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Topological and nontopological solitons

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1 Introduction

In this thesis the phenomena called solitons will be reviewed, furthermore some methods in obtaining them are discussed. In 1834 John Scott Russell first observed a solitary wave [1]. *"I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion ; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an* *hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height."*

Russell thought that these waves were important and should be studied more. However many of his contemporaries, including George Stokes, thought that the wave that Russell observed was impossible. They thought the wave would disperse. This problem was solved in 1895 when Diederik Korteweg and Gustav de Vries derived the shallow water equation; we call this equation the Korteweg-de Vries (KdV) equation. Of this equation the wave profile observed by Russell is an exact solution.

Because solitons were observed before there was a mathematical description of solitons, there is no universally accepted definition of solitons. Only a short list of qualities defines what we see as solitons:

- Wave profile
- Finite energy
- Localized phenomenon
- Stable in time
- Stable under interaction processes

That the soliton is a localized phenomena means that the soliton goes towards a vacuum state at least exponentially fast. There are more phenomena that are named solitons than we will discuss in this thesis. Examples of these are phenomena with, periodic boundary conditions, discontinuous derivative and many more. However in this thesis we will discus only the most basic examples.

In section 2 we will describe dispersive effects on a solitary wave by a higher order derivative term in the wave equation. Furthermore the steepening effects of adding a non-linear term will be discussed. In the case of the KdV-equation these effects cancel out, therefore stable solitary waves can be obtained.

In 1965 Norman Zabusky and Martin Kruskal showed by numerical calculation that when two of the solitary waves (obtained from the KdV-equation) interacted, they did not merge or combine but the faster wave simply overtaking the slower one. Furthermore the shapes of the waves were unchanged, incurring only a phase difference. Because of this particle-like interaction, Zabusky and Kruskal named these waves solitons.

In section 3 we study more details of the KdV-equation, as this equation is central in the historical development of the subject. Moreover many of the qualities of the KdV-equation are shared with other soliton equations [1]. In section 6 we will introduce the Hirota method which is used to obtain soliton solution in many equations. We will use the KdV-equation as an example and re-derive the single soliton solution and also derive an explicit form for the 2-soliton solution.

A more powerful and general method of finding soliton solutions is the inverse scattering transform. A full description of this method is beyond the scope of this thesis but we will explain how this works in case of the KdV-equation in section 4.

[1]soliton equations are equations that have solitons as solutions

Not only in this mathematical context are soliton solutions interesting, in field theories soliton solution can also occur. In these theories, when there is enough non-linearity, stable bound states can exist. However in section 7 Derricks theorem provides restrictions on these theories.

To circumvent Derricks theorem we look at two types of solitons in local field theories, non-topological and topological solitons. In section 8 we will discuss Q-balls, a non-topological soliton. Finally in sections 9 and 10 we will discuss topological solitons.

2 Solitary waves in wave equations

In this section we will attempt to obtain the solitary waves observed by Russell. Although waves, that resemble the wave observed by Russell, can be obtained in the most basic wave equation, they will periodic and therefore not solitary. Instead we study the effect of dispersion caused by an higher order derivative term and the effect of making the equation nonlinear. We try to obtain the solitary wave in each case, however we the wave will not be stable unless these two effect cancel. At the end of the section some results are shown where these effect are balanced [2].

We start our discussion with the most basic wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right) f(x,t) = 0 \tag{1}$$

we assume that the velocity v is constant in time. The wave equation can be rewritten as:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right) f(x,t) = 0 \\ \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f(x,t) = 0\end{aligned} \tag{2}$$

This splits the left moving part and the right moving part of the wave. If we restrict ourselves to right moving waves, we remain with:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f(x,t) = 0 \tag{3}$$

Any solution of equation (3) still satisfies equation (1). Because $f(x,t)$ is a right moving wave, we can rewrite its argument:

$$f(x,t) = f(x - vt) \tag{4}$$

Solving this for a periodic wave gives the plane wave result:

$$f(x,t) = a e^{i(\omega t - kx)} \tag{5}$$

Definition 2.1. The relation between the angular frequency ω and the wave number k is called the **dispersion relation**.

In this case the dispersion relation is $\omega = \nu k$ which is linear. Because the dispersion relation is linear, the phase velocity $\nu_p = \frac{\omega}{k}$ and the group velocity $\nu_g = \frac{\partial \omega}{\partial k}$ are equal. Therefore in any superposition of waves, all the waves move at the same speed and the composed wave stays together. Waves with a linear dispersion relation are called non-dispersive.

By adding a third order derivative term we make the dispersion relation non-linear:

$$\left(\frac{\partial}{\partial t} + \nu \frac{\partial}{\partial x} + \delta \frac{\partial^3}{\partial x^3} \right) f(x,t) = 0 \tag{6}$$

Using the plane wave ansatz (5), we obtain the non-linear dispersion relation:

$$\omega = \nu k - \delta k^3 \tag{7}$$

The phase velocity is given by:

$$\nu_p = \frac{\omega}{k} = \nu - \delta k^2 \tag{8}$$

While the group velocity is given by:

$$\nu_g = \frac{\partial \omega}{\partial k} = \nu - 3\delta k^2 \tag{9}$$

The group velocity and phase velocity are different for $\delta \neq 0$ and therefore a wave composed by a superposition of multiple waves with different k values will spread out, as some waves move faster than others.

Now, instead of adding a higher order derivative, we add a non-linear term to our equation. Replace ν as a constant with $\nu(f) = \nu_0 + \alpha f(x)$, where $\nu_0$and α are constants. The wave equation now looks like:

$$\left(\frac{\partial}{\partial t} + \nu(f) \frac{\partial}{\partial x} \right) f(x,t) = 0 \tag{10}$$

With the following solution:

$$f(x,t) = f(x - \nu(f)t) \tag{11}$$

We are interested in the case where the wave is a solitary wave. In that case the wave has a maximum. The speed $\nu(f)$ is increasing with the amplitude of the wave, therefore the apex of the wave is moving faster than the rest of the wave, causing the wave to topple over.

One can obtain to obtain an equation which supports stable solitary waves by combining the dispersive properties of the higher order derivative and the non-linear term. The resulting wave equation is:

$$\left(\frac{\partial}{\partial t} + \nu(f) \frac{\partial}{\partial x} + \delta \frac{\partial^3}{\partial x^3} \right) f(x,t) = 0 \tag{12}$$

Example 2.1. As an example we take the Korteweg-deVries (KdV) equation. By writing $\nu(f)$ explicitly we obtain:

$$\left(\frac{\partial}{\partial t} + (\nu_0 + \alpha f(x,t))\frac{\partial}{\partial x} + \delta\frac{\partial^3}{\partial x^3}\right) f(x,t) = 0 \tag{13}$$

To obtain the KdV equation, we set the parameters $\delta = 1$ and $\alpha = 6$ and shift the function: $u \equiv f - \nu_0/6$.

$$u_t + 6uu_x + u_{xxx} = 0 \tag{14}$$

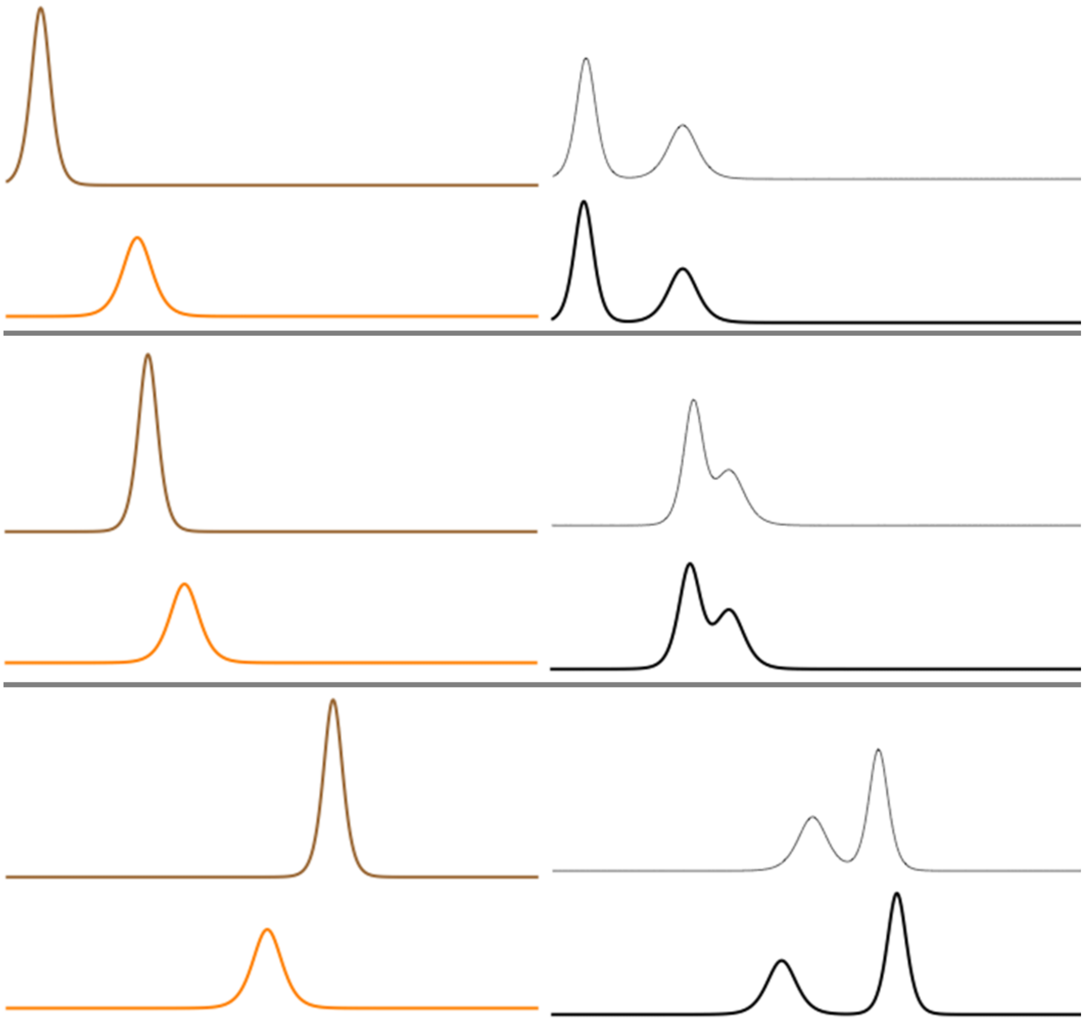


Figure 1: 2-soliton interaction [3]

This equation has solitary wave solutions, in sections 4 and 6 we will obtain these solution. In figure 1 two of these waves are shown. On the left side of the figure the two waves (one large and one small) are shown on three different times. On the other side of figure the gray lines represent the sum of the solution, however as the equation is nonlinear the superposition principle does not apply. The black line shows the time evolution of the initial conditions. When the two waves meet, they do not merge or destroy each other but pass through only incurring a delay, or phase shift, in the interaction.

This behavior was first observed by Kruskal and Zabusky. This conservation under interaction is also the reason why they called these waves solitons, as the *-on* is usually used to name particles (electr*on*, prot*on* and mu*on* for example). As said, the superposition principle does not apply tot nonlinear equation, for solitons however there does appear to be some use for a superposition. The initial condition can be superimposed as long as they are separated. This can be seen by the first of the time steps, there is no difference between sum of the waves or the proper evolution (note: this is not the initial condition of the simulation).

The KdV equation was the first equation of which these soliton solution where discovered, many other equation also have soliton solution, which have the same qualities as the solitons in the KdV equation. However in this thesis the KdV equation will be our favored example, therefore we will discus some of the properties of the KdV equation in the next section.

3 KdV-equation

3.1 properties of the KdV equation

Maybe the most famous non-linear dispersive wave equation is the Korteweg-deVries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{15}$$

First introduced (although not in the form above) by Korteweg and De Vries in 1895 [4] to describe the solitary wave observed by Russell, this equation is also known as the shallow water equation.

Remark 3.1. Although we have taken the KdV equation (15) with these parameters, any equation of the form

$$u_t + auu_x + bu_{xxx} = 0 \tag{16}$$

is a KdV equation. This can be shown by the following scaling

$$x \to b^{-1/3}x \quad u \to \frac{a}{6b^{1/3}}u \tag{17}$$

This transforms (15) into (16). Although all equations of the form (16) are KdV-equations, we will often use the KdV-equation in the form of (15) .

Remark 3.2. [5] In the same way as we have shown that the KdV-equation can be rewritten we can show that it is scale invariant:

$$x \to cx \quad t \to c^3 t \quad u \to c^{-2}u \tag{18}$$

This leads to:

$$\begin{aligned} & u_t + 6uu_x + u_{xxx} = 0 \\ \rightarrow & c^{-5}u_t + c^{-5}6uu_x + c^{-5}u_{xxx} \\ = & c^{-5}(u_t + 6uu_x + u_{xxx}) = 0 \end{aligned} \tag{19}$$

The KdV-equation is also translational invariant for both the x coordinate as the t coordinate:

$$x \rightarrow x + c_1 \quad t \rightarrow t + c_2 \tag{20}$$

Both do not change the equation. Next to scale invariant and transformational invariant, the KdV-equation is also Galilean invariant:

$$\begin{aligned} x \rightarrow & x + vt \\ u \rightarrow & u - \frac{v}{6} \end{aligned} \tag{21}$$

$$\begin{aligned} & u_t + 6uu_x + u_{xxx} \\ \rightarrow & u_t + vu_x + 6\left(u - \frac{v}{6}\right)u_x + u_{xxx} \\ = & u_t + 6uu_x + u_{xxx} \end{aligned} \tag{22}$$

Remark 3.3. If the initial conditions are given the soliton solution is unique [5]. Let v and u be solutions of the KdV-equation, which both satisfy the same initial conditions. We assume that that the initial conditions are at least continuously differentiable.

$$\begin{aligned} u_t + 6uu_x + u_{xxx} = 0 \\ v_t + 6vv_x + v_{xxx} = 0 \end{aligned} \tag{23}$$

Subtracting v from u and setting $w = u - v$ leads to:

$$\begin{aligned} \frac{\partial(u-v)}{\partial t} = \frac{\partial w}{\partial t} &= -6u\frac{\partial u}{\partial x} + 6v\frac{\partial v}{\partial x} - \frac{\partial^3(u-v)}{\partial x^3} \\ &= -6(w+v)\frac{\partial(w+v)}{\partial x} + 6v\frac{\partial v}{\partial x} - \frac{\partial^3 w}{\partial x^3} \\ &= -6u\frac{\partial w}{\partial x} - 6w\frac{\partial v}{\partial x} - \frac{\partial^3 w}{\partial x^3} \end{aligned} \tag{24}$$

We multiply equation (24) with w and integrate over x:

$$\int_{-\infty}^{\infty} \frac{\partial(\frac{1}{2}w^2)}{\partial t} dx = \int_{-\infty}^{\infty} -6wu\frac{\partial w}{\partial x} - 6w^2\frac{\partial v}{\partial x} - w\frac{\partial^3(w)}{\partial x^3} dx \tag{25}$$

The soliton solutions are localized, therefore we assume that u, v, w and their derivatives go to zero sufficiently fast at the boundary. The last part of the previous equation then becomes:

$$\int_{-\infty}^{\infty} w\frac{\partial^3 w}{\partial x^3}dx = ww_{xx}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial x}w_x^2 dx$$
$$= 0 - \frac{1}{2}w_x^2|_{-\infty}^{\infty} = 0 \tag{26}$$

We rewrite the left hand side of equation (25) as follows:

$$\int_{-\infty}^{\infty} \frac{\partial(\frac{1}{2}w^2)}{\partial t}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{1}{2}w^2 dx = \frac{\partial E}{\partial t} \tag{27}$$
$$\text{where} \quad E = \int_{-\infty}^{\infty} \frac{1}{2}w^2 dx$$

The remaining part of the right hand side is:

$$\int_{-\infty}^{\infty} 6wu\frac{\partial w}{\partial x} + 6w^2\frac{\partial v}{\partial x}$$
$$= \int_{-\infty}^{\infty} 6w^2\left(v_x - \frac{1}{2}u_x\right) + 3\frac{\partial(uw^2)}{\partial x}dx$$

The last part is again zero because of the boundary conditions.

$$= \int_{-\infty}^{\infty} 6w^2\left(v_x - \frac{1}{2}u_x\right) dx$$
$$\leqslant \int_{-\infty}^{\infty} 6w^2 m dx = 6mE \tag{28}$$
$$\text{where} \quad m = \sup\left(\left|v_x - \frac{1}{2}u_x\right|\right)$$

This supremum exists because the initial conditions are continuously differentiable and therefore so are the time evolution of those conditions. It would take an infinite amount of energy to do otherwise. Instead of equation (25), we have the inequality:

$$\frac{\partial E}{\partial t} \leqslant 6mE$$
$$\Rightarrow E(t) \leqslant E(0)e^{6mt} \tag{29}$$

From $w(x,0) = u(x,0) - v(x,0) = 0$, we conclude:

$$E(0) = \int_{-\infty}^{\infty} \frac{1}{2}w(0)^2 dx = \int_{-\infty}^{\infty} 0 dx = 0$$
$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}w(t)^2 dx \geqslant 0$$
$$\Rightarrow 0 \leqslant E(t) \leqslant 0 \Rightarrow E(t) = 0$$
$$\Rightarrow w(x,t) = 0$$
$$\Rightarrow u(x,t) = v(x,t) \tag{30}$$

Therefore given the initial conditions the solution is unique.

