D) be a realization of G with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Then if $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$ then $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$.*

Proof. Assume $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$. Without loss of generality we may assume that (A, B, C, D) is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & C_2 & 0 \end{bmatrix}. \tag{21}$$

From the Kalman decomposition explained in Section 3.2 it follows that the subsystem

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} 0 & C_2 \end{bmatrix}, D \right) \tag{22}$$

is reachable so the system (21) is also in controllable form as in (1). As stated in Theorem 4.11 we construct a left coprime factorization

$$L_2^{-1}L_1 = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} \xi I - A_{11} & -A_{12} \\ 0 & \xi I - A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & C_2(\xi I - A_{22})^{-1} \end{bmatrix}. \tag{23}$$

Since L_2 is nonsingular we can split up L_1 into blocks of appropriate sizes: $L_1 = \begin{bmatrix} 0 & L_{12} \end{bmatrix}$. Also stated in Theorem 4.11 we can construct a left coprime factorization

$$K_2^{-1}K_1 = (\begin{bmatrix} 0 & L_{12} \end{bmatrix} \begin{bmatrix} A_{13} \\ 0 \end{bmatrix} + L_2 \cdot 0)(\xi I - A_{33})^{-1} = \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{24}$$

Observe that $(L_1\hat{A}_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ is a polynomial matrix. We can take $K_2 = I$ and $K_1 = (L_1\hat{A}_{12} + L_2C_2)(\xi I - A_{22})^{-1}$, consequently $K_2^{-1}K_1 = (L_1\hat{A}_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ is a left coprime factorization. It follows from equations (17) and (18) that $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$. □

Under the assumption that G has full row rank the converse of Lemma 5.3 is also true. The assumption that G has full row rank is not restrictive since every behavior admits a minimal rational kernel representation, therefore one can always find a $\tilde{G}$ that has full row rank such that $\mathfrak{B} = \ker G(\frac{d}{dt}) = \ker \tilde{G}(\frac{d}{dt})$.

Theorem 5.4. *Let $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{p\times m}$ have full row rank. Let (A,B,C,D) be a realization of G with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Then $\mathfrak{B}_{ON}(A,B,C,D)_{ext} = \ker G(\frac{d}{dt})$ if and only if $\mathcal{R}+\mathcal{N} = \mathbb{R}^n$.*

Proof. ($\Rightarrow$) We want to prove $\mathfrak{B}_{ON}(A,B,C,D)_{ext} = \ker G(\frac{d}{dt}) \Rightarrow \mathcal{R}+\mathcal{N} = \mathbb{R}^n$. This is the same as to prove $\mathcal{R}+\mathcal{N} \neq \mathbb{R}^n \Rightarrow \mathfrak{B}_{ON}(A,B,C,D)_{ext} \neq \ker G(\frac{d}{dt})$.

Assume $\mathcal{R}+\mathcal{N} \neq \mathbb{R}^n$. Without loss of generality we may consider the system as in (4). In the proof we distinguish between two cases:

1. X_2 is absent.
2. X_2 is present.

Case 1: Assume X_2 is absent. From Section 3.2 we may assume that the system is of the form

$$A = \left[\begin{array}{c|cc} A_{11} & A_{13} & A_{14} \\ \hline 0 & A_{33} & A_{34} \\ 0 & 0 & A_{44} \end{array}\right] = \left[\begin{array}{cc} A_{11} & \hat{A}_{12} \\ 0 & A_{22} \end{array}\right], B = \left[\begin{array}{c} B_1 \\ \hline 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right], \tag{25}$$

$$C = \left[\begin{array}{c|cc} 0 & 0 & C_4 \end{array}\right] = \left[\begin{array}{cc} C_1 & C_2 \end{array}\right]. \tag{26}$$

Observe that the system is in controllable form (1). Now let $L_2^{-1}L_1$ be a left coprime factorization of $C_1(\xi I - A_{11})^{-1}$ over $\mathbb{R}[\xi]$. Since

$$L_2^{-1}L_1 = C_1(\xi I - A_{11})^{-1} = 0 \cdot (\xi I - A_{11})^{-1} = 0 \tag{27}$$

we have that $L_2 = I$ and $L_1 = 0$. Let

$$\begin{aligned} K_2^{-1}K_1 &= (L_1\hat{A}_{12} + L_2C_2)(\xi I - A_{22})^{-1} \\ &= \begin{bmatrix} 0 & C_4 \end{bmatrix} \begin{bmatrix} \xi I - A_{33} & -A_{34} \\ 0 & \xi I - A_{44} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & C_4(\xi I - A_{44})^{-1} \end{bmatrix}. \end{aligned} \tag{28}$$

be a left coprime factorization over $\mathbb{R}[\xi]$. Since the pair (C_4, A_{44}) is observable we have that $\left[\begin{smallmatrix} A_{44}-\lambda I \\ C_4 \end{smallmatrix}\right]$ has full column rank for all $\lambda \in \mathbb{C}$ and hence $C_4(\xi I - A_{44})^{-1}$ is a right coprime factorization. Further $C_4(\xi I - A_{44})^{-1}$ is a polynomial matrix iff $(\xi I - A_{44})$ is unimodular which is not the case since $\det(\lambda I - A_{44}) = 0$ for $\lambda \in \sigma(A_{44})$.

Therefore from Lemma 5.2 it follows that $\mathfrak{B}_{ON}(A,B,C,D)_{ext} \neq \ker G(\frac{d}{dt})$.

