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Hydraulic networks represented in a Port-Hamiltonian form

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Abstract

This is a paper about how to write a hydraulic network into a Port-Hamiltonian form using three different kinds of models.

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1 Introduction

In this paper we take a look at hydraulic networks, on which certain assumptions are made. The elements in these networks are pipes, pumps and valves. These elements are explained mathematically in the second section. Our goal is to write these networks in a Port-Hamiltonian form. What this means is also covered in section 2. We are going to use three different models. The first model holds for the simplest cases of hydraulic networks and this is introduced in section 3. Throughout the paper, we will look at the RL-network variant of the hydraulic network. This is because hydraulic networks and RL-networks share the same structure, and in this way we can exploit knowledge of RL-networks, such as Kirchhoff's current and voltage laws. The second model, see section 4, is able to cover larger networks. This model uses Kirchhoff's voltage law to obtain the Port-Hamiltonian form. The last model, section 5, is the so called Brayton-Moser model. This model is better suited for RL-networks with nonlinear resistors. In the conclusion a choice is made which model is best used is which case, when one is rewriting a certain hydraulic network in the Port-Hamiltonian form.

2 The main idea

In this section we are going to introduce hydraulic networks. In subsection 2.1, three assumptions are made on the network and certain definitions are given. In subsection 2.2 the different elements of the network with their mathematical formulation are introduced. This is divided into three subsubsections corresponding to the three different elements; the pipes, pumps and valves. In subsection 2.3, the Dirac structure and the Kirchhoff laws are given, as will be used in our models. In the last subsection, the Port-Hamiltonian model is introduced, which we will use to set up our first model.

2.1 The network and the assumptions

We are going to look at a hydraulic network which consists of pumps, pipes and valves. These are called the elements of the network. A few assumptions are made on the network. First of all, we must give some definitions regarding graph theory, because we will use some terms of this field of mathematics in the follows sections.

2.1.1 Graph theory

A graph F consists of edges and nodes or vertices. F is called connected if for each pair of nodes in the graph, there exists a path which connects them. One calls T a spanning tree of F, if T is a subgraph of F which has no cycles and contains all the nodes of F. If one adds any edge of F which is not in T to T, a cycle is created. The edges of F which are not in the spanning tree T are called the *chords* of T. Because there are, usually, different spanning trees possible for a certain F, the chords are not unique. The cycles that are created by adding a chord are called *fundamental cycles* or *fundamental loops*. The last thing we will use from graph theory is that we will denote by B the incidence matrix corresponding to the network one is modeling. Let the rows of this matrix correspond to the nodes of the network and the columns to the edges. Then, the entry B_{ij} of the matrix is either zero, when the node i is not on the edge j, or it is ± 1, if the node is on the corresponding edge. The plus sign is for flows entering an edge through the specific node and the minus sine is for leaving the edge.

2.1.2 The assumptions

The first assumption is that the network is a connected graph. The elements correpond to the edges of the graph, the vertices define the ends of the elements. The assumption is logical, because otherwise the network falls apart in different connected components.

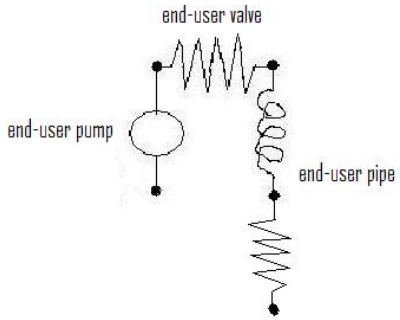

Figure 1: The situation at the end-users

The second assumption is a bit more difficult. In the network, you have certain end-users. You can explain this as houses who can get water from the network. So the total network can be seen as a streetblock full of houses, i.e., end-users. The assumption is, that each end-user valve is in series with a (end-user) pump and a (end-user) pipe. This is depicted in Figure 1.

Recall that, in general, there are different choices for the chords possible. This non-uniqueness is resolved by the second part of Assumption 2, which states that the chords correspond to the end-user pipes.

For each end-user you can construct a fundamental loop. This loop starts at the end-user pipe; hence the pipe that is in series with the end-user valve. It follows the direction of the flow all the way back to the end-user valve. This can be seen in Figure 2.

Assumption 3 states that there is one central heating element which lies in all the fundamental loops and it is represented by a valve. This is again logical, the whole streetblock is getting hot water due to the same heating element.
So, the three assumptions are:

Assumption 1: The network is a connected graph.
Assumption 2: Each end-user valve is in series with a pump and a pipe. Also, the chords of the graph correspond to end-user pipes.
Assumption 3: There is one central heating element and it lies in all the fundamental loops.

In the next section it is explained how you can model a pipe, a pump and a valve. The hydraulic network is similar to a RL-network. The flows correspond to the currents and the pressures to the voltages of the RL-network. Because there is more known about RL-networks we will later rewrite the pipes,

pumps and valves to their electric variants.

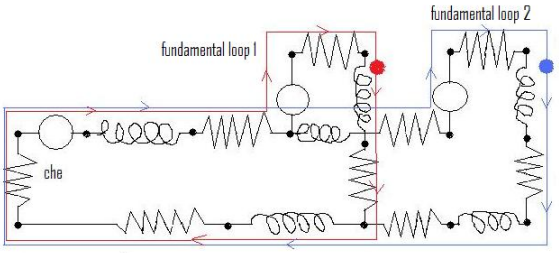

Figure 2: The fundamental loops. The dots indicates the starting positions. CHE means central heating element

2.2 The elements

2.2.1 Valve

For the valves, the following relationship holds,

$$h_i - h_j = \mu_k(K_{vk}, q_k) \tag{1}$$

where $h_i - h_j$ denotes the pressure across the terminals of the valve, q_k is the flow through the valve, K_{vk} is a variable which denotes the change of hydraulic resistance of the valve and $\mu_k(K_{vk}, q_k)$ is a continuously differentiable function which is strictly monotonically increasing and satisfies $\mu_k(K_{vk}, 0) = 0$ for all K_{vk}.

The electrical analog of a valve is a (nonlinear) resistor. In this case $h_i - h_j$ is equal to V_R, which is the voltage across the resistor. One knows that $V_R = RI_R$ in case of an linear resistor. In the case of a nonlinear resistor, one can write $V_R = R(I_R)$, that is,

$$h_i - h_j = V_R = R(I_R) \tag{2}$$

Linking this back to a hydraulic network, one sees that this is equal to (1), since the currents represent the flows and the pressure drops represent the voltages.

2.2.2 Pipe

If one assumes that the fluid flowing through the pipe is incompressible and that the diameter of the pipe is constant along the pipe, one can model the pipe by

$$J_k \frac{dq_k}{dt} = (h_i - h_j) - \lambda_k(K_{pk}, q_k) \tag{3}$$

Here, $h_i - h_j$ denotes the pressure difference between in inlet and the outlet of the pipe, q_k is again the flow (this time through the pipe), the function λ_k describes the pressure loss inside the pipe, K_{pk} is a loss factor and J_k is a constant which depends on the pipe dimensions and the mass density of the fluid.