3.2 the Lax form

Remark 3.4. Instead of introducing the KdV-equation as a model for waves in shallow water, we introduce the equation as follows: given the Schrödinger equation with a parameter a dependent potential [6]:

$$\frac{d^2y(x)}{dx^2} + [\lambda - u(x,a)]y(x) = 0 \tag{31}$$

We rewrite equation (31) as follows:

$$Ly = \lambda y \tag{32}$$

where $L = D^2 - u(x,a)$ and $D = \frac{d}{dx}$ and $\frac{\partial x}{\partial a} = x_a$.

$$\begin{aligned}
&(Ly)_a = Ly_a + L_a y = \lambda_a y + \lambda y_a \\
&(Ly)_a = ((D^2 - u)y)_a = y_{xxa} - uy_a - u_a y = Ly_a - u_a y \\
&\Rightarrow L_a = -u_a
\end{aligned} \tag{33}$$

If we assume that the a dependence of y can be expressed as a differential operator ($y_a = By$), we obtain:

$$\begin{aligned}
(Ly)_a &= Ly_a + L_a y = \lambda_a y + \lambda y_a \\
LBy - u_a y &= \lambda_a y + \lambda By \\
LBy - u_a y &= \lambda_a y + BLy \\
(-u_a + LB - BL)y &= \lambda_a y
\end{aligned} \tag{34}$$

We want to find the shape of the potential that leaves the eigenvalue λ invariant:

$$\lambda_a = 0 \tag{35}$$

Therefore we obtain:

$$-u_a + [L, B] = 0 \tag{36}$$

We will interpret a as a time parameter and write t instead of a.

Example 3.1. Let B_1 be a first order differential operator:

$$B_1 = aD \tag{37}$$

where a is constant

$$\left(u_t - (D^2 - u)(aD) + (aD)(D^2 - u)\right) y = 0 \tag{38}$$

$$\Rightarrow (u_t - au_x)y = 0 \tag{39}$$

This means that the potential satisfies $u_t - au_x = 0$. Therefore the potential is any function of $x + at$.

Example 3.2. Let B_3 be a third order differential operator:

$$B_3 = bD^3 + fD + g \tag{40}$$

$$\text{then} \quad [L, B_3]y = (2f_x + 3bu_x)D^2y + (f_{xx} + 2g_x + 3bu_x x)Dy + (g_{xx} + bu_{xxx} + fu_x)y \tag{41}$$

We require that the terms D^2y and Dy vanish, therefore:

$$2f_x + 3bu_x = 0 \tag{42}$$

$$\Rightarrow f = -\frac{3}{2}bu + c_1 \tag{43}$$

$$f_{xx} + 2g_x + 3bu_{xx} = 0 \tag{44}$$

$$\Rightarrow g = -\frac{3}{4}bu_x + c_2 \tag{45}$$

$$\Rightarrow [L, B_3]y = (g_{xx} + bu_{xxx} + fu_x)y = \frac{1}{4}b(u_{xxx} - 6uu_x) + c_1u_x)y \tag{46}$$

This leads to (after a transformation $x \to x + c_1t$ and $b = -4$) an equation which the potential must satisfy, namely:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{47}$$

Definition 3.1. The KdV-equation can be written using the compatibility in the system of the linear differential equations:

$$L = D^2 + u \tag{48}$$

$$B_3 = bD^3 + fD + g \tag{49}$$

where:

$$b = -4 \tag{50}$$

$$f = -\frac{3}{2}bu + c_1 \tag{51}$$

$$g = -\frac{3}{4}bu_x + c_2 \tag{52}$$

$$\tag{53}$$

and c_1 and c_2 are arbitrary constants.

$$L_t = [B_3, L] \tag{54}$$

This is called the KdV-equation in **Lax form**.

By considering higher order derivatives higher order KdV equations can be obtained, this will be discussed in section 5

Example 3.3. We can use the line of thought of example 3.2 to obtain a explicit solution of the KdV-equation. Assume that there exists a potential for the Schrödinger equation

$$y'' + (k^2 - u)y = 0 \tag{55}$$

such that the solution can be written as:

$$y = e^{ikx} f(k, x) \tag{56}$$

where $f(k, x)$ is polynomial in k. The simplest form is the zeroth order, $y_0 = e^{ikx} a(x)$. We use this as an ansats for the Schrödinger equation:

$$-k^2 e^{ikx} a + 2ike^{ikx} a_x + e^{ikx} a_{xx} + (k^2 - u)e^{ikx} a = 0 \tag{57}$$

$$\Rightarrow 2ika_x + a_{xx} - ua = 0 \tag{58}$$

$a(x)$ is independent of the parameter k, therefore we can separate the part containing k and the part not containing k.

$$k^1 : \qquad 2ia_x = 0 \Rightarrow a_x = 0,\ a_{xx} = 0 \tag{59}$$

$$k^0 : \qquad a_{xx} - ua = 0 \Rightarrow ua = 0 \Rightarrow u = o \vee a = 0 \tag{60}$$

This leads to either the trivial solution, which tells us nothing about the shape of the potential, or leads to the trivial potential, which leads us back to the plane wave solution because $a(x)$ is a constant in this case. Using a first order polynomial in equation (56) gives us more interesting results. The constants are chosen with an eye on future convenience.

$$y_1 = e^{ikx}(2k + ia(x)) \tag{61}$$

This leads to the following solution:

$$-k^2 e^{ikx}(2k + ia) - 2ke^{ikx} a_x + ie^{ikx} a_{xx} + (k^2 - u)e^{ikx}(2k + ia(x)) = 0 \tag{62}$$

$$\Rightarrow -2ka_x + ia_{xx} - 2ku - iua = 0 \tag{63}$$

Again separating k^0 and k^1 we obtain:

$$\begin{aligned} k^1 : &\quad a_x = -u \\ k^0 : &\quad a_{xx} = ua \end{aligned} \tag{64}$$

$$\Rightarrow a_{xx} = -aa_x = -\frac{1}{2}(a^2)_x$$

$$\Rightarrow a_x + \frac{1}{2}a^2 = c_1 \tag{65}$$

When we substitute $a = \frac{2w_x}{w}$ we obtain a linear equation.

$$\Rightarrow \frac{2ww_x - 2w_x^2}{w^2} + \frac{1}{2}\frac{4w_x^2}{w^2} = c_1 \tag{66}$$

$$\Rightarrow 2ww_x - 2w_x^2 + 2w_x^2 = c_1 w^2 \tag{67}$$

$$\Rightarrow w_x - \frac{1}{2}c_1 w = 0 \tag{68}$$

$$\Rightarrow w = \alpha e^{c_2 x} + \beta e^{-c_2 x} \tag{69}$$

where c_1 is a constant of integration and $2c_2^2 = c_1$. We have a solution for w and therefore we also have a solution for u:

$$
\begin{aligned}
a =& \frac{2w_x}{w} = 2(\ln(w))_x && (70)\\
u =& - a_x = -2(\ln(w))_{xx}\\
=& - 2(\ln(\alpha e^{c_2 x} + \beta e^{-c_2 x}))_{xx}\\
=& - 2c_2 \left(\frac{\alpha e^{2c_2 x} - \beta}{\alpha e^{2c_2 x} + \beta}\right)_x\\
=& - 2c_2 \frac{2c_2(\alpha e^{2c_2 x} + \beta)e^{2c_2 x} - 2c_2(\alpha e^{2c_2 x} - \beta)e^{2c_2 x}}{(\alpha e^{2c_2 x} + \beta)^2}\\
=& - 2c_2^2 \left(\frac{2e^{c_2 x - 1/2 \ln(\beta/\alpha)}}{e^{2(c_2 x - 1/2 \ln(\beta/\alpha))} + 1}\right)^2\\
=& - 2c_2^2 (\text{sech} \left(c_2 x - \frac{1}{2}\ln(\beta/\alpha)\right)^2\\
=& - 2c_2^2 \text{sech}^2(c_2 x - c_3) && (71)
\end{aligned}
$$

where $c_3 = \frac{1}{2}\ln(\beta/\alpha)$. In the solution of w the α and β are independent of x but may by a function of t. Therefore c_3 is a function of time. We use the KdV-equation to determine this time dependence. Put this into the KdV-equation:

$$
\begin{aligned}
0 &= u_\tau - 6uu_x + u_{xxx}\\
0 &= -4c_2^2 \dot{c}_3 \text{sech}(c_2 x - c_3)^2 \tanh(c_2 x - c_3) && (72)\\
&\quad + 6 \cdot 2c_2^2 \text{sech}(c_2 x - c_3)^2 \cdot 4c_2^3 \text{sech}(c_2 x - c_3)^2 \tanh(c_2 x - c_3)\\
&\quad + 16c_2^5 \left(\text{sech}(c_2 x - c_3)^2 \tanh(c_2 x - c_3)^3 - 2\text{sech}(c_2 x - c_3)^4 \tanh(c_2 x - c_3)\right)\\
0 &= -\dot{c}_3 + 12c_2^3 \text{sech}(c_2 x - c_3)^2 + 4c_2^3(\tanh(c_2 x - c_3)^2 - 2\text{sech}(c_2 x - c_3)^2)\\
\dot{c}_3 &= 4c_2^3(\tanh(c_2 x - c_3)^2 + \text{sech}(c_2 x - c_3)^2)\\
c_3 &= 4c_2^3 t + \eta_0\\
u &= -2c_2^2 \text{sech}^2(c_2 x - 4c_2^3 t + \eta_0) && (73)
\end{aligned}
$$

this is the one-soliton solution of the KdV-equation.

4 Inverse scattering method

In 1967 a method to obtain solutions to the KdV-equation was found by Gardner, Green, Kruskal and Miura [7]. When a plane wave in the Schrödinger equation (31) interacts with the potential, the wave will be partly reflected and partially transmitted, of which the amplitude can be calculated when the potential is known. Furthermore when the potential is known possible bound states can also be calculated, together with the reflection and transmission this is called the scattering data. Gardner, Green, Kruskal and Miura obtained a method to construct the shape of the potential when the scattering data is known, this is known as the inverse scattering transform. In the case of solitons the reflection coefficient will be zero, making this a powerful method to obtain these solutions.

In the Schrödinger equation when the potential is attractive and finite there will be finite many bound eigenstates. As we are looking for soliton solutions, which are finite and localized, these bound states will be possible. When the energy eigenvalue $\lambda < 0$, a bound state will form, let $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote the eigenvalues of these bound states. As shown in quantum mechanics the eigenstates y_n corresponding to these negative eigenvalues λ_n are square integrable, therefore $|y_n| \to 0$ as $|x| \to \infty$.

Let us write $\lambda_n = (i\kappa_n)^2$, with $\kappa_n > 0$. When $|x| \to \infty$ then $u \to 0$, in that region the Schrödinger equation is

$$y_{xx} - \kappa^2 y = 0 \tag{74}$$

which can be solved, and by requiring that $|y_n| \to 0$ as $x \to \infty$ we obtain:

$$y_n \approx c_n(t)e^{-\kappa x} \tag{75}$$

where $c_n(t)$ is normalization coefficient. In section 3 we derived that in the Schrödinger equation

$$y_{xx} + [\lambda - u(x, a)]y = 0 \tag{76}$$

where the potential satisfies the KdV-equation, the time evolution of y is given by:

$$y_t = B_3 y = (-4D^3 + 6uD + 3u_x)y = -4y_{xxx} + 6uy_x + 3u_x y \tag{77}$$

which reduces to:

$$y_t = -4y_{xxx} \tag{78}$$

when $|x| \to \infty$ because u is localized. With this we obtain the time evolution of the normalization coefficient.

$$\frac{dc_n}{dt} = 4\kappa^3 c_n \tag{79}$$

In the case where $\lambda > 0$, there is a continuum of unbound states. These states will behave like the plane wave solution when $|x| \to \infty$. When a plane wave incident from the left interacts with the potential, part of the wave will reflect and a part will be transmitted. Let $\lambda = k^2$ where $k > 0$.

$$\begin{aligned} y_- &= e^{ikx} + R(k)e^{-ikx} & x &\to -\infty \\ y_+ &= T(k)e^{ikx} & x &\to \infty \end{aligned} \tag{80}$$

where $R(k)$ is the reflection coefficient and $T(k)$ is the transmission coefficient. When the wave is perfectly transmitted $|T| = 1$ and $|R| = 0$,therefore this is a reflectionless potential. Again using the time evolution of y for $x \to \infty$ we obtain:

$$\frac{dT}{dt} = 4ik^3 T \tag{81}$$

Thus if $|T(k, 0)| = 1$ then $T(k, t)| = 1, \forall t$.

Remark 4.1. We have stated that the soliton solution is the reflectionless potential. In figure 2 the transmission coefficient is shown for[2]:

$$y_{xx} + (k^2 - V\text{sech}^2(x))y = 0 \tag{82}$$

This is calculated using the Matlab [11] script [3], shown in appendix A. Figure 2 shows

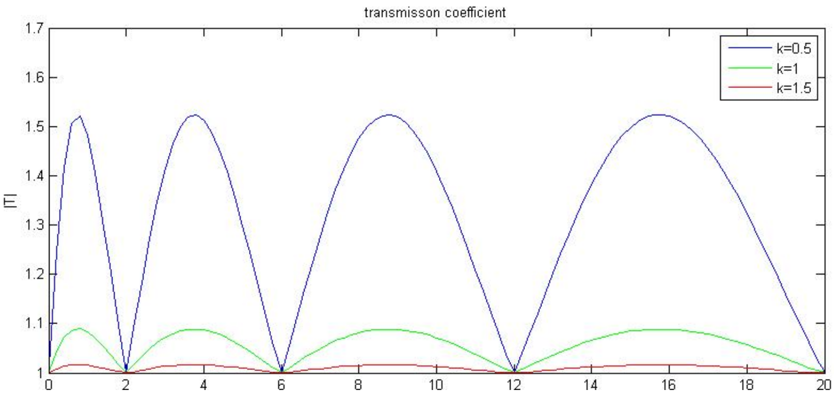


Figure 2: A reflectionless potential is obtained for $V = 0, 2, 6, 12, 20$

that a reflectionless potential is obtained for $V = 0, 2, 6, 12, 20$. Furthermore the value of V for which the potential is reflectionless is k independent. However as k increases $|T|$ will be closer to one everywhere which will cause rounding errors making the graph unreadable.