Case 2: Assume X_2 is present. From Section 3.2 we may assume that the system is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix}. \tag{29}$$

With $A_{11} \in \mathbb{R}^{n_1\times n_1}, A_{22} \in \mathbb{R}^{n_2\times n_2}, A_{33} \in \mathbb{R}^{n_3\times n_3}, A_{44} \in \mathbb{R}^{n_4\times n_4}$ and B, C with appropriate dimensions. In particular it is known from the Kalman decomposition that the subsystem

$$\left(\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \begin{bmatrix} C_2 & C_4. \end{bmatrix}, D \right) \tag{30}$$

is observable. This is equivalent with

$$rank(\mathcal{O}) = rank \begin{bmatrix} A_{22} - \lambda I & A_{24} \\ 0 & A_{44} - \lambda I \\ C_2 & C_4 \end{bmatrix} = n_2 + n_4 \text{ for all } \lambda \in \mathbb{C}. \tag{31}$$

Again, the system (29) is in controllable form (1). Let

$$L_2^{-1}L_1 = \begin{bmatrix}0 & C_2\end{bmatrix}\begin{bmatrix}\xi I - A_{11} & -A_{12} \\ 0 & \xi I - A_{22}\end{bmatrix}^{-1} = \begin{bmatrix}0 & C_2(\xi I - A_{22})^{-1}\end{bmatrix} \tag{32}$$

be a left coprime factorization over $\mathbb{R}[\xi]$. Let $L_1 = \begin{bmatrix}0 & L_{12}\end{bmatrix}$ be split up into appropriate sized blocks. Then $L_2^{-1}L_{12} = C_2(\xi I - A_{22})^{-1}$ is also a left coprime factorization over $\mathbb{R}[\xi]$. Let

$$\begin{aligned} K_2^{-1}K_1 &= \left(\begin{bmatrix}0 & L_{12}\end{bmatrix}\begin{bmatrix}A_{13} & A_{14} \\ 0 & A_{24}\end{bmatrix} + L_2\begin{bmatrix}0 & C_4\end{bmatrix}\right)\begin{bmatrix}\xi I - A_{33} & -A_{34} \\ 0 & \xi I - A_{44}\end{bmatrix}^{-1} \\ &= \begin{bmatrix}0 & L_{12}A_{24} + L_2C_4\end{bmatrix}\begin{bmatrix}\xi I - A_{33} & -A_{34} \\ 0 & \xi I - A_{44}\end{bmatrix}^{-1} \\ &= \begin{bmatrix}0 & (L_{12}A_{24} + L_2C_4)(\xi I - A_{44})^{-1}\end{bmatrix} \end{aligned} \tag{33}$$

be also a left coprime factorization over $\mathbb{R}[\xi]$. Let $K_1 = \begin{bmatrix}0 & K_{12}\end{bmatrix}$ be split up into appropriate sized blocks. Then

$$K_2^{-1}K_{12} = (L_{12}A_{24} + L_2C_4)(\xi I - A_{44})^{-1} \tag{34}$$

is also a left coprime factorization over $\mathbb{R}[\xi]$.

Since $\begin{bmatrix}L_{12} & L_2\end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$, from [4] we can construct polynomial matrices $M_1 \in \mathbb{R}[\xi]^{n_2 \times n_2}, M_2 \in \mathbb{R}[\xi]^{n_2 \times p}$ such that the matrix

$$U_1 = \begin{bmatrix}L_{12} & 0 & L_2 \\ 0 & I & 0 \\ M_1 & 0 & M_2\end{bmatrix} \tag{35}$$

is unimodular. It can be verified that the rank of a polynomial matrix $R \in \mathbb{R}[\xi]$ does not change for any $\lambda \in \mathbb{C}$ when we premultiply it with an unimodular matrix. Thus $rank(U_1(\lambda)\mathcal{O}(\lambda)) = n_2 + n_4$ for all $\lambda \in \mathbb{C}$. Furthermore this matrix is given by

$$U_1\mathcal{O} = \begin{bmatrix}L_{12} & 0 & L_2 \\ 0 & I & 0 \\ M_1 & 0 & M_2\end{bmatrix}\begin{bmatrix}A_{22} - \xi I & A_{24} \\ 0 & A_{44} - \xi I \\ C_2 & C_4\end{bmatrix} = \begin{bmatrix}0 & L_{12}A_{24} + L_2C_4 \\ 0 & A_{44} - \xi I \\ M_1A_{24} + M_2C_2 & M_1A_{24} + M_2C_4\end{bmatrix}. \tag{36}$$

Now assume that $\mathfrak{B}_{ON}(A, B, C, D)_{ext} = \ker G(\frac{d}{dt})$. Then from Lemma 5.2 we know that $K_2^{-1}K_1$ given in (33) is a polynomial matrix and hence $K_2^{-1}K_{12}$ is also a polynomial matrix. Thus we can choose

$$K_{12} = (L_{12}A_{24} + L_2C_4)(\xi I - A_{44})^{-1}, K_2 = I.$$

In particular $K_{12}(\xi I - A_{44}) = L_{12}A_{24} + L_2C_4$. Then (36) simplifies to

$$U_1\mathcal{O} = \begin{bmatrix}0 & K_{12}(A_{44} - \xi I) \\ 0 & A_{44} - \xi I \\ M_1A_{24} + M_2C_2 & M_1A_{24} + M_2C_4\end{bmatrix}. \tag{37}$$

