Robust Synchronization of Coprime Factor Perturbed Multi-Agent Systems

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Abstract

This report deals with robust synchronization of undirected multi-agent networks with uncertain agent dynamics. Given an undirected network with identical nominal dynamics for each agent, we allow uncertainty in the form of coprime factor perturbations of the transfer matrix of the agent dynamics. We assume that these perturbations are stable and have \mathcal{H}_{∞} -norm that is bounded by some a priori given desired tolerance. In this report, we derive state space equations for dynamic observer based protocols that achieve robust synchronization for all such perturbations. We show that robust synchronization of a single linear system by all controllers from a related finite set of feedback controllers. Our protocols are expressed in terms of real symmetric solutions to certain algebraic Riccati equations, and contain weighting factors depending on the eigenvalues of the graph Laplacian. We show that in this class of dynamic protocols, one can achieve a guaranteed tolerance that is proportional to the square root of the quotient of the smallest and the largest eigenvalue of the graph Laplacian.

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Introduction

Recent years have shown an increased interest in networks of systems and distributed control. Much research has been done on control of networked multi-agent systems using only locally available information. A multi-agent system is a dynamical system that consists of multiple input-output systems that interact by interchanging information locally. The input-output systems are called the *agents* of the network. The possibility of interconnection between the agents of the network is represented by a graph, called the *network graph*. The vertices of the network graph represent the agents, while the edges of the graph represent the interaction topology. The network graph can be either undirected or directed, depending on the context. In this report, we assume that the network graph is undirected. An important object in the theory of multi-agent systems is the *Laplacian matrix* of the network graph. Important properties of networked systems can be expressed in terms of the spectrum of the Laplacian, see [4], [12].

In the network, each agent exchanges information with its neighbors. For a given form of this information, the overall dynamics of the network is determined by the dynamics of the individual agents together with the interconnection with their neighbors. A form of information exchange is called a *protocol*. It is desired that these protocols use only locally available information. Such a protocol works as a feedback controller on the overall network, while the feedback processor for each agent uses only information available from its neighbors. In the theory of networked multi-agent systems, an important problem is the design of protocols that yield a desired behavior of the network as a whole.

Different problem formulations from various areas of application involving interconnected dynamical systems that exchange output information can be cast into the multiagent system framework. One such well-known problem formulation is the *consensus problem*. This problem has been extensively studied in [7] and [8]. More recent work considering this problem can be found in [5], [1]. In this set-up, the agents only exchange information with their neighbors. The aim of this exchange of information is to reach agreement on certain quantities that depend on the internal states of all agents. A communication protocol that achieves this agreement is said to achieve consensus. Strongly related to the consensus problem is the *synchronization problem*. In this problem, the agents have identical dynamics and the goal is to establish conditions under which the states of all agents converge to a single trajectory. If this is indeed the case, the network is said to be synchronized.

The protocols used to obtain synchronization are only allowed to use relative state or output information of the neighboring agents to achieve synchronization. If the state or relative state of each agent is available, a static protocol is sufficient to obtain synchronization. However, sometimes the state or the relative state can not be obtained directly. In this case, one can use an *observer based protocol*, which consists of a dynamic part that acts as an observer for the state or relative state of each agent and a static part that feeds back the estimated state or relative state to the agents. In this report, we will provide necessary and sufficient conditions for the existence of such protocols. We will give state space representations for observers that estimate either the state or the relative state of the agents and will give protocols that use these observers to achieve synchronization.

Next, we will extend the results on synchronization using observer based protocols to the problem of *robust synchronization* of linear multi-agent systems. In this situation, the agents of the system share identical *nominal* dynamics. However, the dynamics of each agent contains an *uncertainty* in the sense that the transfer matrix of each agent is a *coprime factor perturbed* variation of the nominal transfer matrix. We assume that this perturbation is stable and is bounded in \mathcal{H}_{∞} -norm by an a priori given tolerance. The problem of robust synchronization is to find conditions under which there exist dynamic protocols that synchronize the network under *all* such allowed perturbations of the agent dynamics.

For networks in which the uncertainties are modeled by *additive perturbation* of the agent dynamics, conditions for the existence of robustly synchronizing dynamic protocols and methods to obtain such protocols were established in [10]. In this report, we will extend these results to networks with coprime factor perturbed agent dynamics. We will show that these protocols depend on the nominal agent dynamics as well as the smallest and largest eigenvalue of the Laplacian of the network graph. For a given network and nominal agent dynamics one wants to find a *maximally permitted tolerance* for which there exist robustly synchronizing dynamic protocols. In this report, we will show that for undirected graphs, using this class of dynamic observer based protocols, one can obtain a guaranteed synchronization radius that is proportional to the square root of the quotient of the smallest and largest eigenvalue of the graph Laplacian.

The outline of this report is as follows. In Section 2, we introduce the notation that will be used throughout this report. We will also introduce some basic graph theory, specifically on the graph Laplacian, that will be needed in the rest of this report. Finally, we introduce a version of the bounded real lemma, that will be instrumental in the proof of our main result on robust synchronization. In Section 3 we provide state space representations for observers for the absolute and relative state of the agents. In Section 4 we will use these observers to construct a dynamic protocol that synchronizes the network and establish conditions under which such protocols exist. Next, in Section 5 we explain the theory behind coprime factor perturbation of transfer matrices. In Section 6, we will formulate the problem of robust synchronization under coprime factor perturbation of the nominal agent dynamics. We show that the problem is equivalent to simultaneous robust stabilization of a single linear system by all controllers from a finite set of related controllers. Then, in Section 7 we formulate our main result and give conditions under which there exist dynamic protocols that robustly synchronize the network. We will provide methods to compute such protocols and show how they depend on the nominal agent dynamics and the smallest and largest eigenvalue of the graph Laplacian.

Preliminaries

In this report we consider multi-agent systems whose interconnection structures are described by undirected unweighted graphs. An undirected graph consists of a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, p\}$ is the set of *nodes* or vertices, and where $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of *edges*. A pair $(i, j) \in \mathcal{E}$, with $i, j \in \mathcal{V}$ and $i \neq j$, represents an edge from node *i* to node *j*. For an undirected graph, if $(i, j) \in \mathcal{E}$ then also $(j, i) \in \mathcal{E}$. An undirected graph is said to be connected if for any pair of distinct nodes $i, j \in \mathcal{V}$ there exists a path from *i* to *j*. For a given vertex *i*, the *neighboring set* \mathcal{N}_i is defined by $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. The degree of a vertex *i* is denoted by deg(*i*) and is defined as deg(*i*) = card(\mathcal{N}_i). The Laplacian matrix *L* of a graph \mathcal{G} with *p* nodes has size $p \times p$ and is defined by

$$L_{i,j} = \begin{cases} \deg(i) & \text{if } i = j, \\ -1 & \text{if } (i,j) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian of an undirected graph is symmetric and consequently has real eigenvalues. For an undirected graph all eigenvalues of the Laplacian are non-negative. The graph is connected if and only if zero is a simple eigenvalue, with a corresponding right eigenvector the *p*-dimensional vector with all entries equal to one. We denote this vector by 1_p . We order the eigenvalues λ_i for i = 1, 2, ..., p of the Laplacian of a connected graph as

$$0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_p.$$

Also, since the Laplacian is symmetric, it can be diagonalized by an orthogonal transformation U that brings it to the following form:

$$U^{T}LU = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{p} \end{pmatrix}.$$

In this report, we denote the set of all proper and stable rational matrices by \mathcal{RH}_{∞} . If $G \in \mathcal{RH}_{\infty}$, then let $||G||_{\infty}$ denote its \mathcal{H}_{∞} -norm, $||G||_{\infty} = \sup_{\mathfrak{R}(\lambda) \geq 0} ||G(\lambda)||$. A square matrix $H \in \mathbb{R}^{n \times n}$ is called Hurwitz if all its eigenvalues have strictly negative real part.

2.1 Notation

Let \mathbb{R} denote the field of real numbers, \mathbb{R}^n the *n*-dimensional Euclidean space and $\mathbb{R}^{n \times n}$ the space of $n \times n$ real matrices. Denote the field of complex numbers by \mathbb{C} and let $\Re(\rho)$ denote the real part of the complex number ρ . Let I_p denote the identity matrix of dimension p and I the identity matrix of appropriate dimension. The tensor or *Kronecker* product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn}B \end{pmatrix}.$$

The Kronecker product satisfies the following properties:

$$(A \otimes B)(C \otimes D) = (AC) \otimes BD,$$

$$(A \otimes B)^T = A^T \otimes B^T,$$

$$A \otimes B + A \otimes C = A \otimes (B + C).$$

This report will use results on the existence of robustly stabilizing controllers from the theory of \mathcal{H}_{∞} control. The \mathcal{H}_{∞} problem was first formulated in [14] and the first solutions in a state space setting were provided in groundbreaking work in [2]. Perhaps the best known lemma in the theory of \mathcal{H}_{∞} control is the *bounded real lemma*. It provides methods to determine the \mathcal{H}_{∞} -norm of a given system. The bounded real lemma can be used in combination with the *small gain theorem*, which has been extensively studied in [13], to show whether the interconnection of systems with a feedback loop is internally stable. The bounded real lemma is as follows:

Lemma 2.1. Assume we have the following system with (C, A) detectable:

$$\dot{x} = Ax + Bu,$$

$$y = Cx + Du.$$

Let $G(s) = C(sI - A)^{-1}B + D$ denote the transfer matrix of the system. Let $\rho > 0$. Then A is Hurwitz and the \mathcal{H}_{∞} -norm of G is strictly less then ρ if and only if $||D|| < \rho$ and there exists a real symmetric solution P of the algebraic Riccati equation

$$A^{T}P + PA + C^{T}C + (PB + C^{T}D)(\rho^{2}I - D^{T}D)^{-1}(B^{T}P + D^{T}C) = 0,$$
(2.1)

such that the following matrix is Hurwitz:

$$A + B(\rho^2 I - D^T D)^{-1}(B^T P + D^T C).$$

For a proof of this lemma we refer to [9]. Now we present a version of the bounded real lemma, adapted for our purposes.