Linking this to a RL-network, one obtains an edge between h_i and h_j with an (nonlinear) resistor and an inductor between them. For the pressure drop $h_i - h_j$, it holds that this is equal to the voltage of the resistor V_R plus the voltage of the inductor V_L. For the resistor, the same holds as for the valve. So $V_R = R(I_R)$. Now we have to rewrite the term V_L. One knows that the stored energy of an inductor is $H(\phi) = \frac{1}{2L}\phi^2$, where L denotes the inductance and ϕ the flux. Because $\frac{dH}{d\phi}(\phi) = I_L$, we see that $I_L = \frac{\phi}{L}$ or $\phi = LI_L$. Then $\dot{\phi} = L\dot{I}_L$ and hence $V_L = L\dot{I}_L$, since by Faraday's law $V_L = \dot{\phi}$. Putting everything together one obtains:

$$h_i - h_j = V_L + V_R = L\dot{I}_L + R(I_R) \tag{4}$$

By Kirchhoff's laws, $I_L = I_R = I$ and, if one relates this to the hydraulic network, $I = q$. So (4) becomes $h_i - h_j = L\frac{dq}{dt} + R(q)$ which is of the form (3). The question remains if the inductor and resistance are in series or in parallel. The answer is that they are in series, but this will be explained at the end of the section.

2.2.3 Pump

If one assumes that the pump is a device that can deliver a desired pressure difference $h_j - h_i$, then one can describe the pressure delivered by the pump by

$$h_i - h_j = -\Delta h_{pk} \tag{5}$$

The pressure difference Δh_{pk} is seen as a control unput. Translating this to a RL-network, $h_i - h_j$ is the difference between two input voltages, so $h_i - h_j = V_{pj} - V_{pi}$, which is of the form (5).

One sees that the corresponding RL-network consists of resistors, inductors and voltage sources. This correspondence was already seen in Figure 2.
In the next section we will introduce the well known Kirchhoff laws of circuit theory.

2.3 Dirac structures and Kirchhoff's laws

Let $f \in \Re^m$ be the flows and $e \in \Re^m$ be the efforts. A Dirac subspace D is an m-dimensional subspace of $\Re^m \times \Re^m$ with the property that $e^T f = 0$ for all (e, f).

Kirchhoff's current law tells us that the sum of the current, entering a node is equal to the sum of the current, leaving the node, for all nodes in the system, i.e.;

$$BI = 0 \tag{6}$$

Here, B is the incidence matrix which was introduced in subsection 2.1.1.

Kirchoff's voltage law says that the voltage around a closed loop must be zero. One can write the voltage as

$$V = B^T \lambda \tag{7}$$

where the λ's are called the potentials. As a consequence, $V^T I = \lambda^T B I = 0$, which corresponds to the information given by the Dirac structure, because the flows correspond to I and the efforts to V, so $e^T f = 0$. Because also dim D = dim I;

$$D = [(I, V) \mid BI = 0, V = B^T \lambda]$$

is a Dirac structure.

2.4 The Port-Hamiltonian model

Port-Hamiltonian systems are defined as:

$$\dot{x} = (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + gu \tag{8}$$

$$y = g^T \frac{\partial H}{\partial x}(x) + du \tag{9}$$

where $J(x)$ is called the structure matrix and is skew-symmetric, $R(x)$ is the resistive structure matrix and it holds that $R(x) = R(x)^T \geq 0$. Furthermore u and y are the flow, respectively the efforts of the system and g is the input matrix. The Hamiltonian is the total energy in the network. In a RL-network, this is just the sum of the energy in all the inductors, i.e.:

$$H(\phi) = \frac{1}{2} \phi^T L^{-1} \phi \tag{10}$$

$$\frac{\partial H}{\partial \phi}(\phi) = L^{-1} \phi \tag{11}$$

where L is the diagonal matrix with i^{th} element the inductance of the i^{th} inductor.
Our goal is to write our model in this form.

Before we will look at an example, we will now prove that the pipes are represented by an inductor and a resistor in series and not by an inductor and resistor in parallel.
Suppose they are in parallel as given in Figure 3.

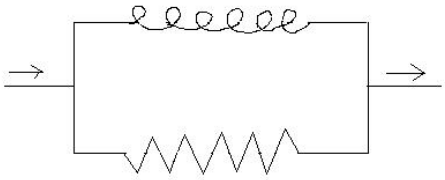

Figure 3: The pipe where the resistor and inductor are parallel

In this case, circuit theory tells us that:

- $I = I_R + I_L$
- $I_L = \frac{\phi}{L}$
- $V = RI_R = R(I - I_L) = R(I - \frac{\phi}{L})$
- $\dot{\phi} = V_L = V$

This gives the system:

$\dot{\phi} = -R\frac{\phi}{L} + RI$
$V = -R\frac{\phi}{L} + RI$

Rewriting this into a form where the voltage is the input:

$\dot{\phi} = V$
$I = \frac{\phi}{L} + \frac{V}{R}$

This is a port-Hamiltonian form, but is does not correspond to the dynamics of a pipe. So the pipes are not represented by a inductor and a resistance in parallel. It looks like the pipes are represented by a series between the inductor and resistance. This is seen in Figure 4. In this case, circuit theory yields the following equations:

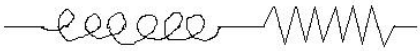

Figure 4: The pipe where the resistor and inductor are in serie

- $I = I_L = I_R$
- $I_L = \frac{\phi}{L}$
- $V_L + V_R = V$
- $V_R = RI_R$
- $\dot{\phi} = V_L = -V_R + V = -RI_R + V$

This yields the system:

$\dot{\phi} = -R\frac{\phi}{L} + V$
$I = \frac{\phi}{L}$

This system is a port-Hamiltonian system and the it corresponds to the dynamics of a pipe, see (4), so the pipes are represented by a series between the inductor and the resistance.

Now we are going to look at a simple RL-network and try to write it in the Port-Hamiltonian form. Assume we have a network depicted as in Figure 5. Here, there is a input voltage, denoted by $u = V$, there are two resistors, R_1 and R_2, and there is a inductor called L.

Kirchhoff's voltage law tells us that:

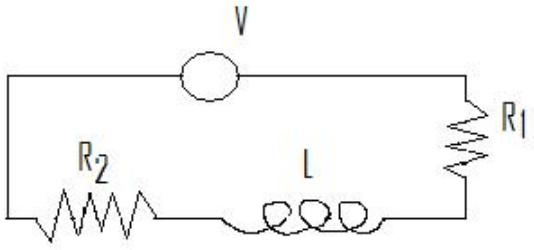

Figure 5: A simple RL-circuit

$$V_{R_1} + V_{R_2} + V_L = V$$

Also, Kirchhoff's current law tells us that

$$I_{R_1} = I_{R_2} = I_L = I$$

Now, one can rewrite all the terms in the voltage law.