Since $\begin{bmatrix}K_{12} & -K_2\end{bmatrix} = \begin{bmatrix}(L_{12}A_{24} + L_2C_4)(\lambda I - A_{44})^{-1} & -I\end{bmatrix}$ has also full row rank for all $\lambda \in \mathbb{C}$ we can construct polynomial matrices $N_1 \in \mathbb{R}[\xi]^{n_4 \times p}, N_2 \in \mathbb{R}[\xi]^{n_4 \times n_4}$ such that the matrix

$$U_2 = \begin{bmatrix}-I & K_{12} & 0 \\ N_1 & N_2 & 0 \\ 0 & 0 & I\end{bmatrix} \tag{38}$$

is unimodular. Now we define

$$\tilde{\mathcal{O}} := U_2U_1\mathcal{O} = \begin{bmatrix}0 & 0 \\ 0 & (N_1K_{12} + N_2)(A_{44} - \xi I) \\ M_1A_{24} + M_2C_2 & M_1A_{24} + M_2C_4\end{bmatrix}.$$

We know that $rank(\tilde{\mathcal{O}}) = rank(\mathcal{O})$ and

$$\begin{aligned} rank(\tilde{\mathcal{O}}) &= rank\begin{bmatrix} 0 & 0 \\ 0 & (N_1K_{12}+N_2)(A_{44}-\xi I) \\ M_1A_{24}+M_2C_2 & M_1A_{24}+M_2C_4 \end{bmatrix} \\ &= rank\begin{bmatrix} 0 & (N_1K_{12}+N_2)(A_{44}-\xi I) \\ M_1A_{24}+M_2C_2 & M_1A_{24}+M_2C_4 \end{bmatrix} = rank(\hat{\mathcal{O}}). \end{aligned}$$

Since $\hat{\mathcal{O}} \in \mathbb{R}[\xi]^{(n_2+n_4)\times(n_2+n_4)}$ is a square polynomial matrix the statement $rank(\hat{\mathcal{O}}(\lambda)) = n_2+n_4$ for all $\lambda \in \mathbb{C}$ is equivalent to $det(\hat{\mathcal{O}}(\lambda)) \neq 0$ for all $\lambda \in \mathbb{C}$. But

$$det(\hat{\mathcal{O}}(\lambda)) = det(M_1A_{24}+M_2C_2)det(N_1K_{12}+N_2)det(A_{44}-\lambda I) = 0, \text{ for } \lambda \in \sigma(A_{44}).$$

This leads to a contradiction so $K_2^{-1}K_1$ is not a polynomial matrix and by Lemma 5.2 we may conclude $\mathfrak{B}_{ON}(A,B,C,D)_{ext} \neq \ker G(\frac{d}{dt})$. Thus $\mathcal{R}+\mathcal{N} = \mathbb{R}^n$.

($\Leftarrow$) This follows immediately from Lemma 5.3. □

It can be noted that Lemma 5.3 holds true though G doesn't have full row rank in contrast to Theorem 5.4.

The following proposition is evident from the proofs of Lemma 5.3 and Theorem 5.4, which is useful in the rest of the paper.

Proposition 5.5. *Let $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{p\times m}$. Let (A,B,C,D) be a realization of G with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ as in (1). Let $L_2^{-1}L_1 = C_1(\xi I - A_{11})^{-1}$ and $K_2^{-1}K_1 = (L_1A_{12}+L_2C_2)(\xi I - A_{22})^{-1}$ be left coprime factorizations over $\mathbb{R}[\xi]$. Then the following statements are equivalent*

1. $\mathcal{R}+\mathcal{N} = \mathbb{R}^n$.
2. $(L_1A_{12}+L_2C_2)(\xi I - A_{22})^{-1}$ *is a polynomial matrix. Equivalently K_2 is unimodular.*

6 Driving variable and rational image representations

In the previous section we have found conditions on a realization (A,B,C,D) of G under which $\mathfrak{B}_{ON}(A,B,C,D)_{ext} = \ker G(\frac{d}{dt})$. In this section we aim at finding conditions on the realization (A,B,C,D) of G under which $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \text{im } G(\frac{d}{dt})$. Before we proceed further we need the following lemma from [4] that states a property of a MLA.

Lemma 6.1. *Let $R \in \mathbb{R}[\xi]^{m\times\bullet}$. Assume $Q \in \mathbb{R}[\xi]^{q\times m}$ is a MLA of R. Then there exists $Q' \in \mathbb{R}[\xi]^{(m-q)\times m}$ such that*

$$U = \begin{bmatrix} Q \\ Q' \end{bmatrix}$$

is unimodular. For every such Q' we have that $Q'R$ has full row rank.

We will prove that the conditions on (A,B,C,D) mentioned in Lemma 5.3 also hold in the case of driving variable representations.