Lemma 2.2. Consider the system $\dot{x} = Ax + Bu$, y = Cx + Du with transfer function $G(s) = C(sI - A)^{-1}B + D$. Assume $D^TD = I$ and that A is Hurwitz. Let $\rho > 1$. The \mathcal{H}_{∞} -norm $||G||_{\infty}$ of the operator from u to y satisfies $||G||_{\infty} \leq \rho$ if there exists a real symmetric positive semi-definite solution P to the Riccati inequality

$$A^{T}P + PA + C^{T}C + \frac{1}{\rho^{2} - 1}(PB + C^{T}D)(B^{T}P + D^{T}C) \le 0.$$
(2.2)

Proof. Assume that P is a solution to (2.2). We have

$$\begin{split} \frac{\mathrm{d}}{\mathrm{dt}} x^T P x &\leq x^T (A^T P + P A) x + u^T B^T P x + x^T P B u, \\ &\leq -\frac{1}{\rho^2 - 1} x^T (P B + C^T D) (B^T P + D^T C) x \\ &+ u^T B^T P x + x^T P B u - x^T C^T C x, \\ &= - ||\sqrt{\rho^2 - 1} u - \frac{1}{\sqrt{\rho^2 - 1}} (B^T P + D^T C) x||^2 \\ &+ \rho^2 ||u||^2 - ||y||^2, \\ &\leq \rho^2 ||u||^2 - ||y||^2. \end{split}$$

Here, the second inequality follows directly from (2.2). Now we take $x(0) = 0, u \in \mathcal{L}_2(\mathbb{R}^+)$ and integrate from 0 to ∞ , which yields $0 \leq \rho^2 ||u||_2^2 - ||y||_2^2$. We obtain that $||y||_2^2 \leq \rho^2 ||u||_2^2$ for all $u \in \mathcal{L}_2(\mathbb{R}^+)$. This implies that the operator norm $||G||_{\infty}$ satisfies $||G||_{\infty} \leq \rho$.

Network observers

In this section, we provide an introduction to the relevant theory of observers for linear systems. Later, we will use these observers in the synthesis of synchronizing protocols for networked multi-agent systems. We provide state space equations for observers of the internal state of each agent, which we call the *absolute state*, and of observers for the *relative state* of each agent, which is the sum of differences in state of an agent with its neighbors.

Consider the system Σ described by

$$\dot{x} = Ax + Bu,$$

$$y = Cx.$$
(3.1)

Here the state x takes its values in \mathbb{R}^n , output y takes its values in \mathbb{R}^r and input u has values in \mathbb{R}^m . The matrices A, B and C are of appropriate dimensions. We want to approximate the state x by the output ξ of an observer, using the input u and the output y of the system. The system that models the observer has the following form:

$$\begin{split} \dot{w} &= Pw + Qu + Ry, \\ \xi &= Sw. \end{split}$$

Interconnecting this system with system (3.1), we obtain the following dynamics:

$$\dot{x} = Ax + Bu,
\dot{w} = Pw + Qu + RCx,
\xi = Sw.$$
(3.2)

Now we introduce the error $e := \xi - x$ as the difference between the estimate ξ and the actual state x. The error dynamics is as follows:

$$\dot{e} = SPw + SQu + SRCx - Ax - Bu,$$

= (SP + SRCS - AS)w - (SRC - A)e + (SQ - B)u. (3.3)

Now, we provide the definition of an observer for a given system. Then we will establish necessary and sufficient conditions for the existence of such an observer. For more information on this topic, we refer to [9].

Definition 3.1. A system Ω is called a state observer for Σ if for any pair of initial values x_0, w_0 satisfying e(0) = 0, for arbitrary input function u, we have e(t) = 0 for all t > 0.

Definition 3.2. An observer Ω is called stable if for each pair of initial values x_0, w_0 we have $e(t) \to 0$ $(t \to \infty)$.

Given initial conditions x(0) = w(0), Definition 3.1 requires that the error e(t) remains zero for all t > 0, for every input u. Thus the error dynamics (3.3) should be independent of u. This implies that SQ = B. The same requirement holds for the coefficient of w. Hence SP = AS - SRCS. This leads to the following, simplified the expression for the dynamics of e:

$$\dot{e} = (A - SRC)e.$$

Substitute this expression into (3.2) to obtain

$$\begin{split} \xi &= S\dot{w} \\ &= SPw + SQu + SRy \\ &= (A - SRC)\xi + Bu + SRy \end{split}$$

Denote G := SR. This leads to the following expression for the observer dynamics.

$$\dot{\xi} = (A - GC)\xi + Bu + Gy, \tag{3.4}$$

Now, the error dynamics is given by

$$\dot{e} = (A - GC)e. \tag{3.5}$$

From this it follows that $e(t) \to 0$ as $t \to \infty$ if and only if A-GC is Hurwitz. Consequently, a necessary and sufficient condition for the existence of a stable observer for x is that there exists a G such that A-GC is Hurwitz. This statement is captured in the following lemma.

Lemma 3.3. There exists a stable observer for the system

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

if and only if (C, A) is detectable.

Next, we introduce network observers. A network observer is a system that observes the aggregate state of all agents in a network. Let $\mathbf{x} = \operatorname{col}(x_1, x_2, \ldots, x_p)$ denote the aggregate state of the individual agents, and $\mathbf{y} = \operatorname{col}(y_1, y_2, \ldots, y_p)$ and $\mathbf{u} = \operatorname{col}(u_1, u_2, \ldots, u_p)$ the aggregate output and input, respectively. The dynamics of \mathbf{x} and \mathbf{y} is given by

$$\dot{\mathbf{x}} = (I \otimes A)\mathbf{x} + (I \otimes B)\mathbf{u},$$
$$\mathbf{y} = (I \otimes C)\mathbf{x}.$$

A network observer for the aggregate *absolute state* \mathbf{x} has the form

$$\dot{\mathbf{w}} = P\mathbf{w} + Q\mathbf{u} + R\mathbf{y},$$

$$\xi = S\mathbf{w},$$

which has the same structure as an observer for standard LTI systems. We can now use (3.4) to simplify the expressions for this observer to

$$\dot{\xi} = ((I \otimes A) - \tilde{G}(I \otimes C))\xi + (I \otimes B)\mathbf{u} + \tilde{G}\mathbf{y},$$

with $\tilde{G} \in \mathbb{R}^{pn \times pr}$. Is is sufficient to consider matrices \tilde{G} such that $\tilde{G} = I \otimes G$ for $G \in \mathbb{R}^{n \times p}$. We will show that the pair $((I \otimes C), (I \otimes A))$ is detectable if and only if the pair (C, A) is detectable. Theorem 3.38 from [9] states that the pair (C, A) is detectable if only every eigenvalue λ of A such that

$$\operatorname{rank} \begin{pmatrix} A - \lambda I_n \\ C \end{pmatrix} < n,$$

lies in the open left half plane. Since $(I \otimes A)$ and A have identical eigenvalues, it follows that the pair $((I \otimes C), (I \otimes A))$ is detectable if and only if every eigenvalue λ of A, such that

$$\operatorname{rank} \begin{pmatrix} (I \otimes A) - \lambda I_{pn} \\ I \otimes C \end{pmatrix} = \operatorname{rank} \begin{pmatrix} I \otimes \begin{pmatrix} A - \lambda I_n \\ C \end{pmatrix} \end{pmatrix}$$
$$= p \cdot \operatorname{rank} \begin{pmatrix} A - \lambda I_n \\ C \end{pmatrix} < pn,$$

also lies in the open left half plane. This holds if and only if (C, A) is detectable. So a necessary and sufficient condition for the existence of a stable network observer is that there exists a G such that A - GC is Hurwitz. In that case we can choose $\tilde{G} = I \otimes G$. It follows that $(I \otimes A) - \tilde{G}(I \otimes C) = I \otimes A - GC$ is Hurwitz. In the next chapters of this report, we will restrict ourselves to observers of the form

$$\dot{\xi} = (I \otimes A - GC)\xi + (I \otimes B)u + (I \otimes G)y.$$

Definition 3.4. A system Ω is called a network observer for the aggregate *absolute state* \mathbf{x} of a multi-agent system Σ with estimation error $\mathbf{e} := \xi - \mathbf{x}$ if for any pair of initial values \mathbf{x}_0 , ξ_0 satisfying $\mathbf{e}(0) = 0$, for arbitrary input function \mathbf{u} , we have $\mathbf{e}(t) = 0$ for all t > 0.

Now, let G be such that A - GC is Hurwitz. Let x_i denote the state of the *i*th agent in the network. Take ξ_i as an observer for the absolute state x_i of agent *i* with dynamics (3.4). As before, let **x** denote the aggregate state vector of the network en let ξ denote the aggregate state of all observers. Then, the dynamics of ξ is given by

$$\xi = (I \otimes A - GC)\xi + (I \otimes B)\mathbf{u} + (I \otimes G)\mathbf{y}$$

We will show that ξ is a stable observer for the overall network. Denote the aggregate error by $\mathbf{e} := \xi - \mathbf{x}$. The error dynamics is given by

$$\begin{split} \dot{\mathbf{e}} &= \dot{\xi} - \dot{\mathbf{x}} \\ &= (I \otimes A - GC)\xi + (I \otimes B)\mathbf{u} + (I \otimes GC)\mathbf{x} - (I \otimes A)\mathbf{x} - (I \otimes B)\mathbf{u} \\ &= (I \otimes A - GC)(\xi - \mathbf{x}) \\ &= (I \otimes A - GC)\mathbf{e}. \end{split}$$

We see that $\mathbf{e}(t) \to 0$ as $t \to \infty$ if (A - GC) is Hurwitz. A necessary and sufficient condition is detectability of (C, A). These observations are captured in the following theorem.

Theorem 3.5. There exists a network observer Ω for the aggregate state \mathbf{x} of a multiagent system Σ if and only if (C, A) is detectable.

3.1 Relative state observers

In the previous section, we provided state space equations for observers for the absolute state x_i . We assumed that the *absolute output* y_i of each agent could be measured. Sometimes this is not the case. Take for instance a group of satellites flying in formation in deep space. Measuring their absolute position with respect to an origin far away could be inaccurate or impossible. Now, they could try to determine their position relative to their neighbors. Denote this *relative state* by

$$\phi_i := \sum_{j \in N} x_i - x_j. \tag{3.6}$$

When this relative state is not directly available for measurement we can try to construct an observer for it using the *relative output*. The relative output of the *i*th agent is given by

$$\zeta_i := \sum_{j \in \mathcal{N}_i} y_i - y_j = C \sum_{j \in \mathcal{N}_i} x_i - x_j = C\phi_i.$$

The dynamics of the relative state ϕ_i of agent *i* is given by

$$\dot{\phi}_i = \sum_{j \in \mathcal{N}_i} \dot{x}_i - \dot{x}_j$$
$$= A \sum_{j \in \mathcal{N}_i} x_i - x_j + B \sum_{j \in \mathcal{N}_i} u_i - u_j$$
$$= A \phi_i + B v_i,$$

where $v_i := \sum_{j \in \mathcal{N}_i} u_i - u_j$ denotes the relative input of the *i*th agent. Next, we construct an observer for the relative state ϕ_i :

$$\dot{w}_i = (A - GC)w_i + Bv_i + G\zeta_i. \tag{3.7}$$

If (C, A) is detectable, there exists a G such that A - GC is Hurwitz and the individual error for agent i defined as $e_i = w_i - \phi_i$ has the dynamics (3.5). Let $\phi := \operatorname{col}(\phi_1, \phi_2, \ldots, \phi_p)$ and $\mathbf{w} := \operatorname{col}(w_1, w_2, \ldots, w_p)$ denote the aggregate relative state of the agents and the aggregate state of the observers, respectively. Denote the aggregate relative output and relative input by $\zeta := \operatorname{col}(\zeta_1, \zeta_2, \ldots, \zeta_p)$ and $v := \operatorname{col}(v_1, v_2, \ldots, v_p)$. The dynamics of the network observer \mathbf{w} is given by

$$\dot{\mathbf{w}} = (I \otimes A - GC)\mathbf{w} + (I \otimes B)v + (I \otimes G)\zeta.$$

Now the error $e := \mathbf{w} - \phi$ has the following dynamics:

$$\dot{e} = (I \otimes A - GC)e.$$

Consequently, $e(t) \to 0$ as $t \to \infty$. We see that w is a stable observer for ϕ and detectability of (C, A) is a sufficient and necessary condition for the existence of such a network observer. Note that not all states $\phi \in \mathbb{R}^{pn}$ are feasible, since ϕ satisfies $\phi = (L \otimes I_n)\mathbf{x}$. However, this poses no problem as $e(t) \to 0$ for all initial conditions ϕ_0, \mathbf{w}_0 .