- $V_{R_1} = R_1 I_{R_1}, V_{R_2} = R_2 I_{R_2}$
- $V_L = \dot{\phi}$
- $I = I_{R_1} = I_{R_2} = I_L = \frac{\phi}{L}$

The voltage law becomes

$$V - R_1 I_{R_1} - R_1 I_{R_2} - \dot{\phi} = V - (R_1 - R_2)\frac{\phi}{L} - \dot{\phi} = 0$$

Rewriting this equation and putt in the equation $I = \frac{\phi}{L}$ yields

1. $\dot{\phi} = -(R_1 + R_2)\frac{\phi}{L} + V$
2. $I = \frac{\phi}{L}$

which is of the form (8) and (9) with $\frac{\phi}{L} = \frac{\partial H}{\partial x}(x)$, $J(x) - R(x) = -(R_1 + R_2)$ and $g = 1$. So we have rewritten this, simple, system in port-Hamiltonian form. In the next section, the model we will be using to get a system of the form (8) and (9) is introduced. This will be the first of the three models we will be using.

3 The first model

We will be using the circuit variant of the hydraulic network first, because the laws are more common. Define the following Dirac subspace

$$D = [(I, V) \mid BI = 0, V = B^T\lambda] \tag{12}$$

Write I in the components who correspond to the currents through the resistors, the inductors and to the voltage sources, i.e. $I = [I_R, I_L, I_P]^T$. One can do the same with the voltage; $V = [V_R, V_L, V_P]^T$.
From circuit theory it is known that:

- $V_R = RI_R$
- $\dot{\phi} = V_L$
- $I_L = \frac{\phi}{L} = \frac{\partial H}{\partial \phi}(\phi)$

Now write the incidence matrix in terms of the part that correspond to the resistors, the part that corresponds to the inductors and the part that corresponds to the voltage sources i.e.:

$$B = (B_R|B_L|B_P)$$

Then, Kirchhoff's current law tells us that $BI = B_RI_R + B_LI_L + B_PI_P = 0$ or

$$B_RI_R + B_L\frac{\partial H}{\partial \phi}(\phi) + B_PI_P = 0, \tag{13}$$

while Kirchhoff's voltage law tells us that

$$\begin{pmatrix} B_R^T\lambda \\ B_L^T\lambda \\ B_P^T\lambda \end{pmatrix} = \begin{pmatrix} V_R \\ V_L \\ -V_P \end{pmatrix} = \begin{pmatrix} RI_R \\ \dot{\phi} \\ B_P^T\lambda \end{pmatrix} \tag{14}$$

Combining (13) and (14) yields the system:

$$\begin{pmatrix} 0 & B_L \\ 0 & 0 \\ -I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \frac{\partial H}{\partial \phi}(\phi) \end{pmatrix} + \begin{pmatrix} B_R & 0 \\ -R & B_R^T \\ 0 & B_L^T \\ 0 & B_P^T \end{pmatrix} \begin{pmatrix} I_R \\ \lambda \end{pmatrix} + \begin{pmatrix} B_P & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I_P \\ V_P \end{pmatrix} = 0 \tag{15}$$

Let us take a look at a simple hydraulic network, where we work with the RL-network variant. The system is given in Figure 6. There are two valves, one end-user and the central heating element. Also, there are four pipes and two pumps. Assumptions 1 and 2 hold is this system, so the end-user valve is in series with a pump and a pipe. The incidence matrices are given by:

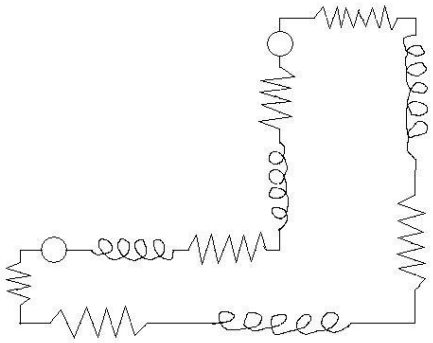

Figure 6: A simple RL-network

$$B_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, B_L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_P = \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, we would want to premultiply equation (15) by a matrix S with maximal rank satisfying

$$S \cdot \begin{pmatrix} B_R & 0 \\ -R & B_R^T \\ 0 & B_L^T \\ 0 & B_P^T \end{pmatrix} = 0$$

Let us denote the matrix $\begin{pmatrix} B_R & 0 \\ -R & B_R^T \\ 0 & B_L^T \\ 0 & B_P^T \end{pmatrix}$ by V for simplicity.

When one looks at the last n columns of V, n being the number of nodes of the network, one sees that the first n rows consist of zeros. The remaining n rows have in each column one and only one 1, and one and only one -1. The rest is zero, so if we premultiply with a matrix S, we know that we can choose

the last n columns of S to be columns full of one's. Here n is the number of components in the network, in this case 12. Now we also know that the first r rows of V, the number of resistors, will only have three nonzero elements and thus the following equations must hold:

- $S_{i1} - S_{i2} = R_1$
- $S_{i3} - S_{i4} = R_2$
- $S_{i5} - S_{i6} = R_3$
- $S_{i6} - S_{i7} = R_4$
- $S_{i9} - S_{i10} = R_5$
- $S_{i11} - S_{i12} = R_6$

This holds for $i = 1, ..., n$. With this in mind, we also have to premultiply the other two matrices by S. When we look at only one row, the equation:

$$S \begin{pmatrix} 0 & B_L \\ 0 & 0 \\ -I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \frac{\partial H}{\partial \phi}(\phi) \end{pmatrix} + S \begin{pmatrix} B_P & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I_P \\ V_P \end{pmatrix} = 0 \tag{16}$$

yields:

$$-\sum_{i=1}^{4} \dot{\phi}_i - R_1 \frac{\partial H}{\partial \phi_1} - R_2 \frac{\partial H}{\partial \phi_2} - R_3 \frac{\partial H}{\partial \phi_3} - R_4 \frac{\partial H}{\partial \phi_4} - R_5 I_{P1} - R_6 I_{P2} + V_{P1} + V_{P2} = 0 \tag{17}$$

Rewriting a bit, one obtains:

$$\sum_{i=1}^{4} \dot{\phi}_i + R_1 \frac{\phi_1}{L_1} + R_2 \frac{\phi_2}{L_2} + R_3 \frac{\phi_3}{L_3} + R_4 \frac{\phi_4}{L_4} + R_5 I_{P1} + R_6 I_{P2} = V_{P1} + V_{P2} \tag{18}$$

But there are four variables and just one equation. One wishes to write this equation in only one variable. Let's take the variable $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$. Then $\dot{\phi} = \dot{\phi_1} + \dot{\phi_2} + \dot{\phi_4} + \dot{\phi_4}$. There are also a few algebraic constraints:

$\frac{\phi_1}{L_1} = \frac{\phi_2}{L_2} = \frac{\phi_3}{L_3} = \frac{\phi_4}{L_4} = I_{p1} + I_{p2}$

Our goal is to write the terms $\frac{\phi_i}{L_i}$ in the variable ϕ.
One knows:

$\phi_1 = \phi - \phi_2 - \phi_3 - \phi_4$.