These results are the same to the results obtained by exact calculations done by Lamb [6]. The potential is reflectionless when $V = n(n+1)$ for $n = 0, 1, 2, 3, \ldots$.

The scattering of waves on a potential is most easily described when scattering pulses. Therefore we return to the most basic wave equation:

$$y_{xx} - \frac{1}{c^2} y_{tt} = 0 \tag{83}$$

Any function of the form $f(x \pm ct)$ is a solution of this equation, for example:

$$\tilde{y} = \delta(t \pm x/c) \tag{84}$$

This is a pulse wave. When we include a potential term to the wave equation,

$$y_{xx} - \frac{1}{c^2} y_{tt} - u(x)y = 0 \tag{85}$$

the pulse wave will pass through while leaving behind some disturbance. A left incident wave, leaves behind a disturbance described by $K(x, ct)$

$$\tilde{y} = \delta(t - x/c) + c\theta(t - x/c)K(x, ct) \tag{86}$$

[2] In appendix A figure 8 shows the points at which the potential is reflectionless till $V = 100$

[3] In the initial conditions the reflection coefficient is omitted, this will lead to an error, however the reflection is relatively small and when $|T| = 1$ the reflection coefficient vanishes, therefore we assume that the error vanishes as $|T| \to 1$.

This $K(x, ct)$ has the information we need to determine the shape of the potential. Substituting (86) into the wave equation (85), we obtain:

$$\delta(t - x/c)\left(2\frac{\partial K(x,ct)}{\partial x} + \frac{2}{c}\frac{\partial K(x,ct)}{\partial t} + u(x)\right)$$
$$-c\theta(t - x/c)\left(\frac{\partial^2 K(x,ct)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 K(x,ct)}{\partial t^2} + u(x)K(x,ct)\right) = 0 \quad (87)$$

Integrating over time from $x/c - \epsilon$ to $x/c + \epsilon$ we obtain:

$$-2\left.\frac{\partial K(x,ct)}{\partial x}\right|_{ct=x} - \frac{2}{c}\left.\frac{\partial K(x,ct)}{\partial t}\right|_{ct=x} = u(x) \quad (88)$$
$$-2\frac{d}{dx}K(x,x) = u(x) \quad (89)$$

We still need to determine this disturbance, for which we will use the Fourier transform of $\tilde{y}$:

$$Y_l(x,\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}\tilde{y}(x,t)$$
$$= e^{ikx} + \int_x^{\infty} dx' K(x,x')e^{ikx'} \quad (90)$$

where $x' = ct$ and $\omega/c = k$. When multiplied with $u(x)$ and using equation (85) for $u\tilde{y}$, we obtain (after integration by parts) the Schrödinger equation.

$$Y_{xx} + \left(k^2 - u(x)\right)Y = 0 \quad (91)$$

A wave incident from the right also leaves behind a disturbance, described by $L(x, ct)$, and in similar manner as before we obtain:

$$Y_r(x,\omega) = e^{-ikx} + \int_x^{\infty} dx' L(x,x')e^{-ikx'} \quad (92)$$

These two solutions are not independent, they are related by the transmission and reflection coefficient [6]:

$$T(k)Y_r(x,k) = R(k)Y_l(x,k) + Y_l(x,-k) \quad (93)$$

again taking the Fourier transform, we obtain the Marchenko equation (see appendix B for the derivation):

$$K(x,y;t) + B(x+y;t) + \int_x^{\infty} K(x,z;t)B(y+z;t)dz = 0 \quad (94)$$

$$\text{where} \quad B(\xi;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)e^{ik\xi}dk + \sum_n c_n(t)^2 e^{-\kappa_n \xi} \quad (95)$$

To solve the inverse scattering, we need to solve the Marchenko equation. Because the reflection is zero $B(\xi; t)$ simplifies to:

$$B(\xi; t) = \sum_n^N c_n(t)^2 e^{-\kappa_n \xi} = \sum_n^N c_n(0)^2 e^{-8ik^3 t - \kappa_n \xi} \tag{96}$$

Therefore $B(x + y)$ can be separated:

$$B(x + y) = \sum_n^N F_n(x) G_n(y) \tag{97}$$

$$\text{where} \quad F_n(x) = c_n(t)^2 e^{-\kappa_n x} \tag{98}$$

$$\text{and} \quad G_n(y) = e^{-\kappa_n y} \tag{99}$$

It follows that $K(x, y)$ can also be separated.

$$K(x, y) = \sum_n^N H_n(x) G_n(y) \tag{100}$$

Instead of giving H_n explicitly, we eliminate it from our equation:

$$\sum_n^N H_n(x) G_n(y) + \sum_n^N F_n(x) G_n(y) + \int_x^\infty \sum_m^N H_m(x) G_m(z) \sum_n^N F_n(z) G_n(y) dz = 0 \tag{101}$$

$$\sum_n^N \left(H_n(x) G_n(y) + F_n(x) G_n(y) + \sum_m^N \int_x^\infty G_m(z) F_n(z) dz H_m(x) G_n(y) \right) = 0 \tag{102}$$

Because the left term is only depending on x and the right term is only depending on y, this equation spits into N equations:

$$\sum_n^N \left(H_n(x) + F_n(x) + \sum_m^N A_{nm}(x) H_m(x) \right) G_n(y) = 0$$

$$\text{where} \quad A_{nm}(x) = \int_x^\infty G_m(z) F_n(z) dz$$

$$\delta_{nm} H_m(x) + F_n(x) + \sum_m^N A_{nm}(x) H_m(x) = 0$$

$$F_n(x) + \sum_m^N \hat{A}_{nm}(x) H_m(x) = 0$$

$$\text{where} \quad \hat{A}_{nm}(x) = \delta_{nm} + A_{nm}(x)$$

$$H_m(x) = -\sum_n^N \hat{A}^{-1}_{mn}(x) F_n(x) \tag{103}$$

$$\Rightarrow K(x, x) = -\sum_n^N \sum_m^N G_n(x) \hat{A}^{-1}_{nm}(x) F_m(x) \tag{104}$$

With this we can obtain an expression for the N-soliton solution

$$\begin{aligned}\hat{A}_{nm}(x) &= \delta_{nm} + \int_x^\infty G_m(z)F_n(z)dz \\ &= \delta_{nm} + c_n^2(t)\int_x^\infty e^{-(\kappa_n+\kappa_m)y}dy \\ &= \delta_{nm} + c_n^2(t)\frac{e^{-(\kappa_n+\kappa_m)x}}{\kappa_n+\kappa_m}\end{aligned} \tag{105}$$

Therefore we obtain an expression for the disturbance:

$$\begin{aligned}K(x,x) &= -\sum_n\sum_m G_n(x)\hat{A}^{-1}_{nm}(x)F_m(x) \\ &= \sum_n^N\sum_m^N e^{-\kappa_n x}\hat{A}^{-1}_{nm}(-c_m^2(t)e^{-\kappa_m x}) \\ &= \sum_n^N\sum_m^N \hat{A}^{-1}_{nm}\frac{d}{dx}\hat{A}_{mn} \\ &= \mathrm{Tr}\left(\hat{A}^{-1}\frac{d}{dx}\hat{A}\right) \\ &= \frac{1}{\det(\hat{A})}\frac{d}{dx}\det(\hat{A}) \\ &= \frac{d}{dx}\log\det\hat{A}\end{aligned} \tag{106}$$

Now we obtain the potential:

$$\begin{aligned}u(x,t) &= -2\frac{d}{dx}K(x,x;t) = -2\frac{d^2}{dx^2}\log\det\hat{A}(x;t) \\ &= -2\frac{d^2}{dx^2}\log\det\left(\delta_{nm} + c_n^2(t)\frac{e^{-(\kappa_n+\kappa_m)x}}{\kappa_n+\kappa_m}\right)\end{aligned} \tag{107}$$

Example 4.1. With this we solve the KdV equation for the single soliton solution. There is only a single eigenvalue k, therefore

$$\begin{aligned}u(x,t) &= -2\frac{d^2}{dx^2}\log\left(1 + c_n^2(0)\frac{e^{-8k^3t-kx}}{k}\right) \\ &= -2\frac{d^2}{dx^2}\log\left(1 + e^{-8k^3t-kx+\eta_0}\right) \\ &= -2\frac{\partial}{\partial x}\frac{ke^{-8k^3t-kx+\eta_0}}{1+e^{-8k^3t-kx+\eta_0}} \\ &= -2\frac{(1+e^{-8k^3t-kx+\eta_0})k^2e^{-8k^3t-kx+\eta_0} - k^2e^{2(-8k^3t-kx+\eta_0)}}{(1+e^{-8k^3t-kx+\eta_0})^2} \\ &= -\frac{2k^2e^{-8k^3t-kx+\eta_0}}{(1+e^{-8k^3t-kx+\eta_0})^2} = -\frac{k^2}{2}\mathrm{sech}^2\left(\frac{-8k^3t-kx+\eta_0}{2}\right)\end{aligned} \tag{108}$$

When we scale $x \to \frac{2}{k}x$, we obtain the the potential for which we have confirmed that it is reflectionless.

Although we have focused on the KdV equation this approach can be generalized, moreover the inverse scattering transform can be used to obtain solutions when the reflection is not zero (this however makes solving the Marchenko equation very difficult). The inverse scattering transform is a powerful tool, however there is no general method to obtain the time dependence of the scattering data.

5 KdV Hierarchy

5.1 pseudo differential operator

Before we continue with our discussion we first introduce an pseudodifferential operator. Which will be useful in the future. Let ∂ denote differentiation to x then we can write:

$$\partial \circ f = (\partial f) + f\partial \tag{109}$$

Therefore we can apply ∂f to a function g:

$$\partial \circ f(g) = ((\partial f) + f\partial)g = (\partial f)g + f(\partial g) \tag{110}$$

which is the Leibniz rule. We generalize this for higher orders:

$$\partial^n \circ f = \sum_{j=0}^{n} \binom{n}{j} (\partial^j f) \cdot \partial^{n-j} \tag{111}$$

The binomial coefficient

$$\binom{n}{j} = \frac{n(n-1)\cdots(n-j+1)}{j(j-1)\cdots 1} \tag{112}$$

is well defined when j is a natural number, furthermore it is set to zero when $j > n$. This means we can extend the summation of equation (111) without any problems:

$$\partial^n \circ f = \sum_{j=0}^{\infty} \binom{n}{j} (\partial^j f) \cdot \partial^{n-j} \tag{113}$$

We obtain differential operators with negative powers, these are pseudodifferential operators. From equation (113) we obtain a expression for ∂^{-1}:

$$\partial^{-1} \circ f = f\partial^{-1} - (\partial f)\partial^{-2} + (\partial^2 f)\partial^{-3} + \dots \tag{114}$$

We know $\partial^n \cdot \partial^m = \partial^{n+m}$, therefore we require $\partial^{-1} \cdot \partial = 1 = \partial \cdot \partial^{-1}$. We will show this for $\partial \cdot \partial^{-1}$.

$$\partial \cdot \partial^{-1} f = \partial(f\partial^{-1} - (\partial f)\partial^{-2} + (\partial^2 f)\partial^{-3} + \dots) \tag{115}$$

$$\partial \cdot \partial^{-1} f = f\partial\partial^{-1} + (\partial f)\partial^{-1} - (\partial f)\partial\partial^{-2} - (\partial^2 f)\partial^{-2} + (\partial^2 f)\partial\partial^{-3} + (\partial^3 f)\partial^{-3} + \dots \tag{116}$$

$$\partial \cdot \partial^{-1} f = f \tag{117}$$

Definition 5.1. In general the expression:

$$P = \sum_{j=0}^{\infty} g_j \partial^{\alpha - j} \tag{118}$$

is called a **pseudodifferential operator of order** $\leqslant \alpha$.

Example 5.1. Using the pseudodifferential operator we calculate the square root of the operator $L = \partial^2 + u$, which we have used in the lax-form of the KdV-equation (54) (although we replace $u \to -u$).

$$X = \partial + \sum_{n=1}^{\infty} f_n \partial^{-n} \tag{119}$$

$$L = X^2 \tag{120}$$

$$X^2 = \partial^2 + 2\sum_{n=1}^{\infty} f_n \partial^{1-n} + \sum_{n=1}^{\infty} (\partial f_n)\partial^{-n} + \sum_{n.m \geqslant 1, l \geqslant 0} \binom{-n}{l} f_n (\partial^l f_m) \partial^{-m-n-l} \tag{121}$$

$$X = \partial + \frac{1}{2} u \partial^{-1} - \frac{1}{4} u_x \partial^{-2} + \frac{1}{8}(u_{xx} - u^2)\partial^{-3} + \dots \tag{122}$$

We split the pseudo-differential operator into two parts, the part containing the nonnegative powers of ∂ and the rest:

$$P_+ = \sum_{j=0}^{\alpha} g_j \partial^{\alpha - j} \tag{123}$$

$$P_- = P - P_+ \tag{124}$$

using this we can reformulate the lax-form of the KdV-equation. When change the parameters of the KdV-equation (to absorb the constants of integration), we obtain:

$$L = \partial^2 + u \tag{125}$$

$$B_3 = \partial^3 + \frac{3}{2} u \partial + \frac{3}{4} u_x \tag{126}$$

We already know $L = X^2$. Moreover we can express B_3 in terms of X:

$$\begin{aligned}
(X^3)_+ &= ((\partial + \frac{1}{2} u \partial^{-1} - \frac{1}{4} u_x \partial^{-2} + \frac{1}{8}(u_{xx} - u^2)\partial^{-3} + (\dots)\partial^{-4} \dots)(\partial^2 + u))_+ \\
&= (\partial^3 + \frac{1}{2} u \partial - \frac{1}{4} u_x + \frac{1}{8}(u_{xx} - u^2)\partial^{-1} + (\dots)\partial^{-2} + u_x + u\partial + \dots)_+ \\
&= \partial^3 + \frac{3}{2} u \partial + \frac{3}{4} u_x = B_3
\end{aligned} \tag{127}$$

Furthermore we know the following:

$$\begin{aligned}
[X^2, X^l] &= X^{2+l} - X^{l+2} = 0 \\
[X^2, X^l] &= [X^2, (X^l)_- + (X^l)_+] \\
\Rightarrow [X^2, (X^l)_+] &= -[X^2, (X^l)_-]
\end{aligned} \tag{128}$$

We reformulate the Lax-form of the KdV-equation:

$$L_t = [L, (L^{3/2})_+] \tag{129}$$

We have stated before (see section 3.2) that including higher order derivatives that we obtain so called higher order KdV-equations. This is done by choosing $l \leqslant 3$ in the following equation:

$$L_t = [L, (L^{l/2})_+] = [(L^{l/2})_-, L] \tag{130}$$

5.2 symmetries

Definition 5.2. An evolutionary equation

$$\frac{\partial u}{\partial t} = K(u) \tag{131}$$

is said to have a **symmetry** of the form:

$$\frac{\partial u}{\partial s} = \hat{K}(u) \tag{132}$$

when we let u depend on two independent time variables s and t, and solving first for s and then for t or solving first for t and then for s give the same result, see figure 3:

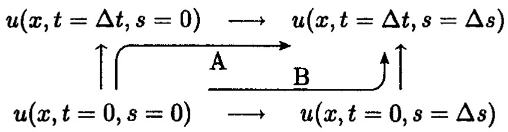


Figure 3: Solving first for s and then for t or the other way around [8].