Synchronization

As before, we consider multi-agent system with p agents and we assume that the underlying network graph G is undirected and connected. The dynamics of agent i is given by

$$\dot{x}_i = Ax_i + Bu_i,$$

$$y_i = Cx_i$$
(4.1)

for i = 1, 2, ..., p. We see that the dynamics of each agent is given by one and the same linear system. We call this the *nominal system*. As before, we assume that the pair (A, B)is stabilizable and the pair (C, A) is detectable. The problem of synchronization is to find a protocol that makes the network synchronized. In this section, we will use protocols based on observers for the relative states of the agents. These observers provide us with estimates of the *relative states* $\sum_{j \in \mathcal{N}_i} (x_i - x_j)$. From Section 3.1, we obtain dynamics (3.7) for an estimate w_i of the relative state of the *i*th agent. We interconnect this estimate with the agent using the static feedback $u_i = Fw_i$. Substituting $v_i = \sum_{j \in \mathcal{N}_i} u_i - u_j$ and $\zeta_i = \sum_{j \in \mathcal{N}_i} y_i - y_j$ in (3.7) results in the protocol

$$\dot{w}_{i} = Aw_{i} + BF \sum_{j \in \mathcal{N}_{i}} (w_{i} - w_{j}) + G(\sum_{j \in \mathcal{N}_{i}} (y_{i} - y_{j}) - Cw_{i}), \ u_{i} = Fw_{i}.$$
(4.2)

Interconnecting the agents with this protocol yields the closed loop system. The dynamics of this system, the overall network dynamics, can be easily represented by taking $\mathbf{x} = \operatorname{col}(x_1, x_2, \ldots, x_p)$ and $\mathbf{w} = \operatorname{col}(w_1, w_2, \ldots, w_p)$ as the aggregate state vectors, $\mathbf{y} = \operatorname{col}(y_1, y_2, \ldots, y_p)$ and $\mathbf{u} = \operatorname{col}(u_1, u_2, \ldots, u_p)$ as the aggregate output and input vectors respectively. We obtain

$$\dot{\mathbf{x}} = (I \otimes A)\mathbf{x} + (I \otimes B)\mathbf{u}, \ \mathbf{y} = (I \otimes C)\mathbf{x}$$
(4.3)

and

$$\dot{\mathbf{w}} = (I \otimes (A - GC) + (L \otimes BF))\mathbf{w} + (L \otimes G)\mathbf{y}, \ \mathbf{u} = (I \otimes F)\mathbf{w}.$$
(4.4)

Then the network dynamics is given by

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ L \otimes GC & I \otimes (A - GC) + (L \otimes BF) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}.$$
(4.5)

Definition 4.1. We say that the protocol synchronizes the network if for all i, j = 1, 2, ..., p

$$x_i(t) - x_j(t) \rightarrow 0, \ w_i(t) - w_j(t) \rightarrow 0$$

as $t \to \infty$.

Now let U be an orthogonal $p \times p$ matrix that diagonalizes L. We define

$$\Lambda := U^T L U = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_p \end{pmatrix}.$$

Applying the state transformation

$$\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} U^T \otimes I & 0 \\ 0 & U^T \otimes I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix},$$
(4.6)

we obtain as new equation for the network dynamics

$$\begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ \Lambda \otimes GC & I \otimes (A - GC) + (\Lambda \otimes BF) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix}.$$
(4.7)

It is well known that synchronization of the network is equivalent with the stability of p-1 systems, see [5], [10]:

Lemma 4.2. The network is synchronized if and only if for p = 2, 3, ..., p the systems

$$\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} = \begin{pmatrix} A & BF \\ \lambda_i GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix}$$
(4.8)

are stable.

Proof. Define a $(p-1) \times p$ matrix H by

$$H := \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix}.$$
 (4.9)

Now ker $(H) = \operatorname{im}(1_p)$, where $1_p = (1, 1, \ldots, 1)^T$ in \mathbb{R}^p . As before, let U be an orthogonal matrix such that $U^T L U = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_p)$. Now the first column of U is equal to $u_1 = \frac{1}{\sqrt{p}} 1_p$, the normalized vector of ones. Let U_2 be such that $U = (u_1 \ U_2)$. Now we have $HU = (0 \ HU_2)$, where HU_2 has full column rank. It is clear that $x_i(t) - x_j(t) \to 0$ for all i, j if and only of $(H \otimes I) \mathbf{x}(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ for all i, j if and only if $(H \otimes I) \mathbf{x}(t) \to 0$ and $w_i(t) - x_j(t) \to 0$ for all i, j if and only if only if $((0 \ HU_2) \otimes I) \mathbf{\tilde{x}}(t) \to 0$, or equivalently, $\mathbf{\tilde{x}}_i(t) \to 0$ for $i = 2, 3, \ldots, p$. The same holds for w_i , with $w_i(t) - w_j(t) \to 0$ for all i, j if and only if $\tilde{w}_i(t) \to 0$ for $i = 2, 3, \ldots, p$.

By applying the state transformation

$$\begin{pmatrix} \bar{x}_i \\ \bar{w}_i \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\lambda_i}I \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix}$$

we obtain that the network is synchronized if and only if for i = 2, 3, ..., p the systems

$$\begin{pmatrix} \dot{\bar{x}}_i \\ \dot{\bar{w}}_i \end{pmatrix} = \begin{pmatrix} A & \lambda_i BF \\ GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} \bar{x}_i \\ \bar{w}_i \end{pmatrix}$$
(4.10)

are stable. We can interpret this closed loop system as the feedback interconnection of the system

$$\dot{\bar{x}}_i = A\bar{x}_i + B\bar{u}_i, \ \bar{y}_i = C\bar{x}_i.$$
 (4.11)

with the controller

$$\bar{w}_i = A\bar{w}_i + B\bar{u}_i + (G\bar{y}_i - C\bar{w}_i), \ \bar{u}_i = \lambda_i F\bar{w}_i.$$
(4.12)

The set of eigenvalues of the system matrix in (4.10) is the union of the sets of eigenvalues of A - GC and $A + \lambda_i BF$. Now the following holds

Lemma 4.3. The dynamic protocol (4.2) synchronizes the network if and only if the system

$$\dot{x} = Ax + Bu, \ y = Cx,\tag{4.13}$$

is stabilized by all p-1 controllers

$$\dot{w} = Aw + Bu + G(y - Cw), \ u = \lambda_i Fw, \ i = 2, 3, \dots, p.$$
(4.14)

This holds if and only if A - GC and $A + \lambda_i BF$ (i = 2, 3, ..., p) are Hurwitz.

In this section, we have obtained results that show the dynamic protocol (4.2) synchronizes the network if and only if the gain F and G are chosen in such a way that the system (4.13) is stabilized by all p-1 controllers (4.14). Such F and G exist if and only if (A, B) is stabilizable and (C, A) is detectable. It is obvious that detectability of (C, A) is a necessary conditions. In [5] and [1], it has been shown that stabilizability of (A, B) is necessary and sufficient for the existence of a single F such that $A + \lambda_i BF$ (i = 2, 3, ..., p)is Hurwitz. In the next chapter, we will show that also the *robust synchronization* problem can be reformulated as a robust stabilization problem using, for a given plant, a set of p-1 controllers.

4.1 Synchronization by state feedback protocol

In the remaining chapters of this report we focus on dynamic protocols using network observers for the relative states of the agents. However, it is worth investigating other forms of protocols, such as static state feedback protocols and observer based protocols using network observers for the absolute state. It has been shown in [5] that under the assumption that the state or relative state of each agent is directly available for measurement, a static protocol which feeds back the relative state to each agent is sufficient for synchronization. We will proof this statement in the current section.

Again, consider the network with agent dynamics

$$\dot{x}_i = Ax_i + Bu_i \ (i = 1, 2, \dots, p) \tag{4.15}$$

We assume that the state x_i of each agent is directly available for use in a protocol. We will show that the following protocol will synchronize the network for a suitable matrix F:

$$u_i = F \sum_{j \in \mathcal{N}_i} x_i - x_j. \tag{4.16}$$

Interconnecting with protocol (4.16) yields the following dynamics for agent *i*:

$$\dot{x}_{i} = Ax_{i} + BF \sum_{j \in \mathcal{N}_{i}} x_{i} - x_{j} = Ax_{i} + BF \sum_{j=1}^{p} L_{ij}x_{j}.$$
(4.17)

As before, let \mathbf{x} denote the aggregate agent state vector and \mathbf{u} the aggregate input vector. The dynamics of the overall network is given by

$$\dot{\mathbf{x}} = (I \otimes A) + (I \otimes B)\mathbf{u},$$

= $(I \otimes A) + (I \otimes B)(L \otimes F)\mathbf{x},$
= $(I \otimes A + L \otimes BF)\mathbf{x}.$

Choose H as in (4.9). Recall that the network is synchronized if and only if $(H \otimes I_n)\mathbf{x} \to 0$ as $t \to \infty$. Again, let U be an orthogonal matrix such that $U^T L U = \Lambda$ is diagonal and $U = (\frac{1}{\sqrt{p}} \mathbf{1}_p \ U_2)$ for a certain U_2 . Recall that HU_2 has full column rank. Apply the state transformation $\tilde{\mathbf{x}} = (U^T \otimes I)\mathbf{x}$ to obtain the following dynamics for $\tilde{\mathbf{x}}$:

$$\dot{\tilde{\mathbf{x}}} = (I \otimes A + \Lambda \otimes BF)\tilde{\mathbf{x}}.$$

From the first part of this chapter we have that the network is synchronized if and only if $(H \otimes I)\mathbf{x} \to 0$ as $t \to \infty$, equivalently $((0 \ HU_2) \otimes I)\tilde{\mathbf{x}} \to 0$ as $t \to \infty$. We obtain that the network is synchronized if and only if for $i = 2, 3, \ldots, p$ the systems

$$\tilde{x}_i = (A + \lambda_i BF) \tilde{x}_i \tag{4.18}$$

are stable. We make the following observation:

Lemma 4.4. Consider the network with agent dynamics as in (4.15). Assume the network graph is undirected en connected. Then the protocol (4.16) synchronizes the network if and only if for i = 2, 3, ..., p each matrix

$$A + \lambda_i BF \tag{4.19}$$

is Hurwitz.