$\Rightarrow \phi_1 = \phi - \frac{L_2}{L_1}\phi_1 - \frac{L_3}{L_1}\phi_1 - \frac{L_4}{L_1}\phi_1$

$\Rightarrow \frac{L_1+L_2+L_3+L_4}{L_1}\phi_1 = \phi$

$\Rightarrow \frac{\phi_1}{L_1} = \frac{\phi}{L_1+L_2+L_3+L_4}$

And because of the algebraic constraints:

$\frac{\phi_2}{L_2} = \frac{\phi_3}{L_3} = \frac{\phi_4}{L_4} = I_{P1} = I_{P2} = \frac{\phi}{L_1+L_2+L_3+L_4}$

Now one can write the answer in only one variable:

$$\dot{\phi} + \frac{R_1 + R_2 + R_3 + R_4 + R_5 + R_6}{L_1 + L_2 + L_3 + L_4}\phi = V_{P1} + V_{P2} \tag{19}$$

This is a Port-Hamiltonian form, where $\frac{\partial H}{\partial \phi}(\phi) = \frac{1}{2(L_1+L_2+L_3+L_4)}\phi^2$, $R(\phi) = -(R_1 + R_2 + R_3 + R_4 + R_5 + R_6)$, $J(\phi) = 0$ and $g = 1$. The second equation that completes the Port-Hamiltonian is $I = \frac{\partial H}{\partial \phi}(\phi) = \frac{1}{2(L_1+L_2+L_3+L_4)}\phi^2$.

This is still the circuit variant. You can rewrite it to the hydraulic variant quite simple. Just note that $I_k = q_k$, $R_k = K_{vk}$ or K_{pk}, $J_k = L_k$ and $\Delta h_{pk} = V_k$. Also note that in this simple series network, the current is equal in each element, so we could write it in only the variable ϕ_1, or q_1. This could be done in the following way:

$\phi = \frac{L_1+L_2+L_3+L_4}{L_1}\phi_1$

$I_1 = \frac{\phi_1}{L_1}$

$\Rightarrow \dot{\phi}_1 = \dot{I}_1 L_1$

$\Rightarrow \frac{L_1+L_2+L_3+L_4}{L_1}\dot{\phi}_1 + \frac{R_1+R_2+R_3+R_4+R_5+R_6}{L_1+L_2+L_3+L_4}\frac{L_1+L_2+L_3+L_4}{L_1}\phi_1 = V_{P1} + V_{P2}$

$\Rightarrow \frac{L_1(L_1+L_2+L_3+L_4)}{L_1}\dot{I}_1 + \frac{L_1(R_1+R_2+R_3+R_4+R_5+R_6)}{L_1}I_1 = V_{P1} + V_{P2}$

$\Rightarrow \dot{I}_1 + \frac{R_1+R_2+R_3+R_4+R_5+R_6}{L_1+L_2+L_3+L_4}q_1 = \frac{V_{P1}+V_{P2}}{L_1+L_2+L_3+L_4}$

Now write $I_k = q_k$, $R_k = K_{vk}$ or K_{pk}, $J_k = L_k$ and $\Delta h_{pk} = V_k$ and the system becomes:

$$\dot{q}_1 + \frac{K_{p1} + K_{p2} + K_{v1} + K_{p3} + K_{p4} + K_{v2}}{J_1 + J_2 + J_3 + J_4}q_1 = \frac{\Delta h_{p1} + \Delta h_{p2}}{J_1 + J_2 + J_3 + J_4} \tag{20}$$

This model does not work for systems with more than one end-user. Because of our assumptions, a system with more than one end-user has always a parallel form. Suppose one looks at a simple RL-network with two end-users on which

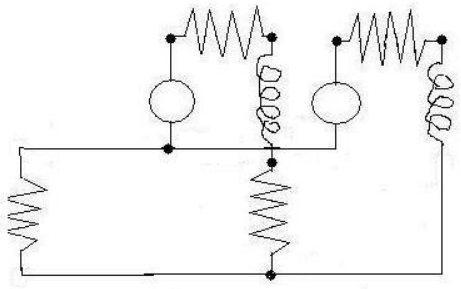

Figure 7: A simple network with two end-users

the assumptions still hold. Take for example Figure 7. The matrices are given by:

$$B_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B_L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, B_P = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If one looks at the last $n+1$ columns of $\begin{pmatrix} B_R & 0 \\ -R & B_R^T \\ 0 & B_L^T \\ 0 & B_P^T \end{pmatrix}$, it becomes clear that if one wants to premultiply by the matrix S to obtain 0, each row of S must satisfy the following equations:

- $S_{i,13} - S_{i,15} = 0$
- $S_{i,15} - S_{i,12} = 0$
- $S_{i,12} - S_{i,17} = 0$
- $S_{i,16} + S_{i,17} - S_{i,11} = 0$
- $S_{i,11} - S_{i,10} - S_{i,14} = 0$
- $S_{i,10} - S_{i,14} = 0$
- $S_{i,14} - S_{i,9} = 0$

- $S_{i,9} - S_{i,16} = 0$

Rewriting a bit yields:

$S_{i,11} = S_{i,16} + S_{i,17} = S_{i,10} + S_{i,14}$

$\Rightarrow S_{i,9} + S_{i,13} = S_{i,10} + S_{i,9}$

$\Rightarrow S_{i,13} = S_{i,10}$

$\Rightarrow S_{i,17} = S_{i,10}$

$\Rightarrow S_{i,11} - S_{i,16} = S_{i,11} + S_{i,14}$

$\Rightarrow -S_{i,16} = S_{i,14}$

$\Rightarrow -S_{i,9} = S_{i,9}$

for $i = 1, .., 17$. This last equation only holds for $S_{i,9} = 0$. But this means that $S_{i,j} = 0$, for $j = 9, .., 17$. From this, it follows that the last $n + 1$ columns of the matrix S are all zero. This only leaves the first n columns. As a matter of fact, one only has to look at B_R of the matrix V. Pre-multiplying this part, such that $S \cdot V = 0$, yields that the first n columns of S consist of one's. So each row of S looks like:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Writing (16) this time yields a $n \times 1$ columns full of zeros. This is not the anser we are looking for. This simple example shows that this method does not work on a network with more than one end-user.