Both $K(u)$ and $\hat{K}(u)$ are differential polynomials, meaning that they are polynomial in u and its derivatives (with respect to a certain variable, in this case x). This is called a symmetry because it resembles an infinitesimal generator. That the order of solving the equation leaves the solution unchanged allows us to state the following:

$$\frac{\partial K(u)}{\partial s} = \frac{\partial^2 u}{\partial s \partial t} = \frac{\partial^2 u}{\partial t \partial s} = \frac{\partial \hat{K}(u)}{\partial t} \tag{133}$$

Example 5.2. The most simple symmetry for the KdV-equation is given by:

$$\frac{\partial u}{\partial t} = K(u) = 6uu_x - u_{xxx} \tag{134}$$

$$\frac{\partial u}{\partial s} = \hat{K}(u) = u_x \tag{135}$$

$$\frac{\partial \hat{K}(u)}{\partial t} = u_{tx} = (6uu_x - u_{xxx})_x = 6(u_x)^2 + 6uu_{xx} - u_{4x} \tag{136}$$

$$\frac{\partial K(u)}{\partial s} = (6uu_x - u_{xxx})_s = 6u_s u_x + 6uu_{sx} - u_{xxxs} = 6(u_x)^2 + 6uu_{xx} - u_{4x} \tag{137}$$

These are the same. Therefore $u_s = u_x$ forms a symmetry of the KdV-equation. This is as we we expected because we have already showed (in remark 3.2) that the KdV-equation is invariant under translations and $u_s = u_x$ corresponds to $u(x, t, s) = u(x + s, t)$ a translation.

Using equation (33) we can rewrite equation (130) as:

$$\frac{\partial u}{\partial x_l} = [L, (L^{l/2})_-] = K_l(u) \quad \text{or} \quad \frac{\partial u}{\partial x_l} = -[L, (L^{l/2})_+] = K_l(u) \tag{138}$$

These are the higher order KdV-equations. The operator L is of differential order 2, while the operator $(L^{l/2})_-$ is of order $-1 \forall l$. Furthermore the commutator adds the orders of the terms and subtracts one from the order because the highest orders always cancel out. Therefore the operator $[L, (L^{l/2})_-]$ is of order 0, because of the construction of both L and $(L^{l/2})_-$, it is a differential polynomial in u with respect to x. We only concern ourselves with equations of odd order as the even orders are all identically zero. Moreover these equation are symmetries of the KdV equation.

Example 5.3. The cases $l = 1, 3$ are not truly "higher order" KdV-equation. However they do give us insight on some of the infinitely many variables that we have introduced by the infinitely many symmetries.

$$\frac{\partial u}{\partial x_1} = -[L, (L^{1/2})_+] \tag{139}$$

$$\frac{\partial u}{\partial x_1} = -[\partial^2 + u, \partial] \tag{140}$$

$$\frac{\partial u}{\partial x_1} = u_x \tag{141}$$

$$x_1 = x \tag{142}$$

And for the $l = 3$ case we obtain:

$$\frac{\partial u}{\partial x_3} = -[L, (L^{3/2})_+] \tag{143}$$

$$\frac{\partial u}{\partial x_3} = -[L, B_3] \tag{144}$$

$$\frac{\partial u}{\partial x_3} = 6uu_x - u_{xxx} \tag{145}$$

$$x_3 = t \tag{146}$$

Now we show that all these higher order equations are symmetries:

$$\frac{\partial K_l}{\partial x_j} = \frac{\partial K_j}{\partial x_l} \tag{147}$$

We know:

$$\frac{\partial L}{\partial x_l} = -[L, (L^{l/2})_+] \tag{148}$$

$$\frac{\partial f(L)}{\partial x_l} = -[f(L), (L^{l/2})_+] \tag{149}$$

and therefore

$$\frac{\partial (L^{j/2})_+}{\partial x_l} = (\frac{\partial L^{j/2}}{\partial x_l})_+ = -([L^{j/2}, (L^{l/2})_+])_+ \tag{150}$$

With this we show that the equations are indeed symmetries.

$$\begin{aligned}
\frac{\partial K_j}{\partial x_l} &= -\frac{\partial}{\partial x_l}[L, (L^{j/2})_+] \\
&= -\left[\frac{\partial}{\partial x_l}L, (L^{j/2})_+\right] - \left[L, \frac{\partial}{\partial x_l}(L^{j/2})_+\right] \\
&= [[L, (L^{l/2})_+], (L^{j/2})_+] + [L, ([L^{j/2}, (L^{l/2})_+])_+] \\
&= [[L, (L^{l/2})_+], (L^{j/2})_+] + [L, ([L_+^{j/2} + L_-^{j/2}, (L^{l/2})_+])_+] \\
&= [[L, (L^{l/2})_+], (L^{j/2})_+] + [L, ([L_+^{j/2}, (L^{l/2})_+])_+ + ([L_-^{j/2}, (L^{l/2})_+])_+] \\
&= [[L, (L^{l/2})_+], (L^{j/2})_+] + [L, ([L_+^{j/2}, (L^{l/2})_+])_+ - ([L_+^{j/2}, L^{l/2}])_+] \\
&= [[L, (L^{j/2})_+], (L^{l/2})_+] + [L, ([L^{l/2}, (L^{j/2})_+])_+] \\
&= -\left[\frac{\partial}{\partial x_j}L, (L^{l/2})_+] - [L, \frac{\partial}{\partial x_j}(L^{l/2})_+\right] \\
&= -\frac{\partial}{\partial x_j}[L, (L^{l/2})_+] = \frac{\partial K_l}{\partial x_j}
\end{aligned} \tag{151}$$

where we have used, in the fifth step of equation (151):

$$\begin{aligned}
0 &= ([L^{j/2}, L^{l/2}])_+ \\
&= ([L_+^{j/2}, L^{l/2}])_+ + ([L_-^{j/2}, L_+^{l/2}])_+ + ([L_-^{j/2}, L_-^{l/2}])_+ \\
&= ([L_+^{j/2}, L^{l/2}])_+ + ([L_-^{j/2}, L_+^{l/2}])_+ \\
&\Rightarrow ([L_-^{j/2}, L_+^{l/2}])_+ = -([L_+^{j/2}, L^{l/2}])_+
\end{aligned} \tag{152}$$

In the last step of equation (152) we have used equation (128). Furthermore in the sixth step of equation (151) we have used the Jacobi identity:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \tag{153}$$

Hence the higher order KdV-equation are symmetries of all other KdV-equations. This infinite system of equations is called the *KdV hierarchy*. Many equations with soliton solution show this structure, however there exists no general treatment and every equation needs to be studied in its own way to determine whether it can be written in this way. Instead of viewing the hierarchy as a system of infinitely many equations, it can be viewed as an infinite set of nonlinear differential equations in a function $\tau(x_1, x_2, x_3, \dots)$ of infinitely many variables. For the KdV-equation this function is related to u as follows:

$$u = 2\frac{\partial^2}{\partial x^2}\log\tau \tag{154}$$

However the discussion of this involves a discussion of hierarchy of the Kadomtsev-Petviashvili (KP) equation, which is beyond the scope of this thesis. The τ function is needed in the next section to obtain a powerful yet simple method for obtaining soliton solutions for our equations.

6 Hirota method

When trying to find soliton solutions in the same way as solutions to "ordinary" wave equations we hit a dead end. The basic approach to wave equations makes use of the superposition principle, when one solution is found, infinitely many (linearly-dependent) solutions are found and by combining solutions the boundary conditions and initial values can be satisfied. But in the case of soliton equations, the equations are non-linear. Therefore the superposition principle does not apply. Furthermore, because of this non-linearity, perturbation theory cannot be used. Perturbation theory would linearize the equations and would negate the non-linearity which governs the soliton solutions.

Definition 6.1. The Hirota method [8] brings the equations to a bilinear form, which simplifies the problem. First we need to introduce the **Hirota derivative** $D_x(f \cdot g)$. Take two single variable functions $f(x)$ and $g(x)$ and make a Taylor expansion around $y = 0$ of the following (x is fixed):

$$h(y) = f(x+y)g(x-y) \tag{155}$$

Normally we would do the following:

$$h(y) = \sum_{j=0}^{\infty} \frac{y^j}{j!} \frac{d^j h(0)}{dy^j} \tag{156}$$

We would like to express this in terms of $f(x)$ and $g(x)$ and derivatives of those.

$$f(x+y)g(x-y) = fg + (f_x g - f g_x)y + (f_{xx} g - 2f_x g_x + f g_{xx})\frac{y^2}{2} + \dots \tag{157}$$

This must be equal to (156) and we introduce the Hirota derivative in the following way:

$$f(x+y)g(x-y) = \sum_{j=0}^{\infty} \frac{y^j}{j!} D_x^j (f \cdot g) \tag{158}$$

Comparing the powers of y in the expansion of $f(x+y)g(x-y)$ we obtain the following:

$$D_x(f \cdot g) = f_x g - f g_x \tag{159}$$

$$D_x^2(f \cdot g) = f_{xx} g - 2f_x g_x + f g_{xx} \tag{160}$$

$$D_x^3(f \cdot g) = f_{xxx} g - 3f_{xx} g_x + 3f_x g_{xx} - f g_{xxx} \tag{161}$$

$$D_x^4(f \cdot g) = f_{xxxx} g - 4f_{xxx} g_x + 6f_{xx} g_{xx} - 4f_x g_{xxx} + f g_{xxxx} \tag{162}$$

In the equations (159)-(162) we observe a pattern, the Hirota derivative follows the Leibniz rule but includes a alternating minus sign. the Leibniz rule can rewritten as:

$$\frac{d^n}{dx^n} f \cdot g = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^n f(x_1) g(x_2)|_{x_2 = x_1 = x} \tag{163}$$

We adept this to include the alternating minus sign and obtain the Hirota derivative for a single variable [10]:

$$D_x^n (f \cdot g) = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^n f(x_1) g(x_2)|_{x_2 = x_1 = x} \tag{164}$$

This can also be done in a multi variable case in exactly the same way. (note: $h(y) = \sum_{j=0}^{\infty} \frac{y^j}{j!} \frac{d^j h(0)}{dx^j} = e^{y \frac{d}{dx}} h(x)|_{x=0}$)

$$f(x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots) g(x_1 - y_1, x_2 - y_2, x_3 - y_3, \dots) = e^{y_1 D_{x_1} + y_2 D_{x_2} + y_3 D_{x_3} + \dots} f \cdot g \tag{165}$$

Again comparing the powers of the variables defines the Hirota derivative, for example:

$$D_{x_1} D_{x_2} (f \cdot g) = -f_{x_1} g_{x_2} - f_{x_2} g_{x_1} + f_{x_1 x_2} g + f g_{x_1 x_2} \tag{166}$$

Putting $g(x) = f(x)$ and $x_1 = t, x_2 = x$ which we will need later on, we obtain:

$$D_t D_x (f \cdot f) = -2 f_t f_x + 2 f_{xt} f \tag{167}$$

Other equations we will need are:

$$\frac{\partial^2}{\partial x \partial t} \log(f) = \frac{\partial}{\partial t} \frac{f_x}{f} = \frac{f f_{xt} - f_x f_t}{f^2} = \frac{1}{2f^2} (D_t D_x (f \cdot f)) \tag{168}$$

And:

$$\begin{aligned} \frac{\partial^4}{\partial x^4} \log(f) = \frac{\partial^3}{\partial x^3} \frac{f_x}{f} &= \frac{\partial^2}{\partial x^2} \left(\frac{f_{xx}}{f} - \left(\frac{f_x}{f} \right)^2 \right) \\ &= \frac{\partial}{\partial x} \left(\frac{f_{xxx}}{f} - \frac{f_{xx} f_x}{f^2} - 2 \frac{f_x}{f} \frac{f f_{xx} - f_x f_x}{f^2} \right) \\ &= \frac{f_{xxxx}}{f} - 4 \frac{f_{xxx} f_x}{f^2} - 3 \frac{f_{xx}^2}{f^2} + 12 \frac{f f_{xx} - f_x^2}{f^3} - 6 \frac{f_x^4}{f^4} \\ &= \frac{1}{2f^2} D_x^4 (f \cdot f) - 6 \left(\frac{1}{2f^2} (D_x^2 (f \cdot f)) \right)^2 \end{aligned} \tag{169}$$

Example 6.1. To bring our equation to bilinear form we need to do a transformation. As an example we are going to bring the KdV-equation (15) to bilinear form using (170).

$$u = 2 \frac{\partial^2}{\partial x^2} \log \tau \tag{170}$$

$$u_t + 6uu_x + u_{xxx} = 0 \tag{171}$$

$$\Rightarrow 2\frac{\partial^3}{\partial t \partial x^2}\log\tau + 6\left(2\frac{\partial^2}{\partial x^2}\log\tau\right)\left(2\frac{\partial^3}{\partial x^3}\log\tau\right) + 2\frac{\partial^5}{\partial x^5}\log\tau = 0 \tag{172}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(2\frac{\partial^2}{\partial t \partial x}\log\tau + 3\left(2\frac{\partial^2}{\partial x^2}\log\tau\right)^2 + 2\frac{\partial^4}{\partial x^4}\log\tau\right) = 0 \tag{173}$$

Integrating once and using (168) and (169) we obtain:

$$2\left(\frac{1}{2\tau^2}(D_tD_x(\tau\cdot\tau))\right)+12\left(\frac{1}{2\tau^2}(D_x^2(\tau\cdot\tau)) + 2\frac{1}{2\tau^2}(D_x^4(\tau\cdot\tau)\right)-12\left(\frac{1}{2\tau^2}(D_x^2(\tau\cdot\tau)\right) = c \tag{174}$$

Which simplifies to:

$$[D_tD_x + D_x^4](\tau\cdot\tau) = c \tag{175}$$

We want τ to give us a solution of the KdV-equation. For example choosing $\tau = 1$ leads to $u = 0$, which is solution of the KdV-equation. We can conclude that $c = 0$.

$$[D_tD_x + D_x^4](\tau\cdot\tau) = 0 \tag{176}$$

This is the KdV-equation in bilinear form.

Remark 6.1. The equation (170) brought the KdV-equation (15) to bilinear form. However (170) does not bring all equations to bilinear form. There is no general method to bring an equation to bilinear form. Sometimes two equations are needed to bring a equation to bilinear form. For example the sine-Gordon equation:

$$u_{xt} = \sin(u) \tag{177}$$

Is transformed by:

$$u = 4\arctan(\frac{F}{G}) \tag{178}$$

With the bilinear system:

$$\begin{cases} [D_xD_t - 1](F\cdot G) = 0 \\ D_x^2[(F\cdot F) - (G\cdot G)] = 0 \end{cases} \tag{179}$$

Moreover not all equation can be transformed into bilinear form. For example the Monge-Ampre equation

$$u_{xy}^2 - u_{xx}u_{yy} = 0 \tag{180}$$

cannot be brought to bilinear form [9].

Definition 6.2. Equation (176) is called a **Hirota equation**. In general we write $(D_1, D_2, \dots)$ for the Hirota derivatives with respect to the variables $(x_1, x_2, \dots)$, then a Hirota equation is of the form:

$$P(D_1, D_2, \dots)(\tau\cdot\tau) = 0 \tag{181}$$

where $P(D_1, D_2, \dots)$ is a polynomial in $(D_1, D_2, \dots)$. Furthermore assume $P(0, 0, \dots) = 0$

Remark 6.2. Before we continue, we look at some qualities of the Hirota derivative. Using equation (164) we obtain the following:

$$D_x(f \cdot g) = -D_x(g \cdot f) \tag{182}$$

Using this we can conclude:

$$P(D_x)(f \cdot g) = P(-D_x)(g \cdot f) \tag{183}$$

Generalizing this to more variables:

$$P(D_1, D_2, \dots)(f \cdot g) = P(-D_1, -D_2, \dots)(g \cdot f) \tag{184}$$

Other useful identities are:

$$D_x^n(f \cdot 1) = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n f(x_1)|_{x_2=x_1=x} = \frac{d^n f}{dx^n} \tag{185}$$

$$\Rightarrow P(D_1, D_2, \dots)(f \cdot 1) = P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots\right) f \tag{186}$$

$$D_x^n(e^{kx} \cdot e^{lx}) = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n e^{kx_1} \cdot e^{lx_2}|_{x_2=x_1=x} = (k-l)^n e^{(k+l)x} \tag{187}$$

$$\Rightarrow P(D_x)(e^{kx} \cdot e^{lx}) = P(k-l)e^{(k+l)x} \tag{188}$$

Let us now restrict ourselves to the case where $g(x) = f(x)$, which is the case for the equations we are interested in. We observe that:

$$P(D_1, D_2, \dots)(f \cdot f) = P(-D_1, -D_2, \dots)(f \cdot f) \tag{189}$$

due to equation (184). Thus P must be an even function when applied to $(f \cdot f)$.