The next question is whether we can find a matrix F that satisfies these requirements. We will now show how to find such matrix F. First choose C such that (C, A) is detectable. Now consider the agent dynamics (4.15) and temporarily denote $y_i = Cx_i$. Since (A, B) is stabilizable and (C, A) is detectable, there exists a stabilizing symmetric positive semidefinite solution P to the algebraic Riccati equation associated with (A, B, C):

$$A^{T}P + PA - PBB^{T}P + C^{T}C = 0, (4.20)$$

such that $A - BB^T P$ is Hurwitz. Choose $F = -\alpha B^T P$, where α is such that $1 - 2\alpha \lambda_i < 0$, equivalently $\frac{1}{2\lambda_i} < \alpha$ for i = 2, 3, ..., p. Since $\lambda_2 \leq \lambda_i$ for all i, we can simply choose $\alpha > \frac{1}{2\lambda_2}$. Now it holds that

$$(A - \alpha\lambda_i BB^T P)^T P + P(A - \alpha\lambda_i BB^T P) = A^T P + PA - 2\alpha\lambda_i PBB^T P$$
$$= -C^T C + (1 - 2\alpha\lambda_i) PBB^T P.$$

The last equality follows directly from (4.20). Let ρ be an eigenvalue of $A - \alpha \lambda_i BB^T P$ with corresponding right eigenvector v. We obtain that $2\Re(\rho)v^*Pv = -||Cv|| + (1 - 2\alpha\lambda_i)||B^TPv||$. First, consider the case that $v^*Pv = 0$. Since α was chosen such that $1 - 2\alpha\lambda_i < 0$ we must have that $||B^TPv|| = 0$. However, from this it follows that $(A - BB^TP)v = \rho v$. Since $A - BB^TP$ is Hurwitz, we have $\Re(\rho) < 0$. In the case that $v^*Pv > 0$ we directly obtain $\Re(\rho) \leq 0$. Now suppose $\Re(\rho) = 0$, it follows that Cv = 0and $B^TPv = 0$. We obtain that $Av = \rho v$ while Cv = 0, so (C, A) has an unstable and unobservable eigenvalue. This contradicts our assumption that (C, A) is detectable. Hence $\Re(\rho) < 0$ and $A - \alpha\lambda_i BB^TP$ is Hurwitz.

Thus, choosing F as above makes the matrices (4.19) Hurwitz. We apply Lemma 4.4 and obtain that the network is synchronized by the protocol (4.16).

4.2 Synchronization by absolute state observer based protocols

We have already provided conditions under which there exists a synchronizing protocol using relative state observers for the network. In this section we will proof that the same conditions hold for the existence of protocols using absolute state observers. From Chapter 3 we have the following dynamics for an *absolute state* observer for agent i:

$$\xi_i = (A - GC)\xi_i + Bu_i - GCx_i,$$

where G is such that A - GC is Hurwitz. Denote the aggregate state of the observers by $\xi = \operatorname{col}(\xi_1, \xi_2, \ldots, \xi_p)$. Applying the feedback $u_i = F \sum_{j \in \mathcal{N}_i} \xi_i - \xi_j$ results in the following dynamics for the aggregate state

 $\dot{\mathbf{x}} = (I \otimes A)\mathbf{x} + (L \otimes BF)\xi.$

We obtain that the dynamics of the aggregate observer state is given by

$$\dot{\xi} = (I \otimes (A - GC) + L \otimes BF)\xi + (I \otimes C)\mathbf{x}$$

Hence, the dynamics of the overall network is given by

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} I \otimes A & L \otimes BF \\ I \otimes GC & I \otimes (A - GC) + (L \otimes BF) \end{pmatrix} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\xi}} \end{pmatrix}.$$
(4.21)

This dynamics shows strong similarities with the network dynamics (4.7) in the case of the relative state observer based protocol, with only different antidiagonal terms. Take U as before and apply state transformation

$$\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} U^T \otimes I & 0 \\ 0 & U^T \otimes I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \xi \end{pmatrix}$$
(4.22)

This leads to the following expression for the network dynamics

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\xi} \end{pmatrix} \begin{pmatrix} I \otimes A & \Lambda \otimes BF \\ I \otimes GC & I \otimes (A - GC) + \Lambda \otimes BF \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\xi} \end{pmatrix}.$$
(4.23)

Next, we will provide a lemma which is analogous with Lemma 4.2. The lemma states that synchronization of the network is equivalent with simultaneous stabilization by p-1 related controllers. The validity of the lemma follows directly from the proof of Lemma 4.2.

Lemma 4.5. The network is synchronized if and only if the systems

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} \tilde{x}_i \\ \tilde{\xi}_i \end{pmatrix} = \begin{pmatrix} A & \lambda_i BF \\ GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{\xi}_i \end{pmatrix}$$
(4.24)

are stable for $i = 2, 3, \ldots, p$.

The systems (4.24) are exactly the same as systems (4.10). Thus, we can directly apply Lemma 4.3 to obtain that the protocol

$$\dot{\xi}_i = (A - GC)\xi_i + BF \sum_{j \in \mathcal{N}_i} (\xi_i - \xi_j) + G(y_i - C\xi_i), \ u_i = F \sum_{j \in \mathcal{N}_i} (\xi_i - \xi_j)$$

synchronizes the network if and only if A - GC and $A + \lambda_i BF$ (i = 2, 3, ..., p) are Hurwitz. These conditions are identical to the ones under which there exists a relative state observer based protocol that synchronizes the network.

Coprime factor perturbations

Next to additive and multiplicative perturbations, coprime factor perturbations are among the best known models that represent uncertainty in the dynamics of linear systems. While it is obvious for both additive and multiplicative perturbations that they result in a linear disturbance, on first sight coprime factor perturbations seem highly nonlinear. This is caused by the way in which the disturbances appear in the transfer matrix of the nominal system. However, in this section we will show how a coprime factor perturbed nominal system can be interpreted as a feedback loop around some linear system associated with the nominal one, see [9]. This makes it easier to reason about the dynamics of the perturbed system and will allow us to consider controllers for linear networks in our next section. Examples of the use of coprime factorizations in control can be found in [11]. Coprime factor perturbations have been investigated in [3]. In the behavioral approach to control theory, the coprime factors is used as a kernel representation, see [6], [13].

Suppose we have the linear system

$$\dot{x} = Ax + Bu,$$
$$y = Cx.$$

The transfer matrix of this system is given by $G(s) = C(sI - A)^{-1}B$. A left-coprime factorization of G over the ring of proper stable rational functions is given by $G = M^{-1}N$, with M and N proper and stable and left-coprime, i.e. there exist proper stable matrices X and Y such that NX + MY = I. Such a coprime factorization is called normalized if $NN^* + MM^* = I$, where $N^*(s) = N^T(-s)$. In the rest of this section, we make the assumption that the system is stabilizable and detectable.

Lemma 5.1. Let Q be the real symmetric solution to the Riccati equation

$$AQ + QA^T - QC^T CQ + BB^T = 0.$$

such that $A - QC^TC$ is Hurwitz. Then $G(s) = M^{-1}(s)N(s)$ is a normalized coprime factorization, where N and M are the transfer matrices of the systems Σ_N and Σ_M respectively, which are given by

$$\Sigma_N = (A - QC^T C, B, C, 0),$$

$$\Sigma_M = (A - QC^T C, -QC^T, C, I).$$

This leads to the following expressions for N and M:

$$N(s) = C(sI - A + QC^{T}C)^{-1}B,$$

$$M(s) = -C(sI - A + QC^{T}C)^{-1}QC^{T} + I,$$
(5.1)

Now the idea is to perturb the transfer matrix G by perturbing M and N with the proper stable matrices Δ_M and Δ_N , respectively. Let G_{Δ} denote the transfer matrix resulting from this perturbation:

$$G_{\Delta} := (M + \Delta_M)^{-1} (N + \Delta_N).$$

We will show that we can represent G_{Δ} as the feedback interconnection around an *auxiliary* plant. We define this plant as follows:

$$\dot{x} = Ax + Bu + QC^{T}d,$$

$$y = Cx + d,$$

$$z = \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d = \begin{pmatrix} y \\ u \end{pmatrix}.$$
(5.2)

Around this system, we will use the following feedback

$$d = \begin{pmatrix} -\Delta_M & \Delta_N \end{pmatrix} z. \tag{5.3}$$

Lemma 5.2. The transfer matrix from u to y obtained by interconnecting (5.2) with (5.3) is equal to $G_{\Delta} = (M + \Delta_M)^{-1}(N + \Delta_N)$.

Proof. We simply rewrite and substitute the equations of (5.2) and (5.3) to obtain different equations that represent the same interconnection:

$$\dot{x} = (A - QC^{T}C)x + Bu + QC^{T}y,$$

$$d = y - Cx,$$

$$z = \begin{pmatrix} y \\ u \end{pmatrix},$$

$$d = -\Delta_{M}y + \Delta_{N}u.$$

(5.4)

Now we apply the Laplace transformation to these equations. Let $\hat{x}, \hat{u}, \hat{y}, \hat{d}$ and \hat{z} denote the Laplace transforms of x, u, y, d and z, respectively. We obtain

$$\hat{x} = (sI - A + QC^T C)^{-1} B\hat{u} + QC^T (sI - A + QC^T C)^{-1} \hat{y},$$

$$\hat{d} = \hat{y} - C\hat{x},$$

$$\hat{z} = \begin{pmatrix} \hat{y} \\ \hat{u} \end{pmatrix},$$

$$\hat{d} = -\Delta_M(s)\hat{y} + \Delta_N(s)\hat{u}.$$

Substituting the equation for \hat{x} in \hat{d} and combining these with (5.1) yields

$$\hat{d} = -C(sI - A + QC^{T}C)^{-1}B\hat{u} + [-C(sI - A + QC^{T}C)^{-1}QC^{T} + I]\hat{y},$$

= -N(s) $\hat{u} + M(s)\hat{y}.$

And finally we combine these last two expressions of \hat{d} to obtain

$$\hat{d} = M(s)\hat{y} - N(s)\hat{u} = -\Delta_M(s)\hat{y} + \Delta_N(s)\hat{u},$$
$$[M(s) + \Delta_M(s)]\hat{y} = [N(s) + \Delta_N(s)]\hat{u}.$$

So the transfer matrix from u to y is given by $\hat{y} = G_{\Delta}(s)\hat{u}$, as was claimed.