In the next section a new model is introduced, which is able so solve systems with more then one end-user.

4 The second model

The problem with the method used in the previous section, is that it only works for networks who only consists of elements in series. In a network with more than one end-user there are always vertices with three edges. Therefore, the previous method of premultiplying with a matrix S does not hold anymore. So we must create another model for solving these systems. A method is constructed in [1] and we will summarize it and explain why it is usefull.

Suppose one has a network with more than one end-user and which is constructed as in Figure 8. Call the number of end-users n, the number of elements b and the number of vertices a. The idea is to construct fundamental loops, which means that for every end-user you construcs a loop, starting at the first element after the end-user valve, which is always a pipe (see Assumpsion 2). From there one takes the shortest way to the central heating element (che), which is an valve (see Assumption 3), and then one follows the shortest way to the end-user valve.
In this way, which is depicted in Figure 9 (where there are two end-users), one can construct n fundamental loops. This method uses the fact that the flows through all of the elements can be described as a combination of at most n different flows. This is, because in the parts of the network which are in series, the flows are the same in all the elements. In the parallel parts, the flow coming out of the parallel part is the same as the sum of the two flows coming in the parallel part. We will call these flows q_{fi} with $i = 1, ..., n$. Now, the main idea is to write the Kirchhoff voltage law for every fundamental loop. So we will get something of the form:

$$V_L + V_R = V$$

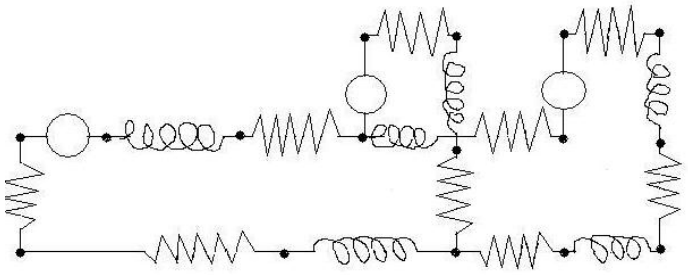

Figure 8: A RL-network with two end users

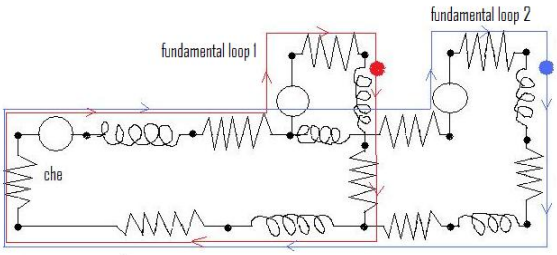

Figure 9: The different fundamental loops

In this model we do not have to look at the circuit theory variant first. We can look at the hydraulic network right away. The way one achieves this is to define certain matrices. First of all, the fundamental loop matrix B. B is a $n \times b$ matrtix and is defined as follows:

$$\mathrm{B}_{ih} = \begin{cases} 1 & \text{if edge } h \text{ is in the } i^{th} \text{ fundamental loop} \\ 0 & \text{if edge } h \text{ is not in the } i^{th} \text{ fundamental loop} \end{cases}$$

The elements are numbered as follows: One starts with numbering the end-user pipes (the inductive part of them that is), and then the first fundamental loop is completed, then the second and so on. This is depicted in Figure 10. Then, some more matrices are constructed. The matrix J is a $b \times b$ diagonal matrix where the diagonal elements are:

$$\mathrm{J}_{ii} = \begin{cases} \mathrm{J}_j & \text{if the } i^{th} \text{ element of the network is a inductor} \\ 0 & \text{if the } i^{th} \text{ element of the network is not a inductor} \end{cases}$$

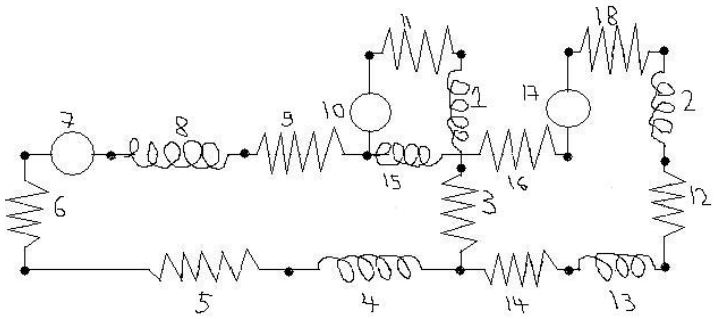

Figure 10: The numbering of the elements

J_j is the inductor constant as was explained in the first section. Here j goes from one to the number of inductors in the network.
The resistors are introduced in a same way. One construct the $b \times 1$ matrix $(\lambda + \mu)$ where the elements are determined in the same way as J:

$$(\lambda+\mu)_{i1} = \begin{cases} \mathrm{K}_{pj} & \text{if the } i^{th} \text{ element of the network is a pipe resistor} \\ \mathrm{K}_{vj} & \text{if the } i^{th} \text{ element of the network is a valve resistor} \\ 0 & \text{if the } i^{th} \text{ element of the network is not a resistor} \end{cases}$$

Also, here K_{pj} and K_{vj} are the resistor constants (in the linear case) and j goes from one to the number of resistors in the network.

The pumps are introduced by the $b \times 1$ matrix Δh_p which is constructed in the same way as J and $(\lambda + \mu)$ but in this case for the pumps;

$$\Delta h_{i1} = \begin{cases} \Delta h_{pj} & \text{if the } i^{th} \text{ element of the network is a pump} \\ 0 & \text{if the } i^{th} \text{ element of the network is not a pump} \end{cases}$$

Here, j goes from one to the number of pumps in the network.

Then, the independent flows q_{fi} are put together in the $n \times 1$ matrix q_f. The flows through each element are then given by $B^T q_f$.
The voltage equation of Kirchhoff then becomes:

$$BJB^T \dot{q_f} + B(\lambda + \mu)(B^T q_f) = B\Delta h_p \tag{21}$$

In this way, one obtains n equations and n variables. This means that one does not need to write down the algebraic constraints anymore to define the system. The system is already solvable.