We are going to construct the solutions, starting with the 0-soliton solution (an n-soliton solution is the solution that has n solitons). We will show that these solution are soliton solutions later. Take $\tau = 1$, which will satisfy the Hirota equation (181):

$$P(D_1, D_2, \dots)(1 \cdot 1) = P(0, 0, \dots)(1 \cdot 1) = 0 \tag{190}$$

We expand τ around this point:

$$\tau = 1 + \varepsilon\tau_1 + \varepsilon^2\tau_2 + \varepsilon^3\tau_3 + \dots \tag{191}$$

For the 1-soliton solution we only need to take the first order term τ_1. Because $P(0, 0, \dots) = 0$ the equations put homogeneous constraints on the other $\tau_2, \tau_3, \dots$ of which $\tau_i = 0$ are solutions and therefore these terms can be disregarded. We remain with the following:

$$P(D_1, D_2, \dots)(1 + \varepsilon\tau_1 \cdot 1 + \varepsilon\tau_1) = P(D_1, D_2, \dots)(1 \cdot 1 + 1 \cdot \varepsilon\tau_1 + \varepsilon\tau_1 \cdot 1 + \varepsilon\tau_1 \cdot \varepsilon\tau_1) \tag{192}$$

We already know the following, using that P is an even function and $P(0, 0, \dots) = 0$:

$$P(D_1, D_2, \dots)(1 \cdot 1) = 0 \tag{193}$$

$$P(D_1, D_2, \dots)(1 \cdot \tau_1) = 0 \Leftrightarrow P(D_1, D_2, \dots)(\tau_1 \cdot 1) = 0 \tag{194}$$

We only need to look at:

$$P(D_1, D_2, \dots)(\tau_1 \cdot 1) = 0 \tag{195}$$

According to equation (186) this is equivalent with:

$$P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots\right)\tau_1 = 0 \tag{196}$$

This is a linear differential equation of which we know the solution. Choose $k_1, k_2, \dots \in \mathbb{C}$ in such a way that $P(k_1, k_2, \dots) = 0$ then:

$$\tau_1 = ce^{k_1 x_1 + k_2 x_2 + \dots} \tag{197}$$

Here $c \in \mathbb{C}$ is a constant. This solves equation (196). We need to make sure that τ_1 also solves the following equation:

$$P(D_1, D_2, \dots)(\tau_1 \cdot \tau_1) = 0 \tag{198}$$

$$P(D_1, D_2, \dots)(\tau_1 \cdot \tau_1) = P(D_1, D_2, \dots)\left(ce^{k_1 x_1 + k_2 x_2 + \dots} \cdot ce^{k_1 x_1 + k_2 x_2 + \dots}\right) \tag{199}$$

Now we can use equation (188):

$$\begin{aligned} P(D_1, D_2, \dots)\left(ce^{k_1 x_1 + k_2 x_2 + \dots} \cdot ce^{k_1 x_1 + k_2 x_2 + \dots}\right) &= P(k_1 - k_1, k_2 - k_2, \dots)e^{2(k_1 x_1 + k_2 x_2 + \dots)} \\ &= P(0, 0, \dots)e^{2(k_1 x_1 + k_2 x_2 + \dots)} = 0 \end{aligned} \tag{200}$$

Therefore τ_1 solves equation (192). The other τ_i are all chosen to be equal to zero. o Therefore our solution of the Hirota equation (181) is:

$$\tau = 1 + ce^{k_1 x_1 + k_2 x_2 + \dots} \tag{201}$$

We can absorb the c into the exponential by writing $\varepsilon = e^{\eta_0}$, therefore our final solution looks like:

$$\tau = 1 + e^{\eta_0 + k_1 x_1 + k_2 x_2 + \dots} \tag{202}$$

Example 6.2. Remember that in the case of the KdV-equation (15) the Hirota type equation is (176):

$$\left[D_t D_x + D_x^4\right](\tau \cdot \tau) = 0$$

Therefore k_1, k_2 need to satisfy:

$$k_1 k_2 + k_2^4 = 0 \Rightarrow k_1 = -k_2^3 \tag{203}$$

$$\Rightarrow \tau = 1 + e^{kx - k^3 t + \eta_0} \tag{204}$$

We transform back to u using equation (170).

$$
\begin{aligned}
u &= 2\frac{\partial^2}{\partial x^2}\log(\tau) = 2\frac{\partial^2}{\partial x^2}\log(1+e^{kx-k^3t+\eta_0}) \\
&= 2\frac{\partial}{\partial x}\frac{ke^{kx-k^3t+\eta_0}}{1+e^{kx-k^3t+\eta_0}} = 2\frac{(1+e^{kx-k^3t+\eta_0})k^2e^{kx-k^3t+\eta_0} - k^2e^{2(kx-k^3t+\eta_0)}}{(1+e^{kx-k^3t+\eta_0})^2} \\
&= \frac{2k^2e^{kx-k^3t+\eta_0}}{(1+e^{kx-k^3t+\eta_0})^2} = \frac{k^2}{2}\mathrm{sech}^2\left(\frac{kx-k^3t+\eta_0}{2}\right)
\end{aligned}
\tag{205}
$$

The Hirota Method can also be used to obtain the N-soliton solution.

$$P(k_1, k_2, \dots) = 0 \tag{206}$$

defined the relation that $k_1, K-2, \dots$ must satisfy to make

$$\tau = 1 + e^{\eta_0 + k_1x_1 + k_2x_2 + \dots} \tag{207}$$

a solution of the equation. When there are multiple sets of solutions $k_1^{(j)}, k_2^{(j)}, \dots$, we can make a superposition of the solutions in the Hirota equation to obtain more then one soliton. Let $k_1^{(1)}, k_2^{(1)}, \dots$ and $k_1^{(1)}, k_2^{(1)}, \dots$ (assume that these sets are different) satisfy the Hirota equation. Then we can write:

$$P(D_1, D_2, \dots)(\tau\cdot\tau) = 0 \tag{208}$$

$$\text{where}\quad \tau = 1 + \epsilon\sum_{j=1}^{2} c_j e^{k_1^{(j)}x_1 + k_2^{(j)}x_2 + \dots} + \epsilon^2\tau_2 + O(\epsilon 3) \tag{209}$$

$$\Rightarrow P(\partial_1, \partial_2, \dots)\tau_2 + c_1c_2P(k_1^{(1)} - k_1^{(2)}, k_2^{(1)} - k_2^{(2)}, \dots)e^{(k_1^{(1)}+k_1^{(2)})x_1 + (k_2^{(1)}+k_2^{(2)})x_2 + \dots} = 0 \tag{210}$$

where we have used equation (188) on the cross terms between the exponentials of $k^{(1)}$ and $k^{(2)}$. The other terms already satisfy the Hirota equation and thus disappear. We obtain an expression for τ_2

$$\tau_2 = -c_1c_2\frac{P(k_1^{(1)} - k_1^{(2)}, k_2^{(1)} - k_2^{(2)}, \dots)}{P(k_1^{(1)} + k_1^{(2)}, k_2^{(1)} + k_2^{(2)}, \dots)}e^{(k_1^{(1)}+k_1^{(2)})x_1 + (k_2^{(1)}+k_2^{(2)})x_2 + \dots} \tag{211}$$

Substitute this back into τ together with the solutions corresponding to $k^{(1)}$ and $k^{(2)}$, then the 2-soliton solution is obtained. This can be generalized to obtain a N-soliton solution. However it is not always possible to find N different sets of $k_1^{(j)}, k_2^{(j)}, \dots$ for a generic Hirota equation,however it turns out that as an empirical fact that the one and two soliton solution can alway be obtain, however the n-soliton solution for $n \leqslant 3$ is equivalent to the integrability of the system [8].

The Hirota method is less calculation heavy than the inverse scattering transform. However where in the inverse scattering transform the difficulties are the calculation of the time dependence of the scattering data and solving Marchenko integral equation, In the case of the Hirota method the problem is finding the transformation to bilinear form. This however is at the cost that the Hirota method only provides soliton solutions, while the inverse scattering transform can also be used to obtain non soliton solutions (in the case that the reflection coefficient is non zero).

7 Derrick's Theorem

Remark 7.1. We will discuss the phenomena of solitons in field theories, remember that the list we of qualities of solitons was:

- Wave profile
- Finite energy
- Localized phenomenon
- Stable in time
- Stable under interaction processes

however the interpretation of this last point is not universal. This leads to solutions that interact very little but still being called solitons. A important notion is that the soliton solution must be a stationary point of the energy, otherwise the solution would decay into a stationary point and would therefore be unstable. Often when studying solitons as solutions in field equation, we restrict ourselves to the stationary solutions. Moving solutions can be obtained by boosting the system.

Remark 7.2. We return to the KdV-equation (15), and look for a field equation with mirrors the behavior of that equation. As we have seen, the magnitude of the wave is related to the speed of the wave. A static wave solution was equal to zero everywhere. When a soliton moves with a certain speed, we can boost our system to follow the soliton, which is not moving after the boost. We have already stated that a static solutions equals zero everywhere, therefore the soliton disappears after the boost. There is no relativistic KdV-soliton, which is no surprise as the KdV-equation is not Lorentz-invariant.

In a large class of field theories there exist bounds on whether soliton solution can exist. The most important of these is Derrick's theorem [13].

Theorem 7.1. *Derrick's theorem. In theories with a Lagrangian of the form:*

$$L(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi) \tag{212}$$

non-trivial time-independent scalar soliton solutions do not exist in more then one spatial dimension in either Euclidean or Minkowski space-time.

The proof of Derrick's theorem is based on spatial rescaling of the energy functional, which is never zero for stationary solutions. Moreover these solutions which are minima of the energy are found by a variational principle and should thus be stationary against all variations of the field, including spatial rescaling. Therefore no stationary solution exist except for the vacuum states. Lorentz boosts imply there are also no moving solutions. Let us study the D-space dimensional energy functional corresponding to the Lagrangian (212)[12]:

$$E[\varphi] = \int d^D x \left(\frac{1}{2} \vec{\nabla}_i \varphi \cdot \vec{\nabla}_i \varphi + U(\varphi(x)) \right) \equiv E_S[\varphi] + E_U[\varphi] \tag{213}$$

$$\text{where} \quad E_S[\varphi] = \int d^D x \left(\frac{1}{2} \vec{\nabla}_i \varphi \cdot \vec{\nabla}_i \varphi \right) \tag{214}$$

$$\text{and} \quad E_U[\varphi] = \int d^D x \left(U(\varphi(x)) \right) \tag{215}$$

Both $E_S[\varphi]$ and $E_U[\varphi]$ are non-negative. Now consider a time-independent solution $\varphi(x)$ and apply a spatial rescaling: $x \to \lambda x$ with $\lambda > 0$. Let $\varphi_\lambda(x)$ be an parametric family of field configurations:

$$\varphi_\lambda(x) \equiv \varphi(\lambda x) \tag{216}$$

In order for the theory to be scale invariant, the energy functional transforms as follows:

$$E[\varphi_\lambda(x)] = E_S[\varphi_\lambda(x)] + E_U[\varphi_\lambda(x)] = \lambda^{2-D} E_S[\varphi(x)] + \lambda^{-D} E_U[\varphi(x)] \tag{217}$$

Because $\varphi(x)$ is a solution, it is a stationary point of the variation of the energy functional around $\lambda = 1$. Therefore we require the following:

$$\left. \frac{d\varphi_\lambda}{d\lambda} \right|_{\lambda=1} = 0 \tag{218}$$

This leads to the following relation:

$$(2 - D)\, E_S[\varphi(x)] = D\, E_U[\varphi(x)] \tag{219}$$

Because both $E_S[\varphi]$ and $E_U[\varphi]$ are non-negative, this equation can only be solved in the case $D = 1$, otherwise only the vacuum solution remains.

Remark 7.3. When we study solitons in field equations for $D > 1$ we need to circumvent this restriction, this can be done in several ways [12]:

By adding an extra field for example gauge fields $A^\mu(x)$. This changes the energy functional in the following way:

$$E = E_S + E_U + E_G \tag{220}$$

$$E_S[\varphi] = \int d^D x T_n(D_i \varphi) \tag{221}$$

$$E_U[\varphi] = \int d^D x U(\varphi) \tag{222}$$

$$E_G[A^\mu] = \int d^D x F_{ij} F_{ij} \tag{223}$$

$$\text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{224}$$

Here T_n is the n power of the derivatives of the scalar kinetic term (usually $n = 2$). The gauge field transforms as a 1-form, $A^\mu_\lambda \equiv \lambda A^\mu(\lambda x)$. The covariant derivative transforms as followed $D^\mu_\lambda \varphi(x) = (d^\mu \varphi_\lambda(x) + A^\mu \varphi_\lambda(x)) \equiv \lambda D^\mu \varphi(\lambda x)$. The field strength transforms as $F^{\mu\nu}_\lambda(x) \equiv \lambda^2 F^{\mu\nu}(x)$, then following the same procedure as before we obtain the constrain:

$$(n - D)E_S - DE_U + (4 - D)E_G = 0 \tag{225}$$

The following solitons use gauge fields to circumvent Derrick's theorem:

- (2) space dimensions: $\mathbb{C}P^N$ model [21]
- (3) space dimensions: 't Hooft-Polyakov monopole [14]
- (4) space dimensions: Instanton [19]

Another way of circumventing Derrick's theorem is to include a non-trivial time dependence $\varphi(x) \to \varphi(x, t)$. Q-balls show this structure by including a time dependent phase. Stability of the soliton is then governed by a Noether charge, which is related to the phase invariance of the Lagrangian. See section 8 for more about Q-balls. By considering more complicated Lagrangians, for example higher powers of the derivative, one can also circumvent Derrick's theorem and thus obtain solitons. Skyrme [15] and Faddeev-Niemi [16] models are examples of this kind.

Remark 7.4. We can split the solitons in field theories in two types, depending on the method which stabilized them. If they are stable because of the topology of the theory then they are called topological solitons. Otherwise they are called non-topological solitons, usually a Noether charge stabilizes these solutions. However it is imaginable that other more exotic theories also allow soliton solutions without a topological or Noether charge, which would also have to be called non-topological solitons.