In this chapter, we have obtained that a coprime factor perturbed system can be represented by an interconnection of a plant (5.2) with the feedback (5.3). This leads us to the conclusion that perturbing a transfer matrix by a coprime factor perturbations again results in a linear system, see [11], [3], [9]. In the next chapter, we will use these results, combined with those from the previous chapters to show how one can construct dynamic observer based protocols that robustly synchronize networks with coprime factor perturbed agents.

Robust synchronization

As noted before, for the given network, we assume that (A, B) is stabilizable and (C, A) is detectable. Again the unperturbed agent dynamics is given by

$$\dot{x}_i = Ax_i + Bu_i, \ y_i = Cx_i, \ i = 1, 2, \dots, p.$$

The agents have identical transfer matrices given by $G(s) = C(sI - A)^{-1}B$. From the previous section, we obtain that there exists a normalized coprime factorization of $G = M^{-1}N$. For each agent, we consider perturbations of the transfer matrix G of the form

$$G_{(-\Delta_M \ \Delta_N)} = (M + \Delta_M)^{-1} (N + \Delta_N), \tag{6.1}$$

where $\Delta_M, \Delta_N \in \mathcal{RH}_{\infty}$ are proper and stable real rational transfer matrices. In this chapter we allow all such perturbations with $||(-\Delta_M \Delta_N)||_{\infty} < \gamma$, where $\gamma > 0$ is a given uncertainty radius. Under these conditions, the agent dynamics remain identical, but are not known exactly. From Lemma 5.2 we obtain that the perturbed dynamics of agent *i* can be represented by the interconnection of the plant

$$\dot{x}_i = Ax_i + Bu_i + QC^T d_i,$$

$$y_i = Cx_i + d_i,$$

$$z_i = \begin{pmatrix} C \\ 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ I \end{pmatrix} u_i + \begin{pmatrix} I \\ 0 \end{pmatrix} d_i.$$

with the feedback

$$d_i = \begin{pmatrix} -\Delta_M & \Delta_N \end{pmatrix} z_i.$$

Definition 6.1. Given a desired tolerance $\gamma > 0$, we say that the protocol robustly synchronizes the network if for all $\Delta_M, \Delta_N \in \mathcal{RH}_{\infty}$ with $|| (-\Delta_M \quad \Delta_N) ||_{\infty} < \gamma$ we have that for all i, j = 1, 2, ..., p

$$x_i(t) - x_j(t) \rightarrow 0, \ w_i(t) - w_j(t) \rightarrow 0$$

as $t \to \infty$. The tolerance γ is called the synchronization radius of the network.

In the rest of this section, we will sometimes denote $\Delta := \begin{pmatrix} -\Delta_M & \Delta_N \end{pmatrix}$ and simply write $d_i = \Delta z_i$.

For dealing with robust synchronization of the network, we will consider a modified version of protocol (4.2). We include a *weighting factor* on the Laplacian L of the network graph. This protocol is of the form

$$\dot{w}_{i} = Aw_{i} + BF \sum_{j \in \mathcal{N}_{i}} \frac{1}{N} (w_{i} - w_{j}) + G(\sum_{j \in \mathcal{N}_{i}} \frac{1}{N} (y_{i} - y_{j}) - Cw_{i}),$$

$$u_{i} = Fw_{i}.$$
(6.2)

Next to the gain matrices F and G, we need to determine the value of the positive real number N. Now, we will derive conditions under which, for a given synchronization radius $\gamma > 0$, there exists a robustly synchronizing protocol. In this report, we are concerned with synchronization of the nominal state components. The state components of the system that represents the perturbation are not under consideration.

Interconnecting the agents with this protocol yields a closed loop system. The dynamics of this system, the overall network dynamics, can be easily represented by taking $\mathbf{x} = \operatorname{col}(x_1, x_2, \ldots, x_p)$ and $\mathbf{w} = \operatorname{col}(w_1, w_2, \ldots, w_p)$ as the aggregate state vectors, $\mathbf{y} = \operatorname{col}(y_1, y_2, \ldots, y_p)$ and $\mathbf{u} = \operatorname{col}(u_1, u_2, \ldots, u_p)$ as the aggregate output and input vectors. The aggregate output and input vectors of the systems describing the perturbations are given by $\mathbf{d} = \operatorname{col}(d_1, d_2, \ldots, d_p)$ and $\mathbf{z} = \operatorname{col}(z_1, z_2, \ldots, z_p)$ respectively. We obtain

$$\dot{\mathbf{x}} = (I \otimes A)\mathbf{x} + (I \otimes B)\mathbf{u} + (I \otimes QC^T)\mathbf{d}$$
$$\dot{\mathbf{y}} = (I \otimes C)\mathbf{x} + (I \otimes I)\mathbf{d}$$

and

$$\mathbf{z} = (I \otimes \begin{pmatrix} C \\ 0 \end{pmatrix})\mathbf{x} + (I \otimes \begin{pmatrix} 0 \\ I \end{pmatrix})\mathbf{u} + (I \otimes \begin{pmatrix} I \\ 0 \end{pmatrix})\mathbf{d}$$

The closed loop dynamics is given by

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ \frac{1}{N}L \otimes GC & I \otimes (A - GC) + (\frac{1}{N}L \otimes BF) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} I \otimes QC^T \\ \frac{1}{N}L \otimes G \end{pmatrix} \mathbf{d},$$
$$\mathbf{z} = \begin{pmatrix} I \otimes \begin{pmatrix} C \\ 0 \end{pmatrix} & I \otimes \begin{pmatrix} 0 \\ F \end{pmatrix} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} I \otimes \begin{pmatrix} I \\ 0 \end{pmatrix} \end{pmatrix} \mathbf{d}.$$

and

$$\mathbf{d} = \begin{pmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta \end{pmatrix} \mathbf{z}$$

We apply the state transformation

$$\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} := \begin{pmatrix} U^T \otimes I & 0 \\ 0 & U^T \otimes I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}, \quad \tilde{\mathbf{z}} = (U^T \otimes I)\mathbf{z}, \text{ and } \tilde{\mathbf{d}} = (U^T \otimes I)\mathbf{d}$$

to obtain

$$\begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ \frac{1}{N}\Lambda \otimes GC & I \otimes (A - GC) + (\frac{1}{N}\Lambda \otimes BF) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} I \otimes QC^T \\ \frac{1}{N}\Lambda \otimes G \end{pmatrix} \tilde{\mathbf{d}}, \quad (6.3)$$

$$\tilde{\mathbf{z}} = \left(I \otimes \begin{pmatrix} C \\ 0 \end{pmatrix} \quad I \otimes \begin{pmatrix} 0 \\ F \end{pmatrix} \right) \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + \left(I \otimes \begin{pmatrix} I \\ 0 \end{pmatrix} \right) \tilde{\mathbf{d}}, \tag{6.4}$$

$$\tilde{\mathbf{d}} = (U^T \otimes I) \begin{pmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta \end{pmatrix} (U \otimes I) \tilde{\mathbf{z}} = (I \otimes \Delta) \tilde{\mathbf{z}}.$$
(6.5)

Analogously as Theorem 4.2 from [10], we have the following theorem:

Theorem 6.2. Let $\gamma > 0$. The following statements are equivalent:

1. The dynamic protocol (6.2) synchronizes the network with perturbed agents

$$\dot{x}_{i} = Ax_{i} + Bu_{i} + QC^{T}d_{i},$$

$$y_{i} = Cx_{i} + d_{i},$$

$$z_{i} = \begin{pmatrix} C \\ 0 \end{pmatrix}x_{i} + \begin{pmatrix} 0 \\ I \end{pmatrix}u_{i} + \begin{pmatrix} I \\ 0 \end{pmatrix}d_{i},$$

$$d_{i} = (-\Delta_{M} \Delta_{N})z_{i},$$
(6.6)

for all $(-\Delta_M \ \Delta_N) \in \mathcal{R}H_{\infty}$ with $||(-\Delta_M \ \Delta_N)||_{\infty} < \gamma$.

2. The perturbed system

$$\dot{x} = Ax + Bu + QC^{T}d,$$

$$y = Cx + d,$$

$$z = \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d,$$

$$d = (-\Delta_{M} \ \Delta_{N})z,$$
(6.7)

is internally stabilized for all $(-\Delta_M \ \Delta_N) \in \mathcal{R}H_{\infty}$ with $||(-\Delta_M \ \Delta_N)||_{\infty} < \gamma$ by all p-1 controllers

$$\dot{w} = Aw + Bu + G(y - Cw),$$

$$u = \frac{1}{N}\lambda_i Fw,$$
(6.8)

where i = 2, ..., p and λ_i is the *i*th eigenvalue of the Laplacian L.

Proof. In this proof, we again use the shorthand notation Δ for $(-\Delta_M \Delta_N)$.

(only if) We want to show that the interconnection with controller (6.8) stabilizes the plant (6.7) if the dynamic protocol (6.2) robustly synchronizes the network. Suppose the network is synchronized by (6.2) for all perturbations Δ with $||\Delta||_{\infty} < \gamma$. Take an arbitrary $\Delta \in \mathcal{R}H_{\infty}$ with $||\Delta||_{\infty} < \gamma$. We perturb each agent in the network (6.6) with Δ . Interconnecting (6.7) and (6.8) yields

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & \frac{1}{N}\lambda_i BF \\ GC & A - GC + \frac{1}{N}\lambda_i BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} QC^T \\ G \end{pmatrix} d,$$

$$z = \begin{pmatrix} C & 0 \\ 0 & \frac{1}{N}\lambda_i F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} d,$$

$$d = \Delta z.$$

$$(6.9)$$

We obtain that $\tilde{d} = (U^T \otimes I)(I \otimes \Delta)(U \otimes I)\tilde{z} = (I \otimes \Delta)\tilde{z}$ in (6.5). Since the network is robustly synchronized by the protocol, we have that $\tilde{x}_i \to 0$, $\tilde{w}_i \to 0$ as $t \to \infty$ for $i = 2, \ldots, p$ in (6.3). This implies that for $i = 2, \ldots, p$ the following systems are internally stable:

$$\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} = \begin{pmatrix} A & BF \\ \frac{1}{N}\lambda_i GC & A - GL + \frac{1}{N}\lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \begin{pmatrix} QC^T \\ \frac{1}{N}\lambda_i G \end{pmatrix} \tilde{d}_i,$$

$$\tilde{z}_i = \begin{pmatrix} C & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{d}_i,$$

$$\tilde{d}_i = \Delta \tilde{z}_i.$$

By the transformation $\tilde{w}_i = \frac{1}{N} \lambda_i \bar{w}_i$, we see that this system is equivalent with (6.9). Therefore, the systems are stable.