Lemma 6.2. *Let $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{p\times m}$. Let (A,B,C,D) be a realization of G with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Then if $\mathcal{R}+\mathcal{N} = \mathbb{R}^n$ then $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \text{im } G(\frac{d}{dt})$.*

Proof. Let Q_1, K_1, K_2, L_1, L_2 be as in the statement of Theorem 4.13. From Theorem 4.13 the external behavior induced by a driving variable representation is equal to

$$\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \ker\ (Q_1K_2L_2)(\frac{d}{dt}),$$

here Q_1 is a MLA of $K_2(L_1B_1 + L_2D)$. We know that $G = L_2^{-1}(L_1B_1 + L_2D)$ is a left coprime factorization and

$$\text{im } G(\frac{d}{dt}) = \{w \mid \exists l \text{ s.t. } L_2(\frac{d}{dt})w = (L_1(\frac{d}{dt})B_1 + L_2(\frac{d}{dt})D)l\}.$$

Clearly the full behavior in (w, l) of this is given by

$$\{(w,l) \mid \begin{bmatrix} -L_2(\frac{d}{dt}) & L_1(\frac{d}{dt})B_1 + L_2(\frac{d}{dt})D \end{bmatrix} \begin{bmatrix} w \\ l \end{bmatrix} = 0\} = \ker \left(\begin{bmatrix} -L_2 & L_1B_1 + L_2D \end{bmatrix} \right)(\frac{d}{dt}).$$

From Theorem 4.9, assuming $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, we know $\ker(R(\frac{d}{dt})) = \ker(U(\frac{d}{dt})R(\frac{d}{dt}))$ iff U is unimodular. Let Q_2 be a left annihilator of $L_1B_1 + L_2D$. Then from Lemma 6.1 there exists Q_2' such that

$$U = \begin{bmatrix} Q_2 \\ Q_2' \end{bmatrix}$$

is unimodular. Now

$$\begin{aligned} \ker \left(\begin{bmatrix} -L_2 & L_1B_1 + L_2D \end{bmatrix} \right)(\frac{d}{dt}) &= \ker \left(\begin{bmatrix} Q_2 \\ Q_2' \end{bmatrix} \begin{bmatrix} -L_2 & L_1B_1 + L_2D \end{bmatrix} \right)(\frac{d}{dt}) \\ &= \ker \left(\begin{bmatrix} -Q_2L_2 & 0 \\ -Q_2'L_2 & Q_2'(L_1B_1 + L_2D) \end{bmatrix} \right)(\frac{d}{dt}). \end{aligned}$$

From Lemma 6.1, we know that $Q_2'(L_1B_1 + L_2D)$ has full row rank. Thus, after eliminating the latent variable l, we find that the external behavior is given by

$$\text{im } G(\frac{d}{dt}) = \ker (Q_2L_2)(\frac{d}{dt}).$$

Since $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$ it follows from Proposition 5.5 that $(L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ is a polynomial matrix. Therefore we have $K_2 = I$ and $K_1 = (L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ such that $K_2^{-1}K_1$ is a left coprime factorization. Since

$$\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \ker (Q_1K_2L_2)(\frac{d}{dt}) \tag{39}$$

$$\text{im } G(\frac{d}{dt}) = \ker (Q_2L_2)(\frac{d}{dt}) \tag{40}$$

where Q_1, Q_2 are MLA's of $K_2(L_1B_1 + L_2D)$ and $L_1B_1 + L_2D$ respectively. It can be verified that $Q_1 = UQ_2$, where U is a polynomial unimodular matrix. Therefore we conclude that $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \text{im } G(\frac{d}{dt})$ □

6.1 Controllability in terms of driving variable representations

In this subsection we will go into more detail about behaviors induced by driving variable representations. We aim at finding necessary conditions under which, given a controllable behavior and a proper rational image representation, a realization of this matrix yields a driving variable representation of the given behavior. In the previous part of this section we have already seen that if $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$ then $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \text{im } G(\frac{d}{dt})$. In particular this means that $\mathfrak{B}_{DV}(A,B,C,D)_{ext}$ is controllable. Also the converse is true, this is stated in Lemma 6.3. Thus, finding conditions under which $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \text{im } G(\frac{d}{dt})$ is equivalent to find conditions under which $\mathfrak{B}_{DV}(A,B,C,D)_{ext}$ is controllable.

Lemma 6.3. *Let $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{\bullet\times\bullet}$. Let (A,B,C,D) be a realization of G. Then the following statements are equivalent.*

1. *$\mathfrak{B}_{DV}(A,B,C,D)_{ext}$ is controllable.*

2. $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \text{im } G(\frac{d}{dt})$.

Proof. (1. $\Rightarrow$ 2.) Assume $\mathfrak{B}_{DV}(A,B,C,D)_{ext}$ is controllable. From Theorem 4.13 it follows that $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \ker Q_2K_2L_2$, where Q_2 is a MLA of $K_2(L_1B_1 + L_2D)$. Since the external behavior $\mathfrak{B}_{DV}(A,B,C,D)_{ext}$ is controllable it follows that $Q_2(\lambda)K_2(\lambda)L_2(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Therefore $Q_2(\lambda)K_2(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

Since Q_2 is a MLA of $K_2(L_1B_1 + L_2D)$, $Q_2K_2(L_1B_1 + L_2D) = 0$. Call $Q_2' = Q_2K_2$ which is LA of $L_1B_1 + L_2D$. Let Q_1 be a MLA of $L_1B_1 + L_2D$. Then $Q_2' = XQ_1$. Since K_2 is square and $Q_2'(\lambda)$ and $Q_1(\lambda)$ have both full row rank for all $\lambda \in \mathbb{C}$, we conclude that X is unimodular. Then we have from equations (39) and (40)

$$\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \ker Q_2K_2L_2 = \ker Q_1L_2 = \text{im } G(\frac{d}{dt}) \tag{41}$$

as required since $Q_2K_2L_2 = XQ_1L_2$.

(2. $\Rightarrow$ 1.) Trivially true. □

Having this result we can state an analogous result as in Lemma 6.2, but with a different proof. The idea of the proof is also used to prove Theorem 6.6.