Let us look at a network with only two end-users as is depicted in Figure 8. Here, $b = 18$ and $n = 2$. The incidence matrix is given by:

$$B = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The matrices J, $(\lambda + \mu)$, Δh_p and $B^T q_f$ are given by:

$$J = \begin{pmatrix} J_1 & & & & & & & & & & & & & & & & & & & & & & \\ & J_2 & & & & & & & & & & & & & & & & & & & & \\ & & 0 & & & & & & & & & & & & & & & & & & & \\ & & & J_3 & & & & & & & & & & & & & & & & & & \\ & & & & 0 & & & & & & & & & & & & & & & & & \\ & & & & & 0 & & & & & & & & & & & & & & & & \\ & & & & & & 0 & & & & & & & & & & & & & & & \\ & & & & & & & J_4 & & & & & & & & & & & & & & \\ & & & & & & & & 0 & & & & & & & & & & & & & \\ & & & & & & & & & 0 & & & & & & & & & & & & \\ & & & & & & & & & & 0 & & & & & & & & & & & \\ & & & & & & & & & & & 0 & & & & & & & & & & \\ & & & & & & & & & & & & J_5 & & & & & & & & & \\ & & & & & & & & & & & & & 0 & & & & & & & & \\ & & & & & & & & & & & & & & J_6 & & & & & & & \\ & & & & & & & & & & & & & & & 0 & & & & & & \\ & & & & & & & & & & & & & & & & 0 & & & & & \\ & & & & & & & & & & & & & & & & & 0 & & & & \end{pmatrix}$$

$$(\lambda + \mu) = \begin{pmatrix} 0 \\ 0 \\ K_{p1} \\ 0 \\ K_{p2} \\ K_{v1} \\ 0 \\ 0 \\ K_{p3} \\ 0 \\ K_{v2} \\ K_{p4} \\ 0 \\ K_{p5} \\ 0 \\ K_{p6} \\ 0 \\ K_{v3} \end{pmatrix}, \Delta h_p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta h_{p1} \\ 0 \\ 0 \\ \Delta h_{p2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta h_{p3} \\ 0 \end{pmatrix}, B^T q_f = \begin{pmatrix} q_{f1} \\ q_{f2} \\ q_{f1} \\ q_{f1} + q_{f2} \\ q_{f1} + q_{f2} \\ q_{f1} + q_{f2} \\ q_{f1} + q_{f2} \\ q_{f1} + q_{f2} \\ q_{f1} + q_{f2} \\ q_{f1} \\ q_{f1} \\ q_{f2} \\ q_{f2} \\ q_{f2} \\ q_{f2} \\ q_{f2} \\ q_{f2} \\ q_{f2} \end{pmatrix}$$

(21) becomes:

$$\begin{pmatrix} (J_1+J_3+J_4)\dot{q}_{f1}+(J_3+J_4)\dot{q}_{f2} \\ (J_3+J_4)\dot{q}_{f1}+(J_2+J_3+J_4+J_5+J_6)\dot{q}_{f2} \end{pmatrix} +$$

$$\begin{pmatrix} (K_{p1}+K_{p2}+K_{v1}+K_{p3}+K_{v2})q_{f1}+(K_{p2}+K_{v1}+K_{p3})q_{f2} \\ (K_{p2}+K_{v1}+K_{p3})q_{f1}+(K_{p2}+K_{v1}+K_{p3}+K_{p4}+K_{p5}+K_{p6}+K_{v3})q_{f2} \end{pmatrix} =$$

$$\begin{pmatrix} \Delta h_{p1}+\Delta h_{p2} \\ \Delta h_{p1}+\Delta h_{p3} \end{pmatrix}$$

So this is a way to solve networks with more then one end-user.
If one tries to solve the system with just one end-user again, see Figure 6, but now with this new method, the same answer should be obtained. The matrices are given by:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} J_1 & & & & & & & & & & & \\ & 0 & & & & & & & & & & \\ & & J_2 & & & & & & & & & \\ & & & 0 & & & & 0 & & & & \\ & & & & 0 & & & & & & & \\ & & & & & 0 & & & & & & \\ & & & & & & J_3 & & & & & \\ & & & & & & & 0 & & & & \\ & & 0 & & & & & & J_4 & & & \\ & & & & & & & & & 0 & & \\ & & & & & & & & & & 0 & \end{pmatrix}$$

$$(\lambda+\mu) = \begin{pmatrix} 0 \\ K_{p1} \\ 0 \\ K_{p2} \\ K_{v1} \\ 0 \\ 0 \\ K_{p3} \\ 0 \\ K_{p4} \\ 0 \\ K_{v2} \end{pmatrix}, \Delta h = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta h_{p1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta h_{p2} \\ 0 \end{pmatrix}$$

Equation (21) becomes:

$$(J_1+J_2+J_3+J_4)\dot{q}_f + (K_{p1}+K_{p2}+K_{v1}+K_{p3}+K_{p4}+K_{v2})q_f = \Delta h_{p1}+\Delta h_{p2}$$

$$\Rightarrow \dot{q}_f + \frac{(K_{p1} + K_{p2} + K_{v1} + K_{p3} + K_{p4} + K_{v2})}{(J_1 + J_2 + J_3 + J_4)} q_f = \frac{\Delta h_{p1} + \Delta h_{p2}}{(J_1 + J_2 + J_3 + J_4)}$$

This is the same as (20). So there are now two models for solving the simple networks with just one end-user. This second model, however, also has the property that it can solve systems with more than just one end-user. In the next section a third method is introduced to solve these kind of networks. This method also has the property that it can solve networks where the resistors are not just linear quite easily.

5 The third model: Brayton-Moser

In circuit theory there are certain equations which deal with possibly nonlinear electrical circuits. From [2] one knows it is necessary that one can split up the circuit in two subcircuits Σ_a and Σ_b where Σ_a contains the current-controlled inductors and current-controlled resistors, and Σ_b contains the voltage-controlled capacitors, conductors and the voltage-controlled resistors. The way the Brayton-Moser equations work is that they have a term called the mixed-potential function which is given by

$$P(I,V) = D(I) - D^*(V) + V^T N I \tag{22}$$

Here,

$$D(I) = \int_0^{N_r I} \hat{V}_r^T(I_r)\,\mathrm{d}I_r$$

represents the resistive content capturing the current-controlled resistors and sources contained in Σ_a. The relations between the currents through the resistors I_r and the voltages is given by $V_r = \hat{V}_r(I_r)$. Also

$$D^*(V) = \int_0^{N_g V} \hat{I_g^T}(V_g)\,\mathrm{d}V_g$$

represents the resistive co-content capturing the voltage-controlled resistors, conductors and sources contained in Σ_b. The relations between the voltages across the conductors V_g and the currents through the conductors is given by $I_g = \hat{I}_g(V_g)$. The term $V^T N I$ represents the instantaneous power delivered from Σ_a to Σ_b. The matrices N, N_r and N_g have entries 1,0 and are obtained from Kirchhoff's voltage and current laws. Now, the dynamics of a possibly nonlinear RLC system can be written as:

$$Q(z)\dot{z} = \nabla_z P(z) \tag{23}$$

with $z = (I,V)^T$ and

$$Q(z) = \begin{pmatrix} -L(I) & O \\ O & C(V) \end{pmatrix} \tag{24}$$

where $L(I)$ and $C(V)$ are the incremental inductance and capacitance matrices. Equations (23) and (24), together with (22) are called the Brayton-Moser equations.
Now we want to use these equations to rewrite our hydraulic network. Let's first look again at the electrical variant of our model. Observe that we don't have any capacitors, so we have a RL-network. This means that:

$$z = I$$

$$Q(I) = -L(I) \tag{25}$$

$$P(I) = D(I) \tag{26}$$

Also, the incremental inductance matrix $L(I)$ times the time derivative of the flows, $\dot{I}$ is equal to the time derivative of the first order gradient of the Hamiltonian. I.e.:

$$Q(z)\dot{z} = -L(I)\dot{I} = -\frac{d}{dt}\nabla_I H(I) \tag{27}$$

Note that I stands for the flows through the inductors corresponding to rhe chords, as in the second model, not the flows through each element. The only thing missing in the Brayton-Moser equations are the voltage sources. This can be quickly solved by extending $D(I)$ by the term $\int_0^{N_p I} V_p^T \, \mathrm{d}I_p$. Here the matrix N_p follows from Kirchhoff's laws and V_p denotes the vector of voltage sources. All together, the Brayton-Moser equations for a RL-circuit are:

$$-\frac{d}{dt}\nabla_I H(I)\dot{I} = \nabla_I \left(\int_0^{N_r I} \hat{V_r^T}(I_r) \, \mathrm{d}I_r - \int_0^{N_p I} V_p^T \, \mathrm{d}I_p \right) \tag{28}$$

Let us again take a look at Figure 6. Now we could try to solve it using the Brayton-Moser equations. Altough the advantage of the Brayton-Moser equations is that it deals with nonlinear resistors, let us assume that they are linear so we can see that this method yields the same answer. So we assume that $\hat{V_r^T}(I_r) = RI_R$. The matrices are given by:

$$N_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, N_p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, I = \begin{pmatrix} I_{L_1} \\ I_{L_2} \\ I_{L_3} \\ I_{L_4} \end{pmatrix}$$

This yields a content function of:

$$P(I) = \frac{1}{2}R_1 I_{L_1}^2 + \frac{1}{2}R_2 I_{L_1}^2 + \frac{1}{2}R_3 I_{L_2}^2 + \frac{1}{2}R_4 I_{L_2}^2 + \frac{1}{2}R_5 I_{L_3}^2 + \frac{1}{2}R_6 I_{L_4}^2 - V_{P1} I_{L_3} - V_{P2} I_{L_4}$$

and the Hamiltonian is given by:

$$H(I) = \frac{1}{2}L_1 I_{L_1}^2 + \frac{1}{2}L_2 I_{L_2}^2 + \frac{1}{2}L_3 I_{L_3}^2 + \frac{1}{2}L_4 I_{L_4}^2$$

(28) becomes:

$$-L_1 \dot{I}_{L_1} = (R_1 + R_2) I_{L_1} \Rightarrow \dot{\phi}_1 = -\frac{R_1 + R_2}{L_1}\phi_1$$

$$-L_2 \dot{I}_{L_2} = (R_3 + R_4) I_{L_2} \Rightarrow \dot{\phi}_2 = -\frac{R_3 + R_4}{L_2}\phi_2$$

$$-L_3\dot{I}_{L_3} = R_5 I_{L_3} - V_{P1} \Rightarrow \dot{\phi}_3 = -\frac{R_5}{L_3}\phi_3$$

$$-L_4\dot{I}_{L_4} = R_6 I_{L_4} - V_{P2} \Rightarrow \dot{\phi}_4 = -\frac{R_6}{L_4}\phi_4$$

Combining all four equations and introducing the variable $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$ again leads to one equation in one variable:

$$\phi + \frac{R_1 + R_2 + R_3 + R_4 + R_5 + R_6}{L_1 + L_2 + L_3 + L_4}\phi = V_{P1} + V_{P2}$$

which is the same as (19). This means that we now have three different models for solving the simple networks with just one end-user.
If we take a look at a network with more then one end-user, we can link this method to our second method. Instead of taking the flows through each inductor we will use our knowledge of the second method and only take the flows of the inductors who correspond to the end-user pipes. So we look at the chords again. The Brayton-Moser method has the benifit that one has to construct lesser matrices then the one needs to following the second method. In this model, one only has to construct N_r, N_p and one has to write down the Hamiltonian.
It should be clear that in this case there really is not much difference between the third and second method. The term BJB^T from the second method corresponds to $\nabla H(I)$ of the third method. The same holds for $B(\lambda + \mu)(B^T q_f) = \nabla_I \int_0^{N_r I} \hat{V}_r^T(I_r)\,\mathrm{d}I_r$ and $B\Delta h_p = \nabla \int_0^{N_p I} V_p^T\,\mathrm{d}I_p$ if one assumes that the resistors are linear. A difference in this case is that the third method could also solve it quite easily if the resistors were not linear.
This last method seems to be the method to use when we look at a network with a certain, large, number of end-users. Assume that there is a network as is depicted in Figure 11. The number of end-users is n. Suppose that the resistors are linear, for simplicity, and suppose that each end-user has the same structure as is depicted in Figure 11. The elements of the network are numbered in the same way as before. The last assumption is that there is only one extra pump, placed after the central heating element. This is also just for simplicity. In this case, there are $3n$ inductors, $n+1$ voltage sources and $4n+1$ resistors. We are only going to look at the flows through the end-user inductors, so there are n unknowns. The Hamiltonian in this system is given by the energy stored in each conductor, expressed in the flows through the chords. So:

$$H(I) = \frac{1}{2}L_1 I_{L_1}^2 + .. + \frac{1}{2}L_n I_{L_n}^2 + \frac{1}{2}(L_{n+1} + L_{n+2})(I_{L_1} + .. + I_{L_n})^2 + .. + \frac{1}{2}(L_{3n-1} + L_{3n})I_{L_n}^2$$

$$\Rightarrow H(I) = \frac{1}{2}\sum_{i=1}^{n} L_i I_{L_i}^2 + \frac{1}{2}\sum_{j=1}^{n}(L_{2j+1} + L_{2j+2})(\sum_{i=j}^{n} I_{L_i})^2$$

The time derivative of the first order gradient of the Hamiltonian becomes:

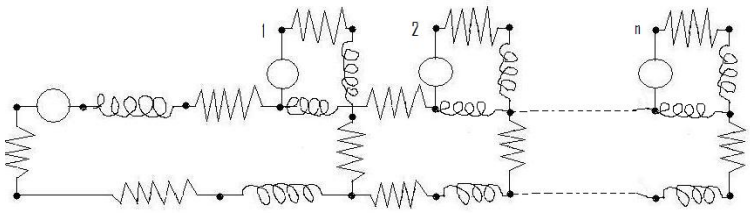

Figure 11: A RL-circuit with n end-users

$$\frac{d}{dt}\nabla_I H(I) = \begin{pmatrix} L_1\dot{I}_{L_1} + \sum_{j=1}^{1}(L_{2j+1}+L_{2j+2})\sum_{i=j}^{n}\dot{I}_{L_i} \\ L_2\dot{I}_{L_2} + \sum_{j=1}^{2}(L_{2j+1}+L_{2j+2})\sum_{i=j}^{n}\dot{I}_{L_i} \\ \vdots \\ L_n\dot{I}_{L_n} + \sum_{j=1}^{n}(L_{2j+1}+L_{2j+2})\sum_{i=j}^{n}\dot{I}_{L_i} \end{pmatrix} \tag{29}$$

The content function $P(I)$ is given by:

$$\begin{aligned} P(I) =& \frac{1}{2}(R_1+R_5)I_{L_1}^2 + \frac{1}{2}\sum_{i=2}^{n}(R_{4i-2}+R_{4i+1})I_{L_i}^2 + \frac{1}{2}(R_2+R_3+R_4)(\sum_{i=1}^{n}I_{L_i})^2 \\ &+\frac{1}{2}\sum_{j=2}^{n}(R_{4j}+R_{4j-1})(\sum_{i=j}^{n}I_{L_i})^2 - V_{p1}\sum_{i=1}^{n}I_{L_i} - \sum_{i=1}^{n}V_{p(i+1)}I_{L_i} \end{aligned}$$

The gradient of $P(I)$ with respect to the flows through the chords is given by:

$$\nabla_I P(I) = \begin{pmatrix} (R_1+R_5)I_{L_1} + (R_2+R_3+R_4)\sum_{i=1}^{n}I_{L_i} - V_{p1} - V_{p2} \\ (R_6+R_9)I_{L_2} + (R_2+R_3+R_4)\sum_{i=1}^{n}I_{L_i} + \sum_{j=2}^{2}(R_{4j}+R_{4j-1})\sum_{i=j}^{n}I_{L_i} - V_{p1} - V_{p3} \\ \vdots \\ (R_{4n-2}+R_{4n+1})I_{L_n} + (R_2+R_3+R_4)\sum_{i=1}^{n}I_{L_i} + \sum_{j=2}^{n}(R_{4j}+R_{4j-1})\sum_{i=j}^{n}I_{L_i} - V_{p1} - V_{pn} \end{pmatrix} \tag{30}$$

Now we have a model for a random number of end-users which form a network as in Figure 11. The equation is:

$$-\frac{d}{dt}\nabla_I H(I) = \nabla_I P(I)$$

with $H(I)$ as in (29) and $P(I)$ as in (30). To complete this section, note that this is just the Port-Hamiltonian related form. If you add the equations:

$$\begin{pmatrix} I_{L_1} \\ I_{L_2} \\ \vdots \\ I_{L_n} \end{pmatrix} = \begin{pmatrix} \frac{\phi_1}{L_1} \\ \frac{\phi_2}{L_2} \\ \vdots \\ \frac{\phi_n}{L_n} \end{pmatrix}$$

one obtains the Port-Hamiltonian model.
The matrices of the Port-Hamiltonian are given by:

$$R(\phi) = -\begin{pmatrix} R_1 + R_5 + R_2 + R_3 + R_4 & \dots & R_2 + R_3 + R_4 \\ R_2 + R_3 + R_4 & \dots & R_2 + R_3 + R_4 + R_8 + R_7 \\ \vdots & \vdots & \vdots \\ R_2 + R_3 + R_4 & \dots & R_{4n-2} + R_{4n+1} + R_2 + R_3 + R_4 + \sum_{j=2}^{n} (R_{4j} + R_{4j-1}) \end{pmatrix}$$

which is diagonal. $\frac{\partial H}{\partial \phi}(\phi) = \begin{pmatrix} \frac{\phi_1}{L_1} \\ \vdots \\ \frac{\phi_n}{L_n} \end{pmatrix}$, $J(\phi) = 0$ and $g = 1$.

6 Conclusion

We have seen three different models for solving hydraulic networks. The first model is a great way to solve the simpler networks with just one end-user, i.e., with no parallel parts. It fails however, when there are more end-users. The second and the third model can be used to solve the networks with more than one end-user. The latter has the benefit that it is quicker and it can solve networks with non-linear resistors more easily. So the conclusion is that the Brayton-Moser way the best way is write solve a hydraulic network in a Port-Hamiltonian form.
In the beginning of the paper, three assumptions were made. But are these assumptions necessary for the different methods to work?
The first assumptions stated that the network one is looking at is a connected graph. This assumption is very logical. If the graph is not connected, the network falls apart in different connected components. The third assumption stated that there is one central heating element that lies in all the fundamental loops. We already mentioned why this was logical; the whole streetblock is getting hot water due to the same heating element.
It is the second assumption that is interesting to look at. It stated that each end-user valve is in series with a pump and a pipe. Also, the chords of the graph correspond to end-user pipes. The structure at the end-user can be explained if one looks at the real world model. The pump in front of the valve is logical, it controlls the pressure at the valves of the household. The pipe after the end-user valve is again logical, the water should return to the central heating element if it has passed the household. So this structure could be explained in this way. However, our three different models would still work when this structure is altered.
Each end-user part of the network starts at a node in the network where the water splits into two different edges and it ends at the node where two flows meet each other again. This end-user part is only in series. This means that the flow through each element in this part is equal to any other flow through an element in this part. So the second part of assumption 2 can be changed; it does not matter which edge of the end-user part one chooses to be the chord. Any of these edges will satisfy, the flow is the same in the whole end-user part. The reason why assumption 2 chooses the end-user pipes is that in this case one expresses the flow through each element in the flows through the end-user pipes (the inductor part). The benefit of this, is that partial derrivative of the Hamiltonian, $\frac{\partial H}{\partial \phi}(\phi)$, is equal to the flow through the end-user pipes (the inductor in the electrical variant). So one obtains the Port-Hamiltonian form, when one chooses the end-user pipes as the chords, quicker then when one chooses a different element of the end-user part to be the chord. When one chooses a different element to be the chord, the models still works, but in the end one has to state that the flow through each element in the end-user part is the same, and then express the flows through the new chords in the flows through the end-user pipes.
So the second assumption is reasonable, but if altered as stated above, the

three models still work.

7 References

1 C. De Persis and C. Skovmose Kallesoe. "Pressure Regulation in Nonlinear Hydraulic Networks by Positive and Quantized Controls." *IEEE Transactions on control systems technology, vol. 19, NO. 6, November 2011.*

2 D. Jeltsema and J. Scherpen. "Multidomain Modeling of Nonlinear Networks and Systems: Energy and power-based perspectives." *IEEE control systems magazine. August 2009.*