(if) Now assume the p-1 controllers (6.8) stabilize system (6.7) for all $\Delta \in \mathcal{RH}_{\infty}$ with $||\Delta||_{\infty} < \gamma$. Using the small gain theorem, we obtain that for $i = 2, 3, \ldots, p$ the closed loop systems (6.9) are internally stable and the transfer matrices G_i from d to zsatisfy $||G_i||_{\infty} \leq \frac{1}{\gamma}$. We show that the protocol (6.2) synchronizes the perturbed network for all perturbations Δ with $||\Delta||_{\infty} < \gamma$. We have to show that for $i = 2, 3, \ldots, p$ we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ as $t \to \infty$, where \tilde{x}_i and \tilde{w}_i satisfy (6.3), (6.4), and (6.5). Note that

$$(U^T \otimes I) \begin{pmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta \end{pmatrix} (U \otimes I) = (I \otimes \Delta).$$

We immediately obtain that the \mathcal{H}_{∞} -norm of the left hand side is less than γ . We will now give equations for the dynamics of $\tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_p$ and $\tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_p$. We denote $\bar{\mathbf{x}} = \operatorname{col}(\tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_p)$, $\bar{\mathbf{w}} = \operatorname{col}(\tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_p)$, $\bar{\mathbf{z}} = \operatorname{col}(\tilde{z}_2, \tilde{z}_3, \ldots, \tilde{z}_p)$, and $\bar{\mathbf{d}} = \operatorname{col}(\tilde{d}_2, \tilde{d}_3, \ldots, \tilde{d}_p)$. From (6.3) we obtain

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} I_{p-1} \otimes A & I_{p-1} \otimes BF \\ \frac{1}{N}\Lambda_1 \otimes GC & I_{p-1} \otimes (A - GC) + (\frac{1}{N}\Lambda_1 \otimes BF) \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} I_{p-1} \otimes QC^T \\ \frac{1}{N}\Lambda_1 \otimes G \end{pmatrix} \bar{\mathbf{d}},$$
$$\bar{\mathbf{z}} = (I_{p-1} \otimes \begin{pmatrix} C \\ 0 \end{pmatrix} & I_{p-1} \begin{pmatrix} 0 \\ F \end{pmatrix}) \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} + (I_{p-1} \otimes \begin{pmatrix} I \\ 0 \end{pmatrix}) d,$$
$$\bar{\mathbf{d}} = (I_{p-1} \otimes \Delta) \bar{\mathbf{z}}.$$

Here $\Lambda_1 := \operatorname{diag}(\lambda_2, \lambda_3, \dots, \lambda_p)$. The transfer matrix of this system from $\overline{\mathbf{d}}$ to $\overline{\mathbf{z}}$ is equal to $G := \operatorname{diag}(G_2, G_3, \dots, G_p)$. We obtain that $||G||_{\infty} \leq \frac{1}{\gamma}$. By applying the small gain

theorem, it follows that the system is internally stable and $\bar{\mathbf{x}}(t) \to 0$ and $\bar{\mathbf{w}}(t) \to 0$ as $t \to \infty$. Hence, the theorem is proved.

To obtain a protocol that robustly synchronizes the network for a desired synchronization radius $\gamma > 0$, it suffices to find the positive real number N and gains F and G such that the single linear system (6.7) is robustly internally stabilized by all p-1 controller (6.8) with stability radius γ . By applying the small gain theorem for this, it is required that all of the controllers (6.8) solve the \mathcal{H}_{∞} -control problem for the system (6.7) such that the closed loop system is internally stable and the transfer matrix G_i from d to zsatisfies $||G_i|| \leq \frac{1}{\gamma}$. In the next chapter, we will give a method to synthesize suitable N, F and G.

Robustly synchronizing protocols

In this section we will establish sufficient conditions for the existence of protocols that robustly synchronize the network for a desired synchronization radius $\gamma > 0$. Furthermore, we will present an algorithm to compute such protocols. From Theorem 6.2 it follows that protocol (6.2) robustly synchronizes the network if the perturbed agent dynamics is robustly internally stabilized by all p-1 controllers given by (6.8). As before, we assume that the pair (A, B) is stabilizable and that the pair (C, A) is detectable.

We consider the following Riccati equations, associated with (A, B, C):

$$A^{T}P + PA - PBB^{T}P + C^{T}C = 0, (7.1)$$

and

$$AQ + QA^T - QC^T CQ + BB^T = 0. ag{7.2}$$

It is well-known that there exist unique real symmetric positive semi-definite solutions P and Q to these equations such that $A - BB^T P$ and $A - QC^T C$ are Hurwitz. These solutions are called the *stabilizing solutions*.

Recall that the second smallest and the largest eigenvalue of the Laplacian L are denoted by λ_2 and λ_p , respectively. Furthermore, under the condition of connectedness, we have $\lambda_2 > 0$. In this section, we will state a theorem that yields a robustly synchronizing protocol for the network with perturbed agents. The synchronization radius that can be attained depends on the eigenvalues of L and the spectral radius $\lambda_{\max}(PQ)$ of the product of the stabilizing solutions to (7.1) and (7.2).

Before doing this, we will prove a lemma that will be instrumental in the proof of our main theorem:

Lemma 7.1. Let $\rho > 0$ be such that $\rho^2 < \frac{1}{1+\lambda_{\max}(PQ)}$. Then $(\frac{1}{\rho^2}-1)I-PQ$ is nonsingular. Temporarily denote

$$\tilde{P} = \left(\left(\frac{1}{\rho^2} - 1\right)I - PQ \right)^{-1} P.$$
(7.3)

Then \tilde{P} is a real symmetric positive semi-definite solution of the algebraic Riccati equation

$$A^{T}\tilde{P} + \tilde{P}A + \rho^{2}C^{T}C - \frac{1}{\rho^{2}}\tilde{P}BB^{T}\tilde{P} + \frac{1}{1-\rho^{2}}(\tilde{P}Q + \rho^{2}I)C^{T}C(Q\tilde{P} + \rho^{2}I) = 0.$$
(7.4)

Proof. First, if $\rho^2 < \frac{1}{1+\lambda_{\max}(PQ)}$, then $\frac{1}{\rho^2} > 1 + \lambda_{\max}(PQ) = \lambda_{\max}(I+PQ)$, so the matrix

$$(\frac{1}{\rho^2} - 1)I - PQ = \frac{1}{\rho^2}I - (I + PQ)$$

is nonsingular and \tilde{P} exists. Furthermore, we have that $\tilde{P} = \tilde{P}^T \ge 0$. Now we show that \tilde{P} is a solution of the algebraic Riccati equation (7.4). Substitute (7.3) in (7.4) and pre-multiply the result with $W := (\frac{1}{\rho^2} - 1)I - PQ$ and post-multiply with W^T to obtain

$$\begin{split} &((\frac{1}{\rho^2}-1)I-PQ)A^TP+PA((\frac{1}{\rho^2}-1)I-QP)-\frac{1}{\rho^2}PBB^TP\\ &+\rho^2((\frac{1}{\rho^2}-1)I-PQ)C^TC(((\frac{1}{\rho^2}-1)I-QP)\\ &+\frac{1}{1-\rho^2}((1-\rho^2)[PQ+I]))C^TC((1-\rho^2)[QP+I]))\\ =\\ &=\\ &(\frac{1}{\rho^2}-1)[A^TP+PA]-P[QA^T+AQ]P-\frac{1}{\rho^2}PBB^TP\\ &+\rho^2[(\frac{1}{\rho^4}-2\frac{1}{\rho^2}+1)C^TC-(\frac{1}{\rho^2}-1)PQC^TC-(\frac{1}{\rho^2}-1)C^TCQP+PQC^TCQP]\\ &+(1-\rho^2)(C^TC+PQC^TC+C^TCQP+PQC^TCQP)\\ =\\ &(\frac{1}{\rho^2}-1)[A^TP+PA]-P[QA^T+AQ]P-\frac{1}{\rho^2}PBB^TP\\ &+(\frac{1}{\rho^2}-2+\rho^2)C^TC-(1-\rho^2)PQC^TC-(1-\rho^2)C^TCQP+\rho^2PQC^TCQP\\ &+(1-\rho^2)(C^TC+PQC^TC+C^TCQP+PQC^TCQP)\\ =\\ &(\frac{1}{\rho^2}-1)[A^TP+PA+C^TC]-P[QA^T+AQ-QC^TCQ]P-\frac{1}{\rho^2}PBB^TP\\ =\\ &(\frac{1}{\rho^2}-1)[A^TP+PA+C^TC]-P[QA^T+AQ-QC^TCQ]P\\ &-\frac{1}{\rho^2}PBB^TP+PBB^TP-PBB^TP\\ =\\ &(\frac{1}{\rho^2}-1)[A^TP+PA-PBB^TP+C^TC]-P[QA^T+AQ-QC^TCQ]P\\ &=\\ &(\frac{1}{\rho^2}-1)[A^TP+PA-PBB^TP+C^TC]-P[QA^T+AQ-QC^TCQ]P$$

We conclude that \tilde{P} in (7.3) is a solution to (7.4).

We now formulate our main result. This theorem provides a robustly synchronizing protocol for the network with perturbed agent dynamics for all perturbations with bounded \mathcal{H}_{∞} -norm. **Theorem 7.2.** Consider the network with p agents, where the perturbed dynamics of agent i is given by

$$\dot{x}_i = Ax_i + Bu_i + QC^T d_i,$$

$$y_i = Cx_i + d_i,$$

$$z_i = \begin{pmatrix} C \\ 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ I \end{pmatrix} u_i + \begin{pmatrix} I \\ 0 \end{pmatrix} d_i$$

with $d_i = (-\Delta^M \ \Delta^N) z_i$. Choose N any real number such that

$$N > \lambda_p.$$

Let $\gamma > 0$ be such that

$$\gamma^2 < \frac{\lambda_2}{N} \frac{1}{1 + \lambda_{\max}(PQ)}.\tag{7.5}$$

Then $(\frac{\lambda_2}{N\gamma^2}-1)I-PQ$ is nonsingular. Define

$$\tilde{P} := \left(\left(\frac{\lambda_2}{N\gamma^2} - 1 \right) I - PQ \right)^{-1} P, \tag{7.6}$$

$$F := -\frac{1}{\gamma^2} B^T \tilde{P} \tag{7.7}$$

$$G := QC^T. (7.8)$$

Then the dynamic protocol (6.2) with N, F and G as above robustly synchronizes the network for all perturbations $(-\Delta^M \ \Delta^N) \in \mathcal{RH}_{\infty}$ with $||(-\Delta^M \ \Delta^N)||_{\infty} < \gamma$.