Lemma 6.4. *Let $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{p\times m}$. Let (A,B,C,D) be a realization of G with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Then $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$ implies that $\mathfrak{B}_{DV}(A,B,C,D)_{ext}$ is controllable.*

Proof. As before we may consider the system in the form (4). Assume $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$. Then the full behavior consists of all solutions of

$$\begin{bmatrix} \xi I - A_{11} & -A_{12} & -A_{13} & -B_1 & 0 \\ 0 & \xi I - A_{22} & 0 & -B_2 & 0 \\ 0 & 0 & \xi I - A_{33} & 0 & 0 \\ 0 & -C_2 & 0 & -D & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v \\ w \end{bmatrix} = 0. \tag{42}$$

Assume that the system matrix

$$P_\Sigma(\xi) = \begin{bmatrix} \xi I - A_{11} & -A_{12} & -A_{13} & -B_1 \\ 0 & \xi I - A_{22} & 0 & -B_2 \\ 0 & 0 & \xi I - A_{33} & 0 \\ 0 & -C_2 & 0 & -D \end{bmatrix}$$

does not have full row rank there exists a matrix $S = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 \end{bmatrix}$ such that it is a MLA of P_Σ. Also from Lemma 6.1 there exists matrix $N = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$ such that $\begin{bmatrix} S \\ N \end{bmatrix}$ is unimodular. We then have

$$\begin{bmatrix} S_1 & S_2 & S_3 & S_4 \\ N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} \xi I - A_{11} & -A_{12} & -A_{13} & -B_1 & 0 \\ 0 & \xi I - A_{22} & 0 & -B_2 & 0 \\ 0 & 0 & \xi I - A_{33} & 0 & 0 \\ 0 & -C_2 & 0 & -D & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v \\ w \end{bmatrix} = \tag{43}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & S_4 \\ M_1 & M_2 & M_3 & M_4 & N_4 \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v \\ w \end{bmatrix} = 0. \tag{44}$$

Here $M = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \end{bmatrix} = NP_\Sigma$. From Lemma 6.1 we know that M has full row rank. Hence we have the external behavior as $\mathfrak{B}_{DV}(A,B,C,D)_{ext} = \ker S_4(\frac{d}{dt})$. We will now prove that

ker $S_4(\frac{d}{dt})$ is controllable. We know from (44) that S_1, S_2, S_3 has to satisfy

$$S_1(\xi I - A_{11}) = 0 \Rightarrow S_1 = 0 \tag{45}$$
$$S_1 A_{12} + S_2(A_{22} - \xi I) + S_4 C_2 = 0 \tag{46}$$
$$S_1 A_{12} + S_3(A_{33} - \xi I) = 0 \Rightarrow S_3 = 0 \tag{47}$$
$$S_1 B_1 + S_2 B_2 + S_4 D = 0. \tag{48}$$

Assume now that $S_4(\lambda)$ does not have full row rank for all $\lambda \in \mathbb{C}$. Than there exists row vector η and $\lambda \in \mathbb{C}$ such that $\eta S_4(\lambda) = 0$. From equations (46), (48) it follows

$$\eta S_2(\lambda) \begin{bmatrix} A_{22} - \lambda I & B_2 \end{bmatrix} = 0. \tag{49}$$

Since $S(\lambda)$ is a MLA it has full row rank for all $\lambda \in \mathbb{C}$ from [4], therefore $\eta S_2(\lambda) \neq 0$. But then equation (49) contradicts with the fact that the pair (A_{22}, B_2) is controllable. Thus $S_4(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, hence $\mathfrak{B}_{DV}(A, B, C, D)_{ext}$ is controllable from Theorem 4.10.

In case that P_Σ does have full row rank then any MLA of P_Σ is void. We can take any unimodular matrix N instead of $\left[\begin{smallmatrix} S \\ N \end{smallmatrix}\right]$ as in (43). Then (44) becomes equal to

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 & N_4 \end{bmatrix} \left(\frac{d}{dt}\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v \\ w \end{bmatrix} = 0. \tag{50}$$

It follows that $\mathfrak{B}_{DV}(A, B, C, D)_{ext} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p)$ and therefore $\mathfrak{B}_{DV}(A, B, C, D)_{ext}$ is controllable. □

The converse of Lemma 6.4 is not true. Consider for example the system where the full behavior is equal to the solution set of

$$\left[\begin{array}{c|c|cc} \frac{d}{dt} - 1 & 0 & 0 & 0 \\ \hline -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ v \\ w_1 \\ w_2 \end{bmatrix} = 0.$$

Obviously $S = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is a MLA of the system matrix. Clearly $\mathfrak{B}_{DV}(A, B, C, D)_{ext} = \ker \begin{bmatrix} 0 & 1 \end{bmatrix}$. Since $\begin{bmatrix} 0 & 1 \end{bmatrix}$ does have constant row rank for all $\lambda \in \mathbb{C}$, $\mathfrak{B}_{DV}(A, B, C, D)_{ext}$ is controllable. It can be noted that in the above system x is observable from (w_1, w_2), therefore $\mathcal{R} + \mathcal{N} = 0 \neq \mathbb{R}^n$.

From the above example it is clear that a further weaker sufficient condition is needed such that is becomes necessary too. After using the 'finer' Kalman decomposition instead of using the controllable form we now look at even a more finer decomposition than the Kalman decomposition called the nine-fold decomposition. For a detailed discussion on the nine-fold decompostion we refer the reader to Section 3.3 and [1].