8 Q-ball

In this section we will discuss the Q-ball [17], a non-topological soliton, which we study using the simple Lagrangian density of a complex scalar field φ in (1+1) dimensions:

$$\mathcal{L} = \partial^\mu \varphi \partial_\mu \varphi^\dagger - U(\varphi^\dagger \varphi) \tag{226}$$

This Lagrangian density is phase invariant: if we substitute $\varphi \to e^{i\theta}\varphi$ the Lagrangian density does not change. This invariance gives rise to a conserved charge by virtue of Noether's theorem. The Lagrangian density gives an equation of motion:

$$\partial^2 \varphi - \varphi \frac{dU(\varphi^\dagger \varphi)}{d(\varphi^\dagger \varphi)} = 0 \tag{227}$$

and the conserved Noether current is:

$$j^\mu = -i(\varphi^\dagger \partial^\mu \varphi - \varphi \partial^\mu \varphi^\dagger) \tag{228}$$

$$
\begin{aligned}
\partial_\mu j^\mu &= -i\partial_\mu(\varphi^\dagger\partial^\mu\varphi - \varphi\partial^\mu\varphi^\dagger) \\
&= -i(\partial_\mu\varphi^\dagger\partial^\mu\varphi + \varphi^\dagger\partial^2\varphi - \varphi\partial^2\varphi^\dagger - \partial^\mu\varphi^\dagger\partial_\mu\varphi) \\
&= -i(\varphi^\dagger\partial^2\varphi - \partial^2\varphi^\dagger\varphi)
\end{aligned}
$$

using the equation of motion

$$
\begin{aligned}
&= -i\left(\varphi^\dagger\varphi\frac{dU(\varphi^\dagger\varphi)}{d(\varphi^\dagger\varphi)} - \varphi\varphi^\dagger\frac{dU(\varphi^\dagger\varphi)}{d(\varphi^\dagger\varphi)}\right) \\
&= -i(\varphi^\dagger\varphi - \varphi^\dagger\varphi)\frac{dU(\varphi^\dagger\varphi)}{d(\varphi^\dagger\varphi)} \\
\partial_\mu j^\mu &= 0 \qquad (229)
\end{aligned}
$$

The conserved current can be used to obtain a conserved charge for the field.

$$
Q \stackrel{\text{def}}{=} \int j^0 dx \qquad (230)
$$

$$
\begin{aligned}
\frac{\partial Q}{\partial t} &= \int \frac{\partial j^0}{\partial t} dx \\
\text{using } \ \partial_\mu j^\mu = 0 &\Rightarrow \frac{\partial j^1}{\partial x} - \frac{\partial j^0}{\partial t} = 0 \\
\frac{\partial Q}{\partial t} &= \int \frac{\partial j^1}{\partial x} dx \\
\frac{\partial Q}{\partial t} &= j^1|_{\text{boundary}} \qquad (231)
\end{aligned}
$$

We are interested in solitons, which are localized. Therefore $\varphi|_{\text{boundary}} = 0$, then $j^1|_{\text{boundary}} = 0$. We can conclude that the charge is conserved in time:

$$
\frac{\partial Q}{\partial t} = 0 \qquad (232)
$$

In order to obtain soliton solutions we must put some restriction on the potential. The potential U must have a minimum at $\varphi = 0$, which we can choose to be zero by shifting the potential. When we have a soliton (or any solution except for the vacuum), $Q \neq 0$. This implies the following:

$$
Q = \int j^0 dx \neq 0 \quad \Rightarrow \quad j^0 \neq 0 \qquad (233)
$$

$$
j^0 = -i(\varphi^\dagger\partial^0\varphi - \partial^0\varphi^\dagger\varphi) \neq 0 \quad \Rightarrow \quad \partial^0\varphi \neq 0 \qquad (234)
$$

Our solution must be time dependent. In order to determine this time dependence, we write:

$$
\varphi = (\varphi_R + i\varphi_I)/\sqrt{2} \qquad (235)
$$

Where φ_R and φ_I are respectively the real and imaginary parts of the field φ. We rewrite the equation for the charge (230) as follows:

$$\begin{aligned} Q = \int j^0 dx &= \int -i(\varphi^\dagger \partial^0 \varphi - \partial^0 \varphi^\dagger \varphi) dx \\ &= \frac{-i}{2} \int (\varphi_R - i\varphi_I) \frac{\partial(\varphi_R + i\varphi_I)}{\partial t} - (\varphi_R + i\varphi_I) \frac{\partial(\varphi_R - i\varphi_I)}{\partial t} dx \\ &= \int \varphi_I \frac{\partial \varphi_R}{\partial t} - \varphi_R \frac{\partial \varphi_I}{\partial t} dx \end{aligned} \tag{236}$$

The solution has minimal energy, satisfying the constraint of the conserved charge (ω is the Lagrange multiplier):

$$\delta(E - \omega Q) = 0 \tag{237}$$

The equation for the energy can be obtained from the Lagrangian density (226) and then be rewritten using (235):

$$\begin{aligned} E &= \int \partial^\mu \varphi \partial_\mu \varphi^\dagger + U(\varphi^\dagger \varphi) dx \\ &= \int \frac{1}{2} \left(\frac{\varphi_R - i\varphi_I}{\partial t} + \frac{\varphi_R - i\varphi_I}{\partial x} \right) \left(\frac{\varphi_R + i\varphi_I}{\partial t} - \frac{\varphi_R + i\varphi_I}{\partial x} \right) + U \left(\frac{\varphi_R^2 + \varphi_I^2}{2} \right) dx \\ &= \int \frac{1}{2} \frac{\partial \varphi_R}{\partial t}^2 + \frac{1}{2} \frac{\partial \varphi_I}{\partial t}^2 + \frac{1}{2} \frac{\partial \varphi_R}{\partial x}^2 + \frac{1}{2} \frac{\partial \varphi_I}{\partial x}^2 + U \left(\frac{\varphi_R^2 + \varphi_I^2}{2} \right) dx \end{aligned} \tag{238}$$

We solve equation (237) assuming that the spatial part already satisfies the minimal energy requirement. Therefore the only variations in the field are of the time derivative.

$$\frac{\partial \varphi_R}{\partial t} \to \frac{\partial \varphi_R}{\partial t} + \delta_R \tag{239}$$

$$\frac{\partial \varphi_I}{\partial t} \to \frac{\partial \varphi_I}{\partial t} + \delta_I \tag{240}$$

Equation (237) then becomes:

$$\begin{aligned} \delta(E - \omega Q) = &\int \frac{1}{2} \frac{\partial \varphi_R}{\partial t}^2 + \frac{\partial \varphi_R}{\partial t} \delta_R + \frac{1}{2} \frac{\partial \varphi_I}{\partial t}^2 + \frac{\partial \varphi_I}{\partial t} \delta_I + \frac{1}{2} \frac{\partial \varphi_R}{\partial x}^2 + \frac{1}{2} \frac{\partial \varphi_I}{\partial x}^2 \\ &+ U \left(\frac{\varphi_R^2 + \varphi_I^2}{2} \right) dx - \omega \left(\int \varphi_I \left(\frac{\partial \varphi_R}{\partial t} + \delta_R \right) - \varphi_R (\frac{\partial \varphi_I}{\partial t} + \delta_I) \right) dx \end{aligned} \tag{241}$$

Collecting the terms δ_R and δ_I yield:

$$\frac{\partial \varphi_R}{\partial t} = \omega \varphi_I \tag{242}$$

$$\frac{\partial \varphi_I}{\partial t} = -\omega \varphi_R \tag{243}$$

The other parts of the equation already satisfy the minimal energy. Solving these equations we obtain:

$$\varphi(x,t) = \sigma(x)e^{-i\omega t} \tag{244}$$

Where $\sigma(x)$ is a real function. We can rewrite the charge (230) using this relation into a more simple form:

$$\begin{aligned} Q = \int j^0 dx &= -i\int (\varphi^\dagger \partial^0 \varphi - \varphi \partial^0 \varphi^\dagger) dx \\ &= \omega \int \sigma(x)^2 dx \end{aligned} \tag{245}$$

To further study the Q-ball we take the potential [18]:

$$U(\varphi^\dagger \varphi) = m^2|\varphi|^2 - \frac{2}{3}a|\varphi|^3 + \frac{1}{2}b|\varphi|^4 \tag{246}$$

By rescaling we simplify the Lagrangian.

$$\varphi \to \frac{m^2}{a}\varphi \tag{247}$$

$$x \to \frac{1}{m}x \tag{248}$$

The Lagrangian density then becomes:

$$\mathcal{L} = \partial^\mu \varphi \partial_\mu \varphi^\dagger - |\varphi|^2 + \frac{2}{3}|\varphi|^3 - \frac{1}{2}B|\varphi|^4 \tag{249}$$

$$\text{where} \quad B = \frac{bm^2}{a^2} \tag{250}$$

The equation of motion then becomes:

$$\partial^2\varphi - \varphi + |\varphi|\varphi - B|\varphi|^2\varphi = 0 \tag{251}$$

Using equation (244) we obtain a differential equation for $\sigma(x)$:

$$\sigma'' + (\omega^2 - 1)\sigma + \sigma^2 - B\sigma^3 = 0 \tag{252}$$

The soliton is localized and is stationary, therefore:

$$\sigma'(0) = 0 \tag{253}$$

$$\sigma(\infty) = 0 \tag{254}$$

This describes exactly the movement of a particle under the potential:

$$V = \frac{1}{2}(\omega^2 - 1)|\sigma|^2 - \frac{1}{3}|\sigma|^3 + \frac{B}{4}|\sigma|^4 \tag{255}$$

To obtain a soliton solution we require that the solution cannot be formed by a superposition of plane-waves. Therefore we require $m_{eff}^2 = \omega^2 - 1 < 0$. Furthermore for the solution to be stable the potential must have a zero different from $\varphi = 0$. These two conditions together give us the constrain:

$$1 - \frac{2}{9B} < \omega^2 < 1 \tag{256}$$

In order to have stability we require that the solution is a minimal energy configuration ($\delta^2 E > 0$). Using the potential we can write equation (238) more explicitly:

$$E = \int \sigma'^2 + \sigma^2 - \frac{2}{3}\sigma^3 + \frac{B}{2}\sigma^4 dx + \frac{Q^2}{\int \sigma^2 dx} \tag{257}$$

Then:

$$\delta^2 E = \int \delta\sigma \hat{O} \delta\sigma dx \tag{258}$$

$$\hat{O} = -\frac{d^2}{dx^2} + (1 - 2\sigma_0 + 3B\sigma_0^2) + 3\omega^2 \tag{259}$$

In order for $\delta^2 E > 0$ it is enough to know that $\hat{O}$ has no negative eigenvalues. To determine whether there are negative eigenvalues a numerical study was done by *M. Axenides, et al* [18] shows the results shown in figure 4. When the parameters are suitably chosen the Q-ball will be stable and will not decay into other Q-balls.

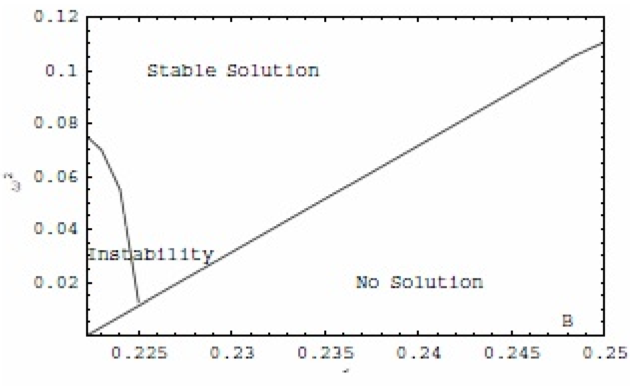
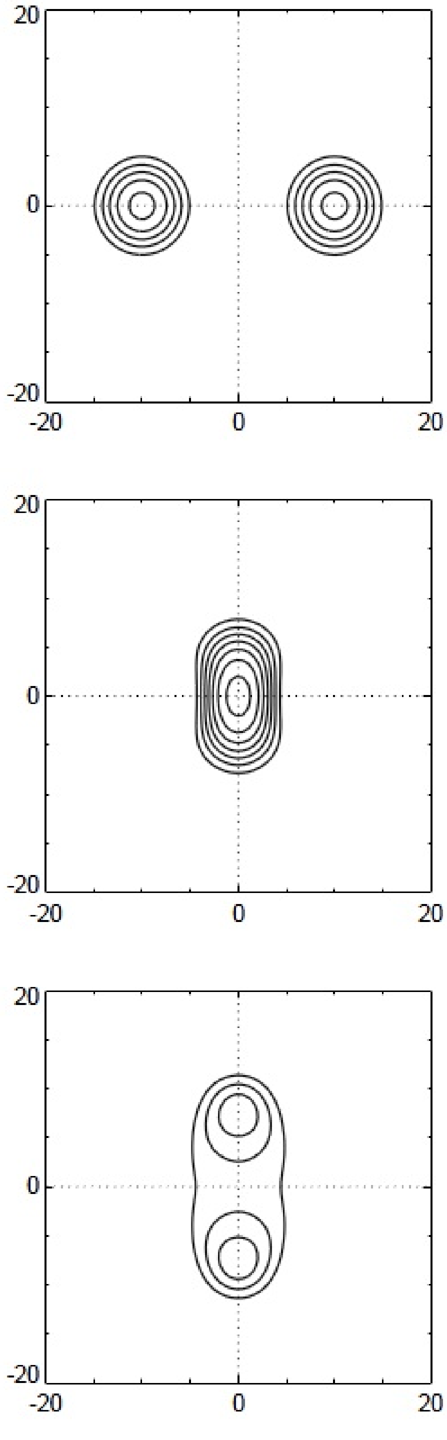


Figure 4: The stability of the Q-ball depending on ω^2 vertical and B horizontal [18]

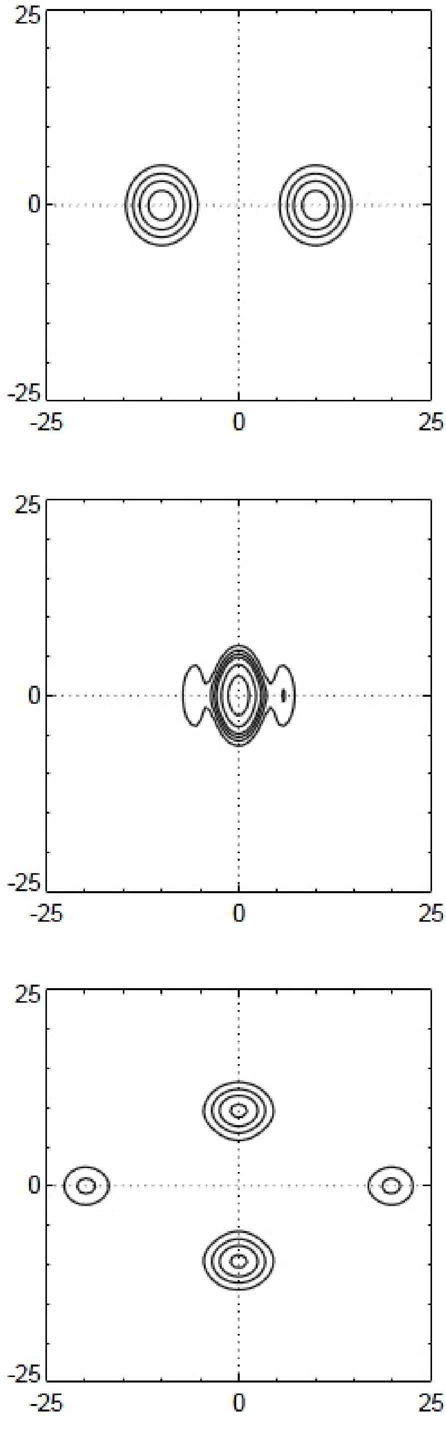
Furthermore their numerical analysis shows the interaction of Q-balls. The figures 5(a), 5(b), 5(c) and 5(d) show the interaction between qballs. At low velocities 5(a) the Q-balls the Q-balls attract and scatter along the x-axis and again attract each other, creating breather type oscillations. At higher velocity 5(b) part of the charge breaks free of the interaction traveling along the x-axis, the rest continues traveling in the y

direction. Each of these Q-balls escape towards infinity. At very high velocity 5(c) only forward scattering occurs and the Q-balls separate from each other, escaping towards infinity. The anti Q-ball , Q-ball interaction is only minimal, they exchange some charge and then travel towards infinity.