Proof. Again denote $(-\Delta^M \ \Delta^N)$ by Δ . By Theorem 6.2, it suffices to prove that the p-1 controllers (6.8) with N, F and G chosen as above, internally stabilize the system

$$\begin{aligned} \dot{x} &= Ax + Bu + QC^{T}d, \\ y &= Cx + d, \\ z &= \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \end{aligned}$$

while $||G_i||_{\infty} \leq \frac{1}{\gamma}$, where G_i is the transfer matrix from d to z (i = 2, 3, ..., p) in the closed loop system given by (6.9). To show that this is indeed the case, we apply the following state transformation to the closed loop system (6.9):

$$\begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}.$$

This results in

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}} \end{pmatrix} = \begin{pmatrix} A + \mu_i BF & -\mu_i BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} QC^T \\ QC^T - G \end{pmatrix} d,$$

$$z = \begin{pmatrix} C & 0 \\ \mu_i F & -\mu_i F \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} d,$$

$$(7.9)$$

where $\mu_i := \frac{\lambda_i}{N}$ for i = 2, 3, ..., p. To proceed, we will apply Lemma 2.2 to the system (7.9). We will show that there exists a real symmetric positive semi-definite solution to the Riccati inequality associated with the system. As before, denote

$$\tilde{P} = \left(\left(\frac{\lambda_2}{N\gamma^2} - 1 \right) I - PQ \right)^{-1} P, \tag{7.10}$$

and apply Lemma 7.1 with $\rho = \frac{\sqrt{N\gamma}}{\sqrt{\lambda_2}}$. We find that \tilde{P} is a solution to the following ARE

$$A^{T}\tilde{P} + \tilde{P}A + \frac{N\gamma^{2}}{\lambda_{2}}C^{T}C - \frac{\lambda_{2}}{N\gamma^{2}}\tilde{P}BB^{T}\tilde{P} + \frac{\frac{\lambda_{2}}{N}}{\frac{\lambda_{2}}{N} - \gamma^{2}}(\tilde{P}Q + \frac{N\gamma^{2}}{\lambda_{2}}I)C^{T}C(Q\tilde{P} + \frac{N\gamma^{2}}{\lambda_{2}}I) = 0.$$

$$(7.11)$$

We now return to the closed loop system (7.9). For ease of notation, we label the system matrices as

$$\tilde{A}_{i} = \begin{pmatrix} A + \mu_{i}BF & -\mu_{i}BF \\ 0 & A - GC \end{pmatrix}, \qquad \tilde{B}_{i} = \begin{pmatrix} QC^{T} \\ QC^{T} - G \end{pmatrix},$$
$$\tilde{C}_{i} = \begin{pmatrix} C & 0 \\ \mu_{i}F & -\mu_{i}F \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Take the controller and observer gains F and G as defined in (7.7) and (7.8) respectively. Now we will apply Lemma 2.2 to show that $||G_i||_{\infty} \leq \frac{1}{\gamma}$, where G_i is the transfer function from d to z (i = 2, 3, ..., p). We will show that there exists a suitable choice of $Z_i \geq 0$ such that Z_i is a solution to the algebraic Riccati inequality (2.2) with $\rho = \frac{1}{\gamma}$:

$$\tilde{A}_i^T Z_i + Z \tilde{A}_i + \tilde{C}_i^T \tilde{C}_i - \frac{\gamma^2}{1 - \gamma^2} (Z_i \tilde{B}_i + \tilde{C}_i^T \tilde{D}) (\tilde{B}_i^T Z_i + \tilde{D}^T \tilde{C}_i) \le 0.$$

$$(7.12)$$

Lemma 2.2 requires that the matrices \tilde{A}_i (i = 2, 3, ..., p) are Hurwitz. The set of eigenvalues of \tilde{A}_i is the union of those of $A + \mu_i BF$ and A - GC. We immediately see that $A - GC = A - QC^T C$ is Hurwitz. Showing that $A + \mu_i BF$ is Hurwitz for i = 2, 3, ..., p can be done in a similar way. Now for i = 2, 3, ..., p, let Y_i be the unique positive semi-definite solution to the Lyapunov equation

$$Y_i(A - QC^T C) + (A - QC^T C)^T Y_i + \frac{\mu_i^2}{\gamma^4} \tilde{P}BB^T \tilde{P} = 0.$$

Next, take

$$Z_i := \begin{pmatrix} \frac{\mu_i}{\gamma^2} \tilde{P} & 0\\ 0 & Y_i \end{pmatrix}.$$
(7.13)

Compute the following terms in (7.12):

$$\begin{split} Z_i \tilde{A}_i &= \begin{pmatrix} \frac{\mu_i}{\gamma^2} \tilde{P} & 0\\ 0 & Y_i \end{pmatrix} \begin{pmatrix} A - \frac{\mu_i}{\gamma^2} BB^T \tilde{P} & \frac{\mu_i}{\gamma^2} BB^T \tilde{P}\\ 0 & A - QC^T C \end{pmatrix};\\ &= \begin{pmatrix} \frac{\mu_i}{\gamma^2} \tilde{P} A - \frac{\mu_i^2}{\gamma^4} \tilde{P} BB^T \tilde{P} & \frac{\mu_i^2}{\gamma^4} \tilde{P} BB^T \tilde{P}\\ 0 & Y_i (A - QC^T C) \end{pmatrix},\\ \tilde{C}_i^T \tilde{C}_i &= \begin{pmatrix} C^T C + \frac{\mu_i^2}{\gamma^4} \tilde{P} BB^T \tilde{P} & -\frac{\mu_i^2}{\gamma^4} \tilde{P} BB^T \tilde{P}\\ -\frac{\mu_i^2}{\gamma^4} \tilde{P} BB^T \tilde{P} & \frac{\mu_i^2}{\gamma^4} \tilde{P} BB^T \tilde{P} \end{pmatrix},\\ Z_i \tilde{B}_i + \tilde{C}_i^T \tilde{D} &= \begin{pmatrix} (\frac{\mu_i}{\gamma^2} \tilde{P} Q + I) C^T\\ 0 \end{pmatrix}. \end{split}$$

Performing the addition in (7.12) we obtain a symmetric 2×2 block matrix. We will show that this matrix is negative semi-definite. In our block matrix the upper right and lower left corners are given by

$$\frac{\mu_i^2}{\gamma^4}\tilde{P}BB^T\tilde{P} - \frac{\mu_i^2}{\gamma^4}\tilde{P}BB^T\tilde{P} = 0$$

and the lower right corner is given by

$$Y_i(A - QC^T C) + (A - QC^T C)^T Y_i + \frac{\mu_i^2}{\gamma^4} \tilde{P} B B^T \tilde{P},$$

which is equal to zero. Since the upper right and lower left corners are zero, the only thing that remains to be shown is that the upper left corner of the matrix is negative semi-definite. This upper left corner is given by

$$\frac{\mu_i}{\gamma^2} \left[\tilde{P}A + A^T \tilde{P} - \frac{\mu_i}{\gamma^2} \tilde{P}BB^T \tilde{P} + \frac{\gamma^2}{\mu_i} C^T C \right]
+ \frac{\gamma^2}{1 - \gamma^2} \left(\frac{\mu_i}{\gamma^2} \tilde{P}Q + I \right) C^T C \left(Q \frac{\mu_i}{\gamma^2} \tilde{P} + I \right)
= \frac{\mu_i}{\gamma^2} \left[\tilde{P}A + A^T \tilde{P} - \frac{\mu_i}{\gamma^2} \tilde{P}BB^T \tilde{P} + \frac{\gamma_2}{\mu_i} C^T C
+ \frac{\mu_i}{\gamma^2} \frac{\gamma^2}{1 - \gamma^2} \left(\tilde{P}Q + \frac{\gamma^2}{\mu_i} I \right) C^T C \left(Q \tilde{P} + \frac{\gamma^2}{\mu_i} I \right) \right].$$
(7.14)

We now show that for N, F and G as chosen above, this matrix is negative semi-definite. For ease of notation, we denote

$$\eta := \frac{\lambda_2}{N}.$$

Now (7.14) can be rewritten as

$$\begin{split} \frac{\mu_i}{\gamma^2} & \left[\tilde{P}A + A^T \tilde{P} + \frac{\gamma^2}{\eta} C^T C + (\frac{\gamma^2}{\mu_i} - \frac{\gamma^2}{\eta}) C^T C \right. \\ & - \frac{\eta}{\gamma^2} \tilde{P}BB^T \tilde{P} + (\frac{\eta - \mu_i}{\gamma^2}) \tilde{P}BB^T \tilde{P} \\ & + \frac{\eta}{\eta - \gamma^2} (\tilde{P}Q + \frac{\gamma^2}{\mu_i} I) C^T C (Q\tilde{P} + \frac{\gamma^2}{\mu_i} I) \\ & + \frac{\mu_i}{1 - \gamma^2} (\tilde{P}Q + \frac{\gamma^2}{\eta} I) C^T C (Q\tilde{P} + \frac{\gamma^2}{\eta} I) \\ & - \frac{\eta}{\eta - \gamma^2} (\tilde{P}Q + \frac{\gamma^2}{\eta} I) C^T C (Q\tilde{P} + \frac{\gamma^2}{\eta} I) \\ \end{split}$$

Applying Lemma 7.1 with $\rho = \frac{\gamma \sqrt{N}}{\sqrt{\lambda_2}}$, it follows from (7.11) that this is equal to

$$\frac{\mu_i}{\gamma^2} \left[\gamma^2 \left(\frac{1}{\mu_i} - \frac{1}{\eta} \right) C^T C + \frac{1}{\gamma^2} (\eta - \mu_i) \tilde{P} B B^T \tilde{P} + \left(\frac{\mu_i}{1 - \gamma^2} \frac{\gamma^4}{\mu_i^2} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^4}{\eta^2} \right) C^T C + \left(\frac{\mu_i}{1 - \gamma^2} \frac{\gamma^2}{\mu_i} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^2}{\eta} \right) (\tilde{P} Q C^T C + C^T C Q \tilde{P}) + \left(\frac{\mu_i}{1 - \gamma^2} - \frac{\eta}{\eta - \gamma^2} \right) \tilde{P} Q C^T C Q \tilde{P} \right].$$
(7.15)