Before we proceed we need the following to prove the main result of this section.

Definition 6.5. *Let $\mathfrak{B} = \mathfrak{B}_{DV}(A, B, C, D)$ be a behavior induced by a driving variable representation. Then we say that x is **strongly observable** from w if $w = 0$ implies $x = 0$.*

Theorem 6.6. *Let $G \in \mathbb{R}(\xi)^{p \times m}_{\mathcal{P}}$. Let (A, B, C, D) be a realization of G with $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}$. Then $\mathcal{R} + \mathscr{V}^* = \mathbb{R}^n$ if and only if $\mathfrak{B}_{DV}(A, B, C, D)_{ext}$ is controllable.*

Proof. ($\Rightarrow$) Assume $\mathcal{R} + \mathscr{V}^* = \mathbb{R}^n$. From using the nine-fold decomposition as explained in Section 3.3 we may assume that the system is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix}.$$

Here A_{22} is equal to A_{88} as in (5), C_{22} is equal to C_{28} as in (5) and A_{99} is absent. The full behavior is given by the solution set of

$$\begin{bmatrix} \xi I - A_{11} & -A_{12} & -B_{11} & -B_{12} & 0 & 0 \\ 0 & \xi I - A_{22} & 0 & 0 & 0 & 0 \\ -C_{11} & 0 & 0 & 0 & I & 0 \\ -C_{21} & -C_{22} & 0 & -D_{22} & 0 & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} = 0.$$

Let $F = -D_{22}^{-1}C_{22}$. We introduce new driving variables $\hat{v}_2 = v_2 - Fx_2$ thus $v_2 = \hat{v}_2 + Fx_2$. Then the full behavior of the transformed system is given by the solution set of

$$\begin{bmatrix} \xi I - A_{11} & -A_{12} & -B_{11} & -B_{12} & 0 & 0 \\ 0 & \xi I - A_{22} & 0 & 0 & 0 & 0 \\ -C_{11} & 0 & 0 & 0 & I & 0 \\ -C_{21} & -C_{22} & 0 & -D_{22} & 0 & I \end{bmatrix} \begin{bmatrix} I & & & & & \\ & I & & & & \\ & & I & & & \\ & F & & I & & \\ & & & & I & \\ & & & & & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ \hat{v}_2 \\ w_1 \\ w_2 \end{bmatrix} = \tag{51}$$

$$\begin{bmatrix} \xi I - A_{11} & -A_{12} - B_{12}F & -B_{11} & -B_{12} & 0 & 0 \\ 0 & \xi I - A_{22} & 0 & 0 & 0 & 0 \\ -C_{11} & 0 & 0 & 0 & I & 0 \\ -C_{21} & 0 & 0 & -D_{22} & 0 & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ \hat{v}_2 \\ w_1 \\ w_2 \end{bmatrix} = 0. \tag{52}$$

As we observe the tranformed system has the property $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$ so the external behavior is controllable from Lemma 6.2. As we have seen in the proof of Lemma 6.4 there exists an unimodular matrix $\left[\begin{smallmatrix} S \\ N \end{smallmatrix}\right]$ such that the solution set of (53) is the same as in (52) where S is void iff P_Σ has full row rank.

$$\begin{bmatrix} S_1 & S_2 & S_3 & S_4 \\ N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} \xi I - A_{11} & -A_{12} - B_{12}F & -B_{11} & -B_{12} & 0 & 0 \\ 0 & \xi I - A_{22} & 0 & 0 & 0 & 0 \\ -C_{11} & 0 & 0 & 0 & I & 0 \\ -C_{21} & 0 & 0 & -D_{22} & 0 & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ \hat{v}_2 \\ w_1 \\ w_2 \end{bmatrix} = \tag{53}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & S_3 & S_4 \\ M_1 & M_2 & M_3 & M_4 & N_3 & N_4 \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ \hat{v}_2 \\ w_1 \\ w_2 \end{bmatrix} = 0. \tag{54}$$

If we apply a transformation on $\hat{v}_2$ such that $\hat{v}_2 = v_2 + Fx_2$ thus $v_2 = \hat{v}_2 - Fx_2$. Then the full

behavior of the original system is given by the solution set of

$$\begin{bmatrix} 0 & 0 & 0 & 0 & S_3 & S_4 \\ M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \end{bmatrix} \begin{bmatrix} I & & & & & \\ & I & & & & \\ & & I & & & \\ & -F & & I & & \\ & & & & I & \\ & & & & & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ \hat{v}_2 \\ w_1 \\ w_2 \end{bmatrix} = \tag{55}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & S_3 & S_4 \\ M_1 & M_2 - M_4F & M_3 & M_4 & M_5 & M_6 \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ \hat{v}_2 \\ w_1 \\ w_2 \end{bmatrix} = 0. \tag{56}$$

From Lemma 6.1 we know that $M = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \end{bmatrix}$ has full row rank. Therefore also $\begin{bmatrix} M_1 & M_2 - M_4F & M_3 & M_4 \end{bmatrix}$ has full row rank. Then we may conclude that the external behaviors are the same of both systems. Especially this means that $w = (w_1, w_2) = \mathfrak{B}_{DV}(A, B, C, D)_{ext}$ is controllable in the original system.

($\Leftarrow$) We proceed with proof by contradiction. Assume $\mathcal{R} + \mathscr{V}^* \neq \mathbb{R}^n$. Then we may consider the system of the form as in (5)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D = D. \tag{57}$$

Here A_{22} is equal to the block A_{99} in (5) and C_2 is equal to $\begin{bmatrix} C_{19} \\ C_{29} \end{bmatrix}$ as in (5). Assume now that P_Σ does not have full row rank. Since the subsystem (A_{11}, B_1, C_1, D) has the property $\mathcal{R} + \mathscr{V}^* = \mathbb{R}^n$ so we know that there exists matrix $\begin{bmatrix} S_1 & S_2 \end{bmatrix}$ such that it is MLA of the system matrix

$$\begin{bmatrix} \xi I - A_{11} & -B_1 \\ -C_1 & -D \end{bmatrix}.$$

Also we can construct N_1, N_2 such that

$$U = \begin{bmatrix} S_1 & 0 & S_2 \\ 0 & I & 0 \\ N_1 & 0 & N_2 \end{bmatrix}$$

is unimodular. The full behavior of system (57) is equal to the solution set of

$$\begin{bmatrix} \xi I - A_{11} & -A_{12} & -B_1 & 0 \\ 0 & \xi I - A_{22} & 0 & 0 \\ -C_1 & -C_2 & -D & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v \\ w \end{bmatrix} = \tag{58}$$

$$\begin{bmatrix} S_1 & 0 & S_2 \\ 0 & I & 0 \\ N_1 & 0 & N_2 \end{bmatrix} \begin{bmatrix} \xi I - A_{11} & -A_{12} & -B_1 & 0 \\ 0 & \xi I - A_{22} & 0 & 0 \\ -C_1 & -C_2 & -D & I \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v \\ w \end{bmatrix} = \tag{59}$$

$$\begin{bmatrix} 0 & -S_1A_{12} - S_2C_2 & 0 & S_2 \\ 0 & \xi I - A_{22} & 0 & 0 \\ N_1(\xi I - A_{11}) - N_2C_1 & -N_1A_{12} - N_2C_2 & -N_1B_1 - N_2D & N_2 \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v \\ w \end{bmatrix} = 0. \tag{60}$$

Since the quadruple $(A_{22}, 0, C_2, D)$ is strongly observable in (57) it follows that $w = 0$ implies $x_2 = 0$. From (60) it then follows that if $w = 0$ then

$$\begin{bmatrix} -S_1A_{12} - S_2C_2 \\ \xi I - A_{22} \end{bmatrix} x_2 = 0 \text{ implies } x_2 = 0.$$

Thus $\begin{bmatrix} -S_1(\lambda)A_{12} - S_2(\lambda)C_2 \\ \lambda I - A_{22} \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, therefore $(S_1A_{12} + S_2C_2)(\xi I - A_{22})^{-1}$ is a right coprime factorization over $\mathbb{R}[\xi]$. Since $(\xi I - A_{22})$ is not unimodular $(S_1A_{12} + S_2C_2)(\xi I - A_{22})^{-1}$ is not a polynomial matrix. Now let $L_2^{-1}L_1 = (S_1A_{12} + S_2C_2)(\xi I - A_{22})^{-1}$ be a left coprime factorization. Because $(S_1A_{12}+S_2C_2)(\xi I-A_{22})^{-1}$ is not a polynomial matrix L_2 is not unimodular. Since $\begin{bmatrix} L_1 & L_2 \end{bmatrix}$ is left prime we can construct matrices M_1, M_2 such that

$$\begin{bmatrix} L_1 & L_2 & 0 \\ M_1 & M_2 & 0 \\ 0 & 0 & I \end{bmatrix}$$

is unimodular. Then the solution set of (60) is equal to the solution set of

$$\begin{bmatrix} L_2 & L_1 & 0 \\ M_1 & M_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & -S_1A_{12} - S_2C_2 & 0 & S_2 \\ 0 & \xi I - A_{22} & 0 & 0 \\ N_1(\xi I - A_{11}) - N_2C_1 & -N_1A_{12} - N_2C_2 & -N_1B_1 - N_2D & N_2 \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v \\ w \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & L_2S_2 \\ 0 & \tilde{M} & 0 & 0 \\ N_1(\xi I - A_{11}) - N_2C_1 & -N_1A_{12} - N_2C_2 & -N_1B_1 - N_2D & N_2 \end{bmatrix} (\frac{d}{dt}) \begin{bmatrix} x_1 \\ x_2 \\ v \\ w \end{bmatrix} = 0.$$

Where $\tilde{M} = -M_1(S_1A_{12} + S_2C_2) + M_2(\xi I - A_{22})$. It can be verified that matrix

$$\begin{bmatrix} 0 & \tilde{M} & 0 & 0 \\ N_1(\xi I - A_{11}) - N_2C_1 & -N_1A_{12} - N_2C_2 & -N_1B_1 - N_2D & N_2 \end{bmatrix}$$

has full row rank. Thus after elimination of x_1, x_2, v we find that the external behavior is equal to $\ker (L_2S_2)(\frac{d}{dt})$. Because L_2 is not unimodular $L_2(\lambda)S_2(\lambda)$ does not have full row rank for all $\lambda \in \mathbb{C}$ thus $\mathfrak{B}_{DV}(A, B, C, D)_{ext}$ is not controllable.

In case that P_Σ does have full row rank we know from [6] that $\mathscr{S}^* + \mathscr{V}^* = \mathbb{R}^n$, hence $\mathcal{R} + \mathscr{V}^* = \mathbb{R}^n$. □

On the controllability of a behavior represented driving variable representation we conclude the following:

Proposition 6.7. *Consider $\mathfrak{B}_{DV}(A, B, C, D)$ to be the full behavior induced by the driving variable representation* (10)*. Then the following statements can be proven.*

1. *$\mathfrak{B}_{DV}(A, B, C, D)$ is controllable if and only if $\mathcal{R} = \mathbb{R}^n$.*
2. *$\mathfrak{B}_{DV}(A, B, C, D)_{(v,w)} := \{(v, w) \mid \exists x \text{ s.t. } (x, v, w) \in \mathfrak{B}_{DV}(A, B, C, D)\}$ is controllable if and only if $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$.*
3. *$\mathfrak{B}_{DV}(A, B, C, D)_{ext}$ is controllable if and only if $\mathcal{R} + \mathscr{V}^* = \mathbb{R}^n$.*

Proof. 1. The full behavior induced by the driving variable representation is equal to

$$\ker R(\frac{d}{dt}) = \ker \begin{bmatrix} \xi I - A & -B & 0 \\ -C & -D & I \end{bmatrix} (\frac{d}{dt})$$

Then $\mathfrak{B}_{DV}(A,B,C,D)$ is controllable if and only if $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$ if and only if $\begin{bmatrix}\lambda I - A & -B\end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$ if and only if the pair (A,B) is controllable if and only if $\mathcal{R} = \mathbb{R}^n$.

2. Assume the system is controllable form (1). Then the full behavior induced by the driving variable representation is equal to

$$\ker R(\frac{d}{dt}) = \ker \begin{bmatrix} \xi I - A_{11} & -A_{12} & -B_1 & 0 \\ 0 & \xi I - A_{22} & 0 & 0 \\ -C_1 & -C_2 & -D & I \end{bmatrix} (\frac{d}{dt}).$$

Let $L_2^{-1}L_1 = C_1(\xi I - A_{11})^{-1}$ and $K_2^{-1}K_1 = (L_1A_{12} + L_2C_2)(\xi I - A_{22})^{-1}$ be left coprime factorizations over $\mathbb{R}[\xi]$. From Theorem 3.5 in [4] it can be proven that after elimination of x the (v,w) behavior is represented by $R_1(\frac{d}{dt})\begin{bmatrix} v \\ w \end{bmatrix} = 0$, where $R_1 = K_2\begin{bmatrix}-(L_1B_1 + L_2D) & L_2\end{bmatrix}$. It can be proven that $\begin{bmatrix}-(L_1B_1 + L_2D) & L_2\end{bmatrix}$ is left prime. We therefore conclude that $\mathfrak{B}_{DV}(A,B,C,D)_{(v,w)}$ is controllable if and only if $R_1(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$ if and only if K_2 is unimodular and from Proposition 5.5 K_2 is unimodular if and only if $\mathcal{R} + \mathcal{N} = \mathbb{R}^n$.

3. This follows from Theorem 6.6. □

7 Conclusion

In this paper we have dealt with the relationship between external behavior of state representations and rational representationsof behaviors. In particular, iIn section 5 we found sufficient conditions under which, given a behavior induced by a proper rational kernel representation, a realization of this matrix yields an output nulling representation of this behavior. Also under the assumption that this matrix has full row rank we concluded that this condition is also necessary.

In section 6 we found sufficient conditions under which, given a behavior induced by a proper rational image representation, a realization of this matrix yields a driving variable representation of this behavior. We also have found necessary and sufficient conditions under which a (full/external) behavior induced by a driving variable representation is controllable.

In future research interesting subjects are such as finding necessary and sufficient conditions under which a behavior induced by an output nulling behavior is controllable. Further the other interesting subject is to find necessary conditions under which, given a behavior induced by a rational kernel representation, a realization of this matrix yields an output nulling representation of this behavior without the assumption that this matrix has full row rank.

References

[1] H. Aling & J.M. Schumacher, *A nine-fold canoncal decomposition for linear systems*, Int. J. Control, 1984, Vol 39, No. 4 779-805

[2] P. J. Antsaklis & A. N. Michel, 'A Linear Systems Primer', Birkhäuser, Boston, 2007

[3] M. Dahleh & M.A. Dahleh & G. Verghese, 'Lectures on Dynamic Systems and Control,' *Dynamic Systems and Control*, Spring 2011, consulted on 29 april 2012, http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-241j-dynamic-systems-and-control-spring-2011/readings/MIT6_241JS11_chap25.pdf

[4] S.V. Gottimukkala & H.L. Trentelman & S. Fiaz, *Realization and elimination in rational representations of behaviors*, Groningen, 2012

[5] J.W. Polderman & J.C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach (Texts in Applied Mathematics, Vol. 26)*, New York: Springer, 1998

[6] H.L. Trentelman & A.A. Stoorvogel & M. Hautus, *Control theory for linear systems*, Groningen, May 2002

[7] M. E. Valcher 'A note on the driving variable realization of behaviors', *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii USA, December 2003

[8] J.C. Willems & Y. Yamamoto, 'Behaviors defined by rational functions', *Lineair algebra and its applications,* Vol. 50, pp 581-608, 1983

[9] E. Zerz, *Topics in Multidimensional Linear Systems Theory,* Springer Lecture Notes in Control and Information Sciences 256, Springer, Londen, 2000