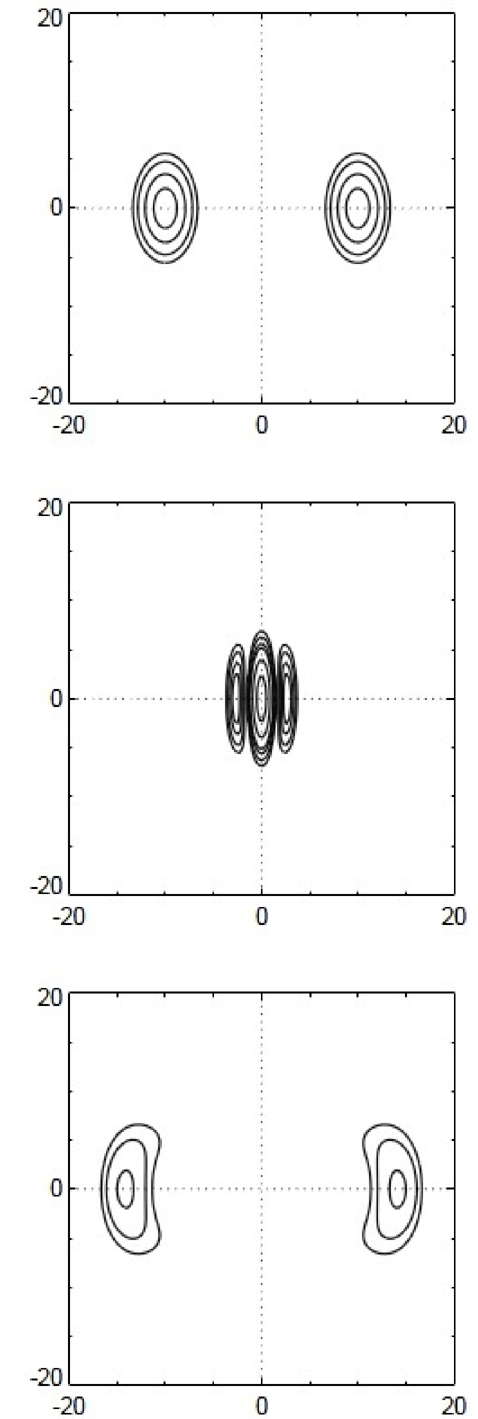
Where the Solitons of the KdV equation did not change their shape under interaction, this is no longer the case for Q-balls. Under interaction Q-balls may decay into other Q-balls depending on their relative velocity. However a single Q-ball is stable in time.



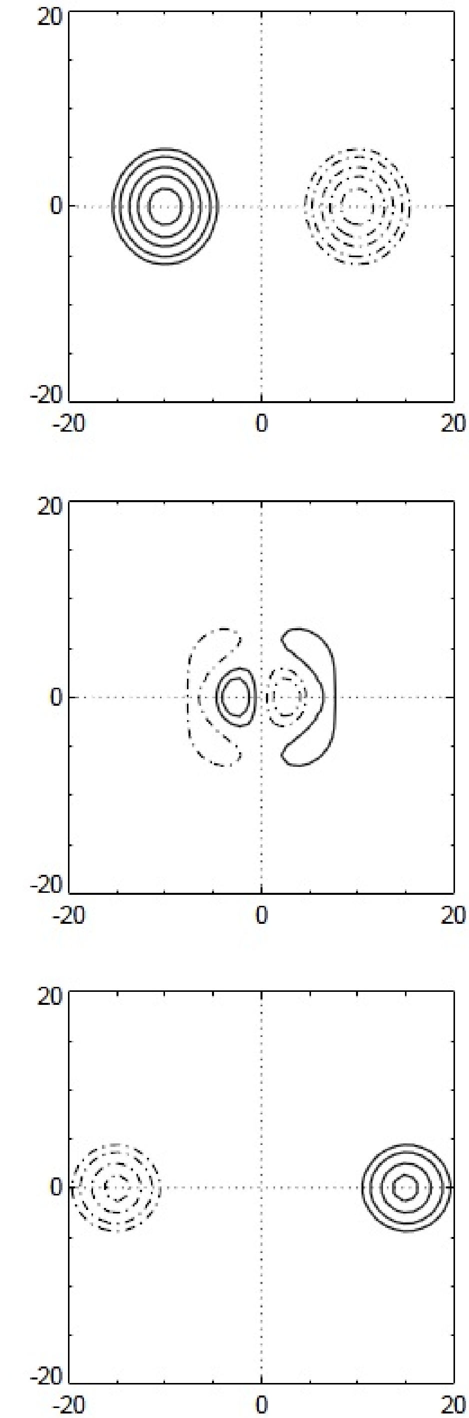
(a) Head-on collision of two Q-balls with the same charge.t = 0, t = 41.9, t = 55.9. Initial velocity of each Q-ball v = 0.2.[18]



(b) Head-on collision of two Q-balls with the same charge.t = 0, t = 19.6, t = 69.8. Initial velocity of each Q-ball v = 0.4. [18]



(c) Head-on collision of two Q-balls with the same charge. t = 0, t = 12.6, t = 29.3. Initial velocity of each Q-ball v = 0.8. [18]



(d) Head-on collision of two Q-balls with opposite charge. Solid lines represent positive values and dashed-dotted lines negative values. : t = 0, t = 28, t = 61.4. Initial velocity of each Q-ball v = 0.4. [18]

Figure 5:

9 Topological conservation laws

Instead of a Noether's charge, which provided stability for the Q-ball, we look at different charges. These charges are not governed by a symmetry however the topology of the theory provides a conserved quantity [19]. The solitons that a stable because of these charges are called topological solitons.

Instead of studying the solutions, we focus on the initial value data. Because the soliton solutions have a finite energy, the initial value date must also have finite energy. Therefore we can conclude that the initial value data must be sufficiently smooth (continuously differentiable). Moreover the initial value data, at boundary

$$\varphi(\pm\infty, t) \stackrel{\text{def}}{=} \lim_{x\to\pm\infty} \varphi(x, t) \tag{260}$$

is a zero of the potential energy. If the potential has a discrete set of zero's, because the initial value data is smooth, this implies):

$$\partial_0 \varphi(\pm\infty, t) = 0 \tag{261}$$

This is a conservation law. When the potential has more then one zero, it is possible that at one end ($x = +\infty$) the solution is connected to a different zero of the potential than at the other end ($x = +\infty$). Between the two boundaries the solution connects them in a smooth manner, which is nonzero because the potential is at least partially nonzero in between two zeros. This topological conserved charge (261) can be used to prove existence of nonzero and non-dissipating solutions.

Example 9.1. As an example, we take the sine-Gordon field-equation. The Lagrangian density is given by:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi \partial_\mu + U(\varphi) \tag{262}$$

$$\text{where} \quad U(\varphi) = 1 - \cos(\varphi) \tag{263}$$

where φ is a real scalar field. In figure 6 we have plotted the shape of the potential.

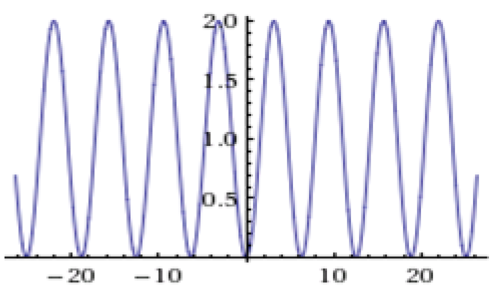


Figure 6: The shape of the potential of the sine-Gordon equation

The zeros of the potential form a discreet set. If a solution is at one zero (for example

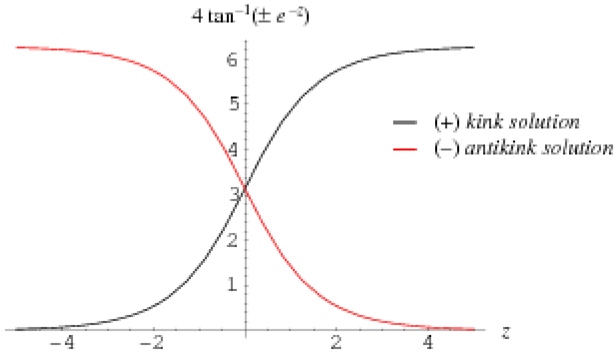


Figure 7: The kink and anti-kink solutions of the sine-Gordon equation

$\varphi = 0$) at $x = -\infty$ and when $x = \infty$ at another zero (for example $\varphi = 2\pi$) somewhere in between a link between these two must form. In the case of the sine-Gordon equation these look like figure 7. Because of the shape these solutions are called kinks (and anti-kinks). In the case of the sine-Gorden equation a kink and a anti-kink will form a breather pair, not dissimilar in behavior as the breather pair observed in section 8 where two slow moving Q-balls formed a breather pair. However in other equations that have kink and anti-kink solutions, the kink and anti-kink may annihilate each other.

As we have observed in section 7 gauge field can be used to circumvent Derrick's theorem 7.1. We will review gauge fields and obtain some properties.

The theories which are of interest to us contain a set of n scalar fields. These we combine into a n-vector, φ. Furthermore these theories contain a gauge group, G, which is a compact connected Lie group with a n-dimensional unitary representation $D(g)$. We assume that G is a simple group, the procedure we follow will still be valid in the case that G is not simple however this would add a notational complexity we would like to avoid. We define a gauge transformation by:

$$g(x)\colon \varphi(x) \to D(g(x))\varphi(x) \tag{264}$$

for any function $g(x)$ from space-time to G. To simplify notation we identify the group G with the representation $D(g)$.

$$g(x)\colon \varphi(x) \to g(x)\varphi(x) \tag{265}$$

Denote the generators of $D(g)$ by T^a, these are $n \times n$ orthogonal Hermitian matrices satisfying the following commutation relations:

$$[T^a, T^b] = ic^{abc}T^c \tag{266}$$

Where c^{abc} are the structure constants. To each of these generators T^a a gauge field A^a_μ is associated. These are real co-vector fields and transform with the gauge transformation given by:

$$A^a_\mu(x)T^a \to g(x)A^a_\mu(x)T^a g(x)^{-1} + ie^{-1}(\partial_\mu g(x))g(x)^{-1} \tag{267}$$

Where e is the gauge coupling constant, which is a real constant. Instead of this gauge transformation it is more common to look at the infinitesimal gauge transformation:

$$g(x) = 1 - iT^b\omega^b(x) \tag{268}$$
$$\delta A^a_\mu = c^{abc}A^b_\mu\delta\omega^c + e^{-1}\partial_\mu\delta\omega^a \tag{269}$$

We define the covariant derivative of φ as:

$$D_\mu\varphi(x) = \left(\partial_\mu + ieA^a_\mu T^a\right)\varphi \tag{270}$$

and we can define a field strength for the gauge fields:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ec^{abc}A^b_\mu A^c_\nu \tag{271}$$

these transform under gauge transformations as follows:

$$D_\mu\varphi(x) \to g(x)D_\mu\varphi(x) \tag{272}$$
$$F^a_{\mu\nu}T^a \to g(x)F^a_{\mu\nu}T^a g(x)^{-1} \tag{273}$$

We construct a Lagrangian density that defines our theory and is invariant under gauge transformations:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + D_\mu\varphi^\dagger D^\mu\varphi - U(\varphi) \tag{274}$$

The potential U is a gauge invariant function of the scalar fields, and we can shift U thereby ensuring that U becomes non-negative.

The ground states φ_0 have zero energy, φ_0 is constant and a zero of U. Furthermore the gauge fields vanish in the ground state. Define H as the subgroup of G that leaves ground states unchanged.

$$h \in H \Leftrightarrow h\varphi_0 = \varphi_0 \tag{275}$$

As the potential is invariant under gauge transformations, if φ_0 is a zero of U then so is $g\varphi_0$. Assume[4] that all zeros of U can be written as $g\varphi_0$ for some $g \in G$. This means that we can identify the zeros of U by the coset space G/H.

Remark 9.1. In some cases it useful to rewrite the gauge fields in such a way that either the time dependence or the radial dependence vanishes. Therefore we want to simplify the gauge fields in such a way that the can be written as:

$$A^a_0 = 0 \tag{276}$$

To this end we need to look at the gauge fields in relation with paths in space-time.

[4]This neglects both accidental degeneracies and internal (non-gauge) symmetries. The accidental degeneracies will disappear when we include quantum theory. Furthermore the results we obtain can be generalized to the case where internal symmetries are included. However both are beyond the scope of this thesis [19].

Take a path P in space-time, traveling from x_0 to x_1. Map a parameter $s \in [0, 1]$ to this path such that $P(0) = x_0$ and $P(1) = x_1$. Consider a field φ such that the covariant derivative vanishes along P:

$$\frac{dx^\mu}{ds} D_\mu \varphi = 0 \tag{277}$$

$$\Rightarrow \frac{d\varphi}{ds} = -ie\frac{dx^\mu}{ds} A^a_\mu T^a \varphi \tag{278}$$

We want to write the final field $\varphi(x_1)$ in terms of the initial field $\varphi(x_0)$ and a transformation $g(P) \in G$:

$$\varphi(x_1) = g(P)\varphi(x_0) \tag{279}$$

This problem is similar to the time evolution given by a potential in quantum mechanics and is solved using Dyson's formula:

$$g(P) = Se^{-ie\int_0^1 \frac{dx^\mu}{ds} A^a_\mu T^a ds} \tag{280}$$

Where S is an ordering based on s, in the same manner that in Dyson's formula there is a ordering based on t. These $g(P)$ transform as [19]:

$$g(P) \to g(x_1)g(P)g(x_0)^{-1} \tag{281}$$

Let P be the path from $(x, 0)$ to x by a straight line, where x is any space-time point. Transform $g(P_x)$ with the gauge transformation $g(x) = g(P_x)^{-1}$.

$$g(P_x) \to g(P_x)^{-1}g(P_x)g(P_0) = g(P_0) = 0 \tag{282}$$

The last part holds because P_0 is the path from and to the same point which changes nothing. After differentiation we obtain:

$$A^a_0 = 0 \tag{283}$$

This fixes the gauge in which we work, so we lose the freedom to choose any gauge if we set $A^a_0 = 0$. A similar argument can be used to obtain $A^a_r = 0, r > \epsilon$, the radial part can be set to zero (everywhere but the origin as there the radial part is not defined there).

Again studying stationary solutions, we can fix the time. We remove the time dependence from our formula. In the two dimensional case[5] we look at the energy of the field:

$$E \geqslant \int_\epsilon^\infty r \int_0^{2\pi} \left(\partial_r\varphi^\dagger\partial_r\varphi + U(\varphi)\right) d\theta dr \tag{284}$$

Note, we have cut out the origin from the integral. Because the solution is smooth this offsets the energy only by a finite amount. We require that the total energy is finite

[5]By including an extra angle this procedure can be adopted for a three dimensional case.

therefore the radial derivative must vanish (for this we use the results of remark 9.1) and

$$\phi(\infty, \theta) \stackrel{\text{def}}{=} \lim_{r \to \infty} \varphi(r, \theta) \tag{285}$$

must be a zero of U. In order for the finite energy condition to hold we need to cancel the energy contribution of the angular dependence of $\varphi(\infty, \theta)$ with the gauge field:

$$\lim_{r \to \infty} r A^a_\theta T^a \varphi(\infty, \theta) = -i e^{-1} \frac{d\varphi(\infty, \theta)}{d\theta} \tag{286}$$

We have obtained a mapping from the boundary (S^1) to the zero's of the potential (G/H) namely $\varphi(\infty, \theta)$.

Definition 9.1. Let $T^s(x)\colon X \to Y$ be a continuous map $\forall s \in [0, 1]$, $x \in X$. If $T^s(x)$ is a continuous function of s, $\forall x$, then for $s_1, s_2 \in [0, 1]$ the maps $T^{s_1}(x)$ and $T^{s_2}(x)$ are **homotopic** to each other.