Define

$$\begin{aligned} \alpha_i &:= \frac{\mu_i}{1 - \gamma^2} \frac{\gamma^4}{\mu_i^2} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^4}{\eta^2} - \gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}), \\ \beta_i &:= \frac{\mu_i}{1 - \gamma^2} \frac{\gamma^2}{\mu_i} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^2}{\eta}, \\ \delta_i &:= \frac{\mu_i}{1 - \gamma^2} - \frac{\eta}{\eta - \gamma^2}. \end{aligned}$$

Then the expression (7.15) can be rewritten as

$$\frac{\mu_i}{\gamma^2} \begin{bmatrix} \frac{1}{\gamma^2} (\eta - \mu_i) \tilde{P} B B^T \tilde{P} \\
+ \begin{pmatrix} I & \tilde{P} Q \end{pmatrix} \begin{pmatrix} \alpha_i C^T C & \beta_i C^T C \\ \beta_i C^T C & \delta_i C^T C \end{pmatrix} \begin{pmatrix} I \\ Q \tilde{P} \end{pmatrix} \end{bmatrix}$$
(7.16)

Note that since $\lambda_2 \leq \lambda_i$, we have that $\eta \leq \mu_i$ for $i = 2, 3, \ldots, p$. For the first term in (7.16) we obtain $\eta - \mu_i \leq 0$ and consequently

$$\frac{1}{\gamma^2}(\eta - \mu_i)\tilde{P}BB^T P \le 0.$$

Thus, to prove that the matrix in expression (7.15) is negative semi-definite, we only need to consider the second term in (7.16). Note that

$$\begin{pmatrix} \alpha_i C^T C & \beta_i C^T C \\ \beta_i C^T C & \delta_i C^T C \end{pmatrix} = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & \delta_i \end{pmatrix} \otimes C^T C.$$

So to prove that the second term in (7.16) is negative semi-definite, it suffices to show that

$$\begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & \delta_i \end{pmatrix} \le 0.$$

We will show this is indeed the case.

First, we show that for all i = 2, 3, ..., p we have $\alpha_i < 0$ and $\delta_i < 0$. Indeed, we have

$$\delta_{i} < 0 \Leftrightarrow \frac{\mu_{i}}{1 - \gamma^{2}} < \frac{\eta}{\eta - \gamma^{2}}$$
$$\Leftrightarrow \mu_{i}(\eta - \gamma^{2}) < \eta(1 - \gamma^{2})$$
$$\Leftrightarrow \mu_{i}\eta - \mu_{i}\gamma^{2} < \eta - \eta\gamma^{2}.$$
(7.17)

Since $\eta \leq \mu_i$ we already have

$$-\mu_i \gamma^2 \le -\eta \gamma^2.$$

And finally, since $N > \lambda_p$, we also have $\mu_i < 1$, which indeed proves $\delta_i < 0$. Next,

$$\begin{aligned} \alpha_{i} &= \frac{\mu_{i}}{1 - \gamma^{2}} \frac{\gamma^{4}}{\mu_{i}^{2}} - \frac{\eta}{\eta - \gamma^{2}} \frac{\gamma^{4}}{\eta^{2}} - \gamma^{2} (\frac{1}{\eta} - \frac{1}{\mu_{i}}), \\ &\leq \frac{\gamma^{4}}{\eta^{2}} (\frac{\mu_{i}}{1 - \gamma^{2}} - \frac{\eta}{\eta - \gamma^{2}}) - \gamma^{2} (\frac{1}{\eta} - \frac{1}{\mu_{i}}), \\ &< 0. \end{aligned}$$

The first inequality uses that $\frac{\gamma^4}{\mu_i^2} \leq \frac{\gamma^2}{\eta^2}$ (which follows from $\eta \leq \mu_i$), and the second, strict, inequality follows again from $\mu_i < 1$.

Having proved that $\alpha_i < 0$, $\delta_i < 0$ for all *i*, we now have that

$$\begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & \delta_i \end{pmatrix} \le 0$$

if and only if $\alpha_i \delta_i - \beta_i^2 \ge 0$. We will check that this indeed holds.

$$\begin{split} \alpha_i \delta_i - \beta_i^2 &= -\gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}) (\frac{\mu_i}{1 - \gamma^2} - \frac{\eta}{\eta - \gamma^2}) - \frac{\mu_i}{1 - \gamma^2} \frac{\eta}{\eta - \gamma^2} \gamma^4 (\frac{1}{\eta} - \frac{1}{\mu_i})^2, \\ &= -\gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}) \left[\frac{\mu_i}{1 - \gamma^2} - \frac{\eta}{\eta - \gamma^2} + \frac{\mu_i}{1 - \gamma^2} \frac{\eta}{\eta - \gamma^2} \gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}) \right], \\ &= -\gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}) \left[\frac{\mu_i (\eta - \gamma^2) - \eta (1 - \gamma^2) + \mu_i \eta \gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i})}{(1 - \gamma^2)(\eta - \gamma^2)} \right], \\ &= -\gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}) \frac{1}{(1 - \gamma^2)(\eta - \gamma^2)} \left[\mu_i (\eta - \gamma^2) - \eta (1 - \gamma^2) + \mu_i \gamma^2 - \eta \gamma^2 \right], \\ &= -\gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}) \frac{1}{(1 - \gamma^2)(\eta - \gamma^2)} \left[\mu_i \eta - \mu_i \gamma^2 - \eta + \eta \gamma^2 + \mu_i \gamma^2 - \eta \gamma^2 \right], \\ &= -\gamma^2 (\frac{1}{\eta} - \frac{1}{\mu_i}) \frac{1}{(1 - \gamma^2)(\eta - \gamma^2)} \eta (\mu_i - 1), \\ &\geq 0, \end{split}$$

since $-\gamma^2(\frac{1}{\eta}-\frac{1}{\mu_i}) \leq 0$, $(1-\gamma^2)(\eta-\gamma^2) > 0$, and $\mu_i - 1 < 0$. Thus, we conclude that our choice for Z_i in (7.13) is indeed a solution of (7.12), so by Lemma 2.2 we obtain that for $i = 2, 3, \ldots, p$ the transfer matrix G_i of (7.9) satisfies $||G_i|| \leq \frac{1}{\gamma}$. Finally, we apply Theorem 6.2 and obtain that the dynamic protocol robustly synchronizes the network. \Box

Remark 7.3. Thus, summing up the steps in the proof above, we now have a conceptual algorithm for constructing a robustly synchronizing protocol. Given a network with coprime factor perturbed agents, we can compute a protocol that robustly synchronizes the network as follows

- 1. Choose N such that $N > \lambda_p$,
- 2. Compute the stabilizing solution P to (7.1) and Q to (7.2),
- 3. Choose a value for the synchronization radius γ such that

$$\gamma^2 < \frac{\lambda_2}{N} \frac{1}{1 + \lambda_{\max}(PQ)}$$

4. Compute F and G as in (7.7) and (7.8), respectively.

Using these steps, we obtain a dynamic protocol that robustly synchronizes the network and achieves the synchronization radius γ .

7.1 Guaranteed synchronization radius

We will now provide a guaranteed synchronization radius for the network. The guaranteed synchronization radius is the supremum over all possible $\gamma > 0$, such that there exist a dynamic protocol of the form (6.2) that synchronizes the network for all Δ with $||\Delta||_{\infty} < \gamma$. Under the above steps, after choosing N and computing P and Q, (7.5) gives an upper bound for γ :

$$\gamma < \sqrt{\frac{\lambda_2}{N} \frac{1}{\sqrt{1 + \lambda_{\max}(PQ)}}}.$$
(7.18)

To maximize our upper bound, we want to choose N as small as possible. However, N is bounded from below by λ_p . So while we would like to choose N as λ_p , we can only choose N arbitrarily close to λ_p . It turns out that this is not a problem. We will show this in the following corollary.

Corollary 7.4. For every γ such that

$$\gamma < \sqrt{\frac{\lambda_2}{\lambda_p}} \frac{1}{\sqrt{1 + \lambda_{\max}(PQ)}},\tag{7.19}$$

there exists a dynamic protocol that robustly synchronizes the network with synchronization radius γ .

Proof. From (7.19) it follows that

$$\lambda_p < \frac{\lambda_2}{\gamma^2} \frac{1}{\sqrt{1 + \lambda_{\max}(PQ)}}.$$

Now choose N such that

$$\lambda_p < N < \frac{\lambda_2}{\gamma^2} \frac{1}{\sqrt{1 + \lambda_{\max}(PQ)}}$$

Then N satisfies (7.18) so the protocol with this choice of N and with F and G as in (7.7) and (7.8) achieves synchronization radius γ .

Conclusions

In this report, we dealt with the problem of robust synchronization of undirected multiagent networks with uncertain agent dynamics. We introduced network observers and provided state space equations for network observers for the absolute and relative state of the agents. We proved that a necessary and sufficient condition for the existence of such observers is detectability of the agent dynamics. These observers where then used in the construction of synchronizing and robustly synchronizing protocols.

Given an undirected network with identical nominal dynamics for each agent, we allow uncertainty in the form of coprime factor perturbations of the transfer matrices of each agent's dynamics. We assume these perturbations are stable and have \mathcal{H}_{∞} -norm that is bounded by some a priori desired tolerance. We provided a method to compute dynamic observer based protocols that, for a given tolerance, robustly synchronize the network for all admissible perturbations. We have shown that robust synchronization of the network by the dynamic protocol is equivalent with robust stabilization of a single linear system by all controllers from a related finite set of feedback controllers. The protocols which we provide are expressed in terms of stabilizing, real symmetric positive semi-definite solutions to certain algebraic Riccati equations, and contain weighting factors depending on the eigenvalues of the graph Laplacian. In this class of dynamic protocols, we have shown that one can achieve a guaranteed tolerance that is proportional to the square root of the quotient of the smallest and the largest eigenvalue of the graph Laplacian.

This report solves the problem of robust synchronization of networked multi-agent systems with coprime factor perturbed agent dynamics. However, the problem of heterogeneity remains to be solved. In this report we assume that all agent dynamics are perturbed by the same coprime factor perturbation, which results in uncertain but identical dynamics for each agent in the network. Allowing different coprime factor perturbations for each agent results in a heterogeneous network with varying agent dynamics. A solution to this problem would provide a generalization of the results in this report. Another interesting problem which we have not touched in our report would be the tracking problem in the context of networked multi-agent systems with uncertain dynamics. The tracking problem, better known as the servo problem, is to construct a controller such that a given system follows an external reference signal. In the context of networks we want all agents to follow the reference signal while only allowing local exchange of information. Finally, an extension of the results obtained in this report to nonlinear perturbations also provides opportunities for further research.

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