If S is homotopic to T we write $S \sim T$. This is an equivalence relation, if $S \sim T$ and $T \sim R$ then $S \sim R$. This can easily be shown be combining the function $T^s(x)$ that deforms S into T with the function that deforms T into R and rescaling the parameter s. These equivalence classes, denoted by T, together with a composition rule, form a group. In the case where $X = [0, 1]$ with the end points identified (effectively $X = S^1$), it is given by (287)

$$(T_1 \cdot T_2) = \begin{cases} T_1(2x) & 0 \geqslant x \geqslant 1/2 \\ T_2(2x-1) & 1/2 \geqslant x \geqslant 1 \end{cases}. \tag{287}$$

The group that is formed in this manner is called the first homotopy group $\pi_1(Y)$. This group is the most interesting in this thesis, since we will mostly be working in two dimensions (where the border is S^1). Therefore we can classify the mappings that we obtained by looking at the boundary (285), this classification is $\pi_1(G/H)$. Solutions cannot go from one class to another as this would mean that they would be deformed discontinuously, which costs an infinite amount of energy [20].

Example 9.2. G is $U(1)$, and the fields consists of a single complex scalar field φ. The potential is given by:

$$U = \frac{\lambda}{2}(\varphi^* \varphi - 1)^2 \tag{288}$$

The zero's of the potential U are:

$$|\varphi| = 1 \tag{289}$$

$$\varphi = e^{i\sigma} \tag{290}$$

where σ is a real number. This forms a circle, thus $G/H = S^1$. Therefore we look at $\pi_1(S^1)$, this the question how can we map a circle on a circle. This characterized by the so called winding number, how often is the circle wrapped around the circle. Without going into the mathematical details, $\pi_1(S^1) = \mathbb{Z}$. This winding number is conserved and therefore a solution with a nonzero winding number will not dissipate into the vacuum solution.

Example 9.3. $G = SO(3)$, and the fields form a three component real isovector field φ. The potential is given by:

$$U = \frac{\lambda}{2}(\varphi^2 - 1)^2 \tag{291}$$

The zero's of the potential U are:

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 1 \tag{292}$$

This forms a sphere, thus $G/H = S^2$. Therefore we look at $\pi_1(S^2)$. When we place circle on the sphere we can always pull the circle together into a point. Thus $\pi_1(S^2) = 0$, therefore no conserved winding number exists and every solution may dissipate into the vacuum solution (unless something else in the theory stops this from happening).

However when considering this theory in three dimensions instead of two, we need to use the second homotopy group as the boundary now is a sphere. In this case we obtain $\pi_2(S^2) = \mathbb{Z}$ therefore in three dimensions solitons are possible.

Topological conservation laws are a useful tool to construct non-dispersive solutions, however they give little information on the explicit form of the solution. Furthermore the result of interactions can not be obtained from these conservation laws only. In the next section we will give an example of a theory with a topological conservation law in which we obtain explicit solutions.

10 $\mathbb{C}P^N$ model

A $\mathbb{C}P^N$ field is set of $N+1$ complex field that are combined in $\mathbb{C}P^N$ spinor [21]:

$$\varphi_\sigma = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \cdots \\ \varphi_{N+1} \end{pmatrix} \tag{293}$$

With the restriction that:

$$\sum_{n=1}^{N+1} |\varphi_n|^2 = 1 \tag{294}$$

Furthermore it is invariant under the local gauge transformation:

$$\varphi_\sigma(x) \to \varphi_\sigma(x) e^{i\Lambda(x)} \tag{295}$$

Because of the presence of this gauge invariance, we introduce a covariant derivative. Otherwise the field equation (which are normally gradient dependent) would change under the gauge transformations.

$$\vec{D}\varphi_\sigma(x) = (\vec{\nabla} + i\vec{A})\varphi_\sigma(x) \tag{296}$$

$$\vec{A} = i(\varphi_\sigma)^\dagger \vec{\nabla}\varphi_\sigma \tag{297}$$

The most simple energy functional that is consistent with the two constrains is:

$$E = \int dx(\vec{D}\varphi_\sigma(x)^\dagger) \cdot (\vec{D}\varphi_\sigma(x)) \tag{298}$$

The equations of motion are easy to obtain, however we need to include a Lagrange multiplier κ to take in account the first constraint (294).

$$\vec{D} \cdot \vec{D}\varphi_\sigma(x) + \kappa\varphi_\sigma(x) = 0 \tag{299}$$

By multiplying with $\varphi_\sigma^\dagger$ we obtain:

$$\varphi_\sigma^\dagger(x)\vec{D} \cdot \vec{D}\varphi_\sigma(x) + \kappa\varphi_\sigma^\dagger(x)\varphi_\sigma(x) = 0 \Rightarrow \varphi_\sigma^\dagger(x)\vec{D} \cdot \vec{D}\varphi_\sigma(x) = -\kappa \tag{300}$$

Which simplifies our equation of motion by removing the Lagrange multiplier.

$$\vec{D} \cdot \vec{D}\varphi_\sigma(x) - \left(\varphi_\tau^\dagger(x)\vec{D} \cdot \vec{D}\varphi_\tau(x)\right)\varphi_\sigma(x) = 0 \tag{301}$$

We will look for solutions in the case where we work in two spatial dimensions. Furthermore we look for static solutions, using boosts to obtain the time dependence. The most basic solution is the zero energy solution:

$$\varphi_\sigma(\vec{r}) = c_\sigma e^{i\Lambda r)} \tag{302}$$

$$\begin{aligned}
\vec{D}\varphi_\sigma &= \vec{D}c_\sigma e^{i\Lambda(r)} \\
&= (\vec{\nabla} + i\vec{A})c_\sigma e^{i\Lambda(r)} \\
&= \vec{\nabla}c_\sigma e^{i\Lambda(r)} + i\vec{A}c_\sigma e^{i\Lambda(r)} \\
&= ic_\sigma e^{i\Lambda(r)}\frac{\partial\Lambda(r)}{\partial r} - i(c_\tau e^{i\Lambda(r)})^\dagger(\vec{\nabla}c_\tau e^{i\Lambda(r)})c_\sigma e^{i\Lambda(r)} \\
&= ic_\sigma e^{i\Lambda(r)}\frac{\partial\Lambda(r)}{\partial r} - ic_\sigma e^{i\Lambda(r)}\frac{\partial\Lambda(\vec{r})}{\partial r} = 0
\end{aligned} \tag{303}$$

where c_σ is a constant and we have used that $c_\sigma^\dagger \cdot c_\sigma = 1$. This solution has zero energy everywhere. To obtain a non-zero but finite energy solution we need to include an angular dependence to the solution, however the energy contribution of this dependence must vanish at the boundary, for which the gauge invariance can be used. Applying the gauge transformation $\varphi_\sigma \to \varphi_\sigma e^{-i\Lambda(\theta)}$ will put the energy contribution of the boundary to zero. Furthermore the gauge invariance of the energy ensures us the boundary will vanish for all solutions of this shape.

$$\lim_{r\to\infty} \varphi_\sigma(r,\theta) = \varphi_\sigma(\infty,\theta) = b_\sigma e^{i\Lambda(\theta)} \tag{304}$$

The $e^{i\Lambda(\theta)}$ term maps a circle on the boundary, which is also a circle. This can be classified with the first homotopy group applied to a circle $\pi_1[S_1] = \mathbb{Z}$. For each $n \in \mathbb{Z}$

there exists a solution. Spinors of the form,

$$\varphi_\sigma(z) = K(z) \begin{pmatrix} 1 \\ \omega_2 \\ \vdots \\ \omega_{N+1} \end{pmatrix} = K(z)\omega_\sigma \tag{305}$$

where $z = x+iy$, $K(z)$ normalizes the spinor and ω_i are holomorphic functions, are exact solution of the equations of motion. This is easily shown because the differentiation of the holomorphic will drop out, therefore when we plug this spinor (305) into the equation of motion (301) we obtain:

$$\begin{aligned} &(\vec{D}\cdot\vec{D}K(z))\omega_\sigma - ((K(z)\omega_\tau)^\dagger(\vec{D}\cdot\vec{D}K(z))\omega_\tau)K(z)\omega_\sigma \\ =&(\vec{D}\cdot\vec{D}K(z))\omega_\sigma - ((K(z)\omega_\tau)^\dagger K(z)\omega_\tau)(\vec{D}\cdot\vec{D}K(z))\omega_\sigma \\ =&(\vec{D}\cdot\vec{D}K(z))\omega_\sigma - (\vec{D}\cdot\vec{D}K(z))\omega_\sigma = 0 \end{aligned} \tag{306}$$

Which proves that (305) is an exact solution. Furthermore the complex conjugate of (305) can be proven to also be a solution.

A more explicit example of such a spinor is, where $n > 0$:

$$\varphi_\sigma(z) = \frac{1}{\sqrt{a^2 + Nr^{2n}}} \begin{pmatrix} a \\ z^n \\ z^n \\ \vdots \\ z^n \end{pmatrix} \tag{307}$$

What is special about this solution is that it clearly shows the mapping of the circle at the boundary.

$$\varphi_\sigma(z) = \frac{e^{in\theta}}{\sqrt{\frac{a^2}{r^{2n}} + N}} \begin{pmatrix} \frac{a}{r^n e^{in\theta}} \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{308}$$

$$\varphi_\sigma(\theta)_{r\to\infty} = \frac{e^{in\theta}}{\sqrt{N}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{309}$$

To obtain solutions with negative winding number, we must use the complex conjugate.

Although we have constructed an explicit solution for the soliton. Under interactions the solitons would need to add their winding number, however construction of a two soliton solution of this model is difficult. A superposition of the initial conditions cannot be made as this would violate the constraint (294). Therefore we cannot compare these solitons with for example the KdV solitons or Q-balls as we can only construct one.

11 Conclusion

A wide variety of phenomena are called solitons. In non-linear partial differential equations solitons are solutions in the form of a solitary wave. These solutions can be obtained by the inverse scattering method and the Hirota method, however both methods cannot be applied to every differential equation and no general treatment of all evolution equations for solitons solutions is known. In spatial dimensions higher than one, Derrick's theorem restricts which field-equation can have soliton solution. There are numerous ways to circumvent Derrick's theorem, we discussed in this thesis Q-balls, a type of nontopological soliton, and topological conservation laws. Q-balls are stable in time both may decay in other solitons under interaction. Topological solitons can annihilate with opposing winding number and are therefore not stable under all interactions. Both of these solitons have different properties than the solitons obtained in partial differential equations like the KdV equation. A more distinguishing naming scheme may be appropriate as the phenomena named solitons are very diverse.

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A Matlab script, transmission coefficient

```
function [Z] = trans
    clear all
    close all
    global k a

    for tel=1:201
        k=10;
        at=.1;
        a=at*(tel-1);

        tstart = 0;
        tend = 60;
        n = 1000;
        tspan = linspace(tstart,tend,n);
```

```
            xinit = [1;1i*k];

            [t,x] = ode45(@integratingfunction, tspan, xinit);
            y = x(:,1);
            B(tel)=max(abs(y));
            atel(tel)=a;
        end

plot(atel,B,'b')
end

function [dydt] = integratingfunction(t,x)
        global k a

        dydt = zeros(size(x));
        dydt(1) = x(2);
        dydt(2) = -(k^2+a*sech(t-30)^2)*x(1);
end
```

Remark A.1. for information about *ode45* see:
http://www.mathworks.nl/help/techdoc/ref/ode45.html

Remark A.2.

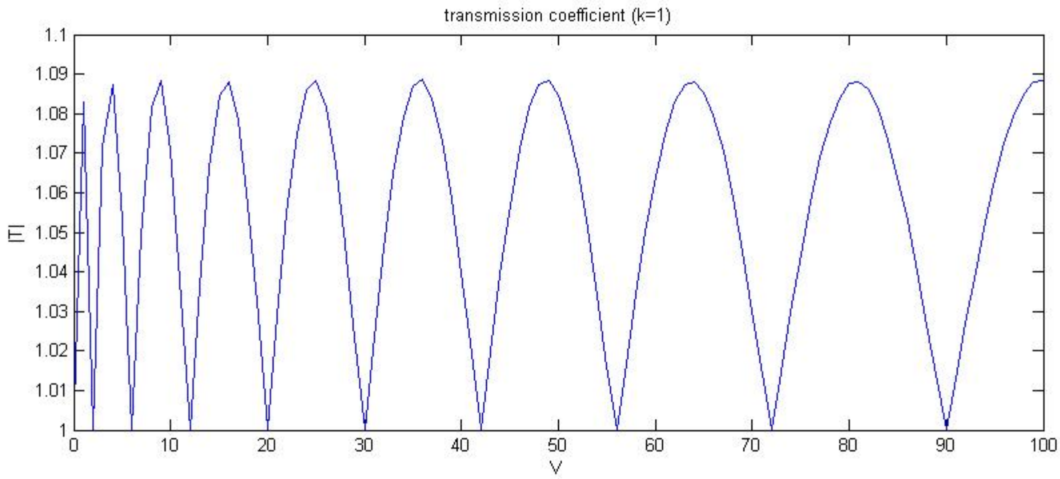

Figure 8: A reflectionless potential is obtained for $V = 0, 2, 6, 12, 20, 30, ..., n(n+1), ...$

B Derivation of the Marchenko equation

In this appendix we preform the Fourier transform which leads to the Marchenko equation.

$$T(k)Y_r(x,k) = R(k)Y_l(x,k) + Y_l(x,-k) \tag{310}$$

We do the Fourier transform:

$$\text{R.H.S.} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int_{-\infty} dz e^{-i\omega z/c} r(z) \int_{-\infty}^{\infty} dt' e^{i\omega t'} \tilde{y}(x,t') + \tilde{y}(x,-t) \tag{311}$$

$$\text{where} \quad r(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikt} R(k) \tag{312}$$

$$\text{R.H.S.} = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dt' r(z) \tilde{y}(x,t') \delta(t' - t - z/c) + \tilde{y}(x,-t) \tag{313}$$

$$= c \int_{-\infty}^{\infty} dt' r(c(t'-t)) \tilde{y}(x,t') + \tilde{y}(x,-t) \tag{314}$$

$$= cr(x - ct) + c^2 \int_{x/c}^{\infty} dt' r(c(t'-t)) K(x, ct') + \delta(t + x/c) + c\theta(-ct - x) K(x, -ct) \tag{315}$$

$$\text{L.H.S.} = \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{-i\omega t} T(k) Y_r(x,k) \tag{316}$$

$$= \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{-i\omega t} T(\omega/c) \int_{-\infty}^{\infty} dt' e^{i\omega t'} \left(\delta(t' + x/c) + c\theta(t' + x/c) L(x, -ct') \right) \tag{317}$$

$$= \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{-i\omega(t + x/c)} (T(\omega/c) - 1 + 1) \tag{318}$$

$$+ c \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{-i\omega(t - t')} (T(\omega/c) - 1 + 1) \theta(t' + x/c) L(x, -ct') \tag{319}$$

$$= c\Gamma(t + x/c) + \delta(t + x/c) + c^2 \int_{-x/c}^{\infty} dt \Gamma(t - t') L(x, -ct') + c \int_{-x/c}^{\infty} dt \delta(t - t') L(x, -ct') \tag{320}$$

$$= c\Gamma(t + x/c) + \delta(t + x/c) + c^2 \int_{-x/c}^{\infty} dt \Gamma(t - t') L(x, -ct') + c\theta(t + x/c) L(x, -ct) \tag{321}$$

$$\text{where} \quad \Gamma(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikt} (T(k) - 1) \tag{322}$$

Because $K(a,b) = 0$ for $a > b$ (it describes the wake behind the pulse) we consider the case $t + x/c < 0$. The integral in $\Gamma(t + x/c)$ can then be calculated as a contour plot, with finite many poles in the upper half plane, these poles correspond to the bounded states [6].

$$\Gamma(t - x/c) = -\sum_n c_n(t)^2 e^{-\kappa_n \xi} \tag{323}$$

equating the left and right hand side, we obtain the Marchenko equation:

$$K(x,y;t) + B(x+y;t) + \int_x^{\infty} K(x,z;t)B(y+z;t)dz = 0 \tag{324}$$

$$\text{where} \quad B(\xi;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)e^{ik\xi}dk + \sum_n c_n(t)^2 e^{-\kappa_n \xi} \tag{325